AMERICAN UNIVERSITY OF BEIRUT

THETA SERIES AND ITS APPLICATION TO SUMS OF SQUARES

by
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As a Mathematics student, I am frequently asked about my choice of study and its practical applications in our daily lives. As with many questions, the answers to these depend on relating disguised Math problems to features of the real world. Once you study Mathematics, you encounter the real meaning of struggling with a hard problem followed by developing hope that you will actually solve it. Although it is known for being the basis of all sciences, it is often a mind-blowing source of joy to the people who understand it, and as Bertrand Russel once wrote ,"Mathematics, rightly viewed, possesses not only truth, but supreme beauty".

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An Abstract of the Thesis of

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Let Q be a positive definite quadratic form on $\mathbb{Z}^k$. Consider the Theta Function defined by $\tilde{\theta}(z) = \sum_{m \in \mathbb{Z}^k} e^{2\pi i Q(m) z}$ for some $z \in \mathbb{H}$. As an interesting application of Modular Forms, we study the number of representations of an integer s by Q. In this regard, we begin by proving the transformation law of $\tilde{\theta}(z)$, following Goro Shimura's approach of the proof which uses some essential techniques such as the Poisson Summation Formula and Fourier Transforms. This shows that $\tilde{\theta}(z)$ is a modular form of weight $\frac{k}{2}$ on the congruence subgroup $\Gamma_0(4)$. After that, we study the Eisenstein series of weight $k \geq 3$ on $\Gamma(M)$ as well as write its Fourier expansion used in expressing bases of the spaces of modular forms accordingly. To end, we approach the growth of Theta's Fourier coefficients to obtain asymptotic formulas for the number of representations mentioned above.

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Chapter 1

Introduction

We denote the number of representations of an integer s by a positive definite quadratic form by

$$r_{Q,h,N}(s) = \#\{(x_1, \cdots, x_k) \in \mathbb{Z}^k \mid Q(x) = s \text{ and } x \equiv h \bmod N\mathbb{Z}^k\}.$$

In this thesis we limit $Q(x)$ to $x_1^2 + \cdots + x_k^2$ and write

$$r_k(s) = \#\{(x_1, \cdots, x_k) \in \mathbb{Z}^k \mid x_1^2 + \cdots + x_k^2 = s\}.$$

It turns out that $r_k(s)$ represents the Fourier coefficient of some function $\tilde{\theta}$.

In his paper, [Shimura, 1973] proved the transformation law of θ which allowed us to show that $\tilde{\theta}$ is a modular form on $\Gamma_0(4)$. Then, according to the dimension of S_k, we were able to write $\tilde{\theta}$ as a combination of Eisenstein Series and cusp forms. Indeed, for $k \leq 8$, the space of cusp forms if trivial. Thus, we get the needed formulas using particular methods and calculations. Furthermore, for $k \geq 10$, we notice that the space of cusp forms is no longer trivial. However, getting $r_k(s) = a(s) + b(s)$ where $a(s)$ is the known Eisenstein part coefficient and $b(s)$ is the cuspidal part coefficient, and using the fact that $\frac{b(s)}{a(s)} \to 0$ as $s \to \infty$, we deduce that the growth of the Fourier coefficient $r_k(s)$ of $\tilde{\theta}$ is the same as that of $a(s)$: $\frac{r_k(s)}{a(s)} \to 1$ as $s \to \infty$.

In chapter 2, we state basic definitions and properties. Furthermore, we establish essential sections such as the Fourier transforms of some functions and Gauss Sums in order to prove the transformation law of the Theta Function. After that, we give a quick overview on the Lipschitz Summation Formula needed for the expansion of the Eisenstein Series.

In chapter 3, we prove that θ is a modular form on the congruence subgroup $\Gamma_0(2, 2N)$. This will allow us to define in the next chapter a new function denoted by $\tilde{\theta}$ and show that it is a modular form on $\Gamma_0(4)$.

We start chapter 4 by obtaining an explicit formula for the Fourier expansion of the Eisenstein Series on $\Gamma(4)$ and consequently on $\Gamma_0(4)$ using some abstract algebra tools. Next, we write the basis of $M_{k/2}(\Gamma_1(4))$, obtained by Magma Calculator, in terms of the Eisenstein Series. In order to reach the main result, we express $\tilde{\theta}$ in terms of the generators of the Eisenstein space and the space of cusp forms, leading to formulas for the number of representations $r_k(s)$.

Chapter 2

Background Theory

In this chapter, we outline the relevant and essential theoretical tools. We first introduce notations and some basic definitions. Besides giving some background on the Poisson Summation Formula on $\mathbb{Z}^n$ and the Fourier transforms of particular functions, we establish relations and properties of Quadratic Residue Symbols. We end the chapter by demonstrating a proof of the Lipschitz Summation Formula.

2.1 Notations

We always consider the following set of notations:

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ denote the ring of rational integers, the rational number field, the real number field, and the complex number field respectively.
- $\mathbb{H}$ denotes the upper half plane.
- All **vectors** in $\mathbb{Z}^n$, $\mathbb{R}^n$ or $\mathbb{C}^n$ will be considered column vectors with n a positive integer.

- The upper right $\boldsymbol{t}$ means "transpose".
- $\boldsymbol{sgn}(a) = \begin{cases} 1 & \text{if } a \geq 0, \\ -1 & \text{otherwise.} \end{cases}$
- For $z \in \mathbb{C}$, we put $\boldsymbol{e(z)} = e^{2\pi i z}$ with $i = \sqrt{-1}$.
- For $z \in \mathbb{C}$, define $\boldsymbol{\sqrt{z}} = z^{\frac{1}{2}}$ such that $-\frac{\pi}{2} < \arg(z^{\frac{1}{2}}) \leq \frac{\pi}{2}$.
- For $z \in \mathbb{C} \setminus \mathbb{R}$ and $\gamma \in GL_2(\mathbb{R})$, $\boldsymbol{\gamma z} = \frac{az+b}{cz+d}$ defines a group action.
- $\boldsymbol{\chi}$ denotes a character modulo N which is defined in details in the section on Gauss Sums.
- The slash operator $\boldsymbol{(f\big|_{k,\chi}\gamma)(z)} = \chi(d)^{-1}(cz+d)^{-k} f(\gamma z)$ with χ a character, $z \in \mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defines an action of $\Gamma_0(4)$.
- $\boldsymbol{M_k}$ denotes the complex vector space of modular forms of weight k.
- $\boldsymbol{S_k}$ denotes the complex vector space of cusp forms of weight k.
- $\boldsymbol{\mathcal{E}_k}$ denotes the Eisenstein space of weight k.

2.2 Basic Definitions

Definition 2.2.1. (Fourier transform) Consider an integrable function

$$f : \mathbb{R}^n \to \mathbb{C}.$$

Then the Fourier transform of f denoted by $\boldsymbol{(\mathcal{F}f)(\zeta)}$ is the function

$$\hat{f} : \mathbb{R}^n \to \mathbb{C}$$

$$\zeta \mapsto \hat{f}(\zeta) = \int_{x \in \mathbb{R}^n} f(x) e(-\zeta^t x)\, dx.$$

For any $u \in \mathbb{R}^n$, $\hat{f}$ satisfies the following property:

$$\text{If } h(x) = f(x+u), \text{ then } \hat{h}(\zeta) = e(\zeta^t u)\hat{f}(\zeta). \tag{2.1}$$

Lemma 2.2.1. *Let $a > 0$ and p a prime number. If a divides $p-1$, then there exists x such that $x^a \equiv 1 \bmod p$ but $x^b \not\equiv 1 \bmod p$ for all $0 < b < a$.*

Lemma 2.2.2. *Let a, b be odd integers. Then*

$$\frac{a-1}{2} + \frac{b-1}{2} \equiv \frac{ab-1}{2} \bmod 2$$

Proof. Since a and b are odd, then $a-1$ and $b-1$ are even. So, we have

$$\begin{aligned} a-1 \text{ and } b-1 \text{ are even } &\Longrightarrow (a-1)(b-1) \equiv 0 \bmod 4 \\ &\Longrightarrow ab - a - b + 1 \equiv 0 \bmod 4 \\ &\Longrightarrow ab - 1 \equiv (a-1) + (b-1) \bmod 4 \\ &\Longrightarrow \frac{ab-1}{2} \equiv \frac{a-1}{2} + \frac{b-1}{2} \bmod 2. \end{aligned}$$ □

Definition 2.2.2. (Spherical Function) Let A be a real symmetric matrix.
A spherical function P of order ν with respect to A is a $\mathbb{C}$-valued function in $\mathbb{R}^n$ such that

$$P(x) = \begin{cases} \text{constant} & \text{if } \nu = 0, \\ \sum \beta_q (q^t A x)^\nu & \text{if } \nu > 0 \end{cases}$$

with finitely many vectors $q \in \mathbb{C}^n$ such that $q^t A q = 0$ if $\nu > 0$ and $\beta_q \in \mathbb{C}$.

Lemma 2.2.3. *Suppose that A is a positive definite real symmetric $n \times n$ matrix.*

Let $\lambda_1, \cdots, \lambda_n$ be the corresponding eigenvalues of A. Then

$$x^t A x \geq \lambda_{min} |x|_2^2$$

where λ_{min} is the smallest eigenvalue of A and $|x|_2$ the Euclidean norm satisfying $x^t x = |x|_2^2$.

Proof. Since A is a real symmetric matrix, then we can write $A = UDU^{-1}$ with U orthogonal $n \times n$ matrix and D a diagonal matrix having $\lambda_i' s$ on its diagonal. Letting $x = Uy$, the term $x^t A x$ can be expressed as:

$$\begin{aligned} x^t A x &= (Uy)^t UDU^{-1}(Uy) \\ &= y^t U^t UDU^{-1} U y \\ &= y^t D y && \text{Since } U \text{ is an orthogonal matrix} \\ &= \sum_{i=1}^{n} \lambda_i y_i^2. \end{aligned}$$

Now notice that

$$\lambda_{min} \sum_{i=1}^{n} y_i^2 \leq \sum_{i=1}^{n} \lambda_i y_i^2 \implies \lambda_{min} |y|_2^2 \leq x^t A x$$

with $|x|_2^2 = |Uy|_2^2 = (Uy)^t(Uy) = y^t U^t U y = y^t y = |y|_2^2$. Thus the needed result. □

Theorem 2.2.4. *(Dirichlet Prime Number Theorem) Let a and m be relatively prime integers with a, $m \geq 1$. Then there exist infinitely many prime numbers p such that $p \equiv a \bmod m$.*

Definition 2.2.3. (Meromorphic Function) A meromorphic function f on $\mathbb{H}$ is a function that is holomorphic on all of $\mathbb{H}$ except for a set of isolated points, which are poles of the function, where the Laurent Series can involve only finitely many terms involving negative powers. Otherwise we have an essential singularity.

Definition 2.2.4. (Weakly Modular for Full Group) Let f be a meromorphic function on $\mathbb{H}$. We say f is weakly modular of weight $k \in \mathbb{Z}$ for the full group, if for

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ we have

$$f(\gamma z) = (cz + d)^k f(z).$$

Definition 2.2.5. (Holomorphic at Infinity) Let f be a meromorphic function that is weakly modular of weight $k \in \mathbb{Z}$. We say that f is holomorphic at infinity if it has a removable singularity at $q = e(z) = 0$, or equivalently f can be written as a Fourier series with $a_n = 0\ \forall n < 0$, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n q^n \text{ with } q = e(z).$$

Definition 2.2.6. (Modular and Cusp Forms for $SL_2(\mathbb{Z})$) A modular form of weight $k \in \mathbb{Z}$ for $SL_2(\mathbb{Z})$ is a holomorphic function f in $\mathbb{H}$, that is weakly modular and holomorphic at infinity. A cusp form of weight k is a modular form of weight k satisfying $f(\infty) = 0$, where $f(\infty)$ represents the constant term a_0 of the Fourier expansion of f.

Definition 2.2.7. (Congruence Subgroup) Consider the Principal Congruence Subgroup of $SL_2(\mathbb{Z})$ of level N defined by:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}.$$

Then, a Congruence Subgroup of $SL_2(\mathbb{Z})$ of level N is a subgroup that contains $\Gamma(N)$.

Example 2.2.1. The two most popular congruence subgroups are $\Gamma_0(N)$ and $\Gamma_1(N)$ which are defined as follows:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); c \equiv 0 \bmod N \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); a \equiv d \equiv 1 \bmod N \text{ and } c \equiv 0 \bmod N \right\}.$$

Definition 2.2.8. (Weakly Modular for Γ) Let Γ be a congruent subgroup and f be a meromorphic function on $\mathbb{H}$. We say f is weakly modular of weight $k \in \mathbb{Z}$ for Γ, if for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$f(\gamma z) = \epsilon(\gamma)(cz + d)^k f(z)$$

where $\epsilon(\gamma)$ is a multiplier system, i.e. a constant written in terms of a, b, c and d.

Definition 2.2.9. (Modular Form on Congruence Subgroup) Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ of finite index and $\gamma_t \in SL_2(\mathbb{Z})$ be such that $\gamma_t \cdot \infty = t$. A modular form of weight k on Γ is a holomorphic function f in $\mathbb{H}$ that is weakly modular for Γ with $|\epsilon(\gamma_t)| = 1$ and has the Fourier expansion given by

$$(f|_k \gamma_t)(z) = \sum_{n=0}^{\infty} a_n q_{\mathcal{C}}^n = \sum_{n=0}^{\infty} a_n e(\frac{nz}{h_{\mathcal{C}}})$$

at all inequivalent cusps $\mathcal{C}$ of width $h_{\mathcal{C}}$ of Γ.

Definition 2.2.10. (Modular Form on Γ with Character) a Modular Form f of weight k and character χ is a holomorphic function satisfying the transformation law:

$$f(\gamma z) = \chi(d)(cz + d)^k f(z) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Proposition 2.2.1. *Every modular form of weight k is a unique linear combination of Eisenstein series and a cusp form. This gives a direct sum decomposition*

$$M_k = S_k \oplus \mathcal{E}_k.$$

2.3 Poisson Summation Formula

Proposition 2.3.1. *For any continuous function $f : \mathbb{R}^n \to \mathbb{C}$ with rapid decay (at least like $|t|^{-c}$ with $c > 1$) as $t \to \infty$, the* ***Poisson Summation Formula*** *may be stated as:*

$$\sum_{m \in \mathbb{Z}^n} f(m) = \sum_{v \in \mathbb{Z}^n} \hat{f}(v)$$

where $\hat{f}$ is the Fourier transform of f.

Proof. For $t \in \mathbb{Z}^n$, define

$$S(t) = \sum_{m \in \mathbb{Z}^n} f(m+t).$$

Notice that

$$S(t+l) = S(t) \quad \forall l \in \mathbb{Z}^n,$$

so $S(t)$ is periodic with respect to $\mathbb{Z}^n$. Furthermore, every infinitely continuous differentiable periodic function admits a Fourier series. Then, we can write

$$S(t) = \sum_{v \in \mathbb{Z}^n} C_v e(v^t t)$$

with

$$\begin{aligned}
C_v &= \int_{0 \le x_1 \le 1} \cdots \int_{0 \le x_n \le 1} S(x) e(-v^t x)\, dx_1 \cdots dx_n \\
&= \int_{0 \le x_1 \le 1} \cdots \int_{0 \le x_n \le 1} \sum_{m \in \mathbb{Z}^n} f(m+x) e(-v^t x)\, dx_1 \cdots dx_n \\
&= \sum_{m \in \mathbb{Z}^n} \int_{0 \le x_1 \le 1} \cdots \int_{0 \le x_n \le 1} f(m+x) e(-v^t x)\, dx_1 \cdots dx_n \\
&= \sum_{m \in \mathbb{Z}^n} \int_{m_1 \le x_1 \le m_1+1} \cdots \int_{m_n \le x_n \le m_n+1} f(x) e(-v^t (x-m))\, dx_1 \cdots dx_n \\
&= \int_{x \in \mathbb{R}^n} f(x) e(-v^t x) e(v^t m)\, dx \\
&= \int_{x \in \mathbb{R}^n} f(x) e(-v^t x)\, dx
\end{aligned}$$

$$= \hat{f}(v).$$

Thus we have,

$$\sum_{m \in \mathbb{Z}^n} f(m+t) = S(t) = \sum_{v \in \mathbb{Z}^n} \hat{f}(v) e(v^t t).$$

Taking $t = 0$ we get the needed result. $\square$

2.4 Fourier Transforms of Particular Functions

Lemma 2.4.1. *For $\zeta \in \mathbb{R}$,*

$$\int_{x \in \mathbb{R}} e^{-\pi(x+i\zeta)^2} \, dx = 1.$$

Proof. Using figure 2.1 on the top of the next page, let γ be the rectangular contour and consider $f(z) = e^{-\pi z^2}$. Notice that f is entire and γ is a closed contour. Then by **Cauchy's Integral Theorem**, we have $\int_\gamma f(z) \, dz = 0$.

On the other hand, we compute the integral along the contour.

- Along $\gamma_1 : z = x, dz = dx, -R_1 \le x \le R_2$. So,

$$\int_{\gamma_1} f(z) \, dz = \int_{-R_1}^{R_2} e^{-\pi x^2} \, dx$$

- Along $\gamma_3 : z = x + i\xi, dz = dx, -R_1 \le x \le R_2$. So,

$$-\int_{\gamma_3} f(z) \, dz = \int_{-R_1}^{R_2} e^{-\pi(x+i\xi)^2} \, dx$$

- Along γ_2 (Similarly γ_4): $z = R_2 + it, 0 \le t \le \xi$ or $\xi \le t \le 0$ according as $\xi > 0$ or

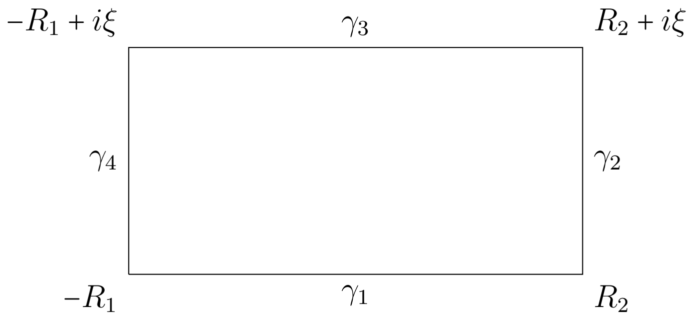

Figure 2.1: Rectangular Contour γ

$\xi < 0$. Then,

$$\begin{aligned}
|f(z)| &= \left|e^{-\pi(R_2+it)^2}\right| \\
&= \left|e^{-\pi(R_2^2-t^2-2R_2it)}\right| \\
&= e^{-\pi(R_2^2-t^2)} \\
&\leq e^{-\pi(R_2^2-\xi^2)}.
\end{aligned}$$

Letting $R_2 \to \infty$, we get that $|f(z)| \to 0$ uniformly. Thus, we get that

$$\lim_{R_2\to\infty} \int_{\gamma_2} f(z)\,dz = \lim_{R_2\to\infty} \int_0^{\xi} e^{-\pi(R_2+it)^2}\,dt = 0.$$

Therefore, we get

$$\begin{aligned}
\int_{\gamma} f(z)\,dz &= \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz \\
&\implies \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx - \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\,dx = 0 \\
&\implies \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\,dx = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx \qquad (2.2)
\end{aligned}$$

Finally, solving the integral on the right side of equation (2.2) by applying the change of variable $u = x\sqrt{\pi}$ we get that

$$\int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\,dx = 1 \qquad \square$$

Proposition 2.4.1. *Consider the function*

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto e^{-\pi x^2}. \end{aligned}$$

Then the Fourier transform of f is the function itself.

Proof.

$$\begin{aligned}
\hat{f}(\zeta) &= \int_{x\in\mathbb{R}} f(x)e(-\zeta x)\,dx \\
&= \int_{x\in\mathbb{R}} e^{-\pi x^2-2\pi i\zeta x}\,dx \\
&= \int_{x\in\mathbb{R}} e^{-\pi(x^2+2ix\zeta)}\,dx \\
&= \int_{x\in\mathbb{R}} e^{-\pi(x+i\zeta)^2}e^{-\pi\zeta^2}\,dx && \text{completing the square} \\
&= e^{-\pi\zeta^2}\int_{x\in\mathbb{R}} e^{-\pi(x+i\zeta)^2}\,dx \\
&= e^{-\pi\zeta^2} && \text{by Lemma 2.4.1} \\
&= f(\zeta).
\end{aligned}$$

□

Proposition 2.4.2. *For $a > 0$, let*

$$f_a(x) = e^{-\pi a x^2}.$$

We have

$$\hat{f}_a(\zeta) = \frac{1}{\sqrt{a}}e^{-\pi\zeta^2/a}.$$

Proof. Notice that $f_a(x) = f(x\sqrt{a})$ where f is the function defined in Propostion 2.4.1. Then,

$$\hat{f}_a(\zeta) = \int_{x\in\mathbb{R}} f_a(x)e(-\zeta x)\,dx$$

$$
\begin{aligned}
&= \int_{x\in\mathbb{R}} e^{-\pi(x\sqrt{a})^2} e(-x\zeta)\,dx \\
&= \frac{1}{\sqrt{a}} \int_{x\in\mathbb{R}} e^{-\pi u^2} e(\frac{-u\zeta}{\sqrt{a}})\,du && \text{applying the change of variable} \\
&= \frac{1}{\sqrt{a}} \int_{x\in\mathbb{R}} f(u) e(\frac{-u\zeta}{\sqrt{a}})\,du \\
&= \frac{1}{\sqrt{a}} \hat{f}(\frac{\zeta}{\sqrt{a}}) \\
&= \frac{1}{\sqrt{a}} f(\frac{\zeta}{\sqrt{a}}) && \text{by Proposition 2.4.1} \\
&= \frac{1}{\sqrt{a}} e^{-\pi\zeta^2/a}. && \square
\end{aligned}
$$

Proposition 2.4.3. *Let P be a Spherical function of order ν, N a positive integer and A a positive definite real symmetric matrix of size n. For $z = \alpha i$ with $\alpha > 0$, we consider a function*

$$
\begin{aligned}
f \,:\, &\mathbb{R}^n \to \mathbb{C} \\
&x \mapsto f(x) = P(Nx) e(\frac{z x^t A x}{2}).
\end{aligned}
$$

Then we have

$$\hat{f}(\zeta) = \sum N^\nu \beta_q (\det A)^{-1/2} \alpha^{-\kappa/2} (-i)^\nu (q^t\zeta)^\nu e^{-\pi\zeta^t A^{-1}\zeta/\alpha}.$$

Proof. Since A is a real symmetric matrix, then A is diagonalizable by an orthogonal n by n matrix U.

i.e,

$$A = U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} U^{-1} = UDU^{-1}.$$

Note that λ_i are positive because A is positive definite. Now let $x = Uy$, with

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}.$$

Take $g(y) = f(Uy)$, then

$$\begin{aligned} g(y) =& f(Uy) \\ &= \sum \beta_q (Nq^t UDU^{-1}Uy)^\nu e(\frac{z}{2}(Uy)^t A(Uy)) \\ &= \sum \beta_q (Nq^t UDy)^\nu e(\frac{z}{2} y^t Dy). \end{aligned}$$

Set $c = q^t U = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix} \in (\mathbb{C}^n)^t$. Notice that

$$\begin{aligned} q^t A q = 0 &\implies q^t (UDU^t) q = 0 \\ &\implies cDc^t = 0 \\ &\implies \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = 0 \\ &\implies \sum_{i=1}^n \lambda_i c_i^2 = 0. \end{aligned}$$

Next, we get

$$g(y) = \sum \beta_q (NcDy)^\nu e(\frac{z}{2} y^t Dy)$$

$$= \sum N^\nu \beta_q [\sum_{i=1}^n \lambda_i c_i y_i]^\nu e(\frac{z}{2}\sum_{i=1}^n \lambda_i y_i^2)$$
$$= \sum N^\nu \beta_q [\sum_{i=1}^n \lambda_i c_i y_i]^\nu e^{-\pi\alpha \sum_{i=1}^n \lambda_i y_i^2}.$$

Hence,

$$\hat{g}(\zeta) = \int_{y_1\cdots y_n} \sum N^\nu \beta_q [\sum_{i=1}^n \lambda_i c_i y_i]^\nu e^{-\pi\alpha \sum_{i=1}^n \lambda_i y_i^2 - 2\pi i \sum_{i=1}^n y_i \zeta_i}\, dy_1 \cdots dy_n. \tag{2.3}$$

Using Proposition 2.4.2, we have that

$$\begin{aligned}\int_{y_1\cdots y_n} e^{-\pi\alpha\lambda_1 y_1^2 - 2\pi i y_1 \zeta_1} \ldots e^{-\pi\alpha\lambda_n y_n^2 - 2\pi i y_n \zeta_n}\, dy_1 \cdots dy_n \\ = \frac{1}{\sqrt{\alpha\lambda_1}} \cdots \frac{1}{\sqrt{\alpha\lambda_n}} e^{-\pi(\zeta_1^2/\alpha\lambda_1 + \cdots + \zeta_n^2/\alpha\lambda_n)}.\end{aligned} \tag{2.4}$$

Now, applying $\sum_{i=1}^n \lambda_i c_i \frac{\partial}{\partial \zeta_i}$ to both sides of equation (2.4) and using the **Leibniz Integral Rule**, we get

$$\int_{y_1\cdots y_n} -2\pi i (\sum_{i=1}^n \lambda_i c_i y_i) e^{-\pi\alpha \sum_{i=1}^n \lambda_i y_i^2 - 2\pi i \sum_{i=1}^n y_i \zeta_i}\, dy_1 \cdots dy_n$$

$$= \frac{1}{\sqrt{\alpha\lambda_1}} \cdots \frac{1}{\sqrt{\alpha\lambda_n}} (\frac{-2\pi}{\alpha} \sum_{i=1}^n c_i \zeta_i) e^{-\pi \sum_{i=1}^n \frac{\zeta_i^2}{\alpha\lambda_i}}.$$

Then, apply $\sum_{i=1}^n \lambda_i c_i \frac{\partial}{\partial \zeta_i}$ to the latter noting the following:

- $(\sum_{i=1}^n \lambda_i c_i \frac{\partial}{\partial \zeta_i})(\sum_{i=1}^n \lambda_i c_i y_i) = 0,$
- $(\sum_{i=1}^n \lambda_i c_i \frac{\partial}{\partial \zeta_i})(\sum_{i=1}^n c_i \zeta_i) = \sum_{i=1}^n \lambda_i c_i^2 = 0.$

So, we get

$$\int_{y_1\cdots y_n} (-2\pi i)^2 (\sum_{i=1}^n \lambda_i c_i y_i)^2 e^{-\pi\alpha \sum_{i=1}^n \lambda_i y_i^2 - 2\pi i \sum_{i=1}^n y_i \zeta_i}\, dy_1 \cdots dy_n$$

$$= \frac{1}{\sqrt{\alpha\lambda_1}} \cdots \frac{1}{\sqrt{\alpha\lambda_n}} (\frac{-2\pi}{\alpha} \sum_{i=1}^n c_i \zeta_i)^2 e^{-\pi \sum_{i=1}^n \frac{\zeta_i^2}{\alpha\lambda_i}}.$$

Thus, applying $\sum_{i=1}^{n} \lambda_i c_i \frac{\partial}{\partial \zeta_i}$ ν times to equation (2.4), we obtain

$$\int_{y_1 \cdots y_n} (\sum_{i=1}^{n} \lambda_i c_i y_i)^{\nu} e^{-\pi\alpha \sum_{i=1}^{n} \lambda_i y_i^2 - 2\pi i \sum_{i=1}^{n} y_i \zeta_i} \, dy_1 \cdots dy_n$$

$$= \frac{1}{\sqrt{\alpha\lambda_1}} \cdots \frac{1}{\sqrt{\alpha\lambda_n}} (\frac{-i}{\alpha} \sum_{i=1}^{n} c_i \zeta_i)^{\nu} e^{-\pi \sum_{i=1}^{n} \frac{\zeta_i^2}{\alpha\lambda_i}}.$$

Replacing in equation (2.3) gives:

$$\begin{aligned}\hat{g}(\zeta) &= \sum N^{\nu} \beta_q \frac{1}{\sqrt{\alpha\lambda_1}} \cdots \frac{1}{\sqrt{\alpha\lambda_n}} (\frac{-i}{\alpha} \sum_{i=1}^{n} c_i \zeta_i)^{\nu} e^{-\pi \sum_{i=1}^{n} \frac{\zeta_i^2}{\alpha\lambda_i}} \\ &= \sum N^{\nu} \beta_q \alpha^{-\kappa/2} (\det A)^{-1/2} (-i)^{\nu} (\sum_{i=1}^{n} c_i \zeta_i)^{\nu} e^{-\pi \sum_{i=1}^{n} \frac{\zeta_i^2}{\alpha\lambda_i}}.\end{aligned}$$

Going back to f, we get

$$\begin{aligned}\hat{f}(\zeta) &= \int_{x \in \mathbb{R}^n} f(x) e(-x^t \zeta) \, dx \\ &= \int_{y \in \mathbb{R}^n} f(Uy) e(-(Uy)^t \zeta) |J| \, dy \\ &= \int_{y \in \mathbb{R}^n} g(y) e(-y^t U^{-1} \zeta) \, dy \\ &= \hat{g}(U^{-1}\zeta)\end{aligned}$$

where J is the transition matrix satisfying $|J| = |\det U| = 1$. Letting $U^{-1}\zeta = \eta \in \mathbb{R}^n$ and using the fact that

$$\sum_{i=1}^{n} \frac{\eta_i^2}{\lambda_i} = \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_n \end{pmatrix} \begin{pmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\lambda_n \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \\ \eta_n \end{pmatrix} = \eta^t D^{-1} \eta,$$

we get

$$\begin{aligned}
\hat{f}(\zeta) &= \sum N^\nu \beta_q \alpha^{-\kappa/2} (\det A)^{-1/2} (-i)^\nu (\sum_{i=1}^n c_i \eta_i)^\nu e^{-\frac{\pi}{\alpha}\sum_{i=1}^n \frac{\eta_i^2}{\lambda_i}} \\
&= \sum N^\nu \beta_q \alpha^{-\kappa/2} (\det A)^{-1/2} (-i)^\nu (\sum_{i=1}^n c_i \eta_i)^\nu e^{-\frac{\pi}{\alpha}\eta^t D^{-1}\eta} \\
&= \sum N^\nu \beta_q \alpha^{-\kappa/2} (\det A)^{-1/2} (-i)^\nu (\sum_{i=1}^n c_i \eta_i)^\nu e^{-\frac{\pi}{\alpha}(U^{-1}\zeta)^t D^{-1}(U^{-1}\zeta)} \\
&= \sum N^\nu \beta_q \alpha^{-\kappa/2} (\det A)^{-1/2} (-i)^\nu (q^t\zeta)^\nu e^{-\pi\zeta^t A^{-1}\zeta/\alpha}.
\end{aligned}$$

□

Proposition 2.4.4. *Consider the shifted function:*

$$f_u(x) = f(x+u) = P(N(x+u))e(\frac{z(x+u)^t A(x+u)}{2})$$

with $u = N^{-1}h$ for $h \in \mathbb{Z}^n$. Then the Fourier transform of f_u is

$$\hat{f}_u(\zeta) = e(\zeta^t u) \sum N^\nu \beta_q \alpha^{-\kappa/2} (\det A)^{-1/2} (-i)^\nu (q^t\zeta)^\nu e^{-\pi\zeta^t A^{-1}\zeta/\alpha}.$$

Proof. This result follows from (2.1) and Proposition 2.4.3. □

2.5 Gauss Sums

The material in this section is mainly taken from [Lang, 1994] pages 83–90 and [Ireland and Rosen, 1990] chapter 5.

Definition 2.5.1. (Character) Let G be a finite abelian group. A Character of G is a homomorphism of G into the multiplicative group $\mathbb{C}^\times$. i.e,

$$\chi : G \to \mathbb{C}^\times.$$

In particular we define the following:

Definition 2.5.2. (Dirichlet Character Modulo M) Let M be a positive integer. A Dirichlet Character modulo M is a function

$$\chi : \mathbb{Z} \to \mathbb{C}$$

with the property that there exists a group homomorphism

$$\chi' : \left(\mathbb{Z}/_{M\mathbb{Z}}\right)^{\times} \to \mathbb{C}^{\times}$$

such that $\chi(d) = \begin{cases} \chi'(d \bmod M) & \text{if } \gcd(d, M) = 1, \\ 0 & \text{Otherwise.} \end{cases}$

Example 2.5.1. Let $M = 4$. We have

$$\chi(m) = \begin{cases} 1 & \text{if } m \equiv 1 \bmod 4, \\ -1 & \text{if } m \equiv 3 \bmod 4, \\ 0 & \text{Otherwise.} \end{cases}$$

Definition 2.5.3. (Quadratic Residue Symbol) For an integer a and an odd integer b, we define the Quadratic Residue Symbol $\left(\frac{\boldsymbol{a}}{\boldsymbol{b}}\right)$ as follows:

(i) If $\gcd(a, b) \neq 1$, then $\left(\frac{a}{b}\right) = 0$

(ii) $\left(\frac{a}{-1}\right) = \boldsymbol{sgn}(a)$

(iii) For b an odd prime, denoted by p, $\left(\frac{a}{b}\right)$ coincides with the ordinary Quadratic Residue Symbol, i.e.

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \equiv \text{non-zero square mod } p, \\ -1 & \text{if } a \equiv \text{non-square mod } p. \end{cases}$$

(iv) For $b > 0$, $\left(\frac{a}{b}\right)$ coincides with the Jacobi Symbol defined next.

Example 2.5.2. We have for p an odd prime

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4, \\ -1 & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

Proof. By Definition 2.5.2. we know that

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } -1 \equiv \text{square} \bmod p, \\ -1 & \text{if } -1 \equiv \text{non-square} \bmod p. \end{cases}$$

Now, notice that

- If $p \equiv 1 \bmod 4$, then by Lemma 2.2.1 we have 4 divides $p-1 \implies \exists x$ such that $x^4 \equiv 1 \bmod p$ but $x^2 \not\equiv 1 \bmod p$.
  On the other hand $(x^2)^2 \equiv 1 \bmod p \implies$ either $x^2 \equiv 1 \bmod p$ or $x^2 \equiv -1 \bmod p$.
  Then $x^2 \equiv -1 \bmod p$ and $\left(\frac{-1}{p}\right) = 1$.

- If $p \equiv 3 \bmod 4$, then we can write $p = 4k + 3$ for some $k \in \mathbb{Z}$. Suppose by contradiction that $\left(\frac{-1}{p}\right) \neq -1$. Then $\exists x$ such that $x^2 \equiv -1 \bmod p$. Squarring both sides, we get $x^4 \equiv 1 \bmod p$. Next, by Fermat's Little Theorem, we have

  $$x^{p-1} = x^{4k+2} \equiv 1 \bmod p.$$

  However,

  $$x^{4k+2} = x^{4k} \cdot x^2 = (x^4)^k \cdot x^2 \equiv -1 \bmod p.$$

  Contradiction. Therefore, $\left(\frac{-1}{p}\right) = -1$. ∎

Example 2.5.3. We have for p an odd prime

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \bmod 8, \\ -1 & \text{if } p \equiv \pm 3 \bmod 8. \end{cases}$$

Proof. The proof of this example uses **Gauss's Lemma** and is found in [Ireland and Rosen, 1990] chapter 5, Proposition 5.1.3. ■

Proposition 2.5.1. *Here are some properties of the Quadratic Residue Symbol for a, b integers and p and odd prime:*

1. $a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right) \bmod p$ ***(Euler Criterion)***
2. $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$
3. *If* $a \equiv b \bmod p$, *then* $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$

Theorem 2.5.1. *(Law of Quadratic Reciprocity) Let p and q be odd primes. Then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\left(\frac{p-1}{2}\right)\cdot\left(\frac{q-1}{2}\right)}.$$

Proof. See [Serre, 2012] chapter 1, section 3.3. □

Lemma 2.5.2. *Note that the above theorem can be restated as follows:*

$$\left(\frac{p}{q}\right) = \begin{cases} +\left(\frac{q}{p}\right) & \textit{if } p \textit{ or } q = 4k+1, \\ -\left(\frac{q}{p}\right) & \textit{if } p \textit{ and } q = 4k+3. \end{cases}$$

Proof.

- p or $q = 4k+1 \implies \left(\frac{p-1}{2}\right)\cdot\left(\frac{q-1}{2}\right) \equiv 0 \bmod 2$. Then by Theorem 2.5.1,

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = +1.$$

- p and $q = 4k+3 \implies (\frac{p-1}{2}) \cdot (\frac{q-1}{2}) \equiv 1 \bmod 2$. Then by Theorem 2.5.1,

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = -1. \qquad \square$$

Definition 2.5.4. (Jacobi Symbol) Let b be an odd positive integer and a an integer. Suppose that $b = p_1 \cdots p_m$ where p_i's are primes. Then define the Jacobi Symbol by

$$\left(\frac{a}{b}\right) = \left(\frac{a}{p_1}\right) \cdots \left(\frac{a}{p_m}\right).$$

Theorem 2.5.3. *(Extended Quadratic Reciprocity) Let a, b be 2 positive odd integers having $gcd(a,b) = 1$. Suppose that $b = p_1 \cdots p_m$, then*

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{(\frac{a-1}{2}) \cdot (\frac{b-1}{2})}.$$

Proof. Using Theorem 2.5.1, we have

$$\begin{aligned}
\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) &= \left(\frac{a}{p_1 \cdots p_m}\right)\left(\frac{p_1 \cdots p_m}{a}\right) \\
&= \left(\frac{a}{p_1}\right) \cdots \left(\frac{a}{p_m}\right)\left(\frac{p_1}{a}\right) \cdots \left(\frac{p_m}{a}\right) \\
&= (-1)^{(\frac{a-1}{2}) \cdot (\frac{p_1-1}{2})} \cdots (-1)^{(\frac{a-1}{2}) \cdot (\frac{p_m-1}{2})} \\
&= (-1)^{(\frac{a-1}{2})(\frac{p_1-1}{2} + \cdots + \frac{p_m-1}{2})} \\
&= (-1)^{(\frac{a-1}{2}) \cdot (\frac{p_1 \cdots p_m - 1}{2})} && \text{by Lemma 2.2.2} \\
&= (-1)^{(\frac{a-1}{2}) \cdot (\frac{b-1}{2})}. \qquad \square
\end{aligned}$$

Proposition 2.5.2. *Let b and c be distinct odd integers and $a \in \mathbb{Z}$ such that $a \geq 1$. Then the extended quadratic reciprocity implies the following statement:*

$$\textit{If } b \equiv \pm c \bmod 4a, \textit{ then } \left(\frac{a}{b}\right) = \left(\frac{a}{c}\right).$$

Proof. Due to multiplicativity and Example 2.5.3, it is enough to prove the result for a an odd prime. By Theorem 2.5.3, we know that

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{(\frac{a-1}{2})\cdot(\frac{b-1}{2})} \implies \left(\frac{a}{b}\right) = (-1)^{(\frac{a-1}{2})\cdot(\frac{b-1}{2})}\left(\frac{b}{a}\right).$$

Now, suppose that $b \equiv +c \bmod 4a$ (Similar calculations apply for $b \equiv -c \bmod 4a$). Then $b \equiv c \bmod a$ and so

$$\begin{aligned}
\left(\frac{a}{b}\right) &= (-1)^{(\frac{a-1}{2})\cdot(\frac{b-1}{2})}\left(\frac{c}{a}\right) \\
&= (-1)^{(\frac{a-1}{2})\cdot(\frac{b-1}{2})}(-1)^{(\frac{c-1}{2})\cdot(\frac{a-1}{2})}\left(\frac{a}{c}\right) \\
&= (-1)^{(\frac{a-1}{2})\cdot(\frac{b+c-2}{2})}\left(\frac{a}{c}\right) \\
&= (-1)^{(a-1)\cdot(\frac{b+c-2}{4})}\left(\frac{a}{c}\right).
\end{aligned}$$

However,

$$\begin{aligned}
b \equiv c \bmod 4a &\implies b \equiv c \bmod 4 \\
&\implies b + c \equiv 2c \bmod 4 \\
&\implies b + c - 2 \equiv 2c - 2 \bmod 4 \equiv 0 \bmod 4 \\
&\implies (a-1)\cdot(\frac{b+c-2}{4}) = (a-1)\frac{4r}{4} = (a-1)r \qquad \text{for some } r \in \mathbb{Z}.
\end{aligned}$$

Hence, using the fact that $a - 1$ is even we get

$$\left(\frac{a}{b}\right) = (-1)^{(a-1)\cdot(\frac{b+c-2}{4})}\left(\frac{a}{c}\right) = (-1)^{(a-1)r}\left(\frac{a}{c}\right) = \left(\frac{a}{c}\right). \qquad \square$$

Definition 2.5.5. (Gauss Sum modulo p) Let p be an odd prime. Then the Gauss Sum modulo p is defined by:

$$G(a) = \sum_{x \bmod p} e(\frac{ax^2}{p}).$$

Remark. If $a \equiv 0 \bmod p$, then

$$G(0) = \sum_{x \bmod p} e(0) = p.$$

Lemma 2.5.4. *Let p be an odd prime number. We have*

$$\sum_{c \bmod p} e(\frac{c}{p}) = 0$$

Proof. Consider the expression

$$e(\frac{1}{p}) \sum_{c \bmod p} e(\frac{c}{p}) = \sum_{c \bmod p} e(\frac{c+1}{p}).$$

Notice that there exists a bijection between the two sets:

$$\begin{aligned} \{0, 1, \cdots, p-1\} &\xleftrightarrow{\sim} \{0, 1, \cdots, p-1\} \\ c &\longmapsto c+1 \bmod p. \end{aligned}$$

Thus the sum on the right hand side of the above expression is a reordering; we get

$$e(\frac{1}{p}) \sum_{c \bmod p} e(\frac{c}{p}) = \sum_{c \bmod p} e(\frac{c+1}{p}) = \sum_{c \bmod p} e(\frac{c}{p}) \tag{2.5}$$

with $e(\frac{1}{p}) \notin \{0, 1\}$ since p is an odd prime. Thus equation (2.5) implies that

$$\sum_{c \bmod p} e(\frac{c}{p}) = 0.$$

□

Proposition 2.5.3. *For $a \not\equiv 0 \bmod p$, we have*

$$G(a) = \left(\frac{a}{p}\right) G(1).$$

Proof. Case 1: If $a \equiv b^2 \bmod p$ for some $b \neq 0$, then

$$G(a) = \sum_{x \bmod p} e(\frac{b^2x^2}{p}).$$

We now have a bijection:

$$\begin{aligned} \mathbb{Z}/p\mathbb{Z} &\overset{\sim}{\longleftrightarrow} \mathbb{Z}/p\mathbb{Z} \\ bx &\longmapsto y \\ b^{-1}y &\longleftarrow\!\shortmid\ x \end{aligned}$$

Thus

$$G(a) = \sum_{y \bmod p} e(\frac{y^2}{p}) = G(1) = \left(\frac{a}{p}\right) G(1).$$

Case 2: If $a \equiv$ non-square mod p, then

$$G(1) + G(a) = \sum_{x \bmod p} e(\frac{x^2}{p}) + \sum_{y \bmod p} e(\frac{ay^2}{p}).$$

Notice that for $x \in \mathbb{Z}/_{p\mathbb{Z}}$, we have $p - x \equiv -x \bmod p$ with $x^2 = (-x)^2$. Thus as x varies mod p, x^2 runs over 0 once and over each *square* mod p twice. Similarly, as y varies mod p, ay^2 runs over 0 once and over each *non-square* twice. Let $\mathfrak{S}$ and $\mathfrak{T}$ denote the set of *squares* and set of *non-squares* mod p respectively. Hence,

$$\begin{aligned} G(1) + G(a) &= e(\frac{0}{p}) + 2\sum_{s \in \mathfrak{S}} e(\frac{s}{p}) + e(\frac{a.0}{p}) + 2\sum_{t \in \mathfrak{T}} e(\frac{t}{p}) \\ &= 2[e(0) + \sum_{\substack{c \neq 0 \\ c \bmod p}} e(\frac{c}{p})] \\ &= 2[\sum_{c \bmod p} e(\frac{c}{p})] \\ &= 0 \qquad \text{by Lemma 2.5.4.} \end{aligned}$$

$$\implies G(a) = -G(1) = \left(\tfrac{a}{p}\right) G(1). \qquad \square$$

Lemma 2.5.5. *For p an odd prime, define the following:*

$$I_p = \int_{-\infty}^{\infty} e(\frac{y^2}{p})\,dy.$$

This integral converges as an improper integral at both ends.

Proof. We have

$$\begin{aligned} I_p &= \int_{-\infty}^{\infty} e(\frac{y^2}{p})\,dy = \int_{-\infty}^{0} e(\frac{y^2}{p})\,dy + \int_{0}^{\infty} e(\frac{y^2}{p})\,dy = 2\int_{0}^{\infty} e(\frac{y^2}{p})\,dy \\ &= 2\lim_{A\to\infty}\int_{0}^{A} e(\frac{y^2}{p})\,dy. \end{aligned}$$

Now, for $A \leq B$ large enough, apply a change of variable $y^2 = t, dy = \frac{dt}{2\sqrt{t}}$. Then

$$\int_{A}^{B} e(\frac{y^2}{p})\,dy = \int_{A^2}^{B^2} e(\frac{t}{p})\frac{1}{2\sqrt{t}}\,dt.$$

Integrate by parts with $u = \frac{1}{2\sqrt{t}}$ and $v' = e(\frac{t}{p})$. So

$$\int_{A}^{B} e(\frac{y^2}{p})\,dy = \frac{p}{4\pi i\sqrt{B}}e(\frac{B^2}{p}) - \frac{p}{4\pi i\sqrt{A}}e(\frac{A^2}{p}) + \int_{A^2}^{B^2}\frac{p}{4\pi i t^{3/2}}e(\frac{t}{p})\,dt$$

$$\begin{aligned} \Longrightarrow \left|\int_{A}^{B} e(\frac{y^2}{p})\,dy\right| &\leq \left|\frac{p}{4\pi i\sqrt{B}}e(\frac{B^2}{p}) - \frac{p}{4\pi i\sqrt{A}}e(\frac{A^2}{p})\right| + \left|\int_{A^2}^{B^2}\frac{p}{4\pi i t^{3/2}}e(\frac{t}{p})\,dt\right| \\ &\leq \left|\frac{p}{4\pi i\sqrt{B}}e(\frac{B^2}{p}) - \frac{p}{4\pi i\sqrt{A}}e(\frac{A^2}{p})\right| + \int_{A^2}^{B^2}\frac{p}{4\pi i t^{3/2}}\,dt \\ &= \left|\frac{p}{4\pi i\sqrt{B}}e(\frac{B^2}{p}) - \frac{p}{4\pi i\sqrt{A}}e(\frac{A^2}{p})\right| + \frac{p}{4\pi i}\int_{A^2}^{B^2}\frac{1}{t^{3/2}}\,dt \\ &= \left|\frac{p}{4\pi i\sqrt{B}}e(\frac{B^2}{p}) - \frac{p}{4\pi i\sqrt{A}}e(\frac{A^2}{p})\right| - \frac{p}{2\pi i}(\frac{1}{\sqrt{B^2}} - \frac{1}{\sqrt{A^2}}). \end{aligned}$$

Taking the limits as $A, B \to \infty$, I_p converges by **Cauchy Criterion**. □

In the following proof, we denote $G(a)$ for $a = 1$ by $G_p(1)$.

Proposition 2.5.4. *For p an odd prime, we have*

$$G_p(1) = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \bmod 4, \\ i\sqrt{p} & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

Proof. We start by stating a property of the convergence of Fourier Series: If ϕ is a function which is continuously differentiable on $[0,1]$, then

$$\frac{\phi(0)+\phi(1)}{2} = \sum_{m\in\mathbb{Z}} c_m$$

where c_m is the m-th Fourier coefficient. i.e.

$$c_m(\phi) = \int_0^1 \phi(x)e(-mx)\,dx.$$

For this reason, consider the function

$$\begin{aligned} f : [0,1] &\to \mathbb{R} \\ x &\mapsto e(\frac{x^2}{p}) \end{aligned}$$

and let $f_k(x) = f(x+k)$ for $k = 0, 1, \cdots, p-1$. Now we have the following:

$$\begin{aligned}
\sum_{k=0}^{p-1} \frac{f_k(0)+f_k(1)}{2} &= \sum_{k=0}^{p-1} \frac{f(k)+f(1+k)}{2} \\
&= \sum_{k=0}^{p-1} \frac{e(\frac{k^2}{p}) + e(\frac{(1+k)^2}{p})}{2} \\
&= \frac{1}{2}\sum_{k=0}^{p-1} e(\frac{k^2}{p}) + \frac{1}{2}\sum_{k=0}^{p-1} e(\frac{(1+k)^2}{p}) \\
&= \frac{1}{2}\sum_{k=0}^{p-1} e(\frac{k^2}{p}) + \frac{1}{2}\sum_{k=1}^{p} e(\frac{k^2}{p}) \\
&= \frac{1}{2}G_p(1) + \frac{1}{2}G_p(1) - \frac{1}{2}e(0) + \frac{1}{2}e(p) \\
&= \frac{1}{2}G_p(1) + \frac{1}{2}G_p(1) - \frac{1}{2} + \frac{1}{2}
\end{aligned}$$

$$= G_p(1).$$

Hence, if $\phi = \sum_{k=0}^{p-1} f_k = f_0 + f_1 + \cdots + f_{p-1}$, then

$$\begin{aligned}
\frac{\phi(0)+\phi(1)}{2} &= f_0(0) + f_1(0) + \cdots + f_{p-1}(0) + f_0(1) + f_1(1) + \cdots + f_{p-1}(1) \\
&= \sum_{k=0}^{p-1} \frac{f_k(0)+f_k(1)}{2} \\
&= G(1).
\end{aligned}$$

Thus by using the property stated above, it would be sufficient to compute the sum of the Fourier coefficients of ϕ to get the value of $G_p(1)$. So now,

$$\begin{aligned}
G_p(1) &= \sum_m \int_0^1 \phi(x)e(-mx)\,dx \\
&= \sum_m \int_0^1 \sum_{k=0}^{p-1} f_k(x)e(-mx)\,dx \\
&= \sum_m \int_0^1 (f_0(x) + f_1(x) + \cdots + f_{p-1}(x))e(-mx)\,dx \\
&= \sum_m \Big(\int_0^1 f_0(x)e(-mx)\,dx + \int_0^1 f_1(x)e(-mx)\,dx + \cdots + \int_0^1 f_{p-1}(x)e(-mx)\,dx \Big) \\
&= \sum_m \Big(\int_0^1 f(x)e(-mx)\,dx + \int_0^1 f(x+1)e(-mx)\,dx + \cdots + \int_0^1 f(x+p-1)e(-mx)\,dx \Big).
\end{aligned}$$

Applying a change of variable $x + k \to x$ inside each integral, we get

$$\begin{aligned}
G_p(1) &= \sum_m \Big(\int_0^1 f(x)e(-mx)\,dx + \int_1^2 f(x)e(-m(x-1))\,dx + \cdots + \int_{p-1}^p f(x)e(-m(x-p+1))\,dx \Big) \\
&= \sum_m \Big(\int_0^1 f(x)e(-mx)\,dx + \int_1^2 f(x)e(-mx)\,dx + \cdots + \int_{p-1}^p f(x)e(-mx)\,dx \Big) \\
&= \sum_m \Big(\int_0^p f(x)e(-mx)\,dx \Big) \\
&= \sum_m \int_0^p e(\frac{x^2}{p})e(-mx)\,dx \\
&= \sum_m \int_0^p e(\frac{x^2 - pmx}{p})\,dx.
\end{aligned}$$

Complete the square $x^2 - pmx = \left(x - \frac{pm}{2}\right)^2 - \frac{p^2m^2}{4}$, hence

$$\begin{aligned} G_p(1) &= \sum_m \int_0^p e(\frac{1}{p}(x - \frac{pm}{2})^2) e(-\frac{1}{p}.\frac{p^2m^2}{4})\, dx \\ &= \sum_m e^{-\frac{\pi i p m^2}{2}} \int_0^p e(\frac{1}{p}(x - \frac{pm}{2})^2)\, dx. \end{aligned}$$

Notice that:

- If m is even $(m = 2r)$, then $e^{-\frac{\pi i m^2 p}{2}} = e^{-\frac{\pi i (2r)^2 p}{2}} = 1$,
- If m is odd $(m = 2r+1)$, then $e^{-\frac{\pi i m^2 p}{2}} = e^{-\frac{\pi i p(2r+1)^2}{2}} = e^{-\frac{4\pi i p r^2}{2}} e^{-\frac{4\pi i r p}{2}} e^{-\frac{\pi i p}{2}}$
  $= (e^{\frac{\pi i}{2}})^{-p} = i^{-p}$.

Therefore, we split the sum between m even and odd.

$$\begin{aligned} G_p(1) &= \sum_{m \text{ even}} e^{-\frac{\pi i p m^2}{2}} \int_0^p e(\frac{1}{p}(x - \frac{pm}{2})^2)\, dx + \sum_{m \text{ odd}} e^{-\frac{\pi i p m^2}{2}} \int_0^p e(\frac{1}{p}(x - \frac{pm}{2})^2)\, dx \\ &= \sum_{r \in \mathbb{Z}} \int_0^p e(\frac{1}{p}(x - \frac{2pr}{2})^2)\, dx + \sum_{r \in \mathbb{Z}} i^{-p} \int_0^p e(\frac{1}{p}(x - \frac{p(2r+1)}{2})^2)\, dx. \end{aligned}$$

Apply the following change of variables $x - pr \to y$ and $x - \frac{p(2r+1)}{2} \to y$ in the two above consecutive integrals to get:

$$\begin{aligned} G_p(1) &= \sum_{r \in \mathbb{Z}} \int_{p(-r)}^{p(-r+1)} e(\frac{y^2}{p})\, dy + \sum_{r \in \mathbb{Z}} i^{-p} \int_{p(-r-\frac{1}{2})}^{p(-r+\frac{1}{2})} e(\frac{y^2}{p})\, dy \\ &= \lim_{R_1, S_1 \to \infty} \int_{p(-R_1)}^{p(S_1)} e(\frac{y^2}{p})\, dy + \lim_{R_2, S_2 \to \infty} i^{-p} \int_{p(-R_2)}^{p(S_2)} e(\frac{y^2}{p})\, dy \\ &= \int_{-\infty}^{\infty} e(\frac{y^2}{p})\, dy + i^{-p} \int_{-\infty}^{\infty} e(\frac{y^2}{p})\, dy. \end{aligned}$$

Therefore,

$$G_p(1) = (1 + i^{-p}) I_p \tag{2.6}$$

which converges by Lemma 2.5.5. Finally, apply a change of variable $\frac{y}{\sqrt{p}} \to u$ to get

$$\begin{aligned} I_p &= \int_{-\infty}^{\infty} e(u^2)\sqrt{p}\, du \\ &= \sqrt{p}\int_{-\infty}^{\infty} e(u^2)\, du \\ &= \sqrt{p} I_1. \end{aligned}$$

Note that the above calculation never used the fact that p is a prime. So now, using equation (2.6) for $p = 1$ we have:

$$G_1(1) = (1+i^{-1})I_1 \implies \sum_{x \bmod 1} e(x^2) = (1+i^{-1})I_1 \implies 1 = (1+i^{-1})I_1 \implies I_1 = \frac{1}{1+i^{-1}}.$$

Thus obtaining the relation:

$$G_p(1) = \sqrt{p}\frac{1+i^{-p}}{1+i^{-1}}.$$

- If $p \equiv 1 \bmod 4$, then $G(1) = \sqrt{p}\frac{1+i^{-1}}{1+i^{-1}} = \sqrt{p}$,
- If $p \equiv 3 \bmod 4$, then $G(1) = \sqrt{p}\frac{1+i^{-3}}{1+i^{-1}} = \sqrt{p}\frac{1+i}{1-i} = i\sqrt{p}$. □

2.6 The Lipschitz Summation Formula

Proposition 2.6.1. *Let $k \geq 2$ be an integer. Then we have*

$$\sum_{n\in\mathbb{Z}} (z+n)^{-k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{d=1}^{\infty} d^{k-1}e(dz).$$

Proof. By Complex Analysis, we have

$$\frac{\pi}{\tan \pi z} = \pi\frac{\cos \pi z}{\sin \pi z} = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right).$$

Moreover,

$$\begin{aligned}
\pi\frac{\cos \pi z}{\sin \pi z} &= \pi i\frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} \\
&= -\pi i\frac{1 + e(z)}{1 - e(z)} \\
&= -\pi i\frac{1 + q}{1 - q} && q = e(z) \\
&= -\pi i\left(1 + \frac{2q}{1 - q}\right) && |q| = e^{-2\pi y} < 1 \\
&= -\pi i\left(1 + 2\sum_{d=1}^{\infty} q^d\right) \\
&= -\pi i - 2\pi i\sum_{d=1}^{\infty} q^d.
\end{aligned}$$

$$\implies \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right) = -\pi i - 2\pi i\sum_{d=1}^{\infty} q^d. \tag{2.7}$$

Notice that $\left(\frac{1}{z-n} + \frac{1}{z+n}\right) = \frac{2}{z^2-n^2} \leq \frac{L}{n^2}$ for large n. This implies convergence. So, taking the derivative with respect to z on both sides of equation (2.7), we get

$$\frac{-1}{z^2} + \sum_{n=1}^{\infty}\left(\frac{-1}{(z-n)^2} + \frac{-1}{(z+n)^2}\right) = -(2\pi i)^2\sum_{d=1}^{\infty} e(zd)d. \tag{2.8}$$

Assuming absolute convergence, we can rearrange and combine the sum on the left hand side to get

$$\frac{-1}{z^2} + \sum_{n=1}^{\infty}\left(\frac{-1}{(z-n)^2} + \frac{-1}{(z+n)^2}\right) = -\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^2}.$$

Therefore, equation (2.8) becomes:

$$-\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^2} = -(2\pi i)^2\sum_{d=1}^{\infty} e(zd)d.$$

Thus, by induction on the derivative we get the general formula. □

Chapter 3

Transformation Law of θ

This chapter aims to prove important transformation formulas in order to reach the fact that Theta is a modular form of half integral weight. Most of this chapter is taken from [Shimura, 1973], pages 440–456.

3.1 Transformation Formulas

We prove now explicit transformation formulas for the Theta Function needed for the next section. We always assume that A and NA^{-1} have coefficients in $\mathbb{Z}$ and that $Ah \in N\mathbb{Z}^n$.

Definition 3.1.1. (Theta Function) Let n be a positive integer, A a positive definite real symmetric matrix of size n, P a spherical function and N a positive integer such that $\det A$ divides N^n. Now, fix an element $h \in \mathbb{Z}^n$ and consider a Theta Function with $z \in \mathbb{H}$, defined by

$$\theta(z; h, A, N, P) = \sum_{m \equiv h \bmod N} P(m) e(\frac{z m^t A m}{2N^2})$$

where the sum is taken over all $m \in \mathbb{Z}^n$ congruent to h mod $N\mathbb{Z}^n$.

Proposition 3.1.1. *The Theta function is holomorphic on* $\mathbb{H}$.

Proof. Consider the subsets of $\mathbb{H}$ of the form

$$\mathcal{R}_{r,s} = \{x + iy; |x| \leq r, y \geq s\}.$$

First, notice that $\mathcal{R}_{r,s}$ form an exhaustion of $\mathbb{H}$ in the following way: $\mathcal{R}_{n',1/n'}$ is contained in the set of interior points $\mathcal{R}^{o}_{n'+1,1/(n'+1)}$ of $\mathcal{R}_{n'+1,1/(n'+1)}$ and thus

$$\mathbb{H} = \bigcup_{n'=1}^{\infty} \mathcal{R}_{n',1/n'} = \bigcup_{n'=1}^{\infty} \mathcal{R}^{o}_{n',1/n'}.$$

Next, by definition of compactness, any compact $K \subset \mathbb{H}$ will be covered by finitely many $\mathcal{R}^{o}_{n',1/n'}$. So it suffices to prove uniform convergence of the series above on $\mathcal{R}_{r,s}$ to get uniform convergence on K. Indeed, let $z = x + iy \in \mathcal{R}_{r,s}$. Then we have

$$\left|e(\frac{zm^tAm}{2N^2})\right| = \left|e(\frac{xm^tAm}{2N^2})\right|.\left|e^{-\pi y m^t Am/N^2}\right| \leq e^{-\pi s m^t Am/N^2}$$

$$\text{with } \sum_{m\in\mathbb{Z}^n} e^{-\pi s m^t Am/N^2} \leq \sum_{m\in\mathbb{Z}^n} e^{-\pi s \lambda_{min}|m|_2^2/N^2} \qquad \text{by Lemma 2.2.3.}$$

We now regroup the terms for each $j = 0, 1, 2\cdots$ and notice that there are at most $(2\sqrt{j} + 1)^n$ choices for m such that $|m|_2^2 = j$. Knowing that $(2\sqrt{j} + 1)^n \leq kj^{n/2}$ for some constant k, we get

$$\sum_{m\in\mathbb{Z}^n} e^{-\pi s \lambda_{min}|m|_2^2/N^2} \leq \sum_{j=0}^{\infty} kj^{n/2} e^{-\pi s \lambda_{min} j/N^2}$$

which converges due to the fast exponential decay. Therefore, the Theta series converges uniformly on $\mathcal{R}_{r,s}$ by **Weierstrass M-test**. Hence the series converges to a holomorphic function leading to the holomorphicity of Theta on $\mathbb{H}$. □

Remark. The spherical function $P(m)$ is also bounded by a power of $|m|$, so this should fit into the convergence framework.

Lemma 3.1.1. *For $h \in \mathbb{Z}^n$, we have a bijection between the following sets:*

$$\left\{(m,g) \in \mathbb{Z}^n \times \left(\mathbb{Z}/_{cN\mathbb{Z}}\right)^n \quad \middle| \quad {\scriptstyle m \equiv g \bmod cN \atop g \equiv h \bmod N}\right\} \xleftrightarrow{\sim} \{m \in \mathbb{Z}^n \quad | \quad m \equiv h \bmod N\}.$$

Proof. Given $g \in \left(\mathbb{Z}/_{cN\mathbb{Z}}\right)^n$ with $g \equiv h \bmod N$, we can write $g = h + Nq$ with $q \in \left(\mathbb{Z}/_{c\mathbb{Z}}\right)^n$. So we get

$$\begin{aligned} m \equiv g \bmod cN &\implies m = cNq' + g && \text{for some } q' \in \mathbb{Z}^n \\ &\implies m = cNq' + h + Nq \\ &\implies m = (cq' + q)N + h \\ &\implies m \equiv h \bmod N. \end{aligned}$$

Conversely, given $m \in \mathbb{Z}^n$ with $m \equiv h \bmod N$, there exists a unique $g \in \left(\mathbb{Z}/_{cN\mathbb{Z}}\right)^n$ such that $m \equiv g \bmod cN$, and we have

$$\begin{aligned} m \equiv g \bmod cN \text{ and } m \equiv h \bmod N &\implies m = g + cNq = h + Nq' \\ &\implies g = h + N(q' - cq) \\ &\implies g \equiv h \bmod N. \end{aligned}$$

□

Lemma 3.1.2. *There exists a bijection between the following sets:*

$$\{p \in \mathbb{Z}^n \quad | \quad Ap \equiv 0 \bmod N\} \xleftrightarrow{\sim} \left\{(k,p) \in \left(\mathbb{Z}/_{N\mathbb{Z}}\right)^n \times \mathbb{Z}^n \quad \middle| \quad {\scriptstyle Ak \equiv 0 \bmod N \atop p \equiv k \bmod N}\right\}.$$

Proof. Given $p \in \mathbb{Z}^n$ such that $Ap \equiv 0 \bmod N$, then there exists a unique $k \in \left(\mathbb{Z}/_{N\mathbb{Z}}\right)^n$ such that $p \equiv k \bmod N\mathbb{Z}^n$ and we have $Ak \equiv 0 \bmod N\mathbb{Z}^n$.

Conversely, given $k \in \left(\mathbb{Z}/_{N\mathbb{Z}}\right)^n$ such that $Ak \equiv 0 \bmod N$, then every $p \equiv k \bmod N$ satisfies $Ap \equiv Ak \bmod N \equiv 0 \bmod N$. □

Denote the Theta function by $\theta(z;h,A,N)$ (dropping P) to prove what follows:

Proposition 3.1.2. *We have the following transformation formulas for $\nu=0$:*

$$\theta(z+2;h,A,N)=e(\frac{h^tAh}{N^2})\theta(z;h,A,N) \tag{3.1}$$

$$\text{For } c\in\mathbb{Z}_+,\ \theta(z;h,A,N)=\sum_{\substack{g\equiv h \bmod N\\ g \bmod cN}}\theta(cz;g,cA,cN) \tag{3.2}$$

$$\theta(\frac{-1}{z};0,A,N)=(\det A)^{-1/2}(-iz)^{n/2}\sum_{\substack{k \bmod N\\ Ak\equiv 0 \bmod N}}\theta(z;k,A,N). \tag{3.3}$$

Proof. (3.1):

$$\begin{aligned}\theta(z+2;h,A,N)&=\sum_{m\equiv h \bmod N}e(\frac{(z+2)m^tAm}{2N^2})\\ &=\sum_{m\equiv h \bmod N}e(\frac{zm^tAm}{2N^2})e(\frac{m^tAm}{N^2}).\end{aligned}$$

Since $m\equiv h \bmod N\mathbb{Z}^n$, then each m can be written as $qN+h$ for some $q\in\mathbb{Z}^n$. So we get

$$\begin{aligned}\theta(z+2;h,A,N)&=\sum_{q\in\mathbb{Z}^n}e(\frac{z(qN+h)^tA(qN+h)}{2N^2})e((q+\frac{h}{N})^tA(q+\frac{h}{N}))\\ &=\sum_{q\in\mathbb{Z}^n}e(\frac{z(qN+h)^tA(qN+h)}{2N^2})e(q^tAq)e(\frac{q^tAh}{N})e(\frac{h^tAq}{N})e(\frac{h^tAh}{N^2}).\end{aligned}$$

Now using the fact that $q\in\mathbb{Z}^n$ and $Ah\in N\mathbb{Z}^n$, we have $q^tAq\in\mathbb{Z}$ and $\frac{q^tAh}{N}\in\mathbb{Z}$. Also, since $A=A^t$, then $\frac{h^tAq}{N}=(\frac{q^tAh}{N})^t\in\mathbb{Z}$. Hence

$$\begin{aligned}\theta(z+2;h,A,N)&=\sum_{m\equiv h \bmod N}e(\frac{zm^tAm}{2N^2})e(\frac{h^tAh}{N^2})\\ &=e(\frac{h^tAh}{N^2})\sum_{m\equiv h \bmod N}e(\frac{zm^tAm}{2N^2})\end{aligned}$$

$$= e(\frac{h^t A h}{N^2})\theta(z; h, A, N).$$

(3.2):

$$\begin{aligned}\theta(cz; g, cA, cN) &= \sum_{m \equiv g \bmod cN} e(\frac{czm^t cAm}{2c^2N^2}) \\ &= \sum_{m \equiv g \bmod cN} e(\frac{zm^t Am}{2N^2})\end{aligned}$$

Then,

$$\begin{aligned}\sum_{g \equiv h \bmod N} \theta(cz; g, cA, cN) &= \sum_{g \equiv h \bmod N} \sum_{m \equiv g \bmod cN} e(\frac{zm^t Am}{2N^2}) \\ &= \sum_{m \equiv h \bmod N} e(\frac{zm^t Am}{2N^2}) \qquad \text{by Lemma 3.1.1} \\ &= \theta(z; h, A, N).\end{aligned}$$

(3.3): Notice that this identity is holomorphic on $\mathbb{H}$ on both sides since

- $\theta(z; 0, A, N)$ is holomorphic on $\mathbb{H}$ by Proposition 3.1.1,
- $\theta(\frac{-1}{z}; 0, A, N)$ is holomorphic on $\mathbb{H}$ as a composition of two holomorphic funtions: $\theta(z; 0, A, N)$ and $\frac{-1}{z}$.

Thus, due to analytic continuation it would be enough to prove it for $z = \alpha i$ with $\alpha > 0$. Consider the function

$$f(x) = e(\frac{zx^t Ax}{2}).$$

Having found the Fourier transform of f in Propostion 2.4.3 for $h = \nu = 0$, apply the

Poisson Summation Formula and set $m = Ns \in N\mathbb{Z}^n$ to get

$$\sum_{s\in\mathbb{Z}^n} e(\frac{zs^tAs}{2}) = (\det A)^{-1/2}(-iz)^{-n/2}\sum_{s\in\mathbb{Z}^n} e(-\frac{s^tA^{-1}s}{2z})$$

$$\implies \sum_{\substack{m\in N\mathbb{Z}^n\\ m\equiv 0 \bmod N}} e(\frac{zm^tAm}{2N^2}) = (\det A)^{-1/2}(-iz)^{-n/2}\sum_{\substack{m\in N\mathbb{Z}^n\\ m\equiv 0 \bmod N}} e(-\frac{m^tA^{-1}m}{2zN^2})$$

$$i.e.\quad \theta(z;0,A,N) = (\det A)^{-1/2}(-iz)^{-n/2}\sum_{m\equiv 0 \bmod N} e(-\frac{m^tA^{-1}m}{2zN^2}).$$

Write $m = Ap$, $p = A^{-1}m$. Note that since $m \equiv 0 \bmod N$ and $NA^{-1} \in M_n(\mathbb{Z})$ then $p \in \mathbb{Z}^n$. Now apply a change of variable $z \to \frac{-1}{z}$ to obtain:

$$\begin{aligned}
\theta(\frac{-1}{z};0,A,N) &= (\det A)^{-1/2}(\frac{i}{z})^{-n/2}\sum_{\substack{p\in\mathbb{Z}^n\\ Ap\equiv 0 \bmod N}} e(\frac{z(p^tA^t)A^{-1}(Ap)}{2N^2})\\
&= (\det A)^{-1/2}(\frac{i}{z})^{-n/2}\sum_{p\in A^{-1}N\mathbb{Z}^n} e(\frac{zp^tAp}{2N^2})\\
&= (\det A)^{-1/2}(\frac{i}{z})^{-n/2}\sum_{\substack{Ak\equiv 0 \bmod N\\ k \bmod N}}\ \sum_{p\equiv k \bmod N} e(\frac{zp^tAp}{2N^2}) \qquad \text{by Lemma 3.1.2}\\
&= (\det A)^{-1/2}(\frac{i}{z})^{-n/2}\sum_{\substack{Ak\equiv 0 \bmod N\\ k \bmod N}} \theta(z;k,A,N). \qquad \square
\end{aligned}$$

Proposition 3.1.3. *According to Definition 3.1.1, we have*

$$\theta(\frac{-1}{z};h,A,N,P) = (-i)^{\nu}(\det A)^{-1/2}(-iz)^{\kappa/2}\sum_{\substack{k \bmod N\\ Ak\equiv 0 \bmod N}} e(\frac{k^tAh}{N^2})\theta(z;k,A,N,P). \tag{3.4}$$

Proof. For the same reason in the proof of (3.3), we prove the result for $z = \alpha i$. Consider the function

$$f_u(x) = P(N(x+u))e(\frac{z(x+u)^tA(x+u)}{2})$$

with $u = N^{-1}h \in \frac{1}{N}\mathbb{Z}^n$. Having found the Fourier transform of f_u in Proposition 2.4.4, apply the Poisson Summation Formula to get

$$\sum_{s\in\mathbb{Z}^n} P(N(s+u))e(\frac{z(s+u)^tA(s+u)}{2}) = \sum N^\nu \beta_q(-iz)^{-\kappa/2}(\det A)^{-1/2}(-i)^\nu \sum_{s\in\mathbb{Z}^n} e(s^tu)\cdot \\ \cdot (q^ts)^\nu e(-\frac{s^tA^{-1}s}{2z}).$$

Notice that for $y = Ns + h$

$$\begin{aligned}\sum_{s\in\mathbb{Z}^n} P(N(s+u))e(\frac{z(s+u)^tA(s+u)}{2}) &= \sum_{y\equiv h \bmod N} P(y)e(\frac{zy^tAy}{2N^2}) \\ &= \theta(z; h, A, N, P)\end{aligned}$$

$$\implies \theta(z; h, A, N, P) = \sum N^\nu \beta_q(-iz)^{-\kappa/2}(\det A)^{-1/2}(-i)^\nu \sum_{s\in\mathbb{Z}^n} e(s^tu)(q^ts)^\nu e(-\frac{s^tA^{-1}s}{2z}).$$

Now, applying a change of variable $z \to \frac{-1}{z}$ and setting $s = \frac{Ap}{N}$,

$$\begin{aligned}\theta(\frac{-1}{z}; h, A, N, P) &= \sum N^\nu \beta_q(-iz)^{\kappa/2}(\det A)^{-1/2}(-i)^\nu \sum_{s\in\mathbb{Z}^n} e(s^tu)(q^ts)^\nu e(\frac{zs^tA^{-1}s}{2}) \\ &= \sum N^\nu \beta_q(-iz)^{\kappa/2}(\det A)^{-1/2}(-i)^\nu \sum_{\substack{k \bmod N \\ Ak\equiv 0 \bmod N}} \sum_{\substack{p\in\mathbb{Z}^n \\ p\equiv k \bmod N}} e(\frac{p^tAh}{N^2})\cdot \\ &\cdot (q^t\frac{Ap}{N})^\nu e(\frac{zp^tAA^{-1}Ap}{2N^2}) \\ &= \sum N^\nu \beta_q(-iz)^{\kappa/2}(\det A)^{-1/2}(-i)^\nu \sum_{\substack{k \bmod N \\ Ak\equiv 0 \bmod N}} \sum_{\substack{p\in\mathbb{Z}^n \\ p\equiv k \bmod N}} e(\frac{p^tAh}{N^2})\cdot \\ &\cdot (q^t\frac{Ap}{N})^\nu e(\frac{zp^tAp}{2N^2}).\end{aligned}$$

In the inner sum over p, p has the form $k + Nl$ for some $l \in \mathbb{Z}^n$. Then

$$\begin{aligned}e(\frac{p^tAh}{N^2}) &= e(\frac{(k+Nl)^tAh}{N^2}) \\ &= e(\frac{k^tAh}{N^2} + \frac{Nl^tAh}{N^2}) \\ &= e(\frac{k^tAh}{N^2}) \qquad \text{given that } Ah \in N\mathbb{Z}^n.\end{aligned}$$

Thus,

$$\begin{aligned}
\theta(\frac{-1}{z};h,A,N,P) &= \sum N^{\nu}\beta_q(-iz)^{\kappa/2}(\det A)^{-1/2}(-i)^{\nu}\sum_{\substack{k \bmod N\\ Ak\equiv 0 \bmod N}}\sum_{\substack{p\in\mathbb{Z}^n\\ p\equiv k \bmod N}} e(\frac{k^tAh}{N^2})\cdot \\
&\cdot (q^t\frac{Ap}{N})^{\nu}e(\frac{zp^tAp}{2N^2}) \\
&= (-i)^{\nu}(\det A)^{-1/2}(-iz)^{\kappa/2}\sum_{\substack{k \bmod N\\ Ak\equiv 0 \bmod N}} e(\frac{k^tAh}{N^2})\sum_{\substack{p\in\mathbb{Z}^n\\ p\equiv k \bmod N}}\sum N^{\nu}\beta_q\cdot \\
&\cdot (q^t\frac{Ap}{N})^{\nu}e(\frac{zp^tAp}{2N^2}) \\
&= (-i)^{\nu}(\det A)^{-1/2}(-iz)^{\kappa/2}\sum_{\substack{k \bmod N\\ Ak\equiv 0 \bmod N}} e(\frac{k^tAh}{N^2})\sum_{\substack{p\in\mathbb{Z}^n\\ p\equiv k \bmod N}}\sum \beta_q\cdot \\
&\cdot (q^tAp)^{\nu}e(\frac{zp^tAp}{2N^2}) \\
&= (-i)^{\nu}(\det A)^{-1/2}(-iz)^{\kappa/2}\sum_{\substack{k \bmod N\\ Ak\equiv 0 \bmod N}} e(\frac{k^tAh}{N^2})\sum_{\substack{p\in\mathbb{Z}^n\\ p\equiv k \bmod N}} P(p)e(\frac{zp^tAp}{2N^2}) \\
&= (-i)^{\nu}(\det A)^{-1/2}(-iz)^{\kappa/2}\sum_{\substack{k \bmod N\\ Ak\equiv 0 \bmod N}} e(\frac{k^tAh}{N^2})\theta(z;k,A,N,P). \qquad \square
\end{aligned}$$

3.2 Transformation Law

In this section, our goal is to prove that the Theta function is a modular form on the congruence subgroup $\Gamma_0(2,2N)$ defined later. For this reason, we use the previous section along with properties of Gauss Sums to obtain the desired result.

Lemma 3.2.1. *For $k \in \left(\mathbb{Z}/N\mathbb{Z}\right)^n$, we have*

$$\sum_{\substack{k \bmod N\\ Ak\equiv 0 \bmod N}} 1 = \det A.$$

Proof. Notice that $\sum_{\substack{k \bmod N\\ Ak\equiv 0 \bmod N}} 1 = \#\ker A$ with A viewed as a homomorphism

$$A : \left(\mathbb{Z}/N\mathbb{Z}\right)^n \to \left(\mathbb{Z}/N\mathbb{Z}\right)^n.$$

Since NA^{-1} has integer coefficients, we have that $N\mathbb{Z}^n \subset A(\mathbb{Z}^n) \subset \mathbb{Z}^n$. So by the **Third Isomorphism Theorem**,

$$\left(\mathbb{Z}/N\mathbb{Z}\right)^n \Big/ A\left(\mathbb{Z}/N\mathbb{Z}\right)^n \simeq \mathbb{Z}^n / A\mathbb{Z}^n.$$

This implies that,

$$\begin{aligned}
[\left(\mathbb{Z}/N\mathbb{Z}\right)^n : A\left(\mathbb{Z}/N\mathbb{Z}\right)^n] &= [\mathbb{Z}^n : A\mathbb{Z}^n] \\
&= |\det A| \\
&= \det A \qquad\qquad \text{since A is positive definite}
\end{aligned}$$

with $A\left(\mathbb{Z}/N\mathbb{Z}\right)^n$ image of A. Next, by the **First Isomorphism Theorem**,

$$A\left(\mathbb{Z}/N\mathbb{Z}\right)^n \simeq \left(\mathbb{Z}/N\mathbb{Z}\right)^n \Big/ \ker A$$

$$\implies \#A\left(\mathbb{Z}/N\mathbb{Z}\right)^n = \frac{N^n}{\#\ker A}$$

$$\implies \#\ker A = \frac{N^n}{\#A\left(\mathbb{Z}/N\mathbb{Z}\right)^n} = \left[\left(\mathbb{Z}/N\mathbb{Z}\right)^n : A\left(\mathbb{Z}/N\mathbb{Z}\right)^n\right] = \det A. \qquad \square$$

In the following lemma, it is enough to assume that A and NA^{-1} are integer matrices, dropping the symmetric and positive definite conditions.

Lemma 3.2.2. *Given $v \in \mathbb{Z}^n$ with $v^tA \equiv 0 \bmod N$, we have*

$$\sum_{\substack{k \in (\mathbb{Z}/N\mathbb{Z})^n \\ Ak \equiv 0 \bmod N}} e\left(\frac{v^tAk}{N^2}\right) = \begin{cases} \det A & \text{if } v \equiv 0 \bmod N, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. • If $v \equiv 0 \bmod N$ (i.e. $v = Nq$ for some $q \in \mathbb{Z}^n$) :

$$\begin{aligned}
\sum_{\substack{k\in(\mathbb{Z}/N\mathbb{Z})^n\\ Ak\equiv 0 \bmod N}} e\Big(\frac{v^tAk}{N^2}\Big) &= \sum_{\substack{k\in(\mathbb{Z}/N\mathbb{Z})^n\\ Ak\equiv 0 \bmod N}} e\Big(\frac{Nq^tAk}{N^2}\Big)\\
&= \sum_{\substack{k\in(\mathbb{Z}/N\mathbb{Z})^n\\ Ak\equiv 0 \bmod N}} e\Big(\frac{q^tAk}{N}\Big)\\
&= \sum_{\substack{k\in(\mathbb{Z}/N\mathbb{Z})^n\\ Ak\equiv 0 \bmod N}} 1 && \text{since } \frac{q^tAk}{N}\in\mathbb{Z}\\
&= \det A && \text{by Lemma 3.2.1.}
\end{aligned}$$

• If $v \not\equiv 0 \bmod N$:

Consider first the case where $A = \begin{pmatrix} a_1 & 0 & \cdots & 0\\ 0 & a_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a_n\end{pmatrix}$ an integer diagonal matrix.

The condition that $NA^{-1} \in M_n(\mathbb{Z})$ means that each a_i is a divisor of N, and we can write $N = a_1d_1 = \cdots = a_nd_n$ with $d_i \in \mathbb{Z}$. Next, we have

$$\begin{aligned}
v^tA \equiv 0 \bmod N &\implies v^t \text{ has the form } \begin{pmatrix} v_1 & v_2 & \cdots & v_n\end{pmatrix} \text{ such that } v_i \equiv 0 \bmod d_i\\
&\implies v^t = \begin{pmatrix} v_1 & v_2 & \cdots & v_n\end{pmatrix} = \begin{pmatrix} w_1d_1 & w_2d_2 & \cdots & w_nd_n\end{pmatrix} \text{ with } w_i \in \mathbb{Z}.
\end{aligned}$$

So, the set of k mod N with $Ak \equiv 0 \bmod N$ is the set

$$k^t = \begin{pmatrix} k_1 & k_2 & \cdots & k_n\end{pmatrix} = \begin{pmatrix} l_1d_1 & l_2d_2 & \cdots & l_nd_n\end{pmatrix}$$

where l_i ranges over integers mod a_i. It follows that

$$v^tAk = v_1a_1k_1 + \cdots + v_na_nk_n = \sum_{i=1}^n w_ia_il_id_i^2$$

$$\implies \frac{v^tAk}{N^2} = \sum_{i=1}^n \frac{w_ia_il_id_i^2}{a_i^2d_i^2} = \sum_{i=1}^n \frac{w_il_i}{a_i}.$$

Therefore,

$$\sum_{\substack{k\in(\mathbb{Z}/N\mathbb{Z})^n \\ Ak\equiv 0 \bmod N}} e\Big(\frac{v^tAk}{N^2}\Big) = \sum_{\substack{l_1 \bmod a_1 \\ \vdots \\ l_n \bmod a_n}} e\Big(\frac{w_1l_1}{a_1}\Big)\cdots e\Big(\frac{w_nl_n}{a_n}\Big).$$

Furthermore, we have

$$\begin{aligned} v \not\equiv 0 \bmod N &\implies N \text{ does not divide at least one } v_i \\ &\implies \exists \text{ at least one } w_i \text{ such that } w_i \not\equiv 0 \bmod a_i. \end{aligned}$$

Assuming that $w_1 \not\equiv 0 \bmod a_1$, we have by Lemma 2.5.3

$$\sum_{l_1 \bmod a_1} e\Big(\frac{w_1l_1}{a_1}\Big) = 0.$$

Hence, for varying l_1 and fixed $l_2,\cdots,l_n$, we get that

$$\begin{aligned} \sum_{\substack{k\in(\mathbb{Z}/N\mathbb{Z})^n \\ Ak\equiv 0 \bmod N}} e\Big(\frac{v^tAk}{N^2}\Big) &= \sum_{\substack{l_1 \bmod a_1 \\ \vdots \\ l_n \bmod a_n}} e\Big(\frac{w_1l_1}{a_1}\Big)\cdots e\Big(\frac{w_nl_n}{a_n}\Big) \\ &= C\cdot \sum_{l_1 \bmod a_1} e\Big(\frac{w_1l_1}{a_1}\Big) \\ &= 0. \end{aligned}$$

Now, for the general case, we use the Smith normal form of A to write $A = UBV$ with B an integer diagonal matrix and U and V integer matrices having $\det U = \det V = \pm 1$. Notice that the following are satisfied:

- U and V are integer matrices with $\det U = \det V = \pm 1 \implies U^{-1}$ and V^{-1} are integer matrices $\implies B = U^{-1}AV^{-1}$ is an integer matrix,
- U, V and NA^{-1} are integer matrices $\implies NB^{-1} = VNA^{-1}U$ is an integer matrix,
- $(U^tv)^tB = v^tUU^{-1}AV^{-1} = v^tAV^{-1} \equiv 0 \bmod N$ since $v^tA \equiv 0 \bmod N$.

Moreover, there exists a bijection between the sets

$$\{k \in \left(\mathbb{Z}/_{N\mathbb{Z}}\right)^n \quad | \quad Ak \equiv 0 \bmod N\} \xleftrightarrow{\sim} \{k \in \left(\mathbb{Z}/_{N\mathbb{Z}}\right)^n \quad | \quad BVk \equiv 0 \bmod N\}.$$

Finally, we get

$$\begin{aligned}
\sum_{\substack{k \in (\mathbb{Z}/N\mathbb{Z})^n \\ Ak \equiv 0 \bmod N}} e\Big(\frac{v^t Ak}{N^2}\Big) &= \sum_{\substack{k \in (\mathbb{Z}/N\mathbb{Z})^n \\ BVk \equiv 0 \bmod N}} e\Big(\frac{v^t UBVk}{N^2}\Big) \\
&= \sum_{\substack{k \in (\mathbb{Z}/N\mathbb{Z})^n \\ BVk \equiv 0 \bmod N}} e\Big(\frac{(U^t v)^t BVk}{N^2}\Big) \\
&= 0 \qquad\qquad \text{by the above case.} \quad \square
\end{aligned}$$

Proposition 3.2.1. *Given α, $\delta \in \mathbb{Z}$, let $a = 2\alpha$, $d = 2\delta$. For $c > 0$, consider the matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then we have $c\gamma(z) = a - (cz+d)^{-1}$ and*

$$\theta(\gamma(z); h, A, N) = (-i)^{\nu} (\det A)^{-1/2} c^{-n/2} (-i(cz+d))^{\kappa/2} \sum_{\substack{Ak \equiv 0 \bmod N \\ k \bmod cN}} \phi(h,k) \theta(cz; k, cA, cN)$$

where

$$\phi(h,k) = \sum_{\substack{g \bmod cN \\ g \equiv h \bmod N}} e\Big(\frac{\alpha g^t Ag + k^t Ag + \delta k^t Ak}{cN^2}\Big).$$

Proof. First,

$$\begin{aligned}
a - \frac{1}{cz+d} &= \frac{acz + ad - 1}{cz+d} \\
&= \frac{acz + bc}{cz+d} \\
&= c\frac{az+b}{cz+d} \\
&= c\gamma(z).
\end{aligned}$$

Next, apply in order the transformation formulas (3.2), (3.1) and (3.4). By (3.2), we have

$$\theta(\gamma(z); h, A, N) = \sum_{\substack{g \bmod cN \\ g \equiv h \bmod N}} \theta(a - \frac{1}{cz+d}; g, cA, cN).$$

By (3.1),

$$\theta(a - \frac{1}{cz+d}; g, cA, cN) = e(\frac{\alpha g^t A g}{cN^2}) \theta(\frac{-1}{cz+d}; g, cA, cN).$$

Then by (3.4) we get,

$$\theta(\frac{-1}{cz+d}; g, cA, cN) = (-i)^\nu (\det cA)^{-1/2} (-i(cz+d)^{\kappa/2}))\cdot$$
$$\cdot \sum_{\substack{k \bmod cN \\ cAk \equiv 0 \bmod cN}} e(\frac{k^t A g}{cN^2}) \theta(cz+d; k, cA, cN).$$

Moreover,

$$\theta(cz+d; k, cA, cN) = e(\frac{\delta k^t A k}{cN^2}) \theta(cz; k, cA, cN).$$

So

$$\theta(\frac{-1}{cz+d}; g, cA, cN) = (-i)^\nu c^{-n/2} (\det A)^{-1/2} (-i(cz+d))^{\kappa/2}\cdot$$
$$\cdot \sum_{\substack{k \bmod cN \\ cAk \equiv 0 \bmod cN}} e(\frac{k^t A g}{cN^2}) e(\frac{\delta k^t A k}{cN^2}) \theta(cz; k, cA, cN).$$

Notice that there is an equality between the two sets:

$$\{k \in (\mathbb{Z}/cN\mathbb{Z})^n \quad | \quad cAk \equiv 0 \bmod cN\} \xleftrightarrow{\sim} \{k \in (\mathbb{Z}/cN\mathbb{Z})^n \quad | \quad Ak \equiv 0 \bmod N\}$$

due to the fact that

$$\begin{aligned} cAk \equiv 0 \bmod cN &\iff cAk = cNq \quad \text{for some } q \in \mathbb{Z}^n \\ &\iff Ak = Nq \quad \text{for } c > 0 \\ &\iff Ak \equiv 0 \bmod N. \end{aligned}$$

Combining the foregoing results we get,

$$\theta(\gamma(z);h,A,N) = \sum_{\substack{g\equiv h \bmod N \\ g \bmod cN}} (-i)^{\nu}(\det A)^{-1/2}c^{-n/2}(-i(cz+d))^{\kappa/2}\cdot$$
$$\cdot \sum_{\substack{k \bmod cN \\ Ak\equiv 0 \bmod N}} e(\frac{k^tAg}{cN^2})\theta(cz;k,cA,cN)e(\frac{\delta k^tAk}{cN^2})e(\frac{\alpha g^tAg}{cN^2}).$$

Finally, interchange the finite sums to obtain:

$$\theta(\gamma(z);h,A,N) = (-i)^{\nu}(\det A)^{-1/2}c^{-n/2}(-i(cz+d))^{\kappa/2}\cdot$$
$$\cdot \sum_{\substack{k \bmod cN \\ Ak\equiv 0 \bmod N}} \sum_{\substack{g\equiv h \bmod N \\ g \bmod cN}} e(\frac{\alpha g^tAg + k^tAg + \delta k^tAk}{cN^2})\theta(cz;k,cA,cN). \square$$

Lemma 3.2.3. *The expression $\phi(h,k)$ in Proposition 3.2.1 can also be written as*

$$\phi(h,k) = e(\frac{-b(\delta k^tAk + k^tAh)}{N^2})\phi(h+2\delta k,0).$$

Proof. Too see this, replace g by $\tilde{g} := g + 2\delta k$ and write

$$\begin{aligned}\phi(h+2\delta k,0) &= \sum_{\substack{\tilde{g} \bmod cN \\ \tilde{g}\equiv h+2\delta k \bmod N}} e(\frac{\alpha\tilde{g}^tA\tilde{g}}{cN^2}) \\ &= \sum_{\substack{g\equiv h \bmod N \\ g \bmod cN}} e(\frac{\alpha(g+2\delta k)^tA(g+2\delta k)}{cN^2}) \\ &= \sum_{\substack{g\equiv h \bmod N \\ g \bmod cN}} e(\frac{\alpha g^tAg}{cN^2})e(\frac{2\alpha g^tA\delta k}{cN^2})e(\frac{2\alpha\delta k^tAg}{cN^2})e(\frac{4\alpha\delta^2k^tAk}{cN^2}).\end{aligned}$$

We have $k^tAg = (k^tAg)^t = g^tAk$ because $k^tAg \in \mathbb{Z}$. In addition, $\gamma \in SL_2(\mathbb{Z})$, so $4\alpha\delta = 1 + bc$. Therefore,

$$\begin{aligned}\phi(h+2\delta k,0) &= \sum_{\substack{g\equiv h \bmod N \\ g \bmod cN}} e(\frac{\alpha g^tAg}{cN^2})e(\frac{4\alpha\delta k^tAg}{cN^2})e(\frac{4\alpha\delta^2k^tAk}{cN^2}) \\ &= \sum_{\substack{g\equiv h \bmod N \\ g \bmod cN}} e(\frac{\alpha g^tAg}{cN^2})e(\frac{(1+bc)k^tAg}{cN^2})e(\frac{\delta k^tAk}{cN^2})e(\frac{\delta bk^tAk}{N^2})\end{aligned}$$

$$= \sum_{\substack{g \equiv h \bmod N \\ g \bmod cN}} e\Big(\frac{\alpha g^t A g + k^t A g + \delta k^t A k}{cN^2}\Big) e\Big(\frac{bck^t A g}{cN^2}\Big) e\Big(\frac{\delta b k^t A k}{N^2}\Big).$$

Use the fact that $g \equiv h \bmod N$ with $g \bmod cN$ to write $g = h + Ng'$ for some $g' \in \left(\mathbb{Z}/_{cN\mathbb{Z}}\right)^n$. Also since $Ak \equiv 0 \bmod N$, we have:

$$e\Big(\frac{bk^t A g}{N^2}\Big) = e\Big(\frac{bk^t A(h + Ng')}{N^2}\Big) = e\Big(\frac{bk^t A h}{N^2}\Big) e\Big(\frac{bk^t A g'}{N}\Big) = e\Big(\frac{bk^t A h}{N^2}\Big).$$

Hence,

$$\begin{aligned}\phi(h + 2\delta k, 0) &= \sum_{\substack{g \equiv h \bmod N \\ g \bmod cN}} e\Big(\frac{\alpha g^t A g + k^t A g + \delta k^t A k}{cN^2}\Big) e\Big(\frac{bk^t A h}{N^2}\Big) e\Big(\frac{\delta b k^t A k}{N^2}\Big) \\ &= \phi(h, k) e\Big(\frac{b(k^t A h + \delta k^t A k)}{N^2}\Big)\end{aligned}$$

$$\Longrightarrow e\Big(\frac{-b(k^t A h + \delta k^t A k)}{N^2}\Big) \phi(h + 2\delta k, 0) = \phi(h, k). \tag{3.5}$$

$\square$

Proposition 3.2.2. *The expression $\phi(h, k)$ depends only on k modulo N and thus*

$$\theta(\gamma(z); h, A, N) i^\nu (\det A)^{1/2} c^{n/2} (-i(cz + d))^{-\kappa/2} = \sum_{\substack{k \bmod N \\ Ak \equiv 0 \bmod N}} \phi(h, k) \theta(z; k, A, N).$$

Proof. Replace k by $k' + Nl$ for some $l \in \mathbb{Z}^n$ in equation (3.5).

- First, using the fact that A has coefficients in $\mathbb{Z}$, $Ah \in N\mathbb{Z}^n$ and $Ak' \in N\mathbb{Z}^n$, we get

$$e\Big(\frac{-b((k' + Nl)^t A h + \delta (k' + Nl)^t A (k' + Nl))}{N^2}\Big) = e\Big(\frac{-b(k'^t A h + \delta k'^t A k')}{N^2}\Big).$$

- Second, we also have

$$\phi(h+2\delta(k'+Nl),0) = \sum_{\substack{g \bmod cN \\ g\equiv h+2\delta(k'+Nl) \bmod N}} e(\frac{\alpha g^t A g}{cN^2}) = \sum_{\substack{g \bmod cN \\ g\equiv h+2\delta k' \bmod N}} e(\frac{\alpha g^t A g}{cN^2})$$
$$= \phi(h+2\delta k',0).$$

Thus, $\phi(h,k'+Nl) = \phi(h,k')$. So, using the latter in Proposition 3.2.1, we can take $\phi(h,k)$ as a common factor for each choice of $k \bmod N$.

$$\theta(\gamma(z);h,A,N)i^{\nu}(\det A)^{1/2}c^{n/2}(-i(cz+d))^{-\kappa/2} = \sum_{\substack{Ak'\equiv 0 \bmod N \\ k' \bmod N}} \phi(h,k')\cdot$$
$$\cdot \sum_{\substack{k\equiv k' \bmod N \\ k \bmod cN}} \theta(cz;k,cA,cN).$$

Finally, by equation (3.2) we have

$$\sum_{\substack{k\equiv k' \bmod N \\ k \bmod cN}} \theta(cz;k,cA,cN) = \theta(z;k',A,N). \qquad \square$$

Proposition 3.2.3. *Suppose that $a = 2\alpha$, $d = 2\delta$ and $c > 0$. Then we have:*

$$\theta(\frac{bz-a}{dz-c};h,A,N)(\det A)c^{n/2}(-\boldsymbol{sgn}(d)i)^{-n}(dz-c)^{-\kappa/2} =$$
$$\sum_{\substack{l \bmod N \\ Al\equiv 0 \bmod N}} \{ \sum_{\substack{k \bmod N \\ Ak\equiv 0 \bmod N}} e(\frac{l^t A k}{N^2})\phi(h,k)\}\theta(z;l,A,N).$$

Proof. Substitute $\frac{-1}{z}$ for z in Proposition 3.2.2, then

$$\theta(\frac{a(\frac{-1}{z})+b}{c(\frac{-1}{z})+d};h,A,N)i^{\nu}(\det A)^{1/2}c^{n/2}(-i(c(\frac{-1}{z})+d))^{-\kappa/2} = \sum_{\substack{k \bmod N \\ Ak\equiv 0 \bmod N}} \phi(h,k)\theta(\frac{-1}{z};k,A,N).$$

By (3.4) we write

$$\theta(\frac{-1}{z};k,A,N)=(-i)^{\nu}(\det A)^{-1/2}(-iz)^{\kappa/2}\sum_{\substack{l \bmod N\\ Al\equiv 0 \bmod N}} e(\frac{l^tAk}{N^2})\theta(z;l,A,N)$$

$$\implies \theta(\frac{bz-a}{dz-c};h,A,N)i^{\nu}(\det A)^{1/2}c^{n/2}(-i(c(\frac{-1}{z})+d))^{-\kappa/2}=$$

$$\sum_{\substack{k \bmod N\\ Ak\equiv 0 \bmod N}} \phi(h,k)(-i)^{\nu}(\det A)^{-1/2}(-iz)^{\kappa/2}\sum_{\substack{l \bmod N\\ Al\equiv 0 \bmod N}} e(\frac{l^tAk}{N^2})\theta(z;l,A,N)$$

$$\implies \theta(\frac{bz-a}{dz-c};h,A,N)(\det A)c^{n/2}i^{\nu}(-i)^{-\nu}(-iz)^{-\kappa/2}(-i(c(\frac{-1}{z})+d))^{-\kappa/2}=$$

$$\sum_{\substack{l \bmod N\\ Al\equiv 0 \bmod N}}\{\sum_{\substack{k \bmod N\\ Ak\equiv 0 \bmod N}} e(\frac{l^tAk}{N^2})\phi(h,k)\}\theta(z;l,A,N).$$

We still need to show that

$$i^{\nu}(-i)^{-\nu}(-iz)^{-\kappa/2}(-i(c(\frac{-1}{z})+d))^{-\kappa/2}=(-\boldsymbol{sgn}(d)i)^{-n}(dz-c)^{-\kappa/2}.$$

Using the appropriate branch of the square root introduced in Chapter 2, we have

- If $d>0: (-dz+c)^{-\kappa/2}=i^{-\kappa}(dz-c)^{-\kappa/2}$,
- If $d<0: (-dz+c)^{-\kappa/2}=(-i)^{-\kappa}(dz-c)^{-\kappa/2}$.

Then,

$$\begin{aligned}
i^{\nu}(-i)^{-\nu}(-iz)^{-\kappa/2}(-i(c(\frac{-1}{z})+d))^{-\kappa/2} &= (-1)^{\nu}(-i)^{-\kappa/2}i^{-\kappa/2}(-dz+c)^{-\kappa/2}\\
&= (-1)^{\nu}(-i)^{-\kappa/2}i^{-\kappa/2}(\boldsymbol{sgn}(d)i)^{-\kappa}(dz-c)^{-\kappa/2}\\
&= (-1)^{\nu}(-1)^{-\kappa/2}i^{-\kappa/2}i^{-\kappa/2}(\boldsymbol{sgn}(d)i)^{-n-2\nu}(dz-c)^{-\kappa/2}\\
&= (-1)^{\nu}(-1)^{-\kappa/2}(-1)^{-\kappa/2}(-1)^{-\nu}(\boldsymbol{sgn}(d)i)^{-n}(dz-c)^{-\kappa/2}\\
&= (-1)^{2\nu-k}(\boldsymbol{sgn}(d)i)^{-n}(dz-c)^{-\kappa/2}\\
&= (-1)^{-n}(\boldsymbol{sgn}(d)i)^{-n}(dz-c)^{-\kappa/2}. \qquad \square
\end{aligned}$$

Lemma 3.2.4. *Suppose now that $d = 2\delta \equiv 0 \bmod 2N$. Then*

$$\phi(h,k) = e(\frac{-bk^tAh}{N^2})\phi(h,0).$$

Proof. Write $\delta = Nq$ with $q \in \mathbb{Z}$ and we have

$$\begin{aligned}\phi(h,k) &= e(\frac{-b(\delta k^tAk + k^tAh)}{N^2})\phi(h+2\delta k, 0)\\ &= e(\frac{-bqNk^tAk}{N^2})e(\frac{-bk^tAh}{N^2}) \sum_{\substack{g \bmod cN\\ g\equiv h+2\delta k \bmod N}} e(\frac{\alpha g^tAg}{cN^2}).\end{aligned}$$

Notice that $e(\frac{-bqNk^tAk}{N^2}) = e(\frac{-bqk^tAk}{N}) = 1$ since $\frac{Ak}{N} \in \mathbb{Z}^n$.

Also, $g \equiv h + 2\delta k \bmod N \iff g \equiv h \bmod N$; thus an identity of sets arises. Hence,

$$\begin{aligned}\phi(h,k) &= e(-\frac{bk^tAh}{N^2}) \sum_{\substack{g \bmod cN\\ g\equiv h \bmod N}} e(\frac{\alpha g^tAg}{cN^2})\\ &= e(\frac{-bk^tAh}{N^2})\phi(h,0).\end{aligned}$$

□

As a consequence of the latter, the right hand side of the equation in Proposition 3.2.3. becomes:

$$\phi(h,0) \sum_{\substack{l \bmod N\\ Al\equiv 0 \bmod N}} \{ \sum_{\substack{k \bmod N\\ Ak\equiv 0 \bmod N}} e(\frac{(l-bh)^tAk}{N^2})\}\theta(z;l,A,N). \tag{3.6}$$

Proposition 3.2.4. *Suppose now that $b = -2\alpha$ with $\alpha \in \mathbb{Z}$, $c \equiv 0 \bmod 2N$ and $d < 0$. Let u be a vector in $\left(\mathbb{Z}/_{d\mathbb{Z}}\right)^n$. We have*

$$\theta(\frac{az+b}{cz+d};h,A,N) = (-\boldsymbol{sgn}(c)i)^n(cz+d)^{\kappa/2}e(\frac{abh^tAh}{2N^2})w(\alpha,|d|)\theta(z;ah,A,N) \tag{3.7}$$

with

$$w(\alpha,|d|) = |d|^{-n/2} \sum_{u \bmod |d|} e(\frac{\alpha u^tAu}{|d|N^2}).$$

Proof. Write $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ instead of $\begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$ in Proposition 3.2.3 and use (3.6) to get:

$$\theta(\frac{az+b}{cz+d}; h, A, N) = \frac{1}{\det A}(-d)^{-n/2}(-\boldsymbol{sgn}(c)i)^n(cz+d)^{\kappa/2}\phi(h,0)\cdot$$
$$\cdot \sum_{\substack{l \bmod N \\ Al \equiv 0 \bmod N}} \{ \sum_{\substack{k \bmod N \\ Ak \equiv 0 \bmod N}} e(\frac{(l-ah)^t Ak}{N^2})\}\theta(z; l, A, N).$$

Now, using Lemma 3.2.2 for $v = l - ah \equiv 0 \bmod N$, we obtain

$$\theta(\frac{az+b}{cz+d}; h, A, N) = (-d)^{-n/2}(-\boldsymbol{sgn}(c)i)^n(cz+d)^{\kappa/2}\phi(h,0) \sum_{\substack{l \bmod N \\ Al \equiv 0 \bmod N \\ l \equiv ah \bmod N}} \theta(z; l, A, N).$$

Note that the set $\{l \bmod N \ | \ Al \equiv 0 \bmod N \text{ and } l \equiv ah \bmod N\}$ is the same as the set $\{l \bmod N \ | \ l \equiv ah \bmod N\}$, because $l \equiv ah \bmod N \implies Al \equiv aAh \bmod N$ but $Ah \equiv 0 \bmod N$ as part of our assumption. Hence, the resulting sum is over just one element $l = ah$. Consequently, we get

$$\begin{aligned}\theta(\frac{az+b}{cz+d}; h, A, N) &= (-d)^{-n/2}(-\boldsymbol{sgn}(c)i)^n(cz+d)^{\kappa/2}\phi(h,0)\theta(z; ah, A, N) \\ &= (-d)^{-n/2}(-\boldsymbol{sgn}(c)i)^n(cz+d)^{\kappa/2} \sum_{\substack{g \bmod dN \\ g \equiv h \bmod N}} e(\frac{\alpha g^t A g}{-dN^2})\theta(z; ah, A, N) \\ &= (-\boldsymbol{sgn}(c)i)^n(cz+d)^{\kappa/2}\theta(z; ah, A, N)|d|^{-n/2} \sum_{\substack{g \bmod dN \\ g \equiv h \bmod N}} e(\frac{\alpha g^t A g}{|d|N^2}).\end{aligned}$$

Set $W = |d|^{-n/2} \sum_{\substack{g \bmod dN \\ g \equiv h \bmod N}} e(\frac{\alpha g^t A g}{|d|N^2})$. Notice that:

- $g \equiv h \bmod N$ and $g \bmod dN \implies g = h + Nu'$ for some $u' \in \left(\mathbb{Z}/d\mathbb{Z}\right)^n$,
- $ad \equiv 1 \bmod N \implies ad = 1 + Nq'$ for some $q' \in \mathbb{Z}^n$.

Therefore, we have $g \equiv h \bmod N \equiv adh \bmod N$. So we can write $g = adh + Nu$ for some $u \in \left(\mathbb{Z}/_{d\mathbb{Z}}\right)^n$. Then

$$\begin{aligned}
W &= |d|^{-n/2} \sum_{\substack{g \bmod dN \\ g \equiv h \bmod N}} e\Big(\frac{\alpha g^t A g}{|d|N^2}\Big) \\
&= |d|^{-n/2} \sum_{u \bmod d} e\Big(\frac{\alpha(adh+Nu)^t A(adh+Nu)}{|d|N^2}\Big) \\
&= |d|^{-n/2} \sum_{u \bmod d} e\Big(\frac{\alpha(ad)^2 h^t Ah}{|d|N^2}\Big) e\Big(\frac{\alpha adNh^t Au}{|d|N^2}\Big) e\Big(\frac{\alpha Nu^t adAh}{|d|N^2}\Big) e\Big(\frac{\alpha N^2 u^t Au}{|d|N^2}\Big).
\end{aligned}$$

Now, we have the following:

$$\begin{aligned}
e\Big(\frac{\alpha(ad)^2 h^t Ah}{|d|N^2}\Big) &= e\Big(\frac{\alpha a^2 dh^t Ah}{-N^2}\Big) && \text{since } d < 0 \\
&= e\Big(\frac{\alpha a(1+Nq')h^t Ah}{-N^2}\Big) && \text{since } ad = 1 + Nq' \\
&= e\Big(\frac{\alpha ah^t Ah}{-N^2}\Big) e\Big(\frac{\alpha aq' h^t Ah}{-N}\Big) \\
&= e\Big(\frac{abh^t Ah}{2N^2}\Big) e\Big(\frac{abq' h^t Ah}{N}\Big) && \text{since } b = -2\alpha \\
&= e\Big(\frac{abh^t Ah}{2N^2}\Big) && \text{since } Ah \in N\mathbb{Z}^n.
\end{aligned}$$

Moreover,

$$e\Big(\frac{\alpha adNh^t Au}{|d|N^2}\Big) = e\Big(\frac{\alpha ah^t Au}{-N}\Big) = 1 \qquad \text{since } Ah \in N\mathbb{Z}^n.$$

Similarly,

$$e\Big(\frac{\alpha Nu^t adAh}{|d|N^2}\Big) = 1.$$

Hence, we obtain

$$\begin{aligned}
W &= |d|^{-n/2} \sum_{u \bmod d} e\Big(\frac{abh^t Ah}{2N^2}\Big) e\Big(\frac{\alpha u^t Au}{|d|}\Big) \\
&= e\Big(\frac{abh^t Ah}{2N^2}\Big) w(\alpha, |d|).
\end{aligned}$$

□

Suppose that $c = 0$, then since $ad - bc = 1$ we get that $d = -1$.

Then $w(\alpha, |d|) = e(0) = 1$.

Therefore we assume in what follows that $c \neq 0$.

Lemma 3.2.5. *For $m \in \mathbb{Z}$ and $mc < 0$, we have*

$$w(\alpha, |d|) = w(\alpha - am, |d + 2cm|).$$

Proof. First, notice that we have the following conditions:

- $2am + b = 2am - 2\alpha = -2(\alpha - am) = -2\alpha'$ with $\alpha' \in \mathbb{Z}$,
- $c \equiv 0 \bmod 2N$,
- $2cm + d < 0$.

These conditions allow us to substitute $z + 2m$ for z in equation (3.7) to get,

$$\theta(\frac{a(z+2m)+b}{c(z+2m)+d}; h, A, N) = (-\boldsymbol{sgn}(c)i)^n (c(z+2m)+d)^{\kappa/2} e(\frac{abh^tAh}{2N^2})\cdot$$
$$\cdot\, w(\alpha, |d|)\theta(z+2m; ah, A, N).$$

On the other hand, we have

$$\theta(\frac{az+(2am+b)}{cz+(2cm+d)}; h, A, N) = (-\boldsymbol{sgn}(c)i)^n (cz+(2cm+d))^{\kappa/2} e(\frac{a(2am+b)h^tAh}{2N^2})\cdot$$
$$\cdot\, w(\alpha - am, |2cm+d|)\theta(z; ah, A, N).$$

Comparing the two equations, we get

$$\theta(z+2m; ah, A, N)w(\alpha, |d|) = e(\frac{a^2mh^tAh}{N^2})\theta(z; ah, A, N)w(\alpha - am, |2cm+d|).$$

However; from equation (3.1) we have

$$\begin{aligned}\theta(z+2m;ah,A,N) &= e(\frac{m(ah)^tA(ah)}{N^2})\theta(z;ah,A,N)\\ &= e(\frac{a^2mh^tAh}{N^2})\theta(z;ah,A,N).\end{aligned}$$

Thus $w(\alpha,|d|) = w(\alpha - am, |d+2cm|)$. □

By Dirichlet Prime Number Theorem, we can take m so that $-d-2cm$ is a positive prime p. Moreover, set $\beta = \alpha - am$. Then,

$$w(\alpha,|d|) = w(\alpha - am, |d+2cm|) = w(\beta, p) = p^{-n/2}\sum_{u \bmod p} e(\frac{\beta u^tAu}{p}). \tag{3.8}$$

Lemma 3.2.6. *Suppose that p is prime to $\det A$. Then there exists an element S of $M_n(\mathbb{Z})$, whose determinant is prime to p, such that S^tAS is congruent modulo p to a diagonal matrix D.*

Proof. Consider the bilinear form

$$\begin{aligned}B : \left(\mathbb{Z}/_{p\mathbb{Z}}\right)^n \times \left(\mathbb{Z}/_{p\mathbb{Z}}\right)^n &\to \mathbb{Z}/_{p\mathbb{Z}}\\ (v_1 \quad , \quad v_2) &\mapsto B(v_1,v_2) = v_1^tAv_2^t.\end{aligned}$$

Let $\{e_1,\cdots,e_n\}$ be the standard basis where $B(e_i,e_j) = A_{ij}$. Furthermore, we have that $B(v_1,v_2) = B(v_2,v_1)$. Indeed, $B(e_i,e_j) = A_{ij} = A_{ji} = B(e_j,e_i)$ since $A = A^t$. This implies that B is a symmetric bilinear form on the vector space $\left(\mathbb{Z}/_{p\mathbb{Z}}\right)^n$. Then, by [Jacobson, 2012] section 6.3, Theorem 6.5, there exists an orthogonal basis $\{b_1,\cdots,b_n\}$ for which the matrix of B relative to this basis is some diagonal matrix D, or equivalently

$$B(v_1,v_2) = \begin{cases}\text{non-zero} & \text{if } i=j,\\ 0 & \text{otherwise.}\end{cases}$$

So now, there exists a change of basis matrix $S' \in GL_n\left(\mathbb{Z}/_{p\mathbb{Z}}\right)$ such that

$$\begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix} = \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix} S'.$$

Finally, lifting S' to some $S \in M_n(\mathbb{Z})$ whose reduction modulo p is S', we get a new matrix representation for B given by $S^t A S \equiv D \bmod p$. $\square$

Proposition 3.2.5. *If $q_1, \cdots, q_n$ are the diagonal elements of D as in Lemma 3.2.6, then*

$$w(\beta, p) = p^{-n/2} \prod_{i=1}^{n} \left(\sum_{x=1}^{p} e\left(\frac{\beta q_i x^2}{p}\right)\right).$$

Proof. Since $\begin{pmatrix} a & -2\beta \\ c & -p \end{pmatrix} \in SL_2(\mathbb{Z})$, we have $2\beta c - ap = 1$. But $c \equiv 0 \bmod 2N$, then p is prime to $2\beta N$. Also, we know that $\det A$ is a divisor of N^n, then p is prime to $\det A$. To see this, suppose that $\gcd(p, \det A) \neq 1$. We have:

$$\begin{aligned} \gcd(p, \det A) \text{ divides } \det A \text{ and } p &\Longrightarrow \gcd(p, \det A) \text{ divides } N^n \text{ and } p \\ &\Longrightarrow \gcd(p, \det A) \text{ divides } \gcd(p, N^n) \\ &\Longrightarrow \gcd(p, N^n) = p \\ &\Longrightarrow \gcd(p, N) = p \\ &\Longrightarrow \gcd(p, 2\beta N) = p \neq 1. \quad \text{Contradiction.} \end{aligned}$$

Thus Lemma 3.2.6 applies. Next, there exists a bijection

$$\begin{aligned} \left(\mathbb{Z}/_{p\mathbb{Z}}\right)^n &\xleftrightarrow{\sim} \left(\mathbb{Z}/_{p\mathbb{Z}}\right)^n \\ u \bmod p &\longmapsto S^{-1} u \bmod p \\ S v \bmod p &\longleftarrow\!\shortmid\; v \bmod p \end{aligned}$$

with $\bar{S}$ invertible matrix in $M_n\left(\mathbb{Z}/_{p\mathbb{Z}}\right)$.

So replacing $u \bmod p$ by $Su \bmod p$ in (3.8) we get:

$$\begin{aligned}
w(\beta,p) &= p^{-n/2} \sum_{Su \bmod p} e(\frac{\beta(Su)^t A(Su)}{p}) \\
&= p^{-n/2} \sum_{u \bmod p} e(\frac{\beta(\bar{S}u)^t A(\bar{S}u)}{p}) \\
&= p^{-n/2} \sum_{u \bmod p} e(\frac{\beta u^t \bar{S}^t A \bar{S} u}{p}) \\
&= p^{-n/2} \sum_{u \bmod p} e(\frac{\beta u^t D u}{p}) && \text{by Lemma 3.2.6} \\
&= p^{-n/2} \sum_{u \bmod p} e(\frac{\beta(u_1^2 q_1 + \cdots + u_n^2 q_n)}{p}) \\
&= p^{-n/2} \{ \sum_{u_1 \bmod p} e(\frac{\beta u_1^2 q_1}{p}) \} \cdots \{ \sum_{u_n \bmod p} e(\frac{\beta u_n^2 q_n}{p}) \} \\
&= p^{-n/2} \prod_{i=1}^{n} (\sum_{x=1}^{p} e(\frac{\beta q_i x^2}{p})). && \square
\end{aligned}$$

Proposition 3.2.6. *Let ε_m be 1 or i according as $m \equiv 1$ or $3 \bmod 4$. Then we have*

$$w(\beta,p) = \varepsilon_p^n \left(\frac{\beta^n \det A}{p}\right).$$

Proof. From Proposition 3.2.5 and the definition of the Gauss Sum, we have

$$\begin{aligned}
w(\beta,p) &= p^{-n/2} \prod_{i=1}^{n} (\sum_{x=1}^{p} e(\frac{\beta q_i x^2}{p})) \\
&= p^{-n/2} \prod_{i=1}^{n} G(\beta q_i).
\end{aligned}$$

By Proposition 2.5.3, $G(\beta q_i) = \left(\frac{\beta q_i}{p}\right) G(1)$, then

$$\begin{aligned}
w(\beta,p) &= p^{-n/2} \prod_{i=1}^{n} [\left(\frac{\beta q_i}{p}\right) G(1)] \\
&= p^{-n/2} [\prod_{i=1}^{n} \left(\frac{\beta q_i}{p}\right)] [\varepsilon_p \sqrt{p}]^n && \text{by Proposition 2.5.4.} \\
&= p^{-n/2} \left(\frac{\beta^n q_1}{p}\right) \cdots \left(\frac{\beta^n q_n}{p}\right) \\
&= p^{-n/2} \left(\frac{\beta^n q_1 \cdots q_n}{p}\right) \varepsilon_p^n p^{n/2} && \text{by Euler Criteria}
\end{aligned}$$

$$= \left(\frac{\beta^n q_1 \cdots q_n}{p}\right) \varepsilon_p^n.$$

Now, by Lemma 3.2.6, we have that

$$S^t A S \equiv \begin{pmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_n \end{pmatrix} \bmod p,$$

$$\implies \det S^t A S \equiv q_1 \cdots q_n \bmod p.$$

Hence,

$$\left(\frac{q_1 \cdots q_n}{p}\right) = \left(\frac{\det S^t A S}{p}\right).$$

Therefore,

$$\begin{aligned} w(\beta, p) &= \varepsilon_p^n \left(\frac{\beta^n (\det S)^2 \det A}{p}\right) \\ &= \varepsilon_p^n \left(\frac{(\det S)^2)}{p}\right)\left(\frac{\beta^n \det A}{p}\right) \\ &= \varepsilon_p^n \left(\frac{\beta^n \det A}{p}\right) \qquad \text{since } (\det S)^2 \text{ is a non-zero square mod } p. \square \end{aligned}$$

Proposition 3.2.7. *We have the following equality:*

$$w(\alpha, |d|) = \varepsilon_d^{-n} (\boldsymbol{sgn}(c) i)^n \left(\frac{2c}{d}\right)^n \left(\frac{\det A}{d}\right). \tag{3.9}$$

Proof.

- First, notice that

$$\varepsilon_p^n = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4, \\ i^n & \text{if } p \equiv 3 \bmod 4 \end{cases}$$

$$= \begin{cases} 1 & \text{if } -d-2cm \equiv 1 \bmod 4, \\ i^n & \text{if } -d-2cm \equiv 3 \bmod 4 \end{cases} \qquad \text{since } p = -d-2cm$$

$$= \begin{cases} 1 & \text{if } -d \equiv 1 \bmod 4, \\ i^n & \text{if } -d \equiv 3 \bmod 4 \end{cases} \qquad \text{since } c \equiv 0 \bmod 2$$

$$= \begin{cases} 1 & \text{if } d \equiv 3 \bmod 4, \\ i^n & \text{if } d \equiv 1 \bmod 4 \end{cases}$$

$$= i^n \cdot \begin{cases} i^{-n} & \text{if } d \equiv 3 \bmod 4, \\ 1 & \text{if } d \equiv 1 \bmod 4 \end{cases}$$

$$= i^n \varepsilon_d^{-n}.$$

- Next, we have that $2\beta c - ap = 1 \implies 2\beta c \equiv 1 \bmod p \implies \beta \equiv (2c)^{-1} \bmod p$ $\implies p$ does not divide $2c$. So,

$$\left(\frac{\beta}{p}\right) = \left(\frac{(2c)^{-1}}{p}\right) = \left(\frac{2c}{p}\right)^{-1} = \left(\frac{2c}{p}\right) = \left(\frac{4c'}{p}\right) = \left(\frac{c'}{p}\right) \text{ for some } c' \in \mathbb{Z}.$$

On the other hand,

$$\left(\frac{2c}{-d}\right) = \left(\frac{4c'}{-d}\right) = \left(\frac{c'}{-d}\right).$$

However,

$$p \equiv -d \bmod 2c \equiv -d \bmod 4c' \implies \left(\frac{c'}{p}\right) = \left(\frac{c'}{-d}\right).$$

Therefore,

$$\left(\frac{\beta}{p}\right) = \left(\frac{2c}{-d}\right) = \left(\frac{2c}{-1}\right)\left(\frac{2c}{d}\right) = \boldsymbol{sgn}(c)\left(\frac{2c}{d}\right).$$

- Next, consider the prime factorization of $\det A$. Then write $\det A = D_0 K^2$ where $D_0 = \prod p_i^{\alpha_i}$ is the product of primes that appear in $\det A$ with an odd power. Furthermore, $\det A$ divides $N^n \implies p_i^{\alpha_i}$ divides $N^n \implies p_i^{\alpha_i}$ divides $N \implies D_o$ is a positive divisor of N. Also, $\gcd(\det A, p) = 1$ $\implies \gcd(K, p) = 1$. So, we get

$$\left(\frac{\det A}{p}\right) = \left(\frac{D_0 K^2}{p}\right) = \left(\frac{D_0}{p}\right)\left(\frac{K^2}{p}\right) = \left(\frac{D_0}{p}\right).$$

Also, for $-d = d_1 d_2 \cdots d_m$, we have

$$\left(\frac{D_0}{-d}\right) = \left(\frac{D_0}{d_1}\right)\cdots\left(\frac{D_0}{d_m}\right)$$

where $\left(\frac{D_0}{d_i}\right)$ depends on $d_i \bmod 4D_0$ by Proposition 2.5.2. Then $\prod_{i=1}^{m}\left(\frac{D_0}{d_i}\right)$ depends on $-d \bmod 4D_0$. Furthermore, we have

$$\begin{aligned} p = -d - 2cm &\implies p = -d - 4c'N \\ &\implies p \equiv -d \bmod 4N \\ &\implies p \equiv -d \bmod 4D_0 \qquad\qquad \text{since } D_0 \text{ divides } N \\ &\implies \left(\frac{D_0}{p}\right) = \left(\frac{D_0}{-d}\right). \end{aligned}$$

Therefore,

$$\left(\frac{D_0}{p}\right) = \left(\frac{D_0}{-d}\right) = \left(\frac{D_0}{-1}\right)\left(\frac{D_0}{d}\right) = \left(\frac{D_0}{d}\right) = \left(\frac{\det A}{d}\right).$$

Hence, replacing in Proposition 3.2.6 we get

$$\begin{aligned} w(\alpha, |d|) = w(\beta, p) = \varepsilon_p^n \left(\frac{\beta^n \det A}{p}\right) &= \varepsilon_p^n \left(\frac{\beta}{p}\right)^n \left(\frac{\det A}{p}\right) \\ &= \varepsilon_d^{-n} (\boldsymbol{sgn}(c) i)^n \left(\frac{2c}{d}\right)^n \left(\frac{\det A}{d}\right). \end{aligned}$$ □

Proposition 3.2.8. *(Key Result) Let γ be an element of the congruence subgroup*

$$\Gamma_0(2, 2N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); b \equiv 0 \bmod 2 \text{ and } c \equiv 0 \bmod 2N \right\}.$$

Then $\theta(z;h,A,N,P)$ is a modular form on $\Gamma_0(2,2N)$ with the transformation law

$$\theta(\gamma(z);h,A,N,P) = e(\frac{abh^tAh}{2N^2})\left(\frac{\det A}{d}\right)\left(\frac{2c}{d}\right)^n \varepsilon_d^{-n}(cz+d)^{\kappa/2}\theta(z;ah,A,N,P)$$

where $\kappa = n+2\nu$ and ν order of P.

Proof. We study two cases:

1. <u>For $d<0$</u>

   Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2,2N)$. Replace equation (3.9) in equation (3.7). Then we get:

$$\begin{aligned}\theta(\frac{az+b}{cz+d};h,A,N,P) &= (-\boldsymbol{sgn}(c)i)^n(cz+d)^{\kappa/2}e(\frac{abh^tAh}{2N^2})w(\alpha,|d|)\theta(z;ah,A,N,P)\\ &= (-\boldsymbol{sgn}(c)i)^n(cz+d)^{\kappa/2}e(\frac{abh^tAh}{2N^2})\varepsilon_d^{-n}(\boldsymbol{sgn}(c)i)^n\left(\frac{2c}{d}\right)^n\cdot\\ &\cdot\left(\frac{\det A}{d}\right)\theta(z;ah,A,N,P)\\ &= e(\frac{abh^tAh}{2N^2})\left(\frac{\det A}{d}\right)\left(\frac{2c}{d}\right)^n\varepsilon_d^{-n}(cz+d)^{\kappa/2}\theta(z;ah,A,N,P).\end{aligned}$$

2. <u>For $d>0$</u>

   Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2,2N)$. Then, by using case 1 we have:

$$\begin{aligned}\theta(\gamma(z);h,A,N,P) &= \theta(\frac{az+b}{cz+d};h,A,N,P) = \theta(\frac{-az-b}{-cz-d};h,A,N,P)\\ &= e(\frac{abh^tAh}{2N^2})\left(\frac{\det A}{-d}\right)\left(\frac{-2c}{-d}\right)^n\varepsilon_{-d}^{-n}(-cz-d)^{\kappa/2}\theta(z;-ah,A,N,P).\end{aligned}$$

   Note that

$$\left(\frac{\det A}{-d}\right) = \left(\frac{\det A}{-1}\right)\left(\frac{\det A}{d}\right) = \boldsymbol{sgn}(\det A)\left(\frac{\det A}{d}\right) = \left(\frac{\det A}{d}\right),$$

$$\left(\frac{-2c}{-d}\right)^n = \left(\frac{-1}{-d}\right)^n \left(\frac{2c}{-d}\right)^n = \left(\frac{-1}{-1}\right)^n \left(\frac{-1}{d}\right)^n \left(\frac{2c}{-1}\right)^n \left(\frac{2c}{d}\right)^n$$
$$= (-1)^n \left(\frac{-1}{d}\right)^n (\boldsymbol{sgn}(c))^n \left(\frac{2c}{d}\right)^n,$$

$$\theta(z; -ah, A, N, P) = \sum_{m \equiv -ah \bmod N} P(m) e\left(\frac{zm^t Am}{2N^2}\right) = \sum_{m \equiv ah \bmod N} P(-m) e\left(\frac{zm^t Am}{2N^2}\right)$$
$$= (-1)^\nu \theta(z; ah, A, N, P).$$

Then, we get

$$\theta(\gamma(z); h, A, N, P) = e\left(\frac{abh^t Ah}{2N^2}\right) \left(\frac{\det A}{d}\right) (-1)^n \left(\frac{-1}{d}\right)^n (\boldsymbol{sgn}(c))^n \cdot$$
$$\cdot \left(\frac{2c}{d}\right)^n \varepsilon_{-d}^{-n} (-cz - d)^{\kappa/2} (-1)^\nu \theta(z; ah, A, N, P). \tag{3.10}$$

Next, use the defined branch of $(cz + d)^{1/2}$ and distinguish among 4 separate cases:

- If $c > 0$ and $d \equiv 1 \bmod 4$:
  $\left(\frac{-1}{d}\right)^n (\boldsymbol{sgn}(c))^n \varepsilon_{-d}^{-n} (-cz - d)^{\kappa/2}$
  $= 1^n 1^n (i\varepsilon_d)^{-n} (-i)^\kappa (cz + d)^{\kappa/2} = (-1)^n (-1)^\nu \varepsilon_d^{-n} (cz + d)^{\kappa/2},$
- If $c > 0$ and $d \equiv 3 \bmod 4$:
  $\left(\frac{-1}{d}\right)^n (\boldsymbol{sgn}(c))^n \varepsilon_{-d}^{-n} (-cz - d)^{\kappa/2}$
  $= (-1)^n 1^n (-i\varepsilon_d)^{-n} (-i)^\kappa (cz + d)^{\kappa/2} = (-1)^n (-1)^\nu \varepsilon_d^{-n} (cz + d)^{\kappa/2},$
- If $c < 0$ and $d \equiv 1 \bmod 4$:
  $\left(\frac{-1}{d}\right)^n (\boldsymbol{sgn}(c))^n \varepsilon_{-d}^{-n} (-cz - d)^{\kappa/2}$
  $= 1^n (-1)^n (i\varepsilon_d)^{-n} i^\kappa (cz + d)^{\kappa/2} = (-1)^n (-1)^\nu \varepsilon_d^{-n} (cz + d)^{\kappa/2},$
- If $c < 0$ and $d \equiv 3 \bmod 4$:
  $\left(\frac{-1}{d}\right)^n (\boldsymbol{sgn}(c))^n \varepsilon_{-d}^{-n} (-cz - d)^{\kappa/2}$
  $= (-1)^n (-1)^n (-i\varepsilon_d)^{-n} i^\kappa (cz + d)^{\kappa/2} = (-1)^n (-1)^\nu \varepsilon_d^{-n} (cz + d)^{\kappa/2}.$

Thus in each case, replacing in equation (3.10) gives the desired result. □

Chapter 4

Applications of Theta Series to Some Quadratic Forms

4.1 Eisenstein Series of weight $k \geq 3$ on $\Gamma(M)$

At this point, in addition to studying relevant actions and orbits, we shall construct the Fourier expansion of $E_{k,\mu,\nu}$ on $\Gamma(M)$ to get an explicit formula for the Eisenstein Series on $\Gamma_0(4)$.

Let M be a positive integer. Recall the **Principal Congruence Subgroup** of $SL_2(\mathbb{Z})$ of level M

$$\Gamma(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod M \right\}.$$

Definition 4.1.1. (Eisenstein Series on Congruence Subgroups) For any positive integer k ≥ 3, we define the Eisenstein Series on $\Gamma(M)$

$$E_{k,\mu,\nu}(z) = \sum_{(m,n)\equiv(\mu,\nu) \bmod M} (mz+n)^{-k}$$

where the sum is taken over all pairs of integers (m,n) except for $(\mu,\nu) = (0,0)$.

Proposition 4.1.1. *The above sum $E_{k,\mu,\nu}$ converges absolutely and uniformly on compact subsets of $\mathbb{H}$ for $k > 2$.*

Proof. See [Diamond and Shurman, 2005] chapter 4, corollary 4.2.2 or [Schoeneberg, 2012] chapter 7, section 1. □

Proposition 4.1.2. *For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(M)$, we have*

$$E_{k,\mu,\nu}(\gamma z) = (cz+d)^k E_{k,\mu,\nu}(z)$$

Proof. Set $m' = am + cn$ and $n' = bm + dn$. Then

$$\begin{aligned}(cz+d)^{-1}(m'z+n') &= (cz+d)^{-1}[(am+cn)z + bm + dn] \\ &= (cz+d)^{-1}[m(az+b) + n(cz+d)] \\ &= m\frac{az+b}{cz+d} + n \\ &= m\gamma z + n. \end{aligned} \tag{4.1}$$

Using that fact that $a \equiv d \equiv 1 \bmod M$ and $b \equiv c \equiv 0 \bmod M$, there exists a bijection between

$$\{(m,n) \in \mathbb{Z}^2 \quad | \quad {}^{m\equiv\mu \bmod M}_{n\equiv\nu \bmod M}\} \xleftrightarrow{\sim} \{(am+cn, bm+dn) \in \mathbb{Z}^2 \quad | \quad {}^{am+cn\equiv\mu \bmod M}_{bm+dn\equiv\nu \bmod M}\}.$$

Proof. Given $(m,n) \equiv (\mu,\nu) \bmod M$, we get

$$m' = am + cn \equiv a\mu + c\nu \bmod M \equiv \mu \bmod M$$

$$n' = bm + dn \equiv b\mu + d\nu \bmod M \equiv \nu \bmod M.$$

Conversely, given $(m',n') \equiv (\mu,\nu) \bmod M$, we need to find m and n in terms of m' and n'.

$$\begin{aligned} m' = am + cn \text{ and } n' = bm + dn &\implies \begin{pmatrix} m' & n' \end{pmatrix} = \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &\implies \begin{pmatrix} m & n \end{pmatrix} = \begin{pmatrix} m' & n' \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \end{aligned}$$

So, we get

$$m = dm' - cn' \equiv d\mu - c\nu \bmod M \equiv \mu \bmod M$$

$$n = -bm' + an \equiv -b\mu + a\nu \bmod M \equiv \nu \bmod M.$$

■

Therefore,

$$\begin{aligned} E_{k,\mu,\nu}(\gamma z) &= \sum_{(m,n)\equiv(\mu,\nu) \bmod M} (m\gamma z + n)^{-k} \\ &= \sum_{\substack{m' \equiv \mu \bmod M \\ n' \equiv \nu \bmod M}} (cz+d)^k (m'z+n')^{-k} \qquad \text{by equation (4.1)} \\ &= (cz+d)^k E_{k,\mu,\nu}(z). \qquad \square \end{aligned}$$

Proposition 4.1.3. *The Fourier expansion of $E_{k,\mu,\nu}(z)$ on $\Gamma(M)$ is:*

$$E_{k,\mu,\nu}(z) = \delta\left(\frac{\mu}{M}\right) \sum_{n\equiv\nu \bmod M} \frac{1}{n^k} + \sum_{\substack{m_0\in\mathbb{Z} \\ m_0+\frac{\mu}{M}>0}} \frac{(-2\pi i)^k}{M^k(k-1)!} \sum_{n_0=1}^{\infty} n_0^{k-1} e(n_0[\frac{\mu}{M} + m_0]z) e(\frac{n_0\nu}{M})$$
$$+ \sum_{\substack{m_0\in\mathbb{Z} \\ m_0-\frac{\mu}{M}>0}} \frac{(2\pi i)^k}{M^k(k-1)!} \sum_{n_0=1}^{\infty} n_0^{k-1} e(n_0[-\frac{\mu}{M} + m_0]z) e(\frac{-n_0\nu}{M})$$

where $\delta(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$

Proof. Starting with the definition of the Eisenstein series, we have

$$\begin{aligned} E_{k,\mu,\nu}(z) &= \sum_{(m,n)\equiv(\mu,\nu) \bmod M} (mz+n)^{-k} \\ &= \sum_{\substack{m=Mm_0+\mu \\ n=Mn_0+\nu}} (mz+n)^{-k} \\ &= \sum_{m_0,n_0\in\mathbb{Z}} \frac{1}{[(Mm_0+\mu)z + (Mn_0+\nu)]^k} \\ &= \sum_{m_0,n_0\in\mathbb{Z}} M^{-k} \frac{1}{[(\frac{\mu}{M}+m_0)z + (\frac{\nu}{M}+n_0)]^k} \\ &= \sum_{m_0\in\mathbb{Z}} M^{-k} \sum_{n_0\in\mathbb{Z}} \frac{1}{[((\frac{\mu}{M}+m_0)z + \frac{\nu}{M}) + n_0]^k}. \end{aligned}$$

Splitting the sum over m_0, we get

$$\begin{aligned} E_{k,\mu,\nu}(z) = &\sum_{\substack{m_0\in\mathbb{Z} \\ m_0+\frac{\mu}{M}>0}} M^{-k} \sum_{n_0\in\mathbb{Z}} \frac{1}{[((\frac{\mu}{M}+m_0)z + \frac{\nu}{M}) + n_0]^k} \\ &+ \sum_{\substack{m_0\in\mathbb{Z} \\ m_0+\frac{\mu}{M}<0}} M^{-k} \sum_{n_0\in\mathbb{Z}} \frac{1}{[((\frac{\mu}{M}+m_0)z + \frac{\nu}{M}) + n_0]^k} \\ &+ \sum_{\substack{m_0\in\mathbb{Z} \\ m_0+\frac{\mu}{M}=0}} M^{-k} \sum_{n_0\in\mathbb{Z}} \frac{1}{[((\frac{\mu}{M}+m_0)z + \frac{\nu}{M}) + n_0]^k}. \end{aligned} \tag{4.2}$$

Notice that

$$\sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{\mu}{M}=0}} M^{-k}\sum_{n_0\in\mathbb{Z}}\frac{1}{[((\frac{\mu}{M}+m_0)z+\frac{\nu}{M})+n_0]^k}=\sum_{n_o\in\mathbb{Z}}\frac{M^{-k}}{(\frac{\nu}{M}+n_0)^k}$$

with $m_0+\frac{\mu}{M}=0$ and $m_0\in\mathbb{Z}$. Thus this term only exists if $\mu\equiv 0 \bmod M$.

In other words,

$$\sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{\mu}{M}=0}} M^{-k}\sum_{n_0\in\mathbb{Z}}\frac{1}{[((\frac{\mu}{M}+m_0)z+\frac{\nu}{M})+n_0]^k}=\delta\left(\frac{\mu}{M}\right)\sum_{n_o\in\mathbb{Z}}\frac{M^{-k}}{(\frac{\nu}{M}+n_0)^k}. \tag{4.3}$$

Using equation (4.3) and the Lipschitz Summation Formula, equation (4.2) becomes:

$$\begin{aligned}
E_{k,\mu,\nu}(z) &= \delta\left(\frac{\mu}{M}\right)\sum_{n_o\in\mathbb{Z}}\frac{M^{-k}}{(\frac{\nu}{M}+n_0)^k}+\sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{\mu}{M}>0}} M^{-k}\sum_{n_0\in\mathbb{Z}}\frac{1}{[((\frac{\mu}{M}+m_0)z+\frac{\nu}{M})+n_0]^k}\\
&+\sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{\mu}{M}<0}} N^{-k}\sum_{n_0\in\mathbb{Z}}\frac{1}{[((\frac{\mu}{M}+m_0)z+\frac{\nu}{M})+n_0]^k}\\
&=\delta\left(\frac{\mu}{M}\right)\sum_{n_o\in\mathbb{Z}}\frac{M^{-k}}{(\frac{\nu}{M}+n_0)^k}+\sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{\mu}{M}>0}} M^{-k}\sum_{n_0\in\mathbb{Z}}\frac{1}{[((\frac{\mu}{M}+m_0)z+\frac{\nu}{M})+n_0]^k}\\
&+\sum_{\substack{m_0\in\mathbb{Z}\\ m_0-\frac{\mu}{M}>0}} M^{-k}\sum_{n_0\in\mathbb{Z}}\frac{1}{[(-(-\frac{\mu}{M}+m_0)z+\frac{\nu}{M})+n_0]^k}\\
&=\delta\left(\frac{\mu}{M}\right)\sum_{n_o\in\mathbb{Z}}\frac{M^{-k}}{(\frac{\nu}{M}+n_0)^k}+\sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{\mu}{M}>0}} M^{-k}\sum_{n_0\in\mathbb{Z}}\frac{1}{[((\frac{\mu}{M}+m_0)z+\frac{\nu}{M})+n_0]^k}\\
&+\sum_{\substack{m_0\in\mathbb{Z}\\ m_0-\frac{\mu}{M}>0}} M^{-k}\sum_{n_0\in\mathbb{Z}}\frac{1}{(-1)^k[((-\frac{\mu}{M}+m_0)z-\frac{\nu}{M})+n_0]^k}\\
&=\delta\left(\frac{\mu}{M}\right)\sum_{n_o\in\mathbb{Z}}\frac{M^{-k}}{(\frac{\nu}{M}+n_0)^k}+\sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{\mu}{M}>0}}\frac{(-2\pi i)^k}{M^k(k-1)!}\sum_{n_0=1}^{\infty}n_0^{k-1}e(n_0[(\frac{\mu}{M}+m_0)z+\frac{\nu}{M}])\\
&+\sum_{\substack{m_0\in\mathbb{Z}\\ m_0-\frac{\mu}{M}>0}}\frac{(2\pi i)^k}{M^k(k-1)!}\sum_{n_0=1}^{\infty}n_0^{k-1}e(n_0[(-\frac{\mu}{M}+m_0)z)-\frac{\nu}{M}])\\
&=\delta\left(\frac{\mu}{M}\right)\sum_{n\equiv\nu \bmod M}\frac{1}{n^k}+\sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{\mu}{M}>0}}\frac{(-2\pi i)^k}{M^k(k-1)!}\sum_{n_0=1}^{\infty}n_0^{k-1}e(n_0[\frac{\mu}{M}+m_0]z)e(\frac{n_0\nu}{M})
\end{aligned}$$

$$+\sum_{\substack{m_0\in\mathbb{Z}\\ m_0-\frac{\mu}{M}>0}}\frac{(2\pi i)^k}{M^k(k-1)!}\sum_{n_0=1}^{\infty}n_0^{k-1}e(n_0[-\frac{\mu}{M}+m_0]z)e(\frac{-n_0\nu}{M}).$$ □

We restrict to $M=4$ in what follows.

Lemma 4.1.1. *We have*

$$\Gamma_0(4)=\dot{\bigcup}_{\gamma_i}\Gamma(4)\gamma_i$$

with the following set of coset representatives γ_i *:*

$$\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&2\\0&1\end{pmatrix},\begin{pmatrix}1&3\\0&1\end{pmatrix},\begin{pmatrix}-1&1\\0&-1\end{pmatrix},\begin{pmatrix}-1&2\\0&-1\end{pmatrix},\begin{pmatrix}-1&3\\0&-1\end{pmatrix}\right\}.$$

Proof. Since $\Gamma(4)$ is a normal subgroup of $\Gamma_0(4)$, define the quotient group ${}_{\Gamma(4)}\backslash^{\Gamma_0(4)}$. Now, consider the morphisms

$$\phi_1\ :\ \Gamma_0(4)\to\left(\mathbb{Z}/_{4\mathbb{Z}}\right)^\times$$
$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\mapsto d \bmod 4$$

with kernel $\Gamma_1(4)$ and $[\Gamma_0(4):\Gamma_1(4)]=2$, and

$$\phi_2\ :\ \Gamma_1(4)\to\mathbb{Z}/_{4\mathbb{Z}}$$
$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\mapsto b \bmod 4$$

with kernel $\Gamma(4)$ and $[\Gamma_1(4):\Gamma(4)]=4$. Therefore we have

$$[\Gamma_0(4):\Gamma(4)]=[\Gamma_0(4):\Gamma_1(4)]\cdot[\Gamma_1(4):\Gamma(4)]=2\cdot 4=8.$$

Finally, it is possible to find the 8 representatives by easy calculations. □

Proposition 4.1.4. *Define*

$$\tilde{E}_{k,\mu,\nu} = \sum_{\gamma_i} E_{k,\mu,\nu}\Big|_{k,\chi}\gamma_i.$$

Then $\tilde{E}_{k,\mu,\nu}$ is a modular form on $\Gamma_0(4)$ with a character, i.e. $\tilde{E}_{k,\mu,\nu} \in M_k(\Gamma_0(4),\chi)$. Moreover,

$$\tilde{E}_{k,\mu,\nu} = \sum_{\gamma_i} E_{k,\mu',\nu'}\chi(d_i)^{-1} \text{ with } \begin{pmatrix}\mu' & \nu'\end{pmatrix} = \begin{pmatrix}\mu & \nu\end{pmatrix}\gamma_i.$$

Proof. Note first that $\gamma_i \in_{\Gamma(4)} \backslash^{\Gamma_0(4)} \subset_{\Gamma(4)} \backslash^{SL_2(\mathbb{Z})}$. Thus it is easy to check that the slash operator is a group action from $_{\Gamma(4)}\backslash^{SL_2(\mathbb{Z})}$ on the set $M_k(\Gamma(4))$ defined by

$$\begin{aligned} M_k(\Gamma(4)) \times_{\Gamma(4)} \backslash^{SL_2(\mathbb{Z})} &\to M_k(\Gamma(4)) \\ (f \quad , \quad \gamma) &\mapsto f.\gamma = (f\Big|_{k,\chi}\gamma)(z). \end{aligned}$$

In other words, $f\Big|_{k,\chi}\gamma$ depends only on cosets $\Gamma(4)\gamma \in_{\Gamma(4)} \backslash^{SL_2(\mathbb{Z})}$ and $f\Big|_{k,\chi}\gamma\gamma'$ corresponds to multiplication in $_{\Gamma(4)}\backslash^{SL_2(\mathbb{Z})}$. Now, let $\alpha \in \Gamma_0(4)$. Then,

$$(\tilde{E}_{k,\mu,\nu}\Big|_{k,\chi}\alpha)(z) = (\sum_{\gamma_i} E_{k,\mu,\nu}\Big|_{k,\chi}\gamma_i)\Big|_{k,\chi}\alpha(z) = (\sum_{\gamma_i} E_{k,\mu,\nu}\Big|_{k,\chi}\gamma_i\alpha)(z).$$

We have that $\alpha \in \Gamma_0(4) = \dot{\bigcup}_{\gamma_i}\Gamma(4)\gamma_i$, then $\Gamma_0(4) = \Gamma_0(4)\alpha = \dot{\bigcup}_{\gamma_i}\Gamma(4)\gamma_i\alpha$. Hence there exists a function j depending on i and α such that $\gamma_i\alpha = \delta_i\gamma_j$ with $\delta_i \in \Gamma(4)$. Thus $\gamma_i\alpha$ is another set of coset representatives for $_{\Gamma(4)}\backslash^{\Gamma_0(4)}$. Therefore,

$$(\tilde{E}_{k,\mu,\nu}\Big|_{k,\chi}\alpha)(z) = (\sum_{\gamma_j} E_{k,\mu,\nu}\Big|_{k,\chi}\gamma_j)(z) = \tilde{E}_{k,\mu,\nu}(z).$$

Moreover, we write

$$\begin{aligned} E_{k,\mu,\nu}\Big|_{k,\chi}\gamma_i(z) &= \chi(d_i)^{-1}(c_iz+d_i)^{-k} \sum_{(m,n)\equiv(\mu,\nu) \bmod 4} (m[\frac{a_iz+b_i}{c_iz+d_i}]+n)^{-k} \\ &= \chi(d_i)^{-1} \sum_{(m,n)\equiv(\mu,\nu) \bmod 4} (m[a_iz+b_i]+n[c_iz+d_i])^{-k} \end{aligned}$$

$$= \chi(d_i)^{-1} \sum_{(m,n)\equiv(\mu,\nu) \bmod 4} ([ma_i + nc_i]z + [mb_i + nd_i])^{-k}$$

$$= \chi(d_i)^{-1} \sum_{(m',n')\equiv(\mu',\nu') \bmod 4} (m'z + n')^{-k}$$

$$= \chi(d_i)^{-1} E_{k,\mu',\nu'}(z)$$

where $m' = ma_i + nc_i \equiv \mu a_i + \nu c_i = \mu'$ and $n' = mb_i + nd_i \equiv \mu b_i + \nu d_i = \nu'$. $\square$

Due to the fact that $\mu, \nu \in \mathbb{Z}/_{4\mathbb{Z}}$, ${}_{\Gamma(4)}\backslash^{\Gamma_0(4)}$ acts on $(\mathbb{Z}/_{4\mathbb{Z}} \times \mathbb{Z}/_{4\mathbb{Z}})^t$. Thus, it would be sufficient to find the set of all orbits of the corresponding elements. Note that the idea is to use these orbits in addition to the Fourier expansion of $E_{k,\mu,\nu}$ to get the Fourier coefficients of the Eisenstein series on $\Gamma_0(4)$.

Proposition 4.1.5. *There are 6 orbits of* $(\mathbb{Z}/_{4\mathbb{Z}} \times \mathbb{Z}/_{4\mathbb{Z}})^t$ *under the action of* ${}_{\Gamma(4)}\backslash^{\Gamma_0(4)}$ *denoted by*

$$\begin{pmatrix}\mu & \nu\end{pmatrix} \cdot {}_{\Gamma(4)}\backslash^{\Gamma_0(4)} = \left\{ \begin{pmatrix}\mu & \nu\end{pmatrix} \cdot \gamma_i \quad \middle| \quad \gamma_i \in {}_{\Gamma(4)}\backslash^{\Gamma_0(4)} \right\}.$$

Proof. Using Lemma 4.1.1, we have the following orbits:

1. Orbit of $\begin{pmatrix}1 & 2\end{pmatrix}$

- $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 2\end{pmatrix}$ $\qquad$ $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}3 & 2\end{pmatrix}$,
- $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 3\end{pmatrix}$ $\qquad$ $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}-1 & 1\\0 & -1\end{pmatrix} = \begin{pmatrix}3 & 3\end{pmatrix}$,
- $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix}$ $\qquad$ $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}-1 & 2\\0 & -1\end{pmatrix} = \begin{pmatrix}3 & 0\end{pmatrix}$,
- $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 3\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix}$ $\qquad$ $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}-1 & 3\\0 & -1\end{pmatrix} = \begin{pmatrix}3 & 1\end{pmatrix}$.

$$\left\{\begin{pmatrix}1 & 2\end{pmatrix}, \begin{pmatrix}3 & 2\end{pmatrix}, \begin{pmatrix}1 & 3\end{pmatrix}, \begin{pmatrix}3 & 3\end{pmatrix}, \begin{pmatrix}1 & 0\end{pmatrix}, \begin{pmatrix}3 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\end{pmatrix}, \begin{pmatrix}3 & 1\end{pmatrix}\right\}.$$

Apply a similar method of calculations in what follows to get

2. Orbit of $\begin{pmatrix}0 & 1\end{pmatrix}$

   $\left\{\begin{pmatrix}0 & 1\end{pmatrix}, \begin{pmatrix}0 & 3\end{pmatrix}\right\}.$

3. Orbit of $\begin{pmatrix}2 & 1\end{pmatrix}$

   $\left\{\begin{pmatrix}2 & 1\end{pmatrix}, \begin{pmatrix}2 & 3\end{pmatrix}\right\}.$

4. Orbit of $\begin{pmatrix}2 & 0\end{pmatrix}$

   $\left\{\begin{pmatrix}2 & 0\end{pmatrix}, \begin{pmatrix}2 & 2\end{pmatrix}\right\}.$

5. Orbit of $\begin{pmatrix}0 & 2\end{pmatrix}$

   $\left\{\begin{pmatrix}0 & 2\end{pmatrix}\right\}.$

6. Orbit of $\begin{pmatrix}0 & 0\end{pmatrix}$

   $\left\{\begin{pmatrix}0 & 0\end{pmatrix}\right\}.$ □

4.2 Fourier Coefficients of a Cusp Form

In this section, we study the growth of the Fourier coefficients of a cusp form. The material is taken from [Miyake, 2006] chapter 2, pages 42-43.

Lemma 4.2.1. *Let $f(z)$ be a cusp form of weight k on Γ, i.e. f has a zero at each cusp. Then, $f(z)\Im(z)^{k/2}$ is bounded on $\mathbb{H}$.*

Proof. Let $f(z)$ be a cusp form and set $\phi(z) = |f(z)|\Im(z)^{k/2}$. Let $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma$, then notice that

$$\phi(\gamma z) = |f(\gamma z)|\Im(\gamma z)^{k/2} = (cz+d)^k|f(z)|[(cz+d)^{-2}]^{k/2}\Im(z)^{k/2} = |f(z)|\Im(z)^{k/2}.$$

So $\phi(z)$ is invariant on Γ and continuous on ${}_{\Gamma}\backslash^{\mathbb{H}}$. Notice that Γ has finitely many inequivalent cusps. Let x_0 be an arbitrary cusp of width h and suppose that γ' is an element of $SL_2(\mathbb{Z})$ such that $\gamma' \cdot \infty = x_0$. Then f has the following Fourier expansion at x_0 :

$$(f\Big|_k \gamma')(z) = \sum_{n=1}^{\infty} a_n e(\frac{nz}{h}).$$

Thus we have

$$\begin{aligned}
\phi(\gamma' z) &= |f(\gamma' z)|\Im(\gamma' z)^{k/2} \\
&= |f(\gamma' z)|[(c'z + d')^{-2}]^{k/2}\Im(z)^{k/2} \\
&= \left|(f\Big|_k \gamma')(z)\right|\Im(z)^{k/2} \\
&= \left|\sum_{n=1}^{\infty} a_n e(\frac{nz}{h})\right|\Im(z)^{k/2} \xrightarrow[\text{as } \Im(z)\to\infty]{} 0.
\end{aligned}$$

Hence, $\phi(z)$ is bounded on a neighborhood of each cusp x_0 (since x_0 was arbitrary), and therefore $\phi(z)$ is bounded on a compact subset of $\mathbb{H}$ defined by $\mathbb{H} \smallsetminus$ union of all these neighborhoods. $\square$

Theorem 4.2.2. *Let $f(z)$ be a modular form and x_0 a cusp of Γ. Suppose that γ' is an element of $SL_2(\mathbb{Z})$ such that $\gamma' \cdot \infty = x_0$. Then we have*

$$a_n = O(n^{k/2}) \quad \textit{where} \quad (f\Big|_k \gamma')(z) = \sum_{n=1}^{\infty} a_n e(\frac{nz}{h}).$$

Proof. Set $g(z) = (f\Big|_k \gamma')(z)$. Then $g(z)$ is a modular form of weight k on $\gamma'^{-1}\Gamma\gamma'$. By the above lemma,

$$\exists M > 0 \quad \text{such that} \quad \left|(f\Big|_k \gamma')(z)\right| \leq M\Im(z)^{-k/2} \quad \forall z \in \mathbb{H}.$$

Therefore, we get

$$|a_n| = \left|\int_0^h g(x + iy)e(-\frac{nz}{h})\, dx\right|$$

$$\leq \int_0^h |g(x+iy)| \left| e(-\frac{nz}{h}) \right| |dx|$$
$$\leq \frac{M}{h} y^{-k/2} e^{2\pi ny/h}.$$

In particular, taking $y = \frac{1}{n}$, we get

$$|a_n| \leq \frac{M}{h} e^{2\pi/h} n^{k/2} \implies a_n = O(n^{k/2}). \qquad \square$$

4.3 Representation of an Integer s by Sums of Squares

As an application of the above, we are interested in the following question: **How to represent an integer s by a positive definite quadratic form?** In other words, given positive integers s and k, in how many ways can s be represented by $Q(x)$ with $x \in \mathbb{Z}^k$. For this reason, we introduce $r_{Q,h,N}(s)$ defined by:

$$r_{Q,h,N}(s) = \#\{(x_1, \cdots, x_k) \in \mathbb{Z}^k \mid Q(x) = s \text{ and } x \equiv h \bmod N\mathbb{Z}^k\}.$$

Then the question becomes if there are any interesting formulas for $r_{Q,h,N}$ and how to efficiently calculate them.

In this section, we let $h = 0$, $N = 1$, $n = k$ and $A = 1_k$ identity matrix. As a result, we get $Q(x) = x_1^2 + x_2^2 + \cdots + x_k^2$ and write

$$r_k(s) = \#\{(x_1, \cdots, x_k) \in \mathbb{Z}^k \mid x_1^2 + \cdots + x_k^2 = s\}.$$

Notice that changing the signs or order of x_i in this case results in giving different representations.

Let us consider a particular example.

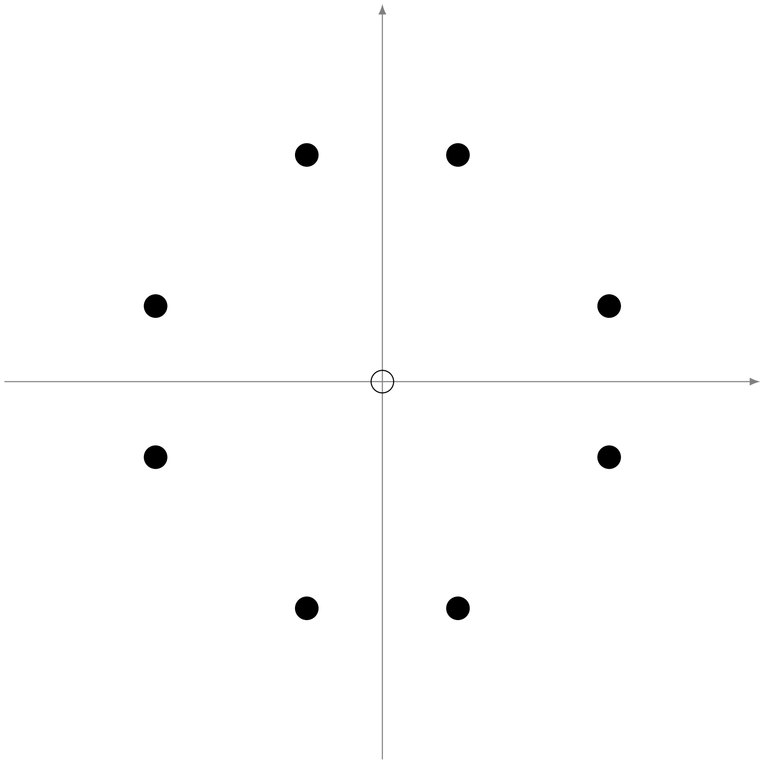

Figure 4.1: Geometric Interpretation of $r_2(10)$

Example 4.3.1. For $k = 2$ and $s = 10$ we have $r_2(10) = 8$. To see this, write

$$\begin{aligned} 10 &= 1^2 + 3^2 = (-1)^2 + 3^2 = 1^2 + (-3)^2 = (-1)^2 + (-3)^2 \\ &= 3^2 + 1^2 = (-3)^2 + 1^2 = 3^2 + (-1)^2 = (-3)^2 + (-1)^2. \end{aligned}$$

Geometrically, this is equivalent to saying that in the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, there are 8 points having distance $\sqrt{10}$ from the origin.

Remark. Since Q is a positive definite quadratic form, we can ensure that the set of points is finite by limiting using an upper bound for each of the x_i in term of x.

Now, denote $\theta(z; 0, 1_k, 1, 1)$ by $\theta(z)$ and consider the following:

Proposition 4.3.1. *For $z \in \mathbb{H}$, let $\tilde{\theta}$ be the function defined by*

$$\begin{aligned}\tilde{\theta} &: \mathbb{H} \to \mathbb{C} \\ z &\mapsto \tilde{\theta}(z) = \theta(2z) = \sum_{m \in \mathbb{Z}^k} e(zm^t m).\end{aligned}$$

Then $\tilde{\theta}$ is a modular form of weight $\frac{k}{2}$ on the congruence subgroup $\Gamma_0(4)$.

Proof. From Proposition 3.2.8, for $\gamma = \begin{pmatrix} a & 2b \\ 2c & d \end{pmatrix} \in \Gamma_0(2,2)$ we have

$$\begin{aligned}\theta(\gamma(z)) &= \theta(\frac{az+2b}{2cz+d}) \\ &= \left(\frac{c}{d}\right)^k \varepsilon_d^{-k}(2cz+d)^{k/2}\theta(z)\end{aligned} \tag{4.4}$$

with multiplier system $\epsilon(\gamma) = \left(\frac{c}{d}\right)^k \varepsilon_d^{-k}$ such that $|\epsilon(\gamma)| = 1$.
Now let $\gamma' = \begin{pmatrix} a' & b' \\ 4c' & d' \end{pmatrix} \in \Gamma_0(4)$, then we have

$$\begin{aligned}\tilde{\theta}(\gamma' z) &= \tilde{\theta}(\frac{a'z+b'}{4c'z+d'}) \\ &= \theta(2(\frac{a'z+b'}{4c'z+d'})) \\ &= \theta(\frac{a'(2z)+2b'}{2c'(2z)+d'}) \\ &= \left(\frac{4c'}{d'}\right)^k \varepsilon_{d'}^{-k}(4c'z+d')^{k/2}\theta(2z) && \text{by equation (4.4)} \\ &= \left(\frac{c'}{d'}\right)^k \varepsilon_{d'}^{-k}(4c'z+d')^{k/2}\tilde{\theta}(z)\end{aligned}$$

with multiplier system $\epsilon(\gamma') = \left(\frac{c'}{d'}\right)^k \varepsilon_{d'}^{-k}$ such that $|\epsilon(\gamma')| = 1$. □

Remark. Notice that we have the following relation between $\tilde{\theta}$ and r_k:

$$\tilde{\theta}(z) = \sum_{m \in \mathbb{Z}^k} e(zm^t m)$$

$$
\begin{aligned}
&= \sum_{m_1,\cdots,m_k\in\mathbb{Z}} e(z(m_1^2+\cdots+m_k^2)) \\
&= \sum_{s\in\mathbb{Z}} \sum_{\substack{m_1,\cdots,m_k\in\mathbb{Z}\\ s=m_1^2+\cdots+m_k^2}} e(sz) && \text{Letting } s = m_1^2+\cdots+m_k^2 \\
&= \sum_{s\in\mathbb{Z}_{\geq 0}} e(sz) \sum_{\substack{m_1,\cdots,m_k\in\mathbb{Z}\\ s=m_1^2+\cdots+m_k^2}} 1 \\
&= \sum_{s\in\mathbb{Z}_{\geq 0}} r_k(s)e(sz).
\end{aligned}
$$

In what follows, we use the fact that $\tilde{E}_{k,\mu,\nu} \in M_k(\Gamma_0(4),\chi)$ in order write $\tilde{\theta}$ as a combination of Eisenstein Series and Cusp Forms. Consequently, we obtain formulas for $r_k(s)$.

Proposition 4.3.2. *For $k = 3$, the Eisenstein space is spanned by $\tilde{E}_{3,1,2}$ and $\tilde{E}_{3,0,1}$.*

Proof. Using Propostion 4.1.4 and the orbits in Propostion 4.1.5, we get the following:

- For $\begin{pmatrix}\mu & \nu\end{pmatrix} = \begin{pmatrix}1 & 2\end{pmatrix}$

$$
\begin{aligned}
\tilde{E}_{3,1,2} &= E_{3,1,2}\chi(1) + E_{3,1,3}\chi(1) + E_{3,3,1}\chi(-1) + E_{3,1,1}\chi(1) + E_{3,3,2}\chi(-1) \\
&\quad + E_{3,3,3}\chi(-1) + E_{3,3,0}\chi(-1) + E_{3,1,0}\chi(1) \\
&= E_{3,1,2} + E_{3,1,3} - E_{3,3,1} + E_{3,1,1} - E_{3,3,2} - E_{3,3,3} - E_{3,3,0} + E_{3,1,0}.
\end{aligned}
$$

Notice that $E_{k,-\mu,-\nu} = (-1)^k E_{k,\mu,\nu}$. Then we get

$$
\begin{aligned}
\tilde{E}_{3,1,2}(z) &= E_{3,1,2}(z) + E_{3,1,3}(z) + E_{3,1,3}(z) + E_{3,1,1}(z) + E_{3,1,2}(z) + E_{3,1,1}(z) \\
&\quad + E_{3,1,0}(z) + E_{3,1,0} \\
&= 2[E_{3,1,0}(z) + E_{3,1,1}(z) + E_{3,1,2}(z) + E_{3,1,3}(z)]
\end{aligned}
$$

$$= C + \frac{2i(2\pi)^3}{4^3 \cdot 2!}\Big[\sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{1}{4}>0}} \sum_{n_0>0} n_0^2 e(n_0(m_0+\frac{1}{4})z) - \sum_{\substack{m_0\in\mathbb{Z}\\ m_0-\frac{1}{4}>0}} \sum_{n_0>0} n_0^2 e(n_0(m_0-\frac{1}{4})z)$$

$$+ \sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{1}{4}>0}} \sum_{n_0>0} n_0^2 e(n_0(m_0+\frac{1}{4})z)e(\frac{n_0}{4}) - \sum_{\substack{m_0\in\mathbb{Z}\\ m_0-\frac{1}{4}>0}} \sum_{n_0>0} n_0^2 e(n_0(m_0-\frac{1}{4})z)e(-\frac{n_0}{4})$$

$$+ \sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{1}{4}>0}} \sum_{n_0>0} n_0^2 e(n_0(m_0+\frac{1}{4})z)e(\frac{n_0}{2}) - \sum_{\substack{m_0\in\mathbb{Z}\\ m_0-\frac{1}{4}>0}} \sum_{n_0>0} n_0^2 e(n_0(m_0-\frac{1}{4})z)e(-\frac{n_0}{2})$$

$$+ \sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{1}{4}>0}} \sum_{n_0>0} n_0^2 e(n_0(m_0+\frac{1}{4})z)e(\frac{3n_0}{4}) - \sum_{\substack{m_0\in\mathbb{Z}\\ m_0-\frac{1}{4}>0}} \sum_{n_0>0} n_0^2 e(n_0(m_0-\frac{1}{4})z)e(-\frac{3n_0}{4})\Big]$$

$$= C + \frac{i\pi^3}{2^3}\Big[\sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{1}{4}>0}} \sum_{n_0>0} n_0^2 e(n_0(m_0+\frac{1}{4})z)\left(1 + e(\frac{n_0}{4}) + e(\frac{n_0}{2}) + e(\frac{3n_0}{4})\right)$$

$$- \sum_{\substack{m_0\in\mathbb{Z}\\ m_0-\frac{1}{4}>0}} \sum_{n_0>0} n_0^2 e(n_0(m_0-\frac{1}{4})z)\left(1 + e(-\frac{n_0}{4}) + e(-\frac{n_0}{2}) + e(-\frac{3n_0}{4})\right)\Big].$$

We distinguish now among 4 cases:

-If $n_0 \equiv 0 \bmod 4$:

$1 + e(\frac{n_0}{4}) + e(\frac{n_0}{2}) + e(\frac{3n_0}{4}) = 1 + e(-\frac{n_0}{4}) + e(-\frac{n_0}{2}) + e(-\frac{3n_0}{4}) = 1 + 1 + 1 + 1 = 4.$

- If $n_0 \equiv 1 \bmod 4$:

$1 + e(\frac{n_0}{4}) + e(\frac{n_0}{2}) + e(\frac{3n_0}{4}) = 1 + i - 1 - i = 0$

and $1 + e(-\frac{n_0}{4}) + e(-\frac{n_0}{2}) + e(-\frac{3n_0}{4}) = 1 - i - 1 + i = 0.$

- If $n_0 \equiv 2 \bmod 4$:

$1 + e(\frac{n_0}{4}) + e(\frac{n_0}{2}) + e(\frac{3n_0}{4}) = 1 - 1 + 1 - 1 = 0$

and $1 + e(-\frac{n_0}{4}) + e(-\frac{n_0}{2}) + e(-\frac{3n_0}{4}) = 1 - 1 + 1 - 1 = 0.$

- If $n_0 \equiv 3 \bmod 4$:

$1 + e(\frac{n_0}{4}) + e(\frac{n_0}{2}) + e(\frac{3n_0}{4}) = 1 - i - 1 + i = 0$

and $1 + e(-\frac{n_0}{4}) + e(-\frac{n_0}{2}) + e(-\frac{3n_0}{4}) = 1 + i - 1 - i = 0.$

Thus, the above sum does not vanish only for the case $n_0 = 4l_0$. So,

$$\tilde{E}_{3,1,2}(z) = C + \frac{i\pi^3}{2^3}\Big[\sum_{\substack{m_0\in\mathbb{Z}\\ m_0+\frac{1}{4}>0\\ n_0=4l_0\geq 4}} 4(4l_0)^2 e(l_0(1+4m_0)z) - \sum_{\substack{m_0\in\mathbb{Z}\\ m_0-\frac{1}{4}>0\\ n_0=4l_0\geq 4}} 4(4l_0)^2 e(l_0(4m_0-1)z)\Big]$$

$$= C + i(2\pi)^3\Big[\sum_{s\geq 1}\sum_{l_0|s} l_0^2\chi\left(\frac{s}{l_0}\right)e(sz)\Big].$$

- For $\begin{pmatrix}\mu & \nu\end{pmatrix} = \begin{pmatrix}0 & 1\end{pmatrix}$

$$\begin{aligned}\tilde{E}_{3,0,1} &= E_{3,0,1}\chi(1) + E_{3,0,1}\chi(1) + E_{3,0,3}\chi(-1) + E_{0,1}\chi(1) + E_{3,0,3}\chi(-1)\\ &+ E_{3,0,3}\chi(-1) + E_{3,0,3}\chi(-1) + E_{3,0,1}\chi(1)\\ &= 4E_{3,0,1} - 4E_{3,0,3}.\end{aligned}$$

Notice that $E_{3,0,1} = -E_{3,0,3}$. Then we get

$$\begin{aligned}\tilde{E}_{3,0,1}(z) &= 8E_{3,0,1}(z)\\ &= C + \frac{i\pi^3}{2}\sum_{n_0>0}\sum_{m_0>0} n_0^2 e(n_0m_0z)\left(e(\frac{n_0}{4}) - e(-\frac{n_0}{4})\right).\end{aligned}$$

We distinguish now among 4 cases:

-If $n_0 \equiv 0 \bmod 4$: $e(\frac{n_0}{4})e(-\frac{n_0}{4}) = 1 - 1 = 0$.

- If $n_0 \equiv 1 \bmod 4$: $e(\frac{n_0}{4})e(-\frac{n_0}{4}) = i + i = 2i$.

- If $n_0 \equiv 2 \bmod 4$: $e(\frac{n_0}{4})e(-\frac{n_0}{4}) = -1 + 1 = 0$.

- If $n_0 \equiv 3 \bmod 4$: $e(\frac{n_0}{4})e(-\frac{n_0}{4}) = -i - i = -2i$.

Thus, the above sum does not vanish only for the cases $n_0 \equiv 1, 3 \bmod 4$. So, set $s = n_0m_0$ to get

$$\begin{aligned}\tilde{E}_{3,0,1}(z) &= C + \frac{i\pi^3}{2}\sum_{s\geq 1}\sum_{n_0|s} n_0^2(2i)\chi(n_0)e(sz)\\ &= C - \pi^3\sum_{s\geq 1}\sum_{n_0|s} n_0^2\chi(n_0)e(sz).\end{aligned}$$

- For $\begin{pmatrix}\mu & \nu\end{pmatrix} = \begin{pmatrix}0 & 2\end{pmatrix}$

$$\tilde{E}_{3,0,2} = 4E_{3,0,2} - 4E_{3,0,2} = 0.$$

- For $\begin{pmatrix}\mu & \nu\end{pmatrix} = \begin{pmatrix}0 & 0\end{pmatrix}$
$$\tilde{E}_{3,0,0} = 0.$$

- For $\begin{pmatrix}\mu & \nu\end{pmatrix} = \begin{pmatrix}2 & 1\end{pmatrix}$
$$\tilde{E}_{3,2,1} = 2E_{3,2,1} + 2E_{3,2,3} - 2E_{3,2,1} - 2E_{3,2,3} = 0.$$

- For $\begin{pmatrix}\mu & \nu\end{pmatrix} = \begin{pmatrix}2 & 0\end{pmatrix}$
$$\tilde{E}_{3,2,0} = 2E_{3,2,0} + 2E_{3,2,2} - 2E_{3,2,0} - 2E_{3,2,2} = 0.$$

Thus, $\tilde{E}_{3,1,2}$ and $\tilde{E}_{3,0,1}$ generate the corresponding Eisenstein space. □

Theorem 4.3.1. *Knowing that the spaces of cusp forms $S_2(\Gamma_1(4)), S_3(\Gamma_1(4))$ and $S_4(\Gamma_1(4))$ are trivial, we have the following identities:*

1. $r_2(s) = 4\sum_{d|s}\chi(d)$,
2. $r_4(s) = 8\sum_{\substack{d|s \\ 4\nmid d}} d$,
3. $r_6(s) = \sum_{d|s}(16\chi\left(\frac{s}{d}\right) - 4\chi(d))d^2$,
4. $r_8(s) = 16\sum_{d|s}(-1)^{s-d}d^3$.

Proof. 1. For $k = 2$, see [Zagier, 1992] part 1.C, page 245.

2. First, for $k = 4$, we have
$$\tilde{\theta}(z) = \sum_{m\in\mathbb{Z}^4} e(zm^tm) \in M_2(\Gamma_0(4)) \subset M_2(\Gamma_1(4)).$$

Knowing that
$$E_2(z) = 1 + \frac{(2\pi i)^2}{\zeta(2)(2-1)!}\sum_{s=1}^{\infty}\sigma_{2-1}(s)q^s = 1 - 24\sum_{s=1}^{\infty}\sigma_1(s)q^s,$$

our goal is to write $\tilde{\theta}$ in terms of E_2. Note that E_2 is not a modular form, but combinations involving $E_2(z), E_2(2z)$ and $E_2(4z)$ are. For this reason, we write:

$$E_2(z) = 1 - 24q - 72q^2 - 96q^3 + \cdots,$$

$$E_2(2z) = 1 - 24q^2 - 72q^4 - 96q^6 + \cdots,$$

$$E_2(4z) = 1 - 24q^4 - 72q^8 + \cdots.$$

Next, by using Magma Calculator, we find a basis for $M_2(\Gamma_1(4)):$

- $J_1 := 1 + 24q^2 + 24q^4 + 96q^6 + 24q^8 ... = 2E_2(4z) - E_2(2z),$
- $J_2 := q + 4q^3 + 6q^5 + 8q^7 ... = \frac{-1}{24}(E_2(z) - 3E_2(2z) + 2E_2(4z)).$

Moreover, by the previous remark we have

$$\begin{aligned}
\tilde{\theta}(z) &= \sum_{s \in \mathbb{Z}_{\geq 0}} r_4(s)e(sz) \\
&= 1 + \sum_{s=1}^{\infty} r_4(s)e(sz) \\
&= 1 + r_4(1)q + r_4(2)q^2 + \cdots \\
&= 1 + 8q + 24q^2 + \cdots \\
&= 1(1 + 24q^2 + \cdots) + 8(q + 4q^3 + \cdots) \\
&= J_1 + 8J_2 \\
&= 2E_2(4z) - E_2(2z) - \frac{1}{3}\left(E_2(z) - 3E_2(2z) + 2E_2(4z)\right) \\
&= \frac{-1}{3}E_2(z) + \frac{4}{3}E_2(4z) \\
&= \frac{-1}{3}(1 - 24\sum_{s=1}^{\infty} \sigma_1(s)q^s) + \frac{4}{3}(1 - 24\sum_{s=1}^{\infty} \sigma_1(s)q^{4s}) \\
&= 1 + 8\sum_{s=1}^{\infty} \sigma_1(s)q^s - 32\sum_{s=1}^{\infty} \sigma_1(s)q^{4s} \\
&= 1 + 8\sum_{s=1}^{\infty} \sigma_1(s)q^s - 32\sum_{s=1}^{\infty} \sigma_1(\frac{s}{4})q^s \\
&= 1 + \sum_{s=1}^{\infty} \left(8\sigma_1(s) - 32\sigma_1(\frac{s}{4})\right) q^s.
\end{aligned}$$

Hence,

$$r_4(s) = 8\sigma_1(s) - 32\sigma_1(\frac{s}{4}).$$

We still need to prove that

$$\sigma_1(s) - 4\sigma_1(\frac{s}{4}) = \sum_{\substack{d|s \\ 4\nmid d}} d.$$

Proof. To see this, we study separate cases:

- If 4 does not divide s :

  Then 4 does not divide any divisor of s. So,

  $$\sigma_1(s) = \sum_{d|s} d = \sum_{\substack{d|s \\ 4\nmid d}} d.$$

- If 4 divides s :

$$\begin{aligned}
\sum_{\substack{d|s \\ 4\nmid d}} d &= \sum_{d|s} d - \sum_{\substack{d|s \\ 4|d}} d \\
&= \sigma_1(s) - \sum_{\substack{d|s \\ 4|d}} d \\
&= \sigma_1(s) - \sum_{l|\frac{s}{4}} 4l && \text{Taking } 4l = d \\
&= \sigma_1(s) - 4\sigma_1(\frac{s}{4}).
\end{aligned}$$

■

Thus, the needed formula.

3. First, for k=6, we have

$$\tilde{\theta}(z) = \sum_{m\in\mathbb{Z}^6} e(zm^t m) \in M_3(\Gamma_0(4)) \subset M_3(\Gamma_1(4)).$$

Next, by using Magma Calculator, we find a basis for $M_3(\Gamma_1(4))$:

- $J_1 := 1 + 12q^2 + 64q^3 + 60q^4...$,
- $J_2 := q + 4q^2 + 8q^3 + 16q^4...$

So, our goal now is to find relations between J_1, J_2 and the elements of the Eisenstein Space. By Proposition 4.3.2, the Eisenstein space as well as $M_3(\Gamma_1(4))$ is spanned by $\tilde{E}_{3,1,2}$ and $\tilde{E}_{3,0,1}$. We write

$$\tilde{E'}_{3,1,2}(z) := \frac{1}{i(2\pi)^3}\tilde{E}_{3,1,2}(z) = C + \sum_{s\geq 1}\sum_{l_0|s} l_0^2\chi\left(\frac{s}{l_0}\right)e(sz) = C + q + 4q^2 + 8q^3 + ...$$

$$\tilde{E'}_{3,0,1}(z) := \frac{1}{-\pi^3}\tilde{E}_{3,0,1}(z) = C + \sum_{s\geq 1}\sum_{n_0|s} n_0^2\chi(n_0)e(sz) = C + q + q^2 - 8q^3 + ...$$

Now we have $J_1 = 4(\tilde{E'}_{3,1,2} - \tilde{E'}_{3,0,1})$ and $J_2 = \tilde{E'}_{3,1,2}$. Moreover, by the previous remark

$$\begin{aligned}
\tilde{\theta}(z) &= \sum_{s\in\mathbb{Z}_{\geq 0}} r_6(s)e(sz) \\
&= 1 + \sum_{s=1}^{\infty} r_6(s)e(sz) \\
&= 1 + r_6(1)q + r_6(2)q^2 + \cdots \\
&= 1 + 12q + 60q^2 + \cdots \\
&= 1(1 + 12q^2 + \cdots) + 12(q + 4q^2 + \cdots) \\
&= J_1 + 12J_2 \\
&= 4(\tilde{E'}_{3,1,2} - \tilde{E'}_{3,0,1}) + 12\tilde{E'}_{3,1,2} \\
&= 16\tilde{E'}_{3,1,2} - 4\tilde{E'}_{3,0,1}
\end{aligned}$$

$$= 16(C + \sum_{s\geq 1}\sum_{l_0|s} l_0^2\chi\left(\frac{s}{l_0}\right)e(sz)) - 4(C + \sum_{s\geq 1}\sum_{n_0|s} n_0^2\chi(n_0)e(sz))$$

$$= C + \sum_{s=1}^{\infty}\left(16\sum_{l_0|s} l_0^2 - 4\sum_{n_0|s} n_0^2\chi(n_0)\right)e(sz).$$

Hence,

$$r_6(s) = 16\sum_{d|s} d^2 - 4\sum_{d|s} d^2\chi(d).$$

4. First, for $k = 8$, we have

$$\tilde{\theta}(z) = \sum_{m\in\mathbb{Z}^8} e(zm^tm) \in M_4(\Gamma_0(4)) \subset M_4(\Gamma_1(4)).$$

Next, by using Magma Calculator, we find a basis for $M_4(\Gamma_1(4))$:

- $J_1 := 1 + 240q^4 + 2160q^8...,$
- $J_2 := q + 28q^3 + 126q^5 + 344q^7...,$
- $J_3 := q^2 + 8q^4 + 28q^6 + 64q^8...$

Knowing that

$$E_4(z) = 1 + \frac{(2\pi i)^4}{\zeta(4)(4-1)!}\sum_{s=1}^{\infty}\sigma_{4-1}(s)q^s = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^s,$$

our goal is to write $\tilde{\theta}$ in terms of E_4. For this reason, we write:

$$E_4(z) = 1 + 240q + 2160q^2 + \cdots = J_1 + 240J_2 + 2160J_3,$$

$$E_4(2z) = 1 + 240q^2 + 2160q^4 + \cdots = J_1 + 240J_3,$$

$$E_4(4z) = 1 + 240q^4 + 2160q^8 + \cdots = J_1.$$

Moreover, by the previous remark we have

$$\tilde{\theta}(z) = \sum_{s\in\mathbb{Z}_{\geq 0}} r_8(s)e(sz)$$

$$
\begin{aligned}
&= 1 + \sum_{s=1}^{\infty} r_8(s) e(sz) \\
&= 1 + r_8(1)q + r_8(2)q^2 + \cdots \\
&= 1 + 16q + 112q^2 + \cdots \\
&= 1(1 + 240q^4 + \cdots) + 16(q + 28q^3 + \cdots) + 112(q^2 + 8q^4 + \cdots) \\
&= J_1 + 16J_2 + 112J_3. \qquad (4.5)
\end{aligned}
$$

Thus,

$$
\begin{pmatrix} E_4(z) \\ E_4(2z) \\ E_4(4z) \end{pmatrix} = \begin{pmatrix} 1 & 240 & 2160 \\ 1 & 0 & 240 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix}
$$

$$
\implies \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{pmatrix} 1 & 240 & 2160 \\ 1 & 0 & 240 \\ 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} E_4(z) \\ E_4(2z) \\ E_4(4z) \end{pmatrix}
$$

$$
\implies \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \frac{1}{240} \begin{pmatrix} 0 & 0 & 240 \\ 1 & -9 & 8 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} E_4(z) \\ E_4(2z) \\ E_4(4z) \end{pmatrix}
$$

Therefore, (4.5) becomes:

$$
\begin{aligned}
\tilde{\theta}(z) &= E_4(4z) + 16\left(\frac{1}{240}E_4(z) - \frac{9}{240}E_4(2z) + \frac{8}{240}E_4(4z)\right) \\
&\quad + 112\left(\frac{1}{240}E_4(2z) - \frac{1}{240}E_4(4z)\right) \\
&= \frac{16}{240}E_4(z) - \frac{32}{240}E_4(2z) + \frac{256}{240}E_4(4z) \\
&= \frac{16}{240}E_4(z) - \frac{32}{240}E_4(2z) + \frac{256}{240}E_4(4z) \\
&= \frac{16}{240}(1 + 240\sum_{s=1}^{\infty}\sigma_3(s)q^s) - \frac{32}{240}(1 + 240\sum_{s=1}^{\infty}\sigma_3(s)q^{2s})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{256}{240}(1+240\sum_{s=1}^{\infty}\sigma_3(s)q^{4s})\\
&=1+16\sum_{s=1}^{\infty}\sigma_3(s)q^s-32\sum_{s=1}^{\infty}\sigma_3(s)q^{2s}+256\sum_{s=1}^{\infty}\sigma_3(s)q^{4s}\\
&=1+16\sum_{s=1}^{\infty}\sigma_3(s)q^s-32\sum_{s=1}^{\infty}\sigma_3(\frac{s}{2})q^s+256\sum_{s=1}^{\infty}\sigma_3(\frac{s}{4})q^s\\
&=1+\sum_{s=1}^{\infty}\left(16\sigma_3(s)-32\sigma_3(\frac{s}{2})+256\sigma_3(\frac{s}{4})\right)q^s.
\end{aligned}
$$

Hence,

$$r_8(s)=16\sigma_3(s)-32\sigma_3(\frac{s}{2})+256\sigma_3(\frac{s}{4}).$$

It remains to prove that

$$\sigma_3(s)-2\sigma_3(\frac{s}{2})+16\sigma_3(\frac{s}{4})=\sum_{d|s}(-1)^{s-d}d^3.$$

Proof. To see this, we study separate cases:

- If 2 does not divide s:

  Then 2 does not divide any divisor of s. Then $s-d$ is even. So,

  $$\sigma_3(s)=\sum_{d|s}d^3=\sum_{d|s}(-1)^{s-d}d^3.$$

- If 2 divides s and 4 does not divide s:

  Apply a change of variable $s=2t$ with t odd. Then, d divides s $\iff$ $d=2^ie$ with $i=0,1$ and e divides t. So,

  $$
  \begin{aligned}
  \sigma_3(s)-2\sigma_3(\frac{s}{2})&=\sum_{d|s}d^3-2\sum_{d|\frac{s}{2}}d^3\\
  &=\sum_{\substack{i=0,1\\ e|t}}(2^ie)^3-2\sum_{e|t}e^3\\
  &=\sum_{e|t}e^3+2^3\sum_{e|t}e^3-2\sum_{e|t}e^3
  \end{aligned}
  $$

$$= -\sum_{e|t} e^3 + \sum_{e|t}(2e)^3$$

$$= \sum_{\substack{d|s \\ d \text{ even}}} d^3 - \sum_{\substack{d|s \\ d \text{ odd}}} d^3$$

$$= \sum_{d|s}(-1)^{s-d} d^3.$$

- If 4 divides s:

  Apply a change of variable $s = 2^i t$ with t odd. Then, d divides s $\iff$ $d = 2^j e$ with $0 \le j \le i$ and e divides t. So,

$$\begin{aligned}
\sigma_3(s) - 2\sigma_3(\frac{s}{2}) + 16\sigma_3(\frac{s}{4}) &= \sum_{d|s} d^3 - 2\sum_{d|\frac{s}{2}} d^3 + 16\sum_{d|\frac{s}{4}} d^3 \\
&= \sum_{\substack{0\le j\le i \\ e|t}} (2^j e)^3 - 2\sum_{\substack{0\le j\le i-1 \\ e|t}} (2^j e)^3 + 16\sum_{\substack{0\le j\le i-2 \\ e|t}} (2^j e)^3 \\
&= \sum_{e|t} e^3 + \sum_{\substack{1\le j\le i \\ e|t}} (2^j e)^3 - 2\sum_{e|t} e^3 - 2\sum_{\substack{1\le j\le i-1 \\ e|t}} (2^j e)^3 \\
&+ 2\cdot 2^3 \sum_{\substack{0\le j\le i-2 \\ e|t}} (2^j e)^3 \\
&= \sum_{\substack{1\le j\le i \\ e|t}} (2^j e)^3 - \sum_{e|t} e^3 - 2\sum_{\substack{1\le j\le i-1 \\ e|t}} (2^j e)^3 + 2\sum_{\substack{0\le j\le i-2 \\ e|t}} (2^{j+1} e)^3 \\
&= \sum_{\substack{1\le j\le i \\ e|t}} (2^j e)^3 - \sum_{e|t} e^3 - 2\sum_{\substack{1\le j\le i-1 \\ e|t}} (2^j e)^3 + 2\sum_{\substack{1\le j\le i-1 \\ e|t}} (2^j e)^3 \\
&= \sum_{\substack{1\le j\le i \\ e|t}} (2^j e)^3 - \sum_{e|t} e^3 \\
&= \sum_{\substack{d|s \\ d \text{ even}}} d^3 - \sum_{\substack{d|s \\ d \text{ odd}}} d^3 \\
&= \sum_{d|s}(-1)^{s-d} d^3.
\end{aligned}$$

$\blacksquare$

Thus we obtain the needed formula for $r_8(s)$. $\square$

Theorem 4.3.2. *We have*

$$r_{10}(s) = \frac{4}{5}\sum_{d|s}\left[\chi(d) + 16\chi\left(\frac{s}{d}\right)\right]d^4 + \frac{8}{5}\sum_{\substack{z\in\mathbb{Z}[i]\\|z|^2=s}} z^4.$$

Proof. For $k = 10$, we have

$$\tilde{\theta}(z) = \sum_{m\in\mathbb{Z}^{10}} e(zm^t m) \in M_5(\Gamma_0(4)) \subset M_5(\Gamma_1(4)).$$

By using Magma Calculator, we find a basis for $M_5(\Gamma_1(4))$:

- $J_1 := 1 - 80q^3 - 60q^4 + ...,$
- $J_2 := q + 216q^3 + 64q^4 + ...,$
- $J_3 := q^2 + 4q^3 + 12q^4 + ...$

  Now, we aim to obtain relations between the above basis and the elements of the Eisenstein space. For this reason, using similar procedure and calculations as in Proposition 4.3.2, we find out that the Eisenstein space is spanned by two elements and we write :

$$\tilde{E'}_{5,1,2}(z) := -\frac{12}{i(2\pi)^5}\tilde{E}_{5,1,2}(z) = C + \sum_{s\geq1}\sum_{l_0|s} l_0^4\chi\left(\frac{s}{l_0}\right)e(sz) = C + q + 16q^2 + 80q^3 + ...$$
$$= J_2 + 16J_3$$

$$\tilde{E}_{5,0,1}(z) := \frac{48}{\pi^5}\tilde{E}_{5,0,1}(z) = C + \sum_{s\geq1}\sum_{n_0|s} n_0^4\chi(n_0)e(sz) = C + q + q^2 - 80q^3 + ...$$
$$= \frac{5}{4}J_1 + J_2 + J_3.$$

Notice that in this case we have a non-trivial space of cusp forms. Furthermore, by Proposition 2.2.1 we have that $M_5 = S_5 \oplus \mathcal{E}_5$. Thus $M_5(\Gamma_1(4))$ is spanned by $\tilde{E}_{5,1,2}, \tilde{E}_{5,0,1}$ and a cusp form $\mathcal{C}$ with the following q-expansion according to Magma:

$$\mathcal{C} := q - 4q^2 + 16q^4 - 14q^5 - 64q^8 + 81q^9 + 56q^10 + O(q^{12}) = J_2 - 4J_3.$$

By the previous remark,

$$\begin{aligned}
\tilde{\theta}(z) &= \sum_{s \in \mathbb{Z}_{\geq 0}} r_{10}(s) e(sz) \\
&= 1 + \sum_{s=1}^{\infty} r_{10}(s) e(sz) \\
&= 1 + r_{10}(1)q + r_{10}(2)q^2 + r_{10}(3)q^3 \cdots \\
&= 1 + 20q + 180q^2 + 960q^3 + \cdots \\
&= J_1 + 20J_2 + 180J_3. \qquad (4.6)
\end{aligned}$$

Thus,

$$\begin{pmatrix} \tilde{E}'_{5,1,2}(z) \\ \tilde{E}'_{5,0,1}(z) \\ \mathcal{C} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 16 \\ \frac{5}{4} & 1 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix}$$

$$\implies \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 16 \\ \frac{5}{4} & 1 & 1 \\ 0 & 1 & -4 \end{pmatrix}^{-1} \begin{pmatrix} \tilde{E}'_{5,1,2}(z) \\ \tilde{E}'_{5,0,1}(z) \\ \mathcal{C} \end{pmatrix}$$

$$\implies \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{4}{5} & -\frac{3}{5} \\ \frac{1}{5} & 0 & \frac{4}{5} \\ \frac{1}{20} & 0 & -\frac{1}{20} \end{pmatrix} \begin{pmatrix} \tilde{E}'_{5,1,2}(z) \\ \tilde{E}'_{5,0,1}(z) \\ \mathcal{C} \end{pmatrix}$$

Therefore, (4.6) becomes:

$$\begin{aligned}
\tilde{\theta}(z) &= -\frac{1}{5}\tilde{E}'_{5,1,2}(z) + \frac{4}{5}\tilde{E}'_{5,0,1}(z) - \frac{3}{5}\mathcal{C} + 20\left(\frac{1}{5}\tilde{E}'_{5,1,2}(z) + \frac{4}{5}\mathcal{C}\right) \\
&+ 180\left(\frac{1}{20}\tilde{E}'_{5,1,2}(z) - \frac{1}{20}\mathcal{C}\right) \\
&= \frac{64}{5}\tilde{E}'_{5,1,2}(z) + \frac{4}{5}\tilde{E}'_{5,0,1}(z) + \frac{32}{5}\mathcal{C} \\
&= \frac{64}{5}\tilde{E}'_{5,1,2}(z) + \frac{4}{5}\tilde{E}'_{5,0,1}(z) + \frac{32}{5}\mathcal{C} \\
&= \frac{64}{5}(C + \sum_{s\geq 1}\sum_{l_0|s} l_0^4\chi\left(\frac{s}{l_0}\right)e(sz)) + \frac{4}{5}(C + \sum_{s\geq 1}\sum_{n_0|s} n_0^4\chi(n_0)e(sz)) + \frac{32}{5}\mathcal{C} \\
&= C + \frac{4}{5}\sum_{s\geq 1}\sum_{d|s}[16\chi\left(\frac{s}{d}\right) + \chi(d)]d^4 e(sz) + \frac{32}{5}\mathcal{C}
\end{aligned}$$

Hence,

$$r_{10}(s) = \frac{4}{5}\sum_{d|s}[16\chi\left(\frac{s}{d}\right) + \chi(d)]d^4 + c_{n,\text{cusp}}$$

where $c_{n,\text{cusp}}$ is the Fourier coefficient of the cusp part of $\tilde{\theta}$. However, by Section 4.2, we know that $c_{n,\text{cusp}} = O(n^{k/2})$ meaning that this term grows less fast than the main term $\frac{4}{5}\sum_{d|s}[16\chi\left(\frac{s}{d}\right) + \chi(d)]d^4$ which defines the Fourier coefficient of the Eisenstein part of $\tilde{\theta}$. To conclude, we can say that $r_{10}(s)$ would be the sum of a main term and some smaller order error. □

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