AMERICAN UNIVERSITY OF BEIRUT

RISK ANALYSIS OF PROJECT FEASIBILITY UNDER RANDOM PAYMENT DELAYS

by
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AN ABSTRACT OF THE THESIS OF

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Net present value has been widely used to evaluate project financial feasibility, along with the amount of risk associated with each alternative. Under certain assumptions, probabilistic information regarding cash flows and their timing can be used to account for the uncertainty surrounding the project.

In our research, we focused on time constrained investments, where cash flows might exceed their due date by a random period specified by the decision maker. For such projects, we derived closed form expression of the net present value distribution under different payment delays, for single period projects. For multiple-period cash inflows with identically distributed payment delays we provide analytical tools to fit suitable distribution to model NPV without relying on the time-consuming Monte Carlo simulation process.

Our aim is to provide an accurate estimation for the distribution of NPV of projects with most encountered payment delay distribution(uniform, triangular, PERT and exponential). This work enhances the decision-making process and provide investors and decisions makers a tool for evaluating the risk of delaying payment on project feasibility.

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To My
Beloved Parents

CHAPTER 1

INTRODUCTION AND MOTIVATION

This chapter is dedicated to present a general background of our research work. Section 1 is an overview on investment decision making and how it is addressed by managers. Then section 2 outlines the main factors behind our motivation to address investment analysis under time uncertainty.

A. Investment Decision Making

Two main factors are relevant in investment decision making: the estimated return as well as the risk associated with it. When doing a feasibility analysis, investors usually rely on discounting future cash flows according to their timing, which is known as the net present value of a project. In general, these cash flows are assumed to be deterministic. However, risk is always present depending on different factors such as the amount of each cash flow, the timing, and the interest rate. In today's challenging business environment, investment analysis must account for the uncertainty in these factors to provide more realism in the decision-making process.

In recent years, several methods have been used to estimate the net present value of a project under uncertainty. When cash flows cannot be estimated with certainty, the most simplified approach bases the present value estimation on the expected value of the sum discounted cash flows. However, analyzing proposed investments as based on expected of present value, does not provide the full picture. For example, a safe investment with a certain return might be preferred over a riskier one with relatively higher expected return. As a result, the variability of the present value must be included in these procedures to allow investors to sort out their preferences according to the risk involved in each investment. Relying on a single expected value rather than computing all possible outcomes with their probability of occurrence may not enough. A probabilistic approach provides a more realistic feasibility analysis regarding the profitability of a given investment because it allows the considering of all possible outcomes with their probability of occurrence. Another frequently used method to analyze uncertainty is the sensitivity analysis which investigates the profitability with by changing the economic parameters of a given project. The limitation of this method is that it provides a range of possible outcomes without their likelihood of occurrence.

Monte Carlo Simulation is commonly used when the parameters of cash flows have their own probability distribution and analytical calculation becomes difficult. It was first introduced by Hertz (1964) to help managers select investments among different alternatives. But the process requires some time to iterate all scenarios and some organizations may lack the finances and the expertise to utilize simulation software.

In this thesis, we propose an efficient alternative to Monte Carlo simulation for analyzing a project having cash flows with certain values but having uncertain timing due for example to delay in payments by customers. Our approach is based on fitting a suitable distribution to the net present value (NPV) of the project by matching the first two moments. We find that the Beta and the Gumbel distribution to provide good fits for most common distribution of payment delays (uniform, triangular, PERT, exponential).

B. Motivation

Cash flows might be generated by different sources; therefore, executives rely on the methods above to account for risk generated by each source. Any change in interest rate, time or return will add uncertainty to the proposed investment. Recently, more attention has been given on uncertain payment times and their impact on investment decision making. Major companies are facing high risk in getting their payments on time. Payment delays have threatened leading businesses in the MENA region specifically in Saudi Arabia and UAE, out of which we name Saudi Oger, one of the leading firms in construction sector. The company has been struggling for the last couple of years due to government payment delays and was hit hard by the slowdown in the construction sector and delays in government payments. It was forced to fire thousands of workers and sell most of its assets. Due to lack of liquidity the firm has not been able to meet its obligations towards suppliers, subcontractors, banks and employees and thus had shut down its operations with a total debt of 83.7 million Saudi riyals which is approximately $22.3 million (Gulf business, 2019).

Another collapse of a major international construction and contracting firm, with many projects in the Gulf Arab states is that of Carillion, a giant British construction firm. The company went bankrupt by taking many risky projects in Gulf States and failed to collect her bills from clients. The straw that broke the proverbial camel's back was the failure of the Qatari government to settle a £200 million bill related to the FIFA world cup 2022 project for almost a year (The Guardian, 2018).

According to a research conducted by Coface, a credit management company in UAE, payment terms rose from 120 to 243 days, on average, for companies operating in metals and construction sector in 2015, while the IT sector average payment terms increased by more than three times (Coface, 2015). Another survey conducted by Moussa (2019) on a sample of manufacturing companies in Arab countries shows that their reliance on credit facilities as a financing vehicle extends the credit period to 180 days.

Our work provides decision makers with an efficient analytical tool to make accurate judgment regarding the risk in their investment opportunities. We consider cash flows that exceed their due date by a certain random period. Given the historical data of client payments track, it is assumed that the decision maker can estimate the parameters of the delay's distribution. We use the individual delay distributions to estimate the distribution of the present value.

The resulting model evaluates the NPV of a series of cash flows with risk of payment delay. This research bridges a gap in the literature regarding investment analysis under uncertainty of cash flows timing.

The reminder of this thesis is organized as follows. In Chapter 2, we summarize the literature related to analyzing uncertainty in investment analysis and how the NPV is evaluated. Then, in Chapter 3, we consider a simple one period investment having one future cash inflow with a certain value but random timing and derive closed-form expressions for the probability distribution of the net present value. In Chapter 4, we extend the study for a series of multiple cash flows and describe the model that fits the beta and the Gumbel distribution into the data, with some illustrating numerical examples. Chapter 5 summarizes our results and explores possible extensions and ideas for future research.

CHAPTER 2

LITERATURE REVIEW

Our research is linked to three areas of literature. Section 1 summarizes the general methods used in NPV evaluation. Section 2 reviews a brief background on project valuation with stochastic cash flows. Section 3 describes the NPV approximation under uncertain timing and provides some mathematical expressions which we use in our analysis.

A. General Methods for NPV Valuation

Risk and return are the two main components of the prism through which an investment is analyzed. Although different strategies can be applied to evaluate the profitability of a given project, the main concern of decision makers is to maximize their earnings while maintaining a low risk level. For deterministic cases, where cash flows values, cash flows timing, and the interest rate are known with certainty, the NPV is simply the sum of discounted cash flows. However, in reality, cash flows and interest rates are subject to uncertainty. In such cases, the net present value becomes a random variable with a distribution that depends on the distributions of the cash flows and interest rates.

NPV has been the most popular tool used in valuation. It consists of discounting future cash flows at an interest rate that considers the cost of capital which typically ranges between 10% and 15% for corporate projects. Hodder & Riggs (1985) and Chapman & Cooper (1987), present many adjustments to the NPV to address the weaknesses of the traditional method and account for risk in long term projects. Stewart, Allison, & Johnson

(2001) develop a risk adjusted NPV obtained by subtracting the risk adjusted cost from the risk adjusted payoff. Another method is the stochastic NPV which allows components of cash flows to be random variable with known probability distribution usually provided by the decision maker. In this case the NPV becomes a random variable with a specific probability distribution. Since it has three main components, the cash flows, their timing, and the discount rate, uncertainty will be evaluated based on the analysis of these parameters. Probabilistic extensions have been published to deal with stochastic information surrounding the net present value of a given project.

B. Stochastic Cash Flow Values

In one of the first contributions to the field of discounted cash flow analysis under stochastic conditions, Hillier (1963) presents an analytical method to determine the probability distribution function of the net present value and internal rate of return of random-value cash flows with certain timing. By estimating the mean and variance for independent and perfectly correlated cash flows, and by using the central limit theorem, the probabilities can be computed using the normal distribution. The limitation of this method was that the decision maker must provide the mean and variance of individual cash flow values. Wagle (1967) addresses this concern and introduces a similar method where these two parameters are computed from data. This method is known as PERT (program evaluation and review technique), where optimistic, pessimistic, and most likely values for cash flows are estimated by the decision maker and the mean and variance are computed accordingly. Hillier (1969) extends his research to derive the correlation between cash flows that are described using a Markov process. Bussey & Stevens (1972) extend these methods to another approach to estimate the correlations coefficient in various periods.

Kim & Elsaid (1985) introduce a safety margin to the value of uncertain cash flows and examine their impact on the net present value of the project. Tung (1992) performs a numerical experiment to identify the appropriate net present value distribution for commonly used distributions describing the probabilistic behavior of cash flows. He concludes that the normal approximation for the net present value distribution is acceptable.

The limitation of these studies is the assumption that cash flows occur at deterministic timing. However, uncertainty is not just related to the amount of return, it is also linked to the timing.

C. Stochastic Timing of Cash Flows

Real life projects are subject to payment delays especially if there is no contract that forces the other party to abide by a certain due date. In general, an investment is feasible if the expected present value is positive. Wagle (1967) considers the life span of a project to be stochastic and combines this information to derive the expected present worth and its variance. He then uses Tchebycheff's inequality to find the probability of the net present value lies with a given range. Young (1983) introduces other extensions to the work of Wagle (1967) by deriving expressions to calculate the moment of the net present value of a project under different cash flow profiles. Zinn, Lesso, & Motazerd (1977) examine the present worth, the variance and semi-variance of several cash flow profiles under continuous compounding. Young & Contreras (1975) extend their contribution to stochastic investment analysis to random lump-sum cash flows occurring at random times, and uniform cash flows with random starting and cessation times. They present formulas for deriving the expected value of cash flows under different timing distribution. Chen & Manes (1986) argues that when cash flows timings are uncertain, they should be addressed as random variable rather than using their expected values to avoid over estimating the expected present worth. But decision makers care about the confidence level of realizing a certain profit. Since they think probabilistically about their cash flows and their timing, the analysis must provide more information regarding the risk of the profitability. Rosenthal (1978) presented new formulas for calculating the variance of cash flows under random timing.

In one of the most recent publications related to this topic, Creemers (2018) studies a project with multiple stages executed in sequence, cash flows are incurred at random times. He derives expressions for the pdf of the NPV of a project having multiple stages with random distributions. When these stages becomes large enough, the duration of the project can be approximated to a normal distribution and the distribution of the NPV becomes lognormal. For multiple cash flows and multiple stages with random durations, he derived the lognormal distribution that fits the NPV by matching the first three moments in order to get a bounded distribution. Our work is in the same vein of Creemers'. However, we seek better ways to approximate the distributions of NPV for cash flows with uncertain timing governed by delays. For single cash flow, we consider a single stage and a delay with specific distribution and derive expressions for the pdf of the NPV, while for multiple cash flows we can seek a better fit by matching the first two moments instead of three. And this has been shown to be valid since the simulation we ran for different distributions of delay suggested that the distribution of the NPV with random payment delays is not always normal and not even lognormal.

CHAPTER 3

SINGLE-WITH RANDOM TIMING DELAY ANALYSIS

In this chapter is dedicated for the analysis of a single cash flow with random payment delay. In section 1 we present the general method for evaluating the distribution of the NPV, then in section 2,3,4 and 5 we derive expressions for the distribution of the NPV of single cash flow with the uniform, triangular, exponential and PERT distributions of delay and support our findings by numerical examples.

A. The General Case

Let P be the initial investment of a project, F the associated cash (in)flow that is due after one period and r the annual interest rate compounded continuously. This payoff can be delayed by a random duration T.

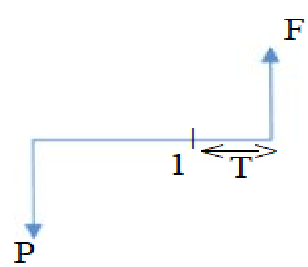


Figure 1: Single period cash flow diagram with random payment delay

Under continuous compounding, the NPV is given by

$$NPV = -P + Fe^{-r(1+T)}$$

The cumulative distribution function (CDF) of NPV, *G*(*x*), can then be derived as follows:

$$G(x) = P(NPV \leq x) = P(-P + Fe^{-r(1+T)} \leq x) = P(T \geq -\frac{1}{r} ln(\frac{x+P}{F}) - 1)$$

Therefore,

$$G(x) = 1 - H(-\frac{1}{r} ln(\frac{x+P}{F}) - 1),$$

(1)

Where, *H*(.) is the CDF of *T*.

The feasibility condition reduces to

$$P(NPV \geq 0) = 1 - G(0) = H(-\frac{1}{r} ln(\frac{P}{F}) - 1)$$

(2)

We consider four of the most common used distributions for the delay in payment, the uniform, triangular, PERT and exponential. Usually, decision makers provide a range (*a*, *b*) within which the cash flow will occur, this is modeled by the uniform distribution (*a*, *b*). They can also add more accuracy to the estimation by providing a most likely value, *c*, which gives a triangular distribution with parameters (*a, c, b*). A smoother alternative to the triangular distribution which can be fit via a three-point estimate is the PERT distribution, which is a special case of the Beta distribution. Finally, when the payment time is highly unpredictable, and waiting the payment for a long time does not imply it is going to occur soon, as in the case of some highly unpredictable (memoryless) clients, the exponential distribution is a goof model for the payment delay.

B. Uniformly Distributed Delay

Assume that the delay T is uniform on (a, b). In this case the probability distribution function (pdf) and CDF of T are given by

$$h(t) = \begin{cases} \dfrac{1}{b-a}, & a \le t \le b \\ 0 \;, & otherwise \end{cases}$$

$$H(t) = \begin{cases} 0, & t < a \\ \dfrac{t-a}{b-a}, & a \le t \le b \\ 1\,, & otherewise \end{cases}$$

Following (1) for and simplifying, the CDF of the NPV is

$$G(x) = \frac{b + \frac{1}{r}\ln\left(\frac{x+P}{F}\right) + 1}{b-a}$$

The pdf of NPV is $g(x) = \frac{dG(x)}{dx}$, which gives,

$$g(x) = \frac{1}{r(b-a)(x+P)}$$

A typical plot of this pdf is given next.

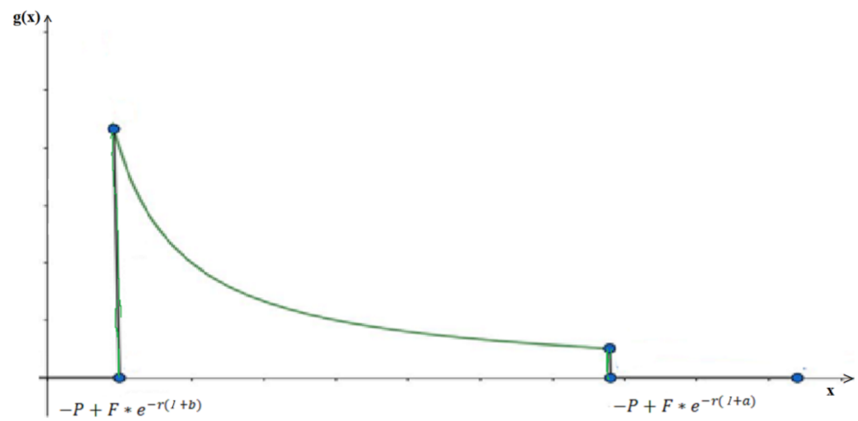


Figure 2: PDF of NPV of a single payment project with a uniform payment delay

$$g(x) = \begin{cases} \dfrac{1}{r(b-a)(x+P)}, & -P + Fe^{-r(1+b)} \le x \le -P + Fe^{-r(1+a)} \\ 0 \quad , & elsewhere \end{cases}$$

The probability of having a feasible investment according to equation (2) is

$$1 - \frac{b + \frac{1}{r}\ln\left(\frac{P}{F}\right) + 1}{b - a}$$

Example 1.
Consider a payment of $1200 due after 1 year from now at an interest rate of 12% compounded yearly. Let the initial investment be $1000 and the payment delay varies uniformly between 0 and 24 months.

The pdf of NPV of this investment opportunity is given by

$$g(x) = \begin{cases} \dfrac{1}{0.24(x+1000)}, & -162.788 \leq x \leq 64.304 \\ 0 \quad , & elsewhere \end{cases}$$

This pdf is plotted in Figure 3. Obviously, this plot indicate a high likelihood of having an infeasible investment, as the pdf is high over the negative range. More specifically, the probability of having a feasible investment is given from (2)

$$1 - \frac{2 + \frac{1}{0.12}\ln\left(\frac{1000}{1200}\right) + 1}{2-0} = 0.259$$

To validate these results, we carried-out a Monte Carlo simulation in @Risk for this investment. The pdf from the simulation is shown in Figure 4, which is very close to the "exact" one in Figure 3. Moreover, the probability that the investment is feasible from the simulation is $P(NPV \geq 0) = 0.26$ which is also very close to the exact value of 0.259.

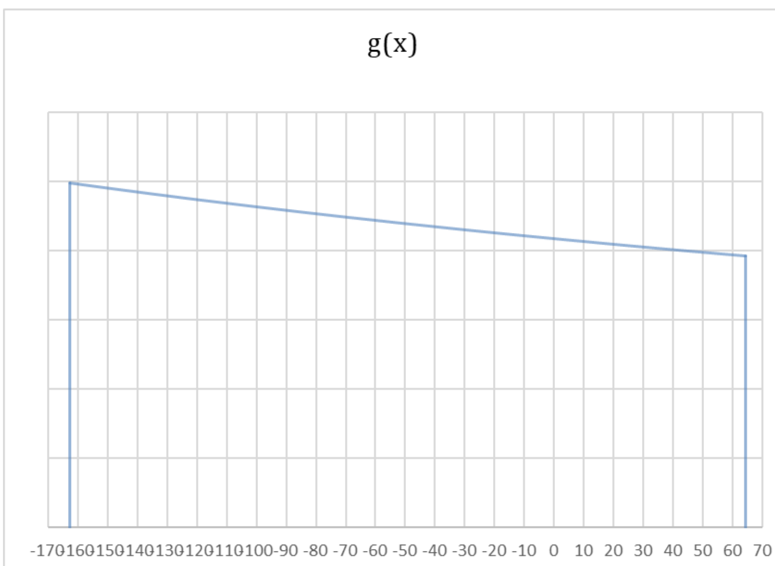


Figure 3: Analytical PDF of NPV of single cash flow with uniform delay

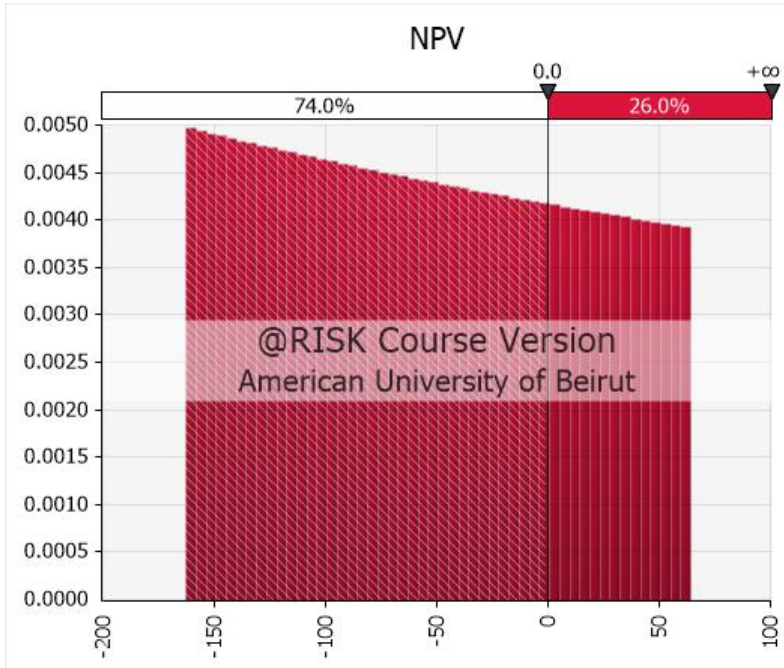


Figure 4: Simulation output for single cash flow with uniform delay

C. Triangular Distribution of Delay

If *T* has a triangular distribution with parameters *a*, *c*, and *b*, then the pdf and CDF of *T* are given by

$$h(t) = \begin{cases} \dfrac{2(t-a)}{(b-a)(c-a)} & , \quad a \le t < c \\ \dfrac{2(b-t)}{(b-a)(b-c)} & , \quad c \le t < b \\ 0 & , \quad elsewhere \end{cases}$$

To evaluate the CDF of NPV, we must take into consideration different ranges for *t*. To simplify the notation, let $w(x) = -\frac{1}{r} ln\left(\frac{x+P}{F}\right) - 1$. Note that (1) can be written as

$G(x) = \mathrm{P}\{T > w(x)\}$.

- If $a \leq -\frac{1}{r} ln\left(\frac{x+P}{F}\right) - 1 \leq c$, or equivalently,

$$-P + Fe^{-r(1+c)} \leq x \leq -P + Fe^{-r(1+a)}$$

Equation (1) then gives

$$G(x) = \int_{w(x)}^{c} \frac{2(t-a)}{(b-a)(c-a)} dt + \int_{c}^{b} \frac{2(b-t)}{(b-a)(b-c)} dt$$

$$= \frac{1}{(b-a)(c-a)}[(c-a)^2 - (w(x)-a)^2] + \frac{1}{(b-a)(b-c)}(b-c)^2$$

$$= 1 - \frac{1}{(b-a)(c-a)}(w(x)-a)^2$$

$$= 1 - \frac{1}{(b-a)(c-a)}[-\frac{1}{r} ln\left(\frac{x+P}{F}\right) - 1 - a]^2$$

The probability density function of x is $g(x) = \frac{dGx}{dx}$,

$$g(x) = \frac{2}{r(b-a)(c-a)(x+P)}(-\frac{1}{r} ln\left(\frac{x+P}{F}\right) - 1 - a),$$

for $-P + Fe^{-r(1+c)} \leq x \leq -P + Fe^{-r(1+a)}$

The probability of having a feasible investment according to equation (2)

$$\frac{1}{(b-a)(c-a)}[(-\frac{1}{r} ln\left(\frac{P}{F}\right) - 1 - a)^2]$$

- If $c \leq -\frac{1}{r} ln\left(\frac{x+P}{F}\right) - 1 \leq b$, or equivalently,

$$-P + Fe^{-r(1+b)} \leq x \leq -P + Fe^{-r(1+c)}$$

Equation (1) gives

$$G(x) = \int_{w(x)}^{b} \frac{2(b-t)}{(b-a)(b-c)} dt$$

$$= \frac{1}{(b-a)(b-c)}[(b - w(x))^2]$$

$$= \frac{1}{(b-a)(b-c)}[(b + \frac{1}{r} ln\left(\frac{x+P}{F}\right) + 1)^2]$$

The probability density function of x is $g(x) = \frac{dGx}{dx}$,

$$g(x) = \frac{2}{r(x+P)(b-a)(b-c)}[(b + \frac{1}{r} ln(\frac{x+P}{F}) + 1)],$$

$$\text{for } -P + Fe^{-r(1+b)} \leq x \leq -P + Fe^{-r(1+c)}.$$

The probability of having a feasible investment according to (2) is

$$1 - \frac{1}{(b-a)(b-c)}[(b + \frac{1}{r} ln\left(\frac{P}{F}\right) + 1)^2]$$

To sum it up, when the payment delay has a triangular distribution the probability distribution of the net present value is

$$g(x) = \begin{cases} \frac{2}{r(x+P)(b-a)(c-a)}[b + \frac{1}{r} ln(\frac{x+P}{F}) + 1] & , -P + Fe^{-r(1+b)} \leq x \leq -P + Fe^{-r(1+c)} \\ \frac{2}{r(b-a)(b-c)(x+P)}[-\frac{1}{r} ln\left(\frac{x+P}{F}\right) - 1 - a], & -P + Fe^{-r(1+c)} \leq x \leq -P + Fe^{-r(1+a)} \\ 0 & , \quad elsewhere \end{cases}$$

A general plot of this pdf is given in Figure 5.

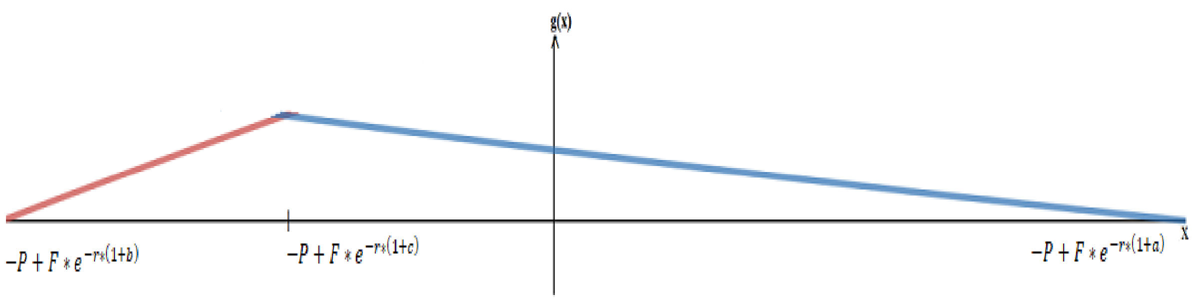


Figure 5: PDF of NPV of a single payment project with a triangular payment delay

Example 2.

Consider a payment of 1200$ due after 12 months from now at an interest rate of 12% compounded monthly. Let the initial investment be 1000$ and the payment delay has a triangular distribution between 0,12 and 24 months.

The pdf of the NPV of this investment opportunity is given by

$$g(x) = \begin{cases} \frac{16.66}{(x+1000)}[3 + 8.33ln(\frac{x+1000}{1200})], -162.788 \le x \le 32.850 \\ \frac{16.66}{(x+1000)}[8.33ln(\frac{x+1000}{1200}) + 2] \;, 32.850 \le x \le 64.304 \\ 0 \qquad\qquad , \quad elsewhere \end{cases}$$

This pdf is plotted in Figure 6. Obviously, this plot indicate a high likelihood of having an infeasible investment, as the pdf is high over the negative range. More specifically, the probability of having a feasible investment is given from (2) as follows

$$w(0) = -\frac{1}{0.12} ln\left(\frac{1000}{1200}\right) - 1 = 0.519 \leftrightarrow 0 \le \mathrm{w} < 1$$

The probability of having a feasible investment according to this value of *w(0)* is

$$\frac{1}{(2-0)(2-1)}(0.519)^2 = 0.135$$

To validate these results we carried-out a Monte Carlo simulation in @Risk for this investment. The pdf from the simulation is shown in Figure 7, which is very close to the "exact" one in Figure 6. Moreover, the probability that the investment is feasible from the simulation is $P(NPV \ge 0) = 0.135$ which the exact value derived analytically.

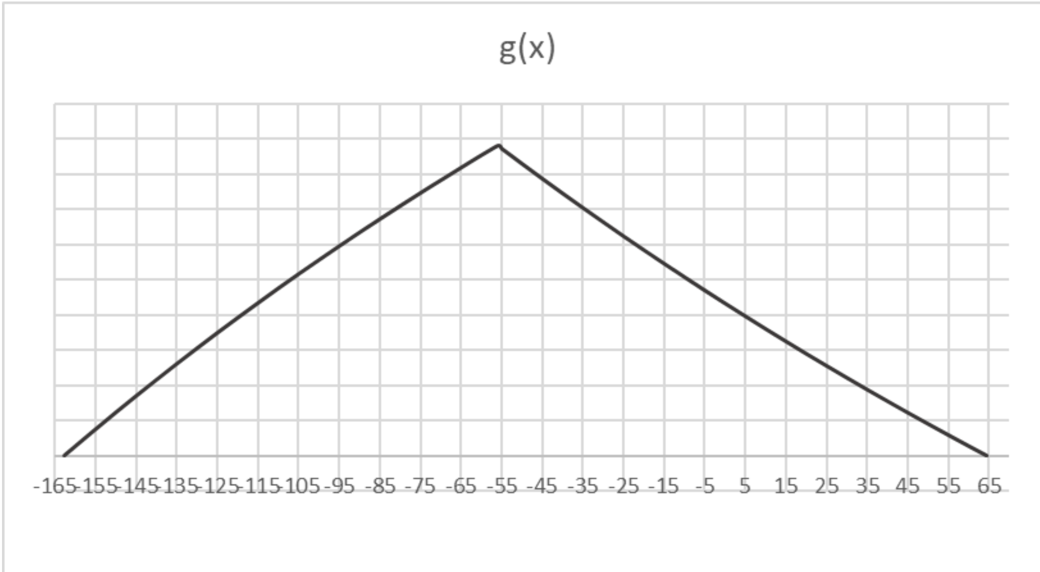


Figure 6: Analytical PDF of NPV of a single cash flow with triangular delay

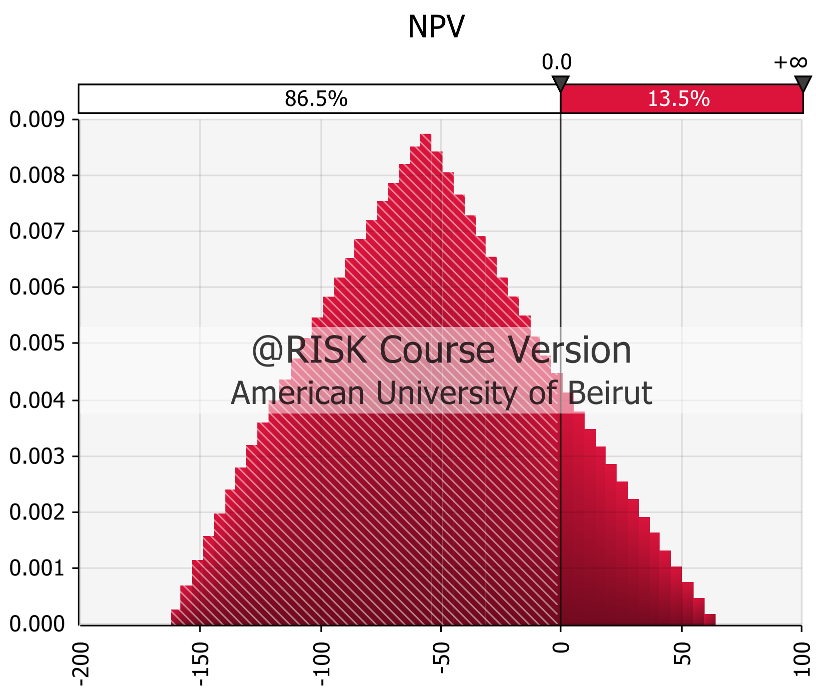


Figure 7: Simulation output for single cash flow with triangular delay

D. Exponentially Distributed Delay

If the time of the cash flow is exponentially distributed with parameter λ.

$$h(t) = \begin{cases} \lambda\, e^{-\lambda\, t}, & t \geq 0 \\ 0 \quad , & elsewhere \end{cases}$$

Following (1), $G(x) \;=\; P(T \;\geq w(x)) \;=\; \int_{w(x)}^{\infty} f(t)dt = \frac{1}{\lambda}\; e^{-\lambda\, w} = e^{\lambda}\; \left(\frac{x+P}{F}\right)^{\frac{\lambda}{r}}$

The probability distribution function of NPV is $g(x) \;=\; \frac{dG(x)}{dx}$, which gives

$$g(x) = \frac{\lambda}{rF} e^{\lambda} \left(\frac{x+P}{F}\right)^{\left(\frac{\lambda}{r}-1\right)}$$

A general plot of this CDF is given in Figure 8.

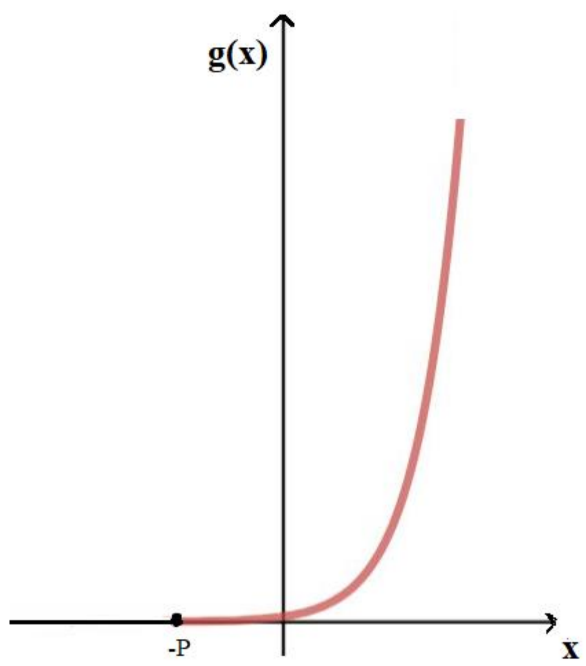


Figure 8: PDF of NPV in a single payment project with exponential payment delay

The probability of having a feasible investment according to (2) is

$$1 - e^{\lambda}\left(\frac{P}{F}\right)^{\left(\frac{\lambda}{r}\right)}$$

Example 3. Consider a payment of 1200$ due after 12 months from now at an interest rate of 12%. Let the initial investment be 1000$ and the average payment delay is 12 months.

The pdf of NPV of this investment opportunity is given by

$$g(x) = \begin{cases} 0.018\left(\dfrac{x+1000}{1200}\right)^{7.33} & , x \geq -1000 \\ 0 & , \quad elsewhere \end{cases}$$

This pdf is plotted in Figure 9. Obviously, this plot indicate a high likelihood of having an infeasible investment, as the pdf is high over the negative range. More specifically, the probability of having a feasible investment is given from (2) is

$$1 - e^{1}\left(\frac{1000}{1200}\right)^{\left(\frac{1}{0.12}\right)} = 0.405$$

To validate these results, we carried-out a Monte Carlo simulation in @Risk for this investment. The pdf from the simulation is shown in Figure 10, which is very close to the “exact” one in Figure 9. Moreover, the probability that the investment is feasible from the simulation is $P(NPV \geq 0) = 0.405$ which is the exactly the same value obtained analytically.

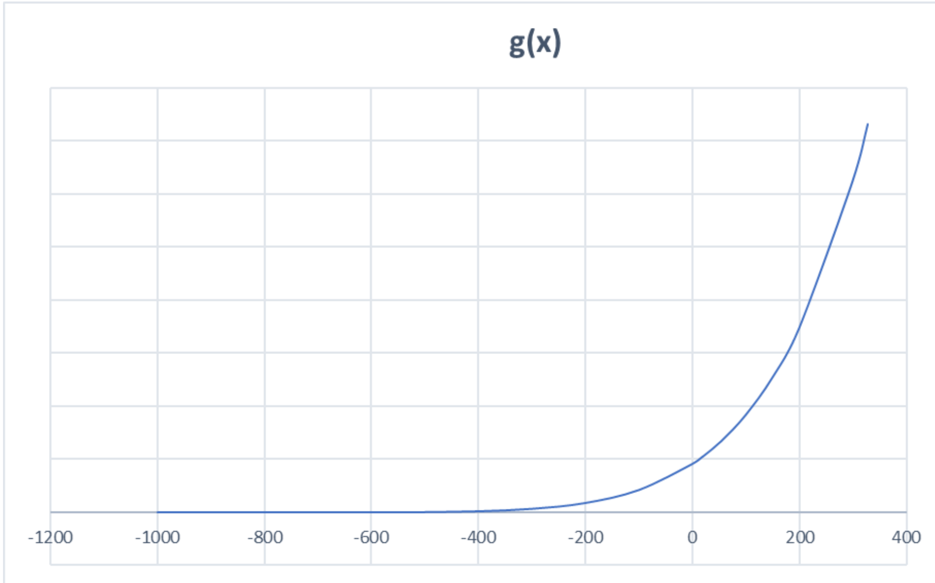


Figure 9: Analytical distribution of NPV of a single cash flow with exponential delay

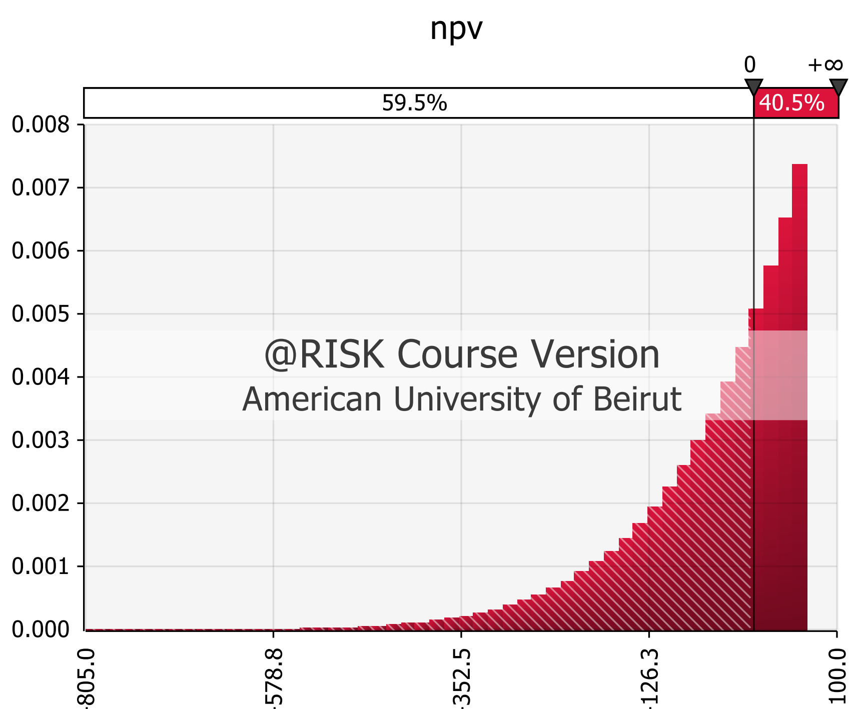


Figure 10: Simulation output for single cash flow with exponential delay

E. PERT Distribution of Delay

If the payment delay has a PERT distribution with parameter (a, m, b) we can derive the p and q, the shape parameters of the underlying Beta-PERT distribution, derived using the following equations from

$$p = \frac{4m + b - 5a}{b - a}$$

$$q = \frac{5b - a - 4m}{b - a}$$

Thus, the PDF of a delay having a PERT distribution is

$$h(t) = \begin{cases} \frac{(t-a)^{p-1}(b-t)^{q-1}}{B(p,q)(b-a)^{p+q-1}}, & a \le t \le b \\ 0 \quad , & elsewhere \end{cases}$$

The CDF of NPV is given from (1) as

$$G(x) = P\{T > w(x)\} = 1 - \int_a^{w(x)} \frac{(t-a)^{p-1}(b-t)^{q-1}}{B(p,q)(b-a)^{p+q-1}} dt = 1 - \int_a^{w(x)} \frac{(a-t)^{p-1}(b-t)^{q-1}}{B(p,q)(b-a)^{p+q-1}} dt$$

Using Leibniz Rule Olver (2000)

$$g(x) = \frac{1}{B(p,q)(b-a)^{p+q-1}} \frac{(-\frac{1}{r} ln\left(\frac{x+P}{F}\right) - 1 - a)^{p-1} \left(b + \frac{1}{r} ln\left(\frac{x+P}{F}\right) + 1\right)^{q-1}}{r(x+P)}$$

$$where - P + Fe^{-r(1+b)} \le x \le -P + Fe^{-r(1+a)}$$

This pdf is plotted in Figure 11. The probability of having a feasible investment according to equation is $G(0)$ which has no closed-form.

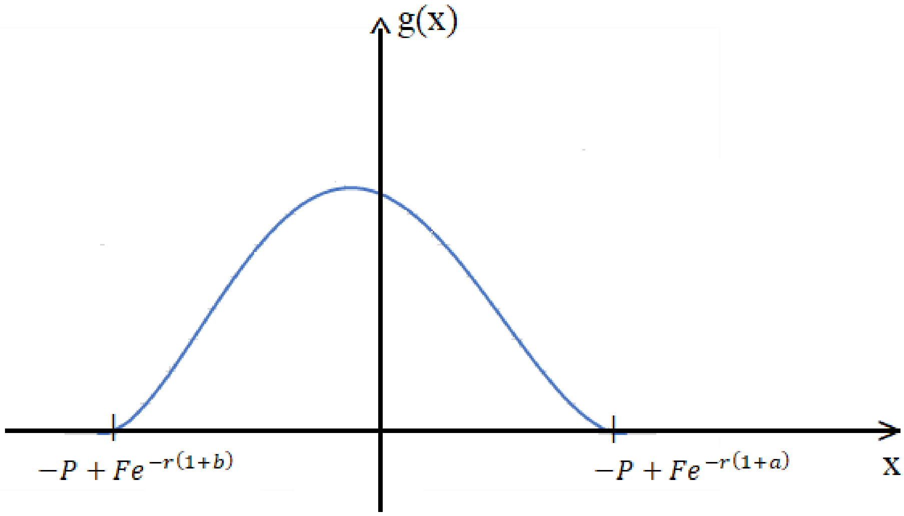


Figure 11: PDF of single cash flow with Pert delay distribution

Example 4.
Consider a payment of 1200$ due after 12 months from now at an interest rate of 12% compounded yearly. Let the initial investment be 1000$ and the payment delay has a Pert distribution between 0, 14 and 16 months.

The pdf of NPV of this investment opportunity is given by

$$g(x) = \frac{(-\frac{1}{0.12} ln\left(\frac{x+1000}{1200}\right) - 1)^{3.5}\left(2.33 + \frac{1}{0.12} ln\left(\frac{x+1000}{1200}\right)\right)^{0.5}}{0.024(x+1000)}$$

This pdf is plotted in Figure 12. Obviously, this plot indicate a high likelihood of having an infeasible investment, as the pdf is high over the negative range. More specifically, the probability of having a feasible investment is given from (2)

$$1 - G(0) = 0.035$$

To validate these results we carried-out a Monte Carlo simulation in @Risk for this investment. The pdf from the simulation is shown in Figure 13, which is very close to the "exact" one in Figure 12. Moreover, the probability that the investment is feasible from the simulation is $P(NPV \geq 0) = 0.031$ which is also very close to the exact value of 0.035.

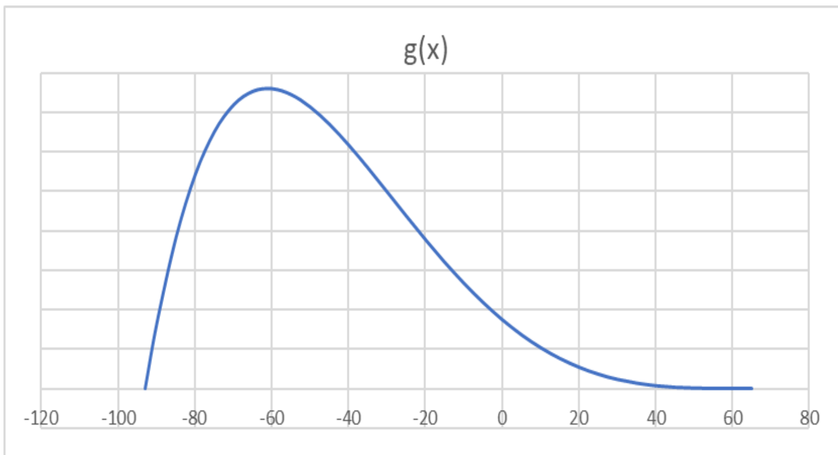


Figure 12: Analytical NPV of single cash flow with PERT distribution of delay

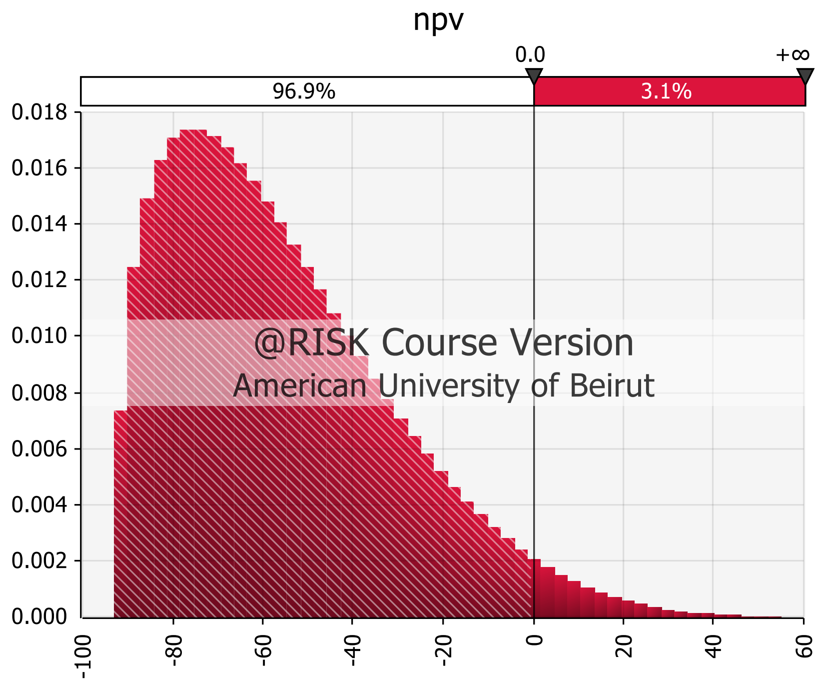


Figure 13: Simulation output for single cash flow with PERT delay

The following table summarizes the results on the distribution of the NPV for single-period investments derived in this chapter.

Table 1 Summary of derived expressions for delay distributions

Distribution	Range of x	pdf of NPV f(x)	Probability of a feasible investment
Uniform	$-P + Fe^{-r(1+b)}$ *and* $-P + Fe^{-r(d+a)}$	$\frac{1}{r(b-a)(x+P)}$	1- $\frac{b+\frac{1}{r}ln\left(\frac{P}{F}\right)+1}{b-a}$
Triangular	$-P + Fe^{-r(1+b)}$ *and* $-P + Fe^{-r(1+c)}$	$\frac{2}{r(x+P)(b-a)(b-c)}[(b+\frac{1}{r}ln(\frac{x+P}{F})+i)]$	$1-\frac{1}{(b-a)(b-c)}[(b$ $\frac{1}{r}ln\left(\frac{P}{F}\right)+$ $1)^2]$
	$-P + Fe^{-r(1+c)}$ *and* $-P + Fe^{-r(1+a)}$	$\frac{2}{r(b-a)(c-a)(x+P)}[(a+\frac{1}{r}ln\left(\frac{x+P}{F}\right)+i)]$	$1-\frac{1}{(b-a)(c-a)}[(c$ $a)^2-$ $(-\frac{1}{r}ln\left(\frac{P}{F}\right)-$ $1-a)^2]$
Exponential	$x \geq -P$	$\frac{\lambda}{rF}e^{\lambda}(\frac{x+P}{F})^{(\frac{\lambda}{r}-1)}$	1- $e^{\lambda}(\frac{P}{F})^{\frac{\lambda}{r}}$
PERT	$-P + Fe^{-r(1+b)}$ *and* $-P + Fe^{-r(1+a)}$	$\frac{1}{B(p,q)(b-a)^{p+q-1}}\frac{(-\frac{1}{r}ln\left(\frac{x+P}{F}\right)-1-a)^{p-1}\left(b+\frac{1}{r}\right.}{r(x+P)}$	1-G(0)

CHAPTER 4

MULTIPLE CASH FLOW ANALYSIS

In this chapter we evaluate the NPV of multiple cash flows under random payment delays. In section 1 we present the methodology for finding the mean and variance of NPV and for fitting a four-parameter beta distribution into the data. Then in section 2 we present the tools we used to validate our estimation. In section 3,4,5 we consider multiple cash flows with uniform, triangular and PERT distribution of delays, we present our findings for each case and support the results with numerical examples.

A. The General Case

Consider a project with initial investment P and the revenues F_i occurring at time i, but can be delayed by a certain random period T_i, where T_i , i = 1, 2, ..., n are independent random variables. Then, the net present value of this project is given by

$$NPV = -P + \sum_{i=1}^{n} e^{-r(i+T_i)} F_i$$

In deriving the distribution of the NPV, we utilize the concept of the moment generating function. The moment generating function of a random variable T is given by $\varphi(r) = E[e^{rT}] = \int_0^\infty e^{rT} f(t)dt$, where $f(t)$ is the probability density function of the random delay T.

The following algorithm provides the steps to fit a beta distribution based on moment matching to the distribution of NPV. The choice of the beta distribution was motivated by simulation experiments indicating that the NPV distribution is generally (slightly) skewed to the left.

Step 1. Using the moment generating function of the delay distribution, estimate the mean and variance of the discounted value of the cash flow at time *i* as

$$m_{i1} = e^{-ir} F_i \phi_i(r) \text{ and } m_{i2} = e^{-2irT_i} F_i^2 \phi_i(2r) - [\phi_i(r)]^2$$

Step 2. Utilizing step (1) set the mean of *NPV* to

$$m_1 = E[NPV] = -P + \mathrm{E}[\sum_{\mathrm{i}=1}^{n} \mathrm{e}^{-\mathrm{r}(i + T_i)} F_i\ ] = -P + \sum_{i=1}^{n} m_{i1}$$

and the variance of *NPV* to

$$m_2 = var[\,NPV] = \sum_{i=1}^{n} m_{i2}$$

Step 3. Assuming that T_i is distributed on (0, b_i), set the minimum and maximum value to of the fitted beta distribution respectively, to

$$A = -P + \sum_{i=1}^{n} e^{-r(i+b_i)} F_i$$

$$B = -P + \sum_{i=1}^{n} e^{-\mathrm{i}r} F_i$$

Step 4. Estimate the parameters *p* and *q* of the beta distribution based on the following equations

$$p = \frac{(\frac{m1-A}{B-A})^2(1-\frac{m1-A}{B-A})}{\frac{m2}{(B-A)^2}} - (\frac{m1-A}{B-A})$$

(3)

$$q = \frac{(\frac{m1-A}{B-A})(1-\frac{m1-A}{B-A})}{\frac{m2}{(B-A)^2}} - p - 1$$

(4)

Then, the fitted beta distribution pdf is given by

$$f(x) = \frac{1}{B(p,q)(b-a)^{p+q-1}}(a-x)^{p-1}(b-x)^{q-1} \quad (5)$$

Skewness is defined as the third moment of a distribution is thought to be an important factor in investor's decision making. Studies on the correlation between skewness and expected return has shown that investors tend to prefer positively skewed distributions since they have longer right tails which indicates that the probability of having extreme high gains is large. The general beta distribution in this case can be a good fit to model different outcomes since it can take different shapes and can model positively skewed data depending on its parameters.

B. Goodness of Fit Analysis

To test the validity of our hypothesis which states that the beta distribution is a good fit for the NPV under random payment delays, we rely on different test to compare the analytical distribution derived using the model described above and the normal distribution commonly used to model uncertainty.

1. Probability Density Plot

We plot the probability density function of the four parameter beta distribution derived using moment matching technique, the beta distribution fitted to the output by the simulation software @Risk and the normal distribution with same mean and variance as those of the data. It is useful to test how good is our estimation of the four parameter and how much discrepancy exists between the output and the normal distribution.

2. The Quantile-Quantile Plot (Q-Q)

A useful way to evaluate the fit of a distribution is to examine the Q-Q plot. This is an ordered representation of the quantiles of a given distribution. If a certain distribution F is a possible representation of the simulation output, then the Q-Q plot is a straight line with a slope of 1, else it will deviate from the straight line. Banks, Carson, Nelson, & Nicol(1984).If multiple distributions are available, in our case the normal and the data distribution, then the decision maker will choose which one is a better approximate based on the plots.

3. The Probability-Probability Plot (P-P)

This is a simple comparison of the CDFs of two or multiple distributions against each other. The difference between the Q-Q plot and the P-P plot is that it amplifies the differences the middles of the distributions Law (2015). This is because in these regions the CDFs of the two compared distributions change rapidly than in regions of low probability density. If the suggested distribution is a good fit, then the points lie on the $x = y$ line.

4. Distribution Function Difference Plot

This plot is a comparison of individual probabilities of a given distribution and the output distribution found by simulation. This graphical representation is useful to show the goodness of fit, if the suggested distribution offers a good approximation then the plot will be close to the horizontal line. Since in our case we’re comparing between the beta distribution and the normal distribution as an approximation to the distribution of the NPV, then the distribution that is closest to the horizontal axis with small deviations will be considered as a better fit.

Our aim is to compare and fit a beta distribution into a series of cash flows with different distribution of delay but with identical cash flows and initial investment and same mean and variance. We assume that the initial investment is about 60 to 70% of the total revenue and the interest rate is 15% which is commonly used by companies for construction and other projects.

The following section outlines the model for fitting the beta distribution for cash flows with uniform, triangular and PERT delays with numerical examples and comparison between simulation and analytical estimation. Using @Risk simulation software, with 100,000 iterations, the distribution of the net present value is given.

C. Uniformly Distributed Delay

If the delay T_i are iid uniformly distributed on (0, b_i) then the moment generating function of T_i is

$$\phi_i(r) = \frac{1 - e^{-rb_i}}{rb_i}$$

Given these values for each cash flow, we can use them to fit a beta distribution based on the steps in the previous section as follows,

$m_1 = \sum_{i=0}^{n} F_i e^{-ir} \ \varphi_i(r) = \sum_{i=0}^{n} F_i e^{-ir} \ \frac{(1- e^{-rb_i})}{rb_i}$

$m_2 = \sum_{i=0}^{n} F_i^{\,2} \, e^{-2rd_i}[\varphi_i(2r) - \varphi_i(r)^2]$

$= \sum_{i=0}^{n} F_i^{\,2} \, e^{-2ir} \frac{(1- e^{-2rb_i})}{2rb_i} - (\sum_{i=0}^{n} F_i e^{-ir} \ \frac{(1- e^{-rb_i})}{rb_i})^2$

$A = \sum_{i=0}^{n} F_i e^{-r(i+b_i)}$ is the minimum value of NPV

$B = \sum_{i=0}^{n} F_i e^{-ir}$ is the maximum value of NPV

Example 5.
Given an initial investment of $100,000 followed by a series of cash inflows due on specific dates but might be delayed by a time which is uniformly distributed between (0, b_i) given in months.

Year	Cash Flow	Value (in $1000)	b_i (months)	Average delay (months)	Delay variance
1	**F_1**	20	4	2	1.33
2	**F_2**	40	2	1	0.33
3	**F_3**	35	6	3	3
4	**F_4**	25	8	4	5.33
5	**F_5**	45	10	5	8.33

We first run a Monte Carlo simulation in @Risk to estimate the NPV distribution. According to the Chi-square test's ranking, the beta distribution is the best fit followed by the normal distribution on the 4th rank. We can visually identify the discrepancies between the distributions in the following graph

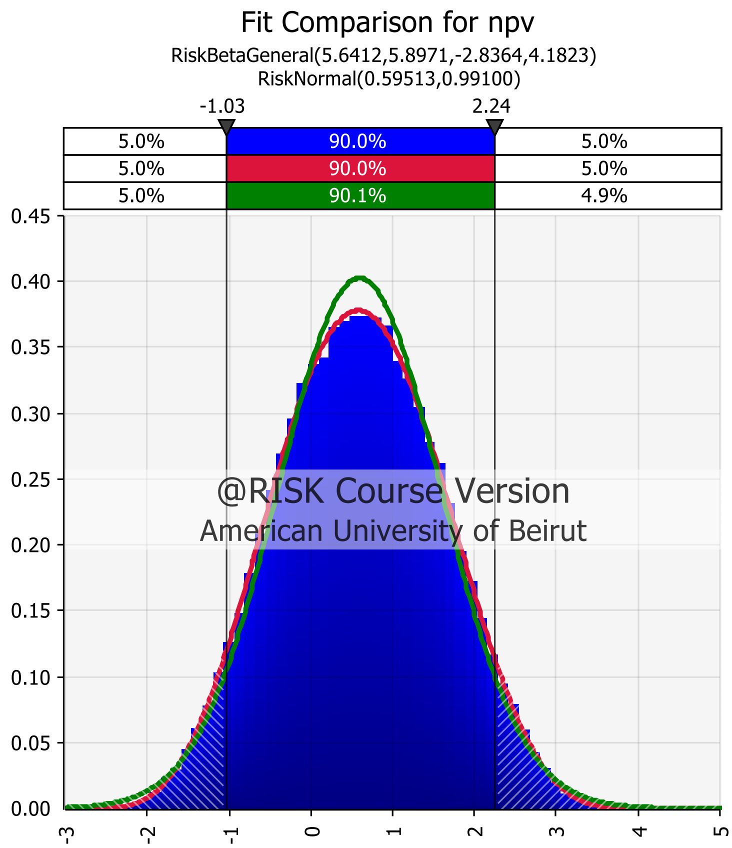


Figure 14: Simulation output for uniform distribution of delay with fitted distributions

The next step is to compare the analytical distribution found by estimating the parameters of the beta distribution to the distribution suggested by the simulation. Since T_i has a uniform distribution, we evaluate the mean and the variance of each return using the equations derived above and then we fit the series into a beta distribution.

$m_1 = 0.59$

$m_2 = 0.98$

$A = -2.85$

B= $= 4.14$

The parameters p and q of the beta distribution are derived using equation (1) and (2). It follows that p = 5.63 and q = 5.79.

These parameters will derive the analytical distribution that will be compared to that suggested by the simulation software to test the validity of our approach.

Using the simulation output, the parameters of the beta distribution are

$A' = -2.84$

$B' = 4.18$

$p' = 5.64$

$q' = 5.89$

We can now plot and compare these two distributions with the normal distribution that has a mean of 0.59 and a standard deviation 0.99 The PDFs are summarized below

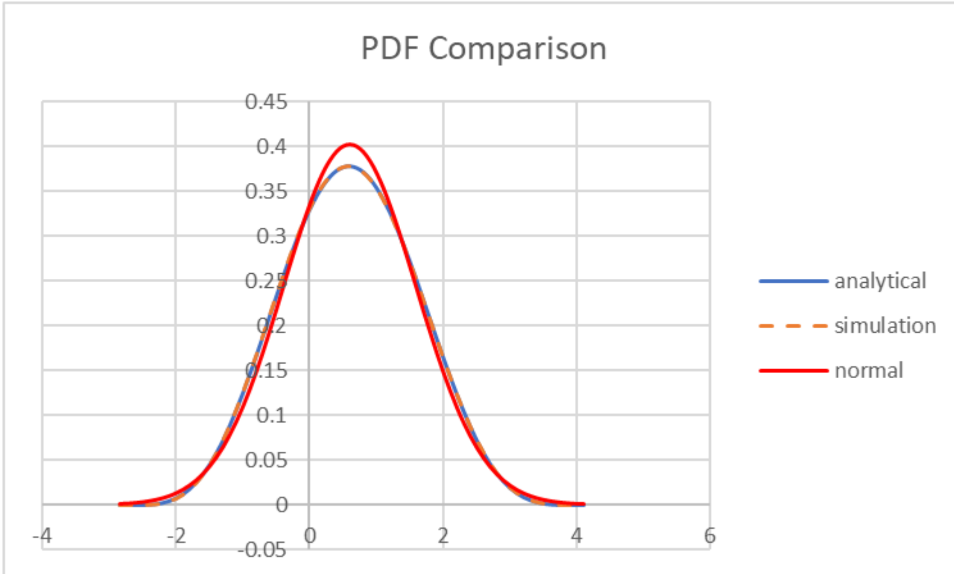


Figure 15: PDF comparison for uniform delay

As we can see, the fitted distribution is very close to the distribution suggested by simulation, while the normal distribution covers an area above the output's distribution.

Another way to verify the goodness of fit is to use a distribution function difference plot, where we compare the differences between the values given by simulation and those given by fitted distribution and simulation at each data point. This allows visual inspection of the goodness of fit.

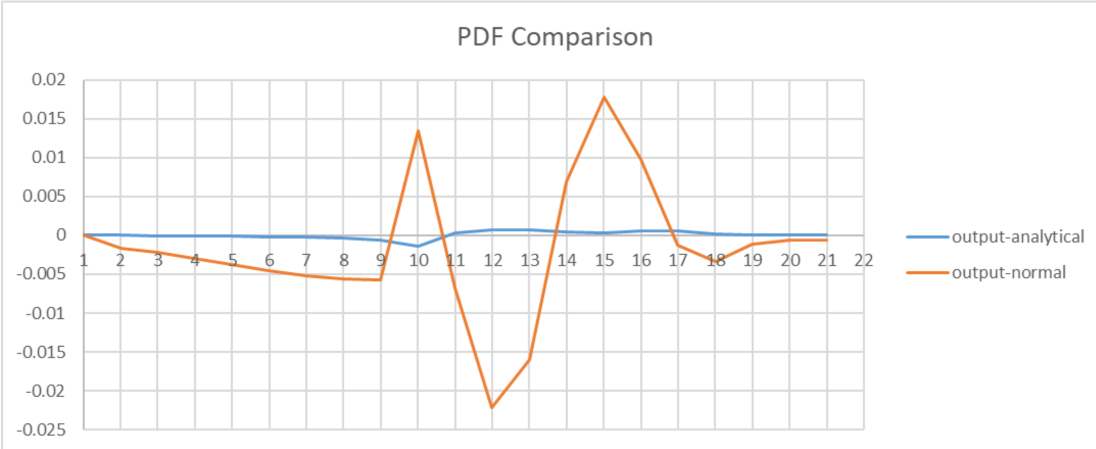


Figure 16: Probability difference plot for uniform distribution of delay

It is obvious that the beta distribution we derived is quite similar to that given by the simulation output. We also notice that at some values there is a difference between the results, but this difference is less than the difference between the results given by the normal distribution and the simulation output. This confirms our hypothesis, that the beta distribution gives better approximation for short time projects with uniform payment delay.

In order to evaluate the feasibility of such investment, decision makers will first evaluate the probability of having a feasible investment, and then make their decision depending on the possible outcomes from such project.

Using the CDF of each distribution, we can compute the probabilities of earning about or below certain values, the results are shown in the table below. We notice that the results given by the fitted beta distribution are approximately the same as those of the output and the beta given by the simulation. The normal distribution's probabilities are slightly different, and these slight differences can make a huge impact on losses for projects with high investment. The average absolute error between the beta fit and simulation is 1.85%.

Probability	Simulation	Normal Fit	Beta Fit
P{NPV < -1}	0.054	0.054	0.054
P{NPV > 0}	0.712	0.726	0.714
P{NPV < 2}	0.917	0.922	0.917
P{NPV > 0.8}	0.423	0.418	0.422
P{NPV > 1}	0.351	0.342	0.349

The Q-Q plot shows that the normal distribution deviates from the simulation output especially around the tails, while the beta distribution presents a better fit along all the region of the NPV.

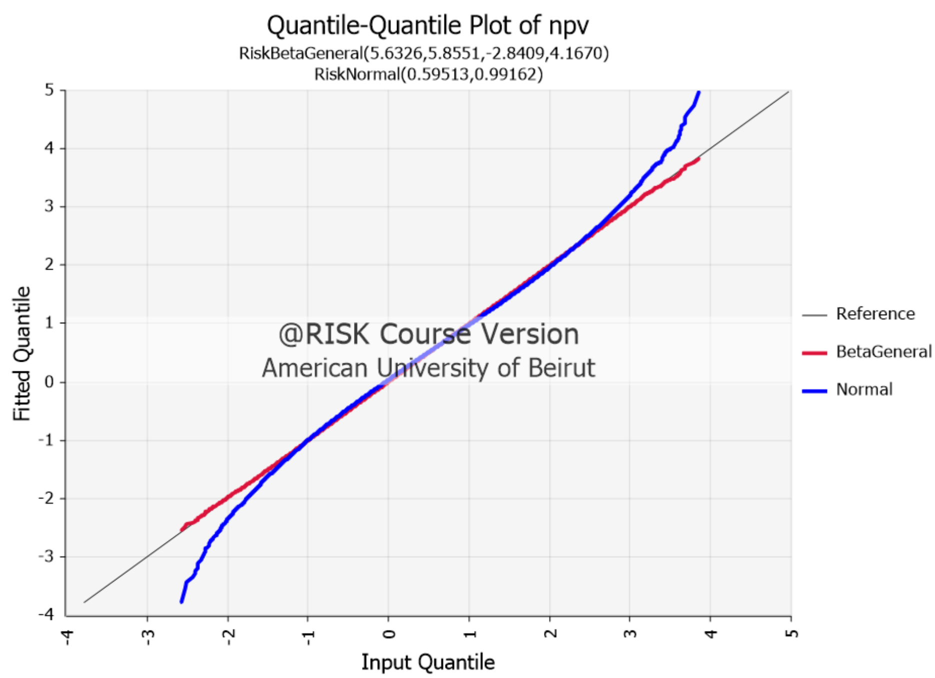


Figure 17: Q-Q plot of uniform delay

On the other side, the P-P plot shows the deviation of the normal distribution from the distribution of the NPV in the region of high probability. It is obvious that the beta distribution is a better fit to model the distribution of NPV with payment delays having a uniform distribution.

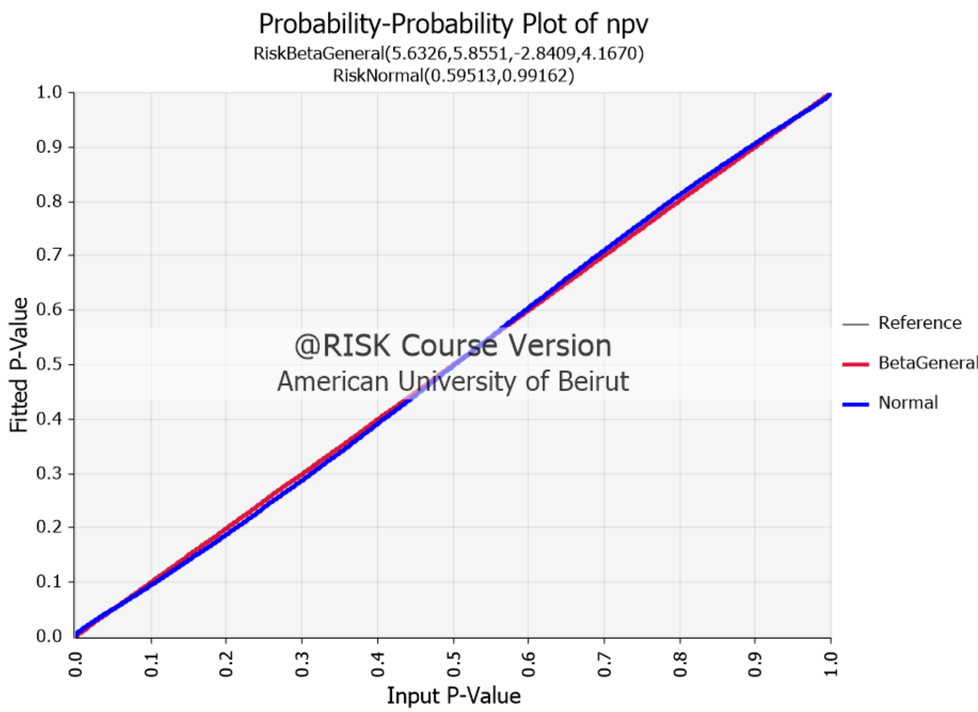


Figure 18: P-P plot of uniform delay

D. Triangular Distribution of Delay

If the delay T_i are iid triangular distributed on $(0, c_i, b_i)$ then the moment generating function of T_i is

$$\phi_i(r) = \frac{1}{r^2 b_i}\left(\frac{e^{-rb_i} - e^{-rc_i}}{(b_i - c_i)} - \frac{e^{-rc_i} - 1}{c_i}\right)$$

Given these values for each cash flow, we can use them to fit a beta distribution based on the steps in the previous section as follows,

$m_1 = \sum_{i=0}^{n} F_i e^{-ir} \; \varphi_i(r) = \sum_{i=0}^{n} F_i e^{-ir} \frac{1}{r^2 b_i} \left(\frac{e^{-rb_i} - e^{-rc_i}}{(b_i - c_i)} - \frac{e^{-rc_i} - 1}{c_i} \right)$

$m_2 = \sum_{i=0}^{n} F_i^{\,2} e^{-2ir} \; [\varphi_i(2r) - \varphi_i(r)^2]$

$= \frac{1}{4r^2 b_i} \left(\frac{e^{-2rb_i} - e^{-2rc_i}}{(b_i - c_i)} - \frac{e^{-2rc_i} - 1}{c_i} \right) - \left\{ \frac{1}{r^2 b_i} \left(\frac{e^{-rb_i} - e^{-rc_i}}{(b_i - c_i)} - \frac{e^{-rc_i} - 1}{c_i} \right) \right\}^2$

$A = \sum_{i=0}^{n} F_i e^{-r(i+b_i)}$ is the minimum value of NPV

$B = \sum_{i=0}^{n} F_i e^{-ir}$ is the maximum value of NPV

Example 6.

Given an initial investment of $100,000 followed by a series of cash inflows due on specific dates but might be delayed by a time which has a triangular distribution between $(0, c_i, b_i)$

Year	Cash flow	Value (in $1000)	c_i	b_i	Average monthly delay	Monthly variance
1	**F1**	20	0.2	5.8	2	1.333
2	**F2**	40	0.4	2.6	1	0.333
3	**F3**	35	1.1	7.9	3	3
4	**F4**	25	1.5	10.5	4	5.333
5	**F5**	45	2	13	5	8.333

We first run a Monte Carlo simulation in @Risk to estimate the NPV distribution. According to the Chi-square test's ranking, the beta distribution is the best fit followed by the normal distribution on the 5th rank. We can visually identify the discrepancies between the distributions in the following graph

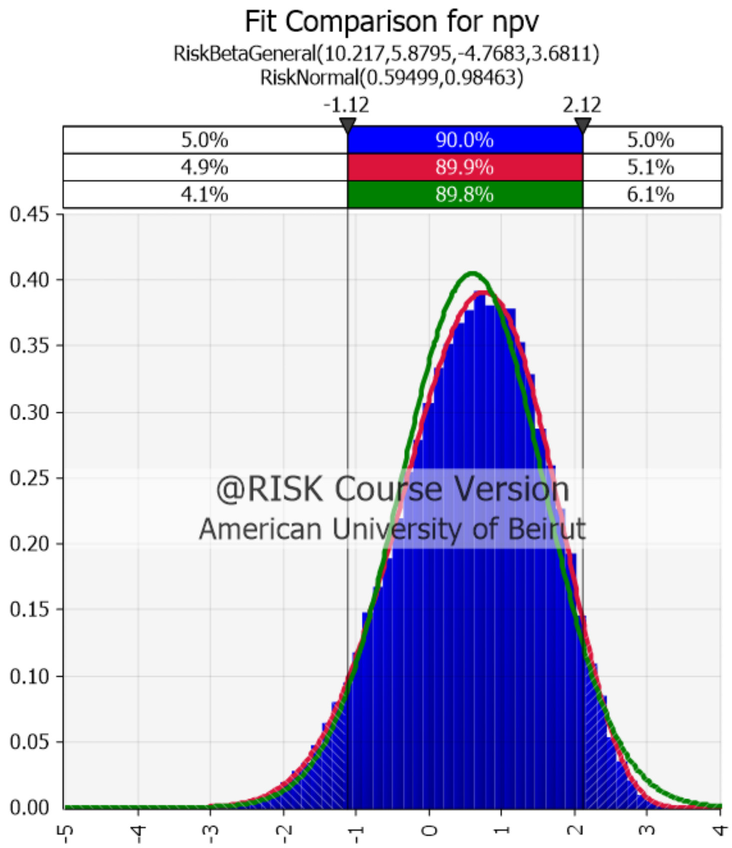


Figure 19: Simulation output for triangular distribution of delay with fitted distributions

The next step is to compare the analytical distribution found by estimating the parameters of the beta distribution to the distribution suggested by the simulation. Since T_i has a uniform distribution, we evaluate the mean and the variance of each return using the equations derived above and then we fit the series into a beta distribution.

$m_1 = 0.59$

$m_2 = 0.97$

$A = -4.88$

$B = 4.14$

The parameters p and q of the beta distribution are derived using equation (1) and (2). It follows that $p = 11.58$ and $q = 7.49$

These parameters will derive the analytical distribution that will be compared to that suggested by the simulation software to test the validity of our approach.

Using the simulation output, the parameters of the beta distribution are

$A' = -4.65$

$B' = 3.95$

$p' = 10.22$

$q' = 5.88$

We can now plot the two distributions and compare them to the normal distribution that has a mean of 0.59 and standard deviation of 0.98. The PDFs are summarized below

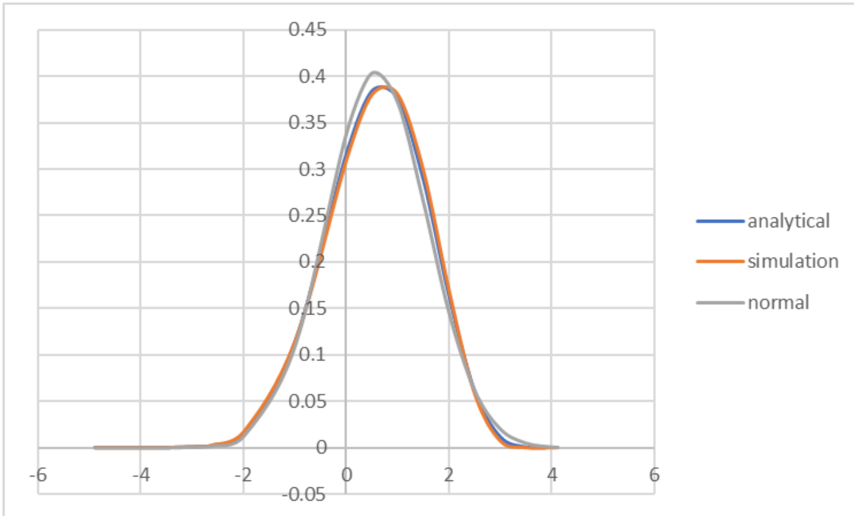


Figure 20: PDF comparison of analytical distribution and simulation output for triangular distribution

As we can see, the fitted distribution is quite the same as the distribution suggested by simulation, while the normal distribution covers a range above the output's curve.

Another way to verify the goodness of fit is to use a distribution function difference plot, where we compare the differences between the values given by simulation and those given by fitted distribution and simulation at each data point. This allows visual inspection of the goodness of fit.

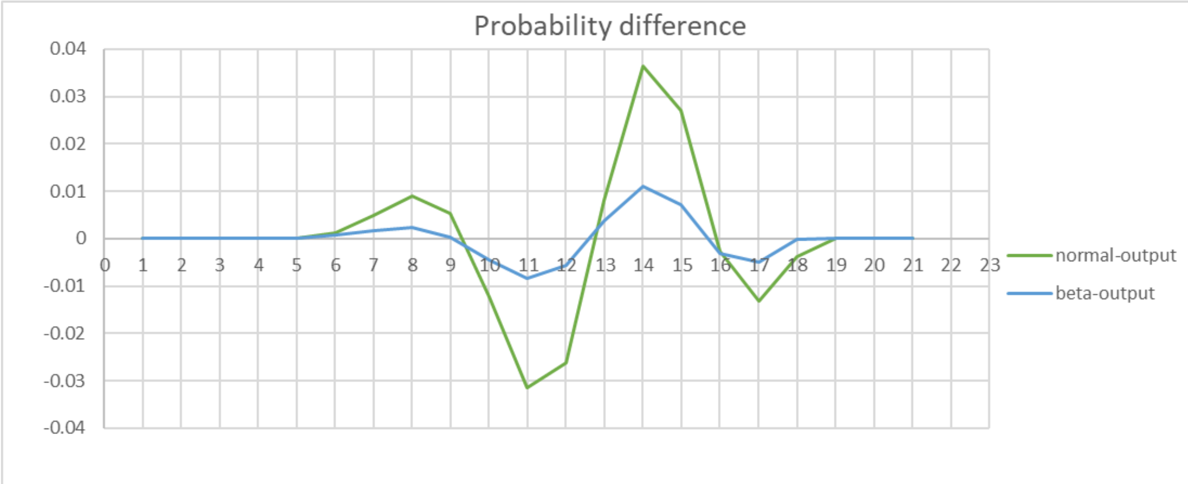


Figure 21: PDF difference plot for triangular delay

Decision makers usually rely on CDF to compute the probability of earning above or below a certain target. In the table below we compare the values given by beta distribution and the normal distribution to the simulation output.

Probability	Simulation	Normal Fit	Beta Fit
P{NPV < -1}	0.062	0.052	0.059
P{NPV > 0}	0.729	0.727	0.728
P{NPV < 2}	0.932	0.923	0.927
P{NPV>0.8}	0.440	0.417	0.433
P{NPV > 1}	0.363	0.340	0.356

We notice that the probability given by the analytical and simulation beta distribution is almost the same as the one of the output. We notice that the normal distribution we can detect a difference in the CDF's. The probability of having a feasible investment increased by approximately 2.38% and the average absolute error between the analytical and simulation results decreased to 1.80%.

We also notice that the range of NPV is larger than that with uniform delays, the mean is quite the same with approximately 595$ and the variance slightly decreased from $989 to $970, approximately by 1%. This might due to the accuracy we added to the estimations by using a triangular distribution. Next, we will try to stick to the same example but by modeling the delay using a Pert distribution rather than triangular to test the difference in the outcome.

The Q-Q plot shows that the normal distribution deviates from the simulation output especially around the tails of positive outcomes, while the beta distribution presents a better fit along all the region of the NPV.

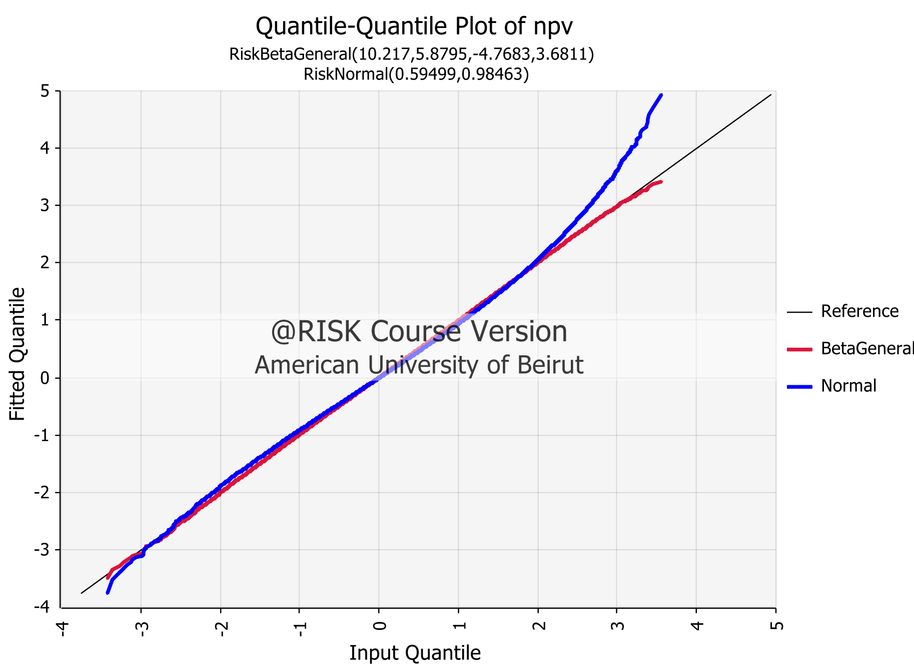


Figure 22: Q-Q plot with triangular delay

On the other side, the P-P plot shows the deviation of the normal distribution from the distribution of the NPV in the region of high probability. It is obvious that the beta distribution is a better fit to model the distribution of NPV with payment delays having a triangular distribution.

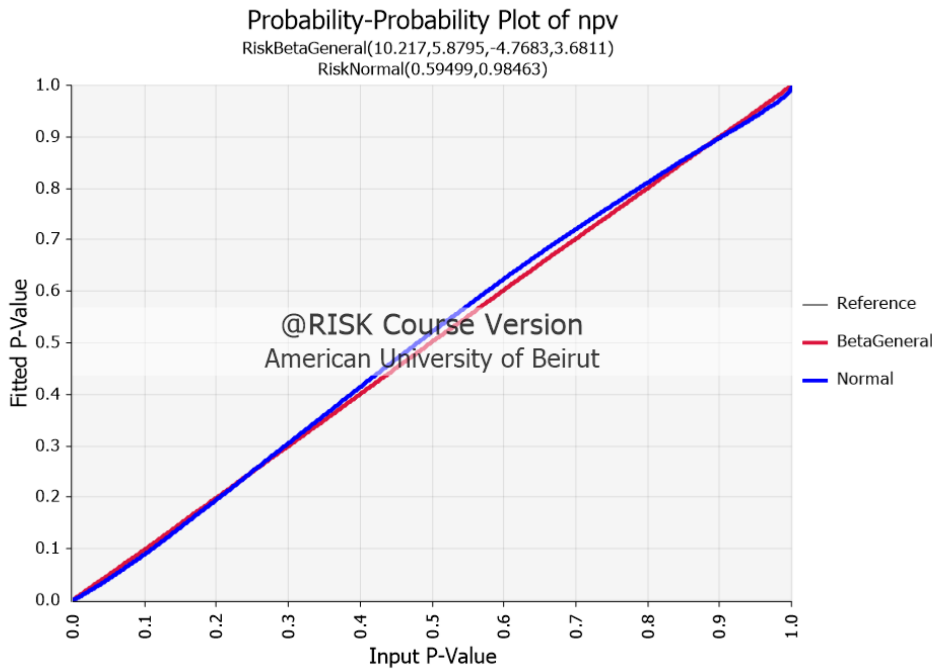


Figure 23: P-P plot with triangular delay

E. PERT Distribution of Delay

We consider the case where the delay T_i are *iid* with PERT distribution on (0, m_i, b_i). In literature, we were able to find the moment generating function of the standard beta distribution. But in our analysis, we consider payment delays that have a pert distribution, with a minimum, maximum and a most likely time for the given payment. To derive the moment generating function of the general beta distribution, we consider a change of variable from the standard to the general form.

If X_i has a standard beta distribution and T_i has a pert distribution with parameter $(0, m_i, b_i)$, $T_i = b_i X_i$

It follows that,

$E[e^{-rT_i}] = E[e^{-rb_iX_i}]$

Then, $\varphi_{iT_i}(r) = \varphi_{iX_i}(b_i r)$

According to JOHNSON(1995)

$$E[e^{-rX_i}] = \varphi_{iX_i}(r) = 1 + \sum_{k=1}^{\infty} \frac{(-r)^k}{k!} \prod_{n=0}^{k-1} \frac{p_i+n}{p_i+q_i+n},$$

Therefore,

$$\varphi_{iT_i}(r) = [\, 1 + \sum_{k=1}^{\infty} \frac{[-rb_i]^k}{k!} \prod_{n=0}^{k-1} \frac{p_i+n}{p_i+q_i+n}],$$

Given these values for each cash flow, we can use them to fit a beta distribution based on the steps in the previous section as follows,

$$m_1 = \sum_{i=0}^{n} F_i e^{-ir}\ \varphi_i(r) = \sum_{i=0}^{n} F_i e^{-ir} [\, 1 + \sum_{k=1}^{\infty} \frac{[-rb_i]^k}{k!} \prod_{n=0}^{k-1} \frac{p_i+n}{p_i+q_i+n}],$$

$$m_2 = \sum_{i=0}^{n} {F_i}^2 e^{-2ri} [\varphi_i(2r) - \varphi_i(r)^2]$$

$$= \sum_{i=0}^{n} {F_i}^2 e^{-2ri} [\, 1 + \sum_{k=1}^{\infty} \frac{[-2rb_i]^k}{k!} \prod_{n=0}^{k-1} \frac{p_i+n}{p_i+q_i+n}] - \{1 + \sum_{k=1}^{\infty} \frac{[-rb_i]^k}{k!} \prod_{n=0}^{k-1} \frac{p_i+n}{p_i+q_i+n}\}^2.$$

$A = \sum_{i=0}^{n} F_i e^{-r(i+b_i)}$ is the minimum value of NPV

$B = \sum_{i=0}^{n} F_i e^{-ir}$ is the maximum value of NPV

Evaluating the expressions of m_1 and m_2 with infinite sums is an arduous process, but we can evaluate the expression in Excel by considering the sum from 1 to a certain value M, and increment the value of M by 1 until we get the same result for two consecutive values, i.e., the sum converges. The steps are detailed in the following algorithm.

- Divide the expression into two components: the product $\prod_{n=0}^{k-1} \frac{p_i+n}{p_i+q_i+n}$ = P_{ik} and the term $\frac{[-rb_i]^k}{k!}$ = s_{ik}. The value of the expression is S_{ik}.
- Define a row for P_{ik}, s_{ik} and S_{ik} for each value of k.
- Set k=1, = $P_{11} = \frac{p_i+n}{p_i+q_i+n}$ and $s_{11} = \frac{[-rb_i]^1}{1!}$. $Set\ S_1 = P_1 s_1$
- Then for k=2, 3, … M , $P_{ik} = P_{ik-1} = \frac{p_i+k+n}{p_i+q_i+k+n}$. Set $S_{ik} = S_{ik-1} + P_{ik}\ S_{ik}$
- Once $k=M$ is reached, compare S_{iM} *to* S_{iM-1} *if* $|S_{iM} - S_{iM-1}| < \epsilon$ where ϵ is a very small number (e.g. ϵ =0.0001) Stop, else continue until this condition is verified.
- Set $\varphi_i(r) = (1 + S_{ik})$

Repeat the same process to evaluate the second moment but set $\frac{[-2rb_i]^k}{k!} = s_k$.

Then,

$$\varphi_i(2r) = (1 + S_{ik})$$

Example 6.
Given an initial investment of $100,000 followed by a series of cash inflows due on specific dates but might be delayed by a time which has a PERT distribution between (a_i, m_i, b_i). given in months.

Year	Cash flow	Value (in $1000)	m_i	b_i	Average delay (months)	Delay variance
1	**F1**	20	1.32	6.96	2	1.33
2	**F2**	40	0.6	1.74	1	0.33
3	**F3**	35	1.92	10.44	3	3
4	**F4**	25	2.52	13.8	4	5.33
5	**F5**	45	3.12	17.28	5	8.33

We first run a Monte Carlo simulation in @Risk to estimate the NPV distribution. According to the Chi-square test's ranking, the beta distribution is the best fit followed by the normal distribution on the 4th rank. We can visually identify the discrepancies between the distributions in the following graph.

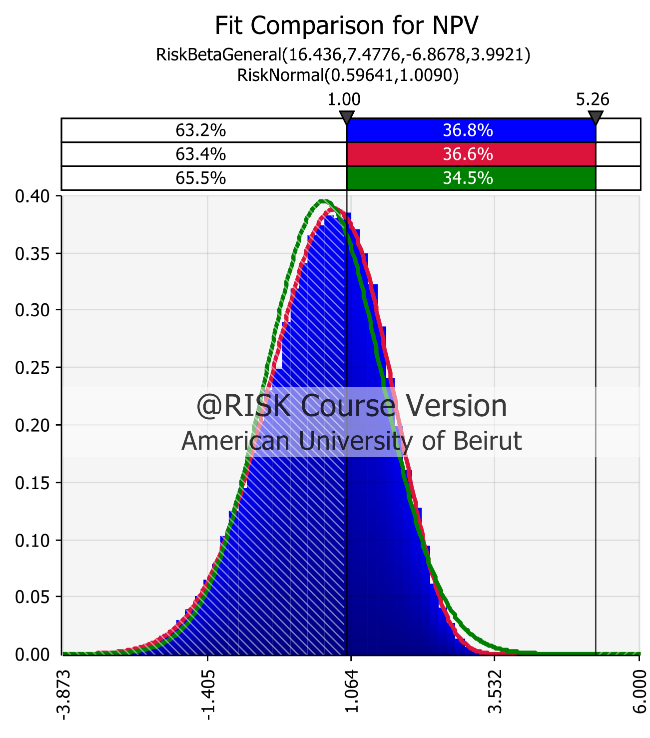


Figure 24: Simulation output for PERT distribution of delay with suggested fitted distributions

The next step is to compare the analytical distribution found by estimating the parameters of the beta distribution to the distribution suggested by the simulation. Since T_i has a PERT distribution, we evaluate the mean and the variance of each return using the equations derived above and then we fit the series into a beta distribution.

$m_1 = 0.59$

$m_2 = 1.02$

$A = -7.58$

$B = 4.14$

The parameters p and q of the beta distribution are derived using equation (1) and (2). It follows that $p = 19.16$ and $q = 8.30$

These parameters will derive the analytical distribution that will be compared to that suggested by the simulation software to test the validity of our approach.

Using the simulation output, the parameters of the beta distribution are

$A' = 6.76$

$B' = 3.95$

$p' = 15.89$

$q' = 7.25$

We can now plot the two distributions and compare them to the normal distribution that has the mean and variance of the output to find the best fit. The PDFs are summarized below

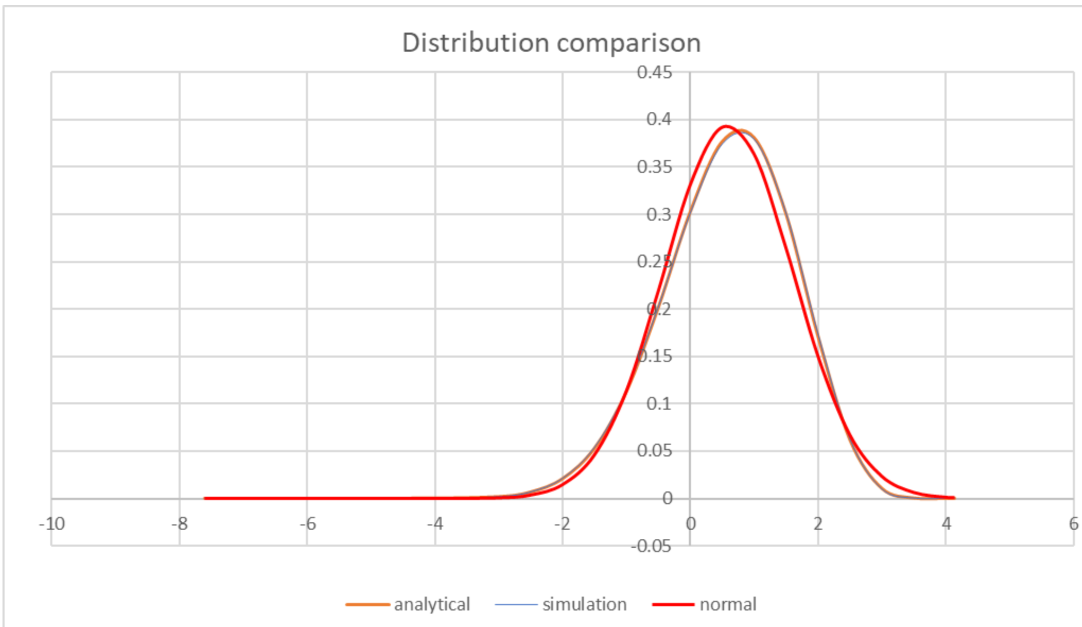


Figure 25: PDF comparison of analytical distribution and simulation output for PERT delay

As we can see, the fitted distribution is quite the same as the distribution suggested by simulation. By comparing this figure to figure 20 we notice that the normal distribution has shifted to the right of the output, which means that it does not seem to be a good fit, relatively the same result as in figure 15 and figure 20 .

Then, we compare the differences between the values given by simulation and those given by fitted distribution and simulation at each data point using the probability difference plot below

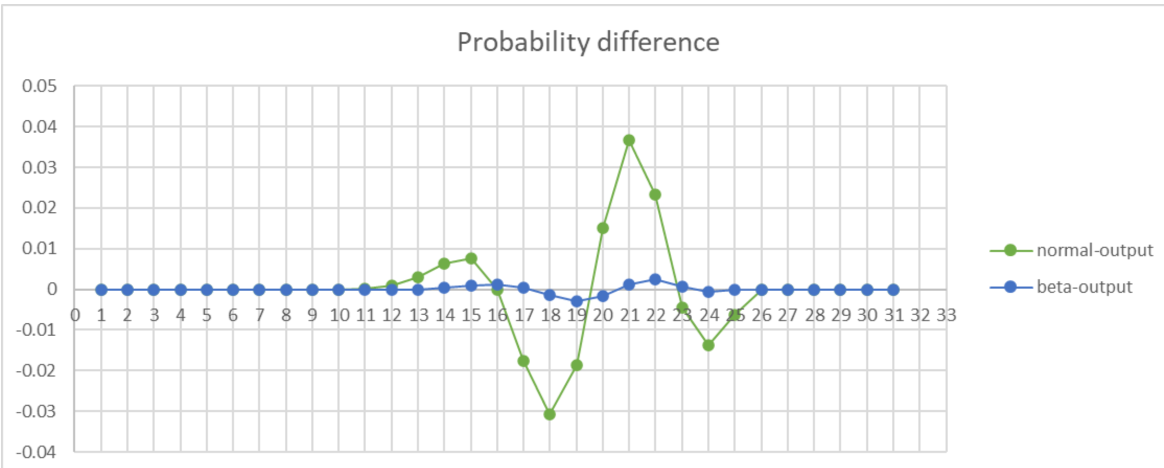


Figure 26: Distribution function difference plot for PERT delay

We can observe that the differences between the output and the normal are larger, while there is almost no discrepancy between the output and the beta distribution. The discrepancy between the beta and the output becomes negligible by changing the distribution of the delay from uniform to triangular to PERT.

Using the CDFs of the simulation output and the analytical distribution we can estimate the probability of having a feasible investment along with other possible outcomes summarized below

Probability	Simulation	Normal Fit	Beta Fit
P{NPV < -1}	0.068	0.057	0.066
P{NPV > 0}	0.729	0.722	0.730

P{NPV < 2}	0.927	0.917	0.926
P{NPV > 0.8}	0.446	0.420	0.443
P{NPV > 1}	0.630	0.655	0.634

We notice that the probability given by the fitted and simulation beta distribution is approximately the same as the one of the output, with an average absolute error of 0.89%. The difference between the normal and the output probabilities become quite larger with PERT delay than with the previously studied distributions.

As we change the distribution of the payment delay from triangular to PERT while keeping the same parameters, we notice that there is not a huge difference in the mean of the NPV but the variance increased from $970 to $1020, which means that the beta distribution has increased the variability of the possible outcomes.

The Q-Q plot shows that the normal distribution deviates from the simulation output especially around the tails, while the beta distribution presents a better fit along all the region of the NPV.

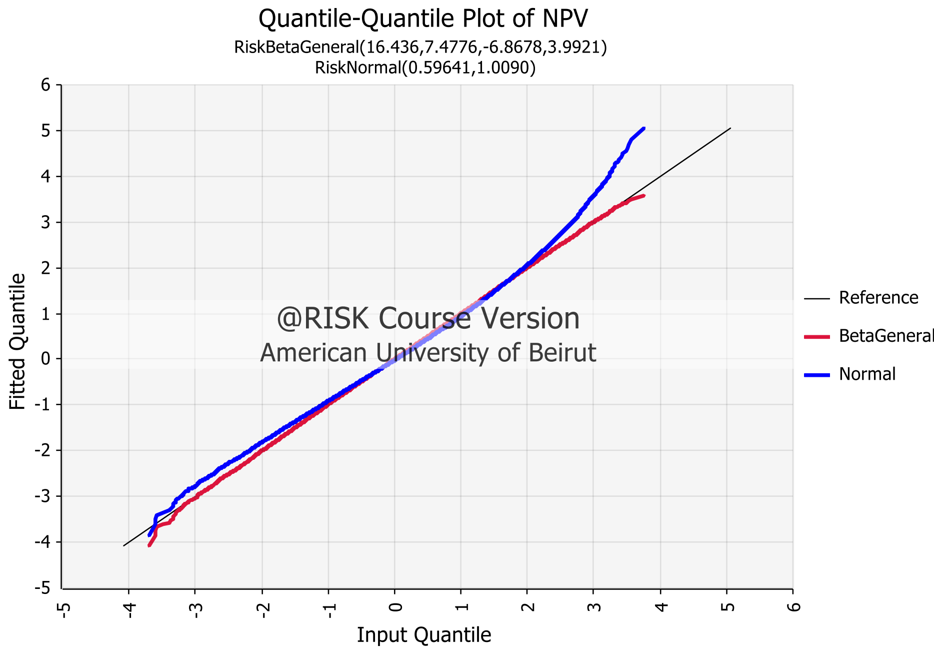


Figure 27: Q-Q plot with PERT delay

On the other side, the P-P plot shows the deviation of the normal distribution from the distribution of the NPV in the region of high probability. It is obvious that the beta distribution is a better fit to model the distribution of NPV with payment delays having a Pert distribution. We can observe that the deviation from the output is more obvious in the case of triangular and PERT delay than the case of uniform delay.

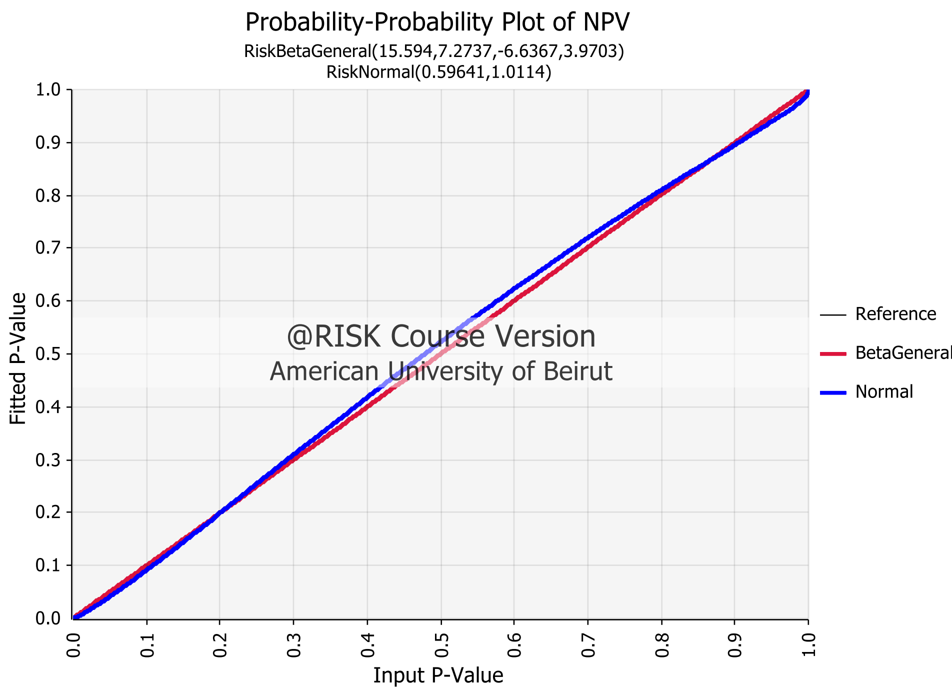


Figure 28: P-P plot with PERT delay

CHAPTER V

EXPONENTIAL DISTRIBUTION OF DELAY

In this chapter we analyze the case of exponential payment delays. In section 1 we derive expressions to estimate the mean and variance of NPV of multiple cash flows with exponential delays and provide an explanation of the difference between exponential and the previously analyzed distributions. In section 2 we provide a brief description about the Extreme Value Minimum distribution which we're going to fit into the NPV. Then, in section 3 we present the methodology to fit this distribution and validate it with a numerical example.

A. Exponentially Distributed Delay

If the delay T_i are iid exponentially distributed with rate λ_i, then the moment generating function of T_i is

$$\varphi_i(r) = \frac{\lambda_i}{\lambda_i + r}$$

Given these values for each cash flow, we can use them to fit a beta distribution based on the steps in the previous section as follows

$$m_1 = \sum_{i=0}^{n} F_i e^{-ir} \ \varphi_i(r) = \sum_{i=0}^{n} F_i e^{-ir} \frac{\lambda_i}{\lambda_i + r},$$

$$m_2 = \sum_{i=0}^{n} F_i^2 \ e^{-2ri}[\varphi_i(2r) - \varphi_i(r)^2]$$

$$= \sum_{i=0}^{n} F_i^2 \ e^{-2ri} \frac{\lambda_i}{\lambda_i + 2r} - (\frac{\lambda_i}{\lambda_i + r})^2.$$

The exponential distribution of delay presents a special case since it entails the risk of default when all payments are delayed to infinity, Ti has support over $(0,\infty)$, $\forall i$. In such situation, it is hard to estimate the range of net present value. The maximum value that NPV can reach occurs when all cash flows are paid at their due date, but the minimum value cannot be estimated since the exponential delay has a memoryless property and makes the estimation of the maximum duration of the delay a bit complicated. Decision makers sometimes consider an acceptable delay to be within the project duration, but if they are dealing with parties where their historical previous behavior show that the exceed the due date of their payment by an exponentially distributed duration, then exponential payment delay, then the left tail of the NPV distribution will vary according to the behavior of each client, since the case where all clients default is very rare to occur.

This has been proven by simulation using the same cash flows and their average delay as in Chapter 4, after running the simulation to model the distribution of the NPV with uniform, triangular and Pert distribution of payment delays, we considered the case where the only information available to the decision maker is the average payment delay of each cash flow and we modeled it y an exponential distribution. Unlike other distributions, the simulation output doesn't suggest that a beta distribution is a good fit for the NPV. The intuition behind it is that the minimum value of NPV needed to estimate the four parameters of the bounded beta distribution cannot be estimated, even by using some approximations, the analytical parameters derived are not valid to model a general beta distribution.

According to the chi-square test, the suggested distributions that can be fitted into the output exclude the beta distribution, and that the normal distribution is not the best fit, rather we have the Extreme value min distribution.

If we estimate the min of NPV using the step of the model described in the previous section,

$A = -P + \lim_{t_{i\to\infty}} \sum_{\mathrm{i}=1}^{5} \mathrm{e}^{-\mathrm{r}(i+T_i)}\mathrm{F}_\mathrm{i} = -P = -100$, while the range of NPV starts at -11 (values given in 1000$).

The change in the payment time modeled by the exponential distribution exhibits significant changes in kurtosis, and this will impact the changes in the NPV distribution so that extreme outcomes occur more frequently than predicted by a normal distribution. On the other hand, the exponential distribution of delay entails the possibility of default, this will impact the tails of distribution of NPV and interpret the use of the extreme value minimum distribution.

B. Extreme Value Minimum Distribution

Extreme value distributions are used to model extreme events that are hard to be observed. The origin of these distributions is the extreme value theory which aimed at first to study events such as flood levels. Now days this theory has been extended to cover different areas (financial crisis, importance of malfunction in industries, …) Charras-Garrido & Lezaud(2013).

The extreme value Type I has one form based on the smallest extreme and the othr based on the largest extreme. These distiribution are commonly known as Gumbel distribution. in our analysis, we are concerned with the extreme value minimum distribution which is the Gumbel minimum and has the following probability density function,

$$f(x) = \frac{1}{\beta} e^{\frac{x-\mu}{\beta}} e^{-e^{\frac{x-\mu}{\beta}}}$$

where μ is the location parameter and β is the scale parameter.

The delay itself is in the form of $e^{-\lambda x}$and using continuous compounding it might be approximated by $e^{-re^{-\lambda x}}$. And since the Extreme Value Minimum is the limiting distribution of a minimum of a set of random variables with unbounded distribution, which in our case the exponential delays, this could be an intuition behind the suggestion that this distribution is a good fit for the NPV distribution under exponential payment delays. The next section outlines the model of deriving the analytical extreme value minimum distribution and compare it to that given by simulation and to the normal distribution.

C. Fitting Technique

The following algorithm provides the steps to derive an analytical expression of a beta distribution that we suggest is a good fit for the distribution of the net present value of project under random payment delays.

Step 1. Using the moment generating function described above, estimate m_1 and m_2.

Step 2. Set $m_2 = \frac{\beta^2\pi^2}{6}$; $\beta = \sqrt{\frac{6m_2}{\pi^2}}$

Step 3. Using the value of β found in step 2, set

$$\mu = m_1 + 0.5772\beta \ .$$

Then, $f(x) = \frac{1}{\beta} e^{\frac{x-\mu}{\beta}} e^{-e^{\frac{x-\mu}{\beta}}}$

Example 7.

Given an initial investment of $100,000 followed by a series of cash inflows due on specific dates but might be delayed by a time which is exponentially distributed with λ_i given in months.

Year	Cash flow	Value (in $1000)	Average delay (months)	Delay variance
1	**F1**	20	2	2
2	**F2**	40	1	1
3	**F3**	35	3	3
4	**F4**	25	4	4
5	**F5**	45	5	5

We first run a Monte Carlo simulation in @Risk to estimate the NPV distribution. According to the Chi-square test's ranking, the Extreme Value minimum is the best fit followed by the normal distribution after 4 ranks. We can visually identify the discrepancies between the distributions in the following graph.

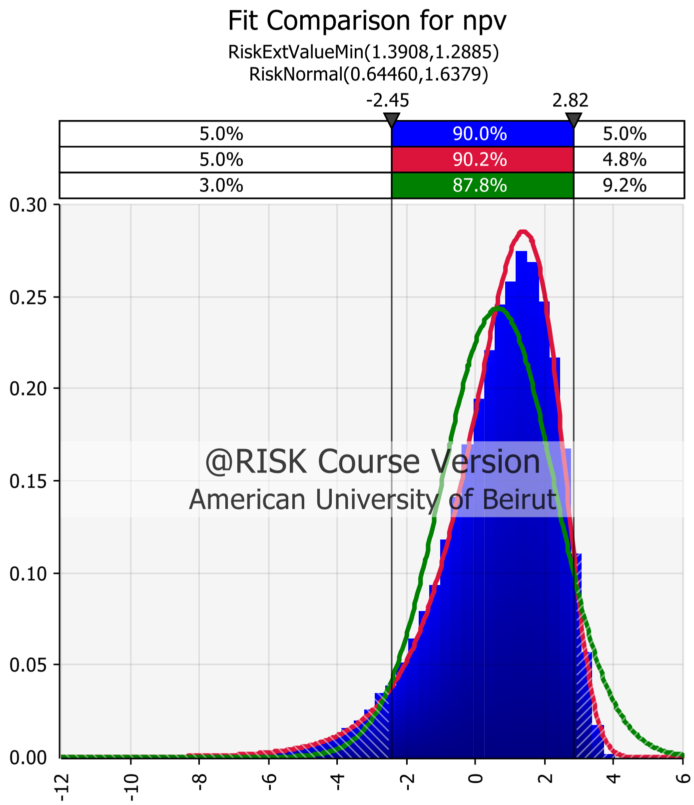


Figure 29: Simulation output for exponential distribution of delay with fitted distributions

The next step is to compare the analytical distribution found by estimating the parameters of the beta distribution to the distribution suggested by the simulation. Since T_i has an exponential distribution, we evaluate the mean and the variance of each return using the equations derived above and then we fit the series into the desired distribution.

.

$m_1 = \sum_{i=0}^{5} F_i e^{-rd_i} \varphi_i(r) = 0.64$

$m_2 = \sum_{i=0}^{5} F_i^{\,2} e^{-2rd_i} [\varphi_i(2r) - \varphi_i(r)^2] = 2.68$

Using Step 2 and 3, $= \sqrt{\frac{6m_2}{\pi^2}} = 1.28$,

and $\mu = m_1 + 0.5772\beta = 1.38$

These parameters will derive the analytical distribution that will be compared to that suggested by the simulation software to test the validity of our approach.

Using the simulation output, the parameters of the Extreme value minimum distribution fitted by the software are $\beta' =$1.29 and $\mu' = 1.39$.

We can now plot and compare these two distributions with the normal distribution that has a mean of 0.64 and a standard deviation 1.64. The PDFs are summarized below and the accuracy of our approximation is shown since the analytical and simulation plots are quite identical, while the normal distribution shows deviation from the plots and in other words deviation from the distribution of the NPV.

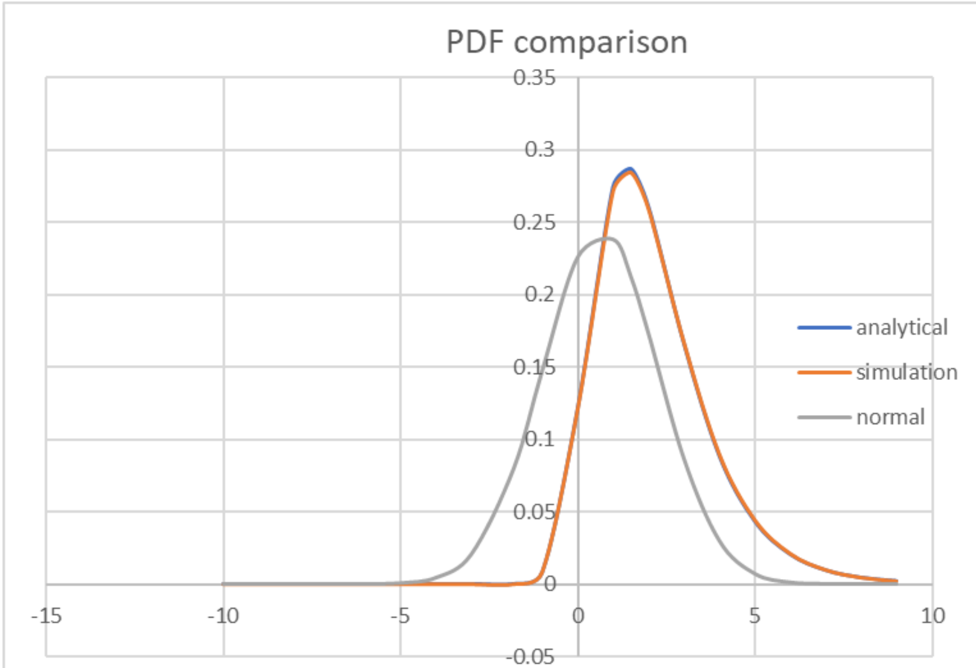


Figure 30: PDF comparison of NPV distribution with exponential delay

It is obvious that the distribution we derived is quite similar to that given by the simulation output. We also notice that at some values there is a difference between the results, but this difference is less than the difference between the results given by the normal distribution and the simulation output. This confirms our hypothesis, that the Extreme Value Minimum distribution gives better approximation for short time projects with exponential payment delay.

These observations are confirmed by the probability difference plot below, where the normal distribution deviates from the simulation distribution.

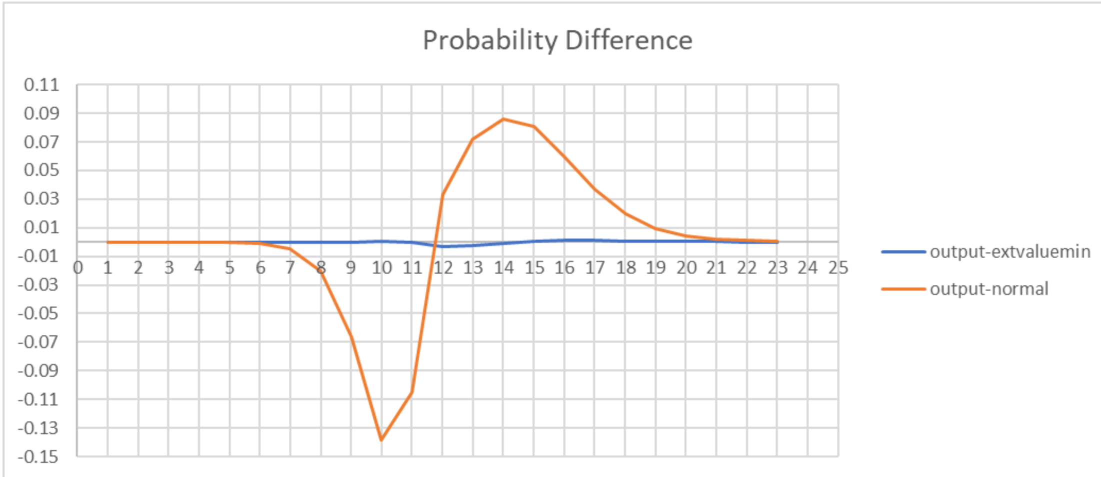


Figure 31: Probability difference plot with exponential delay

Using the CDFs of the distributions discussed above, we can compute the probabilities of earning about or below certain values, and the results are shown in the table below. We notice that the results given by the fitted beta distribution are approximately the same as those of the output and the beta given by the simulation. The normal distribution's probabilities are slightly different, and these slight differences can make a huge impact on losses for projects with high investment.

Probability	Simulation	Normal Fit	ExtValue Fit
P{NPV < -1}	0.151	0.157	0.143
P{NPV > 0}	0.703	0.653	0.712
P{NPV < 2}	0.790	0.796	0.803
P{NPV > 0.8}	0.473	0.536	0.476
P{NPV > 1}	0.524	0.586	0.530

This table confirms that our analytical distribution derived matches the true distribution fitted by the simulation software through the probabilities estimated. We notice also that the discrepancy between the probabilities estimated by the normal distribution and those of the output are quite large and reaches 13% at x>0.8 or for NPV>$800.

Unlike the beta distribution that didn’t show a large deviation from the output in the Q-Q plot, we notice that the extreme value minimum deviates in some regions but not as much as the normal distribution.

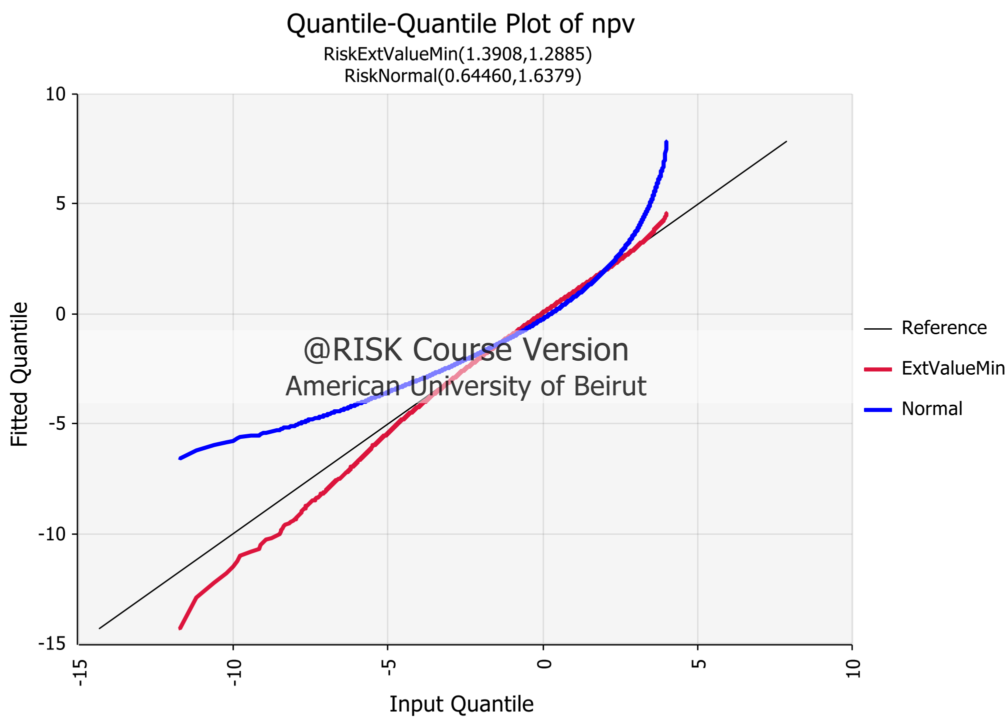


Figure 32: Q-Q plot with exponential delay

On the other hand, the P-P plot doesn’t present any deviation between the *x=y* line and the plot of the extreme value minimum, but an obvious deviation in the central region in the case of normal distribution.

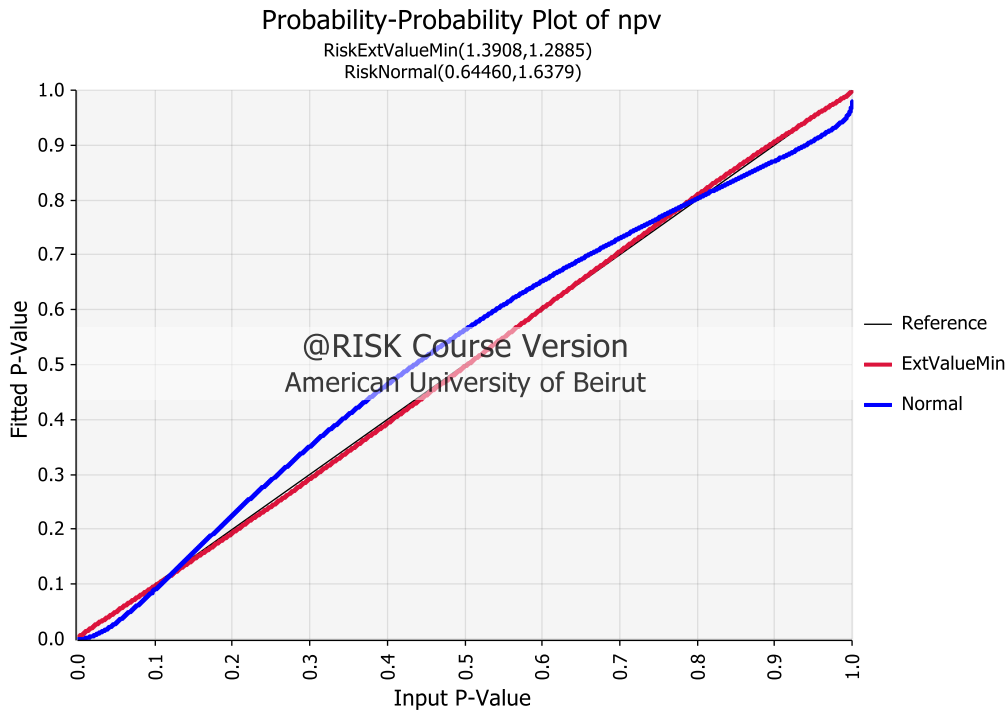


Figure 33: P-P plot with exponential delay

These observations confirm that the normal distribution is not also a good fit to model uncertainty, specifically in the case of investment analysis under payment delays.

CHAPTER VI

CONCLUSION AND FUTURE WORK

The purpose of this thesis is to provide an alternative tool to Monte Carlo Simulation that can be used in investment analysis, specifically investment analysis under time uncertainty. The analytical approach provides efficient estimation for the mean and variance of NPV of projects with payment delays by costumers. These estimations are used to fit a suitable distribution to the NPV according to the delay distribution.

The beta distribution fitted has been proven by numerical example to be a good fit and performs better than the normal distribution for projects with *iid* payment delays that have uniform, triangular and PERT distribution. By matching the first two moments, decision makers can fit a beta distribution to the given data and perform their feasibility analysis accordingly. The accuracy of the estimation compared to Simulation is relatively higher than the results estimated using the traditional approximation that considers the NPV distribution under stochastic conditions to be normal.

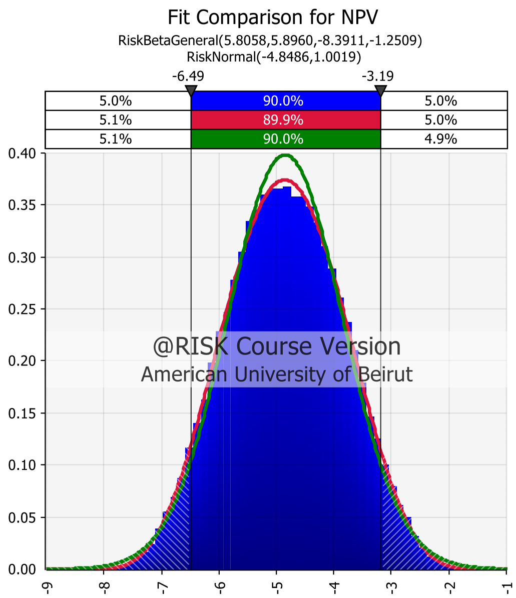
The exponential delay was a special case, since it is not quite often encountered in industries but mainly it can be used to model government' payment delays. The Extreme Value Minimum distribution has been shown to be a better fit than the normal and the methodology used to fit this distribution into the NPV of projects with exponential payment delays has high accuracy.

The validity of our analysis has been tested using statistical tools, and the produced results are statistically indistinguishable than those of @Risk. Moreover, the goodness of fit analysis confirms that assuming that the normal distribution is a good approximation is not valid in the case of payment delays. In the appendix, we show that even by increasing the duration of a project to up to 20 years, the normal distribution doesn't seem to be a good fit.

A possible direction of future work is to evaluate projects with different payment delays distribution, since in our research we focused on iid payment delays having same mean and variance for uniform, triangular and PERT distributions. Another possible extension is to incorporate stochastic cash flows with uncertain timings. For example in some cases cash flows can be paid partially at the due date and the remaining amount is delayed by a certain period. Another worthwhile extension is to perform capital budgeting for projects with uncertain payment timings. And finally performing risk analysis by incorporating cash outflows, where the decision maker is expecting an invoice that has been delayed and at the same time has to pay his bills, this will help managers in better planning and allocating their expenses.

APPENDIX

Uniform delay



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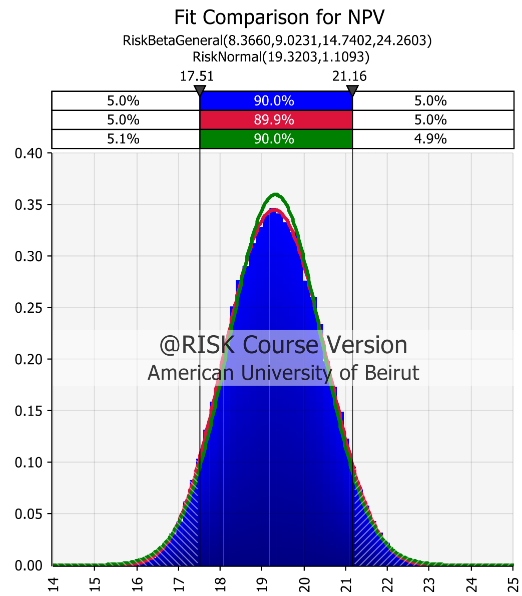

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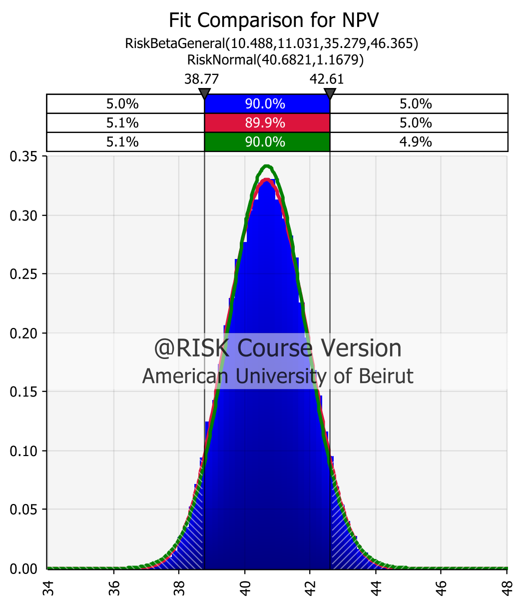

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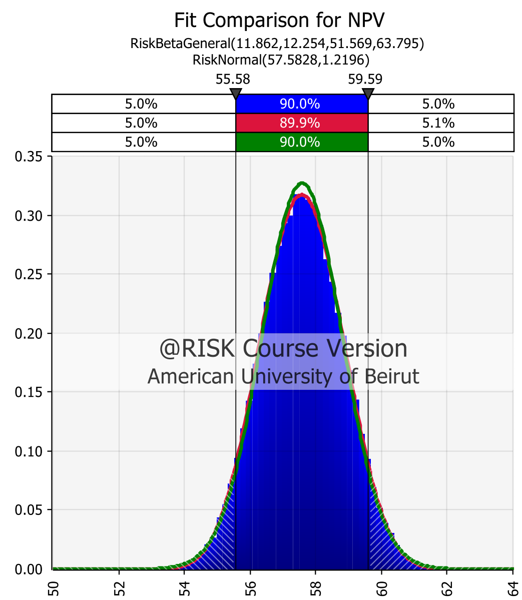

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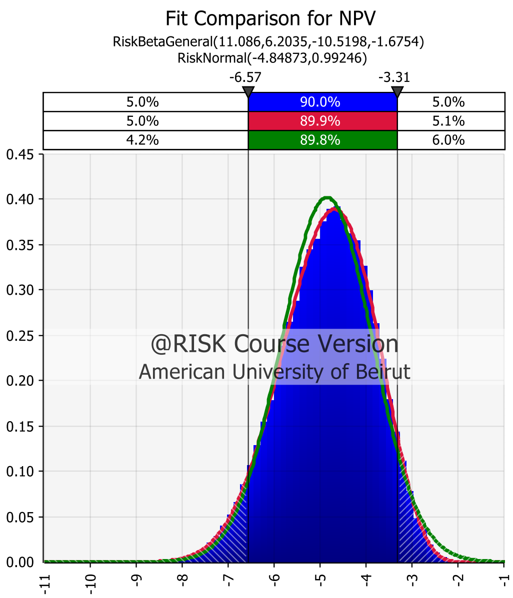
Triangular delay



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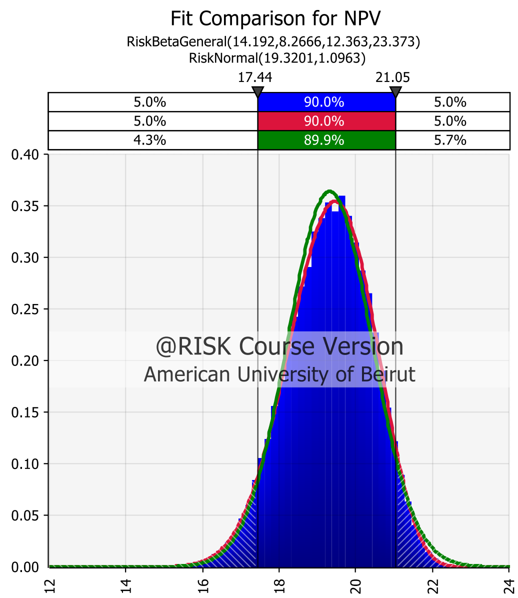

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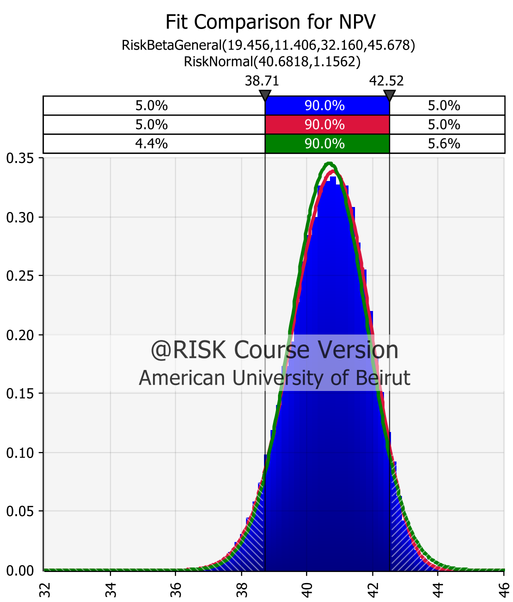

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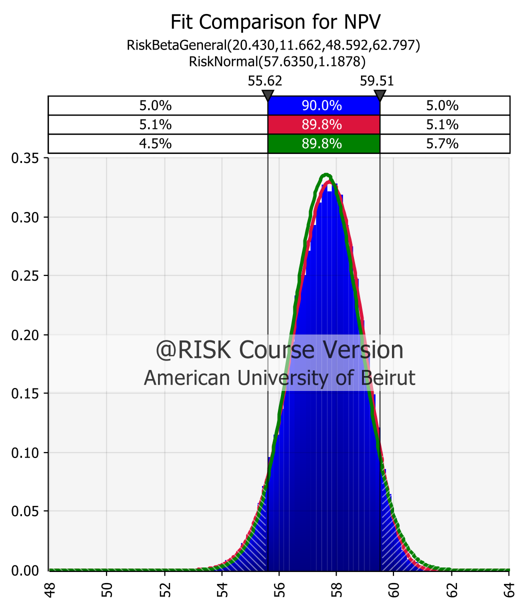

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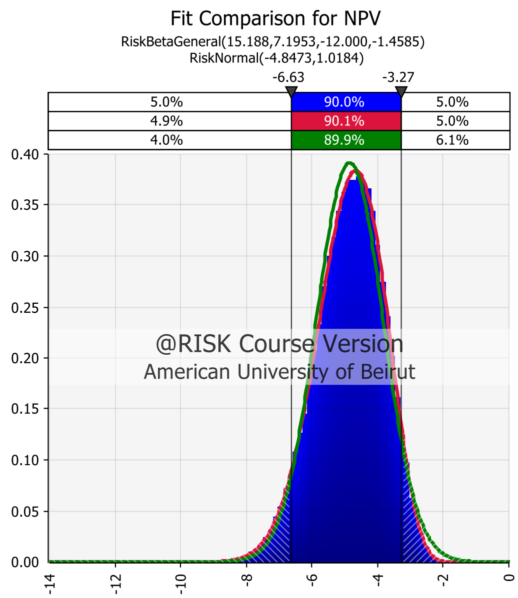
PERT delay



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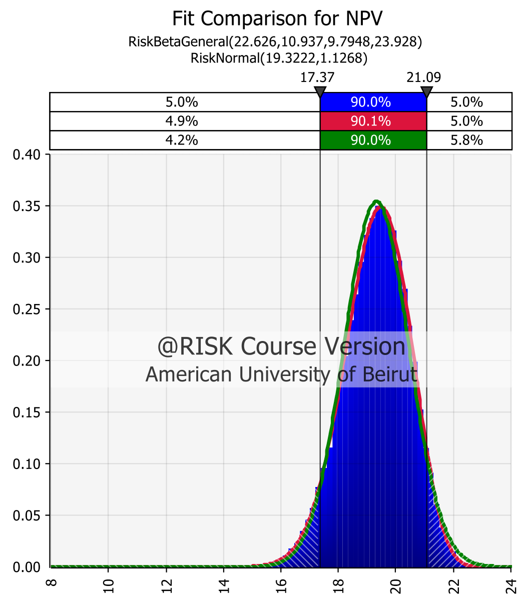

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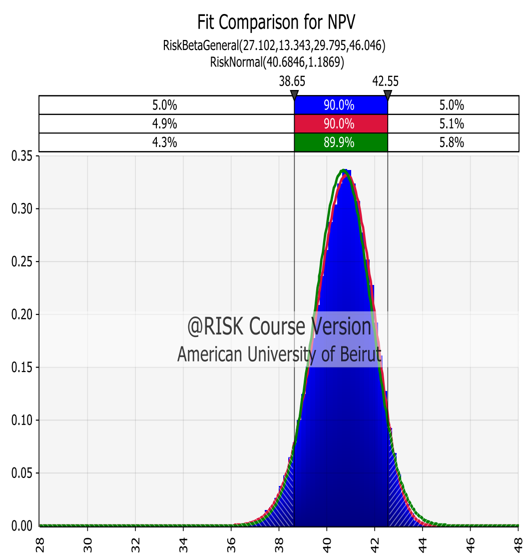

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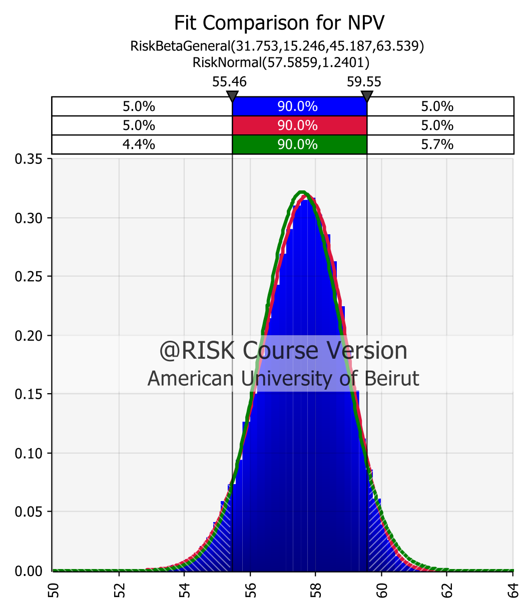

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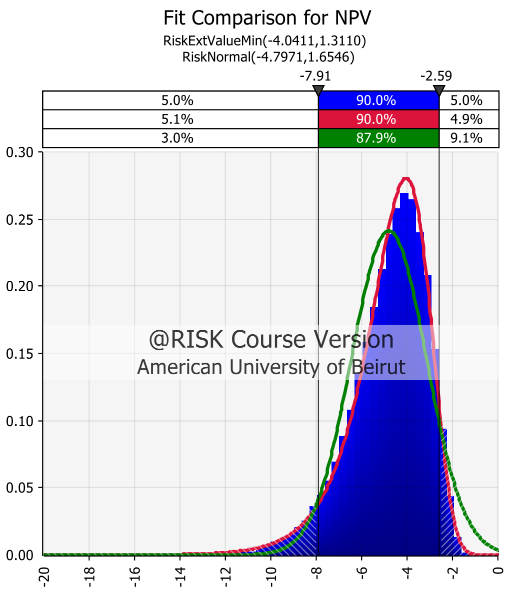
Exponential delay



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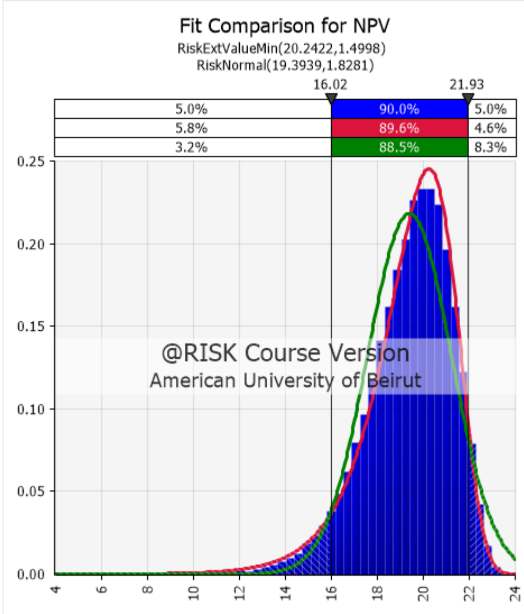

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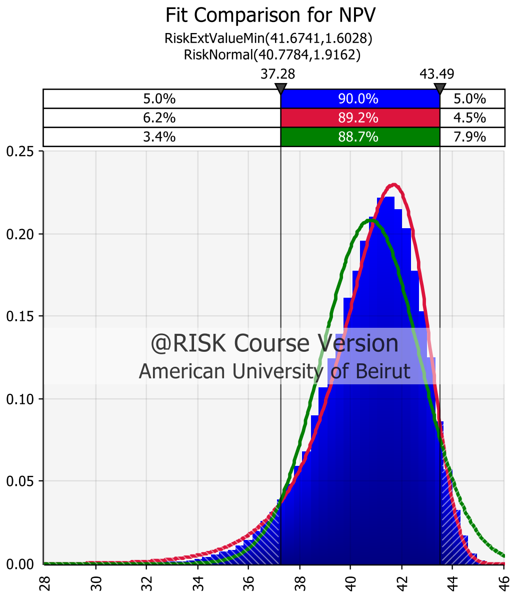

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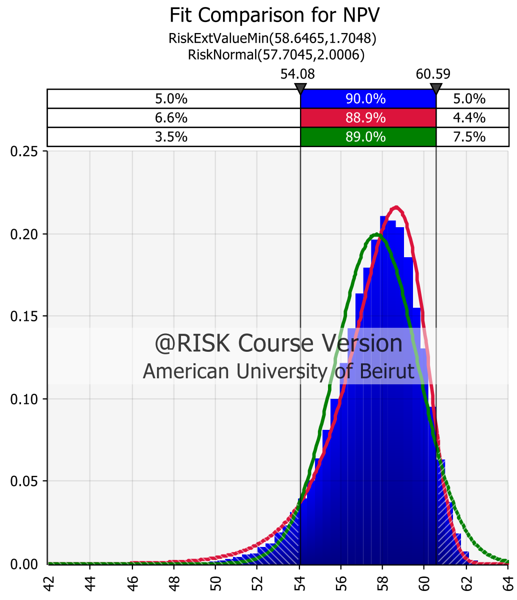

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