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MODIFIED THEORIES OF GRAVITY

by

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An Abstract of the Thesis of

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This dissertation is composed of three parts. The first presents the model of unification of Chern-Simons gauge theories and Chern-Simons gravity in 3D space-time. The second is constructing Hamiltonian formulation of ghost free mimetic massive gravity theory. The third is studying the surface terms of mimetic Horava gravity theory.

By enlarging the tangent group, we are able to unify Chern-Simons gauge theory and Chern-Simons gravity in 3D space-time. Either we start working with pontryagin densities in 4D space-time or we start directly considering Chern-Simons actions in 3D space-time. Such unification leads to the quantization of the coefficients for both Chern-Simons terms for compact groups but not for non-compact groups. Moreover, it leads to a topological invariant quantity of the 3D space-time manifold on which they are defined.

For the second topic, we construct the Hamiltonian of ghost free mimetic massive gravity. The linearized theory is studied and the Hamitlonian equations of

motion are analyzed. Poisson brackets are computed also and closure is proved. The second order scalar Hamiltonian is examined proving that the energy density of mimetic term is indeed positive.

The third topic is related to mimetic Horava gravity. It is shown that the surface terms resulting from the variation of the mimetic Horava action constructed will cancel out; therefore, there is no need for the addition of Gibbons-Hawking-York boundary term.

Contents

Chapter 1

Introduction

General relativity GR, published by Einstein in 1916, has been the best classical theory describing gravity till the moment. It is the geometric description of the gravitational field in terms of the curvature of space-time. The space-time curvature, described by Einstein's tensor $G_{\mu\nu}$, is related directly to the energy momentum tensor $T_{\mu\nu}$ describing matter and energy. This relation is summarized by the below Einstein field equations EFEs

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \tag{1.1}$$

In the Lagrangian formalism, the above equations result from the variation of the below Lagrangian which is linear in the scalar curvature R.

$$S_{GR} = \int d^4x \sqrt{-g} [-\frac{1}{2} R(g_{\mu\nu})] \tag{1.2}$$

Solving the non-linear equations 1.1 gives the metric $g_{\mu\nu}$. The first exact solution to the EFEs was the Schwarzschild metric which gives a great description of black holes. The Reissner–Nordstrom solution generalizes the Schwarzschild's one to deal with electrically charged black holes. The Friedmann-Lemaître-Robertson-Walker (FLRW) metric is another solution that describes the expanding universe. The success of GR has been demonstrated by the deflection of light [Shapiro et al., 2004], perihelion shift of Mercury [Clemence, 1947], and the discovery of merging black hole and neutron stars by the LIGO interference experiment [Abbott et al., 2016]. Despite the clear success, GR receives limitation with the emergence of the "dark universe scenario". In other words, GR doesn't provide a clear explanation for the different phases of acceleration of the universe. Moreover, GR is not able to provide a clear explanation for dark matter. Concerning the early universe, it is well known that the Big Bang Nucleosynthesis (BBN) is important to understand the evolution of the early universe. Despite of this successful theory, a problem is still to be resolved concerning the overproduction of the element ^7Li during the BBN as compared with the observations of the halo stars in the galaxy. In a recent work by [Makki et al., 2019] and references therein, this so called (cosmological Li problem) seems to demand nonstandard input, such as a dark matter component. This means that the standard GR relativity cannot be assumed at this early stage, which motivate a modified theory of relativity and is in support of our effort in the work. At extremely small scale, the classical theory of gravity breaks down. A new model is needed to explain the earlier stage of the universe or the inside of a black hole. This model should be a combination of GR and quantum mechanics.

All these weaknesses points and others push physicists to modify GR. Adding higher order terms to the GR Lagrangian, working with extra dimensions... are ways to construct alternative theories to GR. Chapter 1 reviews Chern-Simons theories that are topological field theories defined in 3-D space-time. Chern-

Simons theories are defined for both gauge theories and gravity.

Attempts of unifying gravity with the other forces started with Einstein himself who tried to unify gravity with electromagnetism. After the formulation of GR by few years, Theodore kaluza (1921) reconstructed GR on five dimensional manifold where it was proven by Oscar Klein (1926), that the gravitational curvature representing the extra spatial direction resembles the electromagnetic force. Thus, their (4+1) GR theory resembles a unified gravity-electromagnetic classical theory in (3+1) dimensional space-time. Based on the concept of Kaluza klein theory, the string theory was built in a way to unify gravity with all the other forces. Few years ago, Chamseddine and Mukhanov proposed a new classical model to unify gravity with the other forces based on the Cartan formulation of GR. Their idea was to enlarge the dimension of tangent space. The SO(1,13) tangent group splits into S(1,3) for gravity and SO(10) describing gauge theory. This model is reviewed in chapter 3. Chapter 4 represents our model of unification in the context of Chern-Simons theories. We are going to prove that both Chern-Simons gravity and Chern-Simons gauge theories are unified in 3D space-time.

Chapters 3 and 4 form the second part of the work. Mimetic gravity is a model introduced by Chamseddine and Mukhanov as a way to describe dark matter. This model has been extended to explain several ideas like dark energy, singularities.... Chapter 3 represents a review of the mimetic dark matter theory with all it's extensions. Ghost free mimetic massive gravity is one of these extensions where the graviton gains mass without the risk of introducing ghosts. Chapter 3 represents a review for this theory. Chapter 4 introduces the canonical formalism of this massive gravity model as an alternative to the Lagrangian one. It's main goal is to count physical degrees of freedom and to offer a possibility to quantize the theory.

Chapter 5 and 6 are concerned mainly with Horava gravity which is a modified theory of gravity that aims to get a quantized theory of gravity. Although this theory is renormalizable, it leads to the emergence of ghosts. Chamseddine, Mukhanov and Russ represent the mimetic Horava gravity which is a recent theory that reproduces Horava gravity using the mimetic field at the synchronous gauge. The theory is ghost free. Chapter 5 represents a review of both the Horava model and the mimetic Horava one. To be able to do the Hamiltonian analysis of any theory, special attention should be taken in dealing with surface terms. Chapter 6 represents our third work which proves that surface terms of mimetic Horava gravity action cancel each others without the need of adding extra terms like the Gibbon-Hawking boundary terms as in the case of pure GR theory.

Chapter 7 represents the conclusion and the future work.

Chapter 2

Topological Field Theories: Chern Simon's theories

Topological field theories (TFTs) are quantum field theories that define topological invariants. These theories are metric independent. After being influenced by Michael Attiyah [Atiyah, 1988], Witten constructed the first TFT which is the Donaldson Witten theory that defines the Donaldson invariant [Witten, 1988]. In general, TFTs can be divided into two groups: Witten or the cohmological type and the Schwarts type. Witten type TFT, like the Donldson Witten one, are theories with action of BRST form with functional average equal zero. The other type of TFTs are the Schwartz type like Chern-Simons theories and the BF theories. These theories have actions that are explicitly independent of the metric

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0 \tag{2.1}$$

and any metric independent operatior have a metric independent expectation value. Chern-Simons theory is a TFT defined on 3-dimensional space-time.

(2+l)-dimensional models have became an active field of research, especially the (2+1) gravity models. As we have mentioned before, quantizing gravity in (3+1) dimension space-time faces several obstacles [Carlip and Jonathan, 2003]. General relativity in (2+1) dimensions is much simpler, physically and mathematically. Constructing a quantum theory for (2+1) gravity has less obstacles than that for (3+1) one.
Witten was the first to incorporate Chern-Simons action in the domain of TFTs theories [Witten, 1989]. He proved that this theory can provide a great description of knot and links invariants, named jones polynomials. In a 3 dimensional manifold, with a compact lie group G = SU(N), the Chern-Simons action is defined as

$$S = \frac{k}{4\pi} \int Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{2.2}$$

where k is a coupling constant and A is a G-gauge connection on the trivial bundle on M. We define the metric independent partition function as

$$Z(M) = \int DAe^{iS} \tag{2.3}$$

Using non perturbative methods, Witten derived the topological invariant jones polynomial as

$$Z(M, L) = \int DAexp(iS) \prod_{i=1}^{r} W_{R_i}(C_i) \tag{2.4}$$

where W_{R_i} is the Wilson loop defined by

$$W_R(C) = Tr_R P exp \int_C A \tag{2.5}$$

R is a representation of the group G and C is an oriented closed curve on the three dimensional manifold. C can be a circle.

Chern-Simons theories have additional applications in several fields. In the domain of gauge theories, Deser an Jackiw proved that we can generate mass for gauge fields for both gravity and gauge theory using Chern-Simons terms [Deser et al., 2000]. In the context of quantum field theory, the infinite range gravitational field is described by a massless spin 2 particle that should mediates the gravitational interactions. Introducing mass for graviton, is one of the main concerns of most modified theories of gravity because it helps in constructing a quantized theory of gravity. Fierz-Pauli linear massive gravity theory was the first theory to generate mass for graviton [Fierz and Pauli, 1939a]. Using Chern-Simons theories in 3D space-time, Deser , Jackiv and Templeton constructed a new massive theory for gravity by adding the gravitational Chern-Simons term to the gravitational action.

$$I_{CS} = -\frac{1}{4}\int dx X^3 = -\frac{1}{4}\int dx \epsilon^{\mu\nu\alpha}[R_{\mu\nu ab}\omega_\alpha^{ab} + \frac{2}{3}\omega_{\mu b}^{c}\omega_{\nu c}^{a}\omega_{\alpha a}^{b}] \tag{2.6}$$

where X^3 is related to the four dimensional topological invariant, the Hirzebruch Pontryagin density

$$*RR \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu\rho\sigma}R_{\alpha\beta}^{\rho\sigma} = \partial_\mu X^\mu \tag{2.7}$$

and the curvature $R_{\mu\nu ab}$ is defined by

$$R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} + \omega^c_{\mu a}\omega_{\nu cb} - (\mu \to \nu)$$
$$\omega_{\mu ab} = \omega_{\mu ba} \tag{2.8}$$

The gravitational action becomes

$$I = \frac{1}{\kappa^2}\int dx\sqrt{g}R + \frac{1}{\kappa^2\mu}I_{CS} \tag{2.9}$$

Similarly, using Chern-Simons actions we can generate mass terms for non-abelian gauge theories where the topological mass term is given by

$$L_G = \frac{\mu}{2g^2}trF^{\mu\nu}F_{\mu\nu} = \frac{\mu}{2g^2}\epsilon^{\mu\nu\alpha}tr(F_{\mu\nu}A_\alpha - \frac{2}{3}A_\mu A_\nu A_\alpha) \tag{2.10}$$

In addition, Chamseddine and Frohlich showed that the lorentz and mixed lorentz-Weyl anomaly, but not the pure Weyl anomaly, of the two dimensional chiral bosons and fermions is cancelled by the anomalies of the three dimensional gravitational Chern-Simons action [Chamseddine and Fröhlich, 1992]. In the context of condensed matter physics, 3D Chern-Simons theories have been used in constructing lattice field theories. Also, there are numerous applications of CS theories in modern holography like the theory of singletons.

Chapter 3

Unification of Gauge and Gravity Chern Simons theories in 3-D space-time

Attempts for unifying gravity with other gauge theories started directly after the formulation of GR. Kaluza-Klein theory is such a model of unification of gravity with electromagnetism, in the context of classical field theories. It is based on the idea of considering higher dimension space-time [Kaluza, 1921b]. Later, Kaluza-Klein theory was generalized to general non-abelian groups as an attempt to unify gravity with Yang mills gauge theory [Kerner, 1968]. Such classical unified field theories have suffered from several weaknesses like the possibility of promoting it to a well defined complete quantum unified theory. In general, all the unification models depend on the formulation of GR that one starts with. The second order formulation of GR is the usual one that deal with the metric $g_{\mu\nu}$ with the action eq 1.2. The other formulation of GR is the first order known as the Cartan one which replaces the metric by two independent variables: the tetrad e^A_α and the connection $\omega^B_{\beta A}$ [Utiyama, 1956], [Kibble, 1961]. The main goal of the tetrad formalism is to incorporate spinor (Dirac and Weyl) in the context of general relativity. This allows us to study their dynamics on curved space-time. GR, through its initial formalism, can only deal with objects behaving as tensors under Lorentz transformation.

To couple spinors to gravity, it is important to impose the concept of local covariance that carries us from the concept of manifold as a whole to tangent space defined at each point on the manifold. This tangent space resembles Minkowski space with a certain tangent group defined on it. The indices on this tangent space are called flat indices (A, B, C ....) while that on the curved manifold as a whole are the Greek ones α, β.... Similar to the affine connection Γ defined on the whole manifold, which is necessary to define the covariant derivative and parallel transport, we define the spin connection ω_μ^{AB} that is important to construct the covariant kinetic term of the spinors. In addition, it is important to choose a representation of the Clifford algebra which are the gamma matrics [de Wit and Smith, 2012]. Mixing flat indices with curved ones is achieved by the Vielbeins (tetrad) e_μ^A. In other words, we can say that this tetrad formalism is simply a gauge theory of general relativity where the spin connections are the gauge fields. It should be noted that this local lorentz symmetry can be generalized to a supersymmetric one thus leading to supergravity.

In terms of this formulation, Einstein-Hilbert action becomes

$$S = \frac{1}{8\kappa^2} \int d^4x \epsilon^{\mu\nu\alpha\beta} \epsilon_{ABCD} e_\mu^A e_\nu^B R_{\alpha\beta}^{CD}(\omega) \tag{3.1}$$

known as the Hilbert-Palatini action. $R_{\alpha\beta}^{CD}$ is the field strength of the gauge connection ω_μ^{AB} of the local Lorentz group SO(4).

3.1 Unification of Gravity and Gauge interactions

The dimension of the tangent space is usually the same as that of the manifold. However, Chamseddine and Mukhanov proved that dimension of the tangent space can be larger than that of the manifold. For example, choosing the dimension of tangent space to be five with the de-Sitter tangent group SO(1,4) or the anti de Sitter one SO(2,3) recovers general relativity again [Chamseddine and Mukhanov, 2010a]. This step is beneficial in coupling matter to gravity. For example, 4d vector field can be coupled to scalar field in five dimensional tangent space. They also discussed the possibility of having a complex tangent space of same dimension as the manifold with the unitary group as tangent group. As a result, Einstein gravity is recovered again also.

In a subsequent paper, Chamseddine and Mukhanov proved that enlarging the tangent space can give interesting physical results like the unification of both gravity and gauge theories [Chamseddine and Mukhanov, 2016b]. The four dimensional space-time manifold is spanned by the coordinate basis e_α, where α goes from 1 to 4. The N-dimensional tangent space, where N is arbitrary and greater than or equal 4, is spanned by v_A, where A ranges from 1 to N. We choose the tangent group to be SO(1,N-1). As we have mentioned before, the two basis are connected by the veilbiens e^A_α through $e_\alpha = e^A_\alpha v_A$. We define $n_{\hat{J}}$, where $\hat{J}$ goes from 5 to N, as the basis of the $N-4$ subspace that is orthogonal to the subspace spanned by e_α. Thus, we get

$$\begin{aligned} &n_{\hat{J}}.e_\alpha = 0 \\ &n_{\hat{J}}.n_{\hat{I}} = \delta_{\hat{I}\hat{J}}, \hat{I}, \hat{J} = 5.....N. \end{aligned} \tag{3.2}$$

e_α and $n_{\hat{j}}$ form a complete basis in the tangent space. Thus, any vector, defined in the tangent space, can be expressed as a linear combination of them

$$v_A = e^{\alpha}_{A} e_\alpha + n^{\hat{j}}_{A} n_{\hat{j}} \tag{3.3}$$

The covariant derivatives in both spaces are defined by

$$\begin{aligned} \nabla_{e_\beta} e_\alpha &= \nabla_\beta e_\alpha = \Gamma^{\nu}_{\alpha\beta} e_\nu \\ \nabla_\beta v_A &= -\omega^{B}_{\beta A} v_B \end{aligned} \tag{3.4}$$

where $\Gamma^{\mu}_{\alpha\beta}$ and $\omega^{B}_{\beta A}$ are the respective affine and spin connections for both the manifold and the tangent space. They are related by the metricity condition

$$\partial_\beta e_{A\alpha} = -\omega^{B}_{\beta A} e_{B\alpha} + \Gamma^{\nu}_{\alpha\beta} e_{A\nu} \tag{3.5}$$

The corresponding curvature tensors for both the manifold and tangent space are defined respectively as

$$\begin{aligned} R^{\rho}_{\gamma\alpha\beta}(\Gamma) &= \partial_\alpha \Gamma^{\rho}_{\beta\gamma} - \partial_\beta \Gamma^{\rho}_{\alpha\gamma} + \Gamma^{\rho}_{\alpha\kappa}\Gamma^{\kappa}_{\beta\gamma} - \Gamma^{\rho}_{\beta\kappa}\Gamma^{\kappa}_{\alpha\gamma} \\ R^{AB}_{\alpha\beta}(\omega) &= \partial_\alpha \omega^{AB}_{\beta} - \partial_\beta \omega^{AB}_{\alpha} + \omega^{AC}_{\alpha}\omega^{B}_{\beta C} - \omega^{AC}_{\beta}\omega^{B}_{\alpha C} \end{aligned} \tag{3.6}$$

These two curvature tensors are related to each other through

$$R^{AB}_{\alpha\beta}(w) = R^{AC}_{\alpha\beta}(w) n^{\hat{I}}_{C} n^{B}_{\hat{I}} + R^{\rho}_{\gamma\alpha\beta}(\Gamma) e^{A}_{\rho} e^{B\gamma}. \tag{3.7}$$

Chamseddine and Mukhanove proved that the tensor $R^{AC}_{\alpha\beta}(w)$ is related to the curvature tensor $F^{\hat{I}}_{\ \alpha\beta\hat{J}}(A)$ of the N-4 subspace, with SO(N-4) group, through choosing a special gauge

$$e^{\mu}_{\hat{I}} = 0 \tag{3.8}$$

As a result, we get, according to the metricity condition eq 3.5,

$$\omega^{a}_{\ \mu\hat{I}} = 0 \tag{3.9}$$

where a ranges from 1 to 4. In this special gauge the mixed curvature becomes zero

$$R^{a\hat{I}}_{\mu\nu} = 0 \tag{3.10}$$

Thus, the curvature tensor $R^{C}_{\alpha\beta A}(\omega)$ is completely related to the field strength of the (N-4) subspace.

$$R^{C}_{\alpha\beta A}(w) n_{C} = n^{\hat{J}}_{A} F^{\hat{I}}_{\ \alpha\beta\hat{J}}(A) \tag{3.11}$$

where the field strength F can be expressed in terms $A^{\hat{I}}_{\beta\hat{J}}$, the connection of the N-4 subspace, as follow

$$F^{\hat{I}\hat{J}}_{\alpha\beta}(A) = \partial_\alpha A^{\hat{I}\hat{J}}_\beta - \partial_\beta A^{\hat{I}\hat{J}}_\alpha + A^{\hat{I}\hat{L}}_\alpha A^{\hat{J}}_{\beta\hat{L}} - A^{\hat{I}\hat{L}}_\beta A^{\hat{J}}_{\alpha\hat{L}} \tag{3.12}$$

Eq 3.7 can be re-written as

$$R^{AB}_{\alpha\beta}(w) = F^{AC}_{\alpha\beta}(w) n^{\hat{I}}_C n^B_{\hat{I}} + R^\rho_{\gamma\alpha\beta}(\Gamma) e^A_\rho e^{B\gamma} \tag{3.13}$$

Varying the dimension of the tangent space leads to different physical results. As we mentioned before, choosing N to be five recovers Einstein gravity again. For N=6, gravity is unified with electromagnetism with A^{56}_α acting as the Maxwell field. To get a total unification of gauge theories and gravity, the tangent group must be SO(1,13).

3.2 Unification of Gauge and Gravity Chern Simons theories in 3-D space-time

Based on the work of Chamseddine and Mukhanove (section 3.1) and using the idea of Chern-Simons theories, presented in chapter 2, we present a new model of unification, in 3D space-time, for both gravity and gauge theories. Our goal can be achieved in 2 different ways: either we start working in 4D space-time using pontryagin densities and then we deduce the unification in 3D or we directly prove unification in 3 dimensional space-time using Chern-Simons actions. We are going to present both methods. [Saghir and Shamseddine, 2017].

3.2.1 Pontryagin densities

Pontryagin densities appeared first in the context of anomalous Feynman diagrams of gauge theories. On a 2n dimensional manifolds these densities are defined as

$$P^{2n} \propto \epsilon^{\mu_1\mu_2.....\mu_{2n}} Tr(F_{\mu_1\mu_2}....F_{\mu_{2n}\mu_{2n-1}}) \tag{3.14}$$

where F is the field strength which is a curvature 2-form of group G

$$F = dA + A \wedge A \tag{3.15}$$

In 4 dimensional space-time, the pontryagin density of gauge theory is

$$P_4 = -\frac{1}{16\pi^2} Tr({}^*F^{\mu\nu} F_{\mu\nu}) \tag{3.16}$$

where

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + f^{ABC} A_{\mu B} A_{\nu C} \tag{3.17}$$

The 4D pontryagin density for gravity is the (Hirzebruch-Pontryagin) given by

$$*RR = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\rho\sigma} R^{\rho\sigma}_{\alpha\beta} \tag{3.18}$$

Using eq 3.13, we start writing the pontryagin density for the larger group SO(N) in 4-D space-time

$$\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} R^{AB}_{\mu\nu} R_{\alpha\beta AB} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}(F^{\hat{I}\hat{J}}_{\mu\nu} n^A_{\hat{J}} n^B_{\hat{I}} + R^{\rho}_{\gamma\mu\nu} e^A_{\rho} e^{B\gamma})(F^{\hat{K}\hat{L}}_{\alpha\beta} n_{\hat{K}A} n_{\hat{L}B} + R^{\delta}_{\sigma\alpha\beta} e_{\delta A} e^{\sigma}_{B}) \tag{3.19}$$

Using the relation $n^A_{\hat{J}} e^{\alpha}_A = 0$ [Chamseddine and Mukhanov, 2016b], all the mixed terms vanish and we are left with the pontryagin densities for both gauge theory and gravity.

$$\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} R^{AB}_{\mu\nu} R_{\alpha\beta AB} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu\hat{K}\hat{L}} F^{\hat{K}\hat{L}}_{\alpha\beta} + \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} R^{ab}_{\mu\nu} R_{\alpha\beta ab} \tag{3.20}$$

which means

$$\int_{M_4} P_4 = \int_{M_4} P_4^{gauge} + \int_{M_4} P_4^{gravity} \tag{3.21}$$

Up to this equation, we have proven the unification of the pontryagin densities in 4D space-time. The pontryagin densities are the exterior derivatives of the Chern-Simons entities.

$$P_4 = Tr(F \wedge F) = Tr((dA + A \wedge A) \wedge (dA + A \wedge A))$$
$$= Tr(d(AdA + A^3)) = Tr(dw_3) \tag{3.22}$$

Knowing that $I_{CS} = \int w_3$, we get

$$\int_{M_4} P_4 = \int_{M_4} dw_3 = \int_{\partial M_4} w_3 \tag{3.23}$$

Using eq 3.23, eq 3.21 can be translated into unification in 3D space-time

$$\int_{M_3} w_3 = \int_{M_3} w_3^{gauge} + \int_{M_3} w_3^{gravity} \tag{3.24}$$

where

$$w_3^{gauge} = \epsilon^{ijk}(A_i^a \partial_j A_k^a + \frac{1}{3} f^{abc} A_i^a A_j^b A_k^c)$$
$$w_3^{gravity} = \epsilon^{ijk}(R_{ijab}\omega_k^{ab} + \frac{2}{3}\omega_{ib}^c \omega_{jc}^a \omega_{ka}^b) \tag{3.25}$$

3.2.2 3-D Chern-Simons terms

Instead of start working in 4D space-time, we can directly start considering Chern-Simons terms in 3D space-time. The Chern-Simons term of the larger group SO(N) is

$$I_{CS} = \frac{k}{4\pi} \int Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{3.26}$$

Using gamma matrices Γ_{AB}, where A,B.. ranges from 1 to N, the 1-form connection A can be written as [de Wit and Smith, 2012]

$$A = dx^{\mu} \frac{1}{4} A_{\mu}^{AB} \Gamma_{AB} \tag{3.27}$$

Γ-matrices are related to the γ-matrices of the clifford algebra

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\delta_{ab} I \tag{3.28}$$

where a,b,c... ranges from 1 to 4. According to the properties of the Γ matrices, eq 3.26 can be transferred into component form

$$I_{CS} = \frac{k}{4\pi} \int (A^{AC} dA^{CA} + \frac{2}{3} A^{AB} A^{BC} A^{CA}) \tag{3.29}$$

Knowing that the indices A, B, C ranges from 1 to N, the above action splits into 2 actions

$$\begin{aligned} I_{CS} &= \frac{k}{4\pi} \int (A^{AC} dA^{CA} + \frac{2}{3} A^{AB} A^{BC} A^{CA}) \\ &= \frac{k}{4\pi} \int (A^{ac} dA^{ca} + \frac{2}{3} A^{ab} A^{bc} A^{ca}) + \frac{k}{4\pi} \int (A^{\hat{I}\hat{J}} dA^{\hat{J}\hat{I}} + \frac{2}{3} A^{\hat{I}\hat{J}} A^{\hat{J}\hat{K}} A^{\hat{K}\hat{I}}) \end{aligned} \tag{3.30}$$

where the first term corresponds to gravity Chern-Simons term and the second to the gauge theory one. Hence, both gravity and gauge theory are unified in 3D space-time.

3.2.3 Consequences of this Unification

Quantization of the coefficient

The quantization of the coupling constant k depends on the signature of space-time. It is quantized for Euclidean space-time where the compact SO(6) tangent group splits into SO(3) for gauge theory term and SO(3) for gravity term. On the other hand, choosing the lorentzian signature manifold with the non-compact group SO(1,5) leads to a non-quantized coupling constant k. The SO(1,5) group splits into SO(1,2) for gravity and SO(3) for gauge theory. The homotopy group of SO(1,5) is equal to that of SO(5).

$$\pi_5(SO(1,5)) = \pi_5(SO(5)) = Z_2 \tag{3.31}$$

In this case, the coupling constant is not quantized since the winding number is not sensitive to torsion and vanishes.

Topological Invariants

The main goal of topological field theories is to compute topological invariants. Witten proved that, in the week coupling limit, the below partition function can not be a topological invariant quantity unless an extra term, related to the gravitational Chern-Simons partition function, is added [Witten, 1989].

$$Z = \int DAexp(\frac{ik}{4\pi}\int_M (AdA + \frac{2}{3}A \wedge A \wedge A) \tag{3.32}$$

In our case, the group G is the SO(6) group. The topological invariant partition function can be written as

$$\begin{aligned} Z &= \int DAexp(iI_{CS}) = \int DAexp(iI_{gauge} + iI_{gravity}) \\ &= \int Dwexp(iI_{gravity} \int DBexp(iI_{gauge}) = Z_1.Z_2 \end{aligned} \tag{3.33}$$

where Z_1 corresponds to the SO(1,2) partition function of gravity Chern-Simons action with w as a gauge connection and Z_2 is that of SO(3) gauge theory with B as a gauge connection.
Following Witten, the weak coupling limit of Z_2 can be written as

$$Z_2 = \sum_\alpha \mu(B^\alpha) \tag{3.34}$$

where $\mu(B^\alpha)$ is a function of flat connections for which the curvature vanishes. Expanding the gauge field B_i around the flat connection, $B_i = B_i^\alpha + C_i$, the gauge Chern-Simons action becomes

$$I_{CS}^{gauge} = kI(B^\alpha) + \frac{k}{4\pi}\int_M Tr(C \wedge DC) \tag{3.35}$$

D is the covariant derivative with respect to B^α. To carry out the gaussian integral in eq 3.35, a gauge fixing mechanism should be applied. Such a step can't be done without choosing a specific metric. Witten chose the metric to satisfy $D_i C^i = 0$. The gauge fixing mechanism generates ghosts which are represented by the below action.

$$S_{GF} = \int_M Tr(\phi D_i C^i + \bar{c} D_i D^i C) \tag{3.36}$$

ϕ is a lagrange multiplier that enforces the gauge fixing condition $D_i C^i = 0$. c and $\bar{c}$ are anticommuting ghosts.
After integrating out C, ϕ, c and $\bar{c}$, eq 3.36 becomes

$$exp(\frac{i\pi\eta(B^\alpha)}{2} T^\alpha \tag{3.37}$$

$\eta(B^\alpha)$ is the "eta-invariant" defined by

$$\eta(B^\alpha) = \frac{1}{2} \lim_{s\to} \sum_i sign\lambda_i |\lambda_i|^{-s} \tag{3.38}$$

where λ_is are eigenvalues of operator L_i, the restriction of $*D_B + D_B*$ on odd forms, T_α is the torsion invariant of flat connections $B^{(\alpha)}$. According to Atiyah-Patodi-Singer theorem, eq 3.34 becomes

$$Z_2 = exp(i\frac{\pi}{2}\eta(0)) \sum_\alpha e^{i(k+c_2(G)/2)I(B^{(\alpha)})} . T^\alpha \tag{3.39}$$

$\eta(0)$ is the eta invariant of the trivial gauge field and $c_2(G)$ is the Casimir operator of G.

Witten noticed that the above partition function is not a topological invariant quantity since $\eta(0)$ is metric dependent. To restore the invariance, Witten suggested to add a counter term to the above partition function. This counter term is found to be proportional to the gravity Chern-Simons action.

By examining eq 3.33, we noticed that in our case the gravitational Chern-Simons action is already found. So, our partition function Z of the larger group SO(6) is indeed a topological invariant quantity without the need to add any extra term. Substituting the weak coupling limit of the gauge partition function in eq 3.33, we get

$$
\begin{aligned}
Z &= \int Dwexp(iI_{grav}).Z_2 \\
&= \int Dwexp(iI_{grav})exp(i\frac{\pi}{2}\eta(0)) \sum_{\alpha} e^{i(k+c_2(G)/2)I(B^{(\alpha)})}.T^{\alpha}
\end{aligned} \tag{3.40}
$$

Hence, gravity and gauge Chern-Simons theories are unified in 3-D space-time.

Chapter 4

Mimetic Gravity

According to the standard model of cosmology, the universe contains 5% baryonic matter and energy, 27% dark matter and 68% an unknown form of energy called dark energy. Both dark matter and dark energy have not been observed directly. Their presence have been implied from a variety of astrophysical and cosmological observations respectively.
Several cosmological and astrophysical observations push us to admit the presence of dark matter. First, to keep the astrophysical objects(galaxies, planets. . .) bound together, the gravitational pull on a certain object must balance it's average kinetic energy (virial theorem). In 1933, Fritz Zwicky discovered that the mass inferred from the luminous and visible matter in the galaxy is not sufficient to keep the cluster bound. He then deduced that there should be an extra hidden mass, named dark matter, responsible; with the visible matter, of keeping the cluster bound together. Second, within the galaxy itself; it was expected that the speed of the star should decrease as it becomes far from the center of the galaxy. However, Vera Rubin and Kent Ford discovered, after observing the Andromeda galaxy, that the velocity of the star remains constant regardless of it's relative position with respect to the center. Third, observing the patterns of cosmic mi-

crowave background (CMB) is a good evidence for the presence of dark matter. Forth, the merging of the content of 2 galaxy clusters gives the so-called bullet cluster. The determination of the mass of this resultant cluster, based on visible matter, gives contradiction with the experimental data. The last evidence for the presence of dark matter comes from the patterns of the large scale structure which couldn't have happened without the presence of dark matter [Arun et al., 2017]. Referring to primordial nucleosynthesis, the possibility for dark matter to be baryonic is ruled out. This means that either it is not discovered yet or it is a non-standard particle. There are several candidates for dark matter that are ruled out by astrophysical observations. One of the possible candidates for dark matter is the weakly interacting massive particles. Both superymmetry and extra dimension theories can predict such particles. Moreover, some suggests that dark matter is due to primordial black holes with specific astrophysical parameters. Some exotic candidates have been suggested also like: gravitinos, gluinos, Q-balls.... The failure of detecting such particles experimentally pushes physicists to modify previous gravity theories, Newtonian gravity and general relativity, to explain the above cosmological and astrophysical observations without the need of dark matter. MOND, modification of Newtonian mechanics, was initially built to explain the flat rotation curves of spiral galaxies. Modifying General relativity have been a great step not only in predicting dark energy and dark matter but also in explaining some unsolved cosmological phenomena by GR. The Lagrangian for GR is proportional to the Ricci scalar R. One way to modify it is by replacing R by a function of it like $f(R)$ or $f(R,T)$; where T is the trace of the energy momentum tensor $T_{\mu\nu}$. Although such theories have succeeded in explaining the dark side of the universe, they are highly non linear leading to some problems like the emergence of negative energy particles named ghosts.
In general modifying GR theory can occur in three different ways:

- Adding new degrees of freedom other than that of the metric tensor. These degrees of freedom can correspond to a scalar field like the quintessence [Zlatev et al., 1999], vector field like the case of Einstein-aether theory [Eling et al., 2006] or a tensor field as the case of Eddington-Born-Infeld gravity [Born and Infeld, 1934].

- Working with extra dimensions like the Kaluza Klein theory which is a modified GR theory in 5 dimensional space time [Kaluza, 1921a].

- Using higher order terms in Einstein-Hilbert action. Einstein's theory is a second order non-linear theory. One way of modification is to consider higher order derivative terms for the metric. A well known example is the f(R) gravity. Other theories that belongs to the same group are: $f(\Box R), f(R,T)$... [Houndjo et al., 2017].

- Breaking the local invariance. This can be obtained by adding terms containing the inverse of differential operators of curvature invariants like $f(R/\Box)$... as in the case of non local gravity [Vernov, 2012].

In general, a modified theory of gravity is considered successful if it's results are consistent with some cosmological observations [Clifton et al., 2012]. Predicting the correct value for the growth rate of the large-scale structure with cosmic time can be a good test for most alternatives for GR [Guzzo et al., 2008]. In addition, a class of modified $f(R)$ gravity theories give a great description of inflation [Cognola et al., 2008]. In the context of mimetic gravity theory, the late time integrated Sachs Wolfe effect; that predicts the inhomogenity in the cosmic microwave background radiations CMB; results from the addition of extra constrained inhomogenous scalar fields [Chamseddine and Mukhanov, 2016a].

4.1 Mimetic Gravity

Mimetic gravity, a modified theory of gravity by Chamseddine and Mukhanov [Chamseddine and Mukhanov, 2013], is somehow a recent theory that was originally found to deal with dark matter. Unlike other modified theories that add extra degrees of freedom or propagate ghosts, mimetic gravity is a ghost free theory that generates dark matter by adding a constrained scalar field. The idea is to isolate the conformal degree of freedom of the metric by writing the physical metric $g_{\mu\nu}$ in terms of an auxiliary metric $\tilde{g}_{\mu\nu}$ and a scalar field ϕ.

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \tag{4.1}$$

Eq 4.1 can be considered a kind of conformal transformation of the physical metric.
Using the corresponding equations of motion, it can be easily shown that the scalar field ϕ satisfies the below constraint

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1 \tag{4.2}$$

Mimetic gravity can be formulated in a different way [Golovnev, 2014]. Eq 4.2 can be imposed as an extra constraint, on the scalar field ϕ, using a lagrange multiplier λ.

$$S = \int d^4x\sqrt{-g}[-\frac{1}{2}R(g_{\mu\nu}) + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1)] \tag{4.3}$$

The resulting equation of motion, with respect to the metric, has the following form

$$G_{\mu\nu} = T_{\mu\nu} + \tilde{T}_{\mu\nu} \tag{4.4}$$

where $T_{\mu\nu}$ is the matter energy momentum tensor and

$$\tilde{T}_{\mu\nu} = (G - T) g_{\mu\alpha} g_{\nu\beta} \partial^{\alpha}\phi \partial^{\beta}\phi \tag{4.5}$$

The other equation of motion with respect to the field ϕ is

$$\frac{1}{\sqrt{-g}} \partial_k (\sqrt{-g}(G - T) g^{k\lambda} \partial_\lambda \phi - \nabla_k ((G - T)\partial^k \phi) = 0 \tag{4.6}$$

The energy momentum tensor of a perfect fluid with energy density ρ, pressure p and four-velocity vector u_μ is

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \tag{4.7}$$

with the four velocity vector satisfying the below condition

$$u_\mu u^\mu = 1 \tag{4.8}$$

Comparing eq 4.5 with the above energy momentum tensor, we deduce that it corresponds to a perfect fluid with zero pressure and energy density $\rho = -(G - T)$. The 4-velocity vector u_μ of this fluid is the gradient of the scalar field $\partial_\mu \phi$. Eq 4.8 is satisfied due to the mimetic constraint eq 4.2. To assure that eq 4.5 corresponds to dark matter, it is better to work in the synchronous gauge

$$ds^2 = dt^2 - \gamma_{ij}(x^i) dx^i dx^j \tag{4.9}$$

and choose $\phi(x^\mu) = \tau$.
Such a choice means that it is possible to choose the hypersurfaces with constant time to be the same as the hypersurfaces with constant ϕ. We choose to work with a specific metric like the Friedmann-Lemaitre-Robertson-Walker (FLRW) one given by

$$ds^2 = -dt^2 + a(t)^2 dx^2 \tag{4.10}$$

where $a(t)$ is the scale factor. Under this choice, eq 4.6 gives

$$\rho = G - T = \frac{C(x^i)}{\sqrt{\gamma}} = \frac{C}{a^3} \tag{4.11}$$

Thus, we get dark matter without the initial presence of dark matter.

4.2 Extensions of Mimetic Gravity Theories

In a subsequent work [Chamseddine et al., 2014], Chamseddine and Mukhanove extended the mimetic gravity theory to a new theory that can predict further cosmological solutions. This can be done by making the mimetic field dynamical through the addition of an extra term $V(\phi)$, called the potential of ϕ, to the action eq 4.3. Hence, the extended action becomes

$$S = \int d^4x\sqrt{-g}[-\frac{1}{2}R(g_{\mu\nu}) + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) - V(\phi)] \tag{4.12}$$

A specific choice of $V(\phi)$ leads to a specific cosmological solution. For example

- $V(\phi) = \alpha\phi^n = \alpha t^n$.

  α is a constant, where it's sign determine the cosmological state of the universe. For example; for negative α, the mimetic matter leads to an oscillating universe with singularity. For positive α, we get an accelerated and inflationary universe.

- $V(\phi) = \frac{\alpha\phi^2}{exp(\phi)+1}$.

  Using this potential, the scale factor becomes $a \propto exp(-\sqrt{\frac{\alpha}{12}}t^2)$, which gives inflaton.

- $V(\phi) = \frac{4}{3}\frac{1}{(1+\phi^2)^2} = \frac{4}{3}\frac{1}{(1+t^2)^2}$.

  Such a choice may lead, under a certain condition, to a bouncing universe.

Mimetic gravity may lead to dark energy using a similar formulation [Chamseddine and Mukhanov, 2016a]. This is achieved by adding extra non-dynamical constrained scalar fields ϕ^a

$$S = \int d^4x\sqrt{-g}[-\frac{1}{2}R(g_{\mu\nu}) + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) + \lambda_a g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi - V(\phi^a)] \tag{4.13}$$

In the synchronous gauge, the constraints on ϕ and ϕ^a gives respectively

$$\begin{aligned} \phi &= t \\ \phi^a &= \phi^a(x^i) \end{aligned} \tag{4.14}$$

Two cases can be considered here.

- $V(\phi^a) = 0$: Under this choice, no additional degrees of freedom appear. The extra fields ϕ^a allows us to distinguish mimetic dark matter from usual dust in the linearized approximation. These extra fields give us information about the vector perturbation of the mimetic dust.

- if $V(\phi^a) \neq 0$: This choice gives inhomogeneous dark energy if V and ϕ^a are inhomogeneous. In addition, This inhomogeneous dark energy can contribute to the late time integrated Sachs-Wolfe effect and can influence the structure formation of the universe,

The success of the mimetic gravity theory doesn't end here. Explaining singularities was one of the weaknesses that Einstein's theory has faced. It is well known that the energy density becomes infinite at the center of the black hole and during the Big Bang. GR predicted the presence of space like singularities but it was unable to explain it. General relativity alone can't deal with singularities. The best way to explain them is through a quantum gravity theory that

combines both general relativity with quantum mechanics. Such theories usually suffer from problems related to ghost as we are going to see in chapter 6 . In the context of mimetic gravity, Chamseddine and Mukhanov resolved singularities classically. The idea is to add to the Einstein's action a function $f(\chi) = f(\Box\phi)$ instead of the potential term $V(\phi)$ [Chamseddine and Mukhanov, 2017b].
The new variable χ is given by

$$\chi = \Box\phi = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{-g}g^{\mu\nu}\frac{\partial\phi}{\partial x^{\nu}}) \tag{4.15}$$

which becomes in the synchronous gauge

$$\chi = \frac{\dot{\gamma}}{2\gamma} \tag{4.16}$$

A good choice for $f(\chi)$, to avoid singularities, is the Born-Infeld type functions which are bounded by a limiting value χ_m

$$f(\chi) = \chi_m^2\left(1 + \frac{1}{3}\frac{\chi^2}{\chi_m^2} - \sqrt{\frac{2}{3}}\frac{\chi}{\chi_m}arcsin\left(\sqrt{\frac{2}{3}}\frac{\chi}{\chi_m}\right) - \sqrt{1 - \frac{2}{3}\frac{\chi^2}{\chi_m^2}}\right) \tag{4.17}$$

In the isotropic Friedmann universe

$$ds^2 = dt^2 - a^2(t)\delta_{ik}dx^i dx^j \tag{4.18}$$

the resulting equation of motion for this theory becomes

$$3\left(\frac{\dot{a}}{a}\right)^2 = \frac{\epsilon_m}{a^3}\left(1-\frac{1}{a^3}\right) \tag{4.19}$$

which gives, after integration, the following solution

$$a = \left(1+\frac{3}{4}\epsilon_m t^2\right)^{1/3} \tag{4.20}$$

It is clear from eq 4.20 that it describes a bouncing Friedmann universe. For $t < -1/\sqrt{\epsilon_m}$, it passes by a contracting phase dominated by cold dark matter ($a \propto t^{2/3}$). Then, it enters a regular bounce stage, for $-1/\sqrt{\epsilon_m} < t < 1/\sqrt{\epsilon_m}$. For $t > 1/\sqrt{\epsilon_m}$, the universe starts to expand, dominated by normal dusts.
The metric describing a Schwarzchild black hole is given by

$$ds^2 = \left(1-\frac{r_g}{r}\right)dt_S^2 - \frac{dr^2}{(1-\frac{r_g}{r})} - r^2 d\Omega^2 \tag{4.21}$$

r_g is the gravitational radius of the black hole and $d\Omega = d\theta^2 + sin^2\theta d\varphi^2$. The only physical singularity is that at the center of the black hole $r = 0$. $r = r_g$ seems, for the first look, to be a singularity but it is just a coordinate singularity that can disappear by a specific choice of a synchronous coordinate system. Using the idea of limiting curvature, the spacetime of the black hole of radius r_g is geodesically complete. In case of falling in a black hole, an observer will enter a region of limiting curvature and will remain there for a short time. After that, he will find himself inside a Schwarzchild black hole of smaller gravitational radius $r_g^{1/3}$. Again, he will stay in a region of limiting curvature for a short time and then he will find himself near the horizon of a new Schwarzchild black hole of smaller radius $r_g^{1/9}$. The cycle will continue until he remains in a region of limiting curvature forever. This is called the "nesting dull" [Chamseddine and Mukhanov, 2017a]. Mimetic $F(R)$ gravity theory was formulated by Nojiri and Odintsov [Nojiri and Odintsov, 2014]. It's action is given by

$$I = \int d^4x\sqrt{-g}[F(R(g_{\mu\nu})) - V(\phi) + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 1)] \tag{4.22}$$

Specific choices of $F(R)$ can give specific cosmological solutions like inflation.

4.3 Massive Gravity

Gravitational interaction is one of the four fundamental forces in nature in addition to electromagnetic, weak and strong interactions. According to quantum mechanics and gauge theory, the photon is the particle that is exchanged during any electromagnetic interaction between any two charged particles. This virtual particle is massless because the electromagnetic force has a long range. Similarly, the graviton particle is a spin two virtual particle that is expected to propagate during any gravitational interaction [Hinterbichler, 2012]. Like the photon, it is expected to be massless. GR is a classical theory that can be applied in regions where quantum effects can be ignored like the solar system. As we approach quantum regions, GR can no more be applied and we need a quantum gravity theory that mixes both general relativity and quantum mechanics. Quantizing gravity suffers from several problems as we are going to see in chapter 6 because GR is not UV complete. Massive gravity is a modified GR theory that can help in constructing a well defined quantum theory of gravity. In addition, introducing a mass term to the action of classical GR can solve the mystery of the accelerated expansion of the universe without the need of dark energy as in the DGP model, massive gravity in extra dimensions [Deffayet, 2001]. In addition, massive gravity leads to dark matter and gravitational waves [Aoki and Mukohyama, 2016].

A successful massive gravity theory is the one that recovers the classical GR theory as m (mass of graviton) tends to zero. It is well known that introducing a mass term to the theory will increase the number of degrees of freedom. The first attempt to massive gravity was by Fierz and Pauli (1939) [Fierz and Pauli, 1939b]. They constructed the first linear massive theory of gravity where the mass term is given by

$$L_{FP} = -\frac{1}{2}m^2\left(h^{\mu\nu}h_{\mu\nu} - h^2\right) \tag{4.23}$$

where $h_{\mu\nu}$ is a second rank symmetric tensor that appears in the perturbation of the metric around the flat metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

In 1970, Van Dam, Veltman and Zakharov discovered that this linear massive theory suffers from vDVZ discontinuity. This means that massless limit ($m \to 0$) of this theory makes prediction different from the linear GR theory. The reason behind this discontinuity is that in the massless limit a massive graviton, with five degrees of freedom, is a massless graviton coupled to a scalar which is responsible for the vDVZ discontinuity [Veltman and Dam, 1970], [Zakharov, 1970].

The failure of the linear theory pushes Vainstein to formulate a massive non linear theory of gravity using the Vainstein mechanism [Vainshtein, 1972]. The idea is that around any massive object M like the sun we can introduce the Vainstein radius $r_V \sim \left(\frac{M}{m^4 M_P^2}\right)^{1/5}$. At $r < r_V$ non linear effects start to dominate and the

linear theory can no more be applied. At the same year, Boulware and Deser discovered that most higher order massive gravity theories suffer from ghosts [Boulware and Deser, 1972]. Instead of having five degrees of freedom, as in the linear theory, the non linear one possess 6 degrees of freedom, where the extra one represents a ghost with negative kinetic energy. Classically, the presence of ghosts lead to an unbounded Hamiltonian that leads to instabilities. At the quantum level, ghosts must be avoided to achieve unitarity.
Ghosts not only arise for non linear theories but also for linear theories with mass term deviating from eq 4.23. This means ghosts appear if $a \neq 0$ in the following

$$-\frac{1}{2}m^2\left(h_{\mu\nu}h^{\mu\nu}-(1-a)\,h^2\right) \tag{4.24}$$

In abelian and non abelian gauge theories, gauge bosons gain mass through the spontaneous local symmetry breaking. Similarly, graviton can attain mass by spontaneous symmetry breaking SSB of the Lorentz invariance. Few years later, new Higgs mechanism was introduced by t'Hooft where he used four scalar fields to break diffeomorphism invariance through their vacuum expectation value [Hooft, 2007]. After this SSB, we get a massive graviton with five degrees of freedom and a non-unitary propagating scalar field. The problem of unitarity pushes physicists to add higher order derivative terms of the scalar to the Einstein Hilbert action and to tune appropriately the negative cosmological constant [Kakushadze, 2008]. Later on, Chamseddine and Mukhanov [Chamseddine and Mukhanov, 2010b] proposed a new model for massive gravity via Higgs mechanism. The resulting model hasn't suffered from any unitarity problem [Oda, 2010] and hasn't put any constraint on the cosmological constant. The idea is to use four scalar fields ϕ^A, where $A = 0, 1, 2, 3$ to construct the field space tensor

$$H^{AB} = g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B \tag{4.25}$$

which will be used to add higher order derivative terms to Einstein Hilbert action. The resulting theory is ghost free preserving Lorentz and diffeomorphism invariance and doesn't suffer from any unitarity problem . In addition, it restores the Fierz-Pauli theory in the broken phase. Moreover, It doesn't put any constraint on the cosmological constant. A problem arises in case of coupling this theory to non-trivial backgrounds, such as a time dependent background or some used in cosmology, where the ghost state got excited. The dRGT construction manages to decouple the ghost completely except in non-trivial backgrounds. The Higgs mechanism for graviton in 4D space-time is generalized for an arbitrary D-dimensional space-time [Oda, 2010].

4.4 Ghost Free Mimetic Massive Gravity

In a subsequent work, Chamseddine and Mukhanov constructed a new well-behaved ghost free massive gravity theory using the Brout-Englert-Higgs mechanism using four scalar fields $\phi^A, A = 0, 1, 2, 3$ [Chamseddine and Mukhanov, 2018a]. One of the four scalar fields will be the mimetic scalar field ϕ^0, obeying the mimetic constraint eq 4.2

.

$$g^{\mu\nu}\partial_\mu\phi^0\partial_\nu\phi^0 - 1 = 0 \tag{4.26}$$

The four scalars ϕ^A, s acquire vacuum expectation value in the broken symmetry phase, in Minkowski space-time

$$< \phi^A > \equiv x^A \tag{4.27}$$

Using eq 4.25, we can build the diffeomorphism invariant set of scalars

$$\bar{h}^{AB} = H^{AB} - \eta^{AB} \tag{4.28}$$

Latin indices $A, B...$ are raised and lowered by the Minkownsi metric $\eta^{AB} = (1, -1, -1, -1)$ and Greek indices $\mu, \nu, \alpha...$ are raised and lowered using the metric $g^{\mu\nu}$.

Using $\bar{h}^{AB}$, we can construct the mass term for graviton. The mass term will be different from that of Fierz-Pauli one. The ghost free mimetic massive gravity theory action becomes

$$S = \int d^4x\sqrt{g}\left[-\frac{1}{2}R + \frac{m^2}{8}\left(\frac{1}{2}\bar{h}^2 - \bar{h}^{AB}\bar{h}_{AB}\right) + \lambda\left(g^{\mu\nu}\partial_\mu\phi^0\partial_\nu\phi^0 - 1\right)\right] \tag{4.29}$$

where $\bar{h} = \bar{h}^A_A = \eta_{AB}\bar{h}^{AB}$. The equation of motion with respect to $g^{\mu\nu}$ is

$$\begin{aligned} G_{\mu\nu} = & -\frac{m^2}{8}\left(\frac{1}{2}\bar{h}^2 - \bar{h}^{AB}\bar{h}_{AB}\right)g_{\mu\nu} + \lambda\left(2\partial_\mu\phi^0\partial_\nu\phi^0\right) \\ & + \frac{m^2}{2}\left(\frac{1}{2}\bar{h}\partial_\mu\phi_A\partial_\nu\phi^A - \bar{h}_{AB}\partial_\mu\phi^A\partial_\nu\phi^B\right) \end{aligned} \tag{4.30}$$

and that of ϕ^A

$$\nabla^\mu \left(m^2 \left(\frac{1}{2}\bar{h}\partial_\mu \phi_A - \bar{h}_{AB}\partial_\mu \phi^B \right) + 4\delta_{0A}\partial_\mu \phi^0 \right) = 0 \tag{4.31}$$

The mimetic constraint eq 4.26 can be written as $\bar{h}^{00} = 0$ and is obtained by varying the action with respect to λ.

To study the degrees of freedom of the massive graviton and to prove that this theory is ghost free, linear perturbation around the Minkowski background is performed

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \\ \phi^A &= x^A + \chi^A \end{aligned} \tag{4.32}$$

To linear order in $h_{\mu\nu}$ and χ, Einstein tensor $G_{\mu\nu}$ becomes

$$\begin{aligned} G_{\mu\nu}(h_\rho) = & -\frac{1}{2}\left(\partial^2 h_{\mu\nu} - \partial_\mu \partial^\rho h_{\rho\nu} - \partial_\nu \partial^\rho h_{\rho\mu} + \partial_\mu \partial_\nu h\right) \\ & +\frac{1}{2}\eta_{\mu\nu}\left(\partial^2 h - \partial^\sigma \partial^\rho h_{\rho\sigma}\right) \end{aligned} \tag{4.33}$$

where $\partial^2 \equiv \partial^\mu \partial_\mu$ and $h \equiv \eta^{\mu\nu} h_{\mu\nu}$.

The linearized $\bar{h}^{AB}$ becomes

$$\bar{h}^{AB} = \delta^A_\mu \delta^B_\nu h^{\mu\nu} + \partial^A \chi^B + \partial^B \chi^A \tag{4.34}$$

Then the $i-j$ tensor equation of eq 4.30 becomes

$$\partial^2\bar{h}_{ij}-\eta_{ij}\left(\frac{1}{2}\partial^2\bar{h}-\frac{4\ddot{h}}{m^2}\right)=-m^2\left(\bar{h}_{ij}-\frac{1}{2}\eta_{ij}\bar{h}\right) \tag{4.35}$$

which gives the following wave equation for a massive field $\bar{h}^T_{ij}$

$$\left(\Box+m^2\right)\bar{h}^T_{ij}=0 \tag{4.36}$$

It is obvious that eq 4.36 describes a massive graviton with five degrees of freedom and characterized by the traceless tensor

$$\bar{h}^T_{ih}\equiv\bar{h}_{ij}-\frac{1}{3}\eta_{ij}\bar{h} \tag{4.37}$$

To assure the absence of ghosts we need to examine the other linearized vector and scalar equation of eq 4.30. The Linearized scalar $0-0$ equation of eq 4.30 is

$$\triangle\bar{h}+\partial^i\partial^j\bar{h}_{ij}=4\lambda+\frac{m^2}{2}\bar{h} \tag{4.38}$$

where $\triangle=-\partial^i\partial_i$. This allows us to write $\bar{h}$ in terms of $\bar{h}^T_{ij}$

$$\bar{h}=6\left(\frac{\partial^i\partial^j\bar{h}^T_{ij}-4\lambda}{3m^2-4\triangle}\right) \tag{4.39}$$

Equation of motion of λ is obtained by combining the scalar equation with the trace of the tensorial one eq 4.35

$$\ddot{\lambda} + \frac{m^2}{4}\lambda = 0 \tag{4.40}$$

The $0-i$ equation doesn't give any new degree of freedom. It help us in expressing $\bar{h}_{0i}$ in terms of $\bar{h}^T_{ij}$ and λ. As a result, this theory is ghost free not only at linear level but to all higher order. Unlike the Fierz-Pauli massive theory, vDVZ discontinuity is abscent in this model. The true physical degrees of freedom of this theory is shown in another way using the cosmological perturbation theory [Chamseddine and Mukhanov, 2018b]. The cosmological implications of mimetic massive gravity is studied in [Solomon et al., 2019]. The effects of the extra mass term on Friedmann-Lemaıtre-Robertson-Walker cosmological backgrounds are to introduce effective radiation, curvature, and cosmological constant terms....

Chapter 5

Hamiltonian formulation of Ghost Free Mimetic Massive Gravity Theory

Hamiltonian formalism forms an alternative to the Lagrangian one. It plays an important role in counting physical degrees of freedom for a certain system. In modern physics, it forms the basis of the canonical quantization of any dynamical system especially those that possess constraints and symmetries, where the canonical variables q_i and p_i are replaced by linear operators $\hat{q}_i$ and $\hat{p}_i$ acting on Hilbert space of states. Canonical formulation forms the basis for some quantum gravity theories like the loop quantum gravity theory as we are going to discuss in chapter 6. Hamiltonian analysis and canonical quantization was first formulated by Dirac [Dirac, 1950],[Dirac, 1951],[Dirac, 1958] and then followed by Bergmann and collaborators [Anderson and Bergmann, 1951],[Bergmann and Goldberg, 1955], [Bergmann, 1956]. Since then, Hamiltonian formulation has been considered the best setup for constructing gauge theories. Later Arnowitt, Deser and Misner constructed the canonical formulation of gravity (ADM formalism) in 1962

[Arnowitt et al., 1959],[Arnowitt et al., 1960a],[Arnowitt et al., 1960b], [Arnowitt et al., 2008].

5.1 Hamiltonian Formalism

In Hamiltonian formalism, a dynamical system is described by set of 2n canonical coordinates that contains: n generalized coordinates q^i and n generalized momentum p^i. i ranges from 1 to n, where n is the number of degrees of freedom of the dynamical system. These canonical coordinates form a 2n dimensional phase space. Instead of being described by the scalar Lagrangian, a system is described by the scalar Hamiltonian H that is the sum of both the kinetic energy and the potential energy. Starting from the action

$$S = \int L(q^i, \dot{q}^i) dt \tag{5.1}$$

where $L(q^i, \dot{q}^i)$ is the total Lagrangian of the system, we can define the conjugate momentum p^i as

$$p^i(q_i, \dot{q}_i) = \frac{\partial L}{\partial \dot{q}^i} \tag{5.2}$$

Then the total Hamiltonian will be

$$H = \sum_i^n p_i \dot{q}^i - L \tag{5.3}$$

Each degree of freedom, described by the couple (q_i, p_i), will have two equations

$$
\begin{aligned}
\dot{q}_i &= \frac{\partial H}{\partial p_i} \\
\dot{p}_i &= -\frac{\partial H}{\partial q_i}
\end{aligned}
\tag{5.4}
$$

Thus, we get a set of 2n Hamiltonian equations of motion describing the dynamical system. Eq's 5.4 can be written in terms of Poisson bracket

$$
\begin{aligned}
\dot{q}^i &= \{q^i, H\} \\
\dot{p}^i &= \{p^i, H\}
\end{aligned}
\tag{5.5}
$$

where the Poisson bracket is defined as

$$
\{f(q,p), g(q,p)\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \right) \tag{5.6}
$$

It should be noted that the canonical coordinates satisfy the following identities

$$
\begin{aligned}
\{q_i, p_j\} &= \delta_{ij} \\
\{q_i, q_j\} &= 0 \\
\{p_i, p_j\} &= 0
\end{aligned}
\tag{5.7}
$$

In the case of field theory, the coordinates q and p are functions of a continuous parameter x. Similar strategy is followed but you should be carefull in deriving the equations of motion where the Kronecker delta δ_{ij} is replaced by the Dirac delta function $\delta(x-y)$. Hence, the sum in the Poisson bracket formula will be replaced by integral over x.

5.2 ADM Formalism

The ADM formalism of gravity, done by Arnowitt, Deser and Misner [Arnowitt et al., 2008], is based on the idea of separating time and space. To achieve this, we define the three dimensional hyperspace $\sum$, labeled with time t and spanned by the metric $h_{\mu\nu}$, to be embedded in the four dimensional manifold M, spanned by the metric $g_{\mu\nu}$. The two metrics are related to each other by

$$h_{\mu\nu} = g_{\mu\nu} - \epsilon n_\mu n_\nu \tag{5.8}$$

where n_μ is a unit vector normal to $\sum$ and $\epsilon = n_\mu n^\mu$ is it's norm. For a space-like hypersurface we have $\epsilon = -1$ while for time-like one we have $\epsilon = 1$. The proper interval between any two hypersurfaces $\sum_t(t, x^i)$ and $\sum_{t+dt}(t + dt, x^i + dx^i)$ is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{5.9}$$

where N and N_i are the lapse and the shift functions respectively. They represent the translation between the two hypersurfaces. The metric $g_{\mu\nu}$ is

$$\begin{bmatrix} N_i N^i - N^2 & N_k \\ N_k & h_{ik} \end{bmatrix} \tag{5.10}$$

and it's inverse $g^{\mu\nu}$ becomes

$$\begin{bmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^k}{N^2} & h^{ik} - \frac{N^i N^k}{N^2} \end{bmatrix} \tag{5.11}$$

The momenta corresponding to the lapse and shift functions are zero while that for $h_{ij}$is given by

$$\pi_{ij} = \frac{\partial}{\partial h^{ij}} \left(\sqrt{-g} L_{ADM}\right) = \sqrt{h} \left(h_{ij} K - K_{ij}\right) \tag{5.12}$$

where K_{ij} is the extrinsic curvature

$$K_{ij} = \frac{1}{2N} \left(N_{i|j} + N_{j|i} - \dot{h}_{ij}\right) \tag{5.13}$$

The $(3+1)$ GR action, written in the ADM formalism, becomes

$$\begin{aligned} S_{ADM} = \int d^4x \left(\pi_{ij} \dot{h}^{ij} - N^\mu H_\mu\right) = \\ \int d^4x \left(\pi_{ij} \dot{h}^{ij} - N R_0 - N^i R_i\right) \end{aligned} \tag{5.14}$$

where

$$R_0 = -\sqrt{h}\left[{}^3R + h^{-1}\left(\frac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}\right)\right]$$
$$R_i = -2h_{ik}\pi^{kj}_{|j} \tag{5.15}$$

The equation of motion with respect to $\pi_{ij}$is

$$\dot{h}_{ij} = 2Nh^{-1/2}\left(\pi_{ij} - \frac{1}{2}h_{ij}\pi\right) + N_{i|j} + N_{j|i} \tag{5.16}$$

Using the below identity

$$\frac{\delta h_{kl}(x)}{\delta h_{ij}(y)} = \frac{1}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right)\delta(x,y) \tag{5.17}$$

The equation of motion with respect to h_{ij} becomes

$$\begin{aligned}\dot{\pi}^{ij} = & -N\sqrt{h}\left({}^3R^{ij} - \frac{1}{2}h^{ij\,3}R\right) + \frac{1}{2}Nh^{-1}h^{ij}\left(\pi^{mn}\pi_{mn} - \frac{1}{2}\pi^2\right) \\ & -2Nh^{-1}\left(\pi^{im}\pi^j_m - \frac{1}{2}\pi\pi^{ij}\right) + \sqrt{h}\left(N^{|ij} - h^{ij}N^{|m}_{|m}\right) + \left(\pi^{ij}N^m\right)_{|m} - N^i_{|m}\pi^{mj} - N^j_{|m}\pi^{mi}\end{aligned} \tag{5.18}$$

The equations of motion corresponding to the lapse and shift functions gives the four constraints

$$H_{\mu} = 0 \tag{5.19}$$

The dynamical variables here are the symmetric h_{ij} and π^{ij} tensors. This gives us 12 independent components. Four of them are cancelled by the constraints eq 5.19 and another four are cancelled by gauge transformations. Thus, we are left with four degrees of freedom giving us the two physical degrees of freedom of the massless graviton.

5.3 Hamiltonian formulation of Mimetic Gravity

The canonical formulation of the extended mimetic gravity theory, with a potential term for ϕ, was performed by Ola Malaeb [Malaeb, 2015]. Using the $(+,-,-,-)$ signature for the metric, the $(3+1)$ Hamiltonian corresponding to the ϕ part of the Lagrangian eq 4.3 is

$$H_{\phi} = \frac{Np^2}{2\sqrt{h}\lambda} + \frac{1}{2}N\sqrt{h}\lambda[1 + h^{ij}\partial_i\phi\partial_j\phi] + pN^i\partial_i\phi + N\sqrt{h}V(\phi) \tag{5.20}$$

where p is the momentum corresponding to ϕ

$$p = N\sqrt{h}\lambda(g^{00}\partial_0\phi + g^{0i}\partial_i\phi) \tag{5.21}$$

λ is a Lagrange multiplier that gives a primary constraint $p_\lambda = 0$ and a corresponding secondary constraint

$$\{p_\lambda, H_\phi\} = 0 \tag{5.22}$$

Using this secondary constraint we can exclude λ from the Hamiltonian 5.20. Hence, the total action in the $(3+1)$ formalism can be written as

$$S = S_g + S_\phi = \int d^4x \left(\pi_{ij}\dot{h}^{ij} + \dot{p} - N\left(R^0 + p\sqrt{h^{ij}\partial_i\phi\partial_j\phi + 1}\right)\right.$$
$$\left. -N^i\left(R_i + p\partial_i\phi\right) - N\sqrt{h}V(\phi)\right) \tag{5.23}$$

To check that this theory doesn't introduce any new degree of freedom, it is important to check the equations of motion.

The equation of motion with respect to π_{ij} is the same as eq 5.16 since the ϕ part of the Hamiltonian is independent of π_{ij}. The equation of motion with respect to h^{ij} becomes

$$\begin{aligned}\dot{\pi}_{ij} = & -N\sqrt{h}\left({}^3R_{ij} - \frac{1}{2}h^3_{ij}R\right) + \frac{1}{2}Nh^{-1}h_{ij}\left(\pi^{mn}\pi_{mn} - \frac{1}{2}\pi^2\right) \\ & -2Nh^{-1}\left(\pi_{im}\pi^m_j - \frac{1}{2}\pi\pi_{ij}\right) + \sqrt{h}\left(N_{|ij} - h_{ij}N^{|m}_{|m}\right) + \left(\pi_{ij}N^m\right)_{|m} \\ & -N^{|m}_i\pi_{mj} - N^{|m}_j\pi_{mi} + \frac{Np\partial_i\phi\partial_j\phi}{2\sqrt{h^{kl}\partial_k\phi\partial_l\phi + 1}} - \frac{1}{2}N\sqrt{h}V(\phi)h_{ij}\end{aligned} \tag{5.24}$$

The equations of motion resulting from the variation with respect to N and N^i gives the four constraints $H_\mu = 0$ similar to eq 5.19.

The equation of motion with respect to p

$$\dot{\phi} - N\sqrt{h^{ij}\partial_i\phi\partial_j\phi + 1} - N^i\partial_i\phi = 0 \tag{5.25}$$

gives the mimetic constraint eq 4.2. The equation of motion with respect to ϕ is

$$\dot{p} - \partial_k\left(\frac{Np\partial^k\phi}{\sqrt{h^{ij}\partial_i\phi\partial_j\phi + 1}} + N^k p\right) + N\sqrt{h}\frac{dV(\phi)}{d\phi} = 0 \tag{5.26}$$

which is just the conservation of the energy momentum tensorT^μ_ν (Bianchi identity). Hence, both Einstein's theory and the mimetic one shares the same number of equations and so the same number of degrees of freedom.

5.4 Hamitlonian Formulation of Ghost Free Mimetic Massive Gravity

5.4.1 Canonical Form

As mentioned in section 4.4, the ghost free mimetic massive gravity, developed by Chamseddine and Mukhanov, is a recent theory that predicts a mass term for graviton, different from that of Fierz Pauli's one, without generating ghosts. The action representing this theory is eq 4.29

$$
\begin{aligned}
S &= \int d^4 x L \\
&= \int d^4 x \sqrt{g} \left[-\frac{1}{2} R + \frac{m^2}{8} \left(\frac{1}{2} \bar{h}^2 - \bar{h}^{AB} \bar{h}_{AB} \right) + \lambda \left(g^{\mu\nu} \partial_\mu \phi^0 \partial_\nu \phi^0 - 1 \right) \right]
\end{aligned}
\tag{5.27}
$$

The induced metric perturbation $\bar{h}^{AB}$ is given by

$$
\bar{h}^{AB} = g^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^B - \eta^{AB} \tag{5.28}
$$

The Minkowski metric η^{AB} is chosen as $(+, -, -, -)$.

The action eq 5.27 contains higher order terms. Our goal is to formulate the Hamiltonian analysis of the linearized version of this theory. To achieve this, we consider the ϕ terms of the above action, S_ϕ, and make a small perturbation of the fields around a broken symmetry phase [Malaeb and Saghir, 2019]

$$
\phi^A = x^A + \chi^A \tag{5.29}
$$

The $(3+1)$ S_ϕ action becomes

$$
\begin{aligned}
S_\phi = & \int d^4x\sqrt{g}\frac{m^2}{8}g^{00}\left(-2-4\partial_0\chi^0-2\partial_0\chi^0\partial_0\chi^0-2\partial_0\chi^i\partial_0\chi^k\eta_{ik}\right)\\
& +\sqrt{g}\frac{m^2}{8}g^{0k}\left(-4\partial_k\chi^0-4\partial_k\chi^0\partial_0\chi^0-4\eta_{mk}\partial_0\chi^m-4\partial_0\chi^m\partial_k\chi^n\eta_{mn}\right)\\
& \sqrt{g}\frac{m^2}{8}+g^{ik}\left(-2\eta_{ik}-4\eta_{mk}\partial_i\chi^m-2\partial_i\chi^0\partial_k\chi^0-2\eta_{mn}\partial_i\chi^m\partial_k\chi^n\right)\\
& +\sqrt{g}\frac{m^2}{8}g^{00}g^{00}\left(\frac{-1}{2}-2\partial_0\chi^0-3\partial_0\chi^0\partial_0\chi^0-\eta_{rs}\partial_0\chi^r\partial_0\chi^s\right)\\
& +\sqrt{g}\frac{m^2}{8}g^{00}g^{0k}\left(-2\partial_k\chi^0-6\partial_k\chi^0\partial_0\chi^0-2\eta_{rk}\partial_0\chi^r-4\eta_{rk}\partial_0\chi^r\partial_0\chi^0-2\eta_{ir}\partial_0\chi^i\partial_k\chi^r\right)\\
& +\sqrt{g}\frac{m^2}{8}g^{00}g^{ik}\left(\eta_{ik}-\partial_i\chi^0\partial_k\chi^0+2\partial_0\chi^0\eta_{ik}+2\partial_i\chi^r\eta_{rk}+4\partial_0\chi^0\partial_i\chi^r\eta_{rk}+\partial_i\chi^r\partial_k\chi^s\eta_{rs}\right.\\
& \left.+\partial_0\chi^0\partial_0\chi^0\eta_{ik}+\eta_{mn}\eta_{ik}\partial_0\chi^m\partial_0\chi^n-2\eta_{mi}\eta_{nk}\partial_0\chi^m\partial_0\chi^n-4\eta_{rk}\partial_0\chi^r\partial_i\chi^0\right)\\
& +\sqrt{g}\frac{m^2}{8}g^{0i}g^{0k}\left(-2\eta_{ik}-2\partial_k\chi^0\partial_i\chi^0-4\eta_{ik}\partial_0\chi^0-4\eta_{ri}\partial_k\chi^r-2\eta_{mr}\eta_{ki}\partial_0\chi^m\partial_0\chi^r\right.\\
& \left.-2\eta_{ki}\partial_0\chi^0\partial_0\chi^0-8\eta_{ri}\partial_0\chi^0\partial_k\chi^r-2\eta_{rs}\partial_i\chi^s\partial_k\chi^r\right)\\
& +\sqrt{g}\frac{m^2}{8}g^{0k}g^{lj}\left(2\partial_k\chi^0\eta_{jl}-4\eta_{kl}\partial_j\chi^0+2\eta_{mk}\eta_{lj}\partial_0\chi^m-4\partial_0\chi^m\eta_{ml}\eta_{kj}+2\partial_k\chi^0\partial_0\chi^0\eta_{lj}\right.\\
& -4\eta_{kl}\partial_0\chi^0\partial_j\chi^0+4\partial_k\chi^0\partial_j\chi^r\eta_{rl}-4\eta_{rl}\partial_j\chi^0\partial_k\chi^r-4\eta_{ks}\partial_j\chi^0\partial_l\chi^s+2\eta_{mn}\eta_{lj}\partial_0\chi^m\partial_k\chi^n\\
& \left.-4\partial_0\chi^m\partial_l\chi^r\eta_{mr}\eta_{kj}-4\partial_0\chi^m\partial_k\chi^n\eta_{ml}\eta_{nj}-4\partial_0\chi^m\partial_j\chi^s\eta_{ml}\eta_{ks}+4\eta_{mk}\eta_{ls}\partial_0\chi^m\partial_j\chi^s\right)\\
& +\sqrt{g}\frac{m^2}{8}g^{ik}g^{lj}\left(\frac{1}{2}\eta_{ik}\eta_{lj}-\eta_{il}\eta_{kj}+2\eta_{lj}\eta_{mk}\partial_i\chi^m-4\partial_i\chi^m\eta_{ml}\eta_{kj}+\partial_l\chi^0\partial_j\chi^0\eta_{ik}\right.\\
& -2\eta_{kj}\partial_i\chi^0\partial_l\chi^0+\eta_{mn}\eta_{lj}\partial_i\chi^m\partial_k\chi^n-2\partial_i\chi^m\partial_l\chi^r\eta_{mr}\eta_{kj}-2\partial_i\chi^m\partial_k\chi^n\eta_{ml}\eta_{nj}\\
& \left.+2\eta_{mk}\eta_{rj}\partial_i\chi^m\partial_l\chi^r-2\eta_{ml}\right)\\
& +\lambda\left(g^{00}\left(1+2\partial_0\chi^0+\partial_0\chi^0\partial_0\chi^0\right)+2g^{0k}\partial_k\chi^0\left(1+\partial_0\chi^0\right)+g^{ik}\partial_i\chi^0\partial_k\chi^0-1\right)
\end{aligned}
\tag{5.30}
$$

The second step is to make small perturbation of the metric around the flat one

$$
g^{\mu\nu}=h^{\mu\nu}+\eta^{\mu\nu} \tag{5.31}
$$

Then, to first order in perturbation, $\bar{h}^{AB}$ becomes

$$\bar{h}_{\mu\nu} = -h_{\mu\nu} + \partial_\mu \chi_\nu + \partial_\nu \chi_\mu \tag{5.32}$$

The linearized theory possess a second order action in $h_{\mu\nu}$ and χ^A, knowing that λ is of first order in perturbation, S_ϕ becomes

$$\begin{aligned}
S_\phi = \int d^4x\sqrt{g}\Bigg(&\partial_0\chi^0\left(\frac{m^2}{8}\left(-2h^{00} - 2\partial_0\chi^0 + 4\partial_i\chi^i + 2h^{ik}\eta_{ik}\right) + 2\lambda\right)\\
&+\partial_0\chi^i\left(\frac{m^2}{8}\left(-4\partial_i\chi^0 - 2\partial_0\chi_i - 4h^{0j}\eta_{ij}\right)\right)\\
&+h^{0k}\left(\frac{m^2}{8}\left(-4\partial_k\chi^0 - 2h^{0i}\eta_{ik}\right)\right)\\
&+h^{00}\left(\frac{m^2}{8}\left(\frac{-1}{2}h^{00} + 2\partial_i\chi^i + \eta_{ik}h^{ik}\right) + \lambda\right)\\
&+h^{ik}\frac{m^2}{8}\left(2\eta_{ik}\partial^j\chi_j - 4\partial_k\chi_i\right) + h^{ik}h^{lj}\frac{m^2}{8}\left(\frac{1}{2}\eta_{ik}\eta_{lj} - \eta_{il}\eta_{kj}\right)\\
&+\frac{m^2}{8}\left(-2\partial_i\chi^0\partial^i\chi^0 + 2\partial_i\chi^i\partial_k\chi^k - 2\partial^k\chi^j\partial_j\chi_k - 2\partial^i\chi^j\partial_i\chi_j\right)\Bigg)
\end{aligned} \tag{5.33}$$

The second order linearized Einstein Hilbert action S_g

$$\begin{aligned}
S_g &= \int d^4x L_g\\
&= -\frac{1}{4}\int d^4x\left(\partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h^{\mu\sigma}\partial_\nu h^\nu_\sigma + \frac{1}{2}\partial_\sigma h^{\mu\nu}\partial^\sigma h_{\mu\nu} + \frac{1}{2}\partial_\mu h\partial^\mu h\right)
\end{aligned} \tag{5.34}$$

To construct the complete Hamiltonian of this theory, it is important to find the conjugate momenta of the variables. The conjugate momenta of χ^0 and χ^i are respectively

$$P = \frac{\partial L}{\partial \dot{\chi^0}} = \frac{m^2}{8}\sqrt{g}\left(-2h^{00} + 4\partial_i\chi^i + 2h^{ik}\eta_{ik}\right) + 2\sqrt{g}\lambda - 4\sqrt{g}\frac{m^2}{8}\dot{\chi^0} \tag{5.35}$$

$$P_i = \frac{\partial L}{\partial \dot{\chi^i}} = \frac{m^2}{8}\sqrt{g}\left(-4\partial_i\chi^0 - 4h^{0j}\eta_{ij}\right) - \frac{m^2}{2}\sqrt{g}\partial_0\chi_i \tag{5.36}$$

The conjugate momenta of h^{00}, h^{oi} and h^{mn} are respectively

$$\Pi = -\frac{1}{4}\partial_m h^{0m} \tag{5.37}$$

$$\Pi_l = \frac{1}{4}\partial_l h^{00} + \frac{1}{4}\partial_l\eta_{rs}h^{rs} - \frac{1}{2}\partial_i\eta_{lr}h^{ri} \tag{5.38}$$

$$\Pi_{mn} = \frac{1}{4}\partial_i h^{i0}\eta_{mn} - \frac{1}{4}\eta_{mn}\partial^0\eta_{rs}h^{rs} + \frac{1}{4}\eta_{mr}\eta_{ns}\partial^0 h^{rs} \tag{5.39}$$

Inverting equations eq 5.35 and 5.36 allows us to express χ^0 and χ^i in terms of their momenta respectively

$$\dot{\chi^0} = \frac{-2}{\sqrt{g}m^2}P + \frac{1}{4}\left(-2h^{00} + 4\partial_i\chi^i + 2h^{ik}\eta_{ik}\right) + \frac{4\lambda}{m^2} \tag{5.40}$$

$$\dot{\chi^i} = \frac{-2P^i}{m^2\sqrt{g}} - \partial_j\chi^0\eta^{ij} - h^{0i} \tag{5.41}$$

Inverting eq 5.39 allows us to write the following

$$\partial^0 h^{qt} = 4\eta^{mq}\eta^{nt}\Pi_{mn} - 2\eta^{qt}\eta^{ij}\Pi_{ij} + \frac{1}{2}\partial_i h^{i0}\eta^{qt} \tag{5.42}$$

The ϕ dependent Hamitlonian H_ϕ of the theory becomes

$$\begin{aligned}
H_\phi &= P\dot{\chi}^0 + P_i\dot{\chi}^i - L_\phi \\
&= -\sqrt{g}\left(h^{ij}\frac{m^2}{8}\left(4\eta_{ij}\partial_k\chi^k - 4\partial_j\chi_i\right)\right. \\
&+\frac{m^2}{8}h^{ij}h^{kl}\left(-\eta_{ik}\eta_{jl} + \frac{1}{2}\eta_{ij}\eta_{kl}\right) \\
&+\frac{m^2}{8}\left(-2\partial_i\chi^j\partial^i\chi^j - 2\partial_i\chi^j\partial_j\chi^i + 4\partial_i\chi^i\partial_j\chi^j\right) \\
&\left.+\lambda\eta_{ij}h^{ij} + 2\lambda\partial_i\chi^i\right) - \frac{Ph^{00}}{2} + \eta_{ij}h^{ij}\frac{P}{2} + P\partial_i\chi^i - \frac{4\lambda^2\sqrt{g}}{m^2} \\
&+\frac{4P\lambda}{m^2} - \frac{P^2}{m^2\sqrt{g}} - \frac{P_iP^i}{m^2\sqrt{g}} - P_ih^{0i} - P^i\partial_i\chi^0
\end{aligned} \tag{5.43}$$

Eq 5.43 shows the presence of λ which is just a Lagrange multiplier. To get rid of it, we use the secondary constraint resulting from the primary one $p_\lambda = 0$

$$0 = \dot{p}_\lambda = \{p_\lambda, H\} \tag{5.44}$$

Then, λ can be expressed as

$$\lambda = \frac{P}{2\sqrt{g}} - \frac{m^2}{8}\eta_{ij}h^{ij} - \frac{m^2}{4}\partial_i\chi^i \tag{5.45}$$

The total Hamiltonian becomes

$$
\begin{aligned}
&H = H_\phi + H_g \\
&H = \Pi \dot{h^{00}} + \Pi_i \dot{h^{0i}} + \Pi_{ij}\dot{h^{ij}} + P\dot{\chi^0} + P_i\dot{\chi^i} - L \\
&= \int d^4x \left(+\frac{1}{2}\eta^{ij}\Pi_{ij}\partial_k h^{k0} + 2\Pi_{ij}\Pi_{kl}\eta^{ik}\eta^{jl} - \Pi_{ij}\Pi_{kl}\eta^{ij}\eta^{kl} \right. \\
&+\frac{1}{16}\partial_i h^{i0}\partial_j h^{j0} - \frac{1}{4}\partial_i h^{ij}\partial_j h^{kl}\eta_{kl} + \frac{1}{4}\partial_i h^{ij}\partial_l h^{kl}\eta_{jk} \\
&-\frac{1}{4}\partial_i h^{0k}\partial^i h^{0l}\eta_{kl} - \frac{1}{8}\partial_i h^{km}\partial^i h^{ln}\eta_{kl}\eta_{mn} + \frac{1}{8}\partial_i h^{kl}\partial^i h^{mn}\eta_{kl}\eta_{mn} \\
&-\frac{1}{m^2\sqrt{g}}P_iP^i - P_ih^{0i} - P^i\partial_i\chi^0 + h^{00}\left(\frac{1}{4}\partial_j\partial_i h^{ij} - \frac{1}{4}\partial_i\partial^i h^{kl}\eta_{kl} - \frac{1}{2}P\right) \\
&-\frac{m^2}{8}h^{ij}(2\eta_{ij}\partial_k\chi^k - 4\partial_j\chi_i) - \frac{m^2}{8}h^{ij}h^{kl}\left(-\eta_{ik}\eta_{jl} + \frac{1}{2}\eta_{ij}\eta_{kl}\right) \\
&\left. -\frac{m^2}{8}\left(-2\partial_i\chi_j\partial^i\chi^j - 2\partial_i\chi^j\partial_j\chi^i + 2\partial_i\chi^i\partial_k\chi^k\right)\right)
\end{aligned} \tag{5.46}
$$

5.4.2 Equations of Motion

Studying the equations of motion is important the determine the characteristics of the physical fields in this theory. The equations of motion with respect to χ^0 and χ^i are respectively

$$\dot{P} = -\partial_i P^i \tag{5.47}$$

$$\dot{P}_i = -\frac{m^2}{4}\partial_i h^{kj}\eta_{kj} + \frac{m^2}{2}\partial_j h^{kj}\eta_{ki} + \frac{m^2}{2}\partial_j\partial^j\chi_i \tag{5.48}$$

The equations of motion with respect to P and P_i are respectively

$$\dot{\chi}^0 = -\frac{h^{00}}{2} \tag{5.49}$$

$$\dot{\chi}_i + \frac{2}{m^2\sqrt{g}}P_i + \partial_i\chi^0 + h^{0j}\eta_{ji} = 0 \tag{5.50}$$

Linearizing the mimetic constraint, eq 4.26, gives

$$\begin{aligned}
\bar{h}^{00} &= g^{\mu\nu}\partial_\mu\phi^0\partial_\nu\phi^0 - 1 \\
&= \left(h^{00}+1\right)\left(1+\partial_0\chi^0\right)\left(1+\partial_0\chi^0\right) - 2h^{0i}\left(1+\partial_0\chi^0\right)\left(\partial_i\chi^0\right) \\
&+h^{ij}\left(\partial_i\chi^0\right)\left(\partial_j\chi^0\right) - 1 \\
&= h^{00} + 2\partial_0\chi^0 = 0
\end{aligned} \tag{5.51}$$

which is exactly eq 5.49.

By substituting eq 5.36, for P_i, in eq 5.50, we get

$$m^2\left(\partial^\rho\bar{h}_{\rho k} - \frac{1}{2}\partial_k\bar{h}\right) = 0 \tag{5.52}$$

Up to linear order, eq 5.47 gives

$$
\begin{aligned}
&\frac{m^2\sqrt{g}}{8}\left(-2\dot{h}^{00}+4\partial_0\partial_i\chi^i+2\partial_0 h^{ik}\eta_{ik}\right)\\
&+2\sqrt{g}\dot{\lambda}-4\sqrt{g}\frac{m^2}{8}\ddot{\chi}^0+\frac{m^2\sqrt{g}}{8}\left(-4\partial_i\partial_j\chi^0\eta^{ij}-4\partial_i h^{0i}\right)\\
&-\frac{m^2\sqrt{g}}{2}\partial_i\partial_0\chi^i=0
\end{aligned}
\tag{5.53}
$$

which is, upon replacing h by $\bar{h}$, exactly the same as

$$
\partial_0\lambda-\frac{m^2}{4}\left(\partial^\rho\bar{h}_{\rho 0}-\frac{1}{2}\partial_0\bar{h}\right)=0 \tag{5.54}
$$

The remaining equation of motion eq 5.48 gives the expression of the conjugate momentum P_m.

The linearized equation of motion of h_{00} is

$$
G_{00}(-\bar{h}_{\rho\sigma})=2\lambda+\frac{m^2}{4}\bar{h} \tag{5.55}
$$

which is equivalent to

$$
\triangle\bar{h}+\partial^i\partial^j\bar{h}_{ij}=4\lambda+\frac{m^2}{2}\bar{h} \tag{5.56}
$$

where G_{00} is the $(0-0)$ component of the below linearized Einstein tensor

$$
\begin{aligned}
G_{\mu\nu}(h_{\rho\sigma})=&-\frac{1}{2}\left(\partial^2 h_{\mu\nu}-\partial_\mu\partial^\rho h_{\rho\nu}-\partial_\nu\partial^\rho h_{\rho\mu}+\partial_\mu\partial_\nu h\right)\\
&+\frac{1}{2}\eta_{\mu\nu}\left(\partial^2 h-\partial^\sigma\partial^\rho h_{\rho\sigma}\right)
\end{aligned}
\tag{5.57}
$$

The Linearized equation of motion of h_{0i} is

$$G_{0i}(-\bar{h}_{\rho\sigma}) = -\frac{m^2}{2}\bar{h}_{0i} \tag{5.58}$$

which is equivalent to

$$\triangle\bar{h}_{0i} + \partial_0\partial^k\bar{h}_{ki} + \partial_0\partial_i\left(\frac{4}{m^2}\lambda - \frac{1}{2}\bar{h}\right) = m^2\bar{h}_{0i} \tag{5.59}$$

Finally, the linearized equation of motion of h_{ij} is

$$G_{ij}(-\bar{h}_{\rho\sigma}) = -\frac{m^2}{2}\left(\bar{h}_{ij} - \frac{1}{2}\eta_{ij}\bar{h}\right) \tag{5.60}$$

using eq 5.57, the $(i-j)$ equation becomes

$$\partial^2\bar{h}_{ij} - \eta_{ij}\left(\frac{1}{2}\partial^2\bar{h} - \frac{4\ddot{\lambda}}{m^2}\right) = -m^2\left(\bar{h}_{ij} - \frac{1}{2}\eta_{ij}\bar{h}\right) \tag{5.61}$$

The equations of motions obtained here, are exactly the same as that obtained in the paper of Chamseddine and Mukhanov [Chamseddine and Mukhanov, 2018a].

5.4.3 Poisson Bracket

To examine the number of physical degrees of freedom, we need to study the constraints and construct the Poisson brackets.

The conjugate momenta Π and Π_i, eqs 5.37 and 5.38 respectively, leads to the below four first primary constraints

$$\begin{aligned} T &= \Pi + \frac{1}{4}\partial_k h^{0k} \\ T_i &= \Pi - \frac{1}{4}\partial_i h^{00} - \frac{1}{4}\partial_i \eta_{rs} h^{rs} + \frac{1}{2}\partial_l \eta_{ir} h^{rl} \end{aligned} \tag{5.62}$$

The total Hamiltonian, eq 5.46, shows that h^{00} is just a Lagrange multiplier. This leads to an extra primary constraint N

$$N = -\frac{1}{4}\partial_j \partial_i h^{ij} + \frac{1}{4}\partial_i \partial^i h^{kl}\eta_{kl} + \frac{1}{2}P \tag{5.63}$$

The Poisson brackets of these five first primary constraints with the total Hamiltonian, lead to a set of secondary constraints. The time change of T is

$$\{T, H\} = +\frac{1}{4}\partial_j \partial_i h^{ij} - \frac{1}{4}\partial_i \partial^i h^{kl}\eta_{kl} - \frac{1}{2}P = 0 \tag{5.64}$$

Up to linear order, $\frac{P}{2}$ can be written as $\frac{m^2}{8}\sqrt{g}\bar{h} + \lambda\sqrt{g}$. After substitution, we notice that the time change of T is exactly the same as the equation of h_{00} eqs 5.55, 5.56.

Similarly, the time change of T_i is given by the Poisson bracket

$$\begin{aligned} \{T_k, H\} = & -\left(+\frac{1}{8}\partial_k \partial_j h^{0j} - \frac{1}{2}\partial_i \partial^i h^{0j}\eta_{kj} - P_k - \frac{1}{2}\partial_k\left(\eta^{ij}\Pi_{ij}\right)\right) \\ & +\frac{1}{4}\eta_{ij}\partial_k\left(h^{ij}\right) - \frac{1}{2}\eta_{kj}\partial_i\left(h^{ij}\right) = 0 \end{aligned} \tag{5.65}$$

which is the equation of motion of h_{0i} eqs 5.58,5.59. The time change of N is

$$\{N, H\} = 0 \tag{5.66}$$

$h_{\mu\nu}$ and χ^A represent fourteen independent fields. The presence of four primary and four secondary constraint reduces the number of degrees of freedom to six. The additional primary constraint leaves us with five independent physical degrees of freedom representing the massive graviton. Thus, this theory is indeed ghost free.

5.4.4 Looking at the Mimetic Term

To make sure that the mimetic term doesn't represent ghosts, we need to examine it's energy density. To achieve this, we start expanding our Hamiltonian eq 5.46 to second order in scalar perturbations. For small perturbations, different fields are expanded as follows

$$\begin{aligned}
\chi^0 &= \chi^0 \\
\chi^i &= \tilde{\chi}^i - \partial^i \pi \\
h^{00} &= -2\phi \\
h^{0i} &= 0 \\
h^{ij} &= 2\psi\eta^{ij}
\end{aligned} \tag{5.67}$$

Substituting these perturbations in the Hamiltonian and using the mimetic constraint $\bar{h}^{00} = 0$, the scalar Hamiltonian becomes, up to second order,

$$H_{scalar} = \int d^4x \left(\psi \triangle \psi - \frac{3m^2}{4}\psi^2 - 2\dot{\chi}^0 \triangle \psi - \frac{m^2}{2}\psi \triangle \pi + \frac{m^2}{4} \triangle \pi \triangle \pi \right.$$
$$\left. + \dot{\chi}^0 P - P^i \partial_i \chi^0 - 3\dot{\psi}^2 - \frac{1}{m^2} P_i P^i \right) \tag{5.68}$$

Up to first order in scalar perturbations, the momenta P eq 5.35 and P_i eq 5.36 become

$$P = 2 \triangle \psi$$
$$P_i = -\frac{m^2}{2}\partial_i \chi^0 + \frac{m^2}{2}\partial_i \dot{\pi} \ , \tag{5.69}$$

Varying the scalar Hamiltonian eq 5.68 with respect to χ^0 and substituting the above expressions for P and P_i, we get

$$\dot{\psi} = \frac{-m^2}{4}\left(\chi^0 - \dot{\pi}\right). \tag{5.70}$$

which is exactly the scalar perturbation of eq 5.47

$$\dot{P} = -\partial_i P^i. \tag{5.71}$$

Keeping two variables only, ψ and π, the above scalar Hamiltonian becomes

$$H_{scalar} = \int d^4x \left(\psi\left(\triangle - \frac{3m^2}{4}\right)\psi + \frac{4}{m^2}\dot{\psi}\left(\triangle - \frac{3m^2}{4}\right)\dot{\psi} - \frac{m^2}{4}\left(2\psi \triangle \pi - \triangle\pi \triangle \pi\right) - 2\dot{\pi} \triangle \dot{\psi}\right) \tag{5.72}$$

To get the energy density of the mimetic term λ, we need to diagonalize the Hamiltonian by finding an expression of ψ in terms of λ and π. Expanding the equation of motion of h_{00} eq 5.56 or the equation of λ eq 5.45, up to first order in scalar perturbation, we get

$$\triangle\psi - \frac{3m^2}{4}\psi = \lambda + \frac{m^2}{4} \triangle \pi \tag{5.73}$$

which gives

$$\psi = \left(\triangle - \frac{3m^2}{4}\right)^{-1}\left(\lambda + \frac{m^2}{4} \triangle \pi\right) \tag{5.74}$$

Upon substituting the expression of ψ in eq 5.72, the Hamiltonian turns to be diagonlaized and function of the mimetic term only

$$H_{\lambda} = \int d^4x \left(\frac{-16\dot{\lambda}^2}{m^2\left(3m^2 - 4\triangle\right)} - \frac{4\lambda^2}{\left(3m^2 - 4\triangle\right)}\right). \tag{5.75}$$

The momentum of λ, p_{λ}, is

$$p_\lambda = \frac{-32\dot{\lambda}}{m^2\left(3m^2 - 4\triangle\right)} \tag{5.76}$$

In terms of p_λ, the Hamiltonian becomes

$$H_\lambda = -\frac{p_\lambda^2 m^2\left(3m^2 - 4\triangle\right)}{64} - \frac{4\lambda^2}{3m^2 - 4\triangle} \tag{5.77}$$

To check the sign of the energy density of λ, we consider different modes. For plane wave modes of wave number $\vec{k}$, $\triangle$ is $-k^2$. For modes with $k \gg m$, the above energy density reduces to

$$\frac{-4}{m^2k^2}\left(\dot{\lambda}^2 + \frac{m^2}{4}\lambda^2\right). \tag{5.78}$$

This appears to be negative and singular as m^2 goes to zero. However, looking at the equation of motion that we get from H_λ

$$\ddot{\lambda} + \frac{m^2}{4}\lambda = 0,$$

we deduce that $\dot{\lambda} \propto m\lambda$. This fact avoids the singularity $m^2 \to 0$. For $m^2 \to 0$ the energy density of the λ term becomes

$$\epsilon_{mim} \simeq \lambda - \frac{\lambda^2}{k^2} \tag{5.79}$$

where the second term is due to the gravitational self interaction which is much smaller than the first term for $\lambda \ll k^2$. In this limit, the energy density is positive. The problem arises if we consider $\lambda > k^2$. In this limit, the linear perturbation theory will be no more valid. Hence, we need to re-examine the non-linear theory again. [Chamseddine and Mukhanov, 2018b]. It is well known that the self interaction between matter results in a negative gravitational energy which reduces the total energy density. If this negative energy becomes comparable to the linear term then we might end with a universe with zero total energy!

The Hamiltonian formulation of ghost free mimetic massive gravity theory shows that this theory is indeed ghost free with five physical degrees of freedom describing graviton. In contrary to the linear Fierz Pauli's massive gravity theory that suffers from the vDVZ discontinuity, this theory doesn't since the constrained mimetic scalar field mixes in a different way leading to a different mass term.

Chapter 6

Mimetic Horava Gravity

Strong, weak, electromagnetic and gravitational forces are the three fundamental forces in nature that govern the interaction between the constituents of this universe. They differ between each others according to their strength and range. Weak, electromagnetic and strong interactions are well defined in the context of standard model as gauge theories respecting a certain local symmetry. The discovery of the Higgs bozon (2012) was the best proof for it's success. The standard model is the best paradigm to construct a quantum field theory for these interactions. Incorporating gravity in it has not been a successful step since GR describes gravity in terms of the metric, which is a dynamical quantity. In addition, the symmetries in GR is the diffeomorphism and Local Lorentz ones that are related to the coordinates of space-time. Such issues makes the quantization of gravity a challenging one. As mentioned before, several reasons have pushed physicists to quantize gravity. Quantizing gravity is essential to explain the early universe and singularities. Moreover, using the classical GR theory to unify gravity with the other interactions, described by a well defined quantum field theory, is impossible. The idea of quantum gravity started with Rosenfeld who talked about the need for quantizing gravity [Rosenfeld, 1930]. For a detailed review of the history of quantum gravity review [Rovelli, 2000]. In general, several theories have been formulated to quantize gravity, directly or indirectly. The main goal of direct quantum gravity theories is to quantize gravity. They start from a given classical theory of gravity and apply certain quantization rules. The covariant and canonical quantum gravity theories fall into this class of quantum gravity theories. In the covariant theory, we usually start from the gravitational path integral

$$Z[g] = \int Dg_{\mu\nu}(x) e^{iS_{EH}[g_{\mu\nu}(x)]} \tag{6.1}$$

and they apply the perturbation strategy by expanding the metric around a background one $\bar{g}_{\mu\nu}$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sqrt{32\pi G} f_{\mu\nu} \tag{6.2}$$

where $f_{\mu\nu}$ denotes the quantized graviton field.

Things resemble the perturbation theory applied for the other forces. Unfortunately, the resulting theory is not renormalizable because the coupling constant $[G_N]$ has negative mass dimension $[G_N]$. In addition, the graviton propagator is proportional to $\frac{1}{k^2}$.To remove divergences, physicists start adding higher order derivative terms to the Einstein-Hilbert action [Stelle, 1977].

$$S_{R^2}[g_{\mu\nu}] = \int_M d^4x \sqrt{-g} \left(\Lambda + \frac{R}{2} + \alpha R_{\mu\nu} R^{\mu\nu} + \beta R^2 \right) \tag{6.3}$$

Although such combination renders the theory renormalizable, it makes it unstable because the higher derivatives introduce additional ghost-like degrees of freedom. These negative energy particles are due to the presence of orders of time derivative higher than two in the modified action eq 6.3. The second direct approach to quantum gravity is the canonical one. The Hamiltonian formulation of gravity is constructed at the classical level and then everything is translated into the quantum.
String theory belongs to the indirect theories of quantum gravity. It's main goal is to construct a unified quantum theory of all interactions.

6.1 Horava-Lifshitz (HL) Gravity

Higher order spatial derivatives are important to improve the UV behavior of the theory while the higher order time derivatives are introducing ghosts!! A possible resolution was suggested by Horava who chose to break Lorentz invariance LI only at very high energies. Horava's action is restricted to contain time order derivative terms only up to second order while the order of spatial derivative is kept arbitrary. The restriction on the time derivatives solves the problem of ghosts. Such strategy resembles the usual technique of quantum field theory QFT of usual interactions. The only difference is that Horava chose to break Lorentz invariance by a Lifshitz-type of anisotropic scaling between space and time. [Wang, 2017].

$$\begin{aligned} x &\to bx \\ t &\to b^z t \end{aligned} \tag{6.4}$$

where z is the dynamical critical exponent. At low energies the invariance is restored again.

According to dimensional and power counting analysis, Horava found that his theory is renormalizable when $z = 3$. The rescaling choice eq 6.4 not only break LI but also the four dimensional diffeomorphism invariance. The spatial invariance is still respected. Based on these requirements, it is easy to construct the Horava's action. Working in the ADM formalism and using the variables (N, N^i, g_{ij}), Horava's action will be made up of terms build up of the below tensors and scalars and their linear combinations.

$$R_{ij}, K_{ij}, \nabla_i, a_i \tag{6.5}$$

R_{ij} is the three dimensional Ricci tensor constructed from the 3D metric g_{ij}.
K_{ij} is the Extrinsic curvature defined as

$$K_{ij} \equiv \frac{1}{2N}(-\dot{g_{ij}} + \nabla_i N_j + \nabla_J N_i) \tag{6.6}$$

∇_i is the Covariant derivative with respect to g_{ij} and a_i is defined as

$$a_i \equiv \frac{N, i}{N} \tag{6.7}$$

One of the terms constructing the Horava's Lagrangian is the gravitational Chern-Simons term given by

$$\begin{aligned} \omega_3(\Gamma) &\equiv Tr(\Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma) \\ &= \epsilon^{ijk}(\Gamma^m_{il}\partial_j\Gamma^l_{km} + \frac{2}{3}\Gamma^n_{il}\Gamma^l_{jm}\Gamma^m_{kn}) \end{aligned} \tag{6.8}$$

where Γ is the christoffel symbol defined using the metric g_{ij}. The second term constructing the Horava's action is $\propto C_{ij}C^{ij}$ where C_{ij} is the Cotton-York tensor defined by

$$C^{ij} = \epsilon_k^{ikl}(R_l^j - \frac{1}{4}R\delta_l^j) \tag{6.9}$$

Other terms are like:

$$K_{ij}K^{ij}, K^2, RR_{ij}R^{ij}, K_{ij}R^{ij}, a_ia^i, R^2, R_{ij}R^{ij}... \tag{6.10}$$

To summarize, Horava's Lagrangian, as any Lagrangian, can be written as the sum of 2 terms: the kinetic one L_K and the potential one L_V. Only the extrinsic curvature K_{ij} contains time derivative of the metric g_{ij}. Thus, the kinetic term of the Lagrangian is made up of only two terms $K_{ij}K^{ij}$ and K^2. All the other terms constitute the potential terms of the Lagrangian L_V.

There are plenty of these terms. To reduce them, Horava imposed extra condition on the lapse function, the projectable condition $N = N(t)$, which eliminates all the terms proportional to a_i and it's derivatives. Thus, ending up with the minimal theory. Extra conditions have been imposed like the detailed balance one on the potential part of the Lagrangian [Mukohyama, 2010].

Although Horava's theory have several cosmological applications concerning the horizon issue and the flatness one, it still suffers from several problems [Hořava, 2011] even after imposing the 2 extra conditions, the projectable and the detailed balance ones. First of all, the Newtonian limit of the theory doesn't exist

[Lü et al., 2009]. Moreover, the breaking of the diffeomorphism invariance generates new dynamical degrees of freedom which remain strongly coupled as we approach the "GR limit" [Charmousis et al., 2009], [Blas et al., 2009], [Li and Pang, 2009]. To solve these issues, several modifications to the initial theory have been proposed. Unfortunately, these modified Horava's theories prove to suffer from additional problems [Henneaux et al., 2010].

6.2 Mimetic Horava Gravity

Recently, Chamseddine, Mukhanov and Russ were able to regenerate the Horava quantum gravity model in a diffeomorphic invariant way, without the risk of introducing ghosts. There work was based on mimetic gravity. Using the mimetic scalar field ϕ, all the terms of Horava gravity action, respecting the space diffeomorphism, are constructed from four dimensional tensors subjected to synchronous gauge [Chamseddine et al., 2019].
In the ADM formalism, the metric is written as

$$ds^2 = N^2 dt^2 - \gamma(dx^i + N^i dt)(dx^j + N^j dt), i = 1, 2, 3 \tag{6.11}$$

where $\gamma_{ij} = -g_{ij}$. In the synchronous gauge, the solution to the mimetic constraint eq 4.26 is

$$\phi = t + A \tag{6.12}$$

Thus, ϕ can represent a specific hypersurface with a timelike unit vector $n_\mu = \partial_\mu \phi$. In addition, we define the projection operator P^ν_μ as

$$P^\nu_\mu = \delta^\nu_\mu - \partial_\mu \phi \partial_k \phi g^{\nu k} \tag{6.13}$$

satisfying the following relations

$$\begin{aligned} &P^\rho_\mu P^\nu_\rho = P^\nu_\mu \\ &P^\nu_\mu \phi = 0 \end{aligned} \tag{6.14}$$

In the synchronous gauge, the projection operator have the following components

$$\begin{aligned} &P^0_0 = 0 \\ &P^i_0 = 0 \\ &P^0_i = 0 \\ &P^j_i = \delta^j_i \end{aligned} \tag{6.15}$$

Using n_μ and P^ν_μ, it is easy to construct tensors with the only non zero components are along the space direction.
The extrinsic curvature K_{ij}, used to construct the terms of Horava's action, can now be easily defined in terms of ϕ in the synchronous gauge as .

$$
\begin{aligned}
K_{ij} &= -\frac{1}{2}\dot{\gamma}_{ij} = -\nabla_i \nabla_j \phi \\
K_i^j &= \gamma^{jl} K_{il} = \nabla_i \nabla^j \phi \\
K &= K_i^i = (ln\sqrt{\gamma}) = \Box\phi
\end{aligned} \tag{6.16}
$$

In the synchronous gauge and using the projection operator, the components of the 3 dimensional Ricci tensor ${}^3R_{ij}$, coincides with $i-j$ components of the below tensor

$$
\tilde{R}_{\mu\nu} = P_\mu^\alpha P_\nu^\beta R_{\alpha\beta} + \Box\phi \nabla_\mu \nabla_\nu \phi - \nabla_\mu \nabla^\rho \phi \nabla_\nu \nabla_\rho \phi - R^\gamma_{\mu\delta\nu} \nabla^\delta \phi \nabla_\gamma \phi \tag{6.17}
$$

Thus, the 3-dimensional Ricci scalar ${}^3R$, which coincides with Ricci scalar $\tilde{R}$, can be easily found by contracting $\tilde{R}_{\mu\nu}$ with $g_{\mu\nu}$

$$
\tilde{R} = 2R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - R - (\Box\phi)^2 + \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi \tag{6.18}
$$

Writing the components of the extrinsic curvature K_{ij} and that of the three dimensional Ricci tensor ${}^3R_{ij}$ and the Ricci scalar ${}^3R$ in terms of ϕ, now we can start constructing the basic terms of the Horava's action in terms of the mimetic field. For example, the Cotton-York tensor eq 6.9 is now defined as

$$
\tilde{C}^\mu_\nu = -\frac{1}{\sqrt{-g}} \epsilon^{\mu\rho k\lambda} \nabla_\lambda \phi \nabla_\rho (\tilde{R}_{\nu k} - \frac{1}{4} g_{\nu k} \tilde{R}) \tag{6.19}
$$

The Chern-Simons form is added to Horava's action through the below term

$$\int d\phi \wedge \tilde{\omega}_P = \int d\phi \wedge (\omega_P - \nabla^{\lambda} d\phi \wedge R^{\tau}_{\lambda} \nabla_{\tau}\phi) \tag{6.20}$$

where $\omega_P = \Gamma^{\nu}_{\ \mu} \wedge d\Gamma^{\mu}_{\ \nu} + \frac{2}{3}\Gamma^{\mu}_{\ \nu}\Gamma^{\nu}_{\ \rho}\Gamma^{\rho}_{\ \mu}$

Gathering all the terms together, we can write the mimetic Horava action as

$$I = \int \sqrt{-g}(\nabla_{\mu}\nabla_{\nu}\phi\nabla^{\mu}\nabla^{\nu}\phi - (\Box\phi)^2 + \tilde{R})d^4x \tag{6.21}$$

where $\tilde{R}$ is given by eq 6.18.

Chapter 7

Mimetic Horava Gravity and Surface Terms

The Hamiltonian analysis is usually the second step after the construction of any field theory. The existence of surface terms in the action formalism makes it hard to perform the canonical analysis. Einstein Hilbert (EH) theory is an example of field theories with the surface terms problem. The reason behind the presence of these surface terms, in EH Lagrangian, is the second order derivative of the dynamical variable, the metric g_{ab}, hidden in the Ricci scalar R. These second order derivatives may lead to equations with third order derivatives that leads to the presence of ghosts. To avoid such a problem, integration by part can be applied to remove the second order derivative terms from the main Lagrangian. Thus, the main EH Lagrangian (Ricci Scalar R) will divide into two parts, the first contains first order derivative terms that are responsible for deriving the field equations and the second is the surface one that hides the second order derivative terms in it. Surface terms don't contribute to the field equations. Ignoring the presence of these surface terms,in canonical formulation, creates problems unless certain boundary terms is added to the constraint to give a well defined equations of motion. It is better to remove these surface terms from the beginning. Einstein proposed to remove the surface term by subtracting it from the main action [Einstein, 1952]. Thus, the new Lagrangian for general relativity becomes $L = R - L_{surface}$. Unfortunately, the new action is not a diffeomorphic invariant. The important property of the boundary term that should be added to the EH action is to kill all the normal derivatives of the metric on the boundary surface. There are plenty of boundary terms satisfying the above condition as shown by Charap and Nelson in [Charap and Nelson, 1983]. This is done by adding to the EH action a boundary term that keeps the action invariant under diffeomorphism. The well known boundary term is the Gibbons-Hawking-York term which depend on the extrinsic curvature K [York Jr, 1972], [Gibbons and Hawking, 1977], [York, 1986]. Thus, the new surface-independent GR action becomes

$$I = (16\pi)^{-1} \int_M R\sqrt{g} d^4x - \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{h} K \qquad (7.1)$$

where ∂M is the boundary of M, h_{ab} is the induced metric on ∂M and K is the trace of the second fundamental form on ∂M.

7.1 Mimetic Horava Gravity and Surface Terms

Before performing the Hamiltonian analysis of the mimetic Horava theory, it is essential to check if there exist surface terms in the action eq 6.21. We are going to show that the surface terms in the mimetic Horava action will cancel among each other [Malaeb and Saghir, 2020]. The mimetic Horava action eq 6.21 can be rewritten as

$$I = \int \sqrt{-g}(-R - 2\nabla_\mu(\Box\phi\nabla^\mu\phi) + 2\nabla_\sigma(\nabla_\mu\nabla^\sigma\phi\nabla^\mu\phi))d^4x \tag{7.2}$$

We define the following actions

$$I_H = \int \sqrt{-g}Rd^4x \tag{7.3}$$

$$I_1 = \int \sqrt{-g}(-2\nabla_\mu(\Box\phi\nabla^\mu\phi)d^4x) \tag{7.4}$$

and

$$I_2 = \int \sqrt{-g}(2\nabla_\sigma(\nabla_\mu\nabla^\sigma\phi\nabla^\mu\phi)d^4x) \tag{7.5}$$

To check if the action eq 7.2 suffers from any surface term, we need to vary each action eqs 7.3,7.4 and 7.5 alone. The variation of the Einstein Hilbert action gives

$$\delta I_H = \int_\gamma G_{\alpha\beta}\delta g^{\alpha\beta}\sqrt{-g}d^4x - \oint_{\partial\gamma} \epsilon h^{\alpha\beta}\delta g_{\alpha\beta,\mu}n^\mu|h|^{\frac{1}{2}}d^3y \tag{7.6}$$

Where

$$\begin{aligned} n_\mu n^\mu &= \epsilon \\ g^{\mu\nu} &= \epsilon n^\mu n^\nu + h^{\mu\nu} \end{aligned} \tag{7.7}$$

The variation of I_1 gives

$$\delta I_1 = I_1'(\delta\phi) + 2\oint_{\partial\gamma} \nabla_\mu\phi\frac{1}{2}g^{\rho\sigma}g^{\alpha\lambda}(2\delta g_{\rho\lambda,\sigma} - \delta g_{\rho\sigma,\lambda})\partial_\alpha\phi d\Sigma^\mu \tag{7.8}$$

where

$$I_1'(\delta\phi) = 2\oint_{\partial\gamma} \nabla_\mu\phi\Box\delta\phi d\Sigma^\mu + 2\oint_{\partial\gamma} \partial_\mu\delta\phi\Box\phi d\Sigma^\mu. \tag{7.9}$$

The variation of I_2 term gives

$$\delta I_2 = I_2'(\delta\phi) - 2\oint_{\partial\gamma} g^{\rho\tau}(\delta g_{\mu\tau,\sigma} + \delta g_{\sigma\tau,\mu} - \delta g_{\mu\sigma,\tau})\nabla_\rho\phi\nabla^\mu\phi d\Sigma^\sigma \tag{7.10}$$

Where

$$I_2'(\delta\phi) = 2\oint_{\partial\gamma} \nabla_\mu\nabla_\sigma\delta\phi\nabla^\mu\phi d\Sigma^\sigma + 2\oint_{\partial\gamma} \nabla_\mu\nabla_\sigma\phi\nabla^\mu\delta\phi d\Sigma^\sigma \tag{7.11}$$

By setting $\delta\phi = 0$ on the boundaries, $I_1'(\delta\phi)$ and $I_2'(\delta\phi)$, could be integrated out (surface of a surface) and will vanish .

Our aim is to study under what condition (if any) the surface terms, over $\delta g_{\mu\nu}$, of δI_H, δI_1 and δI_2 will cancel out upon addition. Knowing that

$$d\Sigma^{\mu} = n^{\mu}\epsilon|h|^{\frac{1}{2}}d^{3}y \tag{7.12}$$

and the completeness relation of the metric is

$$g^{\alpha\beta} = \epsilon n^{\alpha}n^{\beta} + h^{\alpha\beta} \tag{7.13}$$

it turns out that the surface terms will cancel out upon the choice of

$$n_{\mu} = \partial_{\mu}\phi. \tag{7.14}$$

Hence, under the condition 7.14, the surface terms in the mimetic Horava action cancel each others without the need to add extra term as in the case of pure Einstein gravity theory,

Chapter 8

Conclusion and Future Work

In this dissertation, three topics were considered, unification of gauge and gravity Chern-Simons theories in 3D space-time, canonical formulation of ghost free mimetic massive gravity and the mimetic Horava gravity and surface terms.
To reach the unification of both gravity and gauge theories in 3D space-time, we based our work [Saghir and Shamseddine, 2017] on the work of Chamseddine and Mukhanov [Chamseddine and Mukhanov, 2016b] for the unification of gravity and gauge theories. This could be achieved in two different ways based on the starting point. In the first method, we proved that the pontryagin density of the larger group SO(6) splits into the pontryagin density of gauge theory with SO(3) group and that of gravity with SO(3) group. Since the pontryagin density is the divergence of the Chern-Simons form, the unification can be translated from 4D to 3D space-time. The other method is to start working directly with Chern-Simons actions and make the splitting using the characteristics of gamma matrices [de Wit and Smith, 2012]. The quantization of the coupling constant κ depends on the chosen group. If we start using the compact SO(6) group, the coupling constant is quantized while it is not for the SO(1,5) group. After considering the weak coupling limit, we deduced that the partition function of the

larger group SO(6) or SO(1,5) is indeed a topological invariant quantity without the need to add any extra term as Witten did [Witten, 1989]. Further work can be done by generalizing this work to consider supersymmetric groups. It should be noted that by gauging the supergroups, or by extending the space-time manifold to a supermanifold, or both we can extend our Chern-Simons theory into a supersymmetric one. The idea is to choose the suitable supergroup corresponding to SO(1,5). According to [Chamseddine, 1990], we can choose our supergroup to be

$$O(6,1) \oplus SU(2), (8,2) \tag{8.1}$$

The second topic is related to mimetic gravity theory which predicts dark matter without the addition of any extra degree of freedom [Chamseddine and Mukhanov, 2013]. There exits several extensions to this theory. A recent one is the ghost free mimetic massive gravity [Chamseddine and Mukhanov, 2018a], [Chamseddine and Mukhanov, 2018b]. The graviton gain mass using the BEH mechanism using four scalar field where one of them acts as the mimetic field. The theory is free from any negative energy particles. Our work is to perform the Hamiltonian analysis of this theory [Malaeb and Saghir, 2019]. After defining the momenta of the variables and constructing the Hamiltonian up to second order in perturbation, we continue to find the equations of motion. One of them describes a massive graviton h_{ij}^T and another one describes the mimetic dark matter λ. The other fields can be expressed as linear combinations of these two. Poisson brackets are computed and the physical degrees of freedom are found to be six, five for graviton and one for the mimetic field. To make sure that λ doesn't represent ghosts, we found the scalar part of the Hamiltonian up to second order and we deduced that the energy density of λ is indeed positive. Further work can be done by formulating the Hamiltonian analysis of the non linear theory to prove that it is indeed ghost free.

The third topic is related to mimetic Horava gravity, another extension of mimetic gravity [Chamseddine et al., 2019]. Using the mimetic field, Chamseddine and Mukhanov regenerated the Horava quantum gravity model in a diffeomorphism invariant way without the presence of ghosts. We proved that the mimetic Horava action doesn't suffer from any surface terms, as the case of pure Einstein Hilbert action. The surface term resulting from the variation of Einstein Hilbert action is canceled with that resulting from the variation of the added terms in the mimetic Horava action [Malaeb and Saghir, 2020]. Further work can be done by formulating the Hamiltonian analysis of this theory .

Bibliography

[Abbott et al., 2016] Abbott, B. P., Abbott, R., Abbott, T., Abernathy, M., Acernese, F., Ackley, K., Adams, C., Adams, T., Addesso, P., Adhikari, R., et al. (2016). Observation of gravitational waves from a binary black hole merger. *Physical review letters*, 116(6):061102.

[Anderson and Bergmann, 1951] Anderson, J. L. and Bergmann, P. G. (1951). Constraints in covariant field theories. *Physical Review*, 83(5):1018.

[Aoki and Mukohyama, 2016] Aoki, K. and Mukohyama, S. (2016). Massive gravitons as dark matter and gravitational waves. *Physical Review D*, 94(2):024001.

[Arnowitt et al., 1959] Arnowitt, R., Deser, S., and Misner, C. W. (1959). Dynamical structure and definition of energy in general relativity. *Physical Review*, 116(5):1322.

[Arnowitt et al., 1960a] Arnowitt, R., Deser, S., and Misner, C. W. (1960a). Canonical variables for general relativity. *Physical Review*, 117(6):1595.

[Arnowitt et al., 1960b] Arnowitt, R., Deser, S., and Misner, C. W. (1960b). Consistency of the canonical reduction of general relativity. *Journal of Mathematical Physics*, 1(5):434–439.

[Arnowitt et al., 2008] Arnowitt, R., Deser, S., and Misner, C. W. (2008). Republication of: The dynamics of general relativity. *General Relativity and Gravitation*, 40(9):1997–2027.

[Arun et al., 2017] Arun, K., Gudennavar, S., and Sivaram, C. (2017). Dark matter, dark energy, and alternate models: A review. *Advances in Space Research*, 60(1):166–186.

[Atiyah, 1988] Atiyah, M. (1988). New invariants of 3-and 4-dimensional manifolds. *The mathematical heritage of Hermann Weyl (Durham, NC, 1987)*, 48:285–299.

[Bergmann, 1956] Bergmann, P. (1956). Quantisierung allgemein-kovarianter feldtheorien. *Helv. Phys. Acta Suppl*, 4:79.

[Bergmann and Goldberg, 1955] Bergmann, P. G. and Goldberg, I. (1955). Dirac bracket transformations in phase space. *Physical Review*, 98(2):531.

[Blas et al., 2009] Blas, D., Pujolas, O., and Sibiryakov, S. (2009). On the extra mode and inconsistency of hořava gravity. *Journal of High Energy Physics*, 2009(10):029.

[Born and Infeld, 1934] Born, M. and Infeld, L. (1934). Foundations of the new field theory. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 144(852):425–451.

[Boulware and Deser, 1972] Boulware, D. G. and Deser, S. (1972). Can gravitation have a finite range? *Physical Review D*, 6(12):3368.

[Carlip and Jonathan, 2003] Carlip and Jonathan, S. (2003). *Quantum gravity in 2+ 1 dimensions*, volume 50. Cambridge University Press.

[Chamseddine, 1990] Chamseddine, A. H. (1990). Topological gravity and supergravity in various dimensions. *Nuclear Physics B*, 346(1):213–234.

[Chamseddine and Fröhlich, 1992] Chamseddine, A. H. and Fröhlich, J. (1992). Two-dimensional lorentz-weyl anomaly and gravitational chern-simons theory. *Communications in mathematical physics*, 147(3):549–562.

[Chamseddine and Mukhanov, 2010a] Chamseddine, A. H. and Mukhanov, V. (2010a). Gravity with de sitter and unitary tangent groups. *Journal of High Energy Physics*, 2010(3):33.

[Chamseddine and Mukhanov, 2010b] Chamseddine, A. H. and Mukhanov, V. (2010b). Higgs for graviton: simple and elegant solution. *Journal of High Energy Physics*, 2010(8):11.

[Chamseddine and Mukhanov, 2013] Chamseddine, A. H. and Mukhanov, V. (2013). Mimetic dark matter. *Journal of High Energy Physics*, 2013(11):135.

[Chamseddine and Mukhanov, 2016a] Chamseddine, A. H. and Mukhanov, V. (2016a). Inhomogeneous dark energy. *Journal of Cosmology and Astroparticle Physics*, 2016(02):040.

[Chamseddine and Mukhanov, 2016b] Chamseddine, A. H. and Mukhanov, V. (2016b). On unification of gravity and gauge interactions. *Journal of High Energy Physics*, 2016(3):20.

[Chamseddine and Mukhanov, 2017a] Chamseddine, A. H. and Mukhanov, V. (2017a). Nonsingular black hole. *The European Physical Journal C*, 77(3):183.

[Chamseddine and Mukhanov, 2017b] Chamseddine, A. H. and Mukhanov, V. (2017b). Resolving cosmological singularities. *Journal of Cosmology and Astroparticle Physics*, 2017(03):009.

[Chamseddine and Mukhanov, 2018a] Chamseddine, A. H. and Mukhanov, V. (2018a). Ghost free mimetic massive gravity. *Journal of High Energy Physics*, 2018(6):60.

[Chamseddine and Mukhanov, 2018b] Chamseddine, A. H. and Mukhanov, V. (2018b). Mimetic massive gravity: beyond linear approximation. *Journal of High Energy Physics*, 2018(6):62.

[Chamseddine et al., 2019] Chamseddine, A. H., Mukhanov, V., and Russ, T. B. (2019). Mimetic hořava gravity. *Physics Letters B*, 798:134939.

[Chamseddine et al., 2014] Chamseddine, A. H., Mukhanov, V., and Vikman, A. (2014). Cosmology with mimetic matter. *Journal of Cosmology and Astroparticle Physics*, 2014(06):017.

[Charap and Nelson, 1983] Charap, J. and Nelson, J. E. (1983). Surface integrals and the gravitational action. *Journal of Physics A: Mathematical and General*, 16(8):1661.

[Charmousis et al., 2009] Charmousis, C., Niz, G., Padilla, A., and Saffin, P. M. (2009). Strong coupling in hořava gravity. *Journal of High Energy Physics*, 2009(08):070.

[Clemence, 1947] Clemence, G. M. (1947). The relativity effect in planetary motions. *Reviews of Modern Physics*, 19(4):361.

[Clifton et al., 2012] Clifton, T., Ferreira, P. G., Padilla, A., and Skordis, C. (2012). Modified gravity and cosmology. *Physics reports*, 513(1-3):1–189.

[Cognola et al., 2008] Cognola, G., Elizalde, E., Nojiri, S., Odintsov, S., Sebastiani, L., and Zerbini, S. (2008). Class of viable modified f (r) gravities describing inflation and the onset of accelerated expansion. *Physical Review D*, 77(4):046009.

[de Wit and Smith, 2012] de Wit, B. and Smith, J. (2012). *Field theory in particle physics, Volume 1.* Elsevier.

[Deffayet, 2001] Deffayet, C. (2001). Cosmology on a brane in minkowski bulk. *Physics Letters B*, 502(1-4):199–208.

[Deser et al., 2000] Deser, S., Jackiw, R., and Templeton, S. (2000). Topologically massive gauge theories. *Annals of Physics*, 281(1-2):409–449.

[Dirac, 1951] Dirac, P. A. (1951). The hamiltonian form of field dynamics. *Canadian Journal of Mathematics*, 3:1–23.

[Dirac, 1950] Dirac, P. A. M. (1950). Generalized hamiltonian dynamics. *Canadian journal of mathematics*, 2:129–148.

[Dirac, 1958] Dirac, P. A. M. (1958). Generalized hamiltonian dynamics. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 246(1246):326–332.

[Einstein, 1952] Einstein, A. (1952). Hamilton's principle and the general theory of relativity. *prel*, pages 165–173.

[Eling et al., 2006] Eling, C., Jacobson, T., and Mattingly, D. (2006). Einstein-aether theory. In *Deserfest*, pages 163–179. World Scientific.

[Fierz and Pauli, 1939a] Fierz, M. and Pauli, W. E. (1939a). On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 173(953):211–232.

[Fierz and Pauli, 1939b] Fierz, M. and Pauli, W. E. (1939b). On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 173(953):211–232.

[Gibbons and Hawking, 1977] Gibbons, G. W. and Hawking, S. W. (1977). Action integrals and partition functions in quantum gravity. *Physical Review D*, 15(10):2752.

[Golovnev, 2014] Golovnev, A. (2014). On the recently proposed mimetic dark matter. *Physics letters B*, 728:39–40.

[Guzzo et al., 2008] Guzzo, L., Pierleoni, M., Meneux, B., Branchini, E., Le Fèvre, O., Marinoni, C., Garilli, B., Blaizot, J., De Lucia, G., Pollo, A., et al. (2008). A test of the nature of cosmic acceleration using galaxy redshift distortions. *Nature*, 451(7178):541–544.

[Henneaux et al., 2010] Henneaux, M., Kleinschmidt, A., and Gómez, G. L. (2010). A dynamical inconsistency of hořava gravity. *Physical Review D*, 81(6):064002.

[Hinterbichler, 2012] Hinterbichler, K. (2012). Theoretical aspects of massive gravity. *Reviews of Modern Physics*, 84(2):671.

[Hooft, 2007] Hooft, G. (2007). Unitarity in the brout-englert-higgs mechanism for gravity. *arXiv preprint arXiv:0708.3184*.

[Hořava, 2011] Hořava, P. (2011). General covariance in gravity at a lifshitz point. *Classical and Quantum Gravity*, 28(11):114012.

[Houndjo et al., 2017] Houndjo, M., Rodrigues, M., Mazhari, N., Momeni, D., and Myrzakulov, R. (2017). Higher-derivative f (r, r, t) theories of gravity. *International Journal of Modern Physics D*, 26(03):1750024.

[Kakushadze, 2008] Kakushadze, Z. (2008). Gravitational higgs mechanism and massive gravity. *International Journal of Modern Physics A*, 23(10):1581–1591.

[Kaluza, 1921a] Kaluza, T. (1921a). On the unity problem of physics, sitzungse-ber. *Press. Akad. Wiss. Phys. Math. Klasse*, 966.

[Kaluza, 1921b] Kaluza, T. (1921b). Zum unitätsproblem der physik. *Sitzungs-ber. Preuss. Akad. Wiss. Berlin.(Math. Phys.)*, 1921(966972):45–46.

[Kerner, 1968] Kerner, R. (1968). Generalization of the kaluza-klein theory for an arbitrary non-abelian gauge group. In *Annales de l'IHP Physique théorique*, volume 9, pages 143–152.

[Kibble, 1961] Kibble, T. W. (1961). Lorentz invariance and the gravitational field. *Journal of mathematical physics*, 2(2):212–221.

[Li and Pang, 2009] Li, M. and Pang, Y. (2009). A trouble with hořava-lifshitz gravity. *Journal of High Energy Physics*, 2009(08):015.

[Lü et al., 2009] Lü, H., Mei, J., and Pope, C. (2009). Solutions to hořava gravity. *Physical Review Letters*, 103(9):091301.

[Makki et al., 2019] Makki, T. R., El Eid, M. F., and Mathews, G. J. (2019). A critical analysis of the big bang nucleosynthesis. *Modern Physics Letters A*, 34(24):1950194.

[Malaeb, 2015] Malaeb, O. (2015). Hamiltonian formulation of mimetic gravity. *Physical Review D*, 91(10):103526.

[Malaeb and Saghir, 2019] Malaeb, O. and Saghir, C. (2019). Hamiltonian formulation of ghost free mimetic massive gravity. *The European Physical Journal C*, 79(7):584.

[Malaeb and Saghir, 2020] Malaeb, O. and Saghir, C. (2020). Mimetic horava gravity and surface terms. *arXiv preprint arXiv:2005.02469*.

[Mukohyama, 2010] Mukohyama, S. (2010). Hořava–lifshitz cosmology: a review. *Classical and Quantum Gravity*, 27(22):223101.

[Nojiri and Odintsov, 2014] Nojiri, S. and Odintsov, S. D. (2014). Mimetic f (r) gravity: inflation, dark energy and bounce. *Modern Physics Letters A*, 29(40):1450211.

[Oda, 2010] Oda, I. (2010). Higgs mechanism for gravitons. *Modern Physics Letters A*, 25(28):2411–2421.

[Rosenfeld, 1930] Rosenfeld, L. (1930). Zur quantelung der wellenfelder. *Annalen der Physik*, 397(1):113–152.

[Rovelli, 2000] Rovelli, C. (2000). Notes for a brief history of quantum gravity. *arXiv preprint gr-qc/0006061*.

[Saghir and Shamseddine, 2017] Saghir, C. A. and Shamseddine, L. W. (2017). Unification of gauge and gravity chern–simons theories in 3-d space-time. *The European Physical Journal C*, 77(11):780.

[Shapiro et al., 2004] Shapiro, S. S., Davis, J. L., Lebach, D. E., and Gregory, J. (2004). Measurement of the solar gravitational deflection of radio waves using geodetic very-long-baseline interferometry data, 1979–1999. *Physical Review Letters*, 92(12):121101.

[Solomon et al., 2019] Solomon, A. R., Vardanyan, V., and Akrami, Y. (2019). Massive mimetic cosmology. *Physics Letters B*, 794:135–142.

[Stelle, 1977] Stelle, K. (1977). Renormalization of higher-derivative quantum gravity. *Physical Review D*, 16(4):953.

[Utiyama, 1956] Utiyama, R. (1956). Invariant theoretical interpretation of interaction. *Physical Review*, 101(5):1597.

[Vainshtein, 1972] Vainshtein, A. I. (1972). To the problem of nonvanishing gravitation mass. *Physics Letters B*, 39(3):393–394.

[Veltman and Dam, 1970] Veltman, M. and Dam, H. v. (1970). Massive and mass-less yang-mills and gravitational fields. *Nuclear physic: B*, 22(2):397–411.

[Vernov, 2012] Vernov, S. Y. (2012). Nonlocal gravitational models and exact solutions. *Physics of Particles and Nuclei*, 43(5):694–696.

[Wang, 2017] Wang, A. (2017). Hořava gravity at a lifshitz point: a progress report. *International Journal of Modern Physics D*, 26(07):1730014.

[Witten, 1988] Witten, E. (1988). Topological quantum field theory. *Communications in Mathematical Physics*, 117(3):353–386.

[Witten, 1989] Witten, E. (1989). Quantum field theory and the jones polynomial. *Communications in Mathematical Physics*, 121(3):351–399.

[York, 1986] York, J. W. (1986). Boundary terms in the action principles of general relativity. *Foundations of Physics*, 16(3):249–257.

[York Jr, 1972] York Jr, J. W. (1972). Role of conformal three-geometry in the dynamics of gravitation. *Physical review letters*, 28(16):1082.

[Zakharov, 1970] Zakharov, V. I. (1970). Linearized gravitation theory and the graviton mass. *JETP Lett.(USSR)(Engl. Transl.) 12: 312-14 (5 Nov 1970).*

[Zlatev et al., 1999] Zlatev, I., Wang, L., and Steinhardt, P. J. (1999). Quintessence, cosmic coincidence, and the cosmological constant. *Physical Review Letters*, 82(5):896.