AMERICAN UNIVERSITY OF BEIRUT

# LEAST GRADIENT PROBLEM

by

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# An Abstract of the Thesis of

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If $f$ is a given function defined on the boundary $\partial\Omega$ of a domain $\Omega$ in d-dimensional Euclidean space, the least gradient problem (LGP) asks for the following: among all functions $u$ in the space $BV(\Omega)$, and having boundary values equal to $f$, does there exist a function that minimizes the set of all $L^1$ norms of the gradients of such functions? Furthermore, if such a minimizer exists, what further smooth and minimizing properties does it have? The purpose of this thesis is to study this problem in the two dimensional case, where $\Omega$ is strictly convex, and to explore the situation where $\Omega$ is only convex.

The exposition will present a study of level sets of minimizers, as well as the connection, through the co-area theorem, between the properties of those level sets and the minimizing function.

# Contents

# Chapter 1

# Introduction

A lot of research has been done over the years on the least gradient problem.It continues to be a problem of great interest, particularly because there are still subtle questions regarding this problem.

We note that the least gradient problem (LGP) is defined on $\mathbb{R}^n$ for $n \geqslant 1$. In fact, in case of a particular class of boundary data, a particular domain structure, the existence of a continuous solution was achieved upon construction. Indeed, we will later observe in this exposition that the continuity of the solution depends on the continuity of boundary data and structure of domain.

First, uniqueness of the obtained solution was achieved by Stenberg,Williams, and Ziemer, [1], with the fact that Bombieri, De Giorgi, and Giusti,[2], demonstrated that the level sets are minimal surfaces. Since minimal surfaces are solutions of a special differential equation, and because differential equation solutions are unique, the uniqueness of the constructed LGP solution has been defined.

However, a different approach is given in [3] to develop the uniqueness of a solution. This approach focuses on the construction of level sets of solutions and the fact that they are minimal surfaces.

In chapter 2, we will introduce some basic notions that are of great help to understand the set of functions in which we will be working with in the LGP. Furthermore, we will define the notion of perimeter and the co-area formula which will help us in achieving the existence of a solution.

In chapter 3, we first introduce the LGP and observe the construction of a solution obtained for $\Omega \subset \mathbb{R}^2$ plane domain strictly convex with $C^1$ lipchitz boundary and boundary data $f$ continuous on a part $\Gamma$ of the boundary of $\Omega$. After illustrating the proof given in [3] that the constructed function is indeed a solution, we aim on showing its uniqueness when $f$ satisfies some monotonicity condition on $\Gamma$ also given by [3].

We then go further in providing one of the most important results to be given in [3] which is the existence and uniqueness of a solution when $\Omega$ is only convex and not strictly convex.

Moreover, we will show a relation achieved by Gorny,Rybka and Sabra, [3], between the least gradient problem and a problem in free material design.

Then, examples will follow to illustrate this connection.

Lastly, in chapter 4, we define namely the constrained least gradient problem. We now work in $\mathbb{R}^n$, $n \geqslant 1$. Similar to the LGP, the constrained least gradient problem is the LGP with an additional constraint. This constraint requires the solutions to be lipchitz. Also, we will observe the construction of solutions given by [4] while noticing that the additional constraint will affect somehow the choice of sets taken in the construction.

# Chapter 2

# Preliminary

As we aim later on introducing the least gradient problem and tend to solve it as was done in [3], we realize that the set of functions in which the least gradient problem is defined is for functions of bounded variation. Therefore, we first aim on reviewing some basic definitions and some important theorems that will help in solving the least gradient problem.

In this chapter, we review the classical notion of a function of bounded variation, and consider its connection to differentiability. We will review this notion in one dimension and then in higher dimensions. Also, we will define a new notion of bounded variation and establish a relation between the classical notion and the new one. As well, we will define a new notion of perimeter of measurable sets and establish an important formula to be used later in this paper called the coarea formula.

## 2.1 Bounded Variation in $\mathbb{R}$

**Definition 2.1.1.** Let $f : [a, b] \mapsto \mathbb{R}$

We define the essential variation of $f$ to be:

$$essV_a^b f = sup\{\sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|\,; a = t_0 < t_1 < ... < t_{n-1} < t_n = b\, partition\, of\, [a, b]\}$$

where each $t_i$ a point of approximate continuity of $f$ $\forall i = 0, .., n$.

According to Tonelli we say $f$ is of bounded variation and denote it by
$f \in BV(a, b) \iff essV_a^b f < \infty$.

**Example** If $f : [a, b] \mapsto \mathbb{R}$ is monotone then $f \in BV(a, b)$ and
$essV_a^b f = |f(b) - f(a)|$

**Proposition 1.** *Let $f : [a, b] \mapsto \mathbb{R}$. The following implications hold:*
*If f is continuously differentiable $\Rightarrow$ f is lipchitz continuous$\Rightarrow$ f is absolutely continuous $\Rightarrow$ f is of bounded variation $\Rightarrow$ f is differentiable a.e. .*

**Theorem 2.1.1.** [5] *If $f : [a, b] \mapsto \mathbb{R}$ continuous on $[a, b]$ and $f'$ exists and is bounded on $(a, b)$ then $f$ is absolutely continuous on $[a, b]$.*

**Theorem 2.1.2.** [5] *If f is absolutely continuous then $f^{'}$ exists a.e. and is integrable. Also, we have, $essV_a^b(f) = \int_a^b |f^{'}(x)|dx$*

**Definition 2.1.2.** Let $f \in L^1(a, b)$. We now define the number

$$\int_a^b |Df|dx = sup\{\int_a^b fg'\ dx;\ |g| \leqslant 1,\ g \in C_0^1(a, b)\}$$

to be the total variation of $f$.

**Remark 1.** If $f \in C^\infty(a, b)$, then $\int_a^b |Df| = \int_a^b |f'|$

**Theorem 2.1.3.** [6] *Let $f$ defined on $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ $\forall n \geqslant 1$.*
*If $\int_\Omega |Df| < \infty \Rightarrow \exists f_j \in BV(\Omega) \cap C^\infty(\Omega)$; $f_j \longrightarrow f$ in $L^1(\Omega)$ and*
*$\int_\Omega |Df_j| \longrightarrow \int_\Omega |Df|$ as $j \longrightarrow +\infty$*

**Theorem 2.1.4.** [6] *Let $\Omega \subset \mathbb{R}^n$.*
*If $f_j \longrightarrow f$ in $L^1(\Omega)$ then $\exists$ a subsequence $\{f_{j_k}\}$ such that $f_{j_k} \longrightarrow f$ a.e.*

**Theorem 2.1.5.** [6] *Let $f \in L^1(a, b)$ then $\int_a^b |Df| = essV_a^b(f)$*

*Proof.* $\leqslant$) Since each lebesgue point is a point of approximate continuity of $f$, consider the partition $a = t_1 \leqslant t_2 \leqslant ... \leqslant t_n = b$ where each $t_i$ is a lebesgue point. Let $\eta$ be a mollifier satisfying:

1. $0 \leqslant \eta(x) \leqslant 1 \ \forall x \in (a,b)$
2. $supp\eta \subset [-1,1]$
3. $\int_{-1}^{1} \eta(x)dx = 1$

Let $\epsilon > 0$.

Define the function $\eta_\epsilon(x) = \frac{1}{\epsilon}\eta(\frac{x}{\epsilon})$ and the convolution $f^\epsilon := \eta_\epsilon * f \in C^\infty(a,b)$.

$$\begin{aligned}\sum_{j=1}^{m} \mid f^\epsilon(t_{j+1}) - f^\epsilon(t_j) &= \sum_{j=1}^{m} \mid \int_{-\epsilon}^{\epsilon} (\eta_\epsilon(s) f(t_{j+1}-s) - \eta_\epsilon(s) f(t_j - s))ds \mid \\ &= \sum_{j=1}^{m} \mid \int_{-\epsilon}^{\epsilon} \eta_\epsilon(s)(f(t_{j+1}-s) - f(t_j - s))ds \mid \\ &\leqslant \sum_{j=1}^{m} \int_{-\epsilon}^{\epsilon} \eta_\epsilon(s) \mid f(t_{j+1}-s) - f(t_j - s) \mid ds \\ &= \int_{-\epsilon}^{\epsilon} \eta_\epsilon(s) \sum_{j=1}^{m} \mid f(t_{j+1}-s) - f(t_j - s) \mid ds\end{aligned} \tag{2.1}$$

But as $t_j$ is a lebesgue point then $t_j - s$ is a lebesgue point and thus approximate point $\forall j = 1,..,m$

$\Rightarrow \sum |f^\epsilon(t_{j+1}) - f^\epsilon(t_j)| \leqslant essV_a^b(f) \int_{-\epsilon}^{\epsilon} \eta_\epsilon(s)ds$

But$\int_{-\epsilon}^{\epsilon} \eta_\epsilon(s)ds = \int_{-\epsilon}^{\epsilon} \frac{1}{\epsilon}\eta(\frac{s}{\epsilon})ds$

Take $x = \frac{s}{\epsilon} \Rightarrow dx = \frac{1}{\epsilon}ds$

$\Rightarrow \int_{-1}^{1} \eta(x)dx = 1$

$\Rightarrow \sum_{j=1}^{m} \mid f^\epsilon(t_{j+1}) - f^\epsilon(t_j) \mid \leqslant essV_a^b(f)$

Taking sup over all such partitions we get $essV_a^b(f^\epsilon) \leqslant essV_a^b(f)$

Now,$\int_a^b f^\epsilon g' dx = -\int_a^b (f^\epsilon)' g dx \leqslant \mid \int_a^b (f^\epsilon)' dx \mid \leqslant \int_a^b \mid (f^\epsilon)' \mid dx$

Hoewever, as $f^\epsilon$ is continuously differentiable, we have $\int_a^b \mid (f^\epsilon)' \mid dx = essV_a^b f^\epsilon$

$\Rightarrow \int_a^b f^\epsilon g' dx \leqslant essV_a^b f^\epsilon \leqslant essV_a^b f$ .

As $f^\epsilon \longrightarrow f$ in $L^1(a,b)$ as $\epsilon \longrightarrow 0$

Then,$\int_a^b fg' dx = \lim_{\epsilon \to 0} \int_a^b f^\epsilon g' dx \leqslant essV_a^b f$

Taking sup over all such $g$, we finally get the first inequality $\int_a^b |Df| \leqslant essV_a^b(f)$

$\geqslant$) Now suppose $\int_a^b |Df| < \infty \Rightarrow \exists\{f_j\} \subset BV(a,b) \cap C^\infty(a,b)$; $f_j \longrightarrow f$ in $L^1(a,b)$, $\int_a^b |Df_j| \longrightarrow \int_a^b |Df|$, and $\exists$ a subsequence still denoted by $\{f_j\}$ such that $f_j \longrightarrow f$ a.e.

We write $f_j(z) = f_j(y) + \int_y^z f_j'(x)dx$ for $a \leqslant y \leqslant z \leqslant b$

Averaging with respect to y we get,

$⨍_a^b \mid f_j(z) \mid \ dy = ⨍_a^b \mid f_j(y) \mid \ dy + ⨍_a^b |\int_y^z (f_j)'(x)dx|\, dy$

$\Rightarrow |f_j(z)| = ⨍_a^b |f_j(y)|\, dy + \ |\int_y^z f_j'(x)dx|$

$\Rightarrow |f_j(z)| \leqslant ⨍_a^b |f_j(y)|\, dy + \ \int_y^z |f_j'(x)|dx$

$\Rightarrow |f_j(z)| \leqslant ⨍_a^b |f_j(y)|\, dy + \ \int_a^b |f_j'(x)|dx$

But as $f_j \in BV(a,b) \cap C^\infty(a,b)$ we have $\int_a^b |Df_j| = \int_a^b |f_j'| < \infty$ and as $f_j \in L^1(a,b)$ then $\frac{1}{b-a}\int_a^b |f_j(y)|dy < \infty$

$\Rightarrow |f_j(z)| < \infty$

$\Rightarrow f_j$ is uniformly bounded

$\Rightarrow ||f_j||_\infty < \infty \ \forall j$

$\Rightarrow sup_j ||f_j||_\infty < \infty$

But $||\cdot||_\infty$ is continuous and $f_j \longrightarrow f$ a.e. $\Rightarrow ||f||_\infty < \infty$

$\Rightarrow f \in L^\infty(a,b)$

As $f$ is essentially bounded then each approximate point of continuity of $f$ is a lebesgue point and thus $f^\epsilon(x) \longrightarrow f$ a.e. $x$ lebesgue point in (a,b).

Let $a = t_1 < .. < t_m = b$ be a partition of (a,b) with each $t_j$ a point of approximate continuity of f $\forall j = 1, ..m$.

$$\sum_{j=1}^{m} \mid f(t_{j+1}) - f(t_j) \mid = \lim_{\epsilon \to 0} \sum_{j=1}^{m} \mid f^\epsilon(t_{j+1}) - f^\epsilon(t_j) \mid \leqslant \limsup_{\epsilon \to 0} \int_a^b \mid (f^\epsilon)' \mid dx$$

since $f \in C^\infty(a,b)$ and thus $essV_a^b f^\epsilon = \int_a^b |(f^\epsilon)'|dx$.

Claim: $\int_a^b |(f^\epsilon)'|dx \leqslant \int_a^b |Df|$.
In fact, $\int_a^b (f^\epsilon)' g \ dx = -\int_a^b f^\epsilon g' dx = -\int_a^b (\eta_\epsilon * f) g' \ dx = -\int_a^b f(\eta_\epsilon * g)' dx$
now $\eta_\epsilon * g \in C_0^1(a,b)$ and $|\eta_\epsilon * g| \leqslant 1$
Taking sup over all such functions, $\Rightarrow \int_a^b (f^\epsilon)' g dx \leqslant \int_a^b |Df|$
$\Rightarrow \int_a^b |(f^\epsilon)'| \ dx \leqslant \int_a^b |Df|$ since $f \in C^\infty(a,b)$ and thus
$\int_a^b |(f^\epsilon)'|dx = \int_a^b |Df^\epsilon|$

Hence, $\sum_{j=1}^m |f(t_{j+1} - f(t_j)| \leqslant \int_a^b |Df|$
Taking sup over all such partitions we get:
$essV_a^b f \leqslant \int_a^b |Df|$

Conclusion, $\int_a^b |Df| = essV_a^b f$ □

**Remark 2.** Hence, we now say $f \in BV(a,b) \iff \int_a^b |Df| < \infty$.

## 2.2 Bounded Variation in $\mathbb{R}^2$

**Definition 2.2.1.** Consider the rectangle $I = [a,b] \times [c,d]$ and $f$ defined on the rectangle.
Fix $y \in [c,d]$ and define $f_1(t) = f(t,y)$ for $t \in [a,b]$.
Similarly, fix $x \in [a,b]$ and define $f_2(t) = f(x,t)$ for $t \in [c,d]$.
According to Tonelli, $f$ is said to be of bounded variation on I if
$\int_c^d essV_a^b f_1 dy < \infty$ and $\int_a^b essV_c^d f_2 dx < \infty$ where

$$essV_a^b f_1 = sup\{\sum_{j=0}^{m-1} \mid f(t_{j+1}, y) - f(t_j, y); a = t_0 < t_2 < .. < t_m = b\, partition\, of\, [a, b]\}$$

$$essV_c^d f_2 = sup\{\sum_{j=0}^{m-1} \mid f(x, t_{j+1}) - f(x, t_j); c = t_0 < t_2 < .. < t_m = d\, partition\, of\, [c, d]\}$$

with each $t_j$ a point of approximate continuity of f $\forall j = 1, ..m$.

**Remark 3.** Now if f is absolutely continuous, then

$$essV_a^b f_1 = \int_a^b |\frac{\partial f}{\partial x}| dx \ , \ essV_c^d f_2 = \int_c^d |\frac{\partial f}{\partial y}| dy$$

Hence, $f \in BV(I)$ and $\int_c^d \int_a^b |\frac{\partial f}{\partial x}| dxdy < \infty$ and $\int_a^b \int_c^d |\frac{\partial f}{\partial y}| dydx < \infty$ which gives $\int_{[a,b]\times[c,d]} |\nabla f| < \infty$

## 2.3 Bounded variation in $\mathbb{R}^n$ $\forall n > 1$

**Definition 2.3.1.** Denote $x^{'} = (x_1, .., x_{k-1}, x_{k+1}, .., x_n)$ and $f_k(t) = f(x', t) = f(x_1, .., x_{k-1}, t, x_{k+1}, .., x_n)$ as a function of $t \in (a, b)$, $\forall -\infty < a < b < \infty$, $\forall k = 1, .., n$. Let $K \subset \mathbb{R}^{n-1}$ compact, and $L \subset \mathbb{R}^n$ with $L = \{x \in \mathbb{R}^n;\ x' \in K,\ x_k \in (a, b)\}$.

According to Tonelli, $f \in BV(L) \iff \int_K essV_a^b f_k dx^{'} < \infty\ \forall k = 1, ..n$.

**Remark 4.** If $f$ is absolutely continuous then $f \in BV(\mathbb{R}^n)$ and $\int \mid \nabla f \mid < \infty$

**Definition 2.3.2.** For $\Omega \subset \mathbb{R}^n$ open and $f \in L^1(\Omega)$ define:

$$\int_\Omega |Df|\ dx := sup\{\int_\Omega f div g\ dx\ ;\ g = (g_1, .., g_n) \in C_0^1(\Omega), |g| \leqslant 1\}$$

the variation of $f$ in $\Omega$.

**Remark 5.** In fact, for $f \in L^1(\Omega)$, $\int_\Omega |Df| dx = |Df|(\Omega)$ the total variation of $Df$ in $\Omega$ where $Df$ is the distributional derivative of $f$ characterized to be a vector valued radon measure.

However, if $f \in C^k(\Omega)$ for $k \geqslant 1$ we get $\int_\Omega |Df|dx = \int_\Omega |\nabla f|$ where $\nabla f$ is the gradient of $f$ in the usual derivative sense.

If $f \in W^{1,1}(\Omega)$ the sobelev space, then $\int_\Omega |Df|dx = \int_\Omega |gradf|dx$ where now $gradf$ is the gradient of $f$ in the distributional sense.

**Theorem 2.3.1.** [6] *Let $K \subset \mathbb{R}^n$ compact, $x' = (x_1, .., x_{k-1}, x_{k+1}, .., x_n)$ and $C = \{x \in \mathbb{R}^n; x^{'} \in K, a < x_k < b\}$ and $f \in L^1(C)$. Then,*
$\int_K essV_a^b f_k \ dx^{'} < \infty \Leftrightarrow \int_C |Df|dx < \infty \ \forall k = 1, ..n \ , \ \forall -\infty < a < b < \infty$

*Proof.* $\Leftarrow$) Suppose $\int_C |Df|dx < \infty$. We have that

1. $f_k^\epsilon \longrightarrow f_k$ in $L^1(a, b)$

2. $\forall g \in C_0^1(\mathbb{R}^n)$, $|g| \leqslant 1$, we have:
   $\int_C (f^\epsilon)^{'} g dx = -\int_C f^\epsilon divg dx = -\int_C f div(\eta_\epsilon * g)dx \leqslant \int_C |Df|dx$
   $\Rightarrow \int_C |Df^\epsilon|dx \leqslant \int_C |Df|dx.$
   $\Rightarrow \limsup_{\epsilon \longrightarrow 0} \int |Df^\epsilon| \leqslant \int_C |Df|dx < \infty$

Now let $g \in C_0^1(\mathbb{R}^n), |g| \leqslant 1,$
$\int f_k g^{'} dx = \lim_{\epsilon \longrightarrow 0} \int f_k^\epsilon g^{'} dx \leqslant \liminf essV_a^b f_k^\epsilon$ by theorem 2.1.5.
$\Rightarrow essV_a^b f_k \leqslant liminf_{\epsilon \longrightarrow 0} essV_a^b f_k^\epsilon$ for $H^{n-1}$ a.e. $x' \in K$
By Fatou's lemma,

$$\int_K essV_a^b f_k \leqslant \int_K \liminf essV_a^b f_k^\epsilon \leqslant liminf_{\epsilon \longrightarrow 0} \int_K essV_a^b f_k^\epsilon dx' = liminf_{\epsilon \longrightarrow 0} \int_C |(f_k^\epsilon)^{'}|dx$$

But $liminf_{\epsilon \longrightarrow 0} \int_C |(f_k^\epsilon)^{'}|dx \leqslant limsup \int_C |Df^\epsilon|dx < \infty$
Then, $\int_K essV_a^b f_k < \infty$
$\Rightarrow$) Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $\int_K essV_a^b f_k dx^{'} < \infty$.
Let $g \in C_0^1(\mathbb{R}^n)$ , $|g| \leqslant 1$ , $suppg \subset \{x; a < x_k < b\}$
$\int_{\mathbb{R}^n} f \frac{\partial g}{\partial x_k} dx = \int_{suppg} f \frac{\partial g}{\partial x_k} dx = \int_K (\int_a^b f \frac{\partial g}{\partial x_k} dx_k) dx^{'} \leqslant \int_K essV_a^b f_k dx^{'} < \infty$
$\Rightarrow \int_{\mathbb{R}^n} |Df|dx < \infty.$ □

**Remark 6.** Hence, we can now say $f \in BV(\Omega) \iff \int_{\Omega} |Df|dx < \infty$ where $\Omega \subset \mathbb{R}^n$ open.

**Theorem 2.3.2.** [*6*] *Assume* $\Omega \subset \mathbb{R}^n$ *is open and bounded, with* $\partial\Omega$ *lipchitz continuous. There exists a bounded linear mapping*

$$T : BV(\Omega) \mapsto L^1(\partial\Omega;\ H^{n-1})$$

*such that*

$$\int_{\Omega} f divg dx = -\int_{\Omega} g\ d|Df| + \int_{\partial\Omega} (\nu \cdot g) T f dH^{n-1}$$

*for all* $f \in BV(\Omega)$ *and* $g \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ *where* $H^{n-1}$ *is the* $(n-1)$*-dimensional Hausdorff measure.*

**Definition 2.3.3.** The function $Tf$ which is uniquely defined up to sets of $H^{n-1}$-measure zero on $\partial\Omega$, is called the trace of $f$ on $\partial\Omega$.

## 2.4 Perimeter of a measurable set

**Definition 2.4.1.** Now let $f = \mathbb{1}_E$ be the characteristic function for E $\subset \mathbb{R}^n$ a measurable set.

$$\int_{\Omega} |Df|dx = \int_{\Omega} \mid D\mathbb{1}_E \mid dx = sup\{\int_E divgdx\ ;\ g = (g_1, .., g_n) \in C_0^1(\Omega), |g| \leqslant 1\}$$

is said to be the perimeter of E and is denoted by $P(E, \Omega)$.
We say E has finite perimeter if $P(E, \Omega) < \infty$.

**Theorem 2.4.1.** [*7*] *Suppose* $E \subset \mathbb{R}^n$ *has* $C^2$ *boundary, then*

$$P(E, \Omega) = H^{n-1}(\partial E \cap \Omega)$$

*where* $H^{n-1}$ *is the* $(n-1)$*-dimensional Hausdorff measure.*

*Proof.* $\leqslant$ ) Using Gauss Green theorem, we have,

$$\int_E divgdx = \int_{\partial E} g\nu dH^{n-1} \leqslant H^{n-1}(\partial E \cap \Omega)$$

for all $g \in C_0^1(\Omega), |g| \leqslant 1$ where $\nu$ is the outer normal to $\partial E$.
Taking the supremum over all such $g$, we get, $P(E,\Omega) \leqslant H^{n-1}(\partial E \cap \Omega)$

$\geqslant$) As E has $C^2$ boundary, then $\nu_E$(unit outer normal to $\partial E$) exists as a $C^1$-vector valued function.
Let N be the extension of $\nu_E$ to $\mathbb{R}^n$ satisfying:

1. $N = \nu_E$ on E

2. $|N(x)| \leqslant 1 \ \forall x \in \mathbb{R}^n$

3. N$\in C^1(\mathbb{R}^n, \mathbb{R}^n)$

Now for $\eta \in C_0^1(\Omega), |\eta| \leqslant 1$ , define $\phi = N\eta$.
Then, $\phi \in C_0^1(\Omega)$, $|\phi| \leqslant 1$, and $N\nu_E = |\nu_E|^2 = 1$ on E. Then,
$\int_E div\phi dx = \int_{\partial E} \phi\nu_E dH^{n-1} = \int_{\partial E} N\eta\nu_E dH^{n-1} = \int_{\partial E} \eta dH^{n-1}$.
Taking supremum over all such $\phi$ and $\eta$ we get:
$P(E,\Omega) \geqslant sup\{\int_{\partial E} \eta dH^{n-1}, \eta \in C_0^1(\Omega), |\eta| \leqslant 1\} \geqslant H^{n-1}(\partial E \cap \Omega)$ □

To observe the result better in $\mathbb{R}^2$ and $\mathbb{R}^3$ we wil prove the result in another way.
We recall Green's theorem in the plane:
Let E$\subset \mathbb{R}^2$ open, M,N $\in C^1(E)$, $\Omega \subset E$ , $\Omega$ closed , $\partial\Omega$ positively oriented, then
$\int_{\partial\Omega}(Mdx + Ndy) = \iint_\Omega(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dA$

Now let $g = (g_1, g_2) \in C_0^1(\mathbb{R}^2)$ satisfying $|g| \leqslant 1$
Take $N = g_1$, and $M = -g_2$,

$$
\begin{aligned}
\int_\Omega divg dA &= \int_\Omega (\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y}) dA \\
&= \iint_\Omega (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) dA \\
&= \int_{\partial\Omega} (M dx + N dy) \\
&= \int_{\partial\Omega} (-g_2 dx + g_1 dy) \\
&\leqslant \int_{\partial\Omega} \sqrt{(g_1)^2 + (g_2)^2}\sqrt{(dx)^2 + (dy)^2} \\
&\leqslant \int_{\partial\Omega} \sqrt{(dx)^2 + (dy)^2} \\
&= L(\partial\Omega)
\end{aligned}
\tag{2.2}
$$

By a special choice of $g$, we get,

$$sup\{\int_\Omega divgdx \ ; \ g = (g_1, g_2) \in C_0^1(\Omega), |g| \leqslant 1\} = L(\partial\Omega)$$

with $L(\partial\Omega)$ length of the boundary of $\Omega$.

In $\mathbb{R}^3$, suppose $|\overrightarrow{g}| \leqslant 1$
$\int_\Omega divgdx = \int_{\partial\Omega} \overrightarrow{g}.\overrightarrow{n} ds \leqslant \int_{\partial\Omega} |\overrightarrow{g}||\overrightarrow{n}| ds \leqslant \int_{\partial\Omega} ds =$ surface area of $\partial\Omega$.
By a special choice of $g$, we get,

$$sup\{\int_\Omega divgdx \ ; \ g = (g_1, g_2, g_3) \in C_0^1(\Omega), |g| \leqslant 1\} = \ surface\ area\ of\ \partial\Omega$$

## 2.5 Coarea formula

We introduce now the co area formula that permits us to have a relation between the variation of a function in $L^1$ and the perimeter of the superlevel sets of that function. Indeed,

**Theorem 2.5.1.** [*6*] *If* $\Omega \subset \mathbb{R}^n$, $f \in BV(\Omega)$, *and* $E_t = \{x \in \Omega; f(x) \geqslant t\}$ $\forall t \in \mathbb{R}$, *then:*

$$\int_\Omega |Df| = \int_{-\infty}^{+\infty} P(E_t, \Omega) dt$$

*Proof.* We will prove the following using several steps:

Step1: If $f \in L^1(\Omega)$

For $f \geqslant 0$, $f$ can be written as $f(x) = \int_0^\infty \mathbb{1}_{E_t}(x)dt$ for a.e. $x \in \Omega$

For $f \leqslant 0$, f can be written as $f(x) = \int_{-\infty}^0 (\mathbb{1}_{E_t}(x) - 1)dt$ for a.e. $x \in \Omega$.

Now let $g \in C_0^1(\Omega)$ and $|g| \leqslant 1$ then:

$$\begin{aligned}
\int_\Omega f divg dx &= \int_{\Omega \cap \{f \leqslant 0\}} f divg dx + \int_{\Omega \cap \{f \geqslant 0\}} f divg dx \\
&= \int_\Omega (\int_{-\infty}^0 (\mathbb{1}_{E_t}(x) - 1)dt) divg dx + \int_\Omega (\int_0^\infty \mathbb{1}_{E_t}(x)dt) divg dx \\
&= \int_{-\infty}^0 (\int_\Omega (\mathbb{1}_{E_t}(x) - 1) divg dx)dt + \int_0^\infty (\int_\Omega \mathbb{1}_{E_t}(x) divg dx)dt \\
&= \int_{-\infty}^0 (\int_{E_t} divg dx)dt - \int_{-\infty}^0 (\int_\Omega divg dx)dt + \int_0^\infty (\int_{E_t} divg dx)dt \\
&= \int_{-\infty}^{+\infty} (\int_{E_t} divg dx)dt
\end{aligned} \tag{2.3}$$

with $\int_\Omega divg dx = 0$ as $\int_\Omega divg dx = \int_{\partial\Omega} g\nu dH^{n-1} = 0$ as $g|_{\partial\Omega} = 0$.

Hence, taking sup over all such g, we get: $\int_\Omega |Df| \leqslant \int_{-\infty}^\infty P(E_t, \Omega)dt$

It remains to prove $\int_\Omega |Df| \geqslant \int_{-\infty}^\infty P(E_t, \Omega)dt$

Step2: Let $f \in BV(\Omega) \cap C^\infty(\Omega)$

Define $m(t) = \int_{\Omega - E_t} |Df(x)|dx = \int_{\{f \leqslant t\}} |Df|dx \geqslant 0$

$$\begin{aligned}
\int_{-\infty}^\infty m'(t)dt &= \lim_{t \longrightarrow \infty} m(t) - \lim_{t \longrightarrow -\infty} m(t) \\
&= \int_{\Omega - E_\infty} |Df|dx - \int_{\Omega - E_{-\infty}} |Df|dx \\
&= \int_{\Omega \cap \{f \leqslant \infty\}} |Df|dx + \int_{\Omega \cap \{f \geqslant \infty\}} |Df|dx \\
&= \int_{\Omega \cap \{-\infty \leqslant f \leqslant \infty\}} |Df|dx \\
&\leqslant \int_\Omega |Df|dx
\end{aligned} \tag{2.4}$$

$$\text{Now, } \lim_{r\longrightarrow 0}\frac{m(t+r)-m(t)}{r} = \lim_{r\longrightarrow 0}\{\frac{1}{r}(\int_{\Omega-E_{t+r}}|Df|dx - \int_{\Omega-E_t}|Df|dx)\}$$

$$= \lim_{r\longrightarrow 0}\{\frac{1}{r}\int_{E_t-E_{t+r}}|Df|dx\} \tag{2.5}$$

$$\geqslant \lim_{r\longrightarrow 0}\frac{1}{r}\int_{E_t-E_{t+r}}Dfgdx$$

As $\int Dfgdx \leqslant |\int Dfgdx| \leqslant \int |Df||g|dx \leqslant \int |Df|dx$.

Define for $-\infty < t < \infty$ , $r > 0$, $\eta(s) = \begin{cases} 0 & s \leqslant t \\ \frac{s-t}{r} & t \leqslant s \leqslant t+r \\ 1 & s \geqslant t+r \end{cases}$

We recognize as $r \longrightarrow 0$, $\eta(s) = \begin{cases} 0 & s \leqslant t \\ 1 & s > t \end{cases}$

$\eta$ is differentiable everywhere except at $s = t{+}r$ and $s = t \Rightarrow \eta^{'}(s) = \begin{cases} 0 & t+r < s < t \\ \frac{1}{r} & t < s < t+r \end{cases}$

$\Rightarrow \frac{1}{r}\int_{E_t-E_{t+r}} Dfgdx = \int_\Omega \eta^{'}(f(x))Dfgdx = \int_\Omega (\eta(f(x)))^{'}gdx = -\int_\Omega \eta(f(x))divgdx.$

Hence,

$$\lim_{r\longrightarrow 0}\frac{m(t+r)-m(t)}{r} \geqslant -\lim_{r\longrightarrow 0}\int_\Omega \eta(f(x))divgdx = -\int_\Omega \lim_{r\longrightarrow 0}\eta(f(x))divgdx = -\int_{E_t} divgdx$$

$\Rightarrow m^{'}(t) \geqslant -\int_{E_t} divgdx$
$\Rightarrow \int_{-\infty}^{\infty} m^{'}(t) \geqslant -\int_{-\infty}^{\infty}(\int_{E_t} divgdx)dt$
$\Rightarrow \int_\Omega \mid Df \mid dx \geqslant -\int_{-\infty}^{\infty}(\int_{E_t} divgdx)dt$

Taking sup over all such g, we get,
$\int_\Omega \mid Df \mid dx \geqslant \int_{-\infty}^{\infty} P(E_t,\Omega)dt$

Step 3: $f \in BV(\Omega)$
$\Rightarrow \exists\{f_k\} \subset BV(\Omega)\cap C^\infty(\Omega)$, $Df_k \longrightarrow$ in $L^1$ and $\int_\Omega \mid f_k \mid \longrightarrow \int_\Omega \mid Df \mid$ as $k \longrightarrow \infty$

Let $E_t^k = \{x \in \Omega; f_k(x) \geqslant t\}$

Claim: $\int_\Omega(\int_{-\infty}^{\infty} \mid (\mathbb{1}_{E_t^k}(x) - \mathbb{1}_{E_t}(x)) \mid dt)dx = \int_\Omega \mid f_k(x) - f(x) \mid dx$

In fact, $\mathbb{1}_{E_t^k}(x) = \begin{cases} 0 & f_k(x) \leqslant t \\ 1 & f_k(x) \geqslant t \end{cases}$

and $\mathbb{1}_{E_t}(x) = \begin{cases} 0 & f(x) \leqslant t \\ 1 & f(x) \geqslant t \end{cases}$

$\Rightarrow \mid \mathbb{1}_{E_t^k}(x) - \mathbb{1}_{E_t}(x)) \mid = \begin{cases} 1 & f_k(x) \geqslant t \geqslant f(x) \text{ or } f(x) \geqslant t \geqslant f_k(x) \\ 0 & otherwise \end{cases}$

$\Rightarrow \int_{-\infty}^{\infty} |\mathbb{1}_{E_t^k}(x) - \mathbb{1}_{E_t}(x))|dt = \int_{min(f(x),f_k(x))}^{max(f(x),f_k(x))} dt = |f_k(x) - f(x)|$

. Now, as $f_k \longrightarrow f$ in $L^1 \Rightarrow \mathbb{1}_{E_t^k}(x) \longrightarrow \mathbb{1}_{E_t}(x)$ in $L^1$

$\Rightarrow \lim_{k\longrightarrow\infty} \int_\Omega \mathbb{1}_{E_t^k} divg = \int_\Omega \mathbb{1}_{E_t} divg$

With the help of Fatou's lemma, hence,

$$\int_\Omega \mid Df \mid = \lim_{k\longrightarrow\infty} \int_\Omega \mid Df_k \mid = \lim_{k\longrightarrow\infty} \int_{-\infty}^{\infty} P(E_t, \Omega) \geqslant \int_{-\infty}^{\infty} \liminf_{k\longrightarrow\infty} P(E_t^k, \Omega) = \int_{-\infty}^{\infty} P(E_t, \Omega)$$

$\square$

# Chapter 3

# Least Gradient Problem

In this chapter, we will first introduce the least gradient problem and prove the existence and uniqueness of a solution. After, we will show relations with the least gradient problem demonstrated with examples.

## 3.1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, and let $f$ be a given function defined on its boundary $\partial\Omega$. We seek a solution $u \in BV(\Omega)$ to the problem:

$$min\{\int_{\Omega} |\nabla u| \ ; \ u = f\}$$

In fact, if we take a minimizing sequence of the LGP, we cannot ensure the limiting function will satisfy the boundary conditions and thus we cannot ensure the limiting function will be an element of the set. Therefore, we cannot guarantee that a minimum does exist for every functions $f$ and every domain $\Omega$.

**Example 1** Consider the case in one dimension. Here $\Omega = [a, b]$ and we are given a boundary function $f$ i.e. given two values $\{f(a), f(b)\}$. We start with the class of all absolutely continuous functions on $\Omega$, and we seek a solution of the

LGP in this class.

We proceed as follows: given a function $u$ absolutely continuous on $[a, b]$, and satisfying $u(a) = f(a)$, $u(b) = f(b)$, how small can $\int_a^b |u'|$ be. Here, we resort to the fact that, since $u$ is absolutely continuous, its derivative exists almost everywhere, is summable, and the integral of the derivative equals the total variation $essV_a^b u$. But $essV_a^b u \geqslant |u(b) - u(a)| = |f(b) - f(a)|$, and this lower bound is independent of the particular function $u$. This gives us that $min\{\int_a^b |u'|\} \geqslant |f(b) - f(a)|$. In addition, any monotone absolutely continuous function $v$ on $[a, b]$, satisfies the equality $\int_a^b |v'| = |v(b) - v(a)| = |f(b) - f(a)|$. Hence, the minimum is indeed $|f(b) - f(a)|$ and is attained by monotone functions.

**Example 2** The previous example can be extended to cover the class of functions of bounded variation. If $u \in BV(a, b)$, then its derivative exists a.e. but does not usually integrate back to the function. It is however, possible to define a generalized derivative $Du$, and then the question becomes that of minimizing $\int |Du|$. In this case, once again, we have $\int |Du|$ equals the variation of $u$ on $[a, b]$, and as in previous example we find the minimum value and also the extremal functions.

**Example 3** It is natural to move from the one dimensional case to the two dimensional. Let $Q$ be the rectangle $[a, b] \times [c, d]$, let $f$ be a function defined on the boundary of $Q$, and let $u$ be defined and absolutely continuous on $Q$, with boundary values equal to $f$. Here again, the gradient $\nabla u$ exists a.e. and we seek to minimize its integral $\int_Q |\nabla u|$. We recall the concept of bounded variation due to Tonelli. For each fixed $x \in (a, b)$, let $essV_c^d u_2(x)$ be the essential variation of $u(x, .)$ on $[c, d]$. If now we integrate this with respect to $x$ on $[a, b]$ and the resulting integral is finite, we say $u$ is of bounded variation on $Q$. Of course we could start with $(essV_a^b u_1)(y)$ being the essential variation of $u(., y)$ on $[a, b]$, and then integrate with respect to $x$.

We point out that one of these integrals may very well be zero as is seen if we start with a function $u$ which depends only on $x$. So we need to concentrate on a class of functions for which both integrals will be finite.

If we start with an absolutely continuous function $u$, then we can express in terms of the partial derivative as follows:

$$(essV_a^b u_1)(y) = \int_a^b |\frac{\partial u(x,y)}{\partial x}|dxdy \ , (essV_c^d u_2)(x) = \int_c^d |\frac{\partial u(x,y)}{\partial y}|dydx$$

So we end up with two integrals namely

$$\int_c^d \int_a^b |\frac{\partial u(x,y)}{\partial x}|dxdy \ , \int_a^b \int_c^d |\frac{\partial u(x,y)}{\partial y}|dydx$$

Thus, if we start with a function having continuous partial derivatives on Q we are guarenteed the finiteness of both of last integrals.

We can find ower bounds for each of these integrals in the most simple way, namely to use trivial lower bounds for the variation.

**Proposition 2.** *Let $u$ be absolutely continuous on the rectangle $Q = [a,b] \times [c,d]$. Suppose that the boundary values of $u$ are given by a function $f$. Then the following lower bound for the integral of $|\nabla u|$ holds*

$$\int_Q |\nabla u| \geqslant \frac{1}{2}\{\int_c^d |f(b,y) - f(a,y)|dy + \int_a^b |f(x,d) - f(x,c)|dx\}$$

A special case

Suppose $Q = [0,a] \times [0,b]$, and the boundary function $f$ is non-negative and satisfies $f(x,0) = 0$ for all $x \in [0,a]$; and $f(0,y) = 0$ for all $y \in [0,b]$. In this case, we have the simplification

$$\int_c^d |f(b,y)-f(a,y)|dy+\int_a^b |f(x,d)-f(x,c)|dx = \int_0^b f(a,y)dy+\int_0^a f(x,b)dx = \int_{\partial Q} f(x,y)ds$$

**Example 4** Let $\Omega$ be any plane domain in $\mathbb{R}^2$ and $u \in BV(\Omega)$
For $Du$ defined in the distributional sense, we always have $\int_\Omega \mid Du \mid \geqslant 0$.
Hence, $min \int_\Omega \mid Du \mid \geqslant 0$. We now take $f$ to be any constant function defined on $\partial\Omega$ say $f = k$ with k a positive constant. Among all functions $u$ of bounded variation defined on $\Omega$ and $u = k$ on $\partial\Omega$, one has the constant function $u = k$. We get $\nabla u = 0 \Rightarrow \int_\Omega \mid \nabla u \mid = 0$.
Hence for $f$ constant with $\Omega$ plane domain the solution to the LGP is 0.

The method illustrated above relies on the link between the essential variation and the LGP. However, particular classes of functions were taken. What we need, is to find the least gradient function over all bounded variation functions satisfying the boundary conditions. Therefore, in the modern treatment of the subject, what we seem to need is the notion of perimeter of measurable sets. We will use the help of the coarea formula, which connects the perimeter of a set with $\int |Du|$.

**Remark 7.** When saying $u = f$ on $\partial\Omega$ for $u \in BV(\Omega)$, it is meant in the trace sense; $Tu = f$ on $\partial\Omega$ by definition 2.4.1.

## 3.2 Prerequisites

**Definition 3.2.1.** Suppose $E$ measurable set with $P(E, \mathbb{R}^n) < \infty$.

- The measure theoretic boundary $\partial_M E$ is the set of points $x \in \mathbb{R}^n$ such that:
  $\limsup_{r \longrightarrow 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} > 0$ and
  $\liminf_{r \longrightarrow 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} < 1$
- For $x \in \mathbb{R}^n$, the measure theoretic exterior normal $\nu(x, E)$ at $x$ is a unit vector $\nu$ such that:

$\lim_{r\longrightarrow 0} \frac{|B(x,r)\cap\{y;(y-x)\cdot\nu<0, y\notin E\}|}{r^n} = 0$ and
$\lim_{r\longrightarrow 0} \frac{|B(x,r)\{y;(y-x)\cdot\nu>0, y\in E\}|}{r^n} = 0.$

The reduced boundary $\partial^* E$ is the set of points x such that $\nu(x,E)$ exists.

We have $\partial^* E \subset \partial_M E \subset \partial E$.
We will use the following convention, since sets of finite perimeter are defined up to measure zero, $x \in E \iff \limsup_{r\longrightarrow 0} \frac{|E\cap B(x,r)|}{|B(x,r)|} > 0$

**Definition 3.2.2.** We say a function $u \in BV(\Omega)$ is of least gradient if $\forall w \in BV(\Omega)$, with compact support in $\Omega$, $\int_\Omega |Du| \leqslant \int_\Omega |D(u+w)|$.

**Proposition 3.** [8] *If $u_n$ is a least gradient function $\forall n$, and $u_n \longrightarrow u$ in $L^1$ then $u$ is a least gradient function*

**Definition 3.2.3.** Let $\Omega \subset \mathbb{R}^n$. We say $\partial E$ is a minimal surface if

1. $\mathbb{1}_E \in BV_{loc}(\Omega)$
2. $\mathbb{1}_E$ is a least gradient function.

**Proposition 4.** *In $\mathbb{R}^2$, minimal surfaces are straight lines.*

*Proof.* Let $A = (a_1, a_2), B(b_1, b_2)$ be 2 points in the plane $\mathbb{R}^2$. Let $(C)$ be a path joining $A$ to $B$ parametrized by $x = x(t)$, $y = y(t)$ and
$x(0) = a_1, x(1) = b_1, y(0) = a_2, y(1) = b_2$.

$$\begin{aligned}
L(C) &= \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2} dt \\
&= \int_0^1 \mid x'(t) + iy'(t) \mid dt \\
&\geqslant \mid \int_0^1 (x'(t) + iy'(t)) dt \mid \\
&= \mid x(1) - x(0) + i(y(1) - y(0)) \mid \\
&= \mid b_1 - a_1 + i(b_2 - a_2) \mid \\
&= \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}
\end{aligned} \tag{3.1}$$

$\Rightarrow inf\{L(C)\} \geqslant \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$

Consider the line segment $(\tilde{C})$ joining $A$ to $B$ given by:

$x = x(t) = (1-t)a_1 + tb_1$ and $y = y(t) = (1-t)a_2 + tb_2$.

Then, $L(\tilde{C}) = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}dt = \int_0^1 \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}dt = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$.

$\Rightarrow inf\{L(C)\} = L(\tilde{C})$

$\Rightarrow \tilde{C}$ is the curve of smallest euclidean length joining $A$ to $B$. $\square$

**Theorem 3.2.1.** [2] *If $u$ solution to $min\{\int_\Omega |Du|; u \in BV(\Omega), T_{\partial\Omega}u = f\}$, then $\partial\{u \geqslant t\}$ is a minimal surface for each real $t$.*

*Proof.* Let $u$ solution to $min\{\int_\Omega |Du|; u \in BV(\Omega), T_\Gamma u = f\}$. Take $w \in BV(\Omega)$ with $suppw = K \subset \Omega$ with K compact. Set $v = u + w$.

We have $v \in BV(\Omega)$ being the sum of 2 bounded variation functions in $\Omega$ and as $w$ has compact support, $T_\Gamma w = 0$, so that $T_\Gamma v = T_\Gamma(u+w) = T_\Gamma u = f$.

Then, $\int |Du| \leqslant \int |Dv|$ as $u$ solution.

$\Rightarrow \int |Du| \leqslant \int |D(u+w)|$.

Hence, $u$ is a function of least gradient in $\Omega$ by definition 3.2.2.

Now as $u \in BV(\Omega)$, we have by the coarea formula, $\int_\Omega |Du| = \int_{-\infty}^{+\infty} P(E_\lambda, \Omega)d\lambda$ with $E_\lambda = \{x \in \Omega; u(x) \geqslant \lambda\}$ and $\lambda \in \mathbb{R}$.

We then have $P(E_\lambda, \Omega) < \infty$ for a.e. $\lambda$ since $\int_\Omega |Du| < \infty$.

Also, by the coarea formula we have that, $\forall \lambda \in \mathbb{R}$, $K \subset \Omega$ compact, we have $\int_K |Du| = \int_{-\infty}^{+\infty} P(E_\lambda, K)d\lambda$.

We now define $u_1 = max\{u - t, 0\}$ and $u_2 = min\{u, t\}$ for $t \in \mathbb{R}$.

We have $u_1$ and $u_2 \in BV(\Omega)$ because $u \in BV(\Omega)$, $t \in BV(\Omega)$.

If $u - t \geqslant 0 \Rightarrow u \geqslant t \Rightarrow u_1 = u - t$ and $u_2 = t \Rightarrow u = u_1 + u_2$.

If $u - t \leqslant 0 \Rightarrow u \leqslant t \Rightarrow u_1 = 0$ and $u_2 = u \Rightarrow u = u_1 + u_2$.

Hence $u = u_1 + u_2$ $\forall t \in \mathbb{R}$.

By the coarea formula, we have $\int_K \mid Du \mid = \int_K \mid Du_1 \mid + \int_K \mid Du_2 \mid$.

Now let $w \in BV(\Omega)$ with compact support in $\Omega$.

$\int_K \mid Du_1 \mid + \int_K \mid Du_2 \mid = \int_K \mid Du \mid \leqslant \int_K \mid D(u+w) \mid \leqslant \int_K \mid D(u_1+w) \mid + \int_K \mid Du_2 \mid$.

$\Rightarrow u_1$ is a least gradient function. Similarly, by interchanging $u_1$ and $u_2$ in the last inequality, we get $u_2$ is a least gradient function.

We now define, for $\epsilon > 0$, $\lambda \in \mathbb{R}$,

$u_{\epsilon,\lambda} = \frac{1}{\epsilon} min\{\epsilon, max\{u - \lambda, 0\}\} = \frac{1}{\epsilon} min\{\epsilon, u_1\}$.

If $min\{\epsilon, u_1\} = \epsilon \Rightarrow u_{\epsilon,\lambda} = 1$. Then,$u_{\epsilon,\lambda}$ is a least gradient function being a constant function.

If $min\{\epsilon, u_1\} = u_1 \Rightarrow u_{\epsilon,\lambda} = \frac{1}{\epsilon} u_1$. Then, $u_{\epsilon,\lambda}$ is a least gradient function by proving above $u_1$ is so.

Hence, $u_{\epsilon,\lambda}$ is a least gradient function, $\forall \epsilon > 0, \lambda \in \mathbb{R}$.

If $H_n(\{x \in \Omega; u(x) = \lambda\}) = 0 \Rightarrow u(x) \neq \lambda$ a.e. $x \in \Omega \Rightarrow u - \lambda > 0$ or $u - \lambda < 0$.

If $u - \lambda > 0 \Rightarrow max\{u - \lambda, 0\} = u - \lambda \Rightarrow u_{\epsilon,\lambda} = \frac{1}{\epsilon} min\{\epsilon, u - \lambda\}$

As $\epsilon \longrightarrow 0, min\{\epsilon, u - \lambda\} = \epsilon$ as $u - \lambda > 0$

$\Rightarrow u_{\epsilon,\lambda} \longrightarrow \frac{\epsilon}{\epsilon} = 1$ as $\epsilon \longrightarrow 0$

$\Rightarrow \int_K |u_{\epsilon,\lambda} - \mathbb{1}_{E_\lambda}| = \int_K |1 - 1| = 0$ as $\epsilon \longrightarrow 0^+$.

If $u - \lambda < 0 \Rightarrow max\{u - \lambda, 0\} = 0 \Rightarrow u_{\epsilon,\lambda} = 0$

$\Rightarrow \int_K |u_{\epsilon,\lambda} - \mathbb{1}_{E_\lambda}| = \int_K |0 - 0| = 0$ as $\epsilon \longrightarrow 0$.

Hence, $\int_K |u_{\epsilon,\lambda} - \mathbb{1}_{E_\lambda}| = 0$ as $\epsilon \longrightarrow 0^+$.

As $u_{\epsilon,\lambda} \longrightarrow \mathbb{1}_{E_\lambda}$ in $L^1(K)$ and $u_{\epsilon,\lambda}$ is a function of least gradient,by proposition

3 then $\mathbb{1}_{E_\lambda}$ is a function of least gradient.

If $H_n(\{x \in \Omega; u(x) = \lambda\}) > 0$ then $\exists$ a sequence $\lambda_m, \lambda_m < \lambda, \lambda_m \longrightarrow \lambda$ and $H_n(\{x \in \Omega; u(x) = \lambda_m\}) = 0$.
As the previous case, working with $\lambda_m$, we get $\mathbb{1}_{E_\lambda}$ is a function of least gradient.

Hence, as $\mathbb{1}_{E_\lambda} \in BV_{loc}(\Omega)$, and a function of least gradient, $\partial E_\lambda$ is a minimal surface. $\square$

**Proposition 5.** *Let $\Gamma \subset \partial\Omega$ and $f$ defined on $\Gamma$ bounded and continuous. Let $u$ be any solution to the LGP with $Tu = f$ on $\Gamma$. Then $u(\overline{\Omega}) \subset \overline{f(\Gamma)}$*

*Proof.* Let $M = sup_\Gamma f$ and $m = inf_\Gamma f$.
Let $w = min\{M, max\{m, u\}\}$.
We have $w \in BV(\Omega)$ and $T_\Gamma w = f$.
We have $\int_\Omega \mid Dw \mid \leqslant \int_\Omega \mid Du \mid$.
If $u < m \Rightarrow max\{m, u\} = m \Rightarrow w = m \Rightarrow \int_\Omega \mid Dw \mid = 0$.
If $u > M \Rightarrow max\{m, u\} = u \Rightarrow w = M \Rightarrow \int_\Omega \mid Dw \mid = 0$.
In both cases, for $u < m$ and $u > M$ we get $\int_\Omega \mid Dw \mid < \int_\Omega \mid Du \mid$. But this is impossible as $u$ is a solution to the LGP.
Hence, we get $m < u < M$. $\square$

**Proposition 6.** *Let $\Omega \subset \Omega_0$ domains with lipchitz boundaries. If $u \in BV(\Omega_0)$ is a least gradient function in $\Omega_0$ then $u|_\Omega$ is a least gradient function in $\Omega$.*

## 3.3 Solution to the LGP

**Theorem 3.3.1.** [*1*] *For $\Omega$ strictly convex, $\partial\Omega$ lipchitz continuous, and $f$ bounded continuous on $\partial\Omega$, there exists a unique continuous function $u$ defined on $\overline{\Omega}$ solution to to the problem*

$$min\{\int_\Omega |\nabla u|;\ u \in BV(\Omega),\ u = f \ on\ \partial\Omega\}$$

The solution $u$ is obtained upon construction and is proved to be the minimum of the LGP with the help of the coarea formula. Also, the fact that the boundary of the superlevel sets of the solution is a minimal surface, by theorem 3.2.1, plays an important role.

**Remark 8.** The existence and uniqueness of solution will be proved for the LGP $min\{\int_\Omega |Du|, u \in BV(\Omega), u = f \text{ on } \Gamma\}$ where $\Gamma \subset \partial\Omega$ open given by [3]. However, the construction and existence of solution to the LGP stated in theorem 3.3.1 is very similar to what will be proved. Uniqueness results also applies when $\Gamma = \partial\Omega$.

**Construction of the solution**

Let $\Gamma \subset \partial\Omega$ such that $f$ is bounded and continuous on $\Gamma$.
Let $\Omega_0$ be a bounded domain such that $\Omega \subset \Omega_0$ and $\partial\Omega_0 \cap \partial\Omega = \partial\Omega - \Gamma$.
We will denote $\Lambda := \partial\Omega - \Gamma$.
Let $F$ be the extension of f to $\Omega_0$ so that $F \in BV(\Omega_0 - \overline{\Omega}) \cap C(\Omega_0)$
Let $t \in \overline{f(\Gamma)}$,as $F \in BV(\Omega_0 - \overline{\Omega}) \Rightarrow$ by the coarea formula, $P(L_t, \Omega_0 - \overline{\Omega}) < \infty$ a.e. t where $L_t = \{x \in \Omega_0, F(x) \geqslant t\}$.
Denote by $T := \overline{f(\Gamma)} \cap \{t \in \mathbb{R}; P(L_t, \Omega_0 - \overline{\Omega}) < \infty\}$.

Consider now the following problem: for each $t \in T$

$$min\{P(E, \Omega_0), E - \overline{\Omega} = L_t - \overline{\Omega}\} \tag{3.2}$$

This indeed has a solution. By [9], take a minimizing sequence, $P(E_n, \Omega_0) \longrightarrow m$ where $m = inf\{P(E, \Omega_0), E - \overline{\Omega} = L_t - \overline{\Omega}\}$ and $E_n - \overline{\Omega} = L_t - \overline{\Omega}$
As $\mathbb{1}_{E_n} \in BV(\Omega_0)$ then $\exists$ a subsequence still denoted by $\mathbb{1}_{E_n}$ and $\mathbb{1}_E \in BV(\Omega_0)$ such that $\mathbb{1}_{E_n} \longrightarrow \mathbb{1}_E$ in $L^1(\Omega_0)$ with $E - \overline{\Omega} = L_t - \overline{\Omega}$.
We get, $m \leqslant P(E, \Omega_0) \leqslant \liminf P(E_n, \Omega_0) = m$
$\Rightarrow P(E, \Omega_0) = m$

So the minimum does exist.

Now among all minimizers of (3.2) define the following:

$$max\{|E|;\ E\ solves\ (3.2)\} \tag{3.3}$$

(3.3) has a unique solution. Indeed, by [9], let $M = sup\{|E|;$ E solves problem (3.2)$\}$ and let $|E_n| \longrightarrow M \Rightarrow$ there exists a subsequence still denoted by $\mathbb{1}_{E_n}$ and $\mathbb{1}_E \in BV(\Omega_0)$ such that $\mathbb{1}_{E_n} \longrightarrow \mathbb{1}_E$ in $L^1(\Omega_0)$

First we note that :

1) $|E|+|E\Delta E_n| = |E|+|E-E_n|+|E_n-E| = |E\cup E_n|+|E-E_n| \geqslant |E\cup E_n| \geqslant |E_n|$

2) $|\mathbb{1}_{E_n} - \mathbb{1}_E| = \begin{cases} 1 & (x \in E_n \text{ and } x \notin E) \text{ or } (x \in E \text{ and } x \notin E_n) \\ 0 & x \in E_n \cap E \text{ or } x \notin E_n \cup E \end{cases}$

$\Rightarrow \int |\ \mathbb{1}_{E_n} - \mathbb{1}_E\ | = \int_{(E_n-E)\cup(E-E_n)} d\lambda = |\ E\Delta E_n\ |$

Then we have $M \geqslant |E| \geqslant |E_n| - |E\Delta E_n| = |E_n| - ||\mathbb{1}_{E_n} - \mathbb{1}_E||_1 \longrightarrow M$ as $n \longrightarrow \infty$
$\Rightarrow |E| = M$ and E solves (3.2).
So the maximum does exist.

Now one can claim that the solution to (3.3) is unique. In fact, let $E_1, E_2$ be 2 solutions to (3.3).
One knows that $P(E_1 \cup E_2, \Omega_0) + P(E_1 \cap E_2, \Omega_0) \leqslant P(E_1, \Omega_0) + P(E_2, \Omega_0)$
Then, $E_1 \cup E_2$ and $E_1 \cap E_2$ are solutions to (3.2).
As $E_1$ and $E_2$ are maximizers to (3.3),
$|E_1| \geqslant |E_1 \cup E_2| = |E_1| + |E_2 - E_1|$
$|E_2| \geqslant |E_1 \cup E_2| = |E_2| + |E_1 - E_2|$
$\Rightarrow |E_1 \Delta E_2| = 0 \Rightarrow |E_1| = |E_2|$
Hence, there exists a unique solution to (3.3) which we will denote by $E_t$ for $t \in T$.

Now define:

$$A_t = \overline{E_t \cap \Omega}$$

**Definition 3.3.1.** Define the function $u$ on $\overline{\Omega}$ by

$$u(x) = sup\{t;\ x \in\ A_t\}$$

**Lemma 3.3.1.** *$u$ satisfies the following conditions:*

1. $u = f$ *on* $\Gamma$
2. $u \in C(\Gamma \cup \Omega)$

**Lemma 3.3.2.** *Let* $\Gamma \subset \partial\Omega$,
*We then have,* $\{x \in \Gamma; f(x) > t\} \subset E_t \cap \Gamma \subset A_t \cap \Gamma \subset \{x \in \Gamma; f(x) \geqslant t\}$

**Lemma 3.3.3.** *Let* $v \in BV(\Omega), T_\Gamma v = f$ *and* $\tilde{v}$ *the extension of* $v$ *to* $\Omega_0$*;* $\tilde{v} = F$ *on* $\Omega_0 - \overline{\Omega}$ *and* $v = \tilde{v}$ *on* $\overline{\Omega}$*.For* $t \in T$*, define* $G_t := \{\tilde{v} \geqslant t\}$.
*We have* $\partial^* G_t \cap \Gamma \subset f^{-1}(t)$.

*Proof.* Let $x \in \partial^* G_t \cap \Gamma$. Proceeding by contradiction, suppose $f(x) > t \Rightarrow$ $f(x) = t + \epsilon$ for some $\epsilon > 0$.
By definition of trace, $\lim_{r \longrightarrow 0} \frac{1}{|B(x,r) \cap \Omega|} \int_{B(x,r) \cap \Omega} |\tilde{v}(y) - f(x)| dy = 0$.

$$\Rightarrow \lim_{r \longrightarrow 0} \frac{\int_{B(x,r) \cap \Omega \cap \{\tilde{v} \geqslant t\}} |\tilde{v}(y) - f(x)| dy + \int_{B(x,r) \cap \Omega \cap \{\tilde{v} \leqslant t\}} |\tilde{v}(y) - f(x)| dy}{|B(x,r) \cap \Omega|} = 0.$$

$$\Rightarrow 0 \geqslant \limsup_{r \longrightarrow 0} \frac{1}{|B(x,r) \cap \Omega|} \int_{B(x,r) \cap \Omega \cap \{\tilde{v} \geqslant t\}} |\tilde{v}(y) - f(x)| dy$$

$$\Rightarrow 0 \geqslant \epsilon \limsup_{r \longrightarrow 0} \frac{|B(x,r) \cap \Omega \cap G_t|}{|B(x,r) \cap \Omega|}$$

$$\Rightarrow \limsup_{r \longrightarrow 0} \frac{|B(x,r) \cap \Omega \cap G_t|}{|B(x,r) \cap \Omega|} = 0.$$

Similarly, we obtain $\limsup_{r\longrightarrow 0} \frac{|B(x,r)\cap(\Omega_0-\overline{\Omega})\cap G_t|}{|B(x,r)\cap(\Omega_0-\overline{\Omega})|} = 0$ on $\Omega_0 - \overline{\Omega}$ .

Hence, we get, $\limsup_{r\longrightarrow 0} \frac{|B(x,r)\cap\Omega_0\cap G_t|}{|B(x,r)|} = \limsup_{r\longrightarrow 0} \frac{|B(x,r)\cap G_t|}{|B(x,r)|} = 0.$

Then, $x \notin \partial_M G_t$. Contradiction, as $\partial^* G_t \subset \partial_M G_t$, Definition 3.2.1.

A similar argument is made for $f(x) < t$. Hence, $f(x) = t$ $\square$

**Lemma 3.3.4.** *For $\Gamma, f,$ and $E_t$ given as above, we have $\partial E_t \cap \Gamma \subset f^{-1}(t)$.*

**Lemma 3.3.5.** *If $s < t$ then $E_t \subset E_s$.*

*Proof.* Let $F = E_t \cup E_s$ and $E = E_t \cap E_s$.

We begin by proving that $F$ and $E$ are competitors to $E_t$ and $E_s$ in (3.2) respectively.

We realize that $L_t \subset L_s$ since if $x \in L_t \Rightarrow F(x) \geqslant t > s \Rightarrow F(x) > s \Rightarrow x \in L_s$.

- $F - \Omega = (E_t - \Omega) \cup (E_s - \Omega) = (L_t - \Omega) \cup (L_s - \Omega) = L_s - \Omega$ as $L_t \subset L_s$
- $E - \Omega = (E_t - \Omega) \cap (E_s - \Omega) = (L_t - \Omega) \cap (L_s - \Omega) = L_t - \Omega$ as $L_t \subset L_s$

Hence, $P(F, \Omega_0) \geqslant P(E_s, \Omega_0)$ and $P(E, \Omega_0) \geqslant P(E_t, \Omega_0)$. As $P(E_t \cup E_s, \Omega_0) + P(E_t \cap E_s, \Omega_0) \leqslant P(E_t, \Omega_0) + P(E_s, \Omega_0)$, we then get $P(F, \Omega_0) = P(E_s, \Omega_0)$ and $P(E, \Omega_0) = P(E_t, \Omega_0)$.

By problem 2,$|E_s| \geqslant |F| = |E_s| + |E_t - E_s| \Rightarrow |E_t - E_s| = 0$

and $|E_t| \geqslant |E| = |E_t| + |E_s - E_t| \Rightarrow |E_s - E_t| = 0$.

Now we show $E_t \subset E_s$

Let $x \in E_t \Rightarrow$ by definition 3.2.1, $\limsup_{r\longrightarrow 0} \frac{|E_t\cap B(x,r)|}{|B(x,r)|} > 0$

Write $E_t = (E_t - E_s) \cup (E_t \cap Es)$ union of 2 disjoint sets.

$\Rightarrow \limsup_{r\longrightarrow 0} \frac{|E_t\cap B(x,r)|}{|B(x,r)|} = \limsup_{r\longrightarrow 0} \frac{|(E_t-E_s)\cap B(x,r)|}{|B(x,r)|} + \limsup_{r\longrightarrow 0} \frac{|(E_t\cap E_s)\cap B(x,r)|}{|B(x,r)|}$

But $(E_t - E_s) \cap B(x,r) \subset (E_t - E_s)$

$\Rightarrow | (E_t - E_s) \cap B(x,r) | \leqslant | (E_t - E_s) | = 0$

$\Rightarrow \mid (E_t - E_s) \cap B(x,r) \mid = 0$

$\Rightarrow 0 < \limsup_{r \longrightarrow 0} \frac{|E_t \cap B(x,r)|}{|B(x,r)|} = \limsup_{r \longrightarrow 0} \frac{|E_t \cap E_s \cap B(x,r)|}{|B(x,r)|} \leqslant \limsup_{r \longrightarrow 0} \frac{|Es \cap B(x,r)|}{|B(x,r)|}$

$\Rightarrow \limsup_{r \longrightarrow 0} \frac{|Es \cap B(x,r)|}{|B(x,r)|} > 0$

$\Rightarrow x \in E_s$

Hence, $E_t \subset E_s$

□

We shall now illustrate the proof of lemma 3.3.1 given by [3] :

*Proof.* 1. We prove $Tu = f$ on $\Gamma$ i.e. for $z \in \Gamma, \lim_{y \longrightarrow z,\ y \in \Omega} u(y) = f(z)$.
Let $z \in \Gamma$ and set $f(z) = t$ and let $s < t$, then $f(z) = t > s \Rightarrow$ by lemma 3.3.2 $z \in E_s^0 \cap \Gamma \subset A_s \cap \Gamma$.
As $E_s^0$ is open, $\exists$ neighborhood of $z$, $N_z$, such that $N_z \cap \Omega \subset E_s \cap \Omega = A_s \cap \Omega$.
Now let $x_n \in N_z \cap \Omega; x_n \longrightarrow z \Rightarrow u(x_n) \geqslant s$, $\forall n \Rightarrow \liminf_{x_n \longrightarrow z,\ x_n \in \Omega} u(x_n) \geqslant s$, $\forall s < t \Rightarrow \liminf_{x_n \longrightarrow z,\ x_n \in \Omega} u(x_n) \geqslant t$.

We will now prove that it is not possible for $\limsup_{x_n \longrightarrow z,\ x_n \in \Omega} u(x_n) > t$.
Proceeding by contradiction, suppose $\limsup_{x_n \longrightarrow z,\ x_n \in \Omega} u(x_n) > t$. Let $\limsup_{x_n \longrightarrow z,\ x_n \in \Omega} u(x_n) = \tau \Rightarrow \tau > t$. Then, $\exists r \in T; t < r < \tau$. As $\tau > r \Rightarrow \forall n, r < u(x_n) \Rightarrow x_n \in A_r \cap \Omega$.
As $x_n \longrightarrow z$ and $x_n \in A_r$ closed $\Rightarrow z \in A_r \cap \Gamma \Rightarrow$By lemma 3.3.2, $f(z) \geqslant r$.
Contradiction as $f(z) = t < r$.

Hence, for $z \in \Gamma$, $\lim_{x_n \longrightarrow z,\ x_n \in \Omega} u(x_n) = f(z)$

2. Claim 1: $\{x \in \overline{\Omega};\ u(x) \geqslant t\} = \cap_{s<t,\ s \in T} A_s$
Claim 2: $\{x \in \overline{\Omega}; u(x) > t\} = \cup_{s>t,\ s \in T} A_s$

We now show $u$ is continuous on $\Gamma \cup \Omega$. We will do so by proving claim 1 is a closed set and claim 2 is an open set in $\Omega$.
The first claim is a closed set being a countable intersection of closed sets.

It remains to prove claim 2 is an open set in $\Omega$ i.e. $\forall x \in \cup_{s>t} A_s, \exists r > 0, B(x,r) \subset \cup_{s>t} A_s$

So let $x \in \cup_{s>t} A_s \cap \Omega \Rightarrow \exists s_0 > t, x \in A_{s_0} \Rightarrow u(x) \geqslant s_0 > t$.

Since $x \in \Omega \Rightarrow dist(x, \partial\Omega) > 0$.

Since $x \in A_{s_0} \subset A_t \Rightarrow dist(x, \partial A_t) > 0$

Take $r = \frac{1}{2} min\{dist(x, \partial\Omega), dist(x, \partial A_t)\}$.

We now need to prove $B(x,r) \subset \cup_{s>t} A_s$.

Let $x_0 \in B(x,r) \Rightarrow dist(x, x_0) < r \Rightarrow dist(x_o, \partial A_t) > 0 \Rightarrow x_0 \in A_t \Rightarrow u(x_0) > t \Rightarrow x_0 \in \cup_{s>t} A_s$ □

We now illustrate the proof given by [3] to theorem 3.3.1

*Proof.* It remains to prove $u$ is a solution to the LGP i.e. $\forall v \in BV(\Omega)$, $T_\Gamma v = f$, $\int_\Omega |Du| \leqslant \int_\Omega |Dv|$.

Let $u$ be the solution as constructed, and let $v \in BV(\Omega)$, $T_\Gamma v = f$ and $\tilde{v}$ the extension of $v$ to $\Omega_0$; $\tilde{v} = F$ on $\Omega_0 - \overline{\Omega}$ and $v = \tilde{v}$ on $\overline{\Omega}$. Then , $\tilde{v} \in BV(\Omega_0) \cap C(\Omega_0 - \overline{\Omega})$.

Let $G_t = \{\tilde{v} \geqslant t\}$.

$P(G_t, \Omega_0) = H^1(\partial^* G_t \cap \Omega_0) = H^1(\partial^* G_t \cap (\Omega_0 - \Omega)) + H^1(\partial^* G_t \cap \Gamma) + H^1(\partial^* G_t \cap \Omega)$.

By lemma 3.3.3, $\partial^* G_t \cap \Gamma \subset f^{-1}(t) \Rightarrow H^1(\partial^* G_t \cap \Gamma) \leqslant H^1(f^{-1}(t)) = 0 \Rightarrow H^1(\partial^* G_t \cap \Gamma) = 0$.

$\Rightarrow H^1(\partial^* G_t \cap \Omega_0) = H^1(\partial^* G_t \cap \Gamma) + H^1(\partial^* L_t - \Omega)$.

On the other hand, $P(E_t, \Omega_0) = H^1(\partial^* E_t \cap \Omega_0) = H^1(\partial^* E_t \cap (\Omega_0 - \Omega)) + H^1(\partial^* E_t \cap \Gamma) + H^1(\partial^* E_t \cap \Omega) = P(E_t, \Omega) + H^1(\partial^* L_t - \Omega_0)$ because by lemma 3.3.4, $H^1(\partial^* E_t \cap \Gamma) \leqslant H^1(f^{-1}(t)) = 0 \Rightarrow H^1(\partial^* E_t \cap \Gamma) = 0$.

Since by construction $G_t$ satisfies $G_t - \Omega = L_t - \Omega$ and $E_t$ minimizes the perimeter of all such sets, we have, $P(E_t, \Omega_0) \leqslant P(G_t, \Omega_0)$.

From above, we get, $P(E_t, \Omega) \leqslant P(G_t, \Omega)$.

$\Rightarrow \int_{-\infty}^{\infty} P(E_t, \Omega)dt \leqslant \int_{-\infty}^{\infty} P(G_t, \Omega)dt.$

By the coarea formula, theorem 2.5.1, we get, $\int_\Omega |Du| \leqslant \int_\Omega |Dv|$. □

**Proposition 7.** *Let $\Lambda$ as defined and $\gamma$ a connected component of $\partial E_t$. If $\gamma$ intersects $\Lambda$ then it must intersect it orthogonally.*

*Proof.* We know $\gamma$ is a line segment and $\partial E_t$ in $\Omega$ is a minimal surface by proposition 4 and theorem 3.2.1 Suppose $\gamma = [x^t, y^t]$. We proceed by contradiction, suppose $\gamma$ does not intersect $\Lambda$ orthogonally at $x^t$. Consider the ball $B(x^t, r)$ for $r > 0$ such that $B(x^t, r)$ cuts $\gamma$ at $z^t$. There exists a segment $[z^t, w^t]$ that cuts $\Lambda$ orthogonally at $w^t$ and $d(z^t, w^t) < d(z^t, x^t)$. Contradiction as $\gamma$ is of least length. □

### uniqueness of solution

We proceed in proving uniqueness to the solution by supposing if there is another solution to the LGP $min\{\int_\Omega \mid Du \mid ; u \in BV(\Omega), Tu_\Gamma f\}$, and f satisfies a monotonicity condition, then it's a must that the 2 solutions have the same level sets which will lead to the uniqueness of solution.

Let $u$ be a solution to the LGP. We define $\varepsilon_t = \{u \geqslant t\}$ for $t \in u(\Omega)$.

**Lemma 3.3.6.** *Let $u_0$ be the constructed solution. If $u$ is any other solution to the LGP and $\partial\varepsilon_t = \partial\varepsilon_t^0$ then $u = u_0$ in $L^1$.*

**Lemma 3.3.7.** *If $\Omega$ convex and $u$ solution to the LGP for $f$ continuous and bounded on $\Gamma \subset \partial\Omega$ open, we have $\partial\varepsilon_t \cap \Gamma \subset f^{-1}(t)$*

**Theorem 3.3.2.** [3] *Let $u_0$ be the solution to the LGP constructed above and $\partial\varepsilon_t^0 = \{u_0 \geqslant t\}$.*

1. *Let $\Gamma \subset \partial\Omega$ open with endpoints $a$ and $b$.*

   *Let $x_M \in \Gamma$ such that $f$ attains its maximum at $x_M$.*

*Let $f$ strictly increasing on the arc $\overline{ax_M}$ and strictly decreasing on $\overline{x_M b}$ such that $f$ attains each value exactly twice, except at $x_M$, and $f(a) = f(b) = inf_\Gamma f$.*

*We then get $u_0$ is a unique solution and $u_0$ discontinuous on $a$ and $b$. Also, $\exists \tau \in (inf_\Gamma f, sup_\Gamma f)$ such that $|\{u_0 = \tau\}| > 0$.*

2. *Let $x_0 \in \Gamma$; $d(x_0, \Lambda) = d(x_0, a) = d(x_0, b)$ with $f(a) = f(b) = f(x_0)$. Suppose $f$ attains each value twice on the arc $\overline{ax_0}$ with $x_m$ a local minimum and suppose $f$ attains each value exactly twice on the arc $\overline{x_0 b}$ with $x_M$ local maximum.*

   *Then $u_0$ is unique and continuous and $|\{u_0 = f(a)\}| > 0$.*

3. *Let $S := \{x \in \Gamma; d(x, \Lambda) = d(x, y)$ for some $y \in \mathring{\Lambda}\}$.*

   *$D := \{x \in S; \exists$ atleast two $y \in \Lambda; d(x, \Lambda) = d(x, y)\}$.*

   *Let $\phi : S \mapsto P(\Lambda); \phi(x) = \{y \in \Lambda; d(x, \Lambda) = d(x, y)\}$.*

   *One can prove that $D$ is atmost countable.*

   *If $f$ monotone then $u_0$ is unique and discontinuous at $a$ and $b$. Also, $\exists$ atleast one $\tau \in u(\phi(D)); |\{u = \tau\}| > 0$*

*Proof.* 1. Let $u$ be any solution to the LGP. We will first construct the level sets, then prove the existence of $\tau$. Since $f$ takes each value at exactly 2 points in $\Gamma$ except for $x_M$, then for each $t \in (inf_\Gamma f, sup_\Gamma f)$, there exists 2 points $x^t, y^t \in \Gamma$; $f(x^t) = f(y^t) = t$.

If I have a level set at $t$, $\partial\varepsilon_t$, then $\partial\varepsilon_t \cap \Gamma \subset f^{-1}(t) \Rightarrow \forall x \in \partial\varepsilon_t \cap \Gamma$ we must have $f(x) = t$.

Since $\forall t$, $\exists$ 2 points $x^t, y^t$ such that $f(x^t) = f(y^t) = t$ then if I have a level set at $t$, and as level sets must intersect $\partial\Omega$ and in particular $\Gamma$ so that the solution will be continuous up to $\Gamma$, then $x^t, y^t$ must belong to $\partial\varepsilon_t$.

We now consider the map $h : t \mapsto h(t) = d(x^t, y^t) - d(x^t, a) - d(y^t, b)$.

We suppose $d(x^t, \Lambda) = d(x^t, a)$ and $d(y^t, \Lambda) = d(y^t, b)$.

If $\{x^t, y^t\} \in \Gamma$ are very close to $x_M$ such that $f(x_M) - t > 0$ and very small, we have,$d(x^t, y^t) < d(x^t, a) + d(y^t, b) \Rightarrow h(t) < 0$.
Since $\partial\varepsilon_t$ must contain $\{x^t, y^t\}$ and $\partial\varepsilon_t \cap \Gamma \subset f^{-1}(t)$ then $\partial\varepsilon_t$ must be the line segment $[x^t, y^t]$.

If $\{x^t, y^t\} \in \Gamma$ with $x^t$ very close to a and $y^t$ very close to b with $d(x^t, \Lambda) = d(x^t, a)$ and $d(y^t, \Lambda) = d(y^t, b)$ such that $t - f(a) > 0$ and very small we have
$d(x^t, y^t) > d(x^t, a) + d(y^t, b) \Rightarrow h(t) > 0$.
Since $\partial\varepsilon_t$ must contain $\{x^t, y^t\}$ and $d(x^t, a) + d(y^t, b)$ is smaller than $d(x^t, y^t)$, and $\partial\varepsilon_t$ is a minimal surface i.e. the line segment must be of least length, $\partial\varepsilon_t = [x^t, a] \cup [y^t, b]$.

Since $h$ is continuous and $\exists t$ such that $h(t) > 0$ and $\exists t$ such that $h(t) < 0$, by intermediate value theorem, $\exists! \tau \in (inf_\Gamma f, sup_\Gamma f); h(\tau) = 0$.
$\Rightarrow \exists! \tau \in (inf_\Gamma f, sup_\Gamma f); d(x^\tau, y^\tau) = d(x^\tau, a) + d(y^\tau, b)$.
$\Rightarrow \partial\varepsilon_\tau = [x^\tau, a] \cup [x^\tau, y^\tau] \cup [y^\tau, b]$.
Indeed, $\partial\varepsilon_\tau$ is the boundary of the set $\{u = \tau\}$ with $| \{u = \tau\} | > 0$.
Indeed, the set $\{u = \tau\}$ is unique because otherwise there will be another level set to construct with $\partial\varepsilon_t = [c, d]$ with $c, d \in \Lambda$.
Let $v$ be the function constructed by its level sets the same way as $u$ with an additional level set located in $\{u = \tau\}$. We then get $\int |Dv| > \int |Du| = 0$ on $\{u = \tau\}$. But we require minimizing $\int |Du|$ over all $u \in BV(\Omega)$ with $u = f$. Then the construction of $u$ is the best solution one can get.

If we take a sequence of level sets $\forall t > \tau$ the limiting level set will be $[x^\tau, y^\tau]$ and if we take a sequence of level sets $\forall t < \tau$, the limiting level set will be $[x^\tau, a] \cup [y^\tau, b]$. Since $u$ is continuous, for all that is beneath $\partial\varepsilon_\tau$, we have $u = \tau$. We also have, $\forall x \in \Omega - \{u = \tau\}$, $\exists! t$; $x \in \partial\varepsilon_t$. We then have, $\partial\varepsilon_t = \partial\varepsilon_t^0$.

$\Rightarrow u = u_0$ by lemma 3.3.6.

2. Let $u$ be any solution to the LGP. We know that $\partial\varepsilon_t \cap \Gamma \subset f^{-1}(t)$.
As $f$ attains each value twice on $\Gamma$ then $f^{-1}(t) = \{x^t, y^t\}$.We will prove in this case that $\partial\varepsilon_t \cap \Gamma = f^{-1}(t)$. Suppose $\partial\varepsilon_t \cap \Gamma \neq \{x^t, y^t\}$ then $\partial\varepsilon_t = [x^t, c_1] \cup [y^t, c_2]$ with $c_1, c_2 \in \Lambda$. It is a necessity that $c_1$ and $c_2$ are either a or b as we require $\partial\varepsilon_t$ to be of smallest length and $d(x_0, \Lambda) = d(x_0, a) = d(x_0, b)$. Without loss of generality, suppose $x^t, y^t \in \Gamma \cap \overline{x_0 b}$. If $c_1 = a$, then $[x^t, a]$ cuts $\partial\varepsilon_{f(a)}$ which is impossible. Then, $\partial\varepsilon_t = [x^t, b] \cup [y^t, b]$. By the triangular inequality, $d(x^t, y^t) < d(x^t, b) + d(y^t, b)$. Contradiction as $\partial\varepsilon_t$ must be of least length. Therefore, on the arc $\overline{ax_0}$ the level sets are $\partial\varepsilon_t = [x^t, y^t]$ with the limiting level set $[a, x_0]$ and on the arc $\overline{x_0 b}$ the level sets are $\partial\varepsilon_t = [x^t, y^t]$ with the limiting level set $[x_0, b]$. We then get $\partial\varepsilon_{f(a)} = [a, x_0] \cup [x_0, b]$ boundary of the set $\{u = f(a)\}$ with $|\{u = f(a)\}| > 0$.
By the same argument as the proof of 1. , we have uniqueness of the level sets and uniqueness of the the set $\{u = f(a)\} \Rightarrow u = u_0$.

3. Let $u$ be any continuous solution on $\Omega$ to the LGP. Since $f$ takes each value exactly once then for each $t \in (inf_\Gamma f, sup_\Gamma f)$; there exists a unique $x^t \in \Gamma$; $f(x^t) = t$.
Then,$\partial\varepsilon_t$ must have an endpoint $x^t$, but the other endpoint $y \in \Lambda$. Then,$\partial\varepsilon_t = [x^t, y]$ for $y \in \Lambda$.
For points $x^t$ close to a, $\partial\varepsilon_t = [x^t, a]$. As points get away from a, then $\partial\varepsilon_t = [x^t, y]$ for some $y \in \Lambda$. Similar reasoning for the point b.
Since $D$ is atmost countable, then $\exists x \in D, \exists$ atleast $y_1, y_2 \in \Lambda; d(x, y_1) = d(x, y_2) = d(x, \Lambda)$. Let $f(x) = \tau$. Then $\partial\varepsilon_\tau = [x, y_1] \cup [x, y_2]$ and is the boundary of the set $\{u = \tau\} \Rightarrow | \{u = \tau\} | > 0$. Same argument of the above 2 proofs, we prove the uniqueness of $u_0$. $\square$

**Remark 9.** [10] We know if $u$ solution to LGP and $\Omega$ convex then $\partial\varepsilon_t \cap \partial\Omega \subset f^{-1}(t)$ by lemma 3.3.7. Take $\Omega = [0,1] \times [0,1]$ a square and $f = 0$ on 3 sides of $\partial\Omega$ and $f$ a bell shaped curve on the bottom side of $\partial\Omega$.
Suppose there exist a continuous solution on $\overline{\Omega}$ to LGP. Then, $\forall t > 0$, by lemma 3.3.7, $\partial\varepsilon_t$ will be subintervals of the bottom side of $\partial\Omega$ and they will overlap. This is impossible as $\partial\varepsilon_t \cap \partial\varepsilon_s = \phi$ $\forall s \neq t$. Hence, there exists no continuous solution on $\overline{\Omega}$ to LGP.
The problem was indeed in the convexity of $\Omega$ which led to the overlapping of level sets. Therefore, to ensure a continuous solution for all functions $f$, strict convexity of $\Omega$ is a must.

## 3.4 Special case

The above solution to the LGP exists when $\Omega$ strictly convex with lipchitz boundary, and $f$ continuous and bounded on $\partial\Omega$.
Here, we will take a special case of a convex set and prove the existence of a unique continuous solution for the LGP with $f$ continuous and satisfies a monotonicity condition.
Then, for any convex set, one can proceed in a similar manner as for the special case that we will take and guarantee the existence of the LGP for $f$ continuous and monotone.

We take $\Omega$ be a rectangle; $\Omega = [-L, L] \times [-h, h]$ which is a convex but not strictly convex domain.
Let $h_1 = [-L, L] \times \{h\}$
$h_2 = [-L, L] \times \{-h\}$
$v_1 = \{L\} \times [-h, h]$
$v_2 = \{-L\} \times [-h, h]$
and let $\Gamma_1 = \{h_1, v_1\}$

$\Gamma_2 = \{h_2, v_2\}$

Now let $f$ to be strictly monotone on $\Gamma_1$ and $\Gamma_2$.

Without loss of generality, suppose $f$ is strictly increasing on each $\Gamma_1$ and $\Gamma_2$.

**Theorem 3.4.1.** [*3*] *For $\Omega$ and $f$ as stated, there exists $u \in C(\overline{\Omega})$; $u$ is a unique solution to*

$$min\{\int_{\Omega} |Du|;\ u \in BV(\Omega),\ T_{\partial\Omega}u = f\}$$

Before we illustrate the proof of the theorem given by [3], we will state a lemma:

**Lemma 3.4.1.** *Let $\Omega$ be an open set of class $C^1$. Then there exists a surjective continuous linear map, denoted by $\gamma_0$ that sends $W^{1,1}(\Omega) \longrightarrow L^1(\partial\Omega)$. When $U \in W^{1,1}(\Omega) \cap C(\overline{\Omega})$, this trace coincides with the restriction to the boundary. Also, $\exists C > 0$; $\forall u \in L^1(\partial\Omega)$, $\exists U \in W^{1,1}(\Omega)$, $\gamma_0(U) = u$ and $||U||_{W^{1,1}(\Omega)} \leqslant C||u||_{L^1(\partial\Omega)}$*

*Proof.* We will proceed in the proof for theorem 3.4.1 by several steps:

Step 1 We approximate $\Omega$ by a sequence $\Omega_n$ of bounded strictly convex domains in a way $\Omega_n$ is made up of 4 circular arcs passing through the vertices of $\Omega$ and $d(x, \partial\Omega_n) \leqslant \frac{1}{n}$ $\forall x \in \partial\Omega$.

We then have $\partial\Omega_n = \partial\Omega + \nu\gamma_n$ with $\nu$ unit outer normal to $\partial\Omega$ and $\gamma_n$ a smooth function; $0 \leqslant \gamma_n \leqslant \frac{1}{n}$.

We also define for $x \in \partial\Omega$, $f_n(x + \nu\gamma_n(x)) := f(x)$ which remains to be continuous on $\partial\Omega_n$ since $f$ is so.

We now define the LGP :

$min\{\int_{\Omega_n} \mid Du \mid; u \in BV(\Omega_n), T_{\partial\Omega_n}u = f_n\}$

From theorem 3.3.1 , we know that this problem has a unique continuous solution on $\overline{\Omega_n}$ which we will denote by $v_n$.

Step 2: We now restrict $v_n$ to $\Omega$ by setting $u_n = v_n\mathbb{1}_{\Omega}$.

We now prove $u_n \in BV(\Omega)$ and $u_n \longrightarrow u$ in $L^1(\Omega)$.

As $f_n$ is continuous on $\partial\Omega_n$, then there exists $F_n \in W^{1,1}(\Omega_n); F_n = f_n$ on $\partial\Omega_n$ and $||DF_n||_{L^1(\Omega_n)} \leqslant C_n||f_n||_{L^1(\partial\Omega_n)}$. Then,

$$\int_{\Omega} |Du_n| \leqslant \int_{\Omega_n} |Dv_n| \leqslant ||DF_n||_{L^1(\Omega_n)} \leqslant C_n||f_n||_{L^1(\partial\Omega_n)}$$

But

$$C_n||f_n||_{L^1(\partial\Omega_n)} \leqslant C_n||f_n||_{L^\infty(\partial\Omega_n)}|\partial\Omega_n| \leqslant C||f_n||_{L^\infty(\partial\Omega_n)} < \infty$$

$\Rightarrow \int_\Omega |Du_n| < \infty$

$\Rightarrow u_n \in BV(\Omega)$

By compactness of $BV$ in $L^1$, there exists a subsequence still denoted by $u_n$ and a function $u$ such that $u_n \longrightarrow u$ in $L^1(\Omega)$

By semicontinuity, we get $\int_\Omega |Du| \leqslant \liminf \int_\Omega |Du_n| < \infty$

$\Rightarrow u \in BV(\Omega)$

Step 3 We now prove $u$ obtained in Step 2 is a least gradient function for some function $g_n$ such that $T_{\partial\Omega}u = g_n$ by proving $u_n$ is a least gradient function satisfying $T_{\partial\Omega}u_n = g_n$

In fact, for $z \in \partial\Omega$, $g_n(z) = \lim_{y\longrightarrow z,\ y\in\Omega} u_n(y) = \lim_{y\longrightarrow z,\ y\in\Omega} v_n(y) = v_n(z)$ as $v \in C(\overline{\Omega_n})$.

Then, by section 3.2 proposition 6, as $v_n$ is a least gradient function on $\Omega_n$ by step 1, then $v_n|_\Omega = u_n$ is a least gradient function on $\Omega$.

As $u_n \longrightarrow u$ in $L^1(\Omega)$ then u is a least gradient function on $\Omega$ to $min\{\int_\Omega |Dv|;$ $v \in BV(\Omega),\ T_{\partial\Omega}v = g_n\}$ by proposition 3.

Step 4 We prove $u$ converge uniformly to a continuous function w. From step 2 and 3, we have $u_n \longrightarrow u$ in $L^1(\Omega)$ and u is a least gradient function satisfying $T_{\partial\Omega}u = g_n$.

Then, we get $u = w$ a.e. and thus $w$ is a continuous least gradient function satisfying $T_{\partial\Omega}w = g_n$.

Construction of $w$:

For $t \in (minf, maxf)$, let $l^t$ be the line segment joining $x^t \in \Gamma_1$ and $y^t \in \Gamma_2$; $f(x^t) = f(y^t) = t$.
In fact, the $l^t$ are disjoint because otherwise, suppose $\exists t_1, t_2 \in (minf, maxf), l^{t_1}, l^{t_2}$ 2 line segments joining $x^{t_1}, y^{t_1}$ and $x^{t_2}, y^{t_2}$ respectively such that $l^{t_1}, l^{t_2}$ meet at some point inside $\Omega$.
Without loss of generality, we will get $x^{t_1} > x^{t_2}$ and $y^{t_1} < y^{t_2}$. By continuty of $f$ and being strictly increasing on each $\Gamma_i$ $i = 1, 2$ , we get $f(x^{t_1}) = t_1 > f(x^{t_2}) = t_2$ and $f(y^{t_1}) = t_1 < f(y^{t_2}) = t_2$ . Contradiction.

**Lemma 3.4.2.** *For $z \in \Omega$, there exists a unique $l^t$ passing through $z$ $\forall t \in (minf, maxf)$*

*Proof.* Let $z \in \Omega$. Without loss of generality, take $z$ below diagonal joining the endoints of $\Gamma_1$. Let s be the arc length parameter of $\Gamma_1$. We then get, $0 \leqslant s \leqslant (\int_{-L}^{L} \sqrt{1}dx + \int_{-h}^{h} \sqrt{1}dy) = 2(L+h)$.
Let $x(s)$ be a parametrization of $\Gamma_1$.
For each s, let $l(x(s), z)$ be the line segment passing through z and touching $\Gamma_2$ at $y(s)$.
We then have $y(s)$ continuous.
Now let $\lim_{s \longrightarrow 0^+} f(x(s)) = minf$ , and $\lim_{s \longrightarrow (2(L+h))^+} f(x(s)) = maxf$
Then, $\lim_{s \longrightarrow 0^+} (f(x(s)) - f(y(s))) = minf - \lim_{s \longrightarrow 0^+} f(y(s)) \leqslant 0$
and $\lim_{s \longrightarrow (2(L+h))^+} (f(x(s)) - f(y(s))) = maxf - \lim_{s \longrightarrow (2(L+h))^+} f(y(s)) \geqslant 0$
By intermediate value theorem, as $f(x(s)) - f(y(s))$ is continuous, there exists a unique $s_0$; $l^t = [x(s_0), y(s_0)]$; $z \in l^t$ and $f(x(s_0)) = f(y(s_0)) = t$ $\square$

We now define $w : \Omega \mapsto \mathbb{R}$ defined by:

$$w(x) = t \ for \ every \ x \in \Omega$$

By lemma 3.4.2, this map is well defined and bijective.

Indeed, one can prove $w$ to be continuous on $\Omega$ resulting with the inequality , $\forall x_1, x_2 \in \Omega$,

$|w(x_1) - w(x_2)| = |t_1 - t_2| = |f(x^{t_1}) - f(x^{t_2})| \leqslant \omega(c_1|x_1 - x_2 \mid +c_2\sqrt{|x_1 - x_2|})$

with $\omega$ the continuity modulus of f.

**Lemma 3.4.3.** *$u_n$ converges uniformly to $w$*

*Proof.* We first prove that $u_n$ is a cauchy sequence.

Let $u_l(x) = t_1$ and $u_k(x) = t_2$ for $x \in \Omega$.

Since $Tu_n = v_n \Rightarrow$ level sets of $u_n, \partial\{u_n \geqslant t\}$, are straight lines with endpoints on $\partial\Omega$ at which $v_n$ takes the value $t$.

But $v_n$ takes the value $t$ along the level sets $\partial\{v_n \geqslant t\}$ which has endpoints on $\partial\Omega_n$ at which $f_n$ takes the value $t$.

We have $x \in \partial\{u_l \geqslant t_1\} = l_l^{t_1}$ and $x \in \partial\{u_k \geqslant t_2\} = l_k^{t_2}$.

Let $l^{t_1}$ and $l^{t_2}$ be the 2 line segments as defined in this section such that $f$ takes the value $t_1$ on the endpoints of $l^{t_1}$ and $f$ takes the value $t_2$ on the endpoints of $l^{t_2}$. By construction, as the endpoints of $l^{t_1}$ and $l_l^{t_1}$ and the endpoints of $l^{t_2}$ and $l_k^{t_2} \leqslant \frac{1}{n}$, $n = min\{k, l\}$ then $\forall x_l \in l^{t_1}$ and $x_k \in l^{t_2}$ we have $d(x_l, x) < \frac{1}{n}$ and $d(x_k, x) < \frac{1}{n}$.

$|u_l(x) - u_k(x)| = |t_1 - t_2| = |w(x_l) - w(x_k)| \leqslant \omega(c_1|x_l - x_k| + c_2\sqrt{|x_l - x_k|})$

But $|x_l - x_k| = |x_l - x_k + x - x| \leqslant |x_l - x| + |x_k - x| \leqslant \frac{2}{n}$

and $\sqrt{x_l - x_k} \leqslant \frac{\sqrt{2}}{\sqrt{n}}$ with $n = min\{k, l\}$.

$\Rightarrow |u_l(x) - u_k(x)| \leqslant \omega(\frac{c_1 2}{n} + \frac{c_2\sqrt{2}}{\sqrt{n}}) \longrightarrow 0$ as $k, l \longrightarrow \infty$.

$\Rightarrow u_n$ is a cauchy sequence.

As $u_n$ is a cauchy sequence in $\Omega \subset R^2$ complete $\Rightarrow u_n$ converges uniformly to $w$. □

Step 5 It remains to prove $Tw = f$.

Let $z \in \partial\Omega$. We need to prove $\lim_{y \longrightarrow z,\ y \in \Omega} w(y) = f(z)$.
Without loss of generality, suppose $z \in \Gamma_1$. For $y \in \Omega, \exists! l^{t_1} = [x^{t_1}, y^{t_1}]$ passing through $y$; $w(y) = t_1$.
Since $z \in \partial\Omega$ then there exists a unique $l^{t_2} = [z, z^{'}]$ with $z^{'} \in \Gamma_2$ and $f(z) = f(z^{'}) = t_2$.

$|w(y) - f(z)| = |t_1 - t_2| = |f(x^{t_1}) - f(z)| \leqslant \omega(c_1|y - z| + c_2\sqrt{|y - z|})$
So as $y \longrightarrow z$, we get $w(y) \longrightarrow f(z)$.
Hence, $Tw = f$

Step 6 We prove uniqueness of the solution.
We have that, for $v$ any solution of $min\{\int_\Omega \mid Du \mid; u \in BV(\Omega), Tu = f\}$,
$\partial\{v \geqslant t\} \cap \partial\Omega \subset f^{-1}(t)$.
By minimality of the level sets, then $\partial\{v \geqslant t\} = \partial\{w \geqslant t\}$ and thus $u = w$.

$\square$

**Remark 10.** 1. In step 4, from the inequality resultng from continuity of the the solution $w$, we can see that if $f \in C^\alpha(\partial\Omega)$ then $w \in C^{\frac{\alpha}{2}}(\overline{\Omega})$.

2. One can take $f$ to be strictly increasing on $h_i$ and constant on $v_i$. (vice versa also works) for $i = 1, 2$. There will exist a unique solution by proceeding in a similar manner and the level sets are constructed in the following way: $\forall x \in \Omega$, $\exists!$ $l^t$ line segment with endpoints $x^t$ on $h_1$ and $y^t$ on $h_2$ with $t \in (minf, maxf)$. We define the solution $w(x) = t$ and all other steps are the same as was done.

3. One can now generalize to any convex set by proceeding in a similar way as was done and constructing the level sets $l^t$ that fill up $\Omega$.

## 3.5 Connection between LGP and FMD

Let $\Omega$ be a plane domain with lipchitz boundary.

**Definition 3.5.1.** We define the problem that appears in free material design to be(FMD):

$$inf\{\int_{\Omega} |p|,\ p \in L^1(\Omega, \mathbb{R}^2),\ divp = 0,\ p \cdot \nu|_{\partial\Omega} = g\}$$

FMD is the problem of finding the least material distribution of a body to handle a load applied to its boundary. $\nu$ is the unit outer normal to $\partial\Omega$. For the normal trace to be well defined, we require $\Omega$ to belong to a special class of lipchitz domains called deformable lipchitz domain.This special class contains convex sets. Hence, we will consider $\Omega$ to be convex with $\partial\Omega$ lipchitz continuous. Also, as $p \in L^1(\Omega)$, $divp$ is viewed to be in the distributional sense.
As $L^1$ is not weakly* closed, we cannot ensure the existence of minimizers to the problem.

We now consider the following two problems:
$1 - min\{\int_{\Omega} |Du|,\ u \in BV(\Omega),\ T_{\partial\Omega}u = f\}$
$2 - inf\{\int_{\Omega} |p|,\ p \in L^1(\Omega, \mathbb{R}^2),\ divp = 0,\ p \cdot \nu|_{\partial\Omega} = g\}$

As stated in previous section, Problem 1 is the least gradient problem(LGP). We will take $\partial\Omega$ to be lipchitz continuous and $f \in L^1(\partial\Omega)$.

As we are interested in finding solution to LGP, it is proved in [3] that a relation does exist between problem 1 and problem 2. This relation states that finding an element to one of the 2 problems leads to finding an element of the second.

**Proposition 8.** *Let $p \in L^1(\Omega, \mathbb{R}^2)$ with $divp = 0$ and $\Omega$ convex. Then there exists $u \in W^{1,1}(\Omega) \subset BV(\Omega)$ such that $p = R_{-\frac{\pi}{2}}\nabla u$.*

*Also, if* $p \cdot \nu|_{\partial\Omega} = g$ *and* $T_{\partial\Omega} u = f$*, then* $g = \frac{\partial f}{\partial \tau}$.

*In other words, having an element in the set of problem 2 gives an element in the set of problem 1.*

**Theorem 3.5.1.** [3] *Let* $u$ *be a solution to problem 1 and* $\Omega$ *convex. Then* $q = R_{-\frac{\pi}{2}} \nabla u$ *is a solution to problem 2 with* $q \cdot \nu|_{\partial\Omega} = \frac{\partial f}{\partial \tau} = g$ *and* $divq = 0$.

*Proof.* Let $u$ be a solution to problem and let $q := R_{-\frac{\pi}{2}} \nabla u$. Let M be the solution to problem 2. We need to prove $\int_\Omega |q| dx = M$, $divq = 0$, $q \cdot \nu|_{\partial\Omega} = g$.

Let $p_n$ be a minimizing sequence of problem 2 such that $\int_\Omega |p_n| dx \longrightarrow M$ and $divp = 0$, $p_n \in L^1(\Omega, \mathbb{R}^2)$, $p_n \cdot \nu|_{\partial\Omega} = g$. By Proposition 8, $\exists v_n \in W^{1,1}(\Omega) \subset BV(\Omega); p_n = R_{\frac{-\pi}{2}} Dv_n$ and $Tv_n = f$.
As $u$ solution and $v_n$ is an element of the set of problem 1, $M = \int_\Omega |p_n| dx = \int_\Omega |Dv_n| dx \geqslant \int_\Omega |Du| dx = \int_\Omega |q| dx$.
We get $M = \int_\Omega |q| dx$ as it is impossible for M to be strictly greater than $\int_\Omega |Du| dx$ since by construction it is a minimizing sequence.

Also, as $q$ is the rotation of $Du$ by angle $\frac{-\pi}{2} \Rightarrow divq = 0$.

It remains to show $q \cdot \nu|_{\partial\Omega} = g$.
As $u \in BV(\Omega)$, by approximation of BV functions $\exists w_n \in BV(\Omega) \cap C^\infty(\Omega); w_n \longrightarrow u$ in $L^1(\Omega)$ and $\int_\Omega | Dw_n | \longrightarrow \int_\Omega | Du |$ as $n \longrightarrow \infty$. We set $Tw_n = f = Tu$.
Then, $\exists$ a subsequence, still denoted by $Dw_n; Dw_n \longrightarrow Du$ weakly as measures.
We set $\tilde{p_n} := R_{\frac{-\pi}{2}} Dw_n$. By Proposition 1, $\tilde{p_n} \cdot \nu|_{\partial\Omega} = \frac{\partial T w_n}{\partial \tau} = \frac{\partial f}{\partial \tau}$.
Let $\varphi \in Lip(\gamma, \partial\Omega)$ with $\gamma > 1$ and $\Phi$ the extension of $\varphi$ to $Lip(R^2)$,
$< q \cdot \nu|_{\partial\Omega}, \varphi > = < divq, \Phi > + \int_\Omega q \nabla \Phi dx = \int_\Omega q \nabla \Phi dx$ as $divq = 0$.
But $Dw_n \longrightarrow Du \Rightarrow \tilde{p_n} = R_{\frac{-\pi}{2}} Dw_n \longrightarrow R_{\frac{-\pi}{2}} Du = q$ weakly as measures.
Then, $< q \cdot \nu|_{\partial\Omega}, \varphi > = \int_\Omega q \nabla \Phi dx = \lim_{n \longrightarrow \infty} \int_\Omega \tilde{p_n} \nabla \Phi dx = \lim_{n \longrightarrow \infty} < \tilde{p_n} \cdot$

$\nu|_{\partial\Omega}, \varphi >= \lim_{n \longrightarrow \infty} < g, \varphi >=< g, \varphi >$.
$\Rightarrow q \cdot \nu|_{\partial\Omega} = g$. □

### 3.5.1 Example 1

We now show an example where the relation between FMD and LGP is used resulting in a piecewise constant not continuous function $f$ defined on $\partial\Omega$. We find a solution to the LGP with this $f$.
Suppose $\Omega$ strictly convex, $\partial\Omega$ smooth. Consider the FMD problem with $g$ a distribution $\Rightarrow g = \sum_{i=1}^{3} c_i \delta_{a_i}$ with $a_i \in \partial\Omega$ and $\delta_{a_i}$ the delta function. In other words, we apply a load on some points of $\partial\Omega$ and the rest of $\partial\Omega$ remains free.

We assume $\Omega$ is at rest, then for stability we take $\int_\Omega dg = 0$
$\Rightarrow \int_\Omega d(\sum_{i=1}^{3} c_i \delta_{a_i}) = \sum_{i=1}^{3} c_i \int_\Omega \delta_{a_i} dx = \sum_{i=1}^{3} c_i = 0$.
We take $c_1 = \alpha_1 + \alpha_2, c_2 = -\alpha_1, c_2 = -\alpha_2$ so that $\sum_{i=1}^{3} c_i = 0$.
We parametrize $\partial\Omega$ by $s \mapsto x(s)$ for $s \in [0, L)$.
$\Rightarrow g = \alpha_1 + \alpha_2 x(s_1) - \alpha_2 x(s_2) - \alpha_1 x(0)$ with $x(s_1), x(s_2), x(s_0)$ 3 points on $\partial\Omega$ and $x(s_0) = x(0)$.

By FMD and LGP relation, $g = \frac{\partial f}{\partial \tau}$.

We then get, $g = \begin{cases} 0 & s \in [0, s_1) \\ \alpha_1 + \alpha_2 & s \in [s_1, s_2) \\ \alpha_1 & s \in [s_2, L) \end{cases}$

$f$ is piecewise constant and discontinuous on $x(0), x(s_1), x(s_2)$. We assume $\alpha_1, \alpha_2$ are positive, and $f$ is unique up to a constant.

We now seek a soution to the LGP, $min\{\int_\Omega \mid Du \mid ; u \in BV(\Omega), Tu = f\}$.

We approximate $f$ by a continuous function $f^\epsilon$ such that $f = f^\epsilon$ everywhere except at $\{x; d(x, x(s_i) < \epsilon, i = 0, 1, 2\}$. Then, by theorem 3.3.1, the LGP $min\{\int_\Omega \mid Du \mid; u \in BV(\Omega), Tu = f^\epsilon\}$ has a solution which we will denote by $u^\epsilon$.

Since $u^\epsilon$ and $u^\delta$ differ on a neighborhood of $x(s_i)$ $i = 0, 1, 2$,
$||u^\epsilon - u^\delta||_{L^1(\Omega)} \leqslant C\{\epsilon, \delta\}[\mid x(s_0) - x(s_1) \mid + \mid x(s_1) - x(s_2) \mid + \mid x(s_2) - x(s_0) \mid]$.
As $\epsilon, \delta \longrightarrow 0$, $||u^\epsilon - u^\delta||_{L^1(\Omega)} \longrightarrow 0$.
$\Rightarrow$ $u^\epsilon$ is a cauchy sequence in $L^1(\Omega)$ and as $L^1(\Omega)$ Banach space$\Rightarrow$ $u^\epsilon \longrightarrow u$ in $L^1(\Omega)$.

Since $u^\epsilon$ is a least gradient function and $u^\epsilon \longrightarrow u$ in $L^1(\Omega)$ $\Rightarrow$ by proposition 3. $u$ is a least gradient function with $tu = f^\epsilon$. But as $\epsilon \longrightarrow 0$, $f^\epsilon = f$.
$\Rightarrow Tu = f$
$\Rightarrow$ $u$ is a solution to $min\{\int_\Omega \mid Du \mid; u \in BV(\Omega), Tu = f\}$ .

Uniqueness of the solution is claimed but still not proved rigorously. However, [3] states that uniqueness is expected to be achieved as above by constructing the level sets and claiming their uniqueness.

### 3.5.2 Example 2

We now represent an example given by [3] to find a solution to the LGP with $\Omega$ a rectangle and f, not satisfying a monotonicity condition, being the function resulting from the relation between LGP and FMD with the knowledge that a load $g$ is applied on a part of $\partial\Omega$ and $\Omega$ remains at rest.

Let $\Omega = (-L, L) \times (-h, h)$ a rectangle and $g$ a load applied on $\partial\Omega$ in the following way:

$$g = \begin{cases} l_B & [-b,b] \times \{-h\} \\ l_T & [-t,t] \times \{h\} \\ 0 & \text{on the rest of } \partial\Omega \end{cases}$$

where $[-b,b],[-t,t] \subset [-L,L]$

We suppose $\Omega$ is at rest such that $\int_{\partial\Omega} g dx = 0$
$\Rightarrow \int_{\{h\}} \int_{-t}^{t} l_T dx dy + \int_{\{-h\}} \int_{-b}^{b} l_B dx dy = 0$
$\Rightarrow h l_T x|_{-t}^{t} - h l_B x|_{-b}^{b} = 2t l_T h - 2b l_B h = 0$
But $h \neq 0$ and $h[2t l_T - 2b l_B] = 0$
$\Rightarrow 2t l_T = 2b l_B$.

By FMD and LGP relation, $\exists f \in L^1(\partial\Omega); g = \frac{\partial f}{\partial \tau}$.
As we want $f$ to be continuous, we proceed in the following way:
As $g = 0$ on $\{-L\} \times [-h,h] \Rightarrow \frac{\partial f}{\partial y} = 0 \Rightarrow f(-L,y) = C$ with $C$ constant. We take $C = 0$.
$\Rightarrow f(x,y) = 0$ on $\{-L\} \times [-h,h]$.
By continuity of $f$, and as g remains to be 0 we get $f(x,y) = 0$ on $[-L,-b] \times \{-h\}$.
On $[-b,b] \times \{-h\}, g = l_B \Rightarrow f(x,y) = l_B x + k(y)$. By continuity of $f, f(-L,-h) = f(-b,-h) = 0 \Rightarrow -l_B b + k(y) = 0 \Rightarrow k(y) = l_B b$.
$\Rightarrow f(x,y) = l_B x + b l_B$ on $[-b,b] \times \{-h\}$.
As we require $f$ to be continuous and $g = 0$ on $[b,L] \times \{-h\}$ and on $\{L\} \times [-h,h]$ and $f(b,-h) = 2b l_B \Rightarrow f(x,y) = 2b l_B$ on $[b,L] \times \{-h\}$ and on $\{L\} \times [-h,h]$.
We proceed in a similar way on $[-L,L] \times \{h\}$ to get:
$f(x,y) = 0$ on $[-L,-t] \times \{h\}$,
$f(x,y) = l_T x + t l_T$ on $[-t,t] \times \{h\}$
$f(x,y) = 2t l_T$ on $[t,L] \times \{h\}$ and $\{L\} \times [-h,h]$.
In fact, as $2t l_T = 2b l_B$ we then guarentee the continuity of f.

Conclusion, $f(x,y) = \begin{cases} 0 & \text{on } \{-L\} \times [-h,h] \text{ and } [-L,-t] \times \{h\} \text{ and } [-L,-b] \times \{-h\} \\ l_B x + b l_B & \text{on } [-b,b] \times \{-h\} \\ l_T x + t l_T & \text{on } [-t,t] \times \{h\} \\ 2b l_B & \text{on } [b,L] \times \{-h\} \text{ and } \{L\} \times [-h,h] \\ 2\text{tl}_T & \text{on } [t,L]\{h\} \end{cases}$

We now aim on finding a solution to the LGP, with boundary data $f$.

If we take $t = b = L$ then $f$ is strictly increasing on $h_1, h_2$ and constant on $v_1, v_2$ where $h_1, h_2, v_1, v_2$ are defined in section 3.4.
Then, by theorem 3.4.1, there exists a unique continuous solution to the LGP.
However, we have that $f$ is strictly increasing on $[-t,t]\times\{h\}$ and on $[-b,b]\times\{-h\}$, which implies $f$ attains each value at exactly 2 points, one in $[-t,t]\times\{h\}$ and one in $[-b,b]\times\{-h\}$, except for the minimum and maximum of $f$ which are attained at more than 2 points.
We extend $f$ to a function $f_\epsilon = f + k_\epsilon$ continuous and strictly increasing on $h_1$ and $h_2$ and constant on $v_1$ and $v_2$ with

$$k_\epsilon(x,y) = \begin{cases} -\epsilon & \text{on } \{-L\} \times [-h,h] \\ 0 & \text{on } [-b,b] \times \{-h\} \text{ and } [-t,t] \times \{h\} \\ (x+t)\frac{\epsilon}{L-t} & \text{on } [-L,-t] \times \{h\} \\ (x+b)\frac{\epsilon}{L-b} & \text{on } [-L,-b] \times \{-h\} \\ (x-b)\frac{\epsilon}{L-b} & \text{on } [b,L] \times \{-h\} \\ (x-t)\frac{\epsilon}{L-t} & \text{on } [t,L] \times \{h\} \\ \epsilon & \text{on } \{L\} \times [-h,h] \end{cases}$$

By theorem 3.4.1, there exists unique $u_\epsilon \in C(\overline{\Omega})$, solution to the LGP with $Tu_\epsilon = f_\epsilon$ on $\partial\Omega$.

Denote by $P_1$ the polygon with $\partial P_1 = \{\{-L\} \times [-h, h], [-L, -t] \times \{h\}, [-L, -b] \times \{-h\}$, and the line segment $l_1$ with $\partial l_1 = \{(-t, h), (-b, -h)\}\}$ and $P$ the polygon with $\partial P = \{l_1, [-t, t] \times \{h\}, [-b, b] \times \{-h\}$, and the line segment $l_2$ with $\partial l_2 = \{(b, h), (t, h)\}\}$ and $P_2$ the polygon with $\partial P_2 = \{l_2, [t, L]\{h\}, [b, L] \times \{-h\}, \{L\} \times [-h, h]\}\}$.

By construction of level sets as in section 3.4, we now know that every level set joins a point in $h_1$ to a point in $h_2$ and $u_\epsilon$ takes the value on each level set same as the value of $f_\epsilon$ on the endpoints of the level set. Therefore, as
$-\epsilon \leqslant f_\epsilon \leqslant 0$ on $\partial P_1 - l_1$,
$0 \leqslant f_\epsilon \leqslant 2bl_B$ on $\partial P - \{l_2, l_1\}$ and
$2bl_B \leqslant f_\epsilon \leqslant 2bl_B + \epsilon$ on $\partial P_2 - l_2$.

Then:
$-\epsilon \leqslant u_\epsilon \leqslant 0$ on $P_1$
$0 \leqslant u_\epsilon \leqslant 2bl_B$ on $P$
$2bl_B \leqslant u_\epsilon \leqslant 2bl_B + \epsilon$ on $P_2$

$$\text{We now set } u = \begin{cases} 0 & \text{on } P_1 \\ \mathrm{u}_\epsilon & \text{on } P \\ 2\mathrm{bl}_B & \text{on } P_2 \end{cases}$$

As $u_\epsilon$ is independent of $\epsilon$ on $P$ then it is possible to take $u = u_\epsilon$ on $P$.
It is evident that u is continuous on $\overline{\Omega}$.
I now prove $u_\epsilon$ converge in $L^1$ to $u$ because as $u_\epsilon$ is a least gradient function and if $u_\epsilon \longrightarrow u$ in $L^1(\Omega)$ then $u$ is a least gradient function. In fact,
$-\epsilon \leqslant u_\epsilon - u \leqslant 0$ on $P_1$,
$u_\epsilon - u = 0$ on $P$,

$0 \leqslant u_\epsilon - u \leqslant \epsilon$ on $P_2$

$\Rightarrow \mid u - u_\epsilon \mid \leqslant \epsilon$ in $\Omega$

$\Rightarrow u_\epsilon$ converges uniformly to $u$ in $\Omega$

$\Rightarrow u_\epsilon \longrightarrow u$ in $L^1(\Omega)$

$\Rightarrow u$ is least gradient function.

It remains to show $u$ has the correct trace $Tu = f$ on $\partial\Omega$. In fact,
On $P$: $Tu = Tu_\epsilon = u_\epsilon = f_\epsilon = f$.
On $P_1$: Let $z \in \{-L\} \times [-h, h]$ or $[-L, -t] \times \{h\}$ or $[-L, -b] \times \{-h\}$,
$\lim_{y \longrightarrow z,\ y \in \Omega} u(x, y) = \lim_{y \longrightarrow z,\ y \in \Omega} 0 = 0 = f(z)$
On $P_2$: Let $z \in [t, L] \times \{h\}$ or $[b, L] \times \{-h\}$ or $\{L\} \times [-h, h]$
$\lim_{y \longrightarrow z,\ y \in \Omega} u(x, y) = \lim_{y \longrightarrow z,\ y \in \Omega} 2bl_B = 2bl_B = f(z)$

# Chapter 4

# Constrained Least Gradient Problem

Based on [10], we now consider a rod and an external force applied on the rod. We suppose the rod has an exterior constant cross section $\Omega$. Our aim is to find the cross section with least area that will resist the load without deforming. We can consider this problem as a 2-dimensional problem with stress satisfying a yield condition. Without loss of generality, we suppose the stress never exceeds 1 in magnitude or else the cross section yields plastically.

We may vary the cross sections by removing material from $\Omega$ and aim for the cross section with the least area and the stress not exceeding 1.

However, we instead fix $\Omega$ and vary the stresses in $\Omega$ and remove material where the stress is zero.

We can translate this problem into the following:

We let $u$ be a function defined on $\overline{\Omega}$ that gives a stress $R_{-\frac{\pi}{2}}\nabla u$ i.e. rotation of $\nabla u$ by angle $-\frac{\pi}{2}$. We let $f$ be a function defined on $\partial\Omega$ i.e. the load applied on $\partial\Omega$.

We define $w : [0,1] \mapsto \mathbb{R}$ by $t \mapsto w(t) = \begin{cases} 1 & \text{for t} \neq 0 \\ 0 & \text{for t=0} \end{cases}$

Our problem is then:

$$min\{\int_{\Omega} w(|\nabla u|)\ ; |\nabla u| \leqslant 1\ a.e.\ in\ \Omega\ ,\ u = f\ on\ \partial\Omega\}$$

We know that $\int_{\Omega} w(|\nabla u|) \geqslant 0$ since $|\nabla u| \geqslant 0$.
For $|\nabla u| = 0 \Rightarrow min\{\int_{\Omega} w(|\nabla u|);\ |\nabla u| \leqslant 1$ a.e. in $\Omega$, $u = f$ on $\partial\Omega\} = 0$.
For $|\nabla u| \neq 0$, $w(|\nabla u\ |) = 1 \Rightarrow \int_{\Omega} w(|\nabla u|)dA =$ Area($\Omega$).
However, the integrand is nonconvex, and by a well known phenomenon in mathematics, it is a barrier to finding the existence of a solution. We then convexify $w$ by finding the greatest convex function smaller than $w$. It turns out that the convexification of $w$ is $\tilde{w}(t) = t$.
We now solve the problem:

$$min\{\int_{\Omega} \tilde{w}(|\nabla u|)\ ;\ |\nabla u| \leqslant 1\ a.e.\ in\ \Omega\ , u = f\ on\ \partial\Omega\}$$

As $\tilde{w} \leqslant w \Rightarrow \int_{\Omega} \tilde{w} \leqslant \int_{\Omega} w \Rightarrow$ by attaining a solution to the convexified problem, we attain an infimum, might not be a minimum, to our original problem.
We realize that the convexified problem is the least gradient problem with an additional constraint.

**Definition 4.0.1.** We define the constrained least gradient problem to be:

$$min\{\int_{\Omega} |\nabla u|\ ; |\nabla u| \leqslant 1\ a.e.\ in\ \Omega,\ u = f\ on\ \partial\Omega\}$$

with $f$ lipchitz continuous on $\partial\Omega$ satisfying $|f(p) - f(q)| \leqslant d_{\Omega}(p,q)\ \forall p,q \in \partial\Omega$ where $d_{\Omega}(p,q) = inf\{$ length of $\gamma\}$ with $\gamma$ any path joining $p$ to $q$ lying in $\overline{\Omega}$.

The main ingredient in the method of solution is to start by studying the sets $\{x \in \Omega;\ u(x) \geqslant t\}$, and then go on to studying their boundaries, with a view to showing that those sets solve some minimum problem, in a well-defined notion of perimeter.

Now if $\Omega$ is a convex set, then the condition $|\nabla u| \leqslant 1$ a.e. in $\Omega$ in equivalent to $|u(x) - u(y)| \leqslant |x - y|$ for all $x, y \in \Omega$, as can be easily seen, since the line segment joining $x$ to $y$ lies completely in $\Omega$. If $\Omega$ is not convex, we can consider all paths in $\Omega$ joining $x$ to $y$, and then we will have that $|u(x) - u(y)| \leqslant \{$ length of the shortest path in $\Omega$ joining $x$ to $y\}$.

## 4.1 Characterization of level sets

Unlike the LGP, level sets of the solution to the constrained LGP need not to be minimal surfaces due to the constraint $|\nabla u| \leqslant 1$. Indeed, consider a 2-dimensional case with $\Omega \subset \mathbb{R}^2$. Let $\gamma_t$ denote the level curve of $u$ at $t \in \mathbb{R}$. The boundary points at which $u = f = t$ must belong to $\gamma_t$ and along $\gamma_t$ one has $u = t$. $\gamma_t$ must avoid all balls of center $p \in \partial\Omega$ and radius $|f(p) - t|$ because then the distance between the points that give $u = t$ inside the ball and $p$ is less than $|f(p) - t|$ i.e. $|x - p| \leqslant |u(p) - u(x)|$ for $x \in \gamma_t$, contradicting the fact that $|\nabla u| \leqslant 1$. Hence, $\gamma_t$ may not need to be a straight line and thus a minimal surface. Indeed, each level curve must avoid a set which is a union of open disks.
We shall now illustrate an example in $\mathbb{R}^2$ given by [11] aiming for one to see the difference in the construction of level sets between the LGP and the constrained LGP with boundary data deduced by the connection between FMD and LGP.

**Example** Consider $\Omega = [0, 1] \times [0, 1]$ to be the unit square. Consider the FMD problem with $g$ defined on $\partial\Omega$ in the following way: $g = 1$ on the bottom side, $g = -1$ on the left side, and $g = 0$ on the right and top side.
By the connection between FMD and LGP, one has $g = \frac{\partial f}{\partial \tau} \Rightarrow f = x$ on the bottom side, $f = y$ on the left side, and $f = 1$ on the right and top side.
As seen in previous chapter, without the constraint $|\nabla u| \leqslant 1$, one gets the LGP with level sets as straight lines. As $f$ takes each value exactly twice one on the bottom side and one on the left side, level sets are straight lines joining these 2

points $(t, 0)$ and $(0, t)$ for each $t$ between 0 and 1. The lines equations will then be $y + x = t$. Also, one has $u$ constant with $u = 1$ in the square with $y + x \geqslant 1$. As $u$ takes a constant value $t$ along each level set joining $(t, 0)$ and $(0, t)$, then one deduces the unique solution to the LGP is $u(x, y) = x + y$.
However, $|\nabla u| = \sqrt{2} \geqslant 1$.
Therefore, for the constrained LGP with such boundary data $f$ it is impossible for $u$ to have level sets as straight lines as they must avoid all disks $B(x, |f(x) - t|)$ for each boundary point $x$. One then gets that the level sets are circular arcs joining the 2 points $(t, 0)$ and $(0, t)$ with center 0 and radius $t$. Also, $u$ will be constant in the square above the circular arc center 0 radius 1.

So in case $\Omega$ is convex we may consider what appears to be a slightly weaker condition than $|\nabla u| \leqslant 1$ a.e. in $\Omega$. Namely we suppose that the condition $|u(x) - u(y)| \leqslant |x - y|$ holds for all $x \in \partial\Omega$, and $y \in \Omega$. The corresponding problem is then

$$\{\int_\Omega |\nabla v|,\ v = f \ on\ \partial\Omega,\ |v(x) - v(y)| \leqslant |x - y|\ holds\ for\ all\ x \in \partial\Omega,\ y \in \Omega\}$$

It turns out that an analysis of this problem produces a unique solution of the constrained problem.

Now let $\Omega$ be a bounded domain in $\mathbb{R}^n$, and let $f$ be a given function defined on its boundary $\partial\Omega$. Suppose that there is a function $v$ defined on $\Omega$ and satisfying the following conditions:
(i) $v = f$ on $\partial\Omega$; (ii) if $x \in \partial\Omega$, and $y \in \Omega$, then $|u(x) - u(y)| \leqslant |x - y|$.
For such function $v$ we aim on to study the sets $A_t = \{x \in \Omega;\ v(x) \geqslant t\}$, with $t$ a real number.
Fix $t \in \mathbb{R}$, and compare, for all points $p \in \partial\Omega$, the values $f(p)$ of the boundary function $f$ with the real number $t$.
Case 1: Suppose that there is a point $p \in \partial\Omega$ such that $f(p) < t$. Then $t - f(p) > 0$,

and we concentrate on a neighborhood of $p$ of radius $t - f(p) > 0$. If $x \in \Omega$ and is also in this neighborhood, then $|x-p| < t - f(p)$. Then by condition (ii) on $v$ we have, since $v(p) = f(p)$,

$$v(x) - v(p) = v(x) - f(p) \leqslant |x-p| < t - f(p)$$

which implies that $v(x) < t$, and hence such a point $x \notin A_t$. In particular, the point $p$ itself is not in $A_t$. This also implies that if $M = max\{f(p);\, p \in \partial\Omega\}$ and $t > M$, then $f(p) < t$ for all point $p \in \partial\Omega$, and so the boundary of $\Omega$ can be covered by an open set (a union of neighborhoods) which does not intersect that particular $A_t$. Later on, we shall be interested in the largest such possible set.

Case 2: Suppose there is a point $p \in \partial\Omega$ such that $f(p) \geqslant t$. Then $f(p) - t \geqslant 0$ and if $f(p) - t > 0$, we concentrate on a neighborhood of $p$ of radius $f(p) - t$. If $x \in \Omega$ and is also in this neighborhood, then $|x-p| < f(p) - t$, and again we have

$$v(p) - v(x) = f(p) - v(x) \leqslant |p - x| < f(p) - t$$

which implies that $v(x) \geqslant t$, and so $x \in A_t$. This also implies that if $m = min\{f(p);\, p \in \partial\Omega\}$, and $t \leqslant m$, then $f(p) \geqslant t$ for all points $p \in \partial\Omega$, and so the entire boundary of $\Omega$ along with an open set containing it will lie in $A_t$. Once again we are interested in the largest such possible set.

The previous analysis leads naturally to the introduction of 2 sets as follows: given $t \in R$, define two sets $L_t$ and $M_t$ by

$$L_t := \{x \in \Omega;\, \exists p \in \partial\Omega,\ f(p) - t \geqslant 0,\ |p-x| \leqslant f(p) - t\} = \{\cup_{p\in\partial\Omega}\overline{B}(p, f(p)-t)\ ;\ f(p) \geqslant t\}$$

$$M_t := \{x \in \Omega;\, \exists p \in \partial\Omega,\ f(p) - t < 0,\ |p-x| < t - f(p)\} = \{\cup_{p\in\partial\Omega}\overline{B}(p, t-f(p));\ f(p) < t\}$$

**Proposition 9.** [*10*] *Suppose that $v$ satisfies the conditions (i)$v = f$ on $\partial\Omega$, (ii) if $x \in \partial\Omega$, and $y \in \Omega$, then $|v(x) - v(y)| \leqslant |x - y|$. Then, $L_t \subset A_t$, and $A_t \cap M_t = \varnothing$ for each real $t$.*

Now for a given function $v$ satisfying the conditions of the proposition, and for each real $t$, we seek that subset $E \subset \Omega$, which contains $L_t$ does not intersect $M_t$ and has smallest perimeter i.e. we consider the problem

$$min\{P(E);\ L_t \subset E,\ E \cap M_t = \varnothing,\ E \subset \Omega\} \tag{4.1}$$

One can show that this problem always has a solution, but not necessarily a unique solution. To obtain a unique solution, we search for those sets which have largest possible measure i.e.

$$max\{|E|;\ E\ solves\ the\ above\ problem\} \tag{4.2}$$

Now this problem has a unique solution, denoted by $\varepsilon_t$ and we expect that the boundary of this set corresponds with the level set $v = t$.

Another important characterization of the level sets is that the construction of $\partial\varepsilon_t$ is independent of the construction of $\partial\varepsilon_s$.

Indeed, we we shall present the proof of result based on reference [4] that the distance between $\partial\varepsilon_t$ and $\partial\varepsilon_s$ is no less than $|t - s|$ for $s < t$.

First we extend (4.1) and (4.2) from $\Omega$ to $R^n$ .

We now consider the following extended problems

$$min\{P(E, R^n); L_t \subset E, \mathring{M}_t \cap E = \varnothing, E - \Omega = L_t - \Omega\} \tag{4.3}$$

and

$$max\{|E|;\ E\ solves\ (4.3)\} \tag{4.4}$$

We denote the solution of (4.4) by $E_t$.

**Remark 11.** (4.1) and (4.3) are equivalent since

$$P(E, R^n) = P(E, \Omega) + H^{n-1}(\partial^* L_t - \Omega)$$

such that $E_t \cap \Omega = \varepsilon_t$

**Lemma 4.1.1.** *If $s < t$, then $E_t \subset E_s$*

**Lemma 4.1.2.** *Let $s < t$. Let $\eta \in \mathbb{R}^n, |\eta| \leqslant t - s$. Then $E_t + \eta \subset E_s$.*

*Proof.* We first denote $L_t^{'} := L_t + \eta$, $M_t^{'} := M_t + \eta$, $\Omega^{'} := \Omega + \eta$.
We consider the problem:

$$min\{P(E); L_t^{'} \subset E, \mathring{M}_t^{'} \cap E = E, E - \Omega' = L_t^{'} - \Omega'\} \tag{4.5}$$

$$max\{|E|;\ E\ solves\ (4.5)\} \tag{4.6}$$

(4.5) and (4.6) have a unique solution which we will denote by $E_t^{'}$.
We have $L_t^{'} \subset L_s$ since: for $x \in L_t^{'}$, $x = a + \eta$, $a \in L_t \Rightarrow \exists p \in \partial\Omega; |a - p| \leqslant f(p) - t$ $\Rightarrow \exists p \in \partial\Omega;\ |a-p| + |\eta| \leqslant f(p) - t + |\eta| \Rightarrow \exists p \in \partial\Omega;\ |a - p + \eta| \leqslant f(p) - t + t - s = f(p) - s$. As $f(p) \geqslant t > s \Rightarrow f(p) - s \geqslant 0$
$\Rightarrow \exists p \in \partial\Omega; |x - p| \leqslant f(p) - s \Rightarrow x \in L_s$.

Let $E = E_t^{'} \cap E_s$
We need to prove, $L_t^{'} \subset E, E \cap \mathring{M}_t^{'} = E, E - \Omega' = L_t^{'} - \Omega'$

- $L_t^{'} \subset L_s \subset E_s$
  $L_t^{'} \subset E_t^{'}$
  $\Rightarrow L_t^{'} \subset E_t^{'} \cap E_s = E$

- We have $E \subset E_t^{'}$
  $\Rightarrow E \cap \mathring{M}_t^{'} \subset E_t^{'} \cap \mathring{M}_t^{'}$
  $\Rightarrow E \cap M_t^{'} = \varnothing$ as $E_t^{'} \cap \mathring{M}_t^{'} = \varnothing$

- As $L_t^{'} \subset E \subset E_t^{'}$
  $\Rightarrow L_t^{'} - \Omega^{'} \subset E - \Omega' \subset E_t^{'} - \Omega' = L_t^{'} - \Omega'$
  $\Rightarrow E - \Omega^{'} = L_t^{'} - \Omega'$

Hence, $E = E_t^{'} \cap E_s$ competitor of $E_t^{'}$ in (4.5)
$\Rightarrow P(E, R^n) \geqslant P(E_t^{'}, R^n)$

Let $F = E_t^{'} \cup E_s$

We have $M_s \subset M_t^{'}$ since: let $x \in M_s \Rightarrow \exists p \in \partial\Omega; s - f(p) \geqslant 0$ and $\mid x - p \mid \leqslant s - f(p)$.
Let $a = x - \eta$.
We claim $a \in M_t$. $\mid p - a \mid = \mid p - (x - \eta) \mid = \mid p - x + \eta \mid \leqslant \mid p - x \mid + \mid \eta \mid \leqslant s - f(p) + \mid \eta \mid \leqslant s - f(p) + t - s = t - f(p)$.
$\Rightarrow x = a + \eta$ with $a \in M_t \Rightarrow x \in M_t^{'}$.

As $M_s \subset M_t^{'} \Rightarrow \mathring{M}_s \subset M_t^{'}$.
But $\mathring{M}_t^{'}$ is the biggest open set in $M_t^{'} \Rightarrow \mathring{M}_s \subset \mathring{M}_t^{'}$

We need to prove $L_s \subset F, F \cap \mathring{M}_s = \varnothing, F - \Omega = L_s - \Omega$

- $L_s \subset E_s \subset E_s \cup E_t^{'} = F$

- $F \cap \mathring{M}_s = (E_s \cup E_t^{'}) \cap \mathring{M}_s = (E_s \cap \mathring{M}_s) \cup (E_t^{'} \cap \mathring{M}_s)$
  But $E_s \cap \mathring{M}_s = E$ as $E_s$ solves (4.3).
  Since $\mathring{M}_s \subset \mathring{M}_t^{'} \Rightarrow E_t^{'} \cap \mathring{M}_s \subset E_t^{'} \cap \mathring{M}_t^{'} = \varnothing$
  $\Rightarrow E_t^{'} \cap \mathring{M}_s = \varnothing$
  Hence, $F \cap \mathring{M}_s = \varnothing$

- $L_s \subset E_s \Rightarrow L_s - \Omega \subset E_s - \Omega \subset F - \Omega$
  It remains to show $F - \Omega \subset L_s - \Omega$
  $F - \Omega = (E_t^{'} \cup E_s) - \Omega = (E_t^{'} - \Omega) \cup (E_s - \Omega)$
  So we need to prove $E_t^{'} - \Omega \subset L_s - \Omega$ and $E_s - \Omega \subset L_s - \Omega$
  But we know $E_s - \Omega = L_s - \Omega$ So it remains to prove $E_t^{'} - \Omega \subset L_s - \Omega$
  Let $x \in E_t^{'} - \Omega \Rightarrow x \in E_t^{'}$ and $x \notin \Omega \Rightarrow x = a + \eta$, $a \in E_t$, $x \notin \Omega$
  It is enough now to prove $x \in L_s$
  We consider two cases: $a \in \Omega$ and $a \notin \Omega$.
  If $a \notin \Omega \Rightarrow a \in E_t - \Omega = L_t - \Omega \Rightarrow x \in L_t^{'} \subset L_s$

If $a \in \Omega$ then $\exists y' \in \partial\Omega$ with $y' = a + \gamma\eta, 0 \leqslant \gamma \leqslant 1$ since $x \notin \Omega$.

If $f(y') \leqslant t$ we have $a \in E_t$ and $E_t \cap \mathring{M}_t = \varnothing \Rightarrow a \notin M_t$

$\Rightarrow |a - y'| \geqslant t - f(y') \Rightarrow |a - a - \gamma\eta| \geqslant t - f(y') \Rightarrow |\eta| \geqslant t - f(y')$

$\Rightarrow t - s \geqslant t - f(y') \Rightarrow s \leqslant f(y')$.

If $f(y') \geqslant t$, as $t > s \Rightarrow f(y') > s$.

Then as $f(y') > s$, $|x - y'| = |a + \eta - a - \gamma\eta| = |(1-\gamma)\eta| \leqslant |\eta| \leqslant t - s \leqslant f(y') - s$

$\Rightarrow x \in L_s$

Hence, F is a competitor to $E_s$ in (4.3) $\Rightarrow P(E_s \cup E_t^{'}, R^n) \geqslant P(E_s, R^n)$

But $P(E_s \cup E_t^{'}, \Omega) + P(E_s \cap E_t^{'}, \Omega) \leqslant P(E_s, \Omega) + P(E_t^{'}, \Omega)$

and from above we have, $P(E_s \cup E_t^{'}, \Omega) \leqslant P(E_s, \Omega)$ and $P(E_s \cap E_t^{'}, \Omega) \leqslant P(E_t^{'}, \Omega)$

$\Rightarrow P(E_s \cup E_t^{'}, \Omega) = P(E_s, \Omega)$ and $P(E_s \cap E_t^{'}, \Omega) = P(E_t^{'}, \Omega)$

$\Rightarrow E_s \cup E_t^{'}$ and $E_s \cap E_t^{'}$ solve (P'1) and (P'3) respectively

$\Rightarrow (\mid E_s \cup E_t^{'}) \cap \Omega \mid \leqslant \mid (E_s) \cap \Omega \mid$ and $\mid E_s \cap E_t^{'} \cap \Omega \mid \leqslant \mid E_t^{'} \cap \Omega \mid$

But $\mid (E_s \cup E_t^{'}) \cap \Omega \mid = \mid E_s \cap \Omega \mid + \mid (E_t^{'} - E_s) \cap \Omega \mid$

$\Rightarrow \mid E_s \cap \Omega \mid \geqslant \mid E_s \cap \Omega \mid + \mid (E_t^{'} - E_s) \cap \Omega \mid$

$\Rightarrow \mid (E_t^{'} - E_s) \cap \Omega \mid = 0$

It remains to show $E_t' \subset E_s$

Let $x \in E_t' \cap \Omega \Rightarrow$ by definition 2.2.1, $\limsup_{r \longrightarrow 0} \frac{|E_t' \cap B(x,r) \cap \Omega|}{|B(x,r)|} > 0$

Write $E_t' = (E_t' - E_s) \cup (E_t' \cap Es)$ union of 2 disjoint sets.

$\Rightarrow \limsup_{r \longrightarrow 0} \frac{|E_t' \cap B(x,r) \cap \Omega|}{|B(x,r)|} = \limsup_{r \longrightarrow 0} \frac{|\Omega \cap (E_t' - E_s) \cap B(x,r)|}{|B(x,r)|} + \limsup_{r \longrightarrow 0} \frac{|\Omega \cap (E_t' \cap E_s) \cap B(x,r)|}{|B(x,r)|}$

But $(E_t' - E_s) \cap B(x,r) \cap \Omega \subset (E_t' - E_s) \cap \Omega$

$\Rightarrow \mid (E_t' - E_s) \cap B(x,r) \cap \Omega \mid \leqslant \mid (E_t' - E_s) \cap \Omega \mid = 0$

$\Rightarrow \mid (E_t' - E_s) \cap B(x,r) \cap \Omega \mid = 0$

$\Rightarrow 0 < \limsup_{r \longrightarrow 0} \frac{|E_t' \cap B(x,r) \cap \Omega|}{|B(x,r)|} = \limsup_{r \longrightarrow 0} \frac{|E_t' \cap E_s \cap \Omega \cap B(x,r)|}{|B(x,r)|} \leqslant \limsup_{r \longrightarrow 0} \frac{|Es \cap B(x,r) \cap \Omega|}{|B(x,r)|}$

$\Rightarrow \limsup_{r \longrightarrow 0} \frac{|Es \cap B(x,r) \cap \Omega|}{|B(x,r)|} > 0$

$\Rightarrow x \in E_s$

Hence, $E'_t \cap \Omega \subset E_s \cap \Omega$

Also, $E'_t - \Omega = L'_t - \Omega \subset L_s - \Omega = \varepsilon_s - \Omega$

$\Rightarrow E'_t \subset E_s$ $\square$

**Corollary 4.2.** *Let $s < t$, $dist(\partial E_t, \partial E_s) \geqslant t - s$*

*Proof.* Proceeding by contradiction, suppose $dist(\partial E_t, \partial E_s) < t - s$

Let $x \in \partial E_t$. Then one can find $y \notin E_s$ such that $\mid y - x \mid = t - s$.

Set $\eta = y - x \Rightarrow y = \eta + x \in E'_t$. Contradiction to lemma 4.1.2. . $\square$

**Remark 12.** If $\Omega$ not convex we then have $f$ lipchitz with $\mid f(p) - f(q) \mid \leqslant d_\Omega(p, q)$ $\forall p, q \in \partial\Omega$.

Corollary 4.2 will then be $d_\Omega(\overline{\Omega \cap \partial E_s}, \overline{\Omega \cap \partial E_t}) \geqslant t - s$ for $s < t$.

## 4.3 Existence of a solution

**Definition 4.3.1.** Define a function $u^*$ on $\Omega$ by

$$u^*(x) = sup\{t;\ x \in \varepsilon_t\}$$

**Theorem 4.3.1.** [4] *The function $u^*$ is the unique continuous solution to the problem*

$$min\{\int_\Omega |\nabla u|;\ u = f \ on \ \partial\Omega,\ |\nabla u| \leqslant 1 \ a.e. \ in \ \Omega\}$$

Proposition 9 constitutes a characterization of the level sets of the solution to the problem.

Since by definition of $\varepsilon_t$ one has $L_t \subset \varepsilon_t$ and $\varepsilon_t \cap M_t = \varnothing$, it is then easy to show that $u^* = f$ on $\partial\Omega$ and $|u^*(x) - u^*(y)| \leqslant |x - y|$ for all $x \in \partial\Omega$, $y \in \Omega$.

Indeed, Let $x \in \partial\Omega$ such that $f(x) = t$. If $s < t \Rightarrow f(x) > s \Rightarrow x \in L_s$ $\forall s < t \Rightarrow x \in \varepsilon_s$ $\forall s < t$ since $L_s \subset \varepsilon_s \Rightarrow u(x) \geqslant s$ $\forall s < t \Rightarrow u(x) \geqslant t$.

If $s > t \Rightarrow f(x) < s \Rightarrow s - f(x) > 0 \Rightarrow x \in \mathring{M}_s\ \forall s > t \Rightarrow x \notin \varepsilon_s\ \forall s > t$ since $\varepsilon_s \cap \mathring{M}_s = \varnothing \Rightarrow u(x) < s\ \forall s > t \Rightarrow u(x) \leqslant t$. Hence, $u(x) = t = f(x)$.

Now let $u^*(y) = t$ for $y \in \Omega \Rightarrow y \in \partial\varepsilon_t \Rightarrow y \notin L_t \cup M_t \Rightarrow y \notin B(x, |f(x) - u^*(y)|)$ for all $x \in \partial\Omega \Rightarrow |f(x) - u^*(y)| \leqslant |x - y|$ for $y \in \Omega$, $x \in \partial\Omega \Rightarrow |u^*(x) - u^*(y)| \leqslant |x - y|$ for $y \in \Omega$, $x \in \partial\Omega$.

So to prove that $u^*$ is indeed the solution of the constrained least gradient problem one still has to show that $u^*$ is continuous and that $|u^*(x) - u^*(y)| \leqslant |x - y|$ for $x, y \in \Omega$ i.e. $u^*$ is lipchitz. The proof will be illustrated in what follows according to reference [4].
We first begin by introducing new sets that can help in the characterization of level sets. We set the following:

$$B_t = \cap_{s<t}\varepsilon_s,\ C_t = \cup_{s>t}\varepsilon_s,\ D_t = B_t - C_t = B_t \cap C_t^c$$

**Lemma 4.3.1.** $\varepsilon_t \subset \mathring{\varepsilon}_s\ \forall s < t$

**Lemma 4.3.2.** *For each point $x$ on $\partial D_t \cap \Omega$, one can find a sequence of points on $\cup_{s\neq t}(\partial\varepsilon_s \cap \Omega)$ converging to $x$. In other words, $x$ is a limit point of $\cup_{s\neq t}(\partial\varepsilon_s \cap \Omega)$, $t \in R$*

*Proof.* Let $x \in \partial D_t \cap \Omega$. Consider all $r > 0; B(x, r) \subset \Omega$. Then one can find $y \in D_t \cap B(x, r)$ and $z \in (\Omega - D_t) \cap B(x, r)$.
$\Rightarrow y \in B_t, y \notin C_t$, and $z \notin D_t$
$\Rightarrow y \in B_t, y \in \Omega - C_t, z \notin B_t$ or $z \notin C_t^c$
$\Rightarrow [y \in \cap_{s<t}\varepsilon_s$ and $z \in \cup_{s<t}(\Omega - \varepsilon_s)]$ or $[y \in \cap_{s>t}(\Omega - \varepsilon_s)$ and $z \in \cup_{s>t}\varepsilon_s]$
$\Rightarrow$ by first condition $B(x, r)$ contains an element of $\partial\varepsilon_s\ \forall s < t$ sufficiently close to $t$ knowing $\varepsilon_t \subset \varepsilon_s\ \forall s < t$. Similarly, by second condition $B(x, r)$ contains an element of $\partial\varepsilon_s\ \forall s > t$ sufficiently close to $t$. These points will eventually converge to $x$. □

**Lemma 4.3.3.** $\forall t \in \mathbb{R}$

1. *$D_t$ is a closed set*

2. *$D_t = \overline{\Omega} \cap \{x; u^*(x) = t\}$*

3. *$\Omega \cap \partial\varepsilon_t \subset (u^*)^{-1}(t)$*

4. *$\varepsilon_t \subset \{x; u(x) \geqslant t\} = B_t$*

5. *$u^*$ is lipchitz on $\overline{\Omega}$ with lipchitz constant 1*

*Proof.* 1. To prove $D_t$ is closed in $\overline{\Omega}$ we prove $(\partial D_t \cup \mathring{D}_t) \cap \overline{\Omega} = D_t \cap \overline{\Omega}$.
We know $D_t \subset \overline{D_t}$. So it remains to prove $\overline{D_t} \cap \overline{\Omega} \subset D_t \cap \overline{\Omega}$.
But $\mathring{D}_t \subset D_t \Rightarrow$ it suffices to prove $\partial D_t \cap \overline{\Omega} \subset D_t \cap \overline{\Omega}$ i.e. $\forall x \in \partial D_t \cap \overline{\Omega}, x \in B_t$ and $x \in \overline{\Omega} - C_t$.
Let $x \in \partial D_t \cap \overline{\Omega}$. Then $\exists x_i \in D_t$ such that $x_i \longrightarrow x$
But $D_t \subset B_t \Rightarrow \exists x_i \in B_t; x_i \longrightarrow x$.
As $B_t$ is a closed set ( being the intersection of closed sets ), the limit point $x \in B_t$.

It remains to show $x \notin C_t$.
Proceeding by contradiction, suppose $x \in C_t \Rightarrow \exists s_0 > t$ such that $x \in A_{s_0}$.
We will conder 2 cases:
1) Assume $\exists r > 0; B(x,r) \cap \Omega = B(x,r) \cap D_t$,
as $x \in \varepsilon_{s_0} \cap \overline{\Omega} \Rightarrow$ by definition 3.2.1 $\limsup_{r \longrightarrow 0} \frac{|B(x,r) \cap \Omega \cap \varepsilon_{s_0}|}{|B(x,r)|} > 0 \Rightarrow | B(x,r) \cap \Omega \cap \varepsilon_{s_0} | > 0$
But $B(x,r) \cap \Omega \cap C_t = B(x,r) \cap D_t \cap C_t = B(x,r) \cap B_t \cap C_t^c \cap C_t = \varnothing$
As $\varepsilon_{s_0} \subset \cup_{s>t} \varepsilon_s = C_t$
$\Rightarrow | B(x,r) \cap \Omega \cap \varepsilon_{s_0} | \leqslant | B(x,r) \cap \Omega \cap C_t | = 0$ .Contradiction.
Then, $x \notin C_t$.
2) Assume $\forall r > 0$, $B(x,r) \cap \Omega \neq B(x,r) \cap D_t$
By lemma 4.3.2, $\exists \{y_{s_i}\} \longrightarrow x$ as $s_i \longrightarrow t^-$.

If $x \in \mathring{\varepsilon}_{s_0} \Rightarrow x \in \mathring{C}_t \Rightarrow$ x cannot belong to $\partial D_t$. Contradiction. So $x \in \partial\varepsilon_{s_0}$. But by corollary 4.2, $dist(x, y_{s_i}) \geqslant dist(\partial\varepsilon_{s_0}, \partial\varepsilon_{s_i}) \geqslant s_0 - s_i > 0$

As $i \longrightarrow \infty$ we get a contradiction.

$\Rightarrow x \notin C_t \Rightarrow x \in D_t$

2. $\supset$) Let $x \in \overline{\Omega}; u^*(x) = t \Rightarrow x \in \varepsilon_t$.

If $s < t$ we have $\varepsilon_t \subset \varepsilon_s \Rightarrow x \in \varepsilon_s \ \forall s < t \Rightarrow x \in B_t$.

If $s > t \Rightarrow u^*(x) < s \Rightarrow x \notin \varepsilon_s \ \forall s > t \Rightarrow x \notin C_t$.

Hence, $x \in D_t$.

$\subset$ ) Let $x \in D_t \Rightarrow x \in B_t$ and $x \notin C_t \Rightarrow \forall s < t \ x \in \varepsilon_s$ and $\forall s > t \ x \notin \varepsilon_s \Rightarrow \forall s < t$ $u^*(x) \geqslant s$ and $\forall s > t \ u^*(x) < s$

$\Rightarrow u^*(x) \geqslant t$ and $u^*(x) \leqslant t \Rightarrow u^*(x) = t$.

3. Let $x \in \Omega \cap \partial\varepsilon_t$. As $\varepsilon_t$ is closed then $x \in \varepsilon_t \subset B_t$.

It remains to prove $x \notin C_t$. If $x \in C_t \Rightarrow \exists s_0 > t; x \in \partial\varepsilon_{s_0}$

$\Rightarrow dist(\partial\varepsilon_{s_0}, \partial\varepsilon_t) = 0$. Contradiction with the corollary 4.2.

4. We first prove $B_t = \{x; u^*(x) \geqslant t\}$.

Let $x \in B_t \Rightarrow \forall s < t,\ x \in \varepsilon_s$.

But $\varepsilon_t \subset \varepsilon_s \ \forall s < t \Rightarrow x \in \varepsilon_t \Rightarrow u^*(x) \geqslant t$.

Conversely, if $u^*(x) \geqslant t \Rightarrow x \in \varepsilon_t \subset B_t$.

Now $\forall x \in \varepsilon_t$ we have $u^*(x) \geqslant t$.

Hence, $\varepsilon_t \subset \{x; u^*(x) \geqslant t\} = B_t$.

5. Let $x, y \in \Omega$. Let $u^*(x) = s$ and $u^*(y) = t$. By 3, $x \in D_s$ and $y \in D_t$. We will suppose $x \in \mathring{D}_s$ and $y \in \mathring{D}_t \Rightarrow \exists x' \in \partial D_s$ and a geodesic joining $x$ to $y$ passing through $x'$. Also $\exists y' \in \partial D_t$ such that $y'$ belongs to this geodesic. By lemma 4.3.2, $\exists\{x_{s_i}\} \subset \partial\varepsilon_{s_i}$ and $\exists\{y_{t_i}\} \subset \partial\varepsilon_{t_i}$ such that $x_{s_i} \longrightarrow x'$ and $y_{t_i} \longrightarrow y'$ as $s_i \longrightarrow s$ and $t_i \longrightarrow t$ respectively.

$\Rightarrow dist_\Omega(x_{s_i}, y_{t_i}) \geqslant dist_\Omega(\partial\varepsilon_{s_i}, \partial\varepsilon_{t_i}) \geqslant |s_i - t_i|$ by corollary 4.2.

$\Rightarrow \lim_{i\longrightarrow\infty} dist_\Omega(x_{s_i}, y_{t_i}) \geqslant \lim_{i\longrightarrow\infty} |s_i - t_i|$

$\Rightarrow dist_\Omega(x', y') \geqslant |s - t|$

$\Rightarrow dist_\Omega(x, y) \geqslant |s - t|$ as $d_\Omega(x, y) \geqslant d_\Omega(x', y')$

$\Rightarrow dist_\Omega(x, y) \geqslant |u(x) - u(y)|$.

It remains to show $u^*$ continuous on $\partial\Omega$. Let $x \in \partial\Omega$; $u^*(x) = t = f(x)$.
$\forall s < t$ we have $\overline{\Omega} \cap \{y; dist(x, y) \leqslant f(x) - s\} \subset L_s \subset \varepsilon_s$.
$\Rightarrow \forall y \in \overline{\Omega}, dist(x, y) \leqslant | f(x) - s |$ we have $y \in \varepsilon\_s \Rightarrow u^*(y) \geqslant s \forall s < t$
$\Rightarrow \liminf_{y\longrightarrow x, y\in\Omega} u^*(y) \geqslant s \ \forall s < t \Rightarrow \liminf_{y\longrightarrow x, y\in\Omega} u^*(y) \geqslant t = u^*(x)$.
$\forall s > t$, we have $\overline{\Omega} \cap \{y; dist(x, y) \leqslant s - f(x)\} \subset M_s \Rightarrow \forall y \in \overline{\Omega}, dist(x, y) \leqslant s - f(x)$ we have $y \notin \varepsilon_s \Rightarrow u^*(y) < s \ \forall s > t \Rightarrow \limsup_{y\longrightarrow x, y\in\Omega} u^*(y) < s$ $\forall s > t \Rightarrow \limsup_{y\longrightarrow x, y\in\Omega} u^*(y) \leqslant t = u^*(x)$.

Hence $u^*$ is lipchitz on $\overline{\Omega}$. □

We shall now present the proof that the above $u^*$ is indeed the solution of the constrained LGP given by [4]:

*Proof.* Since $u^*$ is lipchitz on $\overline{\Omega}$, by lemma 4.3.3, $|\nabla u^*| \leqslant 1$ a.e. in $\Omega$.
Also by lemma 4.3.3, we have $u^* \in C^{0,1}(\overline{\Omega})$ on $\partial\Omega$. We shall now show that
$\int_\Omega |\nabla u| \leqslant \int_\Omega |\nabla v|$, for each $v$ competitor of $u$ in the constrained LGP.
So we let $v \in C^{0,1}(\overline{\Omega}), |\nabla v| \leqslant 1$ a.e. in $\Omega, v = f$ on $\partial\Omega$.
We then have $\forall p \in \partial\Omega, x \in \Omega, |v(x) - v(p)| \leqslant |x - p|$.
Setting $\varepsilon_t^{'} = \{v \geqslant t\}$, one indeed has $L_t \subset \varepsilon_t^{'}$ and $\mathring{M}_t \cap \varepsilon_t^{'} = \varnothing$ by proposition 9.
Hence, $\varepsilon_t^{'}$ is a competitor to $\varepsilon_t$ in (4.1) $\Rightarrow P(\varepsilon_t, \Omega) \leqslant P(\varepsilon_t^{'}, \Omega)$
$\Rightarrow \int_{-\infty}^{+\infty} P(\varepsilon_t, \Omega) \leqslant \int_{-\infty}^{+\infty} P(\varepsilon_t^{'}, \Omega)$
$\Rightarrow$ by coarea formula, theorem 2.5.1, $\int_\Omega |\nabla u| \leqslant \int_\Omega |\nabla v|$.

□

# Bibliography


[1] W. Ziemer, G. Williams, and P. Sternberg, "Existence, uniqueness, and regularity for functions of least gradient," *Journal Fur Die Reine Und Angewandte Mathematik - J REINE ANGEW MATH*, vol. 1992, pp. 35–60, 01 1992.

[2] E. Bombieri, E. D. Giorgi, and E. Giusti, "Minimal cones and the bernstein problem," *Inventiones mathematicae*, vol. 7, no. 3, p. 243–268, 1969.

[3] W. Górny, P. Rybka, and A. Sabra, "Special cases of the planar least gradient problem," *Nonlinear Analysis: Theory, Methods  Applications*, vol. 151, p. 66–95, 2017.

[4] P. Sternberg, G. Williams, and W. P. Ziemer, "The constrained least gradient problem in $\mathbf{r}^n$," *Transactions of the American Mathematical Society*, vol. 339, p. 403–432, Jan 1993.

[5] N. Grady, "Functions of bounded variation," 2009.

[6] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions.* CRC Press, 2015.

[7] E. GIUSTI, *Minimal surfaces and functions of bounded variation.* Birkhauser, 1984.

[8] M. Miranda, "Comportamento delle successioni convergenti di frontiere minimali," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 38, pp. 238–257, 1967.

[9] W. Górny, "Planar least gradient problem: existence, regularity and anisotropic case," *Calculus of Variations and Partial Differential Equations*, vol. 57, Dec 2018.

[10] G. Williams, "Least gradient problems," in *Theoretical and Numerical Aspects of Geometric Variational Problems*, (Canberra AUS), pp. 255–264, Centre for Mathematics and its Applications, Mathematical Sciences Institute, The Australian National University, 1991.

[11] R. Kohn and G. Strang, "The constrained least gradient problem," *Non-Classical Continuum Mechanics*, p. 226–243, 1987.