AMERICAN UNIVERSITY OF BEIRUT

THE RIEMANN HYPOTHESIS FOR PERIOD POLYNOMIALS OF MODULAR AND HILBERT MODULAR FORMS

by
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An Abstract of the Thesis of

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Title: The Riemann Hypothesis for Period Polynomials of Modular and Hilbert Modular Forms

We study the location of the zeros of period polynomials of modular forms. For an even weight $k \geq 4$ newform $f \in S_k^{\text{new}}(\Gamma_0(N))$, we show that the zeros of its period polynomial $r_f(z)$ lie on the circle $|z| = 1/\sqrt{N}$. Moreover, we explore further generalizations to the case of Hilbert modular forms. In fact, we prove that the zeros of period polynomials of any parallel weight Hilbert modular eigenform on the full Hilbert modular group lie on the unit circle.

Contents

Introduction

Modular forms are omnipresent in mathematics. They appear in different mathematical disciplines like number theory, algebraic geometry, harmonic analysis; and they have recently turned up in the study of black holes and string theory. P. Sarnak[1] described this as “The unreasonable effectiveness of modular forms”. Their presence usually indicates a deep underlying structure teeming with symmetry.

Their period polynomials also are of central importance. Their structure and properties as objects in their own right have attracted a lot of interest from various perspectives. For instance, their have been a number of recent works on the theory of period polynomials and their zeros. It follows from the Eichler-Shimura Isomorphism that studying the zeros of period polynomials is as natural as studying the zeros of the cusp forms themselves. Moreover, these period polynomials can be viewed as the generating functions for the critical values of L-functions associated to modular forms; i.e. they provide a way of encoding critical values of the modular L-functions that has proven very successful in the uncovering of important arithmetic properties of L-values. Indeed, as L-functions are of fundamental importance in a wide number of areas of mathematics, it is thus useful to study period polynomials.

In this survey, we first give a brief introduction to the theory of modular forms and their period polynomials. We then study the location of the zeros of these period polynomials. The first work on this subject is due to Conrey et al.[2] who showed that the odd part of the period polynomial for any level 1 Hecke cusp form, apart from the so-called “trivial zeros”, all lie on the unit circle. Shortly thereafter, El-Guindy and Raji[3] showed that the zeros of the full period polynomial for any level 1 eigenform lie on the unit circle. And then, not long afterwards, Ono et al.[4] coined the term “Riemann Hypothesis for Period Polynomials”(RHPP) for the assertion that all roots of period polynomials lie on a circle centered at the origin, and showed that RHPP holds for any Hecke eigenform of higher level. Recently, Rolen et al.[5] showed that RHPP holds in the situation of Hilbert modular forms as well. More precisely, they show that the zeros of period polynomials of any parallel weight Hilbert modular eigenform on the full Hilbert modular group lie on the unit circle.

Chapter 1

Definitions and Basic Properties

In this chapter, we introduce the basic objects of study: Modular forms, their Hecke operators, associated L-functions, and period polynomials.

1.1 Modular Forms on $SL_2(\mathbb{Z})$

Denote by $\mathbb{H}$ the upper half plane, i.e. the set of all complex numbers with positive imaginary part:

$$\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}.$$

Define the full modular group

$$\Gamma := SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

$SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ in the standard way by *Möbius* transformations:

$$\text{For } z \in \mathbb{C} \text{ and } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ \gamma.z = \frac{az+b}{cz+d}$$

The action preserves $\mathbb{H}$ since, as a simple calculation shows,

$$\Im(\gamma z) = \frac{\Im(z)}{|cz+d|^2}.$$

Definition 1.1.1. *A modular form of weight $k \in \mathbb{Z}$ on Γ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying*

- $f(\gamma z) = (cz+d)^k f(z)$ *for* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$
- f *is holomorphic at* ∞ $\big($*or* $f(z) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n z}\big)$.

Remark. *For $\gamma = -I$, $f(-Iz) = (-1)^k f(z)$; but $f(-Iz) = f(z)$, then non-zero modular forms must be of even weight.*

Definition 1.1.2. *If $c(0) = 0$ in the preceding definition (i.e. f vanishes at ∞), we say that f is a cusp form.*

We denote by M_k the space of modular forms of weight k on Γ, and by S_k that of cusp forms.

Theorem 1.1.1. *Let $f \in S_k$ with $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$. Then the Fourier coefficients $a(n)$ of f satisfy*

$$a(n) = O\left(n^{\frac{k}{2}}\right).$$

Proof. Let

$$g(z) = y^{\frac{k}{2}} |f(z)|$$

where $z \in \mathbb{H}$ and $y = \Im(z) > 0$. Note that for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$\begin{aligned} g(Mz) &= \Im(Mz)^{\frac{k}{2}} |f(Mz)| \\ &= \left(\frac{\Im(z)}{(cz+d)^2}\right)^{\frac{k}{2}} |(cz+d)^k f(z)| \\ &= y^{\frac{k}{2}} |f(z)| \\ &= g(z). \end{aligned}$$

That is, g is invariant under Γ; which means that the values of g arise on the fundametal domain of Γ (see more on this in [6]). Note also that

$$\lim_{z \to i\infty} g(z) = \lim_{y \to \infty} y^{\frac{k}{2}} \left| \sum_{n=1}^{\infty} a(n) e^{2\pi i n z} \right| \leq \lim_{y \to \infty} y^{\frac{k}{2}} \sum_{n=1}^{\infty} |a(n)| e^{-2\pi n y} = 0. \tag{1.1}$$

Then g is continuous in $\mathbb{H}$ and at ∞, and invariant under Γ. These together with (1.1) give us that g is bounded in $\mathbb{H}$, say $g(z) \leq c$. Hence, we obtain

$$f(z) \leq c y^{\frac{-k}{2}}.$$

For $n \geq 1$ and $z \in \mathbb{H}$ consider

$$\int_z^{z+1} f(t) e^{-2\pi i n t}\, dt.$$

We have

$$\begin{aligned} \int_z^{z+1} f(t) e^{-2\pi i n t}\, dt &= \int_z^{z+1} \sum_{m=1}^{\infty} a(m) e^{2\pi i (m-n) t}\, dt \\ &= \sum_{m=1}^{\infty} a(m) \int_z^{z+1} e^{2\pi i (m-n) t}\, dt \\ &= a(n). \end{aligned}$$

Therefore,

$$\begin{aligned} |a(n)| &= \left| \int_z^{z+1} f(t) e^{-2\pi i n t}\, dt \right| \\ &\leq \int_z^{z+1} |f(t)| e^{2\pi n y}\, |dt| \\ &\leq c y^{\frac{-k}{2}} e^{2\pi n y}. \end{aligned}$$

This is true for any $y > 0$; so taking $y = \frac{1}{n}$ we get

$$|a(n)| \leq ce^{2\pi} n^{\frac{k}{2}}$$

for all n as desired. □

Corollary 1.1.1. *If $k < 0$ and $f \in S_k$, then $f \equiv 0$.*

Proof. Write $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$. Then by Theorem 1.1.1, we have that for any $y > 0$

$$|a(n)| \leq cy^{\frac{-k}{2}} e^{2\pi n y}.$$

Letting y tend to 0, we see that $a(n) = 0$ for all $n \geq 1$. □

1.2 The Hecke operators T_n

Definition 1.2.1. *For a fixed integer k and any $n = 1, 2, ...,$ the operator T_n is defined on M_k by the equation*

$$(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz + bd}{d^2}\right). \tag{1.2}$$

In the special case when $n = p$ is prime, the sum on d contains only two terms, so we get

$$T_p f(z) = p^{k-1} f(pz) + \frac{1}{p} \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right).$$

Theorem 1.2.1. *If f has the Fourier expansion at ∞*

$$f = \sum_{m=0}^{\infty} c(m) e^{2\pi i m z}$$

then

$$T_n f(z) = \sum_{m=0}^{\infty} \gamma_n(m) e^{2\pi i m z}$$

where

$$\gamma_n(m) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right).$$

Proof. Writing $f(z) = \sum_{m=0}^{\infty} c(m) e^{2\pi i m z}$, we get

$$\begin{aligned}
T_n f(z) &= n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} \sum_{m=0}^{\infty} c(m) e^{2\pi i m n z/d^2} e^{2\pi i m b/d} \\
&= \sum_{m=0}^{\infty} n^{k-1} \sum_{d|n} d^{-k} c(m) e^{2\pi i m n z/d^2} \sum_{b=0}^{d-1} (e^{2\pi i m/d})^b \\
&= \sum_{q=0}^{\infty} \sum_{d|n} \left(\frac{n}{d}\right)^{k-1} c(qd) e^{2\pi i q n z/d} \ (d \mid m \text{ so write } m = qd) \\
&= \sum_{q=0}^{\infty} \sum_{d|n} d^{k-1} c\left(\frac{qn}{d}\right) e^{2\pi i q d z} \ (d \text{ runs over all divisors of } n, \text{ then so does } \frac{n}{d}) \\
&= \sum_{m=0}^{\infty} \sum_{d|(m,n)} d^{k-1} c\left(\frac{mn}{d^2}\right) e^{2\pi i m z}.
\end{aligned}$$

□

Observe that writing $n = ad$ and letting $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, equation (1.2) takes the form

$$(T_n f)(z) = n^{k-1} \sum_{\substack{a \geq 1, ad=n \\ 0 \leq b < d}} d^{-k} f(Az) = \frac{1}{n} \sum_{\substack{a \geq 1, ad=n \\ 0 \leq b < d}} a^k f(Az).$$

Let

$$\Delta_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \, a, b, d \in \mathbb{Z}, \, ad = n, \, 0 \leq b < d \right\}.$$

To determine the behavior of $T_n f$ under transformations of Γ, we need to study the set Δ_n. One obtains (See [6] Theorem 6.9):

Lemma 1.2.1. *If $A_1 \in \Delta_n$ and $V_1 \in \Gamma$, then there exist matrices $A_2 \in \Delta_n$ and $V_2 \in \Gamma$ such that*

$$A_1 V_1 = V_2 A_2.$$

Moreover, if

$$A_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} \text{ and } V_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$$

for $i = 1, 2$, then we have

$$a_1(\gamma_2 A_2 z + \delta_2) = a_2(\gamma_1 z + \delta_1).$$

Theorem 1.2.2. *If $f \in M_k$ and $V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, then*

$$T_n f(Vz) = (\gamma z + \delta)^k T_n f(z).$$

Proof. For $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Delta_n$ we have

$$(T_n f)(Vz) = \frac{1}{n} \sum_{\substack{a \geq 1, ad = n \\ 0 \leq b < d}} a^k f(AVz).$$

By Lemma 1.2.1, there exists $A_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} \in \Delta_n$ and $V_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \in \Gamma$ such that $AV = V_2 A_2$. As a result,

$$\begin{aligned}
(T_n f)(Vz) &= \frac{1}{n} \sum_{\substack{a \geq 1, ad = n \\ 0 \leq b < d}} a^k f(V_2(A_2 z)) = \frac{1}{n} \sum_{\substack{a \geq 1, ad = n \\ 0 \leq b < d}} a^k (\gamma_2 (A_2 z) + \delta_2)^k f(A_2 z) \\
&= \frac{1}{n} \sum_{\substack{a_2 \geq 1, a_2 d_2 = n \\ 0 \leq b_2 < d_2}} a_2^k (\gamma z + \delta)^k f(A_2 z) \quad \text{(from the previous lemma)} \\
&= (\gamma z + \delta)^k \frac{1}{n} \sum_{\substack{a_2 \geq 1, a_2 d_2 = n \\ 0 \leq b_2 < d_2}} a_2^k f(A_2 z) \\
&= (\gamma z + \delta)^k T_n f(z).
\end{aligned}$$

□

Thus, from the last two theorems, we obtain:

Theorem 1.2.3. *If $f \in M_k$ then $T_n f \in M_k$. Moreover, if f is a cusp form, then $T_n f$ is also a cusp form.*

Definition 1.2.2. *A non-zero function f satisfying a relation of the form*

$$T_n f = \lambda(n) f$$

for some complex scalar $\lambda(n)$ is called an eigenform of the operator T_n, and the scalar $\lambda(n)$ is called an eigenvalue of T_n. Moreover, if f is an eigenform for every Hecke operator $T_n, n \geq 1$, then f is called a simultaneous eigenform. A simultaneous eigenform is said to be normalized if $c(1) = 1$, where $f(z) = \sum_{m=0}^{\infty} c(m) e^{2\pi i m z}$.

Remark. *If f is an eigenform then so is cf for every constant $c \neq 0$. So, if M_k contains a simultaneous eigenform, then it also contains a normalized eigenform.*

We saw that

$$\gamma_n(m) = \sum_{d | (n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right).$$

From this, we get

$$\gamma_n(0) = \sum_{d|n} d^{k-1} c(0) = \sigma_{k-1}(n) c(0)$$

and

$$\gamma_n(1) = c(n)$$

Theorem 1.2.4. *Let $k > 0$ be even. If the space M_k contains a simultaneous Hecke eigenform f with $f(z) = \sum_{m=0}^{\infty} c(m)e^{2\pi imz}$, then $c(1) \neq 0$.*

Proof. We have

$$T_n f = \lambda(n) f, \ \forall n \geq 1.$$

Equating coefficients in the corresponding Fourier expansions, we get that $\gamma_n(m) = \lambda(n)c(m)$, and in particular $\gamma_n(1) = \lambda(n)c(1)$. But we saw that $\gamma_n(1) = c(n)$. Then we have

$$c(n) = \lambda(n)c(1), \ \forall n \geq 1.$$

If $c(1) = 0$, then $c(n) = 0, \forall n \geq 1$ and so $f(z) = c(0)$. Since $k > 0$ the only constant modular form is the zero function, so $f \equiv 0$. But this contradicts the definition of eigenform. Therefore, $c(1) \neq 0$. □

Remark. *For the case of a normalized simultaneous Hecke eigenform, $c(n) = \lambda(n)c(1) = \lambda(n)$. Hence, the n-th Fourier coefficient of f is the same as its n-th eigenvalue.*

Theorem 1.2.5. *Let k be an even integer and $0 \neq f \in S_k$ with $f(z) = \sum_{m=1}^{\infty} c(m)e^{2\pi imz}$. Then f is a normalized simultaneous eigenform if and only if*

$$c(m)c(n) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right)$$

for all $m, n \geq 1$.

Proof. Given f normalized simultaneous, we saw that $c(n) = \lambda(n)$. We also saw that $\gamma_n(m) = \lambda(n)c(m)$. These give us

$$\gamma_n(m) = c(n)c(m)$$

as desired. Conversely, if $\gamma_n(m) = c(n)c(m)$ then

$$T_n f(z) = \sum_{m=1}^{\infty} \gamma_n(m)x^m = \sum_{m=1}^{\infty} c(n)c(m)x^m = c(n)\sum_{m=1}^{\infty} c(m)x^m = c(n)f(z);$$

i.e. f is a simultaneous eigenform. Moreover,

$$c(1) = \gamma_1(1) = c(1)c(1) = c(1)^2.$$

As $c(1) \neq 0$ by Theorem 1.2.4, the result follows. □

1.3 L-functions of Eigenforms

Definition 1.3.1. *If $f(z) = c(0) + \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$, we define the Dirichlet L-function of f as*

$$L(f, s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

Proposition 1.3.1. *If $f \in S_k$, then its L-function $L(f,s)$ converges absolutely for $\Re(s) > 1 + \frac{k}{2}$.*

Proof. We have $f(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z}$ with $c(n) \leq c n^{\frac{k}{2}}$ for some $c \in \mathbb{R}$, by Theorem 1.1.1. Then,

$$|L(f,s)| = \left| \sum_{n=1}^{\infty} \frac{c(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{|c(n)|}{n^{\Re(s)}} \leq c \sum_{n=1}^{\infty} \frac{1}{n^{\Re(s) - \frac{k}{2}}}.$$

Therefore, $L(f,s)$ converges absolutely for $\Re(s) > 1 + \frac{k}{2}$. □

Theorem 1.3.1. *Let $f \in S_k$ be a cusp form with associated L-series $L(f,s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$. If f is a normalized Hecke eigenform, then*

$$L(f,s) = \prod_{p\, prime} \frac{1}{1 - c(p)p^{-s} + p^{k-1-2s}}.$$

Proof. From Theorem 1.2.5, we have that

$$c(m)c(n) = \sum_{d|(n,m)} d^{k-1} c(\frac{mn}{d^2}). \tag{1.3}$$

Write $n = p_1^{i_1} \dots p_l^{i_l}$ for $l \in \mathbb{N}$ and $p_1, \dots, p_l$ distinct primes. Then

$$\begin{aligned} L(f,s) &= \sum_{n=1}^{\infty} \frac{c(n)}{n^s} \\ &= c(1) + \sum_{l \geq 1} \sum_{\substack{i_1, \dots, i_l \in \mathbb{N} \\ p_1, \dots, p_l \text{ distinct primes}}} \frac{c(p_1^{i_1} \dots p_l^{i_l})}{(p_1^{i_1} \dots p_l^{i_l})^s} \\ &= 1 + \sum_{l \geq 1} \sum_{\substack{i_1, \dots, i_l \in \mathbb{N} \\ p_1, \dots, p_l \text{ distinct primes}}} \frac{c(p_1^{i_1}) \dots c(p_l^{i_l})}{p_1^{i_1 s} \dots p_l^{i_l s}} \\ &= \prod_{p\, prime} \sum_{i \in \mathbb{N}_0} \frac{c(p^i)}{p^{is}} \end{aligned}$$

where we used (1.3) and $c(1) = 1$. So we want to prove that

$$\sum_{i \in \mathbb{N}_0} \frac{c(p^i)}{p^{is}} = \frac{1}{1 - c(p)p^{-s} + p^{k-1-2s}}. \tag{1.4}$$

We will prove (1.4) for any $\sum_{i\in\mathbb{N}_0} c(p^i)\alpha^i$ where in our situation $\alpha = p^{-s}$:

$$\begin{aligned}
&(1 - c(p)\alpha + p^{k-1}\alpha^2) \sum_{i\in\mathbb{N}_0} c(p^i)\alpha^i \\
&= \sum_{i\in\mathbb{N}_0} c(p^i)\alpha^i - \sum_{i\in\mathbb{N}_0} c(p)c(p^i)\alpha^{i+1} + \sum_{i\in\mathbb{N}_0} p^{k-1}c(p^i)\alpha^{i+2} \\
&= c(1) + c(p)\alpha + \sum_{i\geq 2} c(p^i)\alpha^i - c(p)c(1)\alpha - \sum_{i\geq 1} c(p)c(p^i)\alpha^{i+1} + \sum_{i\in\mathbb{N}_0} p^{k-1}c(p^i)\alpha^{i+2} \\
&= 1 + \sum_{i\geq 2} c(p^i)\alpha^i - \sum_{i\geq 1}(c(p)c(p^i) - p^{k-1}c(p^{i-1}))\alpha^{i+1} \quad \text{(since } c(1) = 1) \\
&= 1 + \sum_{i\geq 2} c(p^i)\alpha^i - \sum_{i\geq 1} c(p^{i+1})\alpha^{i+1} \quad \text{(using (1.3))} \\
&= 1 + \sum_{i\geq 2} c(p^i)\alpha^i - \sum_{i\geq 2} c(p^i)\alpha^i \\
&= 1.
\end{aligned}$$

□

Definition 1.3.2. *For $f \in S_k$, define the completed L-function $\Lambda(f, s)$ of f by taking the Mellin transform of f along the upper imaginary axis i.e.*

$$\Lambda(f, s) = \int_0^\infty f(iy)y^{s-1}\, dy.$$

Proposition 1.3.2. *$\Lambda(f, s)$ is well defined for all $s \in \mathbb{C}$.*

Proof. See Lemma 1.207 of [7]. □

Theorem 1.3.2. *We have*

$$\Lambda(f, s) = \frac{\Gamma(s)}{(2\pi)^s} L(f, s)$$

for $\Re(s) > 1 + \frac{k}{2}$, where

$$\Gamma(s) = \int_0^\infty e^{-y}y^{s-1}\, dy$$

is the Euler gamma function.

Proof. Let $s \in \mathbb{H}$ with $\Re(s) > 1 + \frac{k}{2}$. Since $a(n) = O(n^{\frac{k}{2}})$, we can apply dominated convergence to get

$$\begin{aligned}
\Lambda(f, s) &= \int_0^\infty f(iy)y^{s-1}\, dy \\
&= \int_0^\infty \sum_{n=1}^\infty a(n)e^{-2\pi ny}y^{s-1}\, dy \\
&= \int_0^\infty \sum_{n=1}^\infty (2\pi n)^{-s}a(n)e^{-\tau}\tau^{s-1}\, dy \quad \text{(by substituting } \tau = 2\pi ny) \\
&= (2\pi)^{-s} \sum_{n=1}^\infty \frac{a(n)}{n^s} \int_0^\infty e^{-\tau}\tau^{s-1}\, dy \\
&= \frac{\Gamma(s)}{(2\pi)^s} L(f, s).
\end{aligned}$$

□

Theorem 1.3.3. *$\Lambda(f,s)$ extends holomorphically to the complex plane and satisfies the functional equation*

$$\Lambda(f,s) = \epsilon(f)\Lambda(f,k-s)$$

for all $s \in \mathbb{C}$, where $\epsilon(f) = \pm 1$.

Proof. By Proposition 1.3.2, $\Lambda(f,s)$ exists $\forall s \in \mathbb{C}$; and by Theorem 1.3.2, $\Lambda(f,s)$ is holomorphic for $\Re(s) > 1 + \frac{k}{2}$. Also, since $L(f,s)$ and $\Gamma(s)$ are holomorphic on a suitable right half-plane, we get that $\Lambda(f,s)$ extends holomorphically to $\mathbb{C}$.
Now let $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $T \in \Gamma$ and $f(\frac{-1}{z}) = f(Tz) = z^k f(z)$ for all $z \in \mathbb{H}$. Using this, we get

$$\begin{aligned}
\Lambda(f,s) &= \int_0^\infty f(iy)y^{s-1}\,dy \\
&= -\int_\infty^0 f(\frac{-1}{iy})y^{1-s}y^{-2}\,dy \quad \text{(by substituting } y \to \frac{1}{y}) \\
&= \int_0^\infty (iy)^k f(iy) y^{-1-s}\,dy \\
&= i^k \int_0^\infty f(iy)y^{k-1-s}\,dy \\
&= i^k \Lambda(f,k-s).
\end{aligned}$$

□

Corollary 1.3.1. *If $f \in S_k$ and $k \equiv 2 \pmod 4$, then $\Lambda(f,\frac{k}{2}) = 0 = L(f,\frac{k}{2})$.*

Corollary 1.3.2. *$L(f,s)$ extends to a holomorphic function on $\mathbb{C}$ and satisfies the functional equation*

$$\frac{(2\pi)^{k-s}}{\Gamma(k-s)}L(f,s) = i^k \frac{(2\pi)^s}{\Gamma(s)}L(f,k-s)$$

for all $s \in \mathbb{C}$.

From this functional equation, we see that if ρ is a zero of $L(f,s)$ then so is $k-\rho$. Moreover, since $L(f,\bar{s}) = \overline{L(f,s)}$, then if ρ is a zero then so too is $\bar{\rho}$. These symmetrical properties of the zeros suggest that the critical line $\Re(s) = \frac{k}{2}$ is a natural line of symmetry for the L-functions. In fact, all the zeros of $L(f,s)$ lie in the strip $|\Re(s) - \frac{k}{2}| < \frac{1}{2}$, with the grand/generalized Riemann hypothesis (GRH) predicting that all the zeros are on the line $\Re(s) = \frac{k}{2}$ (cf. chapter 4 in [8]).

1.4 Period Polynomials

Definition 1.4.1. *For $X \in \mathbb{C}$ and a cusp form $f \in S_k$ we define the period polynomial of f by the integral transformation*

$$r_f(X) = \int_0^{i\infty} (z-X)^{k-2} f(z)\,dz.$$

Proposition 1.4.1. *The period polynomial is indeed a polynomial of degree less than or equal to $k-2$.*

Proof. Using the binomial formula, we have

$$(z-X)^{k-2} = \sum_{l=0}^{k-2} \binom{k-2}{l} z^l (-X)^{k-2-l}.$$

It follows that

$$\begin{aligned} r_f(X) &= \int_0^{i\infty} \sum_{l=0}^{k-2} \binom{k-2}{l} z^l (-X)^{k-2-l} f(z)\, dz \\ &= \sum_{l=0}^{k-2} \binom{k-2}{l} \int_0^{i\infty} z^l f(z)\, dz\, (-1)^{k-2-l} X^{k-2-l} \\ &= \sum_{l=0}^{k-2} a_l X^{k-2-l} \end{aligned}$$

where $a_l = \sum_{l=0}^{k-2} \binom{k-2}{l} (-1)^{k-2-l} \int_0^{i\infty} z^l f(z)\, dz$. □

Theorem 1.4.1. *For $f \in S_k$ and $X \in \mathbb{C}$ we have*

$$\begin{aligned} r_f(X) &= \sum_{l=0}^{k-2} \binom{k-2}{l} (-X)^{k-l-2} i^{l+1} \Lambda(f, l+1) \\ &= -\sum_{l=0}^{k-2} \binom{k-2}{l} X^l (-i)^{k-1-l} \Lambda(f, k-l-1). \end{aligned} \tag{1.5}$$

Proof. We again use the binomial formula

$$(z-X)^{k-2} = \sum_{l=0}^{k-2} \binom{k-2}{l} z^l (-X)^{k-2-l}$$

to get

$$\begin{aligned} r_f(X) &= \int_0^{i\infty} (z-X)^{k-2} f(z)\, dz \\ &= \int_0^{i\infty} \sum_{l=0}^{k-2} \binom{k-2}{l} z^l (-X)^{k-2-l} f(z)\, dz \\ &= \sum_{l=0}^{k-2} \binom{k-2}{l} (-X)^{k-2-l} \int_0^{i\infty} z^l f(z)\, dz \\ &= \sum_{l=0}^{k-2} \binom{k-2}{l} (-X)^{k-2-l} i^{l+1} \int_0^{\infty} y^l f(iy)\, dy \quad \text{(by substituting } iy = z) \\ &= \sum_{l=0}^{k-2} \binom{k-2}{l} (-X)^{k-l-2} i^{l+1} \Lambda(f, l+1). \end{aligned}$$

The second identity of (1.5) follows similarly using the binomial formula

$$(z-X)^{k-2}=(X-z)^{k-2}=\sum_{l=0}^{k-2}\binom{k-2}{l}X^l(-z)^{k-2-l}$$

since k is even. $\square$

Therefore, period polynomials are in fact the generating functions for the critical values of $L(f,s)$:

Corollary 1.4.1. *For $f\in S_k$ and $X\in\mathbb{C}$ we have*

$$r_f(X)=-\sum_{l=0}^{k-2}\frac{(k-2)!}{l!}\frac{L(f,k-l-1)}{(2\pi i)^{k-l-1}}X^l.$$

Proof. From Theorem 1.3.2, we have that

$$\Lambda(f,k-l-1)=\frac{(k-l-2)!}{(2\pi)^{k-l-1}}L(f,k-1-l).$$

Using this in the second equality of (1.5), the result follows immediately. $\square$

Finally, we show that period polynomials satisfy the following functional equation:

Theorem 1.4.2. *Let $f\in S_k$ and $X\in\mathbb{C}$. Then the period polynomial of f satisfies*

$$r_f(X)=-i^k\epsilon(f)X^{k-2}r_f\left(-\frac{1}{X}\right).$$

Proof. From the second equality of (1.5), we have that

$$\begin{aligned}
r_f(X)&=-\sum_{l=0}^{k-2}\binom{k-2}{l}X^l i^{l+1-k}\Lambda(f,k-l-1)\\
&=-\sum_{l=0}^{k-2}\binom{k-2}{k-2-l}X^{k-2-l}i^{-l-1}\Lambda(f,l+1)\quad(\text{substituting } l\to k-2-l)\\
&=-\epsilon(f)\sum_{l=0}^{k-2}\binom{k-2}{l}X^{k-2-l}i^{-l-1}\Lambda(f,k-l-1)\quad(\text{by the functional equation})\\
&=-i^k\epsilon(f)X^{k-2}\sum_{l=0}^{k-2}\binom{k-2}{l}X^{-l}i^{l+1-k}i^{-2-2l}\Lambda(f,k-l-1)\\
&=-i^k\epsilon(f)X^{k-2}r_f\left(-\frac{1}{X}\right).
\end{aligned}$$

$\square$

This "self-inversive" property of the period polynomial, shows that if ρ is a zero of $r_f(X)$ then so is $-\frac{1}{\rho}$; and so the unit circle is a natural line of symmetry for the period polynomials just as the critical line $\Re(s)=\frac{k}{2}$ is a natural line of symmetry for the completed L-function; and that is what we seek to prove in the next chapter.

Chapter 2

RHPP of Modular Forms

In this chapter, we prove that the zeros of period polynomials of any level N eigenform lie on the circle $|z| = 1/\sqrt{N}$.

2.1 The Case of the Full Modular Group

We treat the case when $N = 1$ separately. We start by recalling the well-known *Rouché's Theorem*, which we will find useful throughout.

Lemma 2.1.1. *(Rouché's Theorem)*
Let f and g be two functions holomorphic inside a simple closed curve C, and continuous on C. If $|f| < |g|$ on C, then g and $f+g$ have the same number of zeros inside C counted with multiplicities.

We next define what it means for a polynomial to be *self-inversive*:

Definition 2.1.1. *A polynomial $P(z) = \sum_{i=0}^{d} c_i z^i$ of degree d is said to be self-inversive if it satisfies*

$$P(z) = \epsilon z^d \bar{P}\Big(\frac{1}{z}\Big)$$

for some constant ϵ of modulus 1, where $\bar{P}(z) := \sum_{i=0}^{d} \bar{c}_i z^i$ and the bar denotes complex conjugation.

Note that the (perhaps more familiar) class of *self-reciprocal* polynomials is the special case of this definition when $\bar{P} = P$ and $\epsilon = 1$.

Lemma 2.1.2. *Let $h(z)$ be a nonzero polynomial of degree n with all its zeros in $|z| \leq 1$. Then for $d \geq n$ and any λ with $|\lambda| = 1$, the self-inversive polynomial*

$$P^{\{\lambda\}}(z) = z^{d-n} h(z) + \lambda z^n \bar{h}\Big(\frac{1}{z}\Big)$$

has all its zeros on the unit circle.

Proof. Write $h^*(z) := z^n \bar{h}(1/z)$, and temporarily assume that all n zeros of h are in the open disc $|z| < 1$. We can then write

$$h(z) = \prod_{i=1}^{n} (z - a_i)$$

with $|a_i| < 1$ for all $i \leq n$. It follows that

$$\bar{h}\left(\frac{1}{z}\right) = \prod_{i=1}^{n}\left(\frac{1}{\bar{z}} - \bar{a}_i\right),$$

and so the n zeros of h^* are $1/a_i$ which are in $|z| > 1$. Note also that $z^{d-n}h(z)$ has all its d zeros inside $|z| < 1$. If $\lambda < 1$, set $f(z) = z^{d-n}h(z)$ and $g(z) = \lambda h^*(z)$. Then for $|z| = 1$,

$$|g(z)| < |h^*(z)| = |z^n \bar{h}(1/z)| = |\bar{h}(\bar{z})| = |\overline{h(z)}| = |h(z)| = |f(z)|.$$

Since f has its d zeros in $|z| < 1$, by Rouché's Theorem we deduce that $P^{\{\lambda\}}$ has all its d zeros in $|z| < 1$. If $\lambda > 1$, we can switch the choice of f and g to deduce that $P^{\{\lambda\}}$ has no zeros in $|z| < 1$ (as f has no zeros there), and hence all d of its zeros must be in $|z| > 1$. As the zeros of $P^{\{\lambda\}}$ are continuous functions of λ, we see that, for $\lambda = 1$, $P^{\{\lambda\}}$ must have all its zeros on the unit circle. The result under the weaker assumption that h has all its zeros in the closed unit disc $|z| \leq 1$ follows from continuity of the zeros of $P^{\{\lambda\}}$ as functions of the zeros of h. □

Given a cusp form $f(\tau) = \sum_{n=1}^{\infty} a(n)q^n \in S_k$ (where $q = e^{2\pi i\tau}$). For $w = k - 2 \in 2\mathbb{N}$, we saw in Corollary 1.4.1 that

$$r_f(X) = -\sum_{n=0}^{w} \frac{w!}{n!}\frac{L(f, w-n+1)}{(2\pi i)^{w-n+1}}X^n = -\frac{w!}{(2\pi i)^{w+1}}\sum_{n=0}^{w} L(f, w-n+1)\frac{(2\pi i X)^n}{n!}.$$

For convenience, we consider the polynomial with real coefficients

$$p_f(X) = -\frac{(2\pi i)^{w+1}}{w!}r_f\left(\frac{X}{i}\right) = \sum_{n=0}^{w} L(f, w-n+1)\frac{(2\pi X)^n}{n!}.$$

Proposition 2.1.1. *$p_f(X)$ is self-inversive and can be written as*

$$p_f(X) = q_f(X) + i^k X^w q_f\left(\frac{1}{X}\right)$$

where

$$q_f(X) = \sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi X)^n}{n!} + \frac{1}{2}L(f, k/2)\frac{(2\pi X)^{w/2}}{(w/2)!}.$$

Proof. Using Theorem 1.4.2, we see that

$$p_f(X) = \frac{(2\pi i)^{w+1}}{w!}i^k\epsilon(f)\left(\frac{X}{i}\right)^{k-2} r_f\left(-\frac{i}{X}\right) = i^{k+w}i^k X^w\frac{(2\pi i)^{w+1}}{w!}r_f\left(\frac{1}{iX}\right) = i^k X^w p_f\left(\frac{1}{X}\right).$$

Hence, $p_f(X)$ is self-inversive. Now, we have that

$$\begin{aligned}
&p_f(X) - X^w q_f\Big(\frac{1}{X}\Big)\\
&= \sum_{n=0}^{w} L(f, w-n+1)\frac{(2\pi X)^n}{n!} - \sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi)^n}{n!}X^{w-n} - \frac{1}{2}L(f, \frac{k}{2})\frac{(2\pi X)^{w/2}}{(w/2)!}\\
&= \sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi X)^n}{n!} + \sum_{n=\frac{w}{2}+1}^{w} L(f, w-n+1)\frac{(2\pi X)^n}{n!}\\
&+ \frac{1}{2}L(f, \frac{k}{2})\frac{(2\pi X)^{w/2}}{(w/2)!} - \sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi)^n}{n!}X^{w-n}\\
&= q_f(X) + \sum_{n=\frac{w}{2}+1}^{w} L(f, w-n+1)\frac{(2\pi X)^n}{n!} - \sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi)^n}{n!}X^{w-n}.
\end{aligned} \tag{2.1}$$

Note that

$$\begin{aligned}
&\sum_{n=\frac{w}{2}+1}^{w} L(f, w-n+1)\frac{(2\pi X)^n}{n!} = \sum_{n=0}^{\frac{w}{2}-1} L(f, n+1)\frac{(2\pi X)^{w-n}}{(w-n)!} \quad \text{(substituting } n \to w-n\text{)}\\
&= \sum_{n=0}^{\frac{w}{2}-1} i^k L(f, w-n+1)\frac{(2\pi)^n}{n!}X^{w-n} \quad \text{(by the functional equation)}
\end{aligned}$$

Using this in (2.1) it follows that, for $i^k = 1$

$$p_f(X) - X^w q_f\Big(\frac{1}{X}\Big) = q_f(X).$$

And for $i^k = -1$, we have by Corollary 1.3.1 that $L(f, \frac{k}{2}) = 0$, and so

$$q_f(X) = \sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi X)^n}{n!},$$

which gives in (2.1)

$$\begin{aligned}
p_f(X) &= q_f(X) - 2\sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1)\frac{(2\pi)^n}{n!}X^{w-n} + X^w q_f\Big(\frac{1}{X}\Big)\\
&= q_f(X) - X^w q_f\Big(\frac{1}{X}\Big).
\end{aligned}$$

□

It is clear that $r_f(X)$ would have all its zeros on $|z| = 1$ if and only if the same is true for $p_f(X)$. By Proposition 2.1.1 and Lemma 2.1.2 it suffices to prove that $q_f(X)$ has all its zeros in $|z| \le 1$. To that end, set

$$H_m(z) = \sum_{n=0}^{m} \frac{(2\pi)^n}{n!} z^{m-n}.$$

Proposition 2.1.2. *For $m \geq 25$, $H_m(z)$ has all its zeros in $|z| < 1$.*

Proof. Write $H_m(z) = z^{m-25} H_{25}(z) + g_m(z)$ where

$$g_m(z) = \sum_{n=26}^{m} \frac{(2\pi)^n}{n!} z^{m-n}.$$

For $|z| = 1$, we have

$$|g_m(z)| \leq \sum_{n=26}^{m} \frac{(2\pi)^n}{n!} \leq \sum_{n=26}^{\infty} \frac{(2\pi)^n}{n!} = e^{2\pi} - H_{25}(1) \leq 0.000001823 < |H_{25}(z)|.$$

Hence, $g_m(z)$ and $z^{m-25}H_{25}(z)$ are holomorphic in $|z| \leq 1$ with $|g_m(z)| < |z^{m-25}H_{25}(z)|$. Then by Rouché's Theorem, $z^{m-25}H_{25}(z)$ and $g_m(z) + z^{m-25}H_{25}(z) = H_m(z)$ have the same number of zeros inside the unit disc. Numerical verification using PARI [16] gives that $H_{25}(z)$ has all its 25 zeros inside the unit disc. Therefore $z^{m-25}H_{25}(z)$ has all its m zeros inside the unit disc, and so does $H_m(z)$. $\square$

Next, we will prove the following useful estimates for L-functions.

Lemma 2.1.3. *Let $f \in S_k$ be a normalized Hecke eigenform and let $L(f, s)$ be its associated L-function. Then, for $s \geq 3k/4$, we have*

$$|L(f, s) - 1| \leq 5 \times 2^{-k/4}$$

and, for $s \geq k/2$, we have

$$L(f, s) \leq 1 + 4\sqrt{k} \log(2k).$$

Proof. We will use the following useful bound due to Deligne (see [9] for a proof):

$$|L(f, s)| = \left| \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{d(n)}{n^{s-(k-1)/2}},$$

where $d(n) = \sum_{d|n} 1$ is the divisor function. We will also use the fact (also found in [9]) that

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s},$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function.

Then, if $s \geq 3k/4$ we get

$$|L(f, s) - 1| = \left| \sum_{n=2}^{\infty} \frac{a(n)}{n^s} \right| \leq \sum_{n=2}^{\infty} \frac{d(n)}{n^{s-(k-1)/2}} \leq \sum_{n=2}^{\infty} \frac{d(n)}{n^{k/4}} = \zeta(k/4)^2 - 1.$$

Now for $k \geq 12$, we see that

$$\zeta(k/4)^2 - 1 = (\zeta(k/4) + 1)(\zeta(k/4) - 1) \leq (\zeta(3) + 1)(\zeta(k/4) - 1) < \frac{5}{2}(\zeta(k/4) - 1);$$

and

$$\zeta(k/4)-1=\sum_{n=2}^{\infty}\frac{1}{n^{k/4}}=2^{-k/4}+\sum_{n=3}^{\infty}\frac{d(n)}{n^{k/4}}\leq 2^{-k/4}+\int_2^{\infty}u^{-k/4}\,du\leq 2\times 2^{-k/4}.$$

Therefore,

$$|L(f,s)-1|\leq 5\times 2^{-k/4}.$$

Next, if $s\geq k/2+1$, we get for $k\geq 2$

$$L(f,s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}\leq\sum_{n=1}^{\infty}\frac{d(n)}{n^{s-(k-1)/2}}\leq\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/2}}=\zeta(3/2)^2<7<4k^{1/2}\log 2k+1.$$

If $k/2\leq s\leq k/2+1$, then

$$|L(f,s)|\leq|L(f,k/2)|.$$

Note that from Theorem 1.3.2, we have

$$\begin{aligned}\Gamma(k/2)L(f,k/2)&=(2\pi)^{k/2}\Lambda(f,k/2)\\&=(2\pi)^{k/2}\int_0^{\infty}f(ix)x^{\frac{k}{2}}\,\frac{dx}{x}\\&=(2\pi)^{k/2}\left(\int_0^1 f(ix)x^{\frac{k}{2}}\,\frac{dx}{x}+\int_1^{\infty}f(ix)x^{\frac{k}{2}}\,\frac{dx}{x}\right).\end{aligned}$$

But substituting x by $\frac{1}{x}$ and since $f(\frac{-1}{x})=x^kf(x)$ we see that the first integral is equal to i^k times the second one

$$\int_0^1 f(ix)x^{\frac{k}{2}}\,\frac{dx}{x}=\int_0^1 f\left(\frac{-1}{ix}\right)x^{-\frac{k}{2}-1}\,dx=i^k\int_1^{\infty}f(ix)(x)^{\frac{k}{2}-1}\,dx;$$

and using the Fourier expansion of f, the second integral becomes

$$\int_1^{\infty}f(ix)x^{\frac{k}{2}}\,\frac{dx}{x}=\sum_{n=1}^{\infty}a_n\int_1^{\infty}e^{-2\pi nx}x^{\frac{k}{2}}\,\frac{dx}{x}=\sum_{n=1}^{\infty}\frac{a_n}{n^{k/2}}(2\pi)^{-k/2}\int_{2\pi n}^{\infty}e^{-x}x^{\frac{k}{2}}\,\frac{dx}{x}.$$

Therefore, we get that

$$\Gamma(k/2)L(f,k/2)=(1+i^k)\sum_{n=1}^{\infty}\frac{a_n}{n^{k/2}}\int_{2\pi n}^{\infty}e^{-x}x^{\frac{k}{2}}\,\frac{dx}{x}.$$

Using this with Deligne's bound, it follows that

$$|L(f,s)|\leq|L(f,k/2)|\leq 2\Gamma(k/2)^{-1}\sum_{n=1}^{\infty}\frac{d(n)}{n^{1/2}}\int_{2\pi n}^{\infty}e^{-x}x^{\frac{k}{2}}\,\frac{dx}{x}.$$

We split the sum over n at k. The terms with $n\leq k$ are

$$\leq 2\sum_{n\leq k}\frac{d(n)}{n^{1/2}}$$

as is seen by completing the integrals down to 0. But notice that

$$\sum_{n\le k}\frac{d(n)}{n^{1/2}}=\sum_{n\le k}\sum_{d|n}\frac{1}{n^{1/2}}=\sum_{ed\le k}\frac{1}{(ed)^{1/2}};$$

and hence

$$\sum_{n\le k}\frac{d(n)}{n^{1/2}}=\sum_{mn\le k}\frac{1}{(mn)^{1/2}}\le\sum_{m\le k}\frac{1}{m^{1/2}}\int_0^{k/m}u^{-1/2}\,du=2k^{1/2}\sum_{m\le k}\frac{1}{m}\le 2k^{1/2}\log(2k)$$

for $k\ge 5$. The tail of the series is

$$\begin{aligned}&=2\Gamma(k/2)^{-1}\sum_{n=k+1}^{\infty}\frac{d(n)}{n^{1/2}}\int_{2\pi n}^{\infty}e^{-x/2}e^{-x/2}x^{\frac{k}{2}}\,\frac{dx}{x}\\&\le 2\Gamma(k/2)^{-1}\sum_{n=k+1}^{\infty}\frac{d(n)}{n^{1/2}}e^{-\pi n}\int_{2\pi n}^{\infty}e^{-x/2}x^{\frac{k}{2}}\,\frac{dx}{x}.\end{aligned}$$

The integral is

$$=2^{k/2}\int_{\pi n}^{\infty}e^{-x}x^{\frac{k}{2}}\,\frac{dx}{x}\le 2^{k/2}\Gamma(k/2),$$

and since $d(n)\le 2\sqrt{n}$ (see [9]) we have that the tail is

$$\le 4\times 2^{k/2}\sum_{n=k+1}^{\infty}e^{-\pi n}\le 4\times 2^{k/2}e^{-\pi k}<1.$$

This completes the proof. □

Theorem 2.1.1. *If $f\in S_k$ is a Hecke eigenform, then $r_f(X)$ has all its zeros on the unit circle.*

Proof. Note that for any $c\in\mathbb{C}$, we have $r_{cf}(X)=c.r_f(X)$; so we can take f to be normalized. We want to show that all the zeros of $q_f(X)$ are inside the unit circle. Let $m=k/2-1=w/2$, then for $|X|=1$ we have

$$\begin{aligned}|q_f(X)-H_m(X)|&=\left|\sum_{n=0}^{m-1}L(f,k-n-1)\frac{(2\pi X)^n}{n!}-\frac{(2\pi)^n}{n!}X^{m-n}+(\frac{1}{2}L(f,k/2)-1)\frac{(2\pi)^m}{m!}\right|\\&\le\sum_{n=0}^{m-1}\frac{(2\pi)^n}{n!}|L(f,k-n-1)-1|+\frac{(2\pi)^m}{m!}\left|\frac{1}{2}L(f,k/2)-1\right|\\&\le\sum_{n=0}^{m-1}\frac{(2\pi)^n}{n!}|L(f,k-n-1)-1|+\frac{(2\pi)^m}{m!}\big(|L(f,k/2)|+1\big)\end{aligned}$$

Now, using Lemma 2.1.3, we know that, for $k-n-1\ge 3k/4$ (i.e. for $n\le k/4-1$)

$$|L(f,k-n-1)-1|\le 5\times 2^{-k/4}$$

and, for $k-n-1 \geq k/2$ (i.e. for $n \leq m$),

$$L(f, k-n-1) \leq 1 + 4\sqrt{k}\log(2k).$$

It follows that

$$\begin{aligned}
|q_f(X) - H_m(X)| &\leq \sum_{n=0}^{[k/4]-1} \frac{(2\pi)^n}{n!}|L(f,k-n-1)-1| + \sum_{[k/4]}^{m-1} \frac{(2\pi)^n}{n!}|L(f,k-n-1)-1| \\
&\quad + \frac{(2\pi)^m}{m!}\big(|L(f,k/2)|+1\big) \\
&\leq \sum_{n=0}^{[k/4]-1} \frac{(2\pi)^n}{n!}|L(f,k-n-1)-1| + \sum_{[k/4]}^{m} \frac{(2\pi)^n}{n!}\big(|L(f,k-n-1)|+1\big) \\
&\leq \sum_{n=0}^{[k/4]-1} 5\times 2^{-k/4}\frac{(2\pi)^n}{n!} + \sum_{[k/4]}^{m} \big(2+4\sqrt{k}\log(2k)\big)\frac{(2\pi)^n}{n!} \\
&\leq 5\times 2^{-k/4}e^{2\pi} + \big(2+4\sqrt{k}\log(2k)\big)R_{[k/4]}(1)
\end{aligned}$$

Since $|X| = 1$ and by the well-known Taylor inequality, we have

$$|R_n(X)| = \Big|\frac{f^{(n+1)}(X)}{(n+1)!}\Big| \leq \frac{(2\pi)^{n+1}}{(n+1)!}e^{2\pi}$$

in particular,

$$R_{[k/4]}(1) \leq e^{2\pi}\frac{(2\pi)^{[k/4]}}{[k/4]!}.$$

Moreover, one can show that, for $k \geq 124$

$$\big(2+4\sqrt{k}\log(2k)\big)e^{2\pi}\frac{(2\pi)^{[k/4]}}{[k/4]!} \leq 0.000045$$

and

$$5e^{2\pi}2^{-k/4} \leq 0.0000025.$$

Therefore, for $k \geq 124$ (so $m > 25$) and $|X| = 1$, we have $|q_f(X) - H_m(X)| < |H_m(X)|$, and it follows from Rouché's theorem that $q_f(X)$ has the same number of zeros as $H_m(X)$ inside the unit circle, namely m by Proposition 2.1.2. For cusp forms with $12 \leq k \leq 122$, the result can be verified directly using PARI. The code is as follows:

```
mf(k)=mfinit([1,k],1)

B(k)=mfbasis(mf(k))

P(k)=mfperiodpol(mf(k),B(k)[1])

Z(k)=polroots(P(k))

default(parisizemax,1G)

ploth(t=0,100,apply(z->z+t*exp(I*t)/10^4,Z(k))~,4096)
```

□

2.2 Eigenforms of Higher Levels

We now turn to the general case (any $N \in \mathbb{N}$). We begin by giving a brief introduction to the theory of modular forms of higher levels. For a more detailed introduction, see [10].

2.2.1 Modular Forms on Congruence Subgroups

The principle subgroup of $SL_2(\mathbb{Z})$ of level $N \in \mathbb{N}$ is given by

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

Note that $\Gamma(1) = SL_2(\mathbb{Z})$. Moreover, $\Gamma(N)$ is normal and has a finite index in $\Gamma(1)$. Note also that if $N' \mid N$, then $\Gamma(N) \subset \Gamma(N') \subset \Gamma(1)$.

Definition 2.2.1. *A congruence subgroup is a subgroup of $SL_2(\mathbb{Z})$ that contains $\Gamma(N)$ for some $N \in \mathbb{N}$.*

We are particularly interested in the subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \mod N \right\},$$

$\Gamma_0(N)$ is a congruence subgroup of $SL_2(\mathbb{Z})$.

Definition 2.2.2. *Let G be a congruence subgroup and $\alpha \in G$. Then α is said to be parabolic if $|tr(\alpha)| = 2$.*

Definition 2.2.3. *A cusp of a congruence subgroup G is an element $z \in \mathbb{R} \cup \{\infty\}$ which is fixed by a parabolic element α of G, i.e. $\exists \alpha \in G$ parabolic such that $\alpha z = z$.*

Definition 2.2.4. *A modular form of weight $k \in \mathbb{Z}$ and level N is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying:*

- *$f(\gamma z) = (cz + d)^k f(z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$*
- *f is holomorphic at all the cusps of $\Gamma_0(N)$.*

Since $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$ for any N, we have that $f(Sz) = f(z+1) = f(z)$. So f has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

We denote by $M_k(\Gamma_0(N))$ the space of modular forms of weight k and level N.

Definition 2.2.5. *If $f \in M_k(\Gamma_0(N))$ and $f(z) \to 0$ as z tends to any cusp, then f is said to be a cusp form and we write $f \in S_k(\Gamma_0(N))$.*

A form $f \in S_k(\Gamma_0(N))$ is an *oldform* if $f(z) = g(dz)$ for some $g \in S_k(\Gamma_0(M))$ with $M \mid N$ and $d \mid N/M$. We say f is a *newform* if f is a normalized eigenform which is orthogonal to the space of oldforms (with respect to the Petersson inner product, cf. chapter 1 in [7]). The space of newforms of level N is denoted by $S_k^{\mathrm{new}}(\Gamma_0(N))$.

Let k be even and $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$. Associated to f is its L-function

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{p\,prime} (1 - a(p)p^{-s} + \mathbf{1}_N(p)p^{k-1-2s})^{-1}$$

where $\mathbf{1}_N(p)$ is 1 when $p \nmid N$ and is 0 when $p \mid N$. Its completed L-function is defined by

$$\Lambda(f,s) = N^{s/2} \int_0^{\infty} f(iy)y^{s-1}\,dy$$

satisfying, as before,

$$\Lambda(f,s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(f,s) \tag{2.2}$$

and the functional equation

$$\Lambda(f,s) = \epsilon(f)\Lambda(f,k-s),$$

with $\epsilon(f) = \pm 1$. The period polynomial associated to f is the degree $k-2$ polynomial

$$r_f(z) = \int_0^{i\infty} f(\tau)(\tau - z)^{k-2}\,d\tau,$$

which is again the generating function for the critical values of the L-function:

Theorem 2.2.1. *The period polynomial of f satisfies*

$$r_f(z) = i^{k-1}N^{-\frac{k-1}{2}} \sum_{n=0}^{k-2} \binom{k-2}{n} (\sqrt{N}iz)^n \Lambda(f,k-1-n).$$

Proof. Using the binomial expansion, we get

$$\begin{aligned}
r_f(X) &= \int_0^{i\infty} (z-X)^{k-2} f(z)\,dz \\
&= \int_0^{i\infty} \sum_{n=0}^{k-2} \binom{k-2}{n} z^n(-X)^{k-2-n} f(z)\,dz \\
&= \sum_{n=0}^{k-2} \binom{k-2}{n} (-X)^{k-2-n} \int_0^{i\infty} z^n f(z)\,dz \\
&= \sum_{n=0}^{k-2} \binom{k-2}{n} (-X)^{k-2-n} i^{n+1} \int_0^{\infty} y^n f(iy)\,dy \quad \text{(by substituting } iy = z\text{)} \\
&= \sum_{n=0}^{k-2} \binom{k-2}{n} (-X)^{k-n-2} i^{n+1} N^{-\frac{n+1}{2}} \Lambda(f,n+1) \\
&= i^{k-1}N^{-\frac{k-1}{2}} \sum_{n=0}^{k-2} \binom{k-2}{n} (\sqrt{N}iX)^n \Lambda(f,k-1-n) \quad \text{(substituting } n \to k-2-n\text{)}.
\end{aligned}$$

□

Corollary 2.2.1. *The period polynomial of f further satisfies*

$$r_f(z) = -\frac{(k-2)!}{(2\pi i)^{k-1}} \sum_{n=0}^{k-2} \frac{(2\pi i z)^n}{n!} L(f, k-n-1).$$

Proof. From 2.2, we have that

$$\Lambda(f,s) = \left(\frac{\sqrt{N}}{2\pi}\right)^{k-n-1} \Gamma(k-n-1)L(f,k-n-1).$$

Using this in the expression of r_f in Theorem 2.2.1, we get our result. □

2.2.2 Zeros of Period Polynomials

For $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$, put $m = \frac{k-2}{2}$ and define

$$P_f(z) = \frac{1}{2}\binom{2m}{m}\Lambda(f, \frac{k}{2}) + \sum_{j=1}^{m} \binom{2m}{m+j} \Lambda(f, \frac{k}{2}+j)z^j.$$

Then we have the following relation:

Proposition 2.2.1. *The period polynomial of f satisfies*

$$r_f\left(\frac{z}{i\sqrt{N}}\right) = i^{k-1}N^{-\frac{k-1}{2}}\epsilon(f)z^m\Big(P_f(z) + \epsilon(f)P_f\Big(\frac{1}{z}\Big)\Big).$$

Proof. From Theorem 2.2.1, we have

$$\begin{aligned}
r_f\left(\frac{z}{i\sqrt{N}}\right) &= i^{k-1}N^{-\frac{k-1}{2}} \sum_{n=0}^{k-2} \binom{k-2}{n} \Lambda(f, k-1-n)z^n \\
&= i^{k-1}N^{-\frac{k-1}{2}} \sum_{j=-m}^{m} \binom{2m}{m+j} \Lambda(f, \frac{k}{2}-j)z^{m+j} \quad \text{(substituting } n \to m+j\text{)} \\
&= i^{k-1}N^{-\frac{k-1}{2}}\epsilon(f)z^m \sum_{j=-m}^{m} \binom{2m}{m+j} \Lambda(f, \frac{k}{2}+j)z^j \quad \text{(by the functional equation)} \\
&= i^{k-1}N^{-\frac{k-1}{2}}\epsilon(f)z^m \Bigg[\sum_{j=-m}^{0} \binom{2m}{m+j} \Lambda(f, \frac{k}{2}+j)z^j + \sum_{j=1}^{m} \binom{2m}{m+j} \Lambda(f, \frac{k}{2}+j)z^j\Bigg] \\
&= i^{k-1}N^{-\frac{k-1}{2}}\epsilon(f)z^m \Bigg[\sum_{j=0}^{m} \binom{2m}{m+j} \Lambda(f, \frac{k}{2}-j)z^{-j} + P_f(z) - \frac{1}{2}\binom{2m}{m}\Lambda(f, \frac{k}{2})\Bigg] \\
&= i^{k-1}N^{-\frac{k-1}{2}}\epsilon(f)z^m \Bigg[\sum_{j=1}^{m} \binom{2m}{m+j} \Lambda(f, \frac{k}{2}-j)z^{-j} + P_f(z) + \frac{1}{2}\binom{2m}{m}\Lambda(f, \frac{k}{2})\Bigg] \\
&= i^{k-1}N^{-\frac{k-1}{2}}\epsilon(f)z^m \Bigg[\epsilon(f)P_f\Big(\frac{1}{z}\Big) - \epsilon(f)\frac{1}{2}\Lambda(f, \frac{k}{2}) + P_f(z) + \frac{1}{2}\binom{2m}{m}\Lambda(f, \frac{k}{2})\Bigg].
\end{aligned}$$

If $\epsilon(f) = 1$, the result is clear. If $\epsilon(f) = -1$, then note that $\Lambda(f, \frac{k}{2}) = 0$, and so the result follows. □

Therefore, $r_f(z)$ would have all its zeros on $|z| = 1/\sqrt{N}$ if and only if $P_f(z) + \epsilon(f)P_f(1/z)$ has all its zeros on the unit circle. For that purpose, we prove the following two lemmas about L-functions that we shall find useful.

Lemma 2.2.1. *Let $f \in S_k^{new}(\Gamma_0(N))$. Then the function $\Lambda(f,s)$ is monotone increasing for $s \geq \frac{k}{2} + \frac{1}{2}$. Moreover, we have*

$$0 \leq \Lambda(f, \frac{k}{2}) \leq \Lambda(f, \frac{k}{2}+1) \leq \Lambda(f, \frac{k}{2}+2) \leq \dots$$

If $\epsilon(f) = -1$, then $\Lambda(f, \frac{k}{2}) = 0$ and

$$0 \leq \Lambda(f, \frac{k}{2}+1) \leq \frac{1}{2}\Lambda(f, \frac{k}{2}+2) \leq \frac{1}{3}\Lambda(f, \frac{k}{2}+3) \leq \dots$$

Proof. $\Lambda(f,s)$ is an entire function of order 1, with its zeros all lying in the strip $|\Re(s) - \frac{k}{2}| < \frac{1}{2}$. Thus, Hadamard's factorization formula applies and we may write

$$\Lambda(f,s) = e^{A+Bs}\prod_{\rho}\left(1 - \frac{s}{\rho}\right)e^{s/\rho}. \tag{2.3}$$

Here the product is over all of the zeros of $\Lambda(f,s)$, and A and B are constants. Since $\Lambda(f,s)$ is real valued for real s, we see that $B \in \mathbb{R}$. Note also that if ρ is a zero then so too are $\bar{\rho}$ and $k-\rho$. Then using the functional equation for the case of $\epsilon(f) = 1$ (similarly if $\epsilon(f) = -1$), we have that

$$\begin{aligned}
&\frac{\Lambda(f,s)}{\Lambda(f,k-s)} = 1\\
&\Rightarrow \frac{e^{A+Bs}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho}}{e^{A+B(k-s)}\prod_\rho\left(1-\frac{k-s}{\rho}\right)e^{(k-s)/\rho}} = 1\\
&\Rightarrow e^{B(2s-k)}\prod_\rho\left(\frac{\rho-s}{\rho-k+s}\right)e^{2s-k/\rho} = 1\\
&\Rightarrow B(2s-k) + \sum_\rho \log\left(\frac{\rho-s}{\rho-k+s}\right) + \sum_\rho \frac{2s-k}{\rho} = 0\\
&\Rightarrow B(2s-k) + \frac{1}{2}\sum_\rho\left[\log\left(\frac{\rho-s}{\rho-k+s}\right) + \log\left(\frac{k-\rho-s}{s-\rho}\right)\right] + \frac{2s-k}{2}\sum_\rho\left(\frac{1}{\rho}+\frac{1}{\bar{\rho}}\right) = 0\\
&\Rightarrow B(2s-k) + \frac{1}{2}\sum_\rho \log\left(\frac{\rho-s}{\rho-k+s}\frac{k-\rho-s}{s-\rho}\right) + (2s-k)\sum_\rho\frac{\Re(\rho)}{|\rho|^2} = 0\\
&\Rightarrow B = -\sum_\rho \frac{\Re(\rho)}{|\rho|^2}.
\end{aligned}$$

Therefore, we get for real s

$$\begin{aligned}
\Lambda(f,s) &= e^A \prod_{\rho} \exp\left(-s\frac{\Re(\rho)}{|\rho|^2}\right)\left(1-\frac{s}{\rho}\right)e^{s/\rho} \\
&= e^A \prod_{\rho\in\mathbb{R}}\left(1-\frac{s}{\rho}\right)\prod_{\Im(\rho)>0}\exp\left(\frac{-s\Re(\rho)}{|\rho|^2}\right)\left(1-\frac{s}{\rho}\right)e^{s/\rho}\prod_{\Im(\rho)<0}\exp\left(\frac{-s\Re(\rho)}{|\rho|^2}\right)\left(1-\frac{s}{\rho}\right)e^{s/\rho} \\
&= e^A \prod_{\rho\in\mathbb{R}}\left(1-\frac{s}{\rho}\right)\prod_{\Im(\rho)>0}\exp\left(\frac{-s\Re(\rho)}{|\rho|^2}\right)\left(1-\frac{s}{\rho}\right)e^{s/\rho}\prod_{\Im(\rho)>0}\exp\left(\frac{-s\Re(\bar\rho)}{|\bar\rho|^2}\right)\left(1-\frac{s}{\bar\rho}\right)e^{s/\bar\rho} \\
&= e^A \prod_{\rho\in\mathbb{R}}\left(1-\frac{s}{\rho}\right)\prod_{\Im(\rho)>0}\exp\left(-2s\frac{\Re(\rho)}{|\rho|^2}\right)\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{\bar\rho}\right)\exp\left(2s\frac{\Re(\rho)}{|\rho|^2}\right) \\
&= e^A \prod_{\rho\in\mathbb{R}}\left(1-\frac{s}{\rho}\right)\prod_{\Im(\rho)>0}\left|1-\frac{s}{\rho}\right|^2.
\end{aligned} \tag{2.4}$$

Now, for $s \geq \frac{k}{2}+\frac{1}{2}$, we have that

$$|\rho - s|^2 = (\Re(\rho)-s)^2 + \Im(\rho)^2,$$

then for $s_1 < s_2$, and because all of the zeros lie in $|\Re(s)-\frac{k}{2}| < \frac{1}{2}$,

$$|\rho-s_1|^2 - |\rho-s_2|^2 = (s_1-s_2)(s_1+s_2-2\Re(\rho)) < 0$$

and thus $|1-s/\rho|$ is increasing in s. For $\Im(\rho)>0$, $|1-s/\rho|^2$ is increasing then so is the product above. For real ρ, since the number of zeros is even and $|1-s/\rho|$ is increasing then so is the product above. It follows that $\Lambda(f,s)$ is increasing for $s \geq \frac{k}{2}+\frac{1}{2}$. Further, we have

$$\left|1-\frac{k/2}{\rho}\right| \leq \left|1-\frac{k/2+1}{\rho}\right|,$$

and so $\Lambda(f,\frac{k}{2}) \leq \Lambda(f,\frac{k}{2}+1)$. In addition, the central value $\Lambda(f,\frac{k}{2})$ is known to be nonnegative by the work of Waldspurger [11].

When $\epsilon(f)=-1$, we additionally have a zero of odd order at $\frac{k}{2}$. In this case, we get the additional factor of

$$\left(1-\frac{s}{k/2}\right)$$

in the first product; and so

$$\left(1-\frac{k/2+\ell}{k/2}\right) = \frac{-2\ell}{k} \leq \frac{1}{\ell+1}\left(\frac{-2(\ell+1)}{k}\right) = \frac{1}{\ell+1}\left(1-\frac{k/2+(\ell+1)}{k/2}\right)$$

for any $\ell \geq 1$. Adding this to what we did above, we get our result. □

Lemma 2.2.2. *If $f \in S_k^{new}(\Gamma_0(N))$ and $0 < a \leq b$, then*

$$\frac{L(f,\frac{k+1}{2}+a)}{L(f,\frac{k+1}{2}+b)} \leq \frac{\zeta(1+a)^2}{\zeta(1+b)^2}$$

where $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function.

Proof. First, we compute

$$-\frac{\zeta'}{\zeta}(s) = -\frac{d}{ds}\log(\zeta(s)) = -\frac{d}{ds}\log\prod_p(1-p^{-s})^{-1} = \sum_p \frac{d}{ds}\log(1-p^{-s})$$
$$= \sum_p \frac{\log p}{p^s-1} = \sum_p \log p \sum_{m=1}^{\infty} p^{-ms} = \sum_{p,m} (p^m)^{-s}\log p = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$$

where $\Lambda(n) = \log p$ for $n = p^k$ and $\Lambda(n) = 0$ otherwise. Similarly, we have that

$$-\frac{L'}{L}(f,s) = \sum_{n=1}^{\infty}\frac{\Lambda_f(n)}{n^s}$$

where $|\Lambda_f(n)| \le 2n^{\frac{k-1}{2}}\Lambda(n)$ for all n. It follows that

$$-\frac{L'}{L}(f,s) = \sum_{n=1}^{\infty}\frac{\Lambda_f(n)}{n^s} \le \sum_{n=1}^{\infty}\frac{2n^{\frac{k-1}{2}}\Lambda(n)}{n^s} = -2\frac{\zeta'}{\zeta}\Big(s-\frac{k-1}{2}\Big).$$

Therefore, we have for $a < b$

$$\frac{L(f,\frac{k+1}{2}+a)}{L(f,\frac{k+1}{2}+b)} = \exp\left(\int_a^b -\frac{L'}{L}(f,\frac{k+1}{2}+t)\,dt\right) \le \exp\left(2\int_a^b -\frac{\zeta'}{\zeta}\Big(1+t\Big)\,dt\right) = \frac{\zeta(1+a)^2}{\zeta(1+b)^2}.$$

□

Remark. *Period polynomials for weight 2 newforms f are constant multiples of $L(f,1)$. Hence, we will consider $k \ge 4$.*

The weight 4 case is straightforward:

Theorem 2.2.2. *For $k = 4$, $P_f(z) + \epsilon(f)P_f(1/z)$ has all its zeros on the unit circle.*

Proof. Here $m = (k-2)/2 = 1$, so $P_f(z) = \Lambda(f,2) + \Lambda(f,3)z$.
If $\epsilon(f) = -1$, then the roots of $P_f(z) - P_f(1/z) = \Lambda(f,3)(z-1/z)$ are at $z = \pm 1$, which lie on the unit circle.
If $\epsilon(f) = 1$, then for $z = e^{i\theta}$ on the unit circle, $P_f(z) + P_f(1/z) = 2\Lambda(f,2) + \Lambda(f,3)(e^{i\theta} + e^{-i\theta}) = 2\Lambda(f,2) + 2\Lambda(f,3)\cos(\theta)$, which vanishes when $\cos(\theta) = -\Lambda(f,2)/\Lambda(f,3)$. By Lemma 2.2.1, $\Lambda(f,2) < \Lambda(f,3)$, and so the equation has two solutions for $\theta \in [0,2\pi)$. □

For the weight 6 case, we will first need the following observation:

Lemma 2.2.3. *Let a_1, a_2, b_1, b_2 and c_1, c_2 be all positive with $a_i \ge max(b_i,c_i)$. If $a_i + \gamma c_i \ge (1+\gamma)b_i$, where $\gamma > 0$, then $a_1a_2 + \gamma c_1c_2 \ge (1+\gamma)b_1b_2$.*

Proof. Since $b_2(a_1+\gamma c_1) \ge (1+\gamma)b_1b_2$, it suffices to show that

$$a_1a_2 + \gamma c_1c_2 \ge a_1b_2 + \gamma c_1b_2;$$

or, rearranging, that $a_1(a_2-b_2) \ge \gamma c_1(b_2-c_2)$.
Note that since $a_1 \ge c_1$ and $a_2 - b_2 \ge 0$, we get $a_1(a_2-b_2) \ge c_1(a_2-b_2) \ge c_1(\gamma b_2 - \gamma c_2)$, as desired. □

Theorem 2.2.3. *For $k = 6$, $P_f(z) + \epsilon(f)P_f(1/z)$ has all its zeros on the unit circle.*

Proof. Here $m = 2$, so $P_f(z) = 3\Lambda(f,3) + 4\Lambda(f,4)z + \Lambda(f,5)z^2$.
If $\epsilon(f) = -1$, then $\Lambda(f,3) = 0$ and so

$$\begin{aligned} P_f(z) - P_f\left(\frac{1}{z}\right) &= 4\Lambda(f,4)\left(z - \frac{1}{z}\right) + \Lambda(f,5)\left(z^2 - \frac{1}{z^2}\right) \\ &= \left(z - \frac{1}{z}\right)\left[4\Lambda(f,4) + \Lambda(f,5)\left(z + \frac{1}{z}\right)\right]. \end{aligned}$$

Clearly $X = \pm 1$ are two solutions. Putting $X = e^{i\theta}$, we find that the other roots are the solutions of $\cos\theta = -\frac{2\Lambda(f,4)}{\Lambda(f,5)}$ for $\theta \in [0, 2\pi)$. From Lemma 3.3.2, we have that $2\Lambda(f,4) < \Lambda(f,5)$ and so their are two more roots which also lie on the unit circle.
If $\epsilon(f) = 1$, letting $z = e^{i\theta}$ we have

$$P_f(z) + P_f\left(\frac{1}{z}\right) = 6\Lambda(f,3) + 8\Lambda(f,4)\cos\theta + 2\Lambda(f,5)\cos 2\theta.$$

We want to show this has two zeros in $[0, \pi)$ and thus four zeros in $[0, 2\pi)$. Note that

$$\frac{d}{d\theta}\left[P_f(e^{i\theta}) + P_f(e^{-i\theta})\right] = -8\sin\theta(\Lambda(f,4) + \Lambda(f,5)\cos\theta),$$

we have critical points at $0, \pi$ and the solution $\theta_0 \in [0, \pi)$ to $\cos\theta = -\frac{\Lambda(f,4)}{\Lambda(f,5)}$. To get two roots in $[0, \pi)$ we need $P_f(e^{i\theta}) + P_f(e^{-i\theta})$ to be positive at $\theta = 0$ and π and negative at $\theta = \theta_0$. At $\theta = 0$, $P_f(e^{i\theta}) + P_f(e^{-i\theta}) = 6\Lambda(f,3) + 8\Lambda(f,4) + 2\Lambda(f,5) > 0$. Positivity at $\theta = \pi$ is equivalent to

$$\Lambda(f,5) + 3\Lambda(f,3) > 4\Lambda(f,4) \tag{2.5}$$

while negativity at $\theta = \theta_0$ is equivalent to

$$\Lambda(f,5)^2 + 2\Lambda(f,4)^2 < 3\Lambda(f,3)\Lambda(f,5). \tag{2.6}$$

We use Lemma 2.2.3 suitably, together with the Hadamard factorization formulas (Eqs. 2.3 and 2.4), proceeding zero by zero. We use the Hadamard formula for $\Lambda(f,3)$, $\Lambda(f,4)$, and $\Lambda(f,5)$; Note that by Lemma 2.2.1 all these values are non-negative, so we can assume that the products are taken with absolute values. Note that all the zeros lie in $|\Re(s) - \frac{3}{2}| < \frac{1}{2}$.

Suppose first that $\rho = 3 + z$ is a real zero, then $6 - \rho = 3 - z$ is also a real zero (if $\rho = 3$, we get zeros of even multiplicity at the center, which may be paired). This pair of zeros contributes to $\Lambda(f,5)$ the amount

$$a = \left(1 - \frac{5}{3+z}\right)\left(1 - \frac{5}{3-z}\right) = \frac{4 - z^2}{9 - z^2},$$

to $\Lambda(f,4)$ the amount $b = \frac{1-z^2}{9-z^2}$, and to $\Lambda(f,3)$ the amount $c = \frac{z^2}{9-z^2}$ (using here the absolute value remark). Then with $\gamma = 3$ (and since $|z| < \frac{1}{2}$ here), we have the inequality $a + 3c \geq 4b$.

Now consider a zero $\rho = 3 + iy$ on the critical line, then $\bar{\rho} = 3 - iy$ is also a zero. This pair of zeros contributes to $\Lambda(f, 5)$ the amount

$$a = \left(1 - \frac{5}{3 + iy}\right)\left(1 - \frac{5}{3 - iy}\right) = \frac{4 + y^2}{9 + y^2},$$

to $\Lambda(f, 4)$ the amount $b = \frac{1+y^2}{9+y^2}$, and to $\Lambda(f, 3)$ the amount $c = \frac{y^2}{9+y^2}$, and we check again that $a + 3c \geq 4b$ (and indeed equality holds).

Finally, consider a zero $\rho = 3 + z$ not on the critical line, with $z = x + iy$ (i.e $x \neq 0$). Then $6 - \rho = 3 - x - iy$, $\bar{\rho} = 3 + x - iy$, $6 - \bar{\rho} = 3 - x + iy$ are also zeros. This set of four zeros contributes to $\Lambda(f, 5)$ the amount

$$a = \left(1 - \frac{5}{3 + x + iy}\right)\left(1 - \frac{5}{3 + x - iy}\right)\left(1 - \frac{5}{3 - x + iy}\right)\left(1 - \frac{5}{3 - x - iy}\right) = \frac{|4 - z^2|^2}{|\rho|^2|6 - \rho|^2},$$

to $\Lambda(f, 4)$ the amount

$$b = \frac{|1 - z^2|^2}{|\rho|^2|6 - \rho|^2},$$

and to $\Lambda(f, 3)$ the amount

$$c = \frac{|z^2|^2}{|\rho|^2|6 - \rho|^2}.$$

We can check again that $a + 3c \geq 4b$.

Thus when grouped as above, each group of zeros appearing in the Hadamard formula satisfies a version of 2.5. By Lemma 2.2.3, taking products of these groups of zeros we again obtain a version of 2.5. Letting these products run over all zeros and taking the limit, we obtain 2.5.

The proof of 2.6 is similar, appealing to Lemma 2.2.3 with $\gamma = 2$, and using Hadamard's formula and grouping zeros as above. $\square$

Using $2\cos(\theta) = e^{i\theta} + e^{-i\theta}$ and $2\sin(\theta) = e^{i\theta} - e^{-i\theta}$, we have that for $z = e^{i\theta}$ on the unit circle

$$P_f(z) + P_f\Big(\frac{1}{z}\Big) = \binom{2m}{m}\Lambda(f, \frac{k}{2}) + 2\sum_{j=1}^{m}\binom{2m}{m+j}\Lambda(f, \frac{k}{2} + j)\cos(j\theta),$$

and

$$P_f(z) - P_f\Big(\frac{1}{z}\Big) = 2\sum_{j=1}^{m}\binom{2m}{m+j}\Lambda(f, \frac{k}{2} + j)\sin(j\theta).$$

Classical work of Pólya [12] and Szegö [13] considers trigonometric polynomials

$$u(\theta) = a_0 + a_1\cos(\theta) + a_2\cos(2\theta) + \cdots + a_n\cos(n\theta),$$
$$v(\theta) = a_1\sin(\theta) + a_2\sin(2\theta) + \cdots + a_n\sin(n\theta).$$

They show that if $0 \leq a_0 \leq a_1 \leq \ldots a_{n-1} \leq a_n$, then u and v both have exactly n zeros in $[0, \pi)$ (and therefore $2n$ zeros in $[0, 2\pi)$) and that these zeros are simple. Each interval $(\frac{l-1/2}{n+1/2}\pi, \frac{l+1/2}{n+1/2}\pi)$ for $1 \leq l \leq n$ has precisely one zero of u, and apart from $\theta = 0$, each interval $(\frac{l}{n+1/2}\pi, \frac{l+1}{n+1/2}\pi)$ for $1 \leq l \leq n-1$ has exactly one zero of v. The proof is a simple sign change argument using the positivity of the Fejér kernel. When the level is suitably large, these results apply and provide a quick proof of our result.

Theorem 2.2.4. *For $8 \leq k \leq 14$, $P_f(z)+\epsilon(f)P_f(1/z)$ has all its zeros on the unit circle.*

Proof. For weight k, for the above to apply we must verify the criteria

$$\binom{2m}{m}\Lambda(f,\frac{k}{2}) \leq 2\binom{2m}{m+1}\Lambda(f,\frac{k}{2}+1) \tag{2.7}$$

and that for all $1 \leq j \leq m-1$

$$\binom{2m}{m+j}\Lambda(f,\frac{k}{2}+j) \leq \binom{2m}{m+j+1}\Lambda(f,\frac{k}{2}+j+1). \tag{2.8}$$

The condition (2.7) can be written as

$$\Lambda(f,\frac{k}{2}) \leq \frac{2m}{m+1}\Lambda(f,\frac{k}{2}+1).$$

Since $\Lambda(f,k/2) \leq \Lambda(f,k/2+1)$ by Lemma 2.2.1, (2.7) is then immediate for all $k \geq 4$. Now suppose $k \geq 6$. Using the definition of $\Lambda(f,s)$, (2.8) is equivalent to

$$\sqrt{N} \geq \frac{2\pi}{(k/2-j-1)}\frac{L(f,k/2+j)}{L(f,k/2+j+1)}$$

for all $1 \leq j \leq m-1$. By Lemma 2.2.2, it suffices to have

$$N \geq \left(\frac{2\pi}{k/2-j-1}\right)^2\frac{\zeta(j+1/2)^4}{\zeta(j+3/2)^4},$$

for all $1 \leq j \leq m-1$; since then we'll get that for all $1 \leq j \leq m-1$

$$N \geq \left(\frac{2\pi}{k/2-j-1}\right)^2\frac{\zeta(j+1/2)^4}{\zeta(j+3/2)^4} \geq \left(\frac{2\pi}{k/2-j-1}\frac{L(f,k/2+j)}{L(f,k/2+j+1)}\right)^2$$

as needed. Therefore, our criterion (2.8) is met if

$$N \geq \max_{1\leq j\leq k/2-2}\left(\frac{2\pi}{k/2-j-1}\right)^2\frac{\zeta(j+1/2)^4}{\zeta(j+3/2)^4}. \tag{2.9}$$

For any given k, we can compute the bound (2.9). Thus, for $k=8$ it suffices to take $N \geq 142$; for $k=10$ it suffices to have $N \geq 64$; for $k=12$ it suffices to have $N \geq 45$; for $k=14$ it suffices to have $N \geq 42$. We can use PARI to check (2.8) for those newforms not covered by (2.9) for weights $8 \leq k \leq 14$. The zeros of those newforms that do not satisfy (2.8) still lie on $|z| = 1/\sqrt{N}$. The code we used is the following:

```
mf(k)=mfinit([N,k],0)

B(k)=mfbasis(mf(k))

P(k)=mfperiodpol(mf(k),B(k)[1])
```

```
Z(k)=polroots(P(k))

default(parisizemax,1G)

ploth(t=0,100,apply(z->z+t*exp(I*t)/10^4,Z(k))~,4096)
```

□

Remark. *Eventually, (2.9) cannot furnish a bound better than $4\pi^2$ for N, and so we must turn to another approach for large k and small N.*

Proposition 2.2.2. *$P_f(z)$ can be written as*

$$P_f(z) = (2m)!\left(\frac{\sqrt{N}}{2\pi}\right)^{2m+1} L(f, 2m+1) Q_f(z)$$

where

$$Q_f(z) = z^m \sum_{j=0}^{m-1} \frac{1}{j!}\left(\frac{2\pi}{z\sqrt{N}}\right)^j \frac{L(f, 2m+1-j)}{L(f, 2m+1)} + \frac{1}{2(m!)^2}\left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} \frac{\Lambda(f, \frac{k}{2})}{L(f, 2m+1)}.$$

Proof.

$$\begin{aligned}
P_f(z) &= \frac{1}{2}\binom{2m}{m}\Lambda(f, \frac{k}{2}) + \sum_{j=1}^{m}\binom{2m}{m+j}\Lambda(f, \frac{k}{2}+j)z^j \\
&= (2m)!\left(\frac{\sqrt{N}}{2\pi}\right)^{2m+1}\left[\frac{1}{2(m!)^2}\left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1}\Lambda(f, \frac{k}{2})\right. \\
&\left. + \sum_{j=1}^{m}\frac{1}{(m+j)!(m-j)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} z^j \left(\frac{\sqrt{N}}{2\pi}\right)^{k/2+j}(k/2+j-1)!\, L(f, \frac{k}{2}+j)\right] \\
&= (2m)!\left(\frac{\sqrt{N}}{2\pi}\right)^{2m+1}\left[\frac{1}{2(m!)^2}\left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1}\Lambda(f, \frac{k}{2}) + \sum_{j=1}^{m}\frac{1}{(m-j)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-j} z^j L(f, \frac{k}{2}+j)\right] \\
&= (2m)!\left(\frac{\sqrt{N}}{2\pi}\right)^{2m+1}\left[\frac{1}{2(m!)^2}\left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1}\Lambda(f, \frac{k}{2})\right. \\
&\left. + z^m \sum_{j=0}^{m-1}\frac{1}{j!}\left(\frac{2\pi}{z\sqrt{N}}\right)^j L(f, 2m+1-j)\right] \quad \text{(substituting } j \to m-j\text{)} \\
&= (2m)!\left(\frac{\sqrt{N}}{2\pi}\right)^{2m+1} L(f, 2m+1) Q_f(z).
\end{aligned}$$

□

Therefore, we need to study the zeros of

$$P_f(z) + \epsilon(f) P_f\left(\frac{1}{z}\right) = (2m)!\left(\frac{\sqrt{N}}{2\pi}\right)^{2m+1} L(f, 2m+1)\left(Q_f(z) + \epsilon(f) Q_f\left(\frac{1}{z}\right)\right).$$

Note that for $z = e^{i\theta}$ on the unit circle,

$$\begin{aligned}
Q_f(z) &= \sum_{j=0}^{m-1} \frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j e^{i(m-j)\theta} \frac{L(f, 2m+1-j)}{L(f, 2m+1)} + \frac{1}{2(m!)^2}\left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} \frac{\Lambda(f, \frac{k}{2})}{L(f, 2m+1)} \\
&= \sum_{j=0}^{m-1} \frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j \cos(m-j)\theta \frac{L(f, 2m+1-j)}{L(f, 2m+1)} + \frac{1}{2(m!)^2}\left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} \frac{\Lambda(f, \frac{k}{2})}{L(f, 2m+1)} \\
&+ i\sum_{j=0}^{m-1} \frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j \sin(m-j)\theta \frac{L(f, 2m+1-j)}{L(f, 2m+1)}.
\end{aligned}$$

Hence for $z = e^{i\theta}$ we have that

$$\begin{aligned}
&Q_f(z) - Q_f\left(\frac{1}{z}\right) \\
&= e^{im\theta}\sum_{j=0}^{m-1} \frac{1}{j!}\left(\frac{2\pi}{e^{i\theta}\sqrt{N}}\right)^j \frac{L(f, 2m+1-j)}{L(f, 2m+1)} - e^{-im\theta}\sum_{j=0}^{m-1} \frac{1}{j!}\left(\frac{2\pi}{e^{-i\theta}\sqrt{N}}\right)^j \frac{L(f, 2m+1-j)}{L(f, 2m+1)} \\
&= \sum_{j=0}^{m-1} \frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j \left(e^{i(m-j)\theta} - e^{-i(m-j)\theta}\right) \frac{L(f, 2m+1-j)}{L(f, 2m+1)} \\
&= 2\sum_{j=0}^{m-1} \frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j \frac{L(f, 2m+1-j)}{L(f, 2m+1)} \sin(m-j)\theta \\
&= 2\Im\Big(Q_f(z)\Big),
\end{aligned}$$

and similarly

$$Q_f(z) + Q_f\left(\frac{1}{z}\right) = 2\Re\Big(Q_f(z)\Big).$$

We wish to show that on the unit circle $|z| = 1$, the real and imaginary parts of $Q_f(z)$ have exactly $2m$ zeros.

Theorem 2.2.5. *For $k \geq 16$, the real and imaginary parts of $Q_f(z)$ have all their zeros on the unit circle.*

Proof. Since

$$\exp\left(\frac{2\pi}{z\sqrt{N}}\right) = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{2\pi}{z\sqrt{N}}\right)^j,$$

We can write

$$\begin{aligned}
Q_f(z) &= \frac{1}{2(m!)^2}\left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} \frac{\Lambda(f, \frac{k}{2})}{L(f, 2m+1)} + z^m \exp\left(\frac{2\pi}{z\sqrt{N}}\right) - z^m \sum_{j=m}^{\infty} \frac{1}{j!}\left(\frac{2\pi}{z\sqrt{N}}\right)^j \\
&- z^m \sum_{j=0}^{m-1} \frac{1}{j!}\left(\frac{2\pi}{z\sqrt{N}}\right)^j + z^m \sum_{j=0}^{m-1} \frac{1}{j!}\left(\frac{2\pi}{z\sqrt{N}}\right)^j \frac{L(f, 2m+1-j)}{L(f, 2m+1)} \\
&= z^m \exp\left(\frac{2\pi}{z\sqrt{N}}\right) + S_1(z) + S_2(z) + S_3(z),
\end{aligned}$$

with

$$S_1(z) = z^m \sum_{j=1}^{m-1} \frac{1}{j!}\left(\frac{2\pi}{z\sqrt{N}}\right)^j \left(\frac{L(f,2m+1-j)}{L(f,2m+1)} - 1\right),$$

$$S_2(z) = -z^m \sum_{j=0}^{m-1} \frac{1}{j!}\left(\frac{2\pi}{z\sqrt{N}}\right)^j,$$

and

$$S_3(z) = \frac{1}{2(m!)^2}\left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} \frac{\Lambda(f,\frac{k}{2})}{L(f,2m+1)}.$$

For $z = e^{i\theta}$ on the unit circle,

$$z^m \exp\left(\frac{2\pi}{z\sqrt{N}}\right) = \exp\left(\frac{2\pi}{\sqrt{N}}\cos\theta + i\Big(m\theta - \frac{2\pi}{\sqrt{N}}\sin\theta\Big)\right).$$

The real and imaginary parts of $z^m \exp(2\pi/(z\sqrt{N}))$ both have exactly $2m$ zeros. To see this, consider first the real part

$$\Re\left(z^m \exp\left(\frac{2\pi}{z\sqrt{N}}\right)\right) = \exp\left(\frac{2\pi}{\sqrt{N}}\cos\theta\right)\cos\left(m\theta - \frac{2\pi}{\sqrt{N}}\sin\theta\right).$$

For $\cos(m\theta - \frac{2\pi}{\sqrt{N}}\sin\theta) = 1$, we should have $m\theta - \frac{2\pi}{\sqrt{N}}\sin\theta = 2k\pi$, for $k \in \mathbb{Z}$. As $m \geq 7$,

$$\frac{d}{d\theta}\left(m\theta - \frac{2\pi}{\sqrt{N}}\sin\theta\right) = m - \frac{2\pi}{\sqrt{N}}\cos\theta \geq m - \frac{2\pi}{\sqrt{N}} > 0$$

and so $m\theta - \frac{2\pi}{\sqrt{N}}\sin\theta$ is monotone increasing as θ varies from 0 to 2π, and changes by $2\pi m$ overall. Therefore, there are m values of θ with $\cos(m\theta - \frac{2\pi}{\sqrt{N}}\sin\theta) = 1$. Similarly, there are m interlacing values of θ with $\cos(m\theta - \frac{2\pi}{\sqrt{N}}\sin\theta) = -1$. Between two such interlacing values, there must be a zero of the real part. And thus the real part has exactly $2m$ zeros. Further, because $\exp(2\pi\cos\theta/\sqrt{N}) \geq \exp(-2\pi/\sqrt{N})$ for all θ, if

$$\Big|S_1(z) + S_2(z) + S_3(z)\Big| < \exp\left(\frac{-2\pi}{\sqrt{N}}\right) \tag{2.10}$$

then the real part of $Q_f(z)$ will also have sign changes and thus a zero in these intervals. That is because

$$\begin{aligned}\Re\Big(Q_f(z)\Big) &= \Re\left(z^m \exp\left(\frac{2\pi}{z\sqrt{N}}\right)\right) + \Re\Big(S_1(z) + S_2(z) + S_3(z)\Big)\\ &\leq \exp\left(\frac{2\pi}{\sqrt{N}}\cos\theta\right)\cos\left(m\theta - \frac{2\pi}{\sqrt{N}}\sin\theta\right) + \Big|S_1(z) + S_2(z) + S_3(z)\Big|\\ &< \exp\left(\frac{2\pi}{\sqrt{N}}\cos\theta\right)\left[\cos\left(m\theta - \frac{2\pi}{\sqrt{N}}\sin\theta\right) + 1\right]\end{aligned}$$

and similarly,

$$\begin{aligned}\Re\Big(Q_f(z)\Big) &= \Re\left(z^m \exp\left(\frac{2\pi}{z\sqrt{N}}\right)\right) + \Re\Big(S_1(z)+S_2(z)+S_3(z)\Big)\\ &\geq \exp\left(\frac{2\pi}{\sqrt{N}}\cos\theta\right)\cos\left(m\theta - \frac{2\pi}{\sqrt{N}}\sin\theta\right) - \Big|S_1(z)+S_2(z)+S_3(z)\Big|\\ &> \exp\left(\frac{2\pi}{\sqrt{N}}\cos\theta\right)\left[\cos\left(m\theta - \frac{2\pi}{\sqrt{N}}\sin\theta\right) - 1\right].\end{aligned}$$

A similar argument applies to the imaginary part of $Q_f(z)$, and so it suffices to check the criterion (2.10). By Lemma 2.2.2, we see that

$$\frac{L(f,2m+1-j)}{L(f,2m+1)} \leq \frac{\zeta(\frac{1}{2}+m-j)^2}{\zeta(\frac{1}{2}+m)^2} \leq \zeta\left(\frac{1}{2}+m-j\right)^2.$$

So that

$$\begin{aligned}|S_1(z)|+|S_2(z)| &\leq \sum_{j=1}^{m-1}\frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j\left(\zeta\left(\frac{1}{2}+m-j\right)^2-1\right) + \sum_{j=m}^{\infty}\frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j\\ &= \sum_{j=1}^{m-2}\frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j\frac{2^{m-j}}{2^{m-j}}\left(\zeta\left(\frac{1}{2}+m-j\right)^2-1\right)\\ &\quad+\frac{1}{(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}\left(\zeta\left(\frac{3}{2}\right)^2-1\right)+\sum_{j=m}^{\infty}\frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j\end{aligned}$$

By direct computation, we see that $\zeta(3/2)^2 - 1 \leq \frac{35}{6}$. Moreover, $2^x(\zeta(\frac{1}{2}+x)^2-1)$ is decreasing for $x \geq 2$ and so may be bounded by $4(\zeta(5/2)^2-1)) \leq \frac{16}{5}$. Using the former bound for the term $j = m-1$, and the latter bound for smaller values of j, we obtain

$$\begin{aligned}|S_1(z)|+|S_2(z)| &\leq \frac{16}{5}\sum_{j=1}^{m-2}\frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j 2^{j-m} + \frac{35}{6}\frac{1}{(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} + \sum_{j=m}^{\infty}\frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j 2^{j-m}\\ &\leq \frac{16}{5}2^{-m}\sum_{j=1}^{m-1}\frac{1}{j!}\left(\frac{4\pi}{\sqrt{N}}\right)^j + \frac{17}{4}\frac{1}{(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} + \frac{16}{5}2^{-m}\sum_{j=m}^{\infty}\frac{1}{j!}\left(\frac{4\pi}{\sqrt{N}}\right)^j\\ &= \frac{16}{5}2^{-m}\left(\exp\left(\frac{4\pi}{\sqrt{N}}\right)-1\right) + \frac{17}{4}\frac{1}{(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}.\end{aligned}$$

To bound $S_3(z)$, note that $\Lambda(f, \frac{k}{2}) \le \Lambda(f, \frac{k}{2}+1) = \left(\frac{\sqrt{N}}{2\pi}\right)^{m+2}(m+1)!L(f, m+2)$, hence

$$\begin{aligned}
|S_3(z)| &\le \frac{m+1}{2m(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}\frac{L(f,m+2)}{L(f,2m+1)} \\
&\le \frac{m+1}{2m(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}\zeta\left(\frac{3}{2}\right)^2 \\
&\le \frac{41(m+1)}{12m}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} \le \frac{4}{(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1},
\end{aligned}$$

where we have used the bounds we utilized earlier, and the fact that $\frac{41(m+1)}{12m} \le 4$ for $m \ge 7$ (i.e. for $k \ge 16$). Now combining the bounds for the S_i's, we conclude that

$$|S_1(z)| + |S_2(z)| + |S_3(z)| \le \frac{16}{5}\frac{1}{2^m}\left(\exp\left(\frac{4\pi}{\sqrt{N}}\right) - 1\right) + \frac{33}{4}\frac{1}{(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}.$$

Thus, to verify the condition (2.10), we need only to ensure that

$$\frac{16}{5}\frac{1}{2^m}\left(\exp\left(\frac{4\pi}{\sqrt{N}}\right) - 1\right) + \frac{33}{4}\frac{1}{(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} < \exp\left(-\frac{2\pi}{\sqrt{N}}\right). \tag{2.11}$$

For values of m at least as large as the figure in the first row, the table below gives a bound $N(m)$ such that the estimate (2.11) holds for all $N \ge N(m)$:

m	7	8	9	10	11	12	13	14	16	18	21	29
$N(m)$	28	20	14	11	9	7	6	5	4	3	2	1

For the finitely many newforms missed by (2.11), we can use PARI to confirm the result. □

Therefore, we have finally proved:

Theorem 2.2.6. *Let $N \in \mathbb{N}$ and $k \ge 4$. If $f \in S_k^{new}(\Gamma_0(N))$, then $r_f(z)$ has all its zeros on the circle $|z| = 1/\sqrt{N}$.*

Chapter 3

RHPP of Hilbert Modular Forms

In this chapter, we prove that the zeros of period polynomials of any parallel weight Hilbert modular eigenform on the full Hilbert modular group lie on the unit circle.

3.1 Algebraic Detour

We start by reviewing some basic algebraic concepts. For more details, refer to [14].

Let $\mathbb{Q} \subset K \subset \mathbb{C}$ be a field. We can consider K as a vector space over $\mathbb{Q}$. K is called an algebraic number field if the dimension of this vector space is finite. This dimension is called the degree of K and denoted by

$$n = [K : \mathbb{Q}] := dim_{\mathbb{Q}} K.$$

In this case, K is called a finite extension of $\mathbb{Q}$. The elements of algebraic number fields are always algebraic numbers and each algebraic number is contained in some algebraic number field K. The smallest K which contains a is denoted by

$$K = \mathbb{Q}(a).$$

An embedding of a number field K in $\mathbb{C}$ is an injective field homomorphism of K into $\mathbb{C}$.

Theorem 3.1.1. *Let K be a number field of degree n. Then there are exactly n different embeddings of K in $\mathbb{C}$.*

We usually arrange the embeddings in a certain order and denote them by

$$K \to K^{(j)} \subset \mathbb{C}$$
$$a \to a^{(j)},\ j = 1, \dots, n.$$

We put the n embeddings together into a single $\mathbb{Q}$-linear injective mapping

$$K \to \mathbb{C}^n,\ \ a \to (a^{(1)}, a^{(2)}, \dots, a^{(n)}).$$

An embedding is called real if its image is contained in $\mathbb{R}$. K is called totally real if it admits only real embeddings.
The *trace* and *norm* of an element $a \in K$ over $\mathbb{Q}$ are given, respectively, by

$$Tr(a) = Tr_{K/\mathbb{Q}}(a) = \sum_{j=1}^{n} a^{(j)}, \quad N(a) = N_{K/\mathbb{Q}}(a) = \prod_{j=1}^{n} a^{(j)}.$$

Definition 3.1.1. *Let K be an algebraic number field. The ring of integers of K is defined as*

$$\mathcal{O}_K = K \cap \overline{\mathbb{Z}},$$

where $\overline{\mathbb{Z}}$ is the algebraic closure of $\mathbb{Z}$.

The group of units (invertible elements) of $\mathcal{O}_K$ is denoted by $\mathcal{O}_K^*$.

Theorem 3.1.2. *Let K be a number field of degree n. Then $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of rank n.*

Write $\mathcal{O}_K = \langle a_1, a_2, \dots, a_n \rangle_{\mathbb{Z}}$ and let

$$A = \begin{pmatrix} a_1^{(1)} & a_1^{(2)} & \dots & a_1^{(n)} \\ a_2^{(1)} & a_2^{(2)} & \dots & a_2^{(n)} \\ . & . & & . \\ . & . & & . \\ a_n^{(1)} & a_n^{(2)} & \dots & a_n^{(n)} \end{pmatrix}.$$

Then the *discriminant* D_K of K is given by $D_K = (\mathrm{detA})^2$.

Theorem 3.1.3. *(Minkowski's bound)*
Let K be a number field of degree n, then

$$D_K \geq \left(\frac{n^n}{n!} \right)^2.$$

A subset $\mathfrak{a} \subset K$ is called an *ideal* of K if $\mathfrak{a}$ is an $\mathcal{O}_K$-submodule of K. An ideal $\mathfrak{a}$ is said to be integral if $\mathfrak{a} \subset \mathcal{O}_K$. It is said to be principal if there is an $a \in K$ with $\mathfrak{a} = \langle a \rangle_{\mathcal{O}_K}$. Further, we say $\mathfrak{a}$ is fractional if there exists a non-zero $r \in \mathcal{O}_K$ such that $r\mathfrak{a} \subset \mathcal{O}_K$. For a fractional ideal $\mathfrak{a}$ of K, let

$$\mathfrak{a}_+ = \{x \in \mathfrak{a} : \ x^{(j)} > 0 \text{ for } j = 1, \dots, n\}.$$

Namely, $\mathfrak{a}_+$ is the set of all totally positive elements of $\mathfrak{a}$.
The product of two ideals is defined by

$$\mathfrak{a}.\mathfrak{b} = \left\{ \sum_{i \in I} a_j b_j : \ a_j \in \mathfrak{a}, \ b_j \in \mathfrak{b}, \ I \text{ is finite} \right\}.$$

The set of all fractional ideals of K is a group under this multiplication. The neutral element of this group is the ideal $\mathcal{O}_K$, and the inverse of a fractional ideal $\mathfrak{a}$ is given by

$$\mathfrak{a}^{-1} := \{x \in K \ \mid \ x\mathfrak{a} \subset \mathcal{O}_K\}.$$

An ideal $\mathfrak{p}$ of K is called prime if it is integral and satisfies for $a, b \in \mathcal{O}_K$:

$$a.b \in \mathfrak{p} \Rightarrow a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}.$$

Theorem 3.1.4. *Every non-zero fractional ideal of K can be factored uniquely into a product of prime ideals.*

Definition 3.1.2. *Let $\mathfrak{a}$ be an integral ideal of K. We define the Norm $\mathfrak{a}$ as*

$$N(\mathfrak{a}) := |\mathcal{O}_K/\mathfrak{a}|.$$

By convention, the norm of the zero ideal is taken to be zero. Note that if $\mathfrak{a} = \langle a \rangle$ is principal, then $N(\mathfrak{a}) = |N(a)|$. Moreover, the norm is completely multiplicative i.e. if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of K, then

$$N(\mathfrak{a}.\mathfrak{b}) = N(\mathfrak{a})N(\mathfrak{b}).$$

Thus, by the above theorem, we can define the norm for all fractional ideals of K.

Finally, we define the *different* ideal $\mathfrak{d}_K$ of K to be the inverse of

$$\mathfrak{e} = \{x \in K : \ Tr(xy) \in \mathbb{Z} \text{ for all } y \in \mathcal{O}_K\},$$

i.e. $\mathfrak{d}_K := \mathfrak{e}^{-1}$. Note that if K is totally real, then $\mathfrak{d}_K = \langle D_K \rangle$ and so $N(\mathfrak{d}_K) = |D_K|$.

3.2 Hilbert Modular Forms

We now give a brief introduction to theory of Hilbert modular forms. For more details on the general theory, we refer the reader to the survey of Bruinier in [15].

Let K be a totally real number field of degree n.
If we attach to the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$$

the tuple $(M_1, \ldots, M_n)$ where

$$M_j = \begin{pmatrix} a^{(j)} & b^{(j)} \\ c^{(j)} & d^{(j)} \end{pmatrix}, \ j = 1, \ldots, n$$

we obtain an embedding of groups

$$GL_2(K) \hookrightarrow GL_2(\mathbb{R})^n.$$

The group

$$GL_2^+(K) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K) : \ \det\gamma_j > 0 \ \text{ for } \ j = 1, \ldots, n \right\}$$

acts on $\mathbb{H}^n$ by coordinate linear fractional transformations, i.e. for $z = (z_1, \ldots, z_n) \in \mathbb{H}^n$

$$z \to \gamma z = (\gamma_i z_i)_i = \left(\frac{a^{(i)} z_i + b^{(i)}}{c^{(i)} z_i + d^{(i)}} \right)_{i=1,\ldots,n} = \left(\frac{a^{(1)} z_1 + b^{(1)}}{c^{(1)} z_1 + d^{(1)}}, \ldots, \frac{a^{(n)} z_n + b^{(n)}}{c^{(n)} z_n + d^{(n)}} \right).$$

For $z \in \mathbb{H}^n$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(K)$, we write

$$N(cz + d) = \prod_{i=1}^{n} \left(c^{(i)} z_i + d^{(i)} \right) \det(\gamma_i)^{\frac{-1}{2}}.$$

We then have that

$$\Im(\gamma z) = \frac{\Im(z)}{|N(cz+d)|^2},$$

and so the action preserves $\mathbb{H}^n$. If $\mathfrak{a}$ is a fractional ideal of K, we define the *Hilbert modular group* corresponding to $\mathfrak{a}$ as

$$\Gamma(\mathcal{O}_K \oplus \mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(K) : a, d \in \mathcal{O}_K,\ b \in \mathfrak{a}^{-1},\ c \in \mathfrak{a} \right\}.$$

Moreover, we define the *full Hilbert modular group* to be

$$\Gamma_K := \Gamma(\mathcal{O}_K \oplus \mathcal{O}_K) = GL_2^+(\mathcal{O}_K).$$

Definition 3.2.1. *A holomorphic function $f : \mathbb{H}^n \to \mathbb{C}$ is called a holomorphic Hilbert modular form of weight $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ for Γ_K, if for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$*

$$f(\gamma z) = \prod_{i=1}^{n} det(\gamma_i)^{-k_i/2} \left(c^{(i)} z_i + d^{(i)}\right)^{k_i} f(z).$$

If $k_1 = k_2 = \cdots = k_n := k$ then f is said to have parallel weight, and is simply called a holomorphic Hilbert modular form of weight $k \in \mathbb{Z}$.

We denote the space holomorphic Hilbert modular forms of weight k on Γ_K by $M_k(\Gamma_K)$. Moreover, If $f \in M_k(\Gamma_K)$ vanishes at the cusps of Γ_K, we call it a cusp form and denote this space by $S_k(\Gamma_K)$ as usual.

Note that if $\mathcal{O}_K$ has a unit of negative norm, then $M_k(\Gamma_K) = \{0\}$ for k odd (this is because of the action of matrices of the form $\operatorname{diag}(u, u^{-1})$, where u is the unit of negative norm). Hence we will suppose that k is even.

Each $f \in M_k(\Gamma_K)$ has a Fourier expansion of the form

$$f(z) = a_0 + \sum_{\mu \in (\mathfrak{d}_K^{-1})_+} a_\mu e^{2\pi i Tr(\mu z)} \tag{3.1}$$

where $Tr(\mu z) = \sum_{i=1}^{n} \mu^{(i)} z_i$. Since $\mu \in \mathfrak{d}_K^{-1}$, each ideal $\mathfrak{n} = \mu \mathfrak{d}_K$ is integral. When the forms have parallel weight, $a(\mu) = a(u\mu)$ for any totally positive unit $u \in (\mathcal{O}_K)_+$ and we may rewrite (3.1) as

$$f(z) = a_0 + \sum_{\substack{\mathfrak{n} \in \mathcal{O}_K \\ \mathfrak{n} \neq 0}} a(\mathfrak{n}) \sum_{\nu \in (\mathcal{O}_K^*)_+} e^{2\pi i Tr(\mu \nu z)},$$

and we may identify each modular form by the coefficients $a(\mathfrak{n})$.

Therefore, $f \in S_k(\Gamma_K)$ has an associated L-function given by

$$L(f, s) := \sum_{\mu \in (\mathfrak{d}_K^{-1}/\mathcal{O}_K^*)_+} \frac{a(\mu)}{N(\mu)^s} = \sum_{\substack{\mathfrak{n} \in \mathcal{O}_K \\ \mathfrak{n} \neq 0}} \frac{a(\mathfrak{n})}{N(\mathfrak{n})^s}.$$

If $U = (\mathcal{O}_K^*)_+$ is the group of totally positive units of K, then letting $f(z) = f(z_1, \ldots, z_n)$, $N(z) = z_1 \ldots z_n$ and $dz = dz_1 \ldots dz_n$, we can define the completed L-function by

$$\Lambda(f,s) := \int_{(\mathbb{R}_+)^n/U} f(iy)N(y)^{s-1}\,dy,$$

which satisfies

$$\Lambda(f,s) = \left(\frac{D_K}{(2\pi)^n}\right)^s \Gamma(s)^n L(f,s) \tag{3.2}$$

and the functional equation

$$\Lambda(f,s) = \epsilon(f)\Lambda(f,k-s)$$

where $\epsilon(f) \in \{\pm 1\}$.

We further define the period polynomial of a parallel weight k Hilbert modular eigenform f as

$$r_f(X) := \int_{i((\mathbb{R}_+)^n/U)} f(\tau)(N(\tau)-X)^{k-2}\,d\tau.$$

In analogy with the classical case, we have that r_f is the generating function for the critical values of the L-function:

Theorem 3.2.1. *The period polynomial r_f of f satisfies*

$$r_f(X) = \sum_{\ell=0}^{k-2}(-1)^\ell i^{n(k-\ell-1)}\binom{k-2}{\ell}X^\ell\Lambda(f,k-\ell-1).$$

Proof. Using the binomial expansion, we get

$$\begin{aligned}
r_f(X) &= \int_{i((\mathbb{R}_+)^n/U)} \sum_{l=0}^{k-2}\binom{k-2}{l}N(\tau)^\ell(-X)^{k-2-\ell}f(\tau)\,d\tau \\
&= \sum_{l=0}^{k-2}\binom{k-2}{l}(-X)^{k-2-l}\int_{i((\mathbb{R}_+)^n/U)} N(\tau)^\ell f(\tau)\,d\tau \\
&= \sum_{l=0}^{k-2}\binom{k-2}{l}(-X)^{k-2-l}\int_{(\mathbb{R}_+)^n/U} i^{n(\ell+1)}N(y)^\ell f(iy)\,d\tau \text{ (substituting } \tau = iy) \\
&= \sum_{l=0}^{k-2}\binom{k-2}{l}(-X)^{k-l-2}i^{n(l+1)}\Lambda(f,l+1) \\
&= \sum_{\ell=0}^{k-2}(-1)^\ell i^{n(k-\ell-1)}\binom{k-2}{\ell}X^\ell\Lambda(f,k-\ell-1) \text{ (substituting } \ell \to k-\ell-2).
\end{aligned}$$

□

Corollary 3.2.1. *The period polynomial r_f of f satisfies*

$$r_f(X) = (-1)^n(k-2)!\left(\frac{D_K}{2\pi i}\right)^{k-1}\sum_{\ell=0}^{k-2}\frac{(-1)^{\ell(n+1)}\Gamma(k-\ell-1)^{n-1}}{\ell!}\left(\frac{(2\pi i)^nX}{D_K}\right)^\ell L(f,k-\ell-1).$$

Proof. From (3.2), we have that

$$\Lambda(f, k-\ell-1) = \left(\frac{D_K}{(2\pi)^n}\right)^{(k-\ell-1)} \Gamma(k-\ell-1)^n L(f, k-\ell-1).$$

Substituting this in the expression we got in Theorem 3.2.1, the result follows. □

3.3 Zeros of Period Polynomials

Recall the result we used for determining whether a polynomial has all of its roots on the unit circle (this is Lemma 2.1.2 from before):

Lemma 3.3.1. *A necessary and sufficient condition for all the zeros of a polynomial $P(z) = \sum_{j=0}^{d} a_j z^j \in \mathbb{C}[z]$ to lie on the unit circle is that there exists a polynomial $Q(z)$ of degree $\ell \leq d$, with all of its zeros inside or on the unit circle, such that*

$$P(z) = z^m Q(z) + \epsilon z^\ell \overline{Q}\left(\frac{1}{z}\right),$$

where $m = d - \ell$ and $\epsilon \in \mathbb{C}$ with $|\epsilon| = 1$.

In order to use this lemma, let K be a number field of degree n and f be a parallel weight k Hilbert modular eigenform. Put $m := \frac{k-2}{2}$ and define the two important polynomials $P_f(X)$ and $Q_f(X)$ by

$$P_f(X) = \frac{1}{2}\binom{2m}{m}\Lambda(f, \frac{k}{2}) + \sum_{j=1}^{m} \binom{2m}{m+j} \Lambda\left(f, \frac{k}{2}+j\right) X^j$$

and

$$Q_f(X) = \frac{1}{\Lambda(f, 2m+1)} P_f(X).$$

Then, similarly as before, we can show the following:

Proposition 3.3.1. *$r_f(i^{n+2}X)$ is self-inversive and can be written as*

$$r_f(i^{n+2}X) = i^{n(2m+1)}\epsilon(f)\Lambda(f, 2m+1)X^m \left[Q_f(X) + \epsilon(f) Q_f\left(\frac{1}{X}\right)\right].$$

Then, by the above Lemma, $r_f(X)$ would have all its zeros on the unit circle if and only if $Q_f(X)$ has all its zeros inside the unit circle. For that purpose, we will need the following results that we proved in the previous chapter, which hold in the case of Hilbert modular forms *mutatis mutandis*:

Lemma 3.3.2. *Let $f \in S_k^{new}(\Gamma_K)$. Then the function $\Lambda(f, s)$ is monotone increasing for $s \geq \frac{k}{2} + \frac{1}{2}$. Moreover, we have*

$$0 \leq \Lambda(f, \frac{k}{2}) \leq \Lambda(f, \frac{k}{2}+1) \leq \Lambda(f, \frac{k}{2}+2) \leq \dots$$

If $\epsilon(f) = -1$, then $\Lambda(f, \frac{k}{2}) = 0$ and

$$0 \leq \Lambda(f, \frac{k}{2}+1) \leq \frac{1}{2}\Lambda(f, \frac{k}{2}+2) \leq \frac{1}{3}\Lambda(f, \frac{k}{2}+3) \leq \dots$$

Lemma 3.3.3. *If $f \in S_k^{new}(\Gamma_K)$ and $0 \le a \le b$, then*

$$\frac{L(f, \frac{k+1}{2} + a)}{L(f, \frac{k+1}{2} + b)} \le \frac{\zeta(1+a)^{2n}}{\zeta(1+b)^{2n}}.$$

Remark. *Period polynomials for Hilbert modular eigenforms f of parallel weight 2, are constant multiples of $L(f,1)$. Hence, we will consider $k \ge 4$.*

Theorem 3.3.1. *For $k = 4$ and $k = 6$, $P_f(X) + \epsilon(f)P_f(1/X)$ has all its zeros on the unit circle.*

Proof. For $k = 4$, we have $m = 1$ so $P_f(X) = \Lambda(f,2) + \Lambda(f,3)X$. If $\epsilon(f) = -1$, then $\Lambda(f,2) = 0$ and so

$$P_f(X) - P_f\left(\frac{1}{X}\right) = \Lambda(f,3)\left(X - \frac{1}{X}\right),$$

which clearly has roots at $X = \pm 1$. If $\epsilon(f) = 1$, then for $X = e^{i\theta}$ on the unit circle,

$$P_f(X) + P_f\left(\frac{1}{X}\right) = 2\Lambda(f,2) + \Lambda(f,3)\left(X + \frac{1}{X}\right) = 2\Lambda(f,2) + 2\Lambda(f,3)\cos\theta.$$

By Lemma 3.3.2 we know that $\Lambda(f,2) < \Lambda(f,3)$, then the equation

$$\cos(\theta) = -\frac{\Lambda(f,2)}{\Lambda(f,3)}$$

has two solutions with $\theta \in [0, 2\pi)$.

For $k = 6$, we have $m = 2$ so $P_f(X) = 3\Lambda(f,3) + 4\Lambda(f,4)X + \Lambda(f,5)X^2$. If $\epsilon(f) = -1$, then $\Lambda(f,3) = 0$ and so

$$\begin{aligned} P_f(X) - P_f\left(\frac{1}{X}\right) &= 4\Lambda(f,4)\left(X - \frac{1}{X}\right) + \Lambda(f,5)\left(X^2 - \frac{1}{X^2}\right) \\ &= \left(X - \frac{1}{X}\right)\left[4\Lambda(f,4) + \Lambda(f,5)\left(X + \frac{1}{X}\right)\right]. \end{aligned}$$

Clearly $X = \pm 1$ are two solutions. Putting $X = e^{i\theta}$, we find that the other roots are the solutions of $\cos\theta = -\frac{2\Lambda(f,4)}{\Lambda(f,5)}$ for $\theta \in [0, 2\pi)$. From Lemma 3.3.2, we have that $2\Lambda(f,4) < \Lambda(f,5)$ and so their are two more roots which also lie on the unit circle. If $\epsilon(f) = 1$, letting $X = e^{i\theta}$ we have

$$P_f(X) + P_f\left(\frac{1}{X}\right) = 6\Lambda(f,3) + 8\Lambda(f,4)\cos\theta + 2\Lambda(f,5)\cos 2\theta.$$

We want to show this has two zeros with $\theta \in [0, \pi)$ and thus four zeros with $\theta \in [0, 2\pi)$. Note that

$$\frac{d}{d\theta}\left[P_f(e^{i\theta}) + P_f(e^{-i\theta})\right] = -8\sin\theta\left(\Lambda(f,4) + \Lambda(f,5)\cos\theta\right),$$

we have critical points at $0, \pi$ and the solution $\theta_0 \in [0, 2\pi)$ to $\cos\theta = -\frac{\Lambda(f,4)}{\Lambda(f,5)}$. To get two roots in $[0, \pi)$ we need $P_f(e^{i\theta}) + P_f(e^{-i\theta})$ to be positive at $\theta = 0$ and π and negative at

$\theta = \theta_0$. At $\theta = 0$, $P_f(e^{i\theta}) + P_f(e^{-i\theta}) = 6\Lambda(f,3) + 8\Lambda(f,4) + 2\Lambda(f,5) > 0$. Positivity at $\theta = \pi$ is equivalent to

$$3\Lambda(f,3) + \Lambda(f,5) > 4\Lambda(f,4)$$

while negativity at $\theta = \theta_0$ is equivalent to

$$2\Lambda(f,4)^2 + \Lambda(f,5)^2 < 3\Lambda(f,3)\Lambda(f,5).$$

The last two inequalities can be proved similarly as we did in Theorem 2.2.3 from before. $\square$

We now move to the case of large weights. We will compare $Q_f(X)$ to X^m and use Rouchés Theorem to show $Q_f(X)$ has all its zeros inside the unit circle. On $|X| = 1$, using equation (3.2), we have

$$\begin{aligned} Q_f(X) - X^m &= \frac{1}{2}\frac{\Gamma(m+1)^{n-2}}{\Gamma(2m+1)^{n-1}}\left(\frac{(2\pi)^n}{D_K}\right)^m \frac{L(f,m+1)}{L(f,2m+1)} \\ &\quad + \sum_{j=1}^{m-1} \frac{1}{(m-j)!}\left(\frac{(2\pi)^n}{D_K}\right)^{m-j}\left(\frac{\Gamma(m+1+j)}{\Gamma(2m+1)}\right)^{n-1}\frac{L(f,m+1+j)}{L(f,2m+1)}X^j \\ &= \frac{1}{2}\frac{\Gamma(m+1)^{n-2}}{\Gamma(2m+1)^{n-1}}\left(\frac{(2\pi)^n}{D_K}\right)^m \frac{L(f,m+1)}{L(f,2m+1)} \\ &\quad + \sum_{j=1}^{m-1} \frac{1}{j!}\left(\frac{(2\pi)^n}{D_K}\right)^{j}\left(\frac{\Gamma(2m+1-j)}{\Gamma(2m+1)}\right)^{n-1}\frac{L(f,2m+1-j)}{L(f,2m+1)}X^{m-j}. \end{aligned} \tag{3.3}$$

where the last equality follows by substituting $j \to m-j$ in the sum. Using Lemma 3.3.3, the fact that $\zeta(1/2)^2 \le \frac{11}{5}$, and Minkowski's bound, we obtain

$$\begin{aligned} |Q_f(X) - X^m| &\le \frac{1}{2}\frac{\Gamma(m+1)^{n-2}}{\Gamma(2m+1)^{n-1}}\left(\frac{(2\pi)^n}{D_K}\right)^m\left(\frac{\zeta(1/2)}{\zeta(1/2+m)}\right)^{2n} \\ &\quad + \sum_{j=1}^{m-1}\frac{1}{j!}\left(\frac{(2\pi)^n}{D_K}\right)^m\left(\frac{\Gamma(2m+1-j)}{\Gamma(2m+1)}\right)^{n-1}\left(\frac{\zeta(1/2+m-j)}{\zeta(1/2+m)}\right)^{2n} \\ &\le \frac{1}{2}\frac{\Gamma(m+1)^{n-2}}{\Gamma(2m+1)^{n-1}}\left(\frac{(2\pi)^n(n!)^2}{n^{2n}}\right)^m\left(\frac{11}{5}\right)^n \\ &\quad + \sum_{j=1}^{m-1}\frac{1}{j!}\left(\frac{(2\pi)^n(n!)^2}{n^{2n}}\right)^j\left(\frac{\Gamma(2m+1-j)}{\Gamma(2m+1)}\right)^{n-1}\left(\frac{\zeta(1/2+m-j)}{\zeta(1/2+m)}\right)^{2n} \\ &=: T_n(m). \end{aligned}$$

Therefore, we need to show that $T_n(m) < |X^m| = 1$ for $n \ge 2$ and m big enough. Since we then get by Rouchés Theorem that X^m and $Q_f(X) - X^m + X^m = Q_f(X)$ both have m zeros inside the unit circle; and thus, $Q_f(X)$ would have all of its zeros inside the unit circle, as required.

The numbers $T_n(m)$ are decreasing as n increases because each individual term is decreasing. We want to show that $T_n(m)$ is also decreasing in m. Therefore, once we have $T_2(m_0) < 1$ for some m_0, we then automatically get that $T_n(m) < 1$ for any $n \geq 2$ and $m \geq m_0$. We will do this by showing the following:

Theorem 3.3.2. *$T_n(m+1) - T_n(m) \leq 0$ for $n \geq 2$ and m big enough.*

Proof. We can write $T_n(m+1) - T_n(m)$ as

$$\begin{aligned}
&\frac{1}{2}\frac{\Gamma(m+1)^{n-2}}{\Gamma(2m+1)^{n-1}}\left(\frac{(2\pi)^n(n!)^2}{n^{2n}}\right)^m\left(\frac{11}{5}\right)^n\left[\frac{(2\pi)^n(n!)^2}{2^{n-1}(m+1)(2m+1)^{n-1}n^{2n}} - 1\right]\\
&+\sum_{j=1}^{m-2}\frac{1}{j!}\left(\frac{(2\pi)^n(n!)^2}{n^{2n}}\right)^j\left(\frac{\Gamma(2m+1-j)}{\Gamma(2m+1)}\right)^{n-1}\left(\frac{\zeta(1/2+m-j)}{\zeta(1/2+m)}\right)^{2n}\\
&\times\left[\left(\frac{(2m+2-j)(2m+1-j)}{(2m+2)(2m+1)}\right)^{n-1}\left(\frac{\zeta(1/2+m)\zeta(3/2+m-j)}{\zeta(3/2+m)\zeta(1/2+m-j)}\right)^{2n} - 1\right]\\
&+\frac{1}{(m-1)!}\left(\frac{(2\pi)^n(n!)^2}{n^{2n}}\right)^{m-1}\left(\frac{\Gamma(m+2)}{\Gamma(2m+1)}\right)^{n-1}\left(\frac{\zeta(3/2)}{\zeta(1/2+m)}\right)^{2n}\\
&\times\left[\frac{(2\pi)^n(n!)^2}{mn^{2n}}\left(\frac{m+2}{(2m+1)(2m+2)}\right)^{n-1}\left(\frac{\zeta(1/2+m)}{\zeta(3/2+m)}\right)^{2n}+\right.\\
&\left.\left(\frac{(m+3)(m+2)}{(2m+2)(2m+1)}\right)^{n-1}\left(\frac{\zeta(1/2+m)\zeta(5/2)}{\zeta(3/2+m)\zeta(3/2)}\right)^{2n} - 1\right]\\
&=: A_{n,m} + \sum_{j=1}^{m-2} B_{n,m,j} + C_{n,m}
\end{aligned}$$

where we have paired the extra $j = m$ factor of the sum in $T_n(m+1)$ with the two $j = m-1$ terms. We see that $A_{n,m} \leq 0$ for $m \geq 1$ and any $n \geq 3$ and as soon as $m \geq 4$ for $n = 2$. Moreover, we can use the facts that

$$\frac{1}{\zeta(3/2+m)} \leq 1, \quad \frac{\zeta(1/2+m)}{\zeta(1/2+m-j)} \leq 1, \quad \zeta(3/2+m-j)^2 \leq \frac{8}{5}2^{j-m} + 1$$

to show that $B_{n,m,j} \leq 0$ once

$$\left(\frac{(2m+2-j)(2m+1-j)}{(2m+2)(2m+1)}\right)^{n-1}\left(\frac{8}{5}2^{j-m} + 1\right)^n \leq 1. \tag{3.4}$$

This expression is decreasing in j, so it suffices to show (3.4) only for the $j = 1$ term. This case is equivalent to $\left(\frac{m}{m+1}\right)^{n-1}\left(\frac{16}{5}2^{-m} + 1\right)^n \leq 1$ which one can check is true once $m \geq 6$ for any $n \geq 2$. Once we know the inequality is satisfied for $m \geq 6$, we can go back to $B_{n,,m,j}$ and check the remaining values of m directly. We find that $B_{n,m,j} \leq 0$ for any $m \geq 1$ for $n \geq 2$. It remains to find when is $C_{n,m} \leq 0$. In this case, we must show that

$$\begin{aligned}
&\frac{(2\pi)^n(n!)^2}{mn^{2n}}\left(\frac{m+2}{(2m+1)(2m+2)}\right)^{n-1}\left(\frac{\zeta(1/2+m)}{\zeta(3/2+m)}\right)^{2n}\\
&+\left(\frac{(m+3)(m+2)}{(2m+2)(2m+1)}\right)^{n-1}\left(\frac{\zeta(1/2+m)\zeta(5/2)}{\zeta(3/2+m)\zeta(3/2)}\right)^{2n} \leq 1,
\end{aligned}$$

which occurs once $m \geq 3$ for $n = 2$ and $m \geq 2$ for $n \geq 3$. $\square$

We have shown that $T_n(m)$ is decreasing in both n and m; so we just need to find an m_0 such that $T_2(m_0) < 1$. A computer calculation shows this first occurs for $m = 8$. For higher degrees we can run this calculation again to reduce the number of cases that need to be checked explicitly. For example, $T_3(m) < 1$ once $m \geq 5$; and $T_n(m) < 1$ for $m \geq 3$ once $n \geq 5$. We further reduce the number of remaining cases by allowing the discriminant to vary. For $n = 2$, we have the following table that shows the inequality is satisfied once m is big enough depending on the discriminant:

D_K	5	8	12	13	17	21	24	29	33	≥ 35
$m \geq$	7	6	5	5	4	4	4	4	4	3

Similarly, for $n = 3$ the inequality is satisfied for $m \geq 3$ once we have $D_K \geq 84$. The only other case we need to check is $n = 4$. The inequality is true for $m \geq 3$ once we have $D_K \geq 209$, and the totally real quartic field K with smallest discriminant has $D_K = 725$.

3.3.1 Remaining Cases

There are finitely many remaining cases from the previous section to check. These are totally real quadratic fields ($n = 2$) with discriminant $D_K < 35$, and totally real cubic fields ($n = 3$) with $D_K < 84$. Note that in the cubic case, we only need to consider the two totally real fields with discriminants 49 and 81, and the only special case is $m = 3$. We check these cases computationally using Magma [17].

The main ingredient for our computations consist of obtaining eigenbases for subspaces of cusp forms and creating L-functions for these forms. Once these are obtained, we check that the roots are on the unit circle by testing the inequality $|Q_f(X) - X^m| < 1$ as in Equation (3.3) from the previous section. All the codes we used can be found in the GitHub repository `https://github.com/ababei/HilbertModularFormsCubic` maintained by Angelica Babei.

The main function is `rfx(f:Precision; Embedding)`, where f is the Hilbert modular form, Precision is the precision in $\mathbb{C}$ of the coefficients, and Embedding is a choice of complex embedding $F \hookrightarrow \mathbb{C}$, where the Fourier coefficients of f are defined over F. We obtain the various complex embeddings via the function `CompEmb(f:Bound)`. Another important function is `Lam(f,s:Precision; Embedding)`, the completed L-function of the Hilbert modular form at s, again given a precision in $\mathbb{C}$ and an embedding $F \hookrightarrow \mathbb{C}$. The function `CheckRoots(f:Prec;Emb)` checks that the Equation (3.3) holds, which is sufficient for the roots to be on the unit circle.

For the quadratic fields case, there already is a Hilbert Modular Forms framework in Magma, see `https://magma.maths.usyd.edu.au/magma/handbook/hilbert_modular_forms`. Although there is already an L-series function `LSeries` implemented in Magma, it only takes one complex embedding $F \hookrightarrow \mathbb{C}$, so we defined a new function `LSeriesH(f:Precision;Emb)` defined in "Elements.m" in the repository above, where we did a small adjustment to the preexisting code in Magma to incorporate any given complex embedding.

The code in "PeriodPolynomialsHMF.m" gives two lists, one of which is `PerPolys` and contains all the period polynomials needed to be checked, and another list called `RemainingPerPolys`. We first verify that the inequality $|Q_f(X) - X^m| < 1$ holds, using `CheckRoots`. The inequality holds for all but 11 polynomials. In such cases, we check that the trigonometric polynomial $P_f(X) + \epsilon(f)Pf(\frac{1}{X})$ with $X = e^{i\theta}$ have the necessary number of roots on the interval $[0, \pi)$ as in Theorem 3.3.1.

For the cubic fields case, their is an algorithm described in [5] for reconstructing full spaces of weight k for a cubic field K. The only source of Hilbert modular forms for cubic fields at that point was via Eisenstein series, which we access with the function `EisensteinSeries(M,N,eta,psi,k)` defined in "Elements.m", and which takes the space of Hilbert modular forms, the level N, two Hecke characters η and ψ , and a sequence $k = [k_1, k_1, k_1]$ of parallel even weights. One can then find the subspace of cusp forms, extract a basis of eigenforms by finding matrices of Hecke operators, and then construct the L-series. To check that the period polynomials have roots on the unit circle, we only needed to check that Equation (3.3) holds. Since it does in all our cases, we did not need to compute the trigonometric polynomials, or the roots of the period polynomial themselves.

We finally note that there is a newer and more complete framework for quadratic fields with more functionality, which can be found in the GitHub repository: `https://github.com/edgarcosta/hilbertmodularforms`.

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