AMERICAN UNIVERSITY OF BEIRUT

POWER APPROXIMATION FOR PRICING AMERICAN OPTIONS

by
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American options are one of the most traded instruments in the financial markets. However, pricing such options is challenging since there is the possibility of early exercise of the option. We propose a robust pricing method based on nonlinear regression over a large representative set of "exact" pricing instances obtained via a binomial lattice. Our nonlinear regression is sought to relate the price and the critical stock price of an American option to its key parameters via a power-type regression. Our "power approximation" approach is inspired from the operations research literature on the well-known (s, S) periodic review inventory system. Our objective is to develop a closed-form approximation for pricing American options that outperforms other existing approximations, in terms of accuracy and simplicity.

Our results include developing a large set of near-exact American option prices over a carefully designed grid of parameter values that are common in practice. In addition, we compile the literature for existing American option pricing approximations, identify suitable ones, and apply the resulting approximations to the set of parameters in the test grid. These approximation serve two purposes, which we address in our work, (i) providing a starting point for our power approximations, and (ii) developing a benchmark to compare our algorithms against.

In our work, we develop two closed-form approximations for the critical stock price and premium of an American put option, respectively. Both approximations are based on the Barone-Adesi & Whaley's results (1987). Correction factors fitted by regression are used to modify the results of Barone-Adesi & Whaley's (1987) to improve the accuracy. The two closed-approximations for the critical stock price and the premium of an American put option perform very well with a median relative absolute error of 0.3764% and 0.0795% respectively. As such, these approximations outperform their counterparts in the literature on both accuracy, computational efficiency (speed), and simplicity.

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Chapter 1

Introduction and Motivation

Nowadays, one of the most important challenges in the financial industry is the valuation of different types of financial instruments. Due to the prevalent dense trading activities, defining robust pricing methods that valuate financial instruments in a fast manner becomes a must. This study focuses on financial derivatives and on American options in particular.

Financial derivatives form a class of financial instruments that derive their values from an underlying asset. Options are among the most traded financial derivatives.. An option is a financial contract that gives the holder the choice, and not the obligation, to exercise this contract by buying or selling an underlying asset at a predefined price and date. The two most-known option types present in the financial markets are the European and the American. The difference between the two is that in the case of the American options, the holder of the contract can exercise it at any point in time prior to maturity, which does not apply for the case of a European option. This added flexibility makes the problem of valuation of an American option more complex, since several exercise strategies are available. In fact, "pricing" an American option requires determining (i) the asset price beyond which the option should be exercised at any point in time (i.e. the critical stock price), and (ii) the premium, which is the selling price of the option that is intimately related to the critical stock price.

The well-known Black-Scholes formula is used to price a European option (Black & Scholes, 1973). As for the American option, solving the Black-Scholes differential equation (Black & Scholes, 1973) does not lead to a closed-form solution. In the literature, numerous methodologies have been suggested including: least square Monte Carlo, binomial trees, trinomial trees, finite difference methods and dynamic programming. It is shown that these methods lead to highly accurate results. However, most methods require considerable computational power and are not easy to implement. This motivates our research on finding a closed-form approximation that provides an accurate valuation of an American option that is both computationally efficient and easy to use.

Several efforts have been made to find analytical approximation for the value of an American option (e.g. Barone-Adesi Whaley 1987, Bjerksund & Stensland (1993), Geske & Johnson (1984)...). These approximations are quicker and more efficient than other numerical methods in this context. However, these approximations still require some computations, and when compared to exact pricing methods, there seems to be room to improve accuracy.

In this study, a new approach is offered to price American options and find the critical stock price based on power approximation. Relying on regression and asymptotic analysis together with optimization procedures, we propose to develop an efficient, accurate, and closed-form pricing approximation for American options. The purpose of this research work is to reduce the computational effort compared to numerical approaches and to minimize the error compared to other approximations in the literature.

Part of the motivations for our proposed work stems from the popularity of American options. According to CBOE (Chicago Board Options Exchanges), the largest U.S. Options Exchange, currently most of the equity options traded on the U.S. option exchanges are of the American type.
Large volumes are being traded everyday. For example, Figure 1.1 summarizes the trading activity on April 2, 2020 (CBOE, n.d). The volume of equity options constitutes 37.61% of the total market, while the open interest (number of outstanding options that have not been settled yet) consists 54.87% of the market.

SUM OF ALL PRODUCTS			
	CALL	PUT	TOTAL
VOLUME	2,016,984	2,440,960	4,457,944
OPEN INTEREST	148,457,691	151,018,351	299,476,042
INDEX OPTIONS			
	CALL	PUT	TOTAL
VOLUME	545,779	854,776	1,400,555
OPEN INTEREST	13,570,067	17,354,905	30,924,972
EXCHANGE TRADED PRODUCTS			
	CALL	PUT	TOTAL
VOLUME	549,931	830,587	1,380,518
OPEN INTEREST	45,501,761	58,722,289	104,224,050
EQUITY OPTIONS			
	CALL	PUT	TOTAL
VOLUME	921,274	755,597	1,676,871
OPEN INTEREST	89,385,863	74,941,157	164,327,020

Figure 1.1: Trading activity on April 2, 2020

Speed in trading is one of the most significant factors in this developed work. When dipping into the realm of financial markets, it is common knowledge that speed is key. It is estimated that around 70 percent of all daily trades are high frequency trades (American Physical Society, 2020). The highly volatile financial market warrants the need to make the right decision at the exact right instant (Finsmes, 2019) as major gains can be reaped based on advantageous milliseconds, and major losses could be faced if those milliseconds were to slip away. The importance of speed can be highlighted through an example about currency exchange markets. Assume that the British pound begins to devalue against the U.S dollar, traders with the prior currency would want to sell their pounds to avoid losses. Traders that sell their pounds faster, will be able to sell at a higher value. However, because of the difference in information travel time between two different locations, Londoners will hear about the currency value change faster than New Yorkers. Londoners are closer to the source of the information, giving them an advantage that would allow them to react faster (American Physical Society, 2020).

Although the information travel time difference is only a matter of seconds, it is no secret that those that are able to receive the information faster, and hence react faster, are placed at an advantage. That fact holds particularly true when trading financial instruments such as options. The value of a financial option and the profit it could potentially bear all rely on a set of dramatically changing parameters pertaining to the fluctuating stock market (Burton, 2012). Because all it takes is milliseconds for market numbers to vary, the need for the fastest option price estimation method is needed. A few lost milliseconds could get in the way of great profit (Burton, 2012). A highly dynamic environment as the financial market, with countless players, demands you to be there first. That is a demand that can be attained once the importance of every millisecond is realized.

The remainder of this thesis is organized as follows. Chapter 2 summarizes the related literature. Chapter 3 presents the proposed power approximation model. Chapter 4 summarizes the grid design and model developed to get the critical stock price for an American put option. Chapter 5 lists the characteristics of an American put option. Chapter 6 summarizes a model developed using nonlinear regression to price the critical stock price of an American put option (PAAPC1: power approximation for the American put option critical stock price). Chapter 7 summarizes a model developed using nonlinear regression to price price of an American put option (PAAP1: power approximation for the American put option premium). Finally, Chapter 8 concludes the work and presents ideas for future research.

Chapter 2

Literature Review

In this chapter, we review the related literature. In Section 2.1, we review the key methods and approximations for pricing American options. In Section 2.2, we discuss the power approximation for the (s,S) inventory systems, which motivates our research.

2.1 American Option Pricing Methods

Several numerical methods are present in the literature proposing approaches to valuate American options.

A common method to price options is the binomial lattice (Cox, Ross, & Rubinstein, 1979). The life span of an option is divided into equally spaced time periods. The underlying assumption is that the stock price follows a discrete time process, which approximates the well-known geometric Brownian motion, after applying a change of measure that excludes the possibility of arbitrage. At each time step, the stock price can move up or can move down based on a "risk-neutral" probability. Then, a backward scheme is followed in order to find the option value at each node of the resulting tree (lattice). One extension of this method is proposed by Boyle (1986) who introduces the trinomial lattice approach. This approach is similar to the binomial lattice but with the additional third possibility of having the stock price unchanged at each node of the lattice.

Finite difference methods can be used to solve the problem of pricing American options by solving the Black-Scholes differential equation (with a change of measure to account for arbitrage) over a finite grid. The three most common finite difference techniques are: the explicit finite difference (Hull & White, 1990), the implicit finite difference and Crank-Nicolson finite difference (Wilmott, 2013).
The explicit finite difference method is more or less a generalization of the trinomial tree. The implicit finite difference method is close to the explicit method

with the only difference of using backward difference instead of forward difference in order to solve the differential equation. The Crank-Nicolson method can be considered as a combination of both the explicit and implicit methods.

An approach is proposed to solve a American-style Asian options (Amerasian) by Ben-Ameur, Breton, & L'Ecuyer, 2002. Similar to an American option, an Amerasian option offers the early exercise opportunity, but the payoff depends on the average stock price. This approach can be applied to price American options as well. The proposed method is based on dynamic programming (DP), more specifically, the Markov decision processes (a stochastic DP problem). The DP value function expresses the value of an Amerasian option as a function of the current time, current price, and current stock average. Solving this equation yields both the option value and the optimal exercise strategy. Using a piecewise polynomial interpolation over rectangular finite elements to approximate the value function, the DP equation is solved (Ben-Ameur, Breton, & L'Ecuyer, 2002).

Longstaff & Schwartz (2001) estimate the price of an American option by following the Least Square Monte Carlo method. Their method consists of valuing American options via simulation techniques by estimating the conditional expectation value from continuation of the option using the least square method approach. This estimation allows a direct comparison between the immediate exercise at time T and the expected payoff if the option is not exercised at time T in order to determine the optimal exercise date. Longstaff & Schwartz (2001) present a general LSM algorithm to price American options that are in the money.

Our proposed power approximation pricing method is different than that of Ben-Ameur, Breton, & L'Ecuyer (2002) and Longstaff & Schwartz (2001). Neither simulation nor numerical integration is used in order to determine an optimal exercise strategy, and accordingly get the price of the option. Our method consists of fitting power-type approximations for the critical stock price and premium of an American put option. This is achieved by using regression methods on a representative set of "exact" prices of American put options.

As stated before, some approximations of an American option price have been derived and are sought to be more efficient than numerical methods. Barone-Adesi & Whaley's (1987) procedure consists of finding an approximation for the solution of the key pricing differential equation derived by Black & Scholes (1973) and Merton (1973). The American option value is expressed as the summation of its corresponding European option value and a certain early exercise premium. Applying the differential equation to the American option value, two differential equations should be solved, one for each term. For the European option value, the well-known Black-Scholes formula is the solution. Concerning the early exercise premium, Barone-Adesi & Whaley (1987) end up with a second-order ordinary differential equation for which the solution's general form is derived. An improvement of the Barone-Adesi Whaley (1987) formula is presented in a follow-up paper by Ju & Zhong (1999). The early exercise premium is modified by introducing a time-independent function which is a solution of an ordinary differential equation.

Bjerksund & Stensland (1993) derive a formula that prices a call option paying dividend and then a put-call symmetry is used to determine the put option value. The optimal exercise boundary is assumed to be a flat boundary. An exercise strategy, based on a stopping rule, is proposed whereby, the first time the underlying asset price hits this flat boundary or goes beyond it, the option is exercised. Bjerksund & Stensland (2002) suggest an improvement of their previous model. A similar procedure is followed except that the time to maturity is divided into two sub periods where each one is assumed to have a flat exercise boundary.

The American put value is approximated using a compound option by Geske & Johnson (1984). The main underlying assumption is that the option can be exercised at discrete points in time. It follows that at each predefined point in time, the decision of exercising the option or not is taken according to specific conditions. The problem is here expressed as a free boundary problem where the Black-Scholes differential equation is solved at each point. Nevertheless, using Richard Extrapolation, the price of an American option is approximated by a polynomial expression. Bunch & Johnson (1992) propose an improvement of Geske & Johnson (1984) that relies on choosing the exercise points in an optimal way such as the put value is maximized.

An approach based on the computation of lower and upper bounds on both call and put American options values on a dividend-paying asset has been introduced by Broadie & Detemple (1996). This method yields two option price approximations: the first one is based on the lower bound (LBA) and the second one is based on both bounds (LUBA). The procedure followed to price an American call option is as follows: first, the derivation of a lower bound for the option price is based on a capped option (an option which limits the maximum possible profit for its holder) characterized with a selected constant cap. Second, based on the same class of capped options, we compute a uniform lower bound $L*$ for the optimal exercise boundary of the option. Third, an upper bound for the option price is computed by using a combination of the integral representation of the early exercise premium with the calculated L^*. Once we get the results of the bounds of the American call option, both the LBA and LUBA pricing approximations can be computed using a weighted regression approach. When it comes to pricing an American put option, the put-call symmetry result for these type of options, permits to price the put options based on the values of the call options with a simple substitution of parameters if the Geometric Brownian motion applies.

None of the above approximations offer a simple one-line formula for pricing American options. They can be seen as presenting approximating algorithms. In our proposed research, we aim at developing a simplified closed-form pricing formula with competitive performance.

2.2 Power Approximation for the (s, S) Inventory Systems

Ehrhardt (1979) uses power approximation to compute “optimal” (s,S) inventory policies. In general, the approach uses regression analysis to fit the approximations, and the mean and variance of the demand function. Knowing the distribution of the demand function and its moments, statistical estimates are used in place of the actual mean and variance of the demand. The setup of the problem is presented next.

The studied system is a single-item inventory system where unsatisfied demand is backlogged at the end of each review period. Between the placement of an order and its delivery, there exists a fixed lead time. During review periods, demands are considered to be independent and identically distributed and are represented by a mean and a variance. In addition, a fixed setup cost is considered every time an order is placed. The optimization procedure aims at minimizing the undiscounted expected cost per period over the infinite horizon.

An (s,S) policy can be defined as follows: when the sum of the inventory on hand and on order drops below a predefined level s, an order is placed to bring the inventory position to S. Roberts (1959) derives approximation for the optimal values of s and S based on the assumption that the demand is normally distributed. The method is called the Normal Approximation. However, it is shown that this method is not adequate for all parameters' ranges.

Ehrhardt (1979) proposes an alternative to Robert's method. The Normal demand distribution is no longer required, and a nonlinear power-type regression analysis is used to fit an approximation. The outcome is a closed-form power approximation that requires only to input the mean and the variance of the demand distribution and the key cost parameters, and it is efficient for a wide range of parameters.

However, the performance of the approximation in Ehrhardt (1979) is affected when the factors characterizing demand are varied and when the demand variance is very small. In this context, Ehrhardt & Mosier (1984) proposes a revision of this approximation to account for this deficiency.

Later on, Schneider & Ringuest (1990) suggests a power approximation for (s,S) policies using a service level (in terms of avoiding shortages). This is motivated by the difficulty in estimating the unit shortage cost in practice. A procedure similar to that of Ehrhardt (1979) is followed with a service level requirement in lieu of a shortage cost. In this case, a γ-service level is defined based the percent of demand met from on-hand inventory.
Maddah, Jaber, & Abboud (2004) analyze the (s,S) inventory model with permissible delays in payments and propose power approximations for different flavors inspired by the works of Ehrhardt & Mosier (1984) and Schneider & Ringuest (1990). Motivated by the promising results of power approximations in approximating the typically hard-to-find optimal (s, S) policies, we propose to utilize these approximations for pricing financial derivatives. We focus on American options in our proposed research. In a recent work, Maddah, Aprahamian, & Sawaya (2015) apply a similar methodology for pricing European-style Asian options and develop closed-form approximations that perform well against approximations in their league in the literature. Our work is in the same spirit of Maddah, Aprhamian & Sawaya (2015). However, it involves developing an additional power-type approximation for the critical stock price that governs the early exercise policy of the American put option that we consider.

Chapter 3

Power Approximation Framework

This chapter presents our results including, in Section 3.1 the general approach of power approximation followed to price financial derivatives presented by Maddah, Aprahamian, & Sawaya (2015) in their working paper, in Section 3.2 a description of American put options, in Section 3.3 the grid design given the set of parameters we plan on using in developing our power approximations, and in Section 3.4 details of the binomial lattice method we use to get "exact" results. Finally, in Section 3.5, we present in detail American options pricing approaches in some analytical approximations found in the literature, which will allow us to develop a benchmark on the accuracy of our approximations.

3.1 Power approximation for pricing derivatives

This section contains the general approach of power approximation followed to price financial derivatives presented by Maddah, Aprahamian, & Sawaya (2015) in their working paper. We denote by P the price of the security such as $P(\alpha) = g(\alpha_1, \alpha_2, ..., \alpha_n)$ where $\alpha = \alpha_1, \alpha_2, ..., \alpha_n$ is the set of input parameters and *g(.)* is the exact pricing formula which is typically unknown and then consider the following approximation of the price

$$P(\alpha) \cong \hat{g}(\alpha, \beta) = CF_{Base}(\alpha) + CF(\alpha; \beta)$$

where $\beta = \beta_1, \beta_2, ..., \beta_n$ is the set of decision variables, determined by regression, and serve to fit $P(\alpha)$ to exact values determined by numerical analysis, $\hat{g}(.)$ is the approximate pricing formula, $CF_{Base}(\alpha)$, which is a function of the set of input parameters, is the price of the security that is typically chosen in a way that makes it highly correlated to $\hat{g}(.)$ and has a closed-form solution. Usually, $CF_{Base}(\alpha)$ will represent a crude approximation of the security which is in closed form. $CF(\alpha; \beta)$ is the added correction factor which is a function of both the set input parameter and the set of decision variables. The functional form of $CF(\alpha;\beta)$ is constructed by using theoretical results and asymptotic analysis specific to a particular product. The mathematical model is defined as follows,

$$\min_{\beta} d[P(\alpha), \hat{g}(\alpha;\beta)]$$

where $d[.]$ represents a distance measure between the exact value, $P(\alpha)$, and the approximated value, $\hat{g}(\alpha;\beta)$. Examples of measures often used in the literature include the root-mean square error RMSE and the absolute average relative error AARE. For a data set of N exact values,

$$\text{AARE} = \min_{\beta} \frac{1}{N} \sum_{i=1}^{N} \frac{\mid P_i(\alpha) - g_i(\alpha;\beta) \mid}{P_i(\alpha)}$$

$$\text{RMSE} = \min_{\beta} \sqrt{\frac{\sum_{i=1}^{N} (P_i(\alpha) - g_i(\alpha;\beta))^2}{N}}$$

3.2 American Put Options

In this Section, a description of the American put options is presented (Luenberger, Chapter 12, 1998). The asset price (e.g. stock price) model is based on the geometric Brownian motion model. The main concept behind this model is that the probability that the asset price will change by a specific percentage during a certain time period is always the same. Two important parameters of this model are: the expected annual rate of return of the asset μ and σ the volatility of the asset price and it measures its variability[1].

An American put option gives its holder the choice, but not the obligation, to sell the underlying asset for a price K. The holder of an American put option must pay a price or a premium P. In exchange, the holder of the option can exercise the option any time prior to maturity, T. The parameters of a put option are: the strike price K, the initial stock price S, the time to expiration T, the volatility of the stock price σ and the risk free interest rate r. Assuming that the stock price at time $t < T$ is S_t, the buyer of a put option will gain a profit of $\max(0, K - S_t) - P$. Accordingly, a put option is in the money if $S_t < K$.

It can be shown that in order to avoid arbitrage, American, and other options, can be priced in a risk-neutral manner by replacing μ with the risk free rate, r. In the context of the binomial lattice model, the risk-neutral pricing implies

[1] σ is the standard deviation of the asset price log return, $\sigma^2 = var[\ln(S_t + 1/S_t)]$, where time t is measured in years

the probability that the stock price goes-up over a small time interval, Δ_t, p is replaced by a risk neutral probability $q = (R - d)/(u - d)$, where R is the one-period risk-free compounding factor, and u and d are the factors by which the asset price can increase or decrease in a period (see Section 4.3 for details on the Binomial lattice model).

3.3 Grid Design

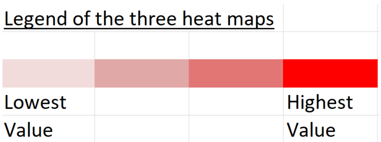
In order to develop the data set of "exact" critical stock price and premium values for calibrating our power approximating method for pricing American put options, we identify common range of parameters by inspecting American options traded over major exchanges. As discussed in Section 3.2, five input parameters are selected: the strike price K, the stock price S, the interest rate r, the volatility σ and the maturity T.

Data addressing the top 20 most active stocks options on major US financial securities market such as NYSE and NASDAQ from the website barchart.com is obtained in order to form the data set. As sated previously, according to CBOE, currently most of the equity options traded on the U.S. option exchanges are American options. The reference date is set to be 31/01/2020. For each one of those stock options, information about available options in the market is collected. For each existing contract, the strike price, the current stock price, the volatility and the maturity are checked along with the density of such contract in the market. The collected data contains contracts with the following maturities (in days): 7, 14, 21, 28, 35, 42, 49, 77, 105, 140, 168, 203, 231, 259, 294, 322, 350, 413, 504, 595, 721, 777, 868. To measure the density of options traded, the open interest measure is taken into consideration (number of unsettled contracts). The open interest varies over a wide range going from 0 to 1,255,090.

This data is used in order to construct three different heat maps. The heat map in Figure 3.1 shows the volatility versus the strike price over stock price (i.e. the moneyness), the second heat map in Figure 3.2 shows maturity versus moneyness. Finally, the third heat map in Figure 3.3 shows maturity versus volatility. Those heat maps are helpful to identify dense areas of the parameter space that will be used in our regression.

Based on the heat maps in Figures 3.1-3.3, one can identify that most dense data falls in the following ranges:
a) Moneyness K/S: concentrated between 0.7 and 1.2. So, a stock price of 100 is picked and then the strike price is obtained accordingly. A step of 0.05 is considered.
b) Volatility σ: concentrated between 0.1 and 0.7. A step of 0.05 is considered.

c) Maturity T: concentrated between 6 months to 1 year with a denser concentration in the area that is less than 6 months. So, a step of one week is considered between 0 and 6 months, and a bigger step of 2 weeks is taken between 6 months and 1 year.
d) Regarding the interest rate r parameter, the US treasury rates are used. They are found in Figure 3.4. Depending on the maturity of each option, the interest rate is chosen.
Table 3.1 represents the ranges of the parameters reflecting the heat maps as stated earlier.
This grid is built by adding all possible combinations between parameters that lie in the above cited ranges, so it will reflect the three constructed heat maps. This grid results in a total number of 5,720 trials.



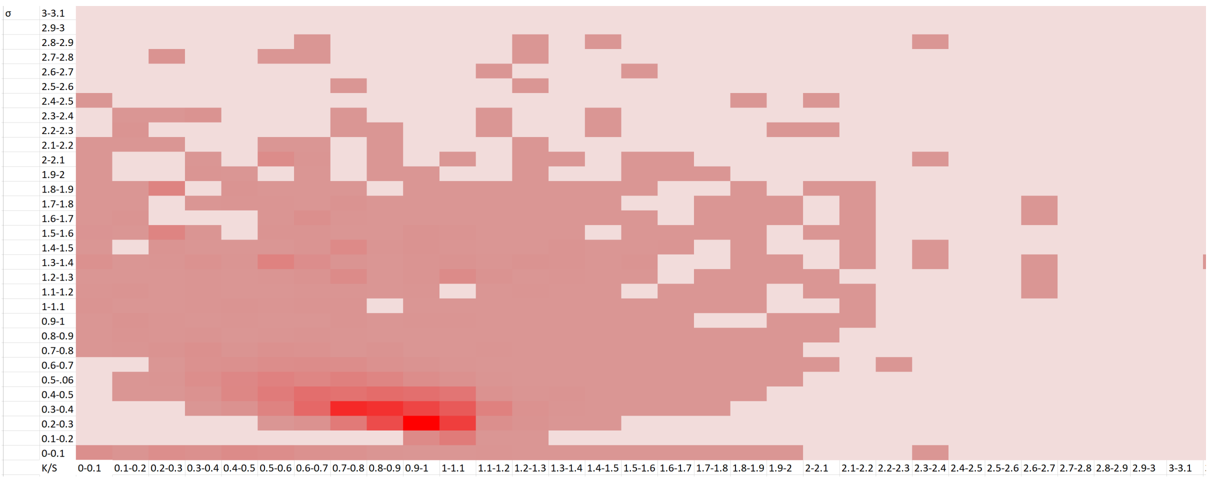


Figure 3.1: Heat map of popular parameter values in terms of σ versus K/S

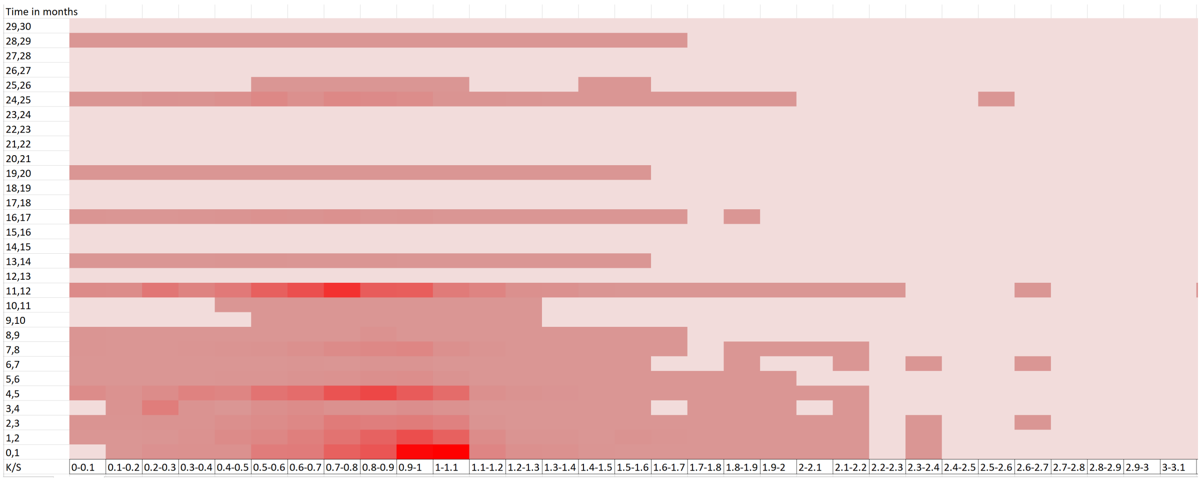


Figure 3.2: Heat map of popular parameter values in terms of T versus K/S

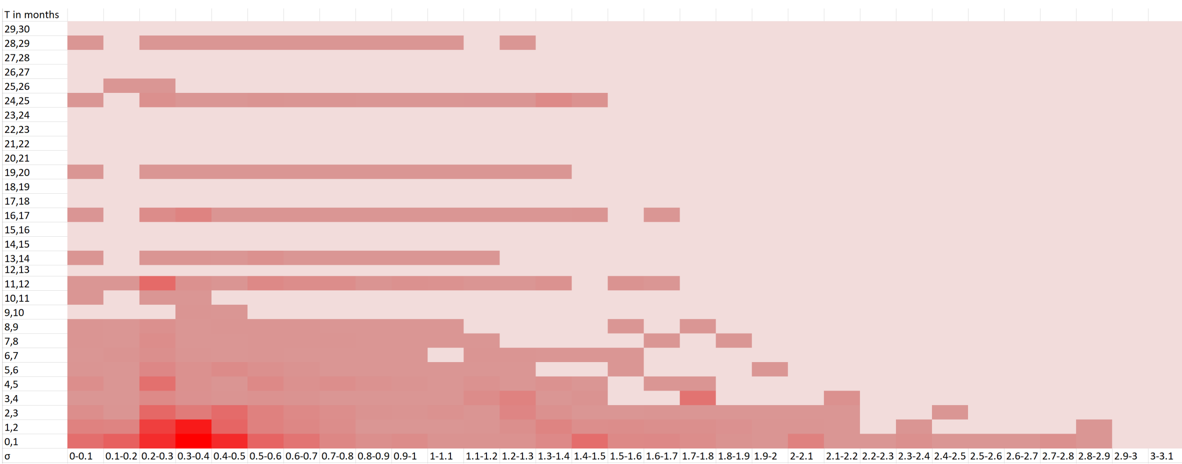


Figure 3.3: Heat map of popular parameter values in terms of T versus σ

Treasury Rates

Name	Last	1M Ago	3M Ago	6M Ago	1Y Ago	Time
1-Month Treasury	1.57%	1.53%	1.57%	2.02%	2.43%	02/07/20
3-Month Treasury	1.56%	1.54%	1.56%	2.02%	2.42%	02/07/20
6-Month Treasury	1.57%	1.56%	1.58%	1.95%	2.49%	02/07/20
1-Year Treasury	1.49%	1.54%	1.58%	1.75%	2.55%	02/07/20
2-Year Treasury	1.41%	1.58%	1.68%	1.59%	2.48%	02/07/20
3-Year Treasury	1.39%	1.59%	1.70%	1.51%	2.46%	02/07/20
5-Year Treasury	1.41%	1.65%	1.74%	1.52%	2.46%	02/07/20
7-Year Treasury	1.51%	1.77%	1.84%	1.60%	2.54%	02/07/20
10-Year Treasury	1.59%	1.85%	1.92%	1.71%	2.65%	02/07/20
20-Year Treasury	1.89%	2.17%	2.24%	2.01%	2.85%	02/07/20
30-Year Treasury	2.05%	2.38%	2.40%	2.22%	3.00%	02/07/20

Figure 3.4: US treasury rates on 02/07/20 (Source: barchart.com)

K	K/S	S	Time (in weeks)	Time (in months)	Interest Rate(in percent)	Volatility
100	0.7	142	1	0.23	1.57	0.1
	0.75	133	2	0.46	1.57	0.15
	0.8	125	3	0.69	1.57	0.2
	0.85	118	4	0.92	1.57	0.25
	0.9	111	5	1.15	1.56	0.3
	0.95	105	6	1.38	1.56	0.35
	1	100	7	1.61	1.56	0.4
	1.05	95	8	1.84	1.56	0.45
	1.1	91	9	2.07	1.56	0.5
	1.15	87	10	2.3	1.56	0.55
	1.2	83	11	2.53	1.56	0.6
			12	2.76	1.56	0.65
			13	2.99	1.56	0.7
			14	3.22	1.57	
			15	3.45	1.57	
			16	3.68	1.57	
			17	3.91	1.57	
			18	4.14	1.57	
			19	4.37	1.57	
			20	4.6	1.57	
			21	4.83	1.57	
			22	5.06	1.57	
			23	5.29	1.57	
			24	5.52	1.57	
			25	5.75	1.57	
			26	5.98	1.57	
			27	6.21	1.49	
			29	6.67	1.49	
			31	7.13	1.49	
			33	7.59	1.49	
			35	8.05	1.49	
			37	8.51	1.49	
			39	8.97	1.49	
			41	9.43	1.49	
			43	9.89	1.49	
			45	10.35	1.49	
			47	10.81	1.49	
			49	11.27	1.49	
			51	11.73	1.49	
			53	12.19	1.49	

Table 3.1: Ranges of parameters reflecting the three heat maps

In developing their model, Broadie & Detemple (1996) use linear regression. Details will be provided insection 4.4.3. They use ranges of parameters very similar to the ones used in this research work. Li (2010) performs a comparison performance of approximations used to compute the critical stock prices of an American put option. The parameters used in the analysis fall in the ranges of our data except for the interest rate. In addition to that, after checking numerical experiments performed by the models cited in the literature review Section, we conclude that our data set shows a very good performance except for the interest rate. In our data set, the interest rate considered falls in the lower ranges common in practice. Consequently, we decide to double our data by taking an interest rate of 0.08. So, our final data set contains 11,440 instances used to calculate the premium and 1,040 instances used to calculate the critical stock price. Constructing a model requires the presence of "in-sample" and "out-sample" testing. In order to be able to do an "out-sample" testing, a subset of the original data set is taken out for later use. It will consist 10 percent of our original data set. To make sure that this chosen sample does not contain any bias, it should be selected randomly. Please see Appendix B for details.

3.4 Binomial Lattice

The Binomial Lattice method is widely used in pricing American options. Sections 4.4.1, 4.4.2 and 4.4.3 describe the procedure to calculate the price of an American put option using a binomial lattice, the procedure to calculate the critical stock price in addition to the number of steps needed to get accurate results.

3.4.1 The Price of an American Put Option

The following steps are followed in order to get the price of an American option using the binomial lattice method:
1) Choose the time step Δt.
2) Calculate the number of steps n: $n = \dfrac{T}{\Delta t}$
3) Get the two factors $u = e^{\sigma\sqrt{dt}}$ and $d = \dfrac{1}{u}$ by which the stock price will increase or decrease respectively. Get the compounding factor $R = e^{r\Delta t}$.
4) Get the risk-neutral probability $q = \dfrac{R-d}{u-d}$ that the stock price will increase by a factor u.
5) Starting by the initial stock price S at time 0, follow a forward process to get the possible stock prices at each node of the lattice: at each node, the stock price can increase by a factor u with a probability q or decrease by a factor u with a

probability $1-q$.
6) Let j the number of upward movements in the asset price. The following equation is used to get the stock price at node i (at time equals to $i\Delta t$): $S(i,j) = Su^j d^{i-j}$
7) Now, starting from the final node, follow a backward process in order to get the option values at each node of the lattice.
8) At the final node, the option value is given by: $P(i,j) = \max(0, K - S(i,j))$.
9) Then, going backward, at each node, two values should be computed: the value of early exercise which is $\max(0, K - S(i,j))$ and the expected value of continuation which is: $E[V_c] = \frac{1}{R}[qP(i+1,j+1) + (1-q)P(i+1,j)]$
10) At every node, the maximum between the continuation and the early exercise value is selected. This gives the value of the option at Time i and price level j, and the exercise policy. Specifically, the option should be exercised if and only if $K - S(i,j) > E[V_c]$.
11) The process continues until reaching the initial node. At the initial node, the value obtained is the price of the American put option.

The higher the number of steps, the more accurate the results are. We develop a Matlab code to compute the price and the critical stock price using the binomial lattice. After some numerical experimentation, we conclude that a convenient convergence is reached with a number of steps equal to 4,200.

3.4.2 The Critical Stock Price of an American Put Option

Another topic covered in the binomial lattice approach is the early exercise boundary. Since the holder of an American option has the possibility of exercising it prior to maturity, it can be shown that at every time i before maturity, there exists a critical price S_i^* under which the early exercise of the option is optimal (Basso, Nardon, & Pianca, 2002). At time i, for a stock price less or equal to the critical stock price, the value of the option is equal to $K - S(i,j)$. In this case, $K - S(i,j) > E[V_c]$ where $E[V_c]$ is the expected value of continuation and not exercising the option. For a stock price greater than the critical stock price, the value of the option is equal to the continuation value, $E[V_c]$. In this case, $E[V_c] > K - S(i,j)$. Thus, a bisection method is implemented in order to calculate the critical stock price, which is the price at which the continuation value and the early exercise value are approximately equal. Please refer to Appendix D for illustration of how the critical stock price is calculated over a "small" lattice.

A series of experiments are conducted to determine the binomial lattice number of steps and the bisection method precision that are going to be used to get the price and the critical stock price of American put options. Those experiments are used to find a trade-off between the accuracy of the critical stock price and the speed of our algorithm measured in terms of CPU time. Two options are taken into consideration; their parameters are presented in Table 3.2. The experiments shown in Table 4.3 consist of trying different combinations of time steps and bisection method precision. The CPU time and and the accuracy of the results are then evaluated. For both options, a convergence of the results is achieved with a binomial lattice with 15,000 steps and a bisection method precision (defined as $|E[V_c] - (K - S0)|$) of 10^{-9}. However, for this case, a high computational effort is needed. According to Table 3.3, a binomial lattice with 4,200 steps and a bisection method precision of 10^{-9} yields accurate results (for the case of option 1 and option 2, the percentage difference in the critical stock price between a binomial lattice with 15,000 steps and a binomial lattice with 4,200 steps is equal to 0.0364% and 0.0238% respectively) while requiring lower computational effort (a lower CPU time). Accordingly, a binomial lattice of 4,200 steps and a precision of 10^{-9} is selected in order to get the critical stock prices of our data. As noted previously, since the initial stock price does not affect the critical stock price, the data used here is reduced from 11,440 instances to 1,040 instances.

Option Number	K	Time (in weeks)	Interest Rate(in percent)	Volatility
1	100	1	1.57	0.5
2	100	3	1.57	03

Table 3.2: Options' parameters used in experiments

Option Number	Number of steps	Precision	CPU time (seconds)	S^*	$AARE^{10^{-8}}_{15,000}$
1	1000	10^{-10}	9.3594	82.51	
		10^{-9}	7.8906	82.51	
		10^{-8}	6.5781	82.51	
		10^{-5}	5.8125	82.51	
		10^{-6}	4.1094	82.52	
		10^{-5}		50.00	
	3000	10^{-10}	86.6563	82.47	
		10^{-9}	70.5156	82.47	
		10^{-8}	58.6250	82.47	
		10^{-7}	42.8750	82.47	
		10^{-6}	31.0625	82.42	
	4200	10^{-10}	158.5000	82.47	
		10^{-9}	132.9063	82.47	0.0364%
		10^{-8}	108.2813	82.46	
		10^{-7}	42.8750	82.47	
		10^{-6}		50	
	6000	10^{-10}	340.2188	82.46	
		10^{-9}	304.6250	82.46	
		10^{-8}	206.8438	82.46	
		10^{-7}	188.7813	82.45	
	10000	10^{-9}	888.3750	82.45	
	15000	10^{-9}	2182.2000	82.44	
	20000	10^{-9}	8420.6875	82.44	
	25000	10^{-9}	26101.8125	82.44	
2	4200	10^{-10}	191.1719	84.12	
		10^{-9}	118.4219	84.12	0.0238%
		10^{-8}	116.2656	84.12	
		10^{-7}	101.8125	84.12	
		10^{-6}	84.0781	84.13	
	15000	10^{-9}	3535.5938	84.10	
	25000	10^{-9}	8491.2031	84.10	

Table 3.3: Experiments reflecting required number of steps and precision of bisection method

3.5 Analytical Approximations

This Section describes in details some analytical approximations found in the academic literature to price an American option and find its critical stock price. The Barone-Adesi & Whaley (1987) formula is simple and seems to serve our purpose in the sense of providing a good starting point for our power approximation. Bjerksund & Stensland (1993) formula and Broadie & Detemple (1996) approximation based on the lower bound (LBA) will be used later on for comparison purposes. In each Section, we will specify whether the specified method is used to price an American put option or to find its critical stock price.

3.5.1 Barone-Adesi & Whaley Formula

Barone-Adesi & Whaley (1987) procedure consists of finding an approximate solution to the Black-Scholes-Merton differential equation,

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial P}{\partial S^2} + bS\frac{\partial P}{\partial S} - rP + \frac{\partial P}{\partial t} = 0 \tag{1}$$

The American option value is expressed as the summation of the price of a European put option with the same parameters, *p(S, T)*, and an early exercise premium, $\epsilon_P(S,T)$,

$$P(S,T) = \epsilon_P(S,T) + p(S,T) \tag{2}$$

Applying (1) to the price in (2), two differential equations should be solved, one for each term. For the European option value, the well-known Black-Scholes formula is the solution, $p(S,T) = Ke^{(-rT)}N(-d_2) - SN(-d_1)$ where $d_1 = (log(S/K) + (r+0.5\sigma^2)T)/(\sigma\sqrt{T}),\ d_2 = d_1 - \sigma\sqrt{T}$ and $N(.)$ is the cumulative univariate normal distribution.

Concerning the early exercise premium, Barone-Adesi & Whaley (1987) end up with a second-order ordinary differential equation for which the solution general form is derived. Based on this solution, an approximation for the put option premium in (2) is given by

$$P(S,T) = \begin{cases} p(S,T) + A_1(\frac{S}{S_T^*})^{q_1} & \text{if } S > S_T^* \\ K - S & \text{if } S \leq S_T^* \end{cases} \tag{3}$$

where S^* is the critical stock price and $A_1 = (-S^*/q_1)(1 - N(-d_1(S_T^*)))$, $q_1 = (-(N-1) - \sqrt{(N-1)^2 + 4k})/(2)$, $N = (2r)/(\sigma^2)$, $k = (2r)/(\sigma^2(1 - e^{-rT}))$, and $d_1(S_T^*) = (ln(S_T^*/K) + (r + \sigma^2/2)T)(\sigma\sqrt{T})$.

The critical price $S*$ is determined by solving the following equation:

$$K - S_T^* = p(S_T^*, T) - \frac{S_T^*}{q_1}(1 - N(-d_1(S_T^*))) \tag{4}$$

Barone-Adesi & Whaley (1987) show that (4) has a unique solution. They also propose a Newton-Raphson algorithm to solve (4). We present this algorithm in Appendix C for completeness. A Matlab code is developed to calculate the price and the critical stock price of an American put option using the Barone-Whaley equation.

3.5.2 Bjerksund & Stensland Formula

In their paper, Bjerksund & Stensland (1993) derive a formula that prices an American call option paying dividend and then a put-call parityy is used to determine the put option value. Bjerksund & Stensland (1993) consider that the call option is equivalent to a European up-and-out call option with a barrier price equal to the trigger price X in addition to a payoff equal to $X-K$ upon exercising the option prior to maturity. They derive the following approximation for the price of an American call option with dividends at the rate δ, where $b = r - \delta$,

$$\begin{aligned} c =&\alpha(X)S^\beta - \alpha(X)\phi(S,T|\beta,X,X) + \phi(S,T|1,X,X) \\ &- \phi(S,T|1,K,X) - K\phi(S,T|0,X,X) + K\phi(S,T|0,K,X) \end{aligned} \tag{7}$$

where,

$\alpha(X) \equiv (X - K)X^{-\beta}$

$\beta = (\frac{1}{2} - \frac{b}{\sigma^2}) + \sqrt{(\frac{b}{\sigma^2} - \frac{1}{2}) + 2\frac{r}{\sigma^2}}$

$\phi(S,T|\gamma,K,X) = e^\lambda S^\gamma (N[d] - (\frac{X}{S})^\kappa N[d - \frac{2ln(X/S)}{\sigma\sqrt{T}}])$

$\lambda \equiv (-r + \gamma b + \frac{1}{2}\gamma(\gamma - 1)\sigma^2)T$

$d \equiv -\frac{ln(S/K) + (b + (\gamma - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}$

$\kappa \equiv \frac{2b}{\sigma^2} + (2\gamma - 1)$

Two methods are presented to determine the trigger price X. The first method consists of calculating the flat boundary as the weighted average of two options with infinite and zero time to maturity,

$$X_T = B_0 + (B_\infty - B_0)(1 - e^{h(T)}) \tag{8}$$
where,
$$h(T) \equiv -(bT + 2\sigma\sqrt{T})(\frac{B_0}{B_\infty - B_0})$$
$$B_\infty = \frac{\beta}{\beta - 1}K$$
$$B_0 = \max\{K, (\frac{r}{r-b})K\}$$

The last two expressions of B_0 and B_∞ are developed by Kim (1990).

In the second method, the trigger price is considered the price at which the call value is maximized. In our work, we will use the first method to get the trigger price X_T from (8).
Note that if $S \geq X$, then value of the option is equal to $S - K$. To obtain the American put value, a put-call symmetry is achieved by changing parameters as follows:
$$p(S, K, T, r, b, \sigma^2) = c(K, S, T, r - b, -b, \sigma^2; X) \tag{10}$$

A Matlab code is developed to calculate the price Bjerksund & Stensland equation.

3.5.3 Broadie & Detemple Formula

An approach based on the computation of lower and upper bounds on both call and put American options values on a dividend paying asset has been introduced by Broadie & Detemple (1996). This method yields two option price approximations: the first one is based on the lower bound (LBA) and the second one is based on both bounds (LUBA). In this Section, we will focus on the approximation based on the lower bound, since the approximation based on both bounds needs some computational effort. In addition to that, Broadie & Detemple (1996) derive a lower bound for the optimal exercise boundary of an American call option. Using a put-call parity, those bounds and approximations for call options can be adjusted for American put options.
The derivation of a lower bound for the American call option price is based on capped call option (an option which limits the maximum possible profit for its holder) characterized by a constant cap L. The value of this capped call option is given by equation (11),

$$\begin{aligned} C(S, L) =& (L - K)[\lambda^{2\phi/\sigma^2} N(d_0) + \lambda^{2\alpha/\sigma^2} N(d_0 + 2f\sqrt{T}/\sigma)] + Se^{-\delta T}[N(d_1^-(L) - \sigma\sqrt{T}) \\ & - N(d_1^-(K) - \sigma\sqrt{T})] - \lambda^{-2(r-\delta)/\sigma^2} Le^{-\delta T}[N(d_1^+(L) - \sigma\sqrt{T}) - N(d_1^+(K) \\ & - \sigma\sqrt{T})] - Ke^{-rT}[N(d_1^-(L)) - N(d_1^-(K)) - \lambda^{1-2(r-\delta)/\sigma^2}[N(d_1^+(L)) \\ & - N(d_1^+(K))]] \end{aligned} \tag{11}$$

where,
$N(.)$ is the cumulative standard normal distribution function
$d_0 = \frac{1}{\sigma\sqrt{T}}[log(\lambda) - f(T)]$
$d^{+/-}(x) = \frac{1}{\sigma\sqrt{T}}[+/- log(\lambda) - log(L) + log(x) + bT]$
$b = \delta - r + 0.5\sigma^2$
$f = \sqrt{b^2 + 2r\sigma^2}$
$\phi = 0.5(b - f)$
$\alpha = 0.5(b + f)$
$\lambda = S/L$
Equation (12) is true for $L \geq \max(S, K)$. To determine L, the optimization problem $L = \max_{L \geq S} C(S, L)$ should be solved. And accordingly, the lower bound on the price of an American call option is $C^l = \max_L C(S, L) \leq C(S)$.
This lower bound is used in order to derive an approximation of the American call option price, called LBA. LBA is defined by $C(S) = \lambda_1 C^l(S)$ and the price of an American put option is defined by $P(K, S, \delta, r, T) = C(S, K, r, \delta, T)$.
Regression techniques are used to obtain λ_1.
$x_1 = T$
$x_2 = \sqrt{T}$
$x_3 = S/K$
$x_4 = r$
$x_5 = \delta$
$x_6 = \min(r/max(\delta, 0.00001), 5)$
$x_7 = x_6^2$
$x_8 = (C^l(S) - c(S))/K$
$x_9 = x_8^2$
$x_10 = C^l(S)/c(S)$
$y_1 = 1.002 \times 10^0 - 1.485 \times 10^{-3} \times x_1 + 6.693 \times 10^{-3} \times x_2 - 1.451 \times 10^{-3} \times x_3 - 3.43 \times 10^{-2} \times x_4 + 6.301 \times 10^{-2} \times x_5 - 1.954 \times 10^{-3} \times x_6 + 2.74 \times 10^{-4} \times x_7 - 1.043 \times 10^{-1} \times x_8 + 5.077 \times 10^{-1} \times x_9 - 2.509 \times 10^{-3} \times x_{10}$

$$\lambda_1 = \begin{cases} 1 & \text{if } C^l(S) = c(S) \text{ or } C^l(S) \leq S - K \\ \max(\min(y_1, 1.0133), 1) & \text{otherwise} \end{cases}$$

Based on the same class of capped options, we can compute the critical stock price on an American call option. To obtain S^*_{call}, equation (12) should be solved in terms of L using Newton's method.
$[1-(\frac{L-K}{L})(2\phi/\sigma^2)]\lambda^{2\phi/\sigma^2}N(d_0)+[1-(\frac{L-K}{L})(2\alpha/\sigma^2)]\lambda^{2\alpha/\sigma^2}N(d_0+2f\sqrt{T}/\sigma)+ e^{-\delta T}\frac{2(b-\sigma^2)}{\sigma^2}\lambda^{-2(r-\delta)/\sigma^2}[N(d_1^+(L)-\sigma\sqrt{T})-N(d_1^+(K)-\sigma]sqrtT)]-e^{-rT}\frac{2bK}{\sigma^2 L}\lambda^{2b/\sigma^2}$

$$[N(d_1^+(L)) - N(d_1^+(K))] = 0 \tag{12}$$

To convert the critical stock price of an American call option to the critical stock price of an American put option, Carr & Chesney (1996) derive the put call parity $S^*_{put}(K, \delta, r) = \dfrac{K^2}{S^*_{call}(K, r, \delta)}$
A combination of a Wolfram and Matlab code is developed to calculate the price and the critical stock price of an American put option using the Broadie & Detemple approach.

Chapter 4

Numerical results on the Binomial Lattice and the Analytical Approximations

In this chapter, we present some of our numerical results on the premium (in Section 4.1) and of the critical stock price values (in Section 4.2) estimated based on the methods presented in Chapter 3. The characteristics of the American put premium are checked for the case of the binomial lattice with 4,200 steps (BL), the Barone-Adesi & Whaley (1987) formula (BW), Bjerksund & Stensland (1993) formula (BS) and Broadie & Detemple (1996) (LBA). On the other hand, the characteristics of the American put critical stock price are checked for the case of BL, BW, and LBA. Those characteristics are used later on to validate the developed models.

4.1 Characteristics of American Put Premium

The premium of an American put option increases when the time to expiration increases, as in Figure 4.1 or when the strike price over the stock price increases, as shown in Figure 4.2, or when the volatility increases as shown in Figure 4.3, or when the interest rate decreases as shown in Figure 4.4.
Trying to draw similar plots on the same graphs in Figures 4.1 - 4.4 using BW, BS and LBA, we obtain overlapping plots. This is due to accurate results that BW, BS and LBA yield with respect to the binomial lattice method, as we will see further in Chapter 5.

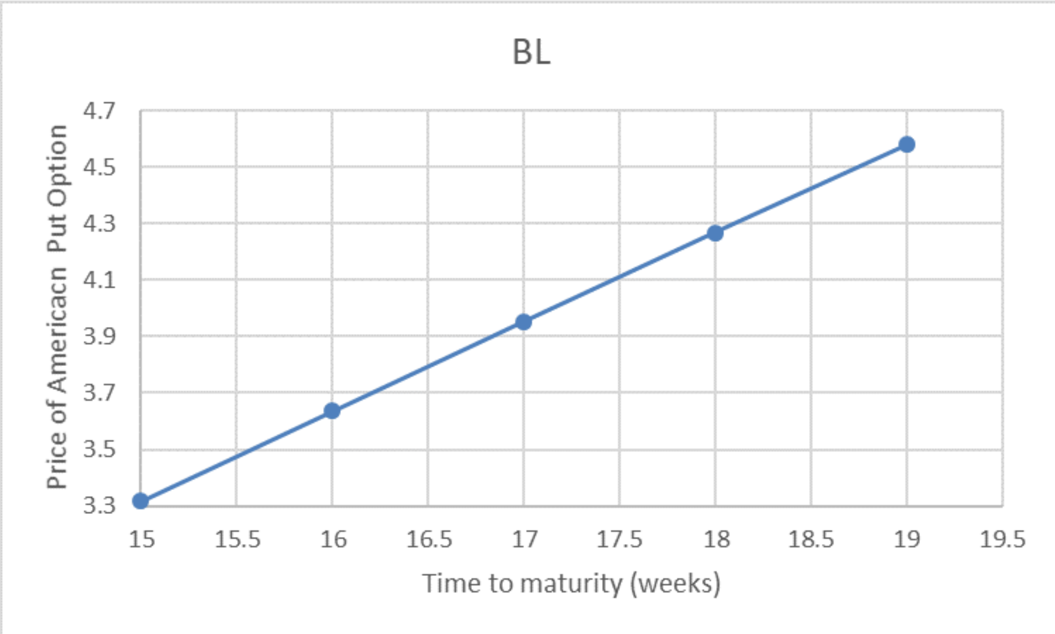


Figure 4.1: Premium of American put option with K =100, S =142, r =0.0157 and σ =0.65 in function of T in weeks for the BL model

Figure 4.2: Premium of American put option with K =100, T =1 week, r =0.0157 and σ =0.6 in function of K/S for the BL model

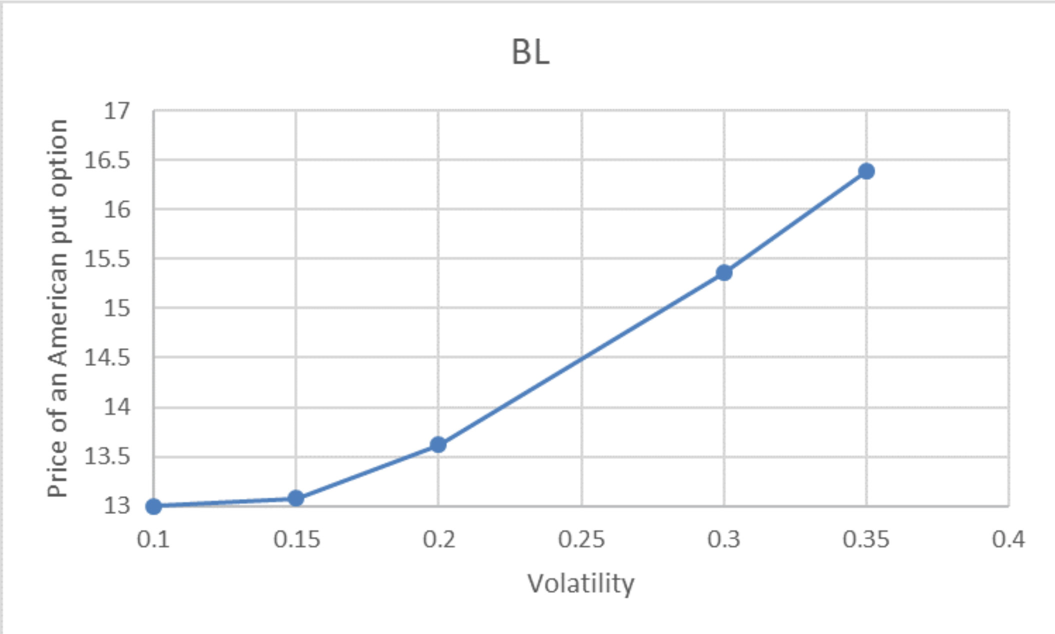


Figure 4.3: Premium of American put option with K =100, T =24 weeks, r =0.0157 and S =87 in function of σ for the BL model

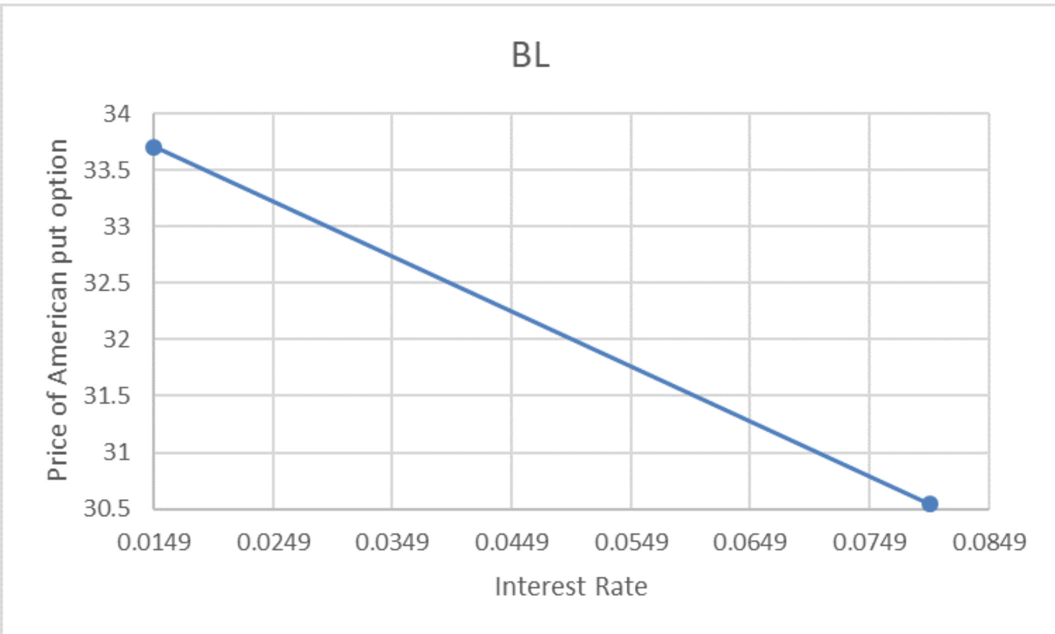


Figure 4.4: Premium of American put option with K =100, S =83, T=53 weeks and σ =0.7 in function of r in weeks for the BL model

4.2 Characteristics of American Put Critical Stock Price

The critical stock price of an American put option decreases when time to expiration increases as shown in Figure 4.5, or when the volatility increases as shown in Figure 4.6, or when the interest rate decreases as in Figure 4.7. In each of Figures 4.5-4.7, four graphs are drawn using the four methods: BL, PAAPC1 (this is our developed model, details about it are found in Chapter 5),BW and LBA, with parameter values shown in the caption of each figure. Unlike the case of the option premium, for the critical stock price, the error of BW and LBA with respect to BL can go up to 6%. This indicates that while BW and LBA exhibit valid behavior in terms of monotinicity in the input parameters, their accuracy can be improved. This is an area where our power approximation PAAPC1 offer the needed improvement, as indicated in Chapter 5 and as shown in Figures 4.5-4.7.

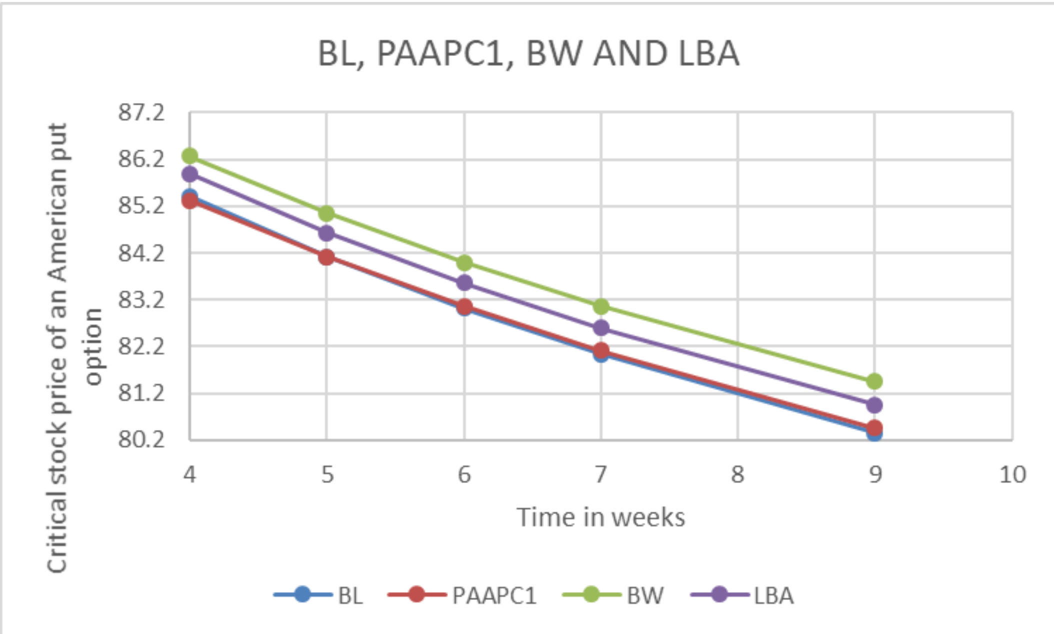


Figure 4.5: Critical Stock Price of American put option with K =100, r =0.0156 and σ =0.25 in function of T in weeks for BL, BW, LBA and PAAPc1 models

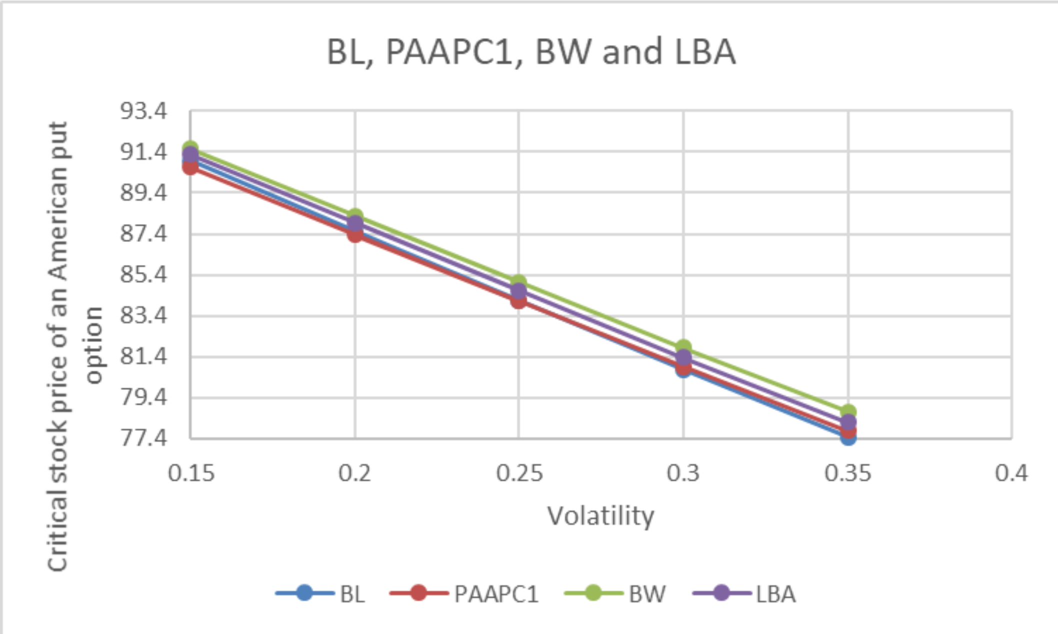


Figure 4.6: Critical Stock Price of American put option with K =100, r =0.0156 and T =5 weeks in function of σ for BL, BW, LBA and PAAPc1 models

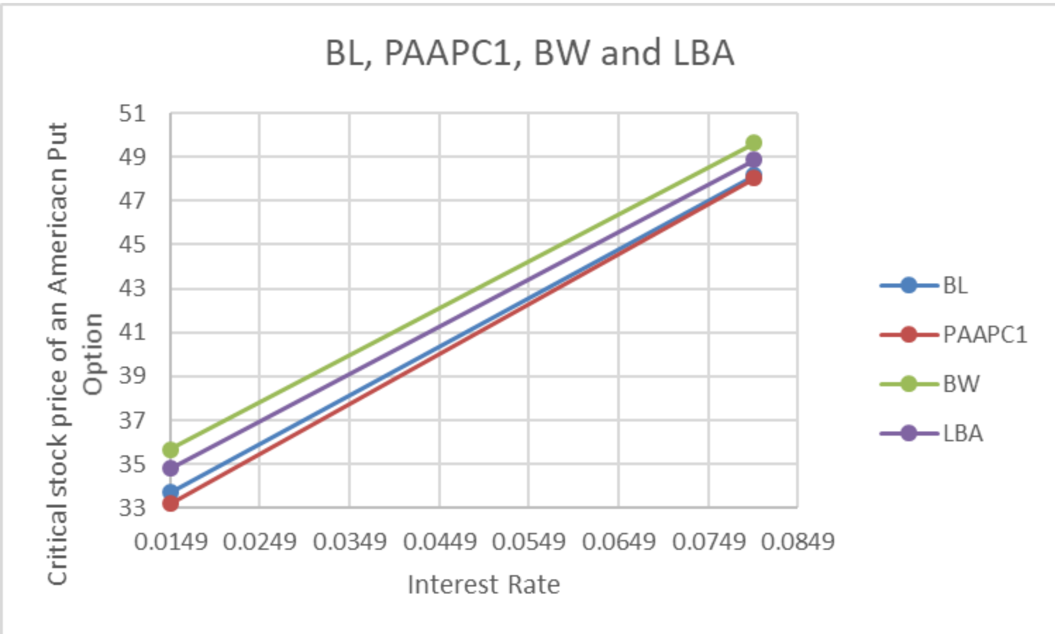


Figure 4.7: Critical Stock Price of American put option with K =100, σ=0.65 and T =33 weeks in function of r for BL, BW, LBA and PAAPc1 models

Chapter 5

The Power Approximations

5.1 PAAPC1: Power Approximation for the American Put Option Critical Stock Price

A non-linear regression model is developed, using 1,040 instances, in order to derive a closed-form approximation for the critical stock price of an American put option. As stated in Chapter 4, Barone-Adesi & Whaley (1987) suggests a Newton-method algorithm that solves an equation in order to get the critical stock price of an American put option. Barone-Adesi & Whaley (1987) provide an approximate expression for the critical stock price that provides the seed value for the iterative procedure of the Newton method algorithm. This initial approximation is shown in (15). This analytical approximation is taken as the starting point of our closed-form approximation, plus a certain regression factor fitted by non-linear regression. The following form of the correction factor is suggested. It includes three terms: one term including a combination between T, r and σ, one term including T and one term including σ. This form is reached after trying to find a compromise between the number of terms added and the accuracy of the approximation.

$$S_T^{PAAPC1} = S_T^{BAW} + CF, \tag{13}$$

where,

$$CF = a_1 T^{a_2} r^{a_3} \sigma^{a_4} + a_5 T^{a_6} + a_7 \sigma^{a_8} \tag{14}$$

$$S_T^{BAW} = S^*(\infty) + (K - S^*(\infty))e^{h_1} \tag{15}$$

$$S^*(\infty) = K/(1 - 1/q_1(\infty)),$$

$$N = 2r/\sigma^2,$$

$$q_1(\infty) = (-(N-1) - \sqrt{(N-1)^2 + 4N})/2,$$

$$h_1 = (rT - 2\sigma\sqrt{T})(K/(K - S^*(\infty))).$$

Excel solver is used to find the parameters a_i, $i = 1, \ldots, 8$, in a way that minimizes the sum of square error (SSE) between the critical stock price in (13)

and that computed "exactly" using the binomial lattice as explained in Chapter 3. The results are shown in Table 5.1. A coefficient of determination R^2 of 0.999220994 is obtained which implies a very good fit of the model.
It is to be noted the formulas of SSE) sum of squares regression (SSR), sum of squares total (SST) and R^2 are
$SSE = \sum_{i=1}^{936}(S^*_{T_i} - S^{PAAPC1}_{T_i})^2$
$SSR = \sum_{i=1}^{936}(S^{PAAPC1}_{T_i} - \bar{S^*_T})^2$
$SST = SSE + SSR$
$R^2 = \dfrac{SSR}{SST}$
where S^*_T is the exact critical stock price obtained from the binomial lattice

Parameters	Values
a_1	-135.608304
a_2	-0.012523
a_3	-0.039998
a_4	1.125138
a_5	0.971170
a_6	1.515302
a_7	152.642029
a_8	1.116479

Table 5.1: Parameters of PAAPC1

In Tables 5.2 and 5.3, a comparison between the developed model and other analytical approximations is performed for in-sample (936 instances) and out-of-sample testing (104 instances). Starting by the in-sample data, the RMSE for PAAPC1 is equal to 0.499 which is less than the RMSE for BW (1.3722) and Detemple (0.7212). The same thing applies to the average and median absolute relative error. PAAPC1 presents an average relative error equal to 0.6014% which is less than the average absolute relative error of the two other models (2.2277% for BW and 1.1459% for Detemple). And finally, also in terms of the median average relative error, PAAPC1 is ranked first with an error equal to 0.4480% (BW and Detemple have an median absolute relative error equal to 1.9693% and 0.9853% respectively). Going to the out-of-sample data, PPAPC1 is ranked first among the other methods according to the RMSE, average and median absolute relative error. It has an RMSE equal to 0.4804 (BW and Detemple have an RMSE equal to 1.4031 and 0.7351 respectively), an average absolute relative error equal to 0.6141% (BW and Detemple have an average absolute relative error equal to 2.3468% and 1.2170% respectively) and a median absolute relative error equal to 0.3764% (BW and Detemple have a median absolute relative error equal to 2.1653% and 1.0594% respectively). We can conclude that PAAPC1 is better than all other approximations in terms of accuracy and simplicity as it is in closed-form.

	BW	Detemple	PAAPC1
Measure 1: RMSE	1.3772	0.7212	0.4900
Measure 2: Average Absolute Relative Error	2.2277%	1.1459%	0.6014%
Measure 3:Median Absolute Relative Error	1.9693%	0.9853%	0.4480%
Rank according to Measure 1	3	2	1
Rank according to Measure 2	3	2	1
Rank according to Measure 3	3	2	1

Table 5.2: In-Sample PAAPC1 Accuracy

	BW	Detemple	PAAPC1
Measure 1: RMSE	1.4031	0.7354	0.4804
Measure 2: Average Absolute Relative Error	2.3468%	1.2170%	0.6141%
Measure 3:Median Absolute Relative Error	2.1653%	1.0594%	0.3764%
Rank according to Measure 1	3	2	1
Rank according to Measure 2	3	2	1
Rank according to Measure 3	3	2	1

Table 5.3: Out-Sample PAAPC1 Accuracy

Figures 4.5 to 4.7 (in chapter 4) show that the characteristics of an American put option critical stock price listed in chapter 4 are verified by PAAPC1.

5.2 PAAP1: Power Approximation for the American Put Option Price

A non linear regression model is developed, using 10,296 instances, in order to derive a closed-form approximation for the price of an American put option. The Barone-Adesi Whaley equation is used as a starting point of our approximation. The first step is to replace, in the Barone-Adesi Whaley formula, the critical stock price obtained by the Newton-Method algorithm by the critical stock price obtained by PAAPC1. The second step is to add a correction factor to q_1 fitted by regression. The correction factor is suggested to be the sum of combinations between the input parameters r, K/S, T and σ. There are 15 possible combinations considering these parameters (1 combination including the 4 for them, 4 combinations including 3 of them each, 6 combinations including two of them each and 4 combinations including each parameter alone). Several experiments are done including a certain number of combinations each time. The purpose of these experiments is to find a compromise between the model accuracy and the number of terms added in the correction factor. The final result is a correction factor including 5 combinations (1 combination including the 4 terms and 4 combinations including 3 parameters each).

$$P(S,T) = \begin{cases} p(S,T) + A_1(\dfrac{S}{S_T^{PAAPC1}})^{q_1+CF_1} & \text{if } S > S_T^{PAAPC1} \\ K - S & \text{if } S \leq S_T^{PAAPC1} \end{cases} \tag{16}$$

where,

$$\begin{aligned} CF_1 &= b_1(K/S)^{b_2}T^{a_3}r^{b_4}\sigma^{b_5} + b_6(K/S)^{b_7}T^{b_8}r^{b_9} + b_{10}T^{b_{11}}r^{b_{12}}\sigma^{b_{13}} \\ &\quad + b_{14}(K/S)^{b_{15}}r^{b_{16}}\sigma^{b_{17}} + b_{18}(K/S)^{b_{19}}T^{b_{20}}\sigma^{b_{21}} \end{aligned} \tag{17}$$

Excel solver is used to find the unknown parameters in the following equation that gives the least sum of square error. The formulas for SSE, SSR, SST and R^2 are presented in the previous section. The results are shown in Table 5.4. A R^2 of 0.999993838 is obtained which implies a very good fit of the model.

Parameters	Values
b_1	0.901197
b_2	19.669270
b_3	-0.079462
b_4	1.994407
b_5	-4.560642
b_6	-0.688482
b_7	0.577829
b_8	1.615944
b_9	0.535818
b_{10}	-0.359980
b_{11}	-0.745330
b_{12}	0.083726
b_{13}	-2.180157
b_{14}	-1.225353
b_{15}	19.433199
b_{16}	2.091176
b_{17}	-4.557230
b_{18}	0.403516
b_{19}	0.950459
b_{20}	-0.782347
b_{21}	-1.988778

Table 5.4: Parameters of PAAP1

In table 5.5 and 5.6, a comparison between the developed model and other analytical approximations is performed for in-sample (10,296 instances) and out-of-sample (1,144 instances) testing. For the in-sample data, PAAP1 is ranked first according to the median absolute relative error with a value equal to 0.0795% (BW, Detemple and BS have a median absolute relative error equal to 0.1944%, 0.1224% and 0.3927% respectively). According to RMSE, PAAP1 is ranked second with a value equal to 0.0184 behind Detemple which has an RMSE equal to 0.0164. (BW and BS have an RMSE equal to 0.0317 and 0.0546 respectively). For the in-sample data, PAAP1 is ranked first according to the median absolute relative error with a value equal to 0.0815% (BW, Detemple and BS have a median absolute relative error equal to 0.2067%, 0.1222% and 0.4049% respectively). According to RMSE, PAAP1 is ranked second with a value equal to 0.0159 slightly behind Detemple which has an RMSE equal to 0.0158. (BW and BS have an RMSE equal to 0.0322 and 0.0548 respectively). We can conclude that for both in-sample and out-of-sample data sets, PAAP1 is ranked first according to the median absolute relative error and second behind Detemple according to RMSE. However, as mentioned in a previous function, Detemple needs a higher computational effort, since an optimization problem should be solved to get the price of the option. Thus, we can conclude that our method performs better, since the error difference between both methods is about 0.002 for the in-sample data and 0.0001 for the out-of-sample data (negligible error). Thus, We can conclude that our model outperforms the other analytical approximations present in the literature.

	BW	Detemple	BS	PAAP1
Measure 1: RMSE	0.0317	0.0164	0.0546	0.0184
Measure 2:Median Absolute Relative Error	0.1947%	0.1224%	0.3927%	0.0795%
Rank according to Measure 1	3	1	4	2
Rank according to Measure 2	3	2	4	1

Table 5.5: In-Sample PAAP1 Accuracy

	BW	Detemple	BS	PAAP1
Measure 1: RMSE	0.0322	0.0158	0.0548	0.0159
Measure 3:Median Absolute Relative Error	0.2067%	0.1222%	0.4049%	0.0815%
Rank according to Measure 1	3	1	4	2
Rank according to Measure 2	3	2	4	1

Table 5.6: Out-Sample PAAP1 Accuracy

Figures 5.1 to 5.4 show that the characteristics of an American put option price listed in section 5.1 are verified by PAAP1.

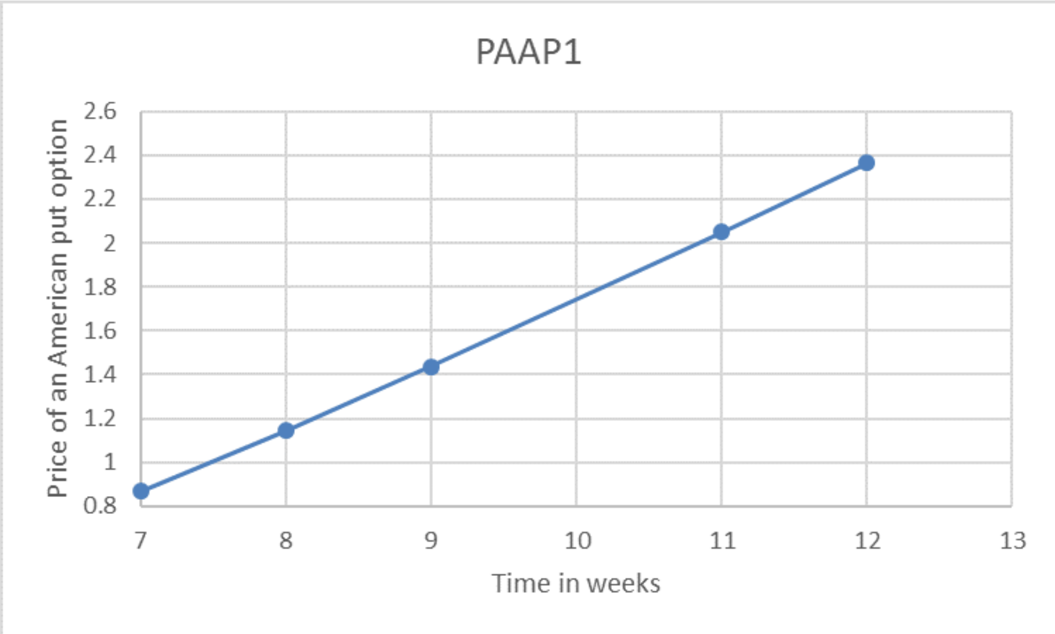


Figure 5.1: Premium of American put option with K =100, S =142, r =0.0157 and σ =0.65 in function of T in weeks for PAAP1 model

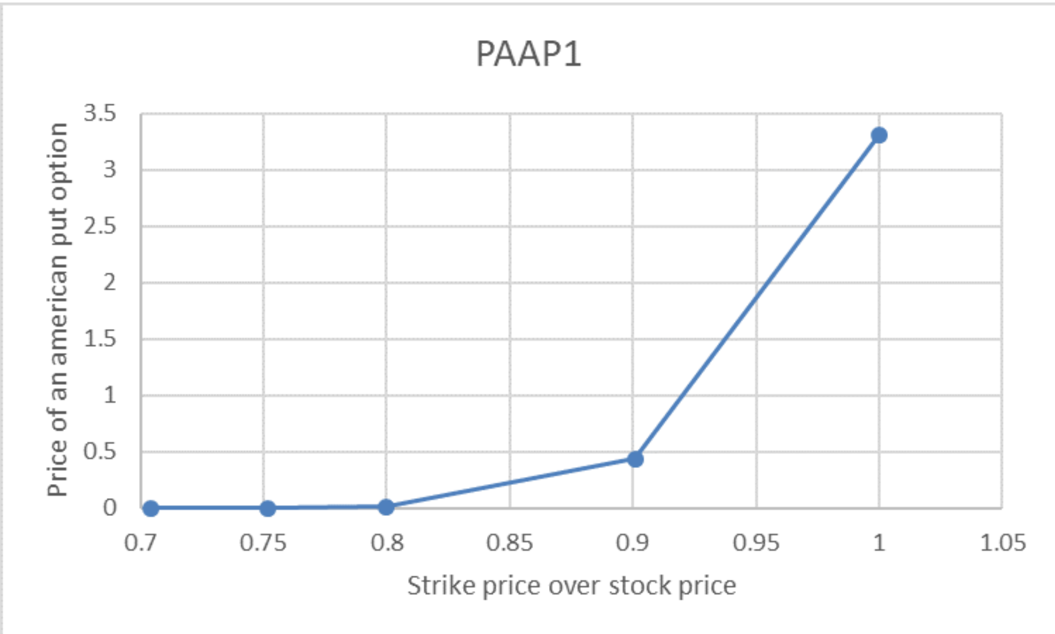


Figure 5.2: Premium of American put option with K =100, T =1 week, r =0.0157 and σ =0.6 in function of K/S for PAAP1 model

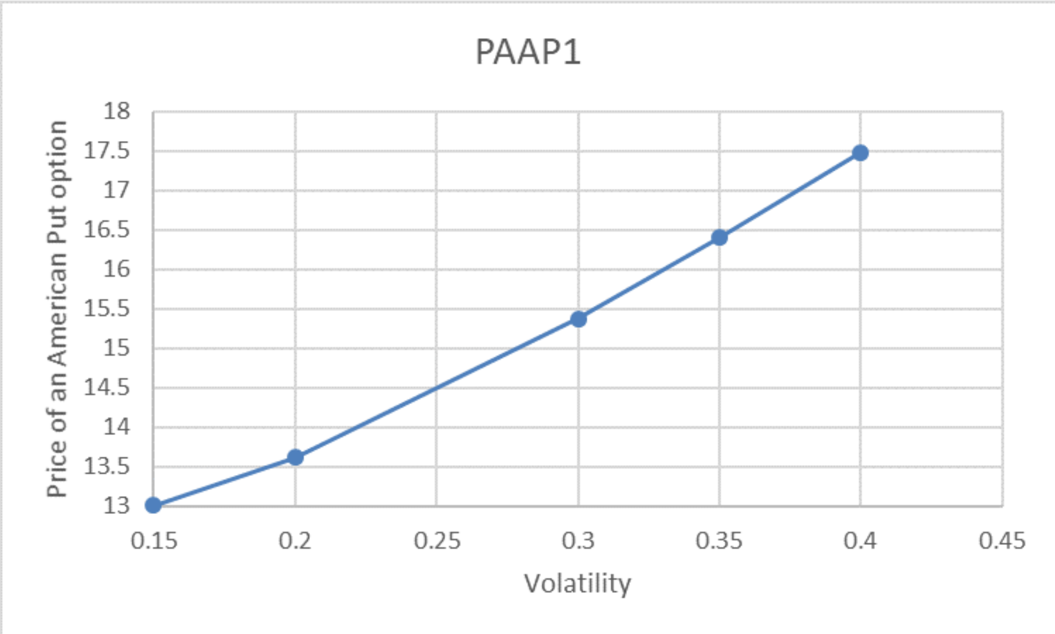


Figure 5.3: Premium of American put option with K =100, T =24 weeks, r =0.0157 and S =87 in function of σ for PAAP1 model

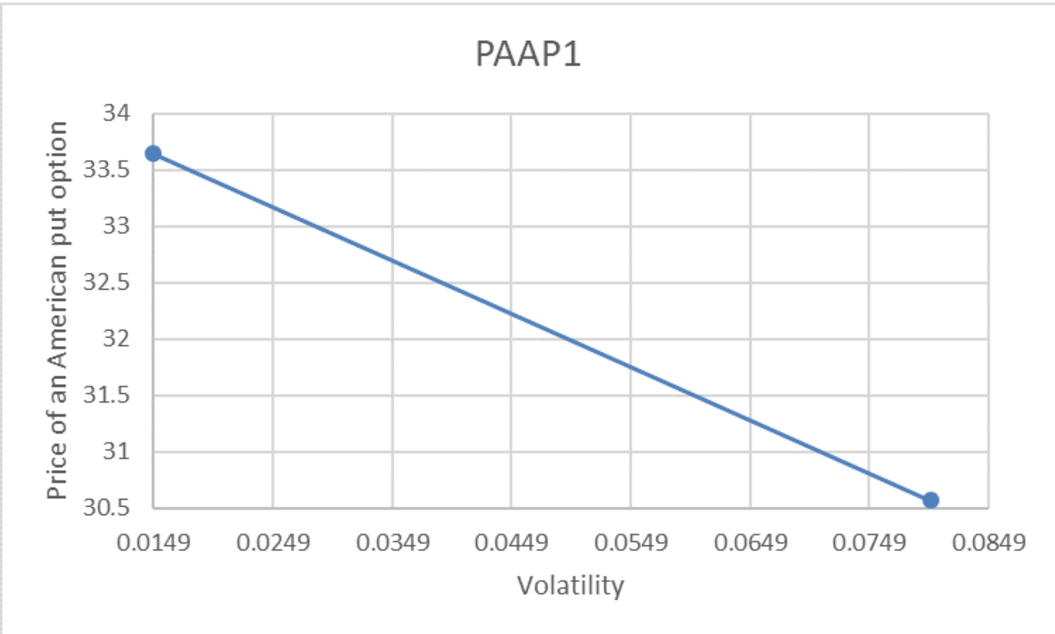


Figure 5.4: Premium of American put option with K =100, S =83, T=53 weeks and σ =0.7 in function of r in weeks for PAAP1 model

Chapter 6

Conclusion and Ideas for Future Work

We propose a method for pricing American put options is based on a power approximation approach. Our aim is to obtain a closed-form approximation of the price of an American put option and its critical stock price, that is easy to compute, and at the same time highly accurate.Our first approximation for determining the critical stock price is developed as follows. PAAPC1 is developed. The approximate analytical expression for the critical stock price derived by Barone-Adesi & Whaley (1987) is taken as the starting point of our approximation, then a regression factor fitted by non-linear regression is added. PAAPC1 beats other approximations (Barone-Adesi & Whaley (1987) and Broadie & Detemple (1996)) in terms of accuracy and computational effort. Our second power approximation for the option premium is developed as follows. Barone-Adesi & Whaley (1987) apprxoimation is improved by initially, replacing the critical stock price obtained by the Newton-Method algorithm by the critical stock price obtained by PAAPC1, and then by adding a correction factor fitted by non-linear regression. It is shown that this formula performs better than the other analytical approximations present in the literature (Barone-Adesi & Whaley (1987), Bjerksund & Stensland (1993) and Broadie & Detemple (1996)).

Future work includes enhancing the developed models to improve their accuracy by trying different form on the correction factors (e.g. trying different polynomial combinations or trying forms that are not even polynomial). Adding to that, this methodology can further developed and tweaked to be implemented on other types of financial instruments such as interest rate derivatives. The pricing problem of an interest rate option is a complicated one. Interest rate behavior over time must be modeled over time in order to price such a contract. Consequently, the price will depend on the model used to describe the volatility and behavior of the interest rate (Antl, 1988).

We acknowledge the limitation of our work of pricing a plane vanilla American options where the underlying asset price follows a stationary Brownian motion. However, this thesis provides a useful illustration on the usefulness of regression-based power approximations in pricing financial derivatives within a tractable and familiar framework. We hope that this will motivate future research on applying such power approximations for calibrating pricing schemes for more complex securities, where the underlying asset follows realistic processes such as those having stochastic volatility, mean reversion, and time dependent parameters, among other ramifications. We believe that power approximations will prove more useful in the context of these sophisticated derivatives, as the existing pricing models are heavily computational.

Appendix A

Abbreviations

AARE	Average absolute relative error
BL	Binomial Lattice
BS	Bjerksund & Stensland
BW	Barone Whaley
LBA	Lower Bound Approximation (Detemple)
SSE	Sum of squares error
SSR	Sum of squares regression
SST	Sum of sqaures error
R^2	Coefficient of determination

Appendix B

Excel Formula

The following Excel function is used to randomly pick an out-of-sample data consisting of 10 % of the of the total exact values,
IF(ROW()>10%*COUNTA($A$2:$A$5721),"",INDEX($A$2:$A$5721,
RANK.AVG(H2,$H$2:$H$5721,0),1))
The goal of the above expression is to select 10 percent of the 5,720 data points existing in the range A2:A5721. The following steps explain the concept behind this expression:
1- Use the random number generator to assign a unique random number between 0 and 1 to each row with the volatile function RAND()
2- Use the RANK.AVG() function in order to rank those numbers in a descending order implicitly. (A random rank will be given for each row)
3- Use INDEX() to select the state corresponding to the rank assigned by RANK.AVG().
4- Use the IF() function to check that the number of required steps is reached i.e. 10 percent of 5720 is reached.
5- Use the ROW() function to get the row number of the cell reference.
6- If the ROW() function returns a number higher than 10 percent of 5720, the value is set to empty.

Appendix C

Newton-Method Algorithm

This algorithm applies the Newton-Raphson technique is developed for solving equation (4):

1) $q_1(\infty) = \dfrac{-(N-1) - \sqrt{(N-1)^2 + 4M}}{2}$

2) $h_1 = (rT - 2\sigma\sqrt{T})(\dfrac{K}{K - S^*(\infty)})$

3) $S^*(\infty) = \dfrac{K}{1 - \dfrac{1}{q_1(\infty)}}$

4) Seed Value: $S^* = S^*(\infty) + (K - S^*(\infty))(e^{h_1}) = S_i$

5) LHS=$K - S_i$

6) RHS=$p(S_i, T) - (1 - N[-d_1(S_i)])\dfrac{S^*}{q_1}$

7) Slope of RHS: $b_i = -N[-d_1(S_i)](1 - \dfrac{1}{q_1}) - \dfrac{(1 + \dfrac{n[-d_1(S_i)]}{\sigma\sqrt{T}}}{q_1}$

8) $S_{i+1} = \dfrac{K - RHS + b_i S_i}{1 + b_i}$

9) Step 8 will give the second guess of S^*. Then, we will have an iterative process until having the following relative absolute error:

$\dfrac{| LHS - RHS |}{K} < 0.0000001$

Appendix D

Binomial Lattice To Get the Critical Stock Price

As mentioned in Chapter 3, a bisection method is implemented in order to calculate the critical stock price using a binomial lattice, price at which the continuation value and the early exercise value are approximately equal. An example is given next to illustrate the idea. Suppose we need to calculate the critical stock price at time $t = 0$ of an an American put option at time to maturity $T = 5/12$ years with $r = 0.1$, $\sigma = 0.2$ and $K = 60$ utilizing a lattice with step $\Delta t = 1/12$ years = 1 month.
Iteration 1: Since the critical stock price is always less than the strike price K, we set a starting value for the stock price at time 0, S_0, equal to strike price value of $K = 60$. Figure D.1 and the first row of Table D.1 report results verifying that it is not optimal to exercise the put option when $S_0 = K$, and that the critical stock price is indeed below 60.
Iteration 2: For this iteration, he stock price at time 0, S_0, is equal to 45. Figure D.1 and the second row of Table D.1 report results verifying that it is optimal to exercise the put option when $S_0 = 45$. However, the difference between $E[V_c]$ and $K - S_0$ is far from zero. Indeed, the critical stock price is above 45.
Iteration 3: For this iteration, he stock price at time 0, S_0, is equal to 52.5. Figure D.2 and the second row of Table D.1 report results verifying that it optimal to exercise the put option when $S_0 = 52.5$. However, the difference between $E[V_c]$ and $K - S_0$ is far from zero. Indeed, the critical stock price is above 52.5.

For the next iterations, the bisection method will search for the critical stock price, this is equivalent to searching for the initial stock price, at which the difference between the exercise value and the continuation value is very small (less than than the precision, equal to 10^{-9}). Thus, for each iteration, a new binomial lattice is built with a new initial stock price and the difference between $K - S_0$ and $E[V_c]$ is checked. As shown in Table D.1, 30 iterations are needed to reach the critical stock price, which is reported in the last row of the table at a value

$S_0^* = 54.2411$. This value is accurate at a precision 10^{-9}. Iteration 1 to 10 and of the final iteration are described in details in Table D.1 with the outcome of each iteration. We can see in Figures D.1 and D.2 four binomial lattices built for the first four iterations. A similar procedure is followed for all other iterations.

Iteration	S_0	Value	$K - S_0$	$E[V_c]$	$\mid E[V_c] - (K - S_0) \mid$	Exercise Option?
1	60.0000	2.3011	0	2.3011	2.3011	NO
2	45.0000	15.0000	15.0000	14.5021	0.4979	YES
3	52.5000	7.5000	7.5000	7.1281	0.3719	YES
4	56.2500	4.2422	3.7500	4.2422	0.4922	NO
5	56.2500	5.6539	5.6250	5.6539	0.0289	NO
6	53.4375	6.5625	6.5625	6.3890	0.1735	YES
7	53.9063	6.0937	6.0937	6.0214	0.0723	YES
8	54.1406	5.8594	5.8594	5.8377	0.0217	YES
9	54.2578	5.8377	5.7422	5.8377	0.0036	NO
10	54.1992	5.8008	5.8008	5.7917	0.0091	YES
...	...	...	...	...	...	...
27	54.2411	5.7589	5.7589	5.7589	$3.0631\ 10^{-8}$	NO
28	54.2411	5.7589	5.7589	5.7589	$6.5050\ 10^{-9}$	NO
29	54.2411	5.7589	5.7589	5.7589	$5.5582\ 10^{-9}$	YES
30	54.2411	5.7589	5.7589	5.7589	$4.7344\ 10^{-10}$	YES

Table D.1: Bisection Method Iterations

Iteration 1

Stock Price Lattice

t = 0	t = 1	t = 2	t = 3	t = 4	t = 5
60.00	63.57	67.34	71.35	75.59	80.08
	56.63	60.00	63.57	67.34	71.35
		53.46	56.63	60.00	63.57
			50.46	53.46	56.63
				47.63	50.46
					44.96

Option Value Lattice

t = 0	t = 1	t = 2	t = 3	t = 4	t = 5
2.30	1.02	0.28	0.00	0.00	0.00
	3.96	1.98	0.65	0.00	0.00
		6.54	3.69	1.48	0.00
			9.54	6.54	3.37
				12.37	9.54
					15.04

Exercise Option?

	t = 1	t = 2	t = 3	t = 4	t = 5
No	No	No	No	No	No
	No	No	No	No	No
		Yes	No	No	No
			Yes	Yes	Yes
				Yes	Yes
					Yes

Iteration 2

Stock Price Lattice

t = 0	t = 1	t = 2	t = 3	t = 4	t = 5
45.00	47.67	50.51	53.51	56.69	60.06
	42.48	45.00	47.67	50.51	53.51
		40.09	42.48	45.00	47.67
			37.84	40.09	42.48
				35.72	37.84
					33.72

Option Value Lattice

t = 0	t = 1	t = 2	t = 3	t = 4	t = 5
15.00	12.33	9.49	6.49	3.31	0.00
	17.52	15.00	12.33	9.49	6.49
		19.91	17.52	15.00	12.33
			22.16	19.91	17.52
				24.28	22.16
					26.28

Exercise Option?

	t = 1	t = 2	t = 3	t = 4	t = 5
Yes	Yes	Yes	Yes	Yes	No
	Yes	Yes	Yes	Yes	Yes
		Yes	Yes	Yes	Yes
			Yes	Yes	Yes
				Yes	Yes
					Yes

Figure D.1: Binomial Lattices corresponding to Iteration 1 and 2

Iteration 3

Stock Price Lattice

t = 0	t = 1	t = 2	t = 3	t = 4	t = 5
52.50	55.62	58.93	62.43	66.14	70.07
	49.55	52.50	55.62	58.93	62.43
		46.77	49.55	52.50	55.62
			44.15	46.77	49.55
				41.67	44.15
					39.34

Option Value Lattice

t = 0	t = 1	t = 2	t = 3	t = 4	t = 5
7.50	4.61	2.39	0.84	0.00	0.00
	10.45	7.50	4.38	1.92	0.00
		13.23	10.45	7.50	4.38
			15.85	13.23	10.45
				18.33	15.85
					20.66

Exercise Option?

	t = 1	t = 2	t = 3	t = 4	t = 5
Yes	No	No	No	No	No
	Yes	Yes	Yes	No	No
		Yes	Yes	Yes	Yes
			Yes	Yes	Yes
				Yes	Yes
					Yes

Iteration 4

Stock Price Lattice

t = 0	t = 1	t = 2	t = 3	t = 4	t = 5
56.25	59.59	63.14	66.89	70.86	75.07
	53.09	56.25	59.59	63.14	66.89
		50.12	53.09	56.25	59.59
			47.30	50.12	53.09
				44.65	47.30
					42.15

Option Value Lattice

t = 0	t = 1	t = 2	t = 3	t = 4	t = 5
4.25	2.20	0.81	0.08	0.00	0.00
	6.91	3.99	1.74	0.18	0.00
		9.88	6.91	3.75	0.41
			12.70	9.88	6.91
				15.35	12.70
					17.85

Exercise Option?

	t = 1	t = 2	t = 3	t = 4	t = 5
No	No	No	No	No	No
	Yes	No	No	No	No
		Yes	Yes	Yes	Yes
			Yes	Yes	Yes
				Yes	Yes
					Yes

Figure D.2: Binomial Lattices corresponding to Iteration 3 and 4

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