AMERICAN UNIVERSITY OF BEIRUT

THE STRUCTURE OF GROUPS OF ANALYTIC TRANSFORMATIONS

by

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An Abstract of the Thesis of

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Title: The Structure of Groups of Analytic Transformations

In this thesis we will review some classical results by H. Cartan, devoted to the study of certain families of real-analytic and holomorphic diffeomorphisms. In particular, it is shown that a local quasi-continuous group of holomorphic transformation has the structure of a local Lie group. In turn, this implies that the automorphism group of a finite dimensional bounded domain in $\mathbb{C}^n$ is a Lie group.

Contents

Chapter 1

Elements of Complex Analysis in Several Variables

1.1 Basic Definitions and Results

In this section we will outline some of the main results of Complex Analysis in several variables which will be needed throughout the thesis.

Part of Complex Analysis in several variables is a generalization of Complex Analysis in one variable. However, other aspects tend to get more complicated and some important results that hold in one variable turn out to not hold in several variables. Most notably, the Riemann Mapping Theorem.

Let $\mathbb{C}^n$ denote the n variables complex space. We will use the standard notation for coordinates in $\mathbb{C}^n$ as follows:

Let $z \in \mathbb{C}^n$, then

$$z = (z_1, ..., z_n) = (x_1 + iy_1, ..., x_n + iy_n)$$

where $z_j \in \mathbb{C}$ and $x_j, y_j \in \mathbb{R}$ for each $1 \leq j \leq n$.

We also need a notion of norm in $\mathbb{C}^n$. Denote by $||z||$ the standard Euclidean norm defined by

$$||z|| = \sqrt{||z_1||^2 + ... + ||z_n||^2}$$

Any theory of functions of several variables requires a multi index notation. We will use the standard multi index notation. Let $\mathbb{Z}^+ = \{0, 1, 2, ...\}$. A multi index α is an element of $(\mathbb{Z}^+)^n$.

If $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi index, and $w = (w_1, ..., w_n)$, then

$$w^\alpha = w_1^{\alpha_1} \cdot w_2^{\alpha_2} \cdots w_n^{\alpha_n}$$

$$\overline{w}^\alpha = \overline{w}_1^{\alpha_1} \cdot \overline{w}_2^{\alpha_2} \cdots \overline{w}_n^{\alpha_n}$$

$$\left(\frac{\partial}{\partial w}\right)^\alpha = \left(\frac{\partial}{\partial w_1}\right)^{\alpha_1} \cdot \left(\frac{\partial}{\partial w_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial w_n}\right)^{\alpha_n}$$

$$\left(\frac{\partial}{\partial \overline{w}}\right)^\alpha = \left(\frac{\partial}{\partial \overline{w_1}}\right)^{\alpha_1} \cdot \left(\frac{\partial}{\partial \overline{w_2}}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial \overline{w_n}}\right)^{\alpha_n}$$

Also $\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_n!$, and $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$

The following partial differential operators defined on $\mathbb{C}^n$ play an important role in Complex Analysis in several variables

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \ j = 1, ..., n$$

$$\frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \ j = 1, ..., n$$

and for $j \neq k$ we have

$$\frac{\partial}{\partial z_j} z_k = 0, \quad \frac{\partial}{\partial \overline{z_j}} \overline{z_k} = 0$$

One of the natural generalizations of the unit disk in several Complex variables is the polydisk, which we will introduced in the following definition.

Definition 1.1.1. Let $p \in \mathbb{C}^n$, let $\rho = (\rho_1, ..., \rho_n)$ with $\rho_1 > 0, ..., \rho_n > 0$. We define the set $\Delta(p, \rho)$ as

$$\Delta(p, \rho) = \{(z_1, ..., z_n) \in \mathbb{C}^n; |z_j - p_j| < \rho_j, \forall j = 1, ..., n\}$$

$\Delta(p, \rho)$ is called the polydisk of center p and polyradius ρ.

Remark: a polydisk is the product of disks in $\mathbb{C}$, i.e.

$$\Delta(p, \rho) = \mathbb{D}(p_1, \rho_1) \times ... \times \mathbb{D}(p_n, \rho_n)$$

Another natural generalization of the disk, is the ball.

Definition 1.1.2. Let $p \in \mathbb{C}^n$, let $\rho > 0$. We define the set $B(p, \rho)$ by

$$B(p, \rho) = \{z \in \mathbb{C}^n; ||z - p|| < \rho\}$$

B is called the Ball of center p and radius ρ

These 2 generalizations of the disk in one variable are completely different sets. One of the notable differences is that the ball has a smooth boundary while the polydisk doesn't. We will also see later on that these sets are not even biholomorphic to each other (see section 1.2).

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a function of class C^1, then f can be expressed as

$$f(z) = u(x, y) + iv(x, y)$$

where u, v are functions from $\mathbb{R}^{2n} \to \mathbb{R}$, and are the real and imaginary part of f. The Cauchy-Riemann equations in n variables can be reformulated by the above operators as

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j} \quad , \quad \frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j}$$

for $j = 1, ..., n$ which implies

$$\frac{\partial f}{\partial \overline{z_j}} = 0$$

for $j = 1, ..., n$

Definition 1.1.3. Let $\Omega \subset \mathbb{C}^n$ be a domain. A function $f : \Omega \to \mathbb{C}$ which is continuously differentiable on Ω is said to be holomorphic on Ω if it satisfies the Cauchy-Riemann equations in each variable separately. i.e.

$$\frac{\partial f}{\partial \overline{z_j}} = 0$$

for each $j = 1, ..., n$.

In other words, f is holomorphic in each variable separately.

The above definition is one of many standard definitions of holomorphicity in several Complex variables which are equivalent [1]. The main feature of those definitions is that they all allow us to prove an integral representation which generalizes the Cauchy integral formula in one variable.

In order to state the formula, we need to introduce a new set called the polytorus.

Definition 1.1.4. Let $p \in \mathbb{C}^n$, let $\rho = (\rho_1, ..., \rho_n)$ with $\rho_1 > 0, ..., \rho_n > 0$

We define the set $T(p, \rho)$ as

$$T(p, \rho) = \{(z_1, ..., z_n) \in \mathbb{C}^n; |z_j - p_j| = \rho_j, \forall j = 1, ..., n\}$$

$T(p, \rho)$ is called the polytorus centered at p with polyradius ρ.

Note that the polytorus is not the boundary of the polydisk. The polytorus is the product of n circles in $\mathbb{C}^n$. In one variable, the boundary of a disk is simple a circle, but in several variables, the boundary of a polydisk is of real dimension $2n - 1$ while the dimension of the polytorus is n (product of n circles). Nevertheless, the polytorus turns out to be the set to integrate over in Cauchy integral formula in several variables. This is a strong result, since this tells us that the information about our holomorphic function is concentrated not only on the boundary of our set, but on a much smaller set.

Theorem 1.1.1. *[1] [The Cauchy Integral Formula] Let $\Omega \subset \mathbb{C}^n$, let $f : \Omega \to \mathbb{C}$. f is said to be holomorphic on Ω if and only if for all p in Ω, for every polyradius ρ such that $\overline{\Delta(p, \rho)} \subset \Omega$ we have*

$$f(z) = \frac{1}{(2i\pi)^n} \int_{T(p,\rho)} \frac{f(w)}{(w_1 - z_1) ... (w_n - z_n)} dw$$

The following two results can be deduced from the Cauchy integral formula

1.1.1 using a similar proof as in one Complex variable. But before that we need to introduce the notion of compact convergence.

Definition 1.1.5. Let $(X, d), (X', d')$ be two metric spaces with distance d, d' respectively, let $\{f_n\}$ be a sequence of functions with $f_k : X \to X'$, $f : X \to X'$ We say $\{f_n\}$ converges compactly to f, if for all compact subsets K of X, we have $\{f_n\}$ converges uniformly to f on K. i.e.

$$\forall \epsilon > 0, \exists N \in \mathbb{N}; k \geq N \implies \sup_{x \in K} d\left(f_k(x) - f(x)\right) < \epsilon$$

Theorem 1.1.2. *Let $\Omega \subset \mathbb{C}^n$, let $f : \Omega \to \mathbb{C}$. f is said to be holomorphic on Ω if and only if f is analytic, i.e. it can be expressed as a power series $f(z) = \sum_{\alpha \in (\mathbb{Z}^+)^n} c_\alpha (z-p)^\alpha$ which converges compactly on $\Delta(p, \rho) \subset \Omega$ with*

$$c_\alpha = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(p) = \frac{1}{(2i\pi)^n} \int_{T(p,\rho)} \frac{f(w)}{(w-p)^{\alpha+(1,\ldots,1)}} dw$$

Theorem 1.1.3 (Cauchy's inequalities)**.** *Let $\Omega \subset \mathbb{C}^n$, let $f : \Omega \to \mathbb{C}$. If f is holomorphic on Ω and $|f| < M$ with $M > 0$ in the polydisk $\Delta(p, \rho)$, then*

$$\left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(p) \right| \leq \frac{\alpha! M}{\rho^\alpha}$$

The next important result, which is Montel's Theorem, will be used later in this thesis and its proof can be derived from Cauchy's Inequalities 1.1.3 and the Arzelà–Ascoli theorem.

Definition 1.1.6. Let $\Omega \subset \mathbb{C}^n$ be a domain, let F be a family of continuous functions on Ω, then F is said to be normal if for every sequence of functions of F, there exists a subsequence that converges compactly on Ω.

Theorem 1.1.4. *[2] [Montel's Theorem] Let $\Omega \subset \mathbb{C}^n$, let F be a family of holomorphic functions on Ω, then the following are equivalent:*

- *The family F is locally bounded.*
- *The family F is normal.*

1.2 Automorphism Groups

The main focus of this thesis will be the study of the automorphism groups of domains in $\mathbb{C}^n$. In this section we will be mainly discussing the automorphism groups of the unit ball and the unit polydisk. The automorphism groups of these 2 sets are interesting because they can be computed explicitly.

Definition 1.2.1. Let $\Omega \subset \mathbb{C}^n$, $\Omega' \subset \mathbb{C}^m$ be open. We say that $f : \Omega \to \Omega'$ is biholomorphic if f is holomorphic, bijective, and its inverse is also holomorphic.

Definition 1.2.2. Let Ω be a domain in $\mathbb{C}^n$, the automorphism group of Ω is the set of all biholomorphic transformations from Ω to Ω and it is denoted by

$$Aut\,(\Omega) = \{f : \Omega \to \Omega; f \text{ is biholomorphic}\}$$

In one variable, the concept of ball and the polydisk are both reduced to the notion of a disk. It is well known that the automorphism group is given by all the transformations of the form

$$f\,(z) = e^{i\theta}\frac{a-z}{1-\overline{a}z}$$

where a is any point in the disk, and $\theta \in \mathbb{R}$. These transformations are actually a composition of 2 transformations, one is the rotation $\left(e^{i\theta}\right)$, and the other is called Blaschke function $\left(\frac{a-z}{1-\bar{a}z}\right)$ The proof of this result is based on the Schwarz Lemma, and this result shows that the automorphism group of the unit disk is a Lie group of dimension 3 since it depends on two parameters, one parameter of real dimension 1 (θ), and another parameter of real dimension 2 (a since it is complex). Note that this group acts transitively on the disk, which means that any point in the disk can be taken to any other point. This result also implies that we can describe the automorphism group of any simply connected domain $\Omega \subsetneq \mathbb{C}$ which is due to the Riemann Mapping Theorem. Knowing this, the only automorphism group left in $\mathbb{C}$ is the one for whole $\mathbb{C}$ which can be also computed explicitly and it is given by

$$Aut\left(\mathbb{C}\right) = \{f; f\left(z\right) = az + b, a \neq 0\}$$

In several variables, we have a generalized version of the Schwarz Lemma which can be used to compute the automorphism groups of the unit ball and unit polydisk.

Theorem 1.2.1. *[3] [Schwarz lemma] Let $B_1 \subset \mathbb{C}^n$, $B_2 \subset \mathbb{C}^m$ be the unit balls with respect to the $\mathbb{C}$ homogeneous norms $||.||_1$ and $||.||_2$ respectively. Let $f : B_1 \to B_2$ be a holomorphic map such that $f\left(0\right) = 0$, then for any $z \in B_1$ we have*

$$||f\left(z\right)||_2 \leq ||z||_1$$

Moreover, if $||f(z_0)||_2 = ||z_0||_1$ *for some* $z_0 \neq 0$*, then* $||f(z)||_2 = ||z||_1$ *for all points on the line passing through* z_0 *and the origin.*

The proof of the Schwarz Lemma in several variables can be carried out in a similar way as the proof in one variable.

However, we will not use the Schwarz Lemma to compute the automorphism group of the unit polydisk, we will instead use the approach employed in [1] which is based on the following result by Cartan.

This result is of crucial importance in its own right, and later it will be applied to study the automorphism groups of more general domains in $\mathbb{C}^n$.

Theorem 1.2.2 (Cartan's Uniqueness Theorem)**.** *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded domain. Let* $p \in \Omega$*, and* $\varphi : \Omega \to \Omega$ *such that* $\varphi(p) = p$*. If* $Jac_{\mathbb{C}}\ \varphi(p) = \mathrm{Id}$*, then* φ *is the identity map.*

Proof. Assume φ is not the identity. Without loss of generality, let $p = 0$ and write φ as a power series around p, notice that the first term of the power series will be 0 since we have $\varphi(0) = 0$, and the second term will be z since $Jac_{\mathbb{C}}$ $\varphi(0) = Id$, then we have

$$\varphi(z) = z + P_k(z) + O\left(z^{k+1}\right)$$

where k is smallest positive integer such that the homogeneous polynomial P_k is not equal to 0 (such P_k exists since otherwise φ would be the identity). Now let $\varphi^j(z) = \varphi \circ ... \circ \varphi(z)$ (φ composed j times with itself). Notice that

$$\varphi^2(z) = z + 2P_k(z) + O\left(z^{k+1}\right)$$

and proceeding by induction on j, one can see that

$$\varphi^j(z) = z + jP_k(z) + O\left(z^{k+1}\right)$$

since φ is defined form Ω to Ω and Ω is bounded, then $|\varphi^j|$ is bounded ($|\varphi^j| < M$ for some $M > 0$). Let $\Delta(0,r)$ be a polydisk such that $\Delta(0,r) \subset \Omega$, then by Cauchy's Inequalities, we get that

$$j\left|\frac{\partial^{|\alpha|}\varphi}{\partial z^\alpha}(0)\right| = \left|\frac{\partial^{|\alpha|}\varphi^j}{\partial z^\alpha}(0)\right| \leq \frac{\alpha! M}{r^\alpha}$$

tending $j \to \infty$ we get a contradiction, unless P_k is equal to 0 which means $\varphi(z) = z$. □

Corollary 1.2.2.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Let $p \in \Omega$, $\varphi : \Omega \to \Omega$, and $\psi : \Omega \to \Omega$ such that $\varphi(p) = \psi(p)$. If $Jac_{\mathbb{C}}\ \psi^{-1}(\varphi(p)) = Id$, then $\varphi = \psi$.*

Proof. Just take $\phi = \psi^{-1}\varphi$ and apply the proposition above to ϕ □

We will use the above corollary in the computation of the automorphism group of the unit polydisk. However, the proof we will be using works for a more general class of domains which will be defined now.

Definition 1.2.3. let $\Omega \subset \mathbb{C}^n$ be a bounded domain. We say Ω is a bounded circular domain if for any z in Ω, we have $w = e^{i\theta}z$ is also in Ω for any $0 \leq \theta < 2\pi$. i.e. if we rotate any point in Ω it stays in Ω.

Proposition 1.2.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded circular domain containing 0. If φ is an automorphism of Ω such that $\varphi(0) = 0$, then φ is linear.*

Proof. Let ϕ_θ be the map $z \to e^{i\theta}$ for $\theta \in [0, 2\pi)$. Define

$$\psi = \phi_{-\theta} \circ \varphi^{-1} \circ \phi_\theta \circ \varphi$$

then we have

$$\psi' = \phi'_{-\theta} \circ \varphi'^{-1} \circ \phi'_\theta \circ \varphi'$$

where φ' and ϕ' are the Jacobian matrices of φ and ϕ respectively. Now ϕ'_θ and $\phi'_{-\theta}$ are diagonal matrices, so they commute with φ' and φ'^{-1}, then $\psi' = \varphi'^{-1} \circ \varphi' = Id$. By Cartan's uniqueness theorem, we get that $\psi = Id$, which means that

$$\varphi \circ \phi_\theta = \phi_\theta \circ \varphi$$

for all θ. Now express $\varphi \circ \phi_\theta$ and $\phi_\theta \circ \varphi$ as power series around 0, since $\varphi(0) = 0$ we get

$$(\varphi \circ \phi_\theta)(z) = \varphi\left(e^{i\theta} z\right) = c_{\alpha_1} e^{i\theta} z + c_{\alpha_2} \left(e^{i\theta} z\right)^2 + ...$$

and

$$(\phi_\theta \circ \varphi)(z) = e^{i\theta}(\varphi(z)) = c_{\alpha_1} e^{i\theta} z + c_{\alpha_2} e^{i\theta} (z)^2 + ...$$

by identification, we get that $c_{\alpha_j} = 0$ for all $j > 1$, which means that φ is linear. $\square$

Now that we have all we need to compute the automorphism group of the unit polydisk:

Theorem 1.2.4 (Automorphisms of the unit polydisk)**.** *We denote the automorphism group of the unit polydisk by Aut($\Delta(0,1)$). Let $\varphi \in$ Aut($\Delta(0,1)$),* *then there exists* $a_1, ..., a_n \in \Delta(0,1)$*,* $\theta_1, ..., \theta_n$ $(0 \leq \theta_i < 2\pi)$*, and a permutation* $\sigma \in S_n$ *such that*

$$\varphi(z) = \left(e^{i\theta_1}\frac{z_{\sigma(1)} - a_1}{1 - \overline{a_1}z_{\sigma(1)}}, ..., e^{i\theta_n}\frac{z_{\sigma(n)} - a_n}{1 - \overline{a_n}z_{\sigma(n)}}\right)$$

Proof. Let $\varphi(0) = \alpha = (\alpha_1, ..., \alpha_n)$ and

$$\psi(z) = \left(\frac{z_{\sigma(1)} - \alpha_1}{1 - \overline{\alpha_1}z_{\sigma(1)}}, ..., \frac{z_{\sigma(n)} - \alpha_n}{1 - \overline{\alpha_n}z_{\sigma(n)}}\right)$$

Then $f = \psi \circ \varphi \in \mathrm{Aut}(\Delta(0,1))$ with $f(0) = 0$, and it will be sufficient to prove

$$f(z) = (e^{i\theta_1}z_{\sigma(1)}, ..., e^{i\theta_n}z_{\sigma(n)})$$

Now by Proposition 1.2.3, we have that $f(z)$ is linear, then $f(z)$ can be written as an $n \times n$ matrix with entries (b_{ij}). For some positive integer $k > 1$, let

$$z^{ik} = \left(\left(1 - \frac{1}{k}\right)\overline{\mathrm{sgn}(b_{i1})}, ..., \left(1 - \frac{1}{k}\right)\overline{\mathrm{sgn}(b_{in})}\right)$$

where $\mathrm{sgn}(z)$ is given by $\mathrm{sgn}(z) = \frac{z}{|z|}$ for $z \neq 0$, and $\mathrm{sgn}(0) = 0$, then the i^{th} component of $f\left(z^{ik}\right)$ is

$$\sum_{j=1}^{n}\left(1 - \frac{1}{k}\right)|b_{ij}|$$

Since f is an automorphism of the unit polydisk, this number should be less than 1. Taking the limit as $k \to \infty$, we get that $\sum_{j=1}^{n}|b_{ij}| \leq 1$.

Now let $w^{jk} = \left(0, .., \left(1 - \frac{1}{k}\right), ..., 0\right)$ where $\left(1 - \frac{1}{k}\right)$ is the j^{th} component, then $w^{jk} \to (0, ..., 1, ..., 0)$ on the boundary of the unit polydisk as $k \to \infty$, then $f\left(w^{jk}\right)$ accumulate on the boundary of the unit polydisk, which means that $\max_i(|b_{ij}|) = 1$. So we have $(|b_{ij}|)$ is a matrix with each row summing to at most

1, and each column having at least one entry of modulus 1, which means that each column has exactly one entry of modulus 1 and all the other entries are 0.

Now for each j, let $|b_{\eta(j),j}| = 1$, if $\eta(j_1) = \eta(j_2)$,then $\sum_{j=1}^{n} |b_{ij}|$ is not less than 1 when $i = \eta(j_1) = \eta(j_2)$, so we get a contradiction. This shows that η is a permutation in S_n and $b_{ij} = 0$ for $i \neq \eta(j)$. Let $\sigma = \eta^{-1}$, then we have $|b_{l,\sigma(l)}| = 1$ and $b_{lj} = 0$ for $j \neq \sigma(l)$. Now let $b_{l,\sigma(l)} = e^{i\theta_l}$,then we get that $f(z) = (b_{lj}).(z) = \left(e^{i\theta_1} z_{\sigma(1)}, ..., e^{i\theta_n} z_{\sigma(n)}\right)$ which finishes the proof. □

Note: A different proof of the previous theorem can be achieved using the Schwarz Lemma, and can be found on page 48 of [3]

Next we will consider the automorphism group of the unit ball. We will limit ourselves to just stating the result without giving a proof. For a more detailed discussion, refer to W. Rudin, Function Theory in the Unit Ball of $\mathbb{C}^n$. [4]

In order to talk about the automorphisms of the unit ball, we need to first define the notion of a unitary matrix.

Definition 1.2.4. Let U be a complex square matrix, then U is said to be unitary if its conjugate transpose U^* is also its inverse, i.e.

$$UU^* = U^*U = I$$

where I is the identity matrix. Note that unitary matrices are the ones that preserves the standard Hermitian inner product in $\mathbb{C}^n$, so they play the same role as orthogonal matrices in $\mathbb{R}^n$. The group of these matrices is called the unitary group and it is a real Lie group of dimension n^2.

Theorem 1.2.5. *[4] Let $B \subset \mathbb{C}^n$ be the unit ball, let $a \in B$, and let P_a be the orthogonal projection to the orthogonal complement of a. Let $Q_a = I - P_a$, let $f : B \to B$ such that*

$$f_a(z) = \frac{a - P_a z - \sqrt{1 - |a|^2} Q_a z}{1 - \langle z, a \rangle}$$

Now any automorphism F of B is given by

$$F(z) = U f(z)$$

where f is of the form

$$f_a(z) = \frac{a - P_a z - \sqrt{1 - |a|^2} Q_a z}{1 - \langle z, a \rangle}$$

and U is an $n \times n$ unitary matrix.

Remark: The functions f above play the role of the Blaschke map and the unitary matrices play the role of the rotations in $\mathbb{C}$.

As a by-product of the explicit computation of the automorphism group of the unit poly-disk and the unit ball in $\mathbb{C}^n$, we can compare the dimensions of these two domains and notice that they are different, which is enough to see that these two domains are not biholomorphic.

As noted before $\Delta(0, 1)$ can be expressed as a product on n unit disks $\mathbb{D} \times \ldots \times \mathbb{D}$ and the automorphsim group of each unit disk depends on 2 parameters θ which is real, and a which is complex, then $\mathrm{Aut}(\mathbb{D})$ is a Lie group of dimension 3, which means that $\Delta(0, 1)$ is of dimension $3n$.

However, the automorphism group unit ball depends on parameter a, which is complex, and the unitary group U, then the dimension of $\mathrm{Aut}(B^n)$ is $2n$ plus the dimension of U which is n^2, which means that the dimension of $\mathrm{Aut}(B^n)$ is $n\,(n+2)$.

We conclude that the dimensions of the automophism groups of the unit ball and unit polydisk is $\mathbb{C}^n$ are not equal, which means that the groups are not isomorphic, and this is enough to show that the unit polydisk and the unit ball in $\mathbb{C}^n$ are not biholomorphic.

Chapter 2

Elements of Lie Theory

In this chapter we will include everything we need to know for this paper about vector fields and Lie theory.

2.1 Basic Definitions and Results

In the study of Lie theory we will be mostly concerned with real analytic manifolds and functions. For convenience, we recall the standard definition of real analytic functions.

A function f is real analytic if it can be expressed locally as a convergent power series. More precisely:

Definition 2.1.1. Let D be an open subset of $\mathbb{R}^n$, then f is real analytic on D if for any $x_0 \in D$, there exists a neighborhood U of x_0 and a sequence P_n of homogeneous polynomials of degree n such that for all x in U

$$f(x) = \sum_{n=0}^{\infty} P_n(x - x_0)$$

Definition 2.1.2. Let G be a group that is also a real analytic manifold. G is called a Lie group if the mapping

$$G \times G \to G$$

given by $(x, y) \to xy^{-1}$ is real analytic.

Some examples of Lie groups are:

1) $(\mathbb{R}^n, +)$ is a Lie group with respect to the addition $(x, y) \to x + y$.

2) The general linear group $GL(n, \mathbb{R})$, which is the group of all invertible matrices with entries in $\mathbb{R}$, with respect to matrix multiplication is a Lie group.

3) The automorphism groups of the polydisk, ball, and $\mathbb{C}$ are also Lie groups with respect to composition.

A very important notion for the study of Lie groups is the notion of vector fields. Indeed historically the concept of Lie groups was arising as an outcome of the study of flows of multiple vector fields.

Definition 2.1.3. A vector field on an open set $\Omega \subset \mathbb{R}^n$ is an n-tuple $V = (v_1, ..., v_n)$ where v_j is a real valued function. V is of class C^k if every v_j is C^k.

Another standard point of view in the study of vector fields is to regard them as differential operators acting on smooth functions.

If f is in $C^\infty(\Omega)$, we define

$$V(f)(x) = \sum_{j=1}^{n} v_j(x) \frac{\partial f}{\partial x_j}(x)$$

for x in Ω.

Because of that, common notation for vector fields is

$$V = \sum_{j=1}^{n} v_j(x) \frac{\partial}{\partial x_j}$$

The previous identification of vector fields with first order differential operators allows us to define the bracket of vector fields.

Definition 2.1.4. Let V, U be two C^2 vector fields, then the bracket $W = [V, U]$ of the vector fields V and U is defined by

$$W(f) = V(U(f)) - U(V(f))$$

for f in $C^\infty(\Omega)$.

A direct computation shows that the bracket is indeed a first order differential operator $W = (w_1, ..., w_n)$ with $w_j = \sum_{k=1}^{n}(v_k \frac{\partial u_j}{\partial v_k} - u_k \frac{\partial v_j}{\partial u_k})$

We shall need the following result from the theory of ordinary differential equations but the proof of it is beyond the scope of this paper.

Proposition 2.1.1. *Let Ω be an open subset of $\mathbb{R}^p$ and Ω_0 be an open relatively compact subset of Ω. Let V be a real analytic vector field on Ω, then there exists*

$\tau > 0$ and a unique real analytic map $g = g_V : \Omega_0 \times I \to \Omega$ where $I = \{t \in \mathbb{R}; |t| < \tau\}$ such that

$$\frac{\partial g(x,t)}{\partial t} = V(g(x,t))$$

with $g(x,0) = x$, $x \in \Omega_0$, and $t \in I$. (the notation of V is defined here as in Definition 2.1.3)

Note: For all $f \in C^\infty(\Omega)$, we have:

$$V(f)(x) = \lim_{t\to 0} \frac{f(g_v(x,t) - f(x)}{t}$$

We call $g = g_V$ the local one-parameter group associated to the vector field V.

The above proposition is a standard result from the theory of ordinary differential equations and it is used on one vector field only. However, since we will be dealing with families of vector fields, it is useful to have a parametric version of the above result.p

Proposition 2.1.2. *Let Ω be an open subset of $\mathbb{R}^p$. If U is open in $\mathbb{R}^q$ and $V : \Omega \times U \to \mathbb{R}^p$ is real analytic, then for U_0 relatively compact in U, there exist $\tau > 0$ such that the following holds: For $\alpha \in U$, let $V_\alpha(x) = V(x, \alpha), x \in \Omega$. Then there exists an analytic map*

$$g : \Omega_0 \times I \times U_0 \to \Omega$$

such that the map $g_\alpha : \Omega_0 \times I \to \Omega$ defined by

$$g_\alpha(x,t) = g(x,t,\alpha)$$

satisfies $g_\alpha = g_{V_\alpha}$.

Moreover, if $t, s, t+s \in I$, *and* $\alpha \in U_0, x \in \Omega_0, g_\alpha \in \Omega_0$, *we have*

$$g_\alpha(g_\alpha(x,t), s) = g_\alpha(x, t+s)$$

The proof of the above propositions can be found in [5]

The above proposition is used to study families of vector fields, a particular important type of families that we study in Lie theory are Lie algebras, which we will define now.

Definition 2.1.5. Let V be a finite dimensional vector space of vector fields on an open set $\Omega \subset \mathbb{R}^p$. We say V is a Lie algebra of vector fields if whenever $U, W \in V$, we have $[U, W] \in V$.

From now on, we will assume that the vector fields belonging to V are real analytic.

2.2 Lie's Theorem

We shall use the following result from classical Lie theory.

Theorem 2.2.1 (Lie's Theorem)**.** *[6] Let* V *be a finite dimensional Lie algebra of real analytic vector fields on an open connected set* $\Omega \subset \mathbb{R}^p$. *Let* Ω_0 *be a relatively compact subset of* Ω, *then there exist a neighborhood* U *of the zero vector field in* V *and a real analytic map*

$$g : \Omega_0 \times U \to \Omega$$

with the following properties: Let $g_u : \Omega_0 \to \Omega$ *be the map* $x \to g(x, u)$

- *For* u, v *sufficiently near* 0*, there exists a unique* $w = w(u, v) \in U$ *such that*

$$g_u \circ g_v = g_w \text{ on } \Omega_0 \cap g^{-1}(\Omega_0)$$

- *For* $u \in U$*, the map* $t \to g_{tu}$ $(t \in \mathbb{R})$ *is the one-parameter group associated to the vector field* u

- *For* u_0, v_0 *sufficiently near* 0 *the maps*

$$u \to w(u, v_0), \quad v \to w(u_0, v)$$

  are analytic isomorphisms of a neighborhood of the zero vector field in V *onto neighborhoods of* v_0, u_0

The map g is the above theorem is called the local Lie group of transformations associated to V.

Note that if we integrate one vector field, we will get back a Lie group locally. However, if we integrate a family of vector fields, in general we will not get back a Lie group locally. The above theorem gives us that if the family of vector fields we are working on is a Lie algebra, then the integration will give us a Lie group locally.

Chapter 3

Proof of Cartan's Theorem

3.1 Groups of Transformations

In his paper, Cartan works in a more general setting. He does not restrict to groups of holomorphic automorphisms, but introduces a notion of local groups of transformations which are not automorphisms but for which it is still possible to define the concept of composition.

Note: For simplicity, the finite dimensional space E we will be working on is going to be $\mathbb{R}^n$ or $\mathbb{C}^n$ with the norm being the standard euclidean norm. However, all the results proved will also work for domains of general complex manifolds in which we can define a distance function in an arbitrary way.

First of all, we start by defining what we mean by a transformation and the notion of distance between two transformations that will be employed.

Definition 3.1.1. Let $D \subset E$ be open. A transformation φ on D is a continuous map from D to E.

Definition 3.1.2. Let G be a set of transformations defined on a domain $D \subset E$, let Δ be a relatively compact subset of D, then the pseudo-distance between two transformations g and g' is defined as $\sup_{p \in \Delta} |g(p) - g'(p)|$ which is the sup norm of $|g - g'|$ on the set Δ.

Remark: This definition does not give a proper distance over the class of continuous transformations since after fixing Δ (arbitrarily) one can have a continuous map which is zero on Δ and non-zero elsewhere, so this will be a non-zero transformation but has a distance (according to the above definition) which is zero with the zero function. This means that one can have two distinct elements having distance zero which contradicts the definition of distance, and that is why it is called a pseudo-distance.

a base for the topology will be given by the set of the pseudo-metric balls (of the form $\{g \in G; d(g, g_0) < \epsilon\}$ for fixed $\epsilon > 0$, $g_0 \in G$, where d is the pseudo-distance defined in Definition 3.1.2)

Now that we defined the topology and the distance on a set of transformation, we can proceed by introducing the groups of transformations Cartan works on. These groups of transformations are sets satisfying the following three properties.

Property (a): Let E be a finite dimensional space, and let $D \subset E$ be a domain. A set of transformations G defined and continuous on D satisfies property (a) if given 2 domains Δ and Δ' relatively compact in D, then

$\forall\ \epsilon > 0$, $\exists\ \epsilon' > 0$ (depending on ϵ and Δ) with $\sup_{p\in\Delta'} |\varphi(p) - p| < \epsilon'$ whenever $\sup_{p\in\Delta} |\varphi(p) - p| < \epsilon$.

Remark: Given property (a) the pseudo-distance given in Definition 3.1 becomes a proper distance, and the topology generated by that distance will be the same topology of compactly convergent maps that was defined in Definition 1.1.5. So the topology on the set of transformations G is the topology of compactly convergent maps in Definition 1.1.5.

Property (b): There exists $\overline{G}$ a neighborhood of the identity transformation, such that for all Δ relatively compact in D, there exists $\eta > 0$ for all $\varphi \in G$ such that $\sup_{p\in\Delta} |\varphi(p) - p| < \eta$ we have:

- $\varphi \in \overline{G}$
- if $p \in \Delta$, then $\varphi(p) \in D$
- $\forall\ \psi \in \overline{G}$, $\exists\ \psi \cdot \varphi \in G$, such that $\psi \cdot \varphi(p) = \psi(\varphi(p))\ \forall\ p \in \Delta$

Remark: $\psi.\varphi(p)$ is defined on the whole of D here but is equal to $\psi(\varphi(p))$ only for $p \in \Delta$ and $\psi \in \overline{G}$.

Property (c): There exists $\overline{\overline{G}} \subset \overline{G}$, such that for all $\varphi \in \overline{\overline{G}}$, there exists $\varphi^{-1} \in \overline{G}$ with $\varphi \cdot \varphi^{-1}(p) = \varphi^{-1} \cdot \varphi(p) = p \quad \forall\ p \in \Delta$

Another property which will be useful is pproperty (a'), since it provides a connection between groups of holomorphic transformations, and the groups satisfying the above properties.

Property (a'): Let G be a set of transformations on a domain D, then G is uniformly bounded on all compact subsets of D. *i.e* for all Δ relatively compact in D, there exists a ball B of finite radius such that $\forall\, p \in \Delta$, $\forall\, \varphi \in G$, we have $\varphi(p) \in B$.

If one is dealing with holomorphic transformations, then property (a') $\implies$ property (a).

Proof. Assume property (a) is not satisfied. Let Δ and Δ' be two relatively compact domains in D, let $\{g_n\}$ be a sequence in G such that

$$\sup_{p\in\Delta} |g_n(p) - p| < \frac{1}{n}$$

and

$$\sup_{p\in\Delta'} |g_n(p) - p| = 1$$

but G is uniformly bounded in D, then by Montel's Theorem, there exists a subsequence $\{g_{n_k}\}$ that converges to some holomorphic function g on all compact subsets of D, in particular on Δ. However, $\{g_n\}$ converges to the identity on Δ, so g is the identity transformation on Δ, and g is holomorphic on D, then by analytic continuation g is the identity transformation on D, and since g_n converges to g alos on Δ', this contradicts the fact that $\sup_{p\in\Delta'} |g_n(p) - p| = 1$. So (a) should be satisfied whenever (a') is satisfied. □

Definition 3.1.3. A set of transformation G, defined on D, is a local group of transformations (in the sense of Cartan) if it satisfies properties ***(a) (b) (c)***

3.2 Quasi-Continuous Groups of Transformations

The notion of quasi-continuous groups plays a crucial role in the results of Cartan. In this section we will start by defining what is a quasi-continuous group of transformations and then discuss some results related to these groups which were proved by Cartan.

Definition 3.2.1. Let G be a group of transformations on $\mathbb{R}^q$, defined and continuous in D. We say G is quasi-continuous of order at most q if we can establish between elements in the neighborhood of the identity transformation and a closed bounded subset of $\mathbb{R}^q$, denoted by Σ, a bijection which is bi-continuous.

One of these results is that the automorphism group of bounded domain in the n-dimensional complex plane turns out to be a quasi-continuous group.

Theorem 3.2.1. *Let $D \subset \mathbb{C}^n$ be a bounded domain, then the group G of all biholomorphic transformations from D to D (automorphism group) is quasi-continuous of order at most $2n(n+1)$*

Proof. Let $S \in G$, S=$(S_1, ..., S_n)$ where each f_i is a holomorphic function in n variables. Let $p \in D$, then we associate to each element $S \in G$, the element in $\mathbb{C}^n \times \mathbb{C}^{n^2}$ given by $S(p)$ (in $\mathbb{C}^n$), and $Jac_p(S)$ (in $\mathbb{C}^{n^2}$). Since $\mathbb{C}^n \times \mathbb{C}^{n^2} = \mathbb{C}^{n(n+1)} \simeq \mathbb{R}^{2n(n+1)}$, then this map associates for all $S \in G$ and point $m \in \mathbb{R}^{2n(n+1)}$. Now let B be a closed ball of center p, let $G_B \subset G$ be the set of transformations such that $S(p) \in B$. Let ϕ: $G_B \to \mathbb{R}^{2n(n+1)}$ with $\phi(S) = q$ for $S \in G_B$ and $q \in \mathbb{R}^{2n(n+1)}$.

Claim 1: ϕ is one-to-one.

Proof. let g_1, $g_2 \in G_B$ such that $\phi(g_1) = \phi(g_2)$, then $g_1(p) = g_2(p)$ and $Jac_p(g_1) = Jac_p(g_2)$, which means $g_2^{-1} \circ g_1(p) = p$ and $Jac_p(g_2^{-1} \circ g_1) = Id$. By Cartan Uniqueness Theorem 1.2.2, we have $g_2^{-1} \circ g_1 = Id$, so $g_1 = g_2$, which shows that

$$\phi : G_B \to \mathbb{R}^{2n(n+1)}$$

is one-to-one. □

Now since D is bounded, then there exists a constant $M > 0$ such that $|g(p)| < M$ for all $g \in G_B$, and by Cauchy's inequality we have $|\frac{\partial g_i}{\partial z_j}(p)| \leq \frac{M}{r}$ where r is the radius of the largest polydisk of center p contained in D, then we can deduce that $Jac_p(g)$ is bounded, which means $\phi(G_B)$ is bounded.

Claim 2: $\phi(G_B)$ closed.

Proof. Let $\{q_n\}$ be a sequence in $\phi(G_B)$, let q be a limit point of $\{q_n\}$ and take the sequence of transformations $\{g_n\}$ corresponding to $\{q_n\}$. Since D is bounded then g_j is bounded for all j. By Montel's Theorem, there exists a subsequence $\{g_{n_k}\}$ which converges to some $g \in G_B$, so $\{q_{n_k}\}$ converges to some q', but $\{q_n\}$ converges to q, then $q' = q$, so $\phi(G_B)$ contains all its limit points, which means $\phi(G_B)$ is closed. □

Claim 3: ϕ is bi-continuous.

Proof. Suppose ϕ^{-1} is not continuous, so there exists a sequence $\{g_n\}$ that does not converge g and $\{q_n\}$ that converges to $q = \phi(g)$, this means that there exists a subsequence $\{g_{n_k}\}$ that contains no elements in some neighborhood of q. However, by Montel's Theorem, there exists a subsequence $\{g_{n_{k_j}}\}$ that converges to some g', but $\{q_{n_{k_j}}\}$ converges to q, then $g'(p) = g(p)$ and $Jac_p(g') = Jac_p(g)$, so by Cartan's Uniqueness Theorem, $g' = g$, which leads to a contradiction. So ϕ is continuous. $\square$

To conclude the theorem, we showed that $\phi(G_B)$ is closed and bounded, and ϕ: $G_B \to \mathbb{R}^{2n(n+1)}$ is one-to-one $\implies$ ϕ: $G_B \to \phi(G_B)$ is a bijection onto a closed bounded subset of $\mathbb{R}^{2n(n+1)}$. We also showed that ϕ: $G_B \to \phi(G_B)$ is continuous, which proves that G is quasi-continuous. $\square$

3.3 Infinitesimal Transformations

In this section, the analytic groups of transformations are considered from a local point of view (Definition 3.1.3).

Definition 3.3.1. Let G be a group of analytic transformations. We say that G admits an infinitesimal transformation

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t}(p,t) = \psi(\Psi(p,t))$$

where ψ is analytic, if G contains the transformations generated by this infinitesimal transformation for $|t| < \tau$ for some $\tau > 0$

Remark: In the above definition, if we let $p = (x_1, ..., x_n)$ for $x_j \in \mathbb{R}^n$ or $\mathbb{C}^n$, the map $\psi = (\psi_1, ..., \psi_n)$, where ψ_j is real analytic or holomorphic, can be seen as a vector field $\psi = \sum_{j=1}^{n} \psi_j \partial x_j$, and then $\dfrac{\mathrm{d}\Psi}{\mathrm{d}t} = \psi(\Psi(p,t))$ represents the system of ODEs $\frac{\partial \Psi_j(p,t)}{\partial t} = \psi_j(\Psi(p,t))$ with initial condition $\Psi(p,0) = p$ where $\Psi = (\psi_1, ..., \psi_n)$. Now for every Δ relatively compact in D, the classical theorems of existence tells us that there exists $\tau > 0$ such that this system admits a unique solution defined for $p \in \Delta$ and $|t| < \tau$. For the case where ψ is real analytic, $\Psi(\cdot, t)$ is actually the flow of the vector field ψ.

After we defined what it means for a group of analytic transformations to admit an infinitesimal transformation G, we can introduce the following theorem whose proof can be found in [7]

Theorem 3.3.1. *[7] Let ψ_1, ..., ψ_k be analytic functions in D which are $\mathbb{R}$-linearly independent. Denote by*

$$\Psi(p, a_1 t, a_2 t, ..., a_k t)$$

the 1-parameter group generated by the infinitesimal transformation

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t}(p,t) = a_1\psi_1(\Psi(p,t)) + a_2\psi_2(\Psi(p,t)) + ... + a_p\psi_k(\Psi(p,t))$$

Let $t_i = a_i t$. Let Δ_0 be a relatively compact subset of D, and suppose Ψ is analytic with respect to p and t_i, for p in the interior of Δ_0 and $|t_i| < \tau$. Suppose that for $|t_i| < \tau$, $\Psi(p, t_1, ..., t_p)$ forms a lie group on the domain D. Now let G be a group of analytic transformations in D. Suppose that $\Psi(p, t_1, ..., t_p) \in \overline{G}$ for $|t_i| < \tau' \leq \tau$, then Ψ is analytic with respect to p, t_i for p in the interior of D and $|t_i| < \tau'$.

Now we will indicate a sufficient condition for a group of analytic transformations G to admit an infinitesimal transformation.

Theorem 3.3.2. *Let G be a locally closed group of analytic transformations. If there exists a sequence $\{S_n\}$ of analytic transformations ($S_i \in G$) that converges to the identity transformation, and a sequence $\{m_k\}$ ($m_i > 0, m_i \in \mathbb{Z}$) such that $m_k(S_k(p) - p) = \psi_k(p)$ converges uniformly on compact subsets of D to an analytic transformation $\psi(p) \neq 0$, then G admits an infinitesimal transformation* $\dfrac{\mathrm{d}\Psi}{\mathrm{d}t}(p,t) = \psi(\Psi(p,t))$

Proof. Let G be a neighborhood of the identity transformation. Let Δ and Δ' be two relatively compact subsets of D such that $\Delta \subset \Delta'$, then $\exists\ r > 0$, $r < min(\eta, d(\Delta, \Delta'))$, where η is coming from property (b) applied to Δ', and $d(\Delta, \Delta')$ is the distance between the boundaries of Δ and Δ', then for all S and T in G with $|S(p) - p| < r, |T(p) - p| < r$, we have $TS(p) = T(S(p))$ for all p in Δ. Let $\Psi(p,t)$ be the 1-parameter group generated by $\dfrac{\mathrm{d}\Psi}{\mathrm{d}t} = \psi(\Psi(p,t))$, then by the classical theorem of existence, Ψ is analytic with respect to p and t, for p in the interior of Δ and $|t_i| < \tau$ for some fixed $\tau > 0$. Now let $\mathrm{A} = \max|\psi(p)|$ in Δ', $\rho = \min(\frac{r}{A}, \tau)$ and $0 < t_0 < \rho$. Let $q_k \in \mathbb{Z}$ such that $|q_k - m_k t_0| < 1$ (q_k is the smallest integer $\leq m_k t_0$), then we have $|\frac{t_0}{q_k} - \frac{1}{m_k}| < \frac{B}{m_k^2}$ for some constant B independent of k.

Claim 1: for a fixed k large enough, $S_k \circ ... \circ S_k$ (q_k times) exist and belongs to G.

Proof. Since $m_k(S_k(p)-(p)) = \psi_k$ and ψ_k converges uniformly to ψ, so $\psi_k = \psi+\eta'_k$ where $\eta'_k \to 0$ as $k \to \infty$, then we have

$$S_k(p) - p = \frac{1}{m_k}(\psi(p) + \eta_k(p))$$

and since $|\frac{t_0}{q_k} - \frac{1}{m_k}| < \frac{B}{m_k^2}$, then

$$S_k(p) - p = \frac{t_0}{q_k}(\psi(p) + \eta'_k(p))$$

where $\eta'_k \to 0$ as $k \to \infty$, and we have $A = \max(\psi_k)$, $t_0 < \rho$, $\rho < \frac{r}{A}$, then as $k \to \infty$ we get

$$|S_k(p) - p| < \frac{\rho}{q_k}A < \frac{rA}{Aq_k} < \frac{r}{q_k}$$

Let $p_i = S_k^i(p)$, then we have

$$|S_k(p_2) - p_2| < \frac{r}{q_k}$$

and so

$$|S_k^2(p) - S_k(p)| < \frac{r}{q_k}$$

Now

$$|S_k^2(p) - S_k(p) + S_k(p) - p| < |S_k^2(p) - S_k(p)| + |S_k(p) - p| < \frac{2r}{q_k}$$

doing this q_k times, we get

$$|S_k^{q_k}(p) - p| < \frac{q_k r}{q_k} = r$$

and since $r < min(\eta, d(\Delta, \Delta')$, then $S_k^{q_k} \in G$

□

Claim 2: $S_k^{q_k}$ converges uniformly to $\Psi(p, t_0)$ for $p \in \Delta$.

Proof. Ψ is analytic, so

$$\Psi(p,t) - p = t(\psi(p) + \eta''(p,t))$$

For $t = \frac{t_0}{q_k}$,

$$\Psi(p, \frac{t_0}{q_k}) - p = \frac{t_0}{q_k}(\psi(p) + \eta''(p, \frac{t_0}{q_k})$$

and since

$$|S_k(p) - p| = \frac{t_0}{q_k}(\psi(p) + \eta'(p))$$

then

$$|\Psi(p, \frac{t_0}{q_k}) - S_k(p)| < \frac{\epsilon_k}{q_k}$$

where $\epsilon_k = t_0(\eta'' - \eta') \to 0$ as $k \to \infty$. Now let p and p' in Δ', we have

$$|\Psi(p', t) - \Psi(p, t)| < |p' - p| + t|\psi(p') + \eta''(p') + \psi(p) + \eta''(p)|$$

and $\psi(p) < A$, then

$$|\Psi(p', t) - \Psi(p, t)| < |p' - p|(1 + Ct)$$

Let

$$p_1 = S_k(p) \quad p_1' = \Psi(p, \frac{t_0}{q_k})$$

$$...$$

$$p_{i+1} = S_k(p_i) \quad p_{i+1}' = \Psi(p_i', \frac{t_0}{q_k})$$

$$...$$

$$p_{q_k} = S_k(p_{q_k-1}) \quad p'_{q_k} = \Psi(p'_{q_k-1}, \frac{t_0}{q_k})$$

then

$$|p'_{i+1} - p_{i+1}| = |p'_{i+1} - \Psi(p_i, \frac{t_0}{q_k}) + \Psi(p_i, \frac{t_0}{q_k}) - p_{i+1}| < |p'_{i+1} - \Psi(p_i, \frac{t_0}{q_k})| + |\Psi(p_i, \frac{t_0}{q_k}) - p_{i+1}|$$

$$|p'_{i+1} - p_{i+1}| < \frac{\epsilon_k}{q_k} + |p'_i - p_i|(1 + C\frac{t_0}{q_k})$$

We have that $|p'_1 - p_1| < \frac{\epsilon_k}{q_k}$, then

$$|p'_2 - p_2| < \frac{\epsilon_k}{q_k} + \frac{\epsilon_k}{q_k}(1 + C\frac{t_0}{q_k})$$

$$\dots$$

$$|p'_i - p_i| < \frac{\epsilon_k}{q_k} + \frac{\epsilon_k}{q_k}(1 + C\frac{t_0}{q_k})^{i-1}$$

then

$$|p'_{q_k} - p_{q_k}| < \frac{\epsilon_k}{q_k}(1 + C\frac{t_0}{q_k})^{q_k-1} < \epsilon_k(1 + C\frac{t_0}{q_k})^{q_k} < \epsilon_k e^{Ct_0}$$

but $\epsilon_k \to 0$ as $k \to \infty$, then $|p'_{q_k} - p_{q_k}| \to 0$ as $k \to \infty$. $\square$

Now since G is closed, then it contains all limit points, so $\Psi(p, t_0) \in G$ which finishes the proof. $\square$

Now that we know when a group of analytic transformations admits an infinitesimal transformation, we will give the following results which will help provide a link between such groups and Lie groups.

The proof of the two following theorem can be found in [7]

Theorem 3.3.3. *[7] If a locally closed group of analytic transformations admits two infinitesimal transformations* $\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \psi_1$, $\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \psi_2$, *then it admits the bracket and any linear combination of* ψ_1 *and* ψ_2.

Corollary 3.3.3.1. *If a locally closed group of analytic transformations admits* k *infinitesimal transformations* $\psi_1, ..., \psi_k$, *then it admits any bracket and any linear combination between them.*

Theorem 3.3.4. *[7] Let* G *be a locally closed group of analytic transformations. Suppose* G *admits the infinitesimal transformations* $\psi_1, \psi_2, ..., \psi_k$. *Denote by* $\Psi(p, a_1 t, ..., a_k t)$ *is the 1-parameter group generated by*

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t}(p,t) = a_1\psi_1(\Psi(p)) + ... + a_k\psi_k(\Psi(p))$$

and $\Delta \subset D$ *is a domain, then* $\Psi(p, t_1, ..., t_k)$ *is analytic with respect to* p *and* t_i *(*$t_i = a_i t$*) for* p *in the interior of* Δ *and* $|t_i| < \tau$ $\forall i$.
Also there exist u *with* $0 < u \leq \tau$ *such that the transformation* $\Psi(p, t_1, ...t_k) \in G$ *when* $|t_i| < u$.

Theorem 3.3.5. *Let* G *be a group of analytic transformations. If* G *is quasi-continuous of order at most* q, *then* G *cannot admit more than* q *infinitesimal transformations which are linearly independent.*

Proof. Assume G admits k infinitesimal transformations $\psi_1, ..., \psi_k$, and let us prove $k \leq q$. By Theorem 3.3.4, G admits a family of infinitesimal transformations

$$\Psi(p, t_1, ..., t_k)$$

for $|t_i| < u$.

Now let $v > 0$, consider the transformations $\Psi(p, t_1, ..., t_k)$ for $|t_i| < u$ and $\Psi(p, t'_1, ..., t'_k)$ for $|t'_i| < v$, for $u \neq v$, these two transformations are distinct in Δ. But G is quasi-continuous of order at most q, then by the definition of quasi continuity, there exists a map $\phi : G \to \mathbb{R}^q$ which is injective. Let $f : \mathbb{R}^k \to G$, and consider the injective map $\phi \circ f : \mathbb{R}^k \to \mathbb{R}^q$. By the classical theorem of Invariance of Domain [8], if U is a subset of $\mathbb{R}^k$, then $(\phi \circ f)(U)$ is open in $\mathbb{R}^q$, but if $q < k$ $(\phi \circ f)(U)$ is a proper subset of $\mathbb{R}^k$ and thus can't be open, which means we have $k \leq q$.

$\square$

Theorem 3.3.6. *If a quasi-continuous group of analytic transformations (of order at most q) in a domain D admits at least one infinitesimal transformation, then the set of all infinitesimal transformations*

$$\Psi(p, t_1, ..., t_k)$$

of G generate a local Lie Group Γ.

Moreover, there exists $u > 0$ such that:

- *The function Ψ is analytic with respect to $p, t_1, ..., t_k$ for p in the interior of D and $|t_i| < u$*
- *The transformation $\Psi(p, t_1, ..., t_k) \in G$ (or $\overline{G}$) for $|t_i| < u$*

Proof. By Theorem 3.3.5 and Theorem 3.3.3, all the infinitesimal transformations of G are a linear combination of a finite number of linearly independent infinitesimal transformations $\psi_1, ..., \psi_k$ $(k < q)$. By Theorem 3.3.4 applied to the group $\overline{G}$, we get that $\overline{G}$ contains a family of transformations

$$\Psi(p, t_1, ..., t_k)$$

which are analytic for $p \in \Delta$ and $|t_i| < u$. By Theorem 3.3.3, the bracket of any 2 transformations of $\psi_1, ..., \psi_k$ is in G, which means these brackets themselves are linear combinations of $\psi_1, ..., \psi_k$, so this family is actually a Lie algebra. Now since the family of transformations

$$\Psi(p, t_1, ..., t_k)$$

is a Lie algebra, then by Lie's Theorem 2.2.1, it generates a Lie group in the domain Δ. Applying Theorem 3.3.1, we get that $\Psi(p, t_1, ..., t_k)$ is analytic with respect to p in the interior of D, and $|t_i| < u$ which finishes the proof. $\square$

3.4 Analytic Transformations and Lie Groups

In this section we will be proving the main result of this paper which states that if D is a bounded domain in $\mathbb{C}^n$, then the automorphism group of D is a Lie group.

We start by introducing a new property, called property [P], which will provide a connection between groups of analytic transformations and Lie groups.

Definition 3.4.1. We say that a group of transformations G satisfies property [P] if for all sequences $\{S_k\}$ with elements in G, such that $\{S_k\} \to Id$ with $S_i \neq Id$ for all i, there exists a subsequence $\{S_{n_k}\}$, and a sequence $m_i (m_i \in \mathbb{N})$ such that $m_i[S_{k_i}(p) - p]$ converges compactly to $\psi(p)$ where $\psi \neq 0$ and is analytic on D.

Any Lie group G of analytic transformations (of dimension k) satisfies property [P], since for every S_k in G, one can choose an element p_k in $\mathbb{R}^k$ which corresponds to a vector field whose flow at $t = 1$ gives S_k, as $S_k \to Id$, $p_k \to 0$, and working on p_k gives us property [P] by taking m_i to be the best approximation of $\frac{1}{|p_i|}$. However the converse is not always true.

Now that we defined what property [P] is, it turns out that a group of holomorphic transformations satisfies property [P], but before proving that, we will need some preliminary results.

Lemma 3.4.1. *Let Δ, Δ' be two polydisks in $\mathbb{C}^n$ of radii ρ, ρ' respectively and $\rho' < \rho$. Let $u, v \in \mathbb{R}^+$ with $u < 1 < v$, then there exists $\alpha > 0$ that satisfies the following property: if for all holomorphic transformation T on Δ and for all $q \in \mathbb{N}$ with*

$$\sup_{p \in \Delta} |T^i(p) - (p)| \leq \alpha$$

for $i < q$, T^q is holomorphic on Δ, and $p \in \Delta'$, then

$$\frac{u}{q}|T^q(p) - p| \leq |T(p) - p| \leq \frac{v}{q}|T^q(p) - p|$$

on Δ'.

Proof. Let Δ'_1 be a polydisk of radius ρ'_1 such that $\rho' < \rho'_1 < \rho$, then there exists $\beta > 0$ such that if for all S holomorphic in Δ and $|S(p) - p| < \beta$, then for all $p_1, p_2 \in \Delta'_1$ we have

$$u < \frac{|p_1 - p_2|}{|S(p_1) - S(p_2)|} < v \tag{3.1}$$

This is true because S is close to the identity, and S is holomorphic, so by Cauchy's inequalities, the derivative of S will also be close to the identity, and this will result in S being Lipschitz with constant very close to 1, *i.e* $|S(p_1) - S(p_2)| < \frac{1}{u}|p_1 - p_2|$ (a similar argument applied to the inverse of S will result in $|p_1 - p_2| < v|S(p_1) - S(p_2))$.

Now let $\alpha = \min(\beta, \rho'_1 - \rho')$. Suppose T and q satisfies condition of Lemma 3.4.1 and let $S(p) = \frac{1}{q}(p + T(p) + ... + T^{q-1}(p))$, then $|S(p) - p| \leq \alpha$ in Δ. Moreover if $p \in \Delta'$, then $T(p) \in {\Delta'_1}^o$ since $\alpha \leq \rho'_1 - \rho'$. Now applying (3.1) to S where $p_1 = p, p_2 = T(M)$ we get

$$u < \frac{|T(p) - p|}{|S(T(p)) - S(p)|} < v$$

but

$$S(T(p)) - S(p) = \frac{1}{q}(T^q(p) - p)$$

so rearranging all the terms, one gets

$$\frac{u}{q}|T^q(p) - p| < |T(p) - p| < \frac{v}{q}|T^q(p) - p|$$

which finishes the proof. □

Lemma 3.4.2. *Let* Δ *be a polydisk in* $\mathbb{C}^n$ *then there exists* $\alpha > 0$ *such that if* T *holomorphic in* Δ *and* $|T^i(p) - p| < \alpha$ *in* Δ *for all* i*, then* T *is the identity.*

Proof. Let Δ, Δ' be two polydisks with the same center and radii r and r' respectively, with $r' < r$. Let $u, v \in \mathbb{R}^+$ such that $u < 1 < v$ and choose α in the same way as Lemma 3.4.1. If T satisfies the condition of Lemma 3.4.2 then we have

$$|T(p) - p| \leq \frac{v}{q}|T^q(p) - p| < \frac{v\alpha}{q}$$

Let $q \to \infty$ then $T \to Id$ on Δ'. But T is analytic, so by analytic continuation, $T(p) = p$ on Δ.

$\square$

Corollary 3.4.2.1. *A group of holomorphic transformations can not admit arbitrarily small subgroups (there exists* $\epsilon > 0$ *such that* $N_\epsilon(Id)$ *does not contain a proper subgroup).*

Now that we have all we need to proceed with the proof of the following theorem.

Theorem 3.4.3. *A group of holomorphic transformations satisfies property [P].*

Proof. Let D be a domain, let G be a group of holomorphic transformations in D. We need to show that if T_k converges to identity and $T_i \neq Id$, then there exists T_{k_i}, m_i such that $m_i(T_{k_i}(p) - p)$ converges compactly to $\psi(p)$ where $\psi \neq 0$. It is enough to show that we can assign for all $T \in G$ an integer $q_T > 0$ such that the family $q_T(T(p) - p)$ is

- Uniformly bounded on all Δ relatively compact in D
- The zero transformation is not a limit point of this family

since due to Montel's Theorem, there will exist a subsequence of this family which will converge to some ψ, and since the zero transformation is not a limit point of this family then we can be sure that $\psi \neq 0$. Now let Δ, Δ' be 2 polydisks with the same center such that $\Delta' \subset \Delta$. Let $u, v \in \mathbb{R}^+$ such that $u < 1 < v$. Let $\alpha' = \min(\alpha, \eta(\Delta))$ (α from Lemma 3.4.1). If there exists $p \in \Delta$ such that

$$|T(p) - p| \geq \alpha'$$

i.e the $\sup_{p \in \Delta} |T(p) - p| \geq \alpha'$ we let $q_T = 1$. If

$$|T(p) - p| < \alpha'$$

then chose q_T such that

$$|T^i(p) - p| < \alpha'$$

$\forall i < q_T$ and $|T^{q_T}(p) - p| \geq \alpha'$. Note that such q_T exists by Lemma 3.4.2 since $T \neq Id$. Now since

$$|T^i(p) - p| < \alpha'$$

and

$$T^{q_T} = T(T^{q_T - 1})$$

then

$$|T^{q_T}(p) - p| < |T^{q_T}(p) - T^{q_T - 1}(p)| + |T^{q_T - 1}(p) - p| < 2\alpha'$$

so T^{q_T} are uniformly bounded on Δ for all T, which means they are uniformly bounded on all relatively compact subsets of D (by property a'). For $p \in \Delta'$ we have $q_T(T(p) - p)$ uniformly bounded on Δ' by (3.1). Let $\Delta'' \subsetneq \Delta'$, since

$$|T^{q_T}(p) - p| \geq \alpha'$$

on Δ, then there exits ϵ' such that

$$|T^{q_T}(p) - p| \geq \epsilon'$$

on Δ'' (by property a). Now by (3.1), for all $T \in G$, $p \in \Delta''$, $q_T(T(p) - p) \geq u\epsilon'$, so we have $q_T(T(p) - p)$ is uniformly bounded on Δ' and 0 is not a limit point of this family. We just need to show that $q_T(T(p) - p)$ is uniformly bounded on all relatively compact subsets on D. Let $\cup \Delta'_n$ be a cover of D such that Δ'_i is a relatively compact polydisk in D and $\Delta'_k \cap \Delta'_{k+1}$ is non-empty for all k. Let $\Delta'_1 = \Delta'$ (Δ' is the one defined above), define D_k to be the union of the first k polydisks. We will proceed by induction, assume that the family $q_T(T(p) - p)$ is uniformly bounded on D_k, and let us prove that it is uniformly bounded on D_{k+1}. By the same work done on Δ' above, one can find a sequence on integers $\overline{q_T}$ such that the family $\overline{q_T}(T(p) - p)$ is uniformly bounded on Δ'_{k+1} and 0 is not a limit point of this family. Then there exists $\alpha > 0$ such that $\frac{q_T}{\overline{q_T}} < \alpha$ for all $T \in G$ since otherwise $\forall \alpha > 0$ there would exist $T \in G$ with $\frac{q_T}{\overline{q_T}} \geq \alpha$ which means there exist a sequence $\{T_k\}$ such that $\frac{\overline{q_{T_k}}}{q_{T_k}} \to 0$, then

$$\overline{q_{T_k}}(T_k(p) - p) = \frac{\overline{q_{T_k}}}{q_{T_k}} q_{T_k}(T_k(p) - p)$$

converges to 0, but 0 is not a limit point of this family, so we get a contradictions. So we have

$$q_T(T(p)-p) = \frac{q_T}{\overline{q_T}}\overline{q_T}(T(p)-p)$$

is uniformly bounded on Δ'_{k+1}, which means $q_T(T(p)-p)$ uniformly bounded on D_{k+1}. □

Corollary 3.4.3.1. *Let G be a locally closed group of holomorphic transformations such that the identity transformation is not isolated, then G admits at least one infinitesimal transformation.*

Proof. By Theorem 3.4.3, a group of holomorphic transformations satisfies property [P], then by Theorem 3.3.2 this group admits an infinitesimal transformation. □

Now that we know a holomorphic group of transformations satisfies property [P], all is left in order to prove Cartan's main theorem in this paper is the following theorem.

Theorem 3.4.4. *If a quasi-continuous group of analytic transformations satisfies property [P], then it is a local Lie group, and in particular a continuous group.*

Proof. If the group G contains transformations which are arbitrarily close to the identity, then it admits at least an infinitesimal transformation by Theorem 3.3.2. By virtue of Theorem 3.3.6, the set of all infinitesimal transformations of G generates a group $\Phi(p, t_1, ..., t_k)$, where Φ is analytic with respect to $p, t_1, ..., t_k$

for $p \in D$ and $|t_i| < u \ \forall\ i$. Furthermore, the transformation $\Phi(p, t_1, ..., t_k) \in \overline{G}$ for $|t_i| < u$ where $\overline{G}$ is a neighborhood of the identity transformation. Let $0 < v < u$ such that the transformations with $|t_i| \leq v$ are all distinct, let Γ be the set of these transformations, Γ is a closed group. We need to show that all transformations $T \in G$ for which $|T(p) - p|$ is small enough are in Γ. Let us proceed by contradiction. Suppose there exists a sequence of transformations $\{T_k\} \in G$ that converges to the identity such that $T_i \notin \Gamma$ for all i. Let $U \in \Gamma$ and consider $UT_k \in G$ (this composition is defined for k large enough, since T_k converges to the identity), then there exists $\epsilon_k > 0$ (in a fixed domain Δ) such that

$$\sup_{p \in \Delta} |UT_k(p) - p| \geq \epsilon_k$$

for all $U \in \Gamma$. This lower bound ϵ_k exists since $T_k \notin \Gamma$ so if $|UT_k(p) - p| \to 0$ then $T_k^{-1} = U \in \Gamma$, so $T_k \in \Gamma$ which is a contradiction. Equality above is reached by some transformation $U_k \in \Gamma$ since Γ is compact (U_k admits its minimum). Now since ϵ_k is at most equal to $\sup_{p \in \Delta} |T_k(p) - p|$ (which occurs when U is the identity), then $\epsilon_k \to 0$ as $k \to \infty$.

Now let $U_k T_k = S_k$, then $U_k = S_k T_k^{-1}$ Let $0 < v' < v$, and denote by Γ' the set of transformations $\Phi(p, t_1, ..., t_p)$ such that $|t_i| < v'$, then

$$|US_k(p) - p| \geq |S_k(p) - p|$$

for all $U \in \Gamma'$, for k large enough. Now by property [P] applied to $\{S_k\}$, there exists a sub-sequence $\{S_k\}$ and m_k such that $m_k(S_k(p) - p) \to \psi(p)$ uniformly on all relatively compact domains in D, such that $\psi(p) \neq 0$ and is analytic on D, then Γ' admits the infinitesimal transformation ψ by Theorem 3.3.2. Let $A = \max|\psi(p)|$ on Δ, then the product $\epsilon_k m_k \to A$ since ϵ_k here is basically $\sup_{p\in\Delta}|S_k(p) - p|$. Now denote by U_t the transformation $\Psi(p,t)$ generated by the infinitesimal transformation

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t}(p,t) = \psi(\Psi(p,t))$$

If t is small enough, then U_t is part of Γ'. Let $U_{-\frac{1}{m_k}} S_k = S'_k$ and apply it to all points p in Δ. Now let

$$\sup_{p\in\Delta}|S'_k(p) - p| = \epsilon'_k$$

We will show that $\epsilon'_k m_k \to 0$ as $k \to \infty$, which implies that $\epsilon'_k < \epsilon_k$ for k large enough since $\epsilon_k m_k \to A > 0$. This will lead to a contradiction since ϵ_k is the smallest number obtained by multiplying U_k (which are in Γ) by T_k and taking the sup norm of $|U_k T_k(p) - (p)|$.

$S'_k = \Psi(S_k(p, -\frac{1}{m_k}))$ and ϵ'_k is the maximum distance between $\Psi(S_k(p, -\frac{1}{m_k}))$ and p, where $p \in \Delta$.

$$\Psi(S_k(p), -\frac{1}{m_k}) - p = S_k(p) - p - \frac{1}{m_k}\psi(S_k(p)) + O(\frac{1}{m_k^2})$$

So now

$$m_k(\Psi(S_k(p), -\frac{1}{m_k}) - p) = (m_k(S_k(p) - p) - \psi(p)) + (\psi(p) - \psi(S_k(p)) + \frac{\lambda(p)}{m_k}$$

as $k \to \infty$, the above expression $\to 0$, which means $\epsilon'_k m_k \to 0$. $\square$

After proving Theorems 3.4.3 and 3.4.4 the main theorem becomes straight forward.

Theorem 3.4.5. *Let $D \subset \mathbb{C}^n$ be a bounded domain, then the group G of all biholomorphic transformations from D to D (automorphism group) is a Lie group.*

Proof. By Theorem 3.2.1, the automorphism group of D is quasi-continuous, and since automorphism group consists of biholomorphic transformation, then by Theorem 3.4.3 it satisfies property [P]. Now by Theorem 3.4.4, the automorphism group of D which is quasi-continuous and satisfies property [P] is a local Lie group. The global result follows from the local result by relatively standard arguments of Lie group theory [7]. $\square$

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