AMERICAN UNIVERSITY OF BEIRUT

STRUCTURE OF CERTAIN RINGS WITH CONDITIONS ON NONPERIODIC ELEMENTS AND CERTAIN VON NEUMANN π-REGULAR RINGS WITH PRIME CENTERS

by
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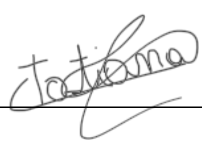
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An Abstract of the Thesis of

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Major: Mathematics

Title: Structure of certain rings with conditions on nonperiodic elements and certain Von Neumann π-regular rings with Prime Centers.

First, we study the structure and commutativity of rings with the property that for each nonperiodic element x, there exists a positive integer K=K(x), such that x^k is central for all $k \geq K$.
Then we study rings with certain conditions on zero divisors, for example we prove that a periodic ring with identity and commuting nilpotents, such that every zero divisor is either idempotent or nilpotent, then N is an ideal and R/N is either Boolean or a field.
We also study rings with prime or semi prime center, in particular we study the structure of certain Von-Neumann π-regular rings with certain constraints such as having prime centers and other constraints.
The structure of rings with other conditions on elements will also be studied.
(Please note that the results in this thesis are essentially based on papers: [1], [2], [3], [4])

Contents

Chapter 1

Introduction and Preliminary

In this chapter we introduce some definitions and preliminary theorems that we will use in the proofs of the results in the next chapters.
Throughout this chapter, we let R be an associative ring.

1.1 Basic Definitions and Theorems:

Definition 1.1.1: An element a $\in$ R is said to be a *nilpotent* element if $a^n = 0$ for some positive integer n. We will denote by N the set of nilpotent elements of R.
A ring R is said to be *Reduced* if N $= \{0\}$.

Definition 1.1.2: An ideal I (left, right or 2 sided ideal) is said to be *nil* if every element in I is nilpotent. An ideal I is said to be *nilpotent* if $\mathrm{I}^n = 0$ for some positive integer n.

Definition 1.1.3: An element a $\in$ R is said to be a right (respectively left) *zero divisor* if there exists an element b $\in$ R; b $\neq 0$ satisfying ba $= 0$ (respectively ab $= 0$). We will denote by D the set of all right zero divisors of R. An element a is said to be a zero divisor it is either a right or a left zero divisor.

Definition 1.1.4: A nonzero element a $\in$ R is said to be *regular* if it is neither a left nor a right zero divisor. A left (respectively right) ideal I of R is said to be *regular* if there is e $\in$ R such that r - re (respectively r - er) $\in$ I for all r $\in$ R.

Definition 1.1.5: An element a $\in$ R is said to be *periodic* if $x^n = x^m$ for some distinct positive integers n and m. We will denote by P the set of all periodic elements of R. An element a $\in$ R is said to be *potent* if $a^n = a$ for some positive integer n. We will denote by P_0 the set of all potent elements of R.

Remark 1.1.1: $P_0 \subseteq$ P.

Definition 1.1.6: An element a $\in$ R is said to be idempotent if $a^2 = a$. A ring R is *Boolean* if every element in R is idempotent.

Lemma 1.1.1: Let R be a ring. If x is both idempotent and nilpotent in R, then
x = 0.

Proof: Let x be an element that is both idempotent and nilpotent in R.
Assume x $\neq$ 0.
Let k $\geq$ 2 be the least positive integer such that $x^k = 0$.
So $x^{k-2}x^2 = 0$.
But x in idempotent, hence $x^{k-1} = 0$ which contradicts the fact that k is the least positive integer satisfying $x^k = 0$.
Therefore x = 0.

Definition 1.1.7: An element a $\in$ R is said to be a central element if it commutes with every element of R. i.e. ab = ba $\forall$ b $\in$ R. We will denote by C the set of all central elements of R.

Lemma 1.1.2: Let R be a ring with no nonzero nilpotent elements, then every idempotent element is central.

Proof: Let e be an idempotent element.
Let x$\in R$, then:
$(xe - exe)^2$ = xexe - xeexe - exexe - exeexe
= xexe - xe^2xe - exexe - exe^2xe
= 0.
Similary for $(ex - exe)^2 = 0$.
So ex - exe and xe - exe are nilpotent elements, hence ex = exe = xe.
Thus ex = xe for all x $\in R$. Therefore e is central.

Definition 1.1.8: We define the commutator [x,y] = xy - yx. The ideal generated by all commutators in R is called the *commutator ideal* of R, it will be denoted by C(R).

Definition 1.1.9: An ideal M of R is said to be a *maximal ideal* if it satisfies the following properties:

- M $\neq$ R,
- If M $\subseteq$ N where N is an ideal of R, then M = N or N = R.

An ideal M of R is said to be a *minimal ideal* if it satisfies the following properties:

- M $\neq$ 0,
- If N $\subseteq$ M where N is an ideal of R, then M = N or N = 0.

Lemma 1.1.3: Let R be a ring with identity, and let I be a proper ideal of R. Then there is a maximal ideal of R containing I.

Proof: Consider $\mathcal{E}$ = {J proper ideal of R such that I $\subseteq$ J}.

- $\mathcal{E}$ is nonempty since I $\in \mathcal{E}$.
- Let $\mathcal{C}$ be an arbitrary chain of ideals in $\mathcal{E}$, then $\bigcup \mathcal{C}$ is an element of $\mathcal{E}$ and hence an upper bound of $\mathcal{C}$ in $\mathcal{E}$.

Thus by Zorn's Lemma, $\mathcal{E}$ has a maximal element M that is a maximal ideal of R containing I.

Definition 1.1.10: A (left) module A over a ring R is said to be a *simple* R-module if RA $\neq$ 0 and A has no proper submodules. A ring R is said be *simple* if $R^2 \neq 0$ and R has no proper ideals (two sided).

Definition 1.1.11: An ideal P of R is said to be *prime* if P $\neq$ R and for any ideal A, B of R, AB $\subseteq$ P implies A $\subseteq$ P or B $\subseteq$ P.

Theorem 1.1.1: If P is an ideal in a ring R such that P $\neq R$ and for all a, b $\in$ R;
ab $\in$ P implies a $\in$ P or b $\in$ P, then P is prime.
Conversely, if P is prime and R is commutative, then P satisfies the above relation.

Proof: See [12, p.127]

Definition 1.1.12: Let M be a left R-module.
Then $\mathcal{A}$(M) = { r $\in$ R; rm = 0 for all m $\in$ M } is called *the left annihilator of M*. If $\mathcal{A}$(M) = 0 then M is said to be a *Faithful* left R-module. The right annihilator is defined analogously

Definition 1.1.13: A ring R is said to be (left) *primitive* if there exists a simple faithful (left) R-module.

Definition 1.1.14: An element a in R is said to be *left (respectively right) quasi-regular* if there exists r $\in$ R such that r + a + ra = 0 (resp. a + r + ar = 0). The element r is called a left (respectively right) quasi-inverse of a. A (left, right or two-sided) ideal I of R is said to be left quasi-regular if every element of I is left quasi-regular. Right quasi-regular ideal is defined analogously.

Lemma 1.1.4: If x is nilpotent, then x is right quasi-regular.

Proof: Since x is nilpotent, then $x^n = 0$ for some positive integer n.

Let r = - x + x^2 - x^3 + ... + $(-1)^{n-1}x^{n-1}$. Then x + r + xr = 0.
Therefore x is right quasi-regular.

1.2 Subdirect Product of a Family of Rings:

Definition 1.2.1: A ring R is said to be a *subdirect product (or sum) of the family of rings* $\{ R_i \mid i \in I \}$ if R is a subring of the direct product $\prod_{i\in I} R_i$ such that for all k $\in$ I, $\amalg_k$: R $\longrightarrow$ R_k is the canonical epimorphism.
If R is isomorphic to a subdirect product T of rings R_i, i$\in$ I, T is called a *representation of R as a subdirect product of rings* R_i.

Example 1.2.1: The direct product $\prod_{i\in I} R_i$ is itself a subdirect product of the rings R_i. There could be other subdirect products of the rings R_i.

Theorem 1.2.1: A ring R has a representation as a subdirect product of rings R_i if and only if for each i$\in$ I, there exists an epimorphism ϕ_i: R $\longrightarrow$ R_i such that if
r $\neq 0$ then $\phi_k(r) \neq 0$ for at least one k $\in$ I.

Proof:
($\Longrightarrow$) Let T be a subdirect product of rings R_i, i $\in$ I.
Let f: R $\longrightarrow$ T be an isomorphism.
Let $\amalg_i$: T $\longrightarrow$ R_i be the canonical epimorphism.
Let ϕ_i: R $\longrightarrow$ R_i be defined by $\phi_i = \amalg_i \circ f$. Then ϕ_i is clearly onto being a composition of 2 onto functions.

Let r $\neq 0$ in R, then f(r)$\neq 0$ (bijective).
Now assume that $\phi_i = 0$ for all i $\in$ I, then $\amalg_i(f(r)) = 0$ for all i $\in$ I. Hence f(r) = 0 since $\amalg_i$ is a projection for all i $\in$ I; a contradiction.
Hence there exists k $\in$ I such that $\phi_k(r) \neq 0$.

($\Longleftarrow$) Assume for each i $\in$ I, there exist epimorphisms
ϕ_i: R $\longrightarrow$ R_i such that if r $\neq 0$, then $\phi_i(r) \neq 0$ for at least one i.
For each r $\in$ R, we assign $f_r \in \prod R_i$ where $f_r = \{\phi_i(r)\}_{i\in I}$, that is, the i^{th} component of f_r is $\phi_i(r)$.
Define ψ: R $\longrightarrow \prod_{i\in I} R_i$ by $\psi(r) = f_r \in \prod R_i$.

(a) ψ is a homomorphism since each ϕ_i is a homomorphism.

(b) ψ is one to one:

$$\begin{aligned} \mathrm{Ker}\psi &= \{r \in R;\ \psi(r) = 0\} \\ &= \{r \in R;\ \psi(r) = 0 = f_r = \{\phi_i(r)\}_{i\in I}\} \\ &= \{r \in R;\ \phi_i(r) = 0\ \forall\ i \in I\} \\ &= \{0\} \text{ since } \phi_i(r) \neq 0 \text{ for at least one i.} \end{aligned}$$

Thus by the first isomorphism theorem we get, R $\cong \psi(R)$,
i.e. R isomorphic to a subring of $\amalg R_i$.

(c) $\psi(R)$ is a subdirect product:
we need to show that $\amalg_i(\psi(R))$=R_i.

Let $r_i \in R_i$, then there exists r ∈ R such that $\phi_i(r) = r_i$, since ϕ_i is onto.
Now consider $f_r = \{\phi_j(r)\}_{j\in I} \in \psi(R)$ where $\phi_j(r) = r_j$ if i = j and $\phi_j(r) = 0$ if $i \neq j$.
Then $\Pi_i(f_r) = \Pi_i(\{\phi_j(r)\}_{j\in I}) = \phi_i(r) = r_i \in R_i$.
So $\Pi_i|_{\psi(R)}$ is onto R_i and thus $\Pi_i(\psi(R)) = R_i$.

Therefore, $\psi(R)$ is a subdirect product.

Theorem 1.2.2: A ring R has a representation as a subdirect product of rings R_i if and only if for each i∈I, there exists an ideal K_i of R such that $R/K_i \cong R_i$ and $\bigcap_{i\in I} K_i = \{0\}$.

Proof:
(⟹) R has a representation as a subdirect product of rings R_i, then for each i ∈ I there exists an epimorphism ϕ_i: R ⟶ R_i such that $\phi_i(r) \neq 0$ for $r \neq 0$ for some i. So by first isomorphism theorem $R/\text{Ker}\phi_i \cong R_i$, and call $\text{Ker}\phi_i = K_i$.
Now let $r \in \bigcap_{i\in I} K_i$, then $r \in \text{Ker}\phi_i$ ∀ i ∈ I. Hence $\phi_i(r) = 0$ for all i, which implies that r = 0 by hypothesis. Therefore $\bigcap_{i\in I} K_i = \{0\}$.

(⟸) Suppose that there are ideals K_i such that $R/K_i \cong R_i$ and $\bigcap_{i\in I} K_i = \{0\}$.
Let $\psi_i : R/K_i \longrightarrow R_i$ be an isomorphism.
Define $\phi_i : R \longrightarrow R_i$ by $\phi_i(r) = \psi_i(r + K_i) \in R_i$.
Clearly $\text{Ker}\phi_i = \{r \in R; \phi_i(r) = 0 = \psi_i(r + K_i)\}$
$= \{r \in R; r + K_i = K_i\}$ since ψ_i is one to one
$= K_i$.
Now let $\phi_i(r) = 0$ for all i, then $\psi_i(r + K_i) = 0$ for all i.
Hence $r \in K_i$ for all i, which implies that $r \in \bigcap_{i\in I} K_i = 0$. Thus r = 0.
So by Theorem 1.1, R has a *representation as a subdirect product of rings R_i.*

Definition 1.2.2: A ring R is said to be *subdirectly irreducible* if the intersection of all nonzero ideals is not zero.

Example 1.2.2: A division ring R has no proper nonzero ideals, and hence the intersection of all nonzero ideals is $R \neq \{0\}$. Thus, any division ring is subdirectly irreducible.

Lemma 1.2.1: Let R be a subdirectly irreducible ring with no nonzero nilpotent elements. Then every idempotent element is either 0 or 1.

Proof: Let e be an idempotent element of R.
Consider the set I = {r - er; r ∈R }, I is an ideal of R.
In fact,

- $0 = 0$ - e.0 so $0 \in I$
- Let x - ex and y - ey be two elements in I with x,y $\in$ R, then
  (x - ex) - (y - ey) = (x - y) - e(y - x) $\in$ I.
- let x - ex $\in$ I with x,r $\in$ R, then
  (x - ex)r = xr - (ex)r = xr - e(xr) $\in$ I.
- let x - ex $\in$ I with x,r $\in$ R, then
  r(x - ex) = rx - r(ex) = rx - (re)x = rx - e(rx) $\in$ I (since e is central by Lemma 1.2)

Consider also the ideal eR
Assume e is neither 0 nor 1, so I and eR are nonzero proper ideals of R.

Now let x $\in$ eR $\bigcap$ I.
Then x = er = y - ey for r, y $\in$ R.
So ex = er = 0 since e is idempotent, which implies that x = er = 0
Hence eR $\bigcap I = \{0\}$ with eR and I two nonzero ideals, then the intersection of two nonzero ideals of R is zero; which is a contradiction since R is subdirectly irreducible.

Therefore e = 0 or e = 1.

Lemma 1.2.2: Let R be a subdirectly irreducible ring with central idempotents. Then every idempotent element is either 0 or 1.

Proof: The proof is a consequence of Lemma 1.2.1.

Theorem 1.2.3: Birkhoff's theorem Every ring R has a representation as a subdirect product of subdirectly irreducible rings.

Proof: Let R $\neq 0$, and take a $\neq 0$ element in R.
Let $\mathcal{U} = \{$ I ; I ideal of R not containing a $\}$
Then $\mathcal{U} \neq \varnothing$ since $\{0\} \in \mathcal{U}$.
Now let $\mathcal{C}$ be a chain of ideals in $\mathcal{U}$. Then $\bigcup \mathcal{C}$ is an ideal of R;

- a,b $\in \bigcup \mathcal{C}$, then a $\in$ C_1 and b $\in$ C_2 with $C_1 \subset C_2$, so a,b $\in$ C_2.
  Hence a - b $\in C_2$, thus a - b $\in \bigcup \mathcal{C}$.
  Also ra $\in$ C_1, then ra $\in \bigcup \mathcal{C}$ for all r $\in$ R.
- a $\notin \bigcup \mathcal{C}$ since all ideals in $\mathcal{C}$ do not contain a.

So every chain in $\mathcal{U}$ has an upper bound in $\mathcal{U}$.
Therefore by Zorn's lemma, $\mathcal{U}$ has a maximal element, call it K_a.
For each a $\neq 0$, choose K_a to be an ideal maximal in the set of ideals not containing a.

Now we will show that R is isomorphic to a subdirect product of the rings R/K_a, for all a $\neq 0$.
Using Theorem 1.2.2, we need to show that $\bigcap\limits_{a \in R} K_a = 0$ and R/K_a is subdirectly irreducible for all a $\in$ R.

Let $0 \neq$ r $\in \bigcap\limits_{a \in R}$ K$_a$, then r $\in$ K$_a$ for all a $\neq 0$ in R. Hence r $\in K_r$, which is a contradiction. So $\bigcap\limits_{a \in R}$ K$_a$ $= 0$ and R is a subdirect product of the rings R/K$_a$.

To show that R/K$_a$ is subdirectly irreducible, we let N/K$_a$ be any nonzero ideal of R/K$_a$, N ideal of R. Then N is an ideal of R containing K$_a$. But K$_a$ is the maximal ideal in the set of ideals not containing a, so a $\in$ N.
Hence a + K$_a$ $\in N/K_a$.This implies that every ideal N/K$_a$ of R/K$_a$ contains a + K$_a$.
Therefore the intersection of all nonzero ideal of R/K$_a$ is nonzero.Thus, R/K$_a$ is subdirectly irreducible.

1.3 Prime and Semiprime Rings:

Definition 1.3.1: An ideal Q is said to be a *semiprime ideal* if I$^2 \subseteq$ Q then I $\subseteq Q$.

Remark 1.3.1: Every prime ideal is semiprime.

Theorem 1.3.1: The following are equivalent;

(a) Q is a semiprime ideal of R.
(b) For a $\in$ R, aRa $\subseteq Q \Rightarrow a \in Q$.
(c) For a $\in R, (a)^2 \subseteq Q \Rightarrow a \in Q$.

Proof:
(a) $\Rightarrow$ (b):
Suppose Q is a semiprime ideal.
Let a $\in$ R, aRa $\subseteq$ Q, then
R(aRa)R $\subseteq$ Q (Q ideal) $\Rightarrow$ (RaR)(RaR) $\subseteq$ Q
$\Rightarrow$ (RaR)$^2 \subseteq$ Q
$\Rightarrow$ (RaR) $\subseteq$ Q (Q semiprime and (RaR) is an ideal)
$\Rightarrow$ a $\in$ Q since a $\in$ (RaR).

(b) $\Rightarrow$ (c):
Suppose (a)$^2 \subseteq Q$ for a $\in$ R, then
(aR)(aR) $\subseteq$ (a)(a) $\subseteq$ (a)$^2 \subseteq$ Q $\Rightarrow$ a $\in$ Q.

(c) $\Rightarrow$ (a):
Suppose I$^2 \subseteq$ Q for I ideal, and let a $\in$ I, then
(a) $\subseteq$ I $\Rightarrow$ (a)$^2 \subseteq$ I^2 $\Rightarrow$ (a)$^2 \subseteq$ Q $\Rightarrow$ a $\in$ Q.
Hence I $\subseteq$ Q, and Q is a semiprime ideal.

Definition 1.3.2: For any ring R and any ideal I of R, $\sqrt{I}$ is the intersection of all prime ideals of R containing I. In particular $\sqrt{I}$ is a prime ideal, called the prime radical of I.

The prime radical P(R) of a ring R, also denoted by $\sqrt{0}$, is the intersection of all prime ideal of R. If R has no prime ideals then P(R) = R.

Definition 1.3.3: A ring is said to be a *prime ring* if $\{0\}$ is a prime ideal.
A ring R is said to be a *semiprime ring* if $\{0\}$ is a semiprime ideal.

Theorem 1.3.2: A ring R is semiprime if and only if R has no nonzero nilpotent ideals.

Proof:
($\Rightarrow$) Suppose R is a semiprime ring, that is $\{0\}$ is a semiprime ideal.
Let N be a nilpotent ideal of R, and k be the least positive integer such that $N^k = 0$.
And assume $N \neq 0$. We consider 2 cases:

If k is even, then $(N^{k/2})^2 = 0 \Rightarrow N^{k/2} = 0$ since $\{0\}$ is a semiprime ideal. This contadicts the fact that k is the smallest positive integers for which $N^k = 0$.

If k is odd then $N^{k+1} = 0 \Rightarrow (N^{k+1/2})^2 = 0 \Rightarrow N^{k+1/2} = 0$ since $\{0\}$ is a semiprime ideal. This also is a contradiction.
Hence N = 0.

($\Leftarrow$) Suppose R has no nonzero nilpotent ideals.
Let $I^2 \subseteq \{0\} \Rightarrow I^2 = 0$. So I a nilpotent ideal, hence I = 0.
Then $I \subseteq \{0\}$.
Therefore $\{0\}$ is a semiprime ideal of R, and R is a semiprime ring.

Example 1.3.1:
(a) Any domain is a prime ring.
(b) Any reduced ring is a semiprime ring

Proof:

(a) Let a, b $\in$ R and suppose aRb = $\{0\}$,
Then arb = 0 $\forall$ r $\in$ R $\Rightarrow$ ar = 0 or b = 0 $\forall$ r $\Rightarrow$ a = 0 or b = 0

(b) R is a reduced ring, so R has no nonzero nilpotent elements
$\Rightarrow$ R has no nonzero nilpotent ideals
$\Rightarrow$ R is semiprime.

Theorem 1.3.3: Let R be a ring, and Q an ideal of R.

(a) Q is a prime ideal if and only if R/Q is a prime ring.
(b) Q is a semiprime ideal if and only if R/Q is a semiprime ring.

Proof:

(a)
R/Q prime ring $\Leftrightarrow$ (a + Q)R/Q(b + Q) = Q implies a + Q = Q or b + Q = Q
$\Leftrightarrow$ aRb $\subseteq$ Q implies a $\in$ Q or b $\in$ Q

$\Leftrightarrow$ Q is a prime ideal.

(b)

R/Q semiprime ring $\Leftrightarrow$ {0 + Q} semiprime ideal of R/Q
$\Leftrightarrow$ (a + Q)R/Q(a + Q) = Q implies a + Q =Q
$\Leftrightarrow$ aRa $\subseteq$ Q implies a $\in$ Q
$\Leftrightarrow$ Q is a semiprime ideal.

Theorem 1.3.4: If P(R) is the prime radical of R, then R/P(R) is a semiprime ring.

Proof: P(R) is a prime ideal hence a semiprime ideal. So by Theorem 1.2.6 R/P(R) is a semiprime ring.

Theorem 1.3.5: The prime ideal P(R) contains every nilpotent right (left) ideal of R.

Proof: If I is a nilpotent right (left) ideal of R, then $I^n = \{0\}$ for some positive integer n. Hence $I^n \subseteq P(R)$ and P(R) is prime, so I $\subseteq$ P(R).

Corollary 1.3.1: If R is commutative, then P(R) is the ideal consisting of all nilpotent elements, and hence P(R) is the largest nil ideal of R.

Proof: If a is a nilpotent element of R, then the ideal (a) is a nilpotent ideal of R (since R is commutative).
So (a) $\subseteq$ P(R) by Theorem 1.3.5, and hence a $\in$ P(R).

Theorem 1.3.6: A ring R is semiprime if and only if R is isomorphic to a subdirect product of prime rings.

Proof: R is semiprime $\Leftrightarrow$ $P(R) = \{0\} = \bigcap \{ P_\alpha; P_\alpha$ is a prime ideal of R }
$\Leftrightarrow$ R is isomorphic to a subdirect product of rings R/P_α by Theorem 1.2.2, with each R/P_α a prime ring by Theorem 1.3.3.

1.4 Jacobson Radical and Semisimple Rings:

Theorem 1.4.1: If R is a ring, then there is an ideal J(R) of R such that:

(a) J(R) is the intersection of all left annihilators of simple R-modules.
(b) J(R) is the intersection of all regular maximal left ideals of R.
(c) J(R) is a left quasi-regular left ideal which contains every left quasi-regular left ideal of R.

Statements (a) and (c) are true if "left" is replaced by "right".

The ideal J(R) is called the *Jacobson Radical* of the ring R.

Proof: See [12, p.426]

Lemma 1.4.1: If $x^2 \in J(R)$, then x is right quasi-regular.

Proof: Since $x^2 \in J(R)$, then $-x^2 \in J(R)$; hence $-x^2$ is right quasi-regular.

So there exists $r \in R$ such that $r - x^2 + (-x^2)r = 0$.
Now let $t = r - x - xr$. Then,

$$\begin{aligned} t + x + xt &= r - x - xr + x + x(r - x - xr) \\ &= r - x^2 - x^2r = 0. \end{aligned}$$

Therefore, x is right quasi-regular.

Lemma 1.4.2: Let $a \in R$.
If ax is right quasi-regular for all $x \in R$, then $a \in J = J(R)$.

Proof: Consider $aR = \{ ax ; x \in R \}$. aR is a right ideal of R.
Since ax is right quasi-regular for all $x \in R$, then aR is a right quasi-regular right ideal.

Now, let $A = \{ ax + na ; x \in R, n \in Z \}$. A is a right ideal that contains a and aR.
Let $s = ax + na \in A$, then

$$\begin{aligned} -s^2 &= -(ax + na)^2 \\ &= -[a(xax) + a(x.na) + a(n.ax) + a(n^2.a)] \\ &= a[-xax + n.xa + n.ax + n^2.a] \end{aligned}$$

Hence $-s^2 \in aR$ which gives $-s^2$ is right quasi regular. Then, by the proof of Lemma 1.4.1, we can get that s is right quasi-regular. But s was arbitrarily chosen from A, and hence A is a right quasi-regular right ideal.

However, by Theorem 1.4.1, the Jacobson radical J(R) contains every right quasi-regular right ideal. That is, $A \subseteq J$.
Since $a \in A$, then $a \in J$ which ends the proof.

Theorem 1.4.2: J(R) contains every left (or right) nil ideal of R.

Proof: Suppose $a^n = 0$.
Let $r = - a + a^2 - \ldots + (-1)^{n-1}a^{n-1}$, then

$$\begin{aligned} r + a + ra &= (- a + a^2 - \ldots + (-1)^{n-1}a^{n-1}) + a + (-a^2 + \ldots + (-1)^n a^n) \\ &= 0. \end{aligned}$$

Similarly for $a + r + ar = 0$.

So every nilpotent element is left quasi-regular and right quasi-regular.

Then every nil left (right) ideal is left (right) quasi-regular ideal and hence contained in J(R).

Definition 1.4.1: A ring R is said to be *semisimple* if $J(R) = 0$.

A ring R is said to be a *radical ring* if $J(R) = R$.

Theorem 1.4.3: Let R be a ring. Then R/J(R) is semisimple.

Proof: Let J = J(R). We want to show that J(R/J) = 0.
We consider the canonical epimorphism,

$$\pi : \text{R} \rightarrow \text{R/J with } \pi(\text{r}) = \text{r} + \text{J} = \bar{r};$$

and let $\mathcal{C}$ be the collection of all regular maximal left ideals of R.

By definition of J(R), J(R) $\subseteq$ I for all I $\in \mathcal{C}$.

I/J is a maximal left ideal of R/J since I is a maximal ideal in R. Since I is regular, then there exists an element e $\in$ R with r - re $\in$ I
$\forall$ r $\in$ R;

We want to show I/J is regular.
Let, r $\in$ R, then

$$\bar{r} - \bar{r}.\bar{e} = \pi(\text{r - re}) \in \pi(\text{I}) = \text{I/J for all r} \in \text{R}.$$

So I/J is a regular maximal left ideal of R/J.

If $\bar{r} \in \bigcap\{\pi(\text{I}); \text{I} \in \mathcal{C}\}$ then $\bar{r} \in \bigcap \{\text{I/J}; \text{I} \in \mathcal{C}\}$.
Hence, r $\in$ I for each I $\in \mathcal{C}$, hence r $\in$ J(R).
So, $\bar{r} = 0$ in R/J, and hence J(R/J) = 0.

Therefore, R/J is semisimple.

Theorem 1.4.4: A ring R is *semisimple* if and only if it has a representation as a subdirect product of *primitive rings.*

Proof: See [12, p.435]

1.5 Chain Conditions:

Definition 1.5.1: A module A is said to satisfy the *ascending chain condition (ACC) on submodules* (also called *Noetheiran*) if for every chain
$A_1 \subseteq A_2 \subseteq ... \subseteq ...$ of submodules of A, there exists an integer n such that
$A_k = A_n$ $\forall k \geq$n (the chain stops).
A module A is said to satisfy the *descending chain condition (DCC) on submodules* (also called *Artinian*) if for every chain $A_1 \supseteq A_2 \supseteq ... \supseteq ...$ of submodules of A, there exists an integer n such that $A_k = A_n$ $\forall k \geq$n (the chain stops).

Definition 1.5.2: A (left) R-modules A is said to satisfy the minimum condition on submodules if every nonempty set of submodules of A has a minimal element with respect to inclusion.
A (left) R-modules A is said to satisfy the maximal condition on submodules if every nonempty set of submodules of A has a maximal element with respect to inclusion.

Theorem 1.5.1: A (left) R-module A satisfies the minimum condition if and only if it satisfies the DCC.And it satisfies the maximum condition if and only if it satisfies the ACC.

Proof:
(⇒) Suppose A satisfies the minimum condition on submodules of A. Let

$$A_1 \supset A_2 \supset ... \supset ...$$

be a strictly decreasing sequence of submodules of A that does not stop. Then the set { A_k ; k positive integer } is a nonempty set of submodules of A without a minimal element.
This is a contradiction since A satisfies the minimum condition on submodules. Hence A is Artinian.

(⇐) Suppose A satisfies the DCC on submodules.
If A does not satisfy the minimum condition, then there is a non empty set $\mathcal{E}$ of submodules of A with no minimal element.
Let A_0 be any element of $\mathcal{E}$ and let

$$A_0 \supset A_1 \supset A_2 \supset ...A_k$$

be a strictly decreasing chain of submodules of A.

Now since $\mathcal{E}$ has no minimal element, there is a submodules A_{k+1} in $\mathcal{E}$ such that $A_k \supset A_{k+1}$. So $A_0 \supset A_1 \supset A_2 \supset ...A_k \supset A_{k+1}$ is a strictly decreasing chain of submodules in $\mathcal{E}$.

Preceding as above, we obtain an infinite descending chain of submodules. This contradicts that A satisfies the DCC.
Hence A satisfies the minimum condition.

Similar proof for ACC and maximal condition.

Theorem 1.5.2: A left R-module A is *noetherian* if and only if every submodule of A is finitely generated.

Proof:
(⇒) Suppose A is a noetheirna R-module, and B be any submodule of A. We want to show B is finitely generated.

let $\mathcal{E}$ be the set of all finitely generated submodules of B.
$\mathcal{E} \neq 0$, since the zero submodule a finitely generated submodule of B.
Since A satisfies the ACC, then $\mathcal{E}$ has a maximal element, say A_0.
we will show that $B = A_0$, hence B will be finitely generated.
Suppose $A_0 \neq B$, then there is an element $x \in B$, such that $x \notin A_0$.
The submodule A_1 generated by $A_0 \bigcup \{x\}$ is clearly finitely generates and so $A_1 \in \mathcal{E}$. But $A_0 \subset A_1$, this contradicts the maximality of A_0 in $\mathcal{E}$.
So $B = A_0$ and B is finitely generated.

(⇐) Suppose that every submodule of A is finitely generated. Let

$$A_1 \subset A_2 \subset ... \subset ...$$

be an inifnite ascending chain of submodules of A.
Let $B = \bigcup\limits_k A_k$, then B is a submodule of A:

- x,y $\in$ B $\Rightarrow$ x $\in A_{n_1}$, y $\in A_{n_2}$. Let n = max{ n_1, n_2 },
  So x,y $\in A_n \Rightarrow$ x + y $\in A_n \subseteq$ B.
- r $\in$ R, x $\in$ B $\Rightarrow$ rx $\in A_{n_1} \subseteq B$.
  So by hypothesis, B is generated by a finite subset { $x_1, x_2, ..., x_r$ } where $x_1, x_2, ..., x_r \in \bigcap_k A_k$.

Then there exist natural numbers $n_1, n_2, ..., n_r$ such that
$x_i \in A_{n_i}$, i = 1, 2, ..., r.
Let k_0 = max{ $n_1, n_2, ..., n_r$ }, so $x_1, x_2, ..., x_r \in A_{k_0}$.
Hence B $\subseteq A_{k_0}$. But $A_{k_0} \subseteq$ B. So A_{k_0} = B.

For k $\geq k_0$, B = $A_{k_0} \subset A_k \subseteq$ B. So $A_{k_0} = A_k$ for every k $\geq k_0$ and the chain stops.

Definition 1.5.3: A ring R is left (right) *Noetherian* if R satisfies the ascending chain condition on left (right) ideal of R.
R is said to be noetherian if it is both left and right noetherian.
A ring R is left (right) *Artinian* if R satisfies the descending chain condition on left (right) ideal of R.
R is said to be artinian if it is both left and right artinian.
So R is left (right) artinian if it is artinian as a left (right) R-module.
Similarly for noetherian.

Theorem 1.5.3: Let A be an artinian (noetherian) R-module and B is a submodule of A. Then B and A/B are artiniann (noetherian).

Proof (Artinian case): If $\mathcal{E}$ is a nonempty set of submdoules of B, then $\mathcal{E}$ is a nonempty set of submodules of A and hence has a minimal element since A is artinian. Therefore B artinian.

To show A/B artinian, suppose

$$A_1/B \supset A_2/B \supset ...$$

is a descending chain of submodules of A/B, then

$$A_1 \supset A_2 \supset ...$$

is a descending chain of submodules of A containing B.
This chain must stop since A is artinian, that is; there exists k such that $A_i = A_k$ for all i $\geq$ k.

So the above series must stop and hence A/B is artinian.

Theorem 1.5.4: Let A be an R-module, and B a submodule of A. If B and A/B are artinian (noetherian) then A is artinian (noetherian).

Proof (Noetherian case): Let

$$A_1 \subset A_2 \subset ... \subset ...$$

be an ascending chain of submodules of A.

Then { B $\bigcap$ A$_k$; k positive integer } is an ascending chain of submodules of B. And { $\pi(A_k)$; k positive integer } is an ascending chain of A/B, where

$$\pi\text{: A} \longrightarrow \text{A/B with } \pi(x) = \text{x + B.}$$

B is noetherian $\Rightarrow$ there exists a positive integer k_1 such that

$$\text{B} \bigcap \text{A}_k = \text{B} \bigcap \text{A}_{k_1} \text{ for all k} \geq \text{k}_1.$$

A/B noetherian $\Rightarrow$ there exists a positive integer k_2 such that

$$\pi(A_k) = \pi(A_{k_2}) \text{ for all k} \geq \text{k}_2.$$

Now let k_0 = max { k_1, k_2 }. We will show $A_k = A_{k_0}$) $\forall k \geq k_0$.
Let $k \geq k_0$, $A_{k0} = A_k$ (ascending chain)
x $\in$ A_k $\Rightarrow$ $\pi(x)$ = x + B $\in$ $A_k = \pi(A_{k_2}) = \pi(A_{k_0})$ since $k_0 \geq k_2$.
So there exists $x_0 \in A_{k_0}$ such that Π(x) = $\pi(x_0)$. Then x + B = x_0 + B, and hence x - $x_0 \in$ B

But $A_{k_0} \subset A_k$ and x - $x_0 \in A_k$ since x $\in A_k$ and $x_0 \in A_{k_0}$
So x - $x_0 \in$ B $\bigcap A_k$ = B $\bigcap A_{k_0}$
Hence x $\in A_{k_0}$ and $A_k = A_{k_0}$ and thus A is noetherian.

Theorem 1.5.5: If R is a left (right) Artinian ring, then the Jacobson radical is a nilpotent ideal.

Proof: Let J = J(R). Consider the chain of left ideals in R:

$$J \supset J^2 \supset J^3 \supset ...$$

Since R is left Artinian then there exists m such that $J^n = J^m$ for all n $\geq$ m.
Suppose $J^m \neq 0$.

Let $\mathcal{E}$ = { I $\neq$ 0; I ideal of R satisfying J^m.I $\neq$ 0 }.
$\mathcal{E}$ is nonempty since $J^m.J^m = J^m \neq 0$.

Since R is left Artinian, then $\mathcal{E}$ has a minimal element M $\neq$ 0 and J^m.a $\neq$ 0 for some a $\neq$ 0 in M.
J^m.a $\neq$ 0 and M is a minimal left ideal, then J^m.a = M. So there exists x $\in J^m$ such that xa = a.

Since x $\in J^m$ then -x $\in J^m$, which implies that -x is a left quasi-regular ($J^m \subseteq$ J).

Then, s - x - sx = 0 for some s $\in$ R and sa - xa - sxa = 0.
But xa = a and hence xa = 0 which implies that a = 0, a contradiction.
Hence, $J^m = \{0\}$ and J is nilpotent.

Corollary 1.5.1: If R is left (right) Artinian, then every nil ideal of R is nilpotent.

Proof: By Theorem 1.4.2, every nil ideal is contained in J(R). Since R is left (right) Artinian, then by Theorem 1.5.5, we have J(R) is nilpotent. Hence, every nil ideal of R will be nilpotent.

Theorem 1.5.6: Wedderburn-Artin he following conditions on a ring R are equivalent;

(a) R is a nonzero semisimple left Artinian ring
(b) There exist division rings $D_1, ..., D_k$, and positive integers $n_1, ..., n_k$ such that R is isomorphic to the ring $Mat_{n_1}D_1 \times Mat_{n_2}D_2 \times \ ... \ \times Mat_{n_k}D_k$.

Proof: See [12, p.436]

Theorem 1.5.7: Let R be a semisimple left Artinian ring, then R has identity.

Proof: If R is a semisimple left Artinian ring then, by Wedderburn-Artin Theorem, there exist division rings $D_1, ..., D_k$, and positive integers $n_1, ..., n_k$ such that R is isomorphic to the ring

$$Mat_{n_1}D_1 \times Mat_{n_2}D_2 \times \ ... \ \times Mat_{n_k}D_k.$$

Since each matrix ring has identity, then R has identity.

Chapter 2

On Rings with Conditions on Nonperiodic Elements

We study structure and commutativity of rings R with the property that for each nonperiodic element x of R there exists a positive integer K = K(x) such that x^k is a central element for all $k \geq K$.

Definition 2.1: A ring R is said to be a *c-ring* if R has at most finitely many noncentral elements.

Theorem 2.1: A ring R is a c-ring if and only if R is either finite or commutative.

Proof:
($\Rightarrow$) Suppose R is a noncommutative c-ring. Then $R\backslash C$ is finite since R is a c-ring.

And for all $x \in R\backslash C$, $x + C \subseteq R\backslash C$ so $x + C$ is finite; hence C is finite. Therefore R is finite.

($\Leftarrow$) Suppose first R is finite, then $R\backslash C$ is finite.
Now if R is commutative then $R\backslash C$ is empty hence finite. So in both cases R is a c-ring.

Definition 2.2: A ring R is said to be a *c*-ring* if for every $x \in R$, either x is periodic or there exists a positive integer K = K(x) such that $x^k \in C$ for all $k \geq K$.

Lemma 2.1: Let x be a periodic element of the ring R. Then

(a) some power of x is idempotent;
(b) there exists an integer $n \geq 2$ such that $x - x^n \in N$;
(c) $x = y + w$, where $y \in P_0$ and $w \in N$.

Proof:

(a) Let a be a periodic element so there exist n and m distinct positive integers such that a$^n = a^m$, assuming n > m.
Then a$^m(a^{n-m}$ - 1)= 0.

For each j $\geq$ m, take j= m + λ where $\lambda \geq 0$.
Multiplying both sides from the left by a $^\lambda$ we get,
a$^\lambda a^m(a^{n-m}$ - 1) = 0; that is a$^{m+\lambda}$(a^{n-m} - 1) = 0.
So a$^j(a^{n-m}$ - 1) = 0 $\forall$ j $\geq$ m.

But,

$$\begin{aligned}
a^j a^{k(n-m)} &= a^j a^{n-m} a^{(k-1)(n-m)} \\
&= a^j (a^{n-m})^{k-1} \\
&= a^j (a^{n-m})^{k-2} \\
&= ... \\
&= a^j a^{n-m} \\
&= a^j.
\end{aligned}$$

Therefore, a^{j}a$^{k(n-m)}$ = a^j $\forall$ k positive integer. (1)

On the other hand,

$$\begin{aligned}
(a^{m(n-m)})^2 &= a^{2m(n-m)} \\
&= a^{(m+m)(n-m)} \\
&= a^{m(n-m)+m(n-m)} \\
&= a^{j+m(n-m)}
\end{aligned}$$

where j = m(n-m) $\geq$ m.
So, (a$^{m(n-m)})^2$ = a$^{j+m(n-m)}$ where j = m(n-m) $\geq$ m.
Therefore, (a$^{m(n-m)})^2$ = a$^{m(n-m)}$ by applying (1) with k = m.
Hence, for every x $\in$ R there exists a positive integer k such that x^k is idempotent.

(b) Let x$^n = x^m$ with n > m > 1. Then x$^{m-1}(x - x^{n-m+1}) = 0$
$\Rightarrow$ x$^{m-2}x(x - x^{n-m+1}) = 0 =$ x$^{m-2}x^{n-m+1}(x - x^{n-m+1})$.
It follows that x$^{m-2}(x - x^{n-m+1})^2 = 0$
Similarly x$^{m-2}(x - x^{n-m+1})^2 =$ x$^{m-3}x(x - x^{n-m+1})^2 = 0$.
So x^{m-3}x^{n-m+1}(x - x$^{n-m+1})^2$ = 0 = x$^{m-3}x(x - x^{n-m+1})^2$.
Hence x$^{m-3}(x - x^{n-m+1})^3 = 0$.

By induction we get
x(x - x$^{n-m+1})^{m-1} = 0 = x^{n-m+1}(x - x^{n-m+1})^{m-1}$.

So (x - x$^{n-m+1})^m$=0. Therefore x - x^{n-m+1} is a nilpotent element of R.

(c) Let x$^n = x^m$ with n $\geq$ n - m + 1 > m.
Take y = x^{n-m+1} and w = x - x^{n-m+1} then w is nilpotent by (b) and y is potent by (a) by choosing n very large such that n - m + 1 $\geq$ m with j = k = n - m + 1.

Lemma 2.2: Let R be reduced ring. Then

(a) if a, b $\in$ R and ab = 0, then arb = 0 for all r $\in$ R.

(b) every periodic element of R is potent.
Proof:
(a) Let a, b $\in$ R such that ab = 0 and let r $\in$ R, then
$(ba)^n$ = bababababa... = 0 for some n positive integer, so ba $\in$ N.

But R reduced so N = {0}, and hence ba = 0. It follows that
$(arb)^2$ = arbarb = 0 so arb $\in$ N. Thus arb = 0.

(b) Let x be a periodic element of R. By lemma 2.1 (b), x - $x^n \in$ N for some positive integer n, but N = {0} so x - x^n = 0.
Therefore x = x^n.

Lemma 2.3: Let R be a ring with 1, and x, y $\in$ R. If there exists a positive integer k such that $x^k[x,y] = (x+1)^k[x,y] = 0$, then [x,y] = 0.

Proof: We will prove in general for some positive integer n > 1, if $x^n y = (x+1)^n y = 0$ then y = 0.

$(x+1)^n y = 0 \Rightarrow x^{n-1}(x+1)^n y = 0$

$\Rightarrow x^{n-1} \sum_{k=0}^{n} \binom{n}{k} x^k y = 0$

$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} x^{k+n-1} y = 0 \Rightarrow x^{n-1} y = 0.$

Now $x^n y = 0 \Rightarrow (-1+(x+1))^n y = 0$

$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (x+1)^k y = 0$

$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (x+1)^{k+n-1} y = 0$ (multiplying both sides by $(x+1)^{n-1}$ on the left)

$\Rightarrow (-1)^n (x+1)^{n-1} y = 0.$

So $x^{n-1}y = 0 = (x+1)^{n-1}y$.

We continue this process to get xy = 0 = (x+1)y, and then xy = xy + y = 0.
Thus y = 0.
In our situation, [x,y] = 0.

Lemma 2.4: Let R be a ring, k a positive integer, and x an element of R such that $x^k \in$ C and $x^{k+1} \in$ C. Then $x^k[x,y] = 0$ for all y $\in$ R.

Proof: Let y be in R,
$x^k[x,y] = x^k(xy - yx) = x^{k+1}y - x^k yx = yx^{k+1} - yx^{k+1} = 0.$

Lemma 2.5: If R is a ring such that R = $P_0 \bigcup C$, then R is commutative.

Proof: See [7]

Lemma 2.6: The class of c*-rings is closed under taking subrings and homomorphic images.

Proof: Let R be a c*-ring, and S be a subring of R.
Let x be an element of S, then x is an element of R. So x is either periodic or $x^k \in C \ \forall \ k \geq K$ for some K positive integer. Hence S is also a c*-ring.

Also, Let

$$f: R \longrightarrow f(R) \text{ be an epimorphism.}$$

Let y be an element of f(R), so there exists $x \in R$ such that $y = f(x)$. And since R is a c*-ring, then x is either periodic or $x^k \in C \ \forall \ k \geq K$ for some K positive integer.

• If x is periodic, so there exist distinct positive integers m and n such that $x^n = x^m$. Then $f(x^n) = f(x^m) \Rightarrow (f(x))^n = (f(x))^m$ since f is a homomorphism

$$\Rightarrow y^n = y^m$$

Hence y is periodic.

• If $x^k \in C \ \forall \ k \geq K$ for some K positive integer, then

$$f(x^k a) = f(ax^k) \text{ for all } a \in R \Rightarrow f(x^k)f(a) = f(a)f(x^k) \text{ for all } a \in R$$
$$\Rightarrow (f(x))^k f(a) = f(a)(f(x))^k \text{ for all } a \in R$$
$$\Rightarrow y^k f(a) = f(a)y^k \text{ for all } a \in R,$$

and this is true $\forall \ k \geq K$ for some K positive integer.

Thus f(R) is a c*-ring.

Theorem 2.2: Let R be a c*-ring. Then

(a) If R is reduced, then R is commutative.
(b) If R is prime, then R is either commutative or periodic.

Proof:
(a) Let x be in R, we consider 2 cases;

- If x is periodic, then by Lemma 2.2(b), x is potent.
- If x is not periodic, then there exists k positive integer such $x^k \in C$ and $x^{k+1} \in C$. Then by Lemma 2.4, $x^k[x,y] = 0 \ \forall \ y \in R$ and $(1 + x)^k[x,y] = 0 \ \forall \ y \in R$, and we deduce using Lemma 2.3 that $[x,y] = 0$ for all $y \in R$, hence $x \in C$. Therefore by Lemma 2.5, R is commutative.

(b) Let R be a prime ring. First we prove that every nonzero element of C is regular.
Let x be a nonzero element of C, and assume x is not regular; that is, there exists b nonzero such that $ab = 0$, hence $arb = 0$ for all $r \in R$ by Lemma 2.2(a). But R is prime so $\{0\}$ is a prime ideal, thus $a = 0$ or $b = 0$. This is a contradiction.

Now we let $x \in R$ that is a c*-ring. If x is not periodic, then there exists a positive integer K such that $x^k \in C \ \forall \ k \geq K$.

- If $x^K = 0$ then $x = 0$ since R is prime.

• If $x^K \neq 0$, then x^K and x^{K+1} are regular central elements.
By Lemma 2.4, $x^K[x,y] = 0$ for all $y \in R$ implies $[x,y] = 0$ since x^K is regular.

Thus xy = yx for all $y \in R$, and $x \in C$. Therefore every element of R is either periodic or central.

Now suppose R is neither commutative nor periodic, and let
$x \in C\backslash P$; we may assume $x \neq 0$.

For each $y \notin C$, we have $xy \notin C$.
Because if $xy \in C$, then (xy)z = z(xy) for all z in R. But $x \in C$, so x(yz) = x(zy) for all z, hence x(yz - zy) = 0 for all z.
Thus yz = zy for all z, since R is prime and $x \neq 0$.

Hence y, $xy \in P$. Thus there exist distinct positive integers m, n such that $y^m = y^n$ and $(xy)^m = (xy)^n$; and it follows that $(x^m - x^n)y^m = 0$. Since $x^m - x^n \in C\backslash\{0\}$, we get $y^m = 0$; that is $y \in N$. We have shown so far that $R = C \bigcup N$.

Since R is not commutative, there exists $y \in N\backslash C$.
For $x \in C$, $x + y \notin C$, so $(x + y)^m = 0$ for some positive integer m; and it follows easily that $x^m + u = 0$, where u is a sum of pairwise commuting nilpotent elements.
Thus $x^m \in N$ and $x \in N$. But this gives that R = N, hence R = P, contradicting our assumption that R is not periodic, and our proof is complete.

Theorem 2.3: If R is a c*-ring, then C(R) is periodic. Moreover, if N is commutative, then C(R) is nil.

Proof: Let P(R) denote the periodic radical; that is, the maximal periodic ideal of ring R. We assume without loss of generality that $P(R) \neq R$.

Then R/P(R) is a subdirect product of a nonempty family
$\{ R_\alpha; \alpha \in J\}$ of prime rings such that each $P(R_\alpha) = \{0\}$ from [8].
Then by Theorem 2.2(b), each R_α is commutative, hence R/P(R) is commutative.

So for any x + P(R), y + P(R) $\in$ R/P(R), we have that xy + P(R) = yx + P(R) then xy - yx + P(R) = 0 + P(R), and then xy - yx $\in$ P(R). Thus $C(R) \subseteq P(R)$.

Now we let N be commutative. If we show that N is an ideal, then by Lemma 2.6, R/N is a c*-ring since R is.

And R /N is reduced; because if $(x + N)^n = 0 + N$ for some positive integer n, then $(x + N)^n \subseteq N$; hence $x + N \subseteq N$ and x + N = 0 + N. Then by Theorem 2.2(a) we get that R/N is commutative.
So $\forall$ x + N, y + N $\in$ R/ N, xy + N = yx + N, hence $[x,y] \in N$. Thus $C(R) \subseteq N$. And this completes the proof.

In order to show that N is an ideal, we let a, b be elements in N and r be an element of R.
$a^m = 0$ and $b^n = 0$ for some positive integers m and n.

Consider

$$\begin{aligned}(a - b)^{m+n} &= a^{m+n} + c_{m-1}a^{m+n-1} + c_{m-2}a^{m+n-2}b^2 + ... + c_1ab^{m+n-1} + b^{m+n} \\ &= 0 + 0 + 0 + ... + 0 \\ &= 0\end{aligned}$$

So a - b $\in$ N.
We still need to show that ar and ra $\in$ N. Let $a \neq 0 \in N$, We will prove by induction that,
$a^n = 0 \Rightarrow (ra)^n = 0$ and $(ar)^n = 0\ \forall\ r \in R$.
Suppose first that $a^{\,2} = 0$, we consider 2 cases for ar;

<u>Case 1:</u> If ar is periodic,

Then by Lemma 2.1(a); $(ar)^j = e$ is idempotent for some positive integer j. Then

$$\begin{aligned}(re - ere)^2 &= rere + ereere - reere - erere \\ &= rere + ere^2re - re^2re - erere \\ &= rere + erere - rere - erere \\ &= 0.\ \text{So } re - ere \in N.\end{aligned}$$

Then a(re - ere) = (re - ere)a since N is commutative.
So $a(re - ere)a = (re - ere)a^2 = 0$.

$$\begin{aligned}&\Rightarrow a(re - ere)a = 0. \\ &\Rightarrow a(r(ar)^j - (ar)^j r(ar)^j)a = 0 \\ &\Rightarrow a(r(ar)^j - (ar)^j r(ar)^j)ar = 0 \\ &\Rightarrow ar(ar)^j(ar) - a(ar)^j r(ar)^j(ar) = 0 \\ &\Rightarrow (ar)^{j+2} - a^2(rar...ar)r(ar)^{j+1} = 0 \\ &\Rightarrow (ar)^{j+2} = 0 \text{ since } a^2 = 0 \\ &\Rightarrow ar \in N. \\ &\Rightarrow (ar)a = a(ar) = a^2r = 0 \text{ (N commutative).} \\ &\Rightarrow (ar)^2 = arar = 0.r = 0\end{aligned}$$

Similar reasoning for ra.
Hence $(ar)^2 = 0$ and $(ra)^2 = 0$.

<u>Case 2:</u> If $(ar)^k \in C$ for all $k \geq K$ for some K positive integer.

So $(ar)^K a = a(ar)^K = 0$ since $a^2 = 0$.
Then $(ar)^{K+1} = (ar)^K ar = 0$.

$$\begin{aligned}&\Rightarrow ar \in N \\ &\Rightarrow ara = (ar)a = a(ar) = a^2r = 0. \text{ (N commutative)} \\ &\Rightarrow (ar)^2 = (ar)ar = (ara)r = 0 \\ &\Rightarrow (ra)^2 = (ra)(ra) = r(ara) = 0.\end{aligned}$$

Hence we proved that $a^2 = 0$ implies that $(ar)^2 = (ra)^2 = 0\ \forall\ r \in R$.

Now suppose $a^m = 0$ implies that $(ar)^m = 0$ and $(ra)^m = 0 \ \forall$ r $\in$ R.
Let $a^n = 0$ (with n > m).
We want to show that $(ar)^n = 0$ and $(ra)^n = 0$ for all r $\in$ R.

Since R is a c*-ring, then we consider 2 case for ar;

Case 1: If ar is periodic,

Then by Lemma 2.1(a), $(ar)^j = e$ is idempotent for some j positive integer.
So as proved above, (re - ere) $\in$ N.
Then a(re - ere) = (re -ere)a since N is commutative.
So ra(re - ere)a = r(re -ere)a^2

$$\Rightarrow ra(r(ar)^j - (ar)^j r(ar)^j)a = r(r(ar)^j - (ar)^j r(ar)^j)a^2$$
$$\Rightarrow rar(ar)^j a - ra(ar)^j r(ar)^j a = r(r(ar)^j - (ar)^j r(ar)^j)a^2$$
$$\Rightarrow (ra)(ra)(ra)^j - ra^2 s = ta^2 \text{ where s, t} \in R.$$
$$\Rightarrow (ra)^{j+2} - ra^2 s = ta^2.$$

Also $(a^2)^{n-1} = 0$ since $a^n = 0$.
Take m = n - 1 < n, then by the induction hypothesis we get $(ra^2)^{n-1} = 0$ and again $(ra^2s)^{n-1} = 0$ by the induction hypothesis.
So $ra^2s \in$ N and $ta^2 \in$ N.

But N is commutative, so $(ra)^{j+2} = ta^2 - ra^2s \in N$,
$\Rightarrow ((ra)^{j+2})^m = 0$ for some m > 0, so $(ra)^{m(j+2)} = 0$. Hence ra $\in$ N
Similar reasoning for ar.
So ar and ra both commute with a, since N is commutative.

$$\begin{aligned}(ra)^n &= (ra)(ra)^{n-1}\\ &= r(ra)^{n-1}a\\ &= r(ra)(ra)^{n-2}a\\ &= r^2a(ra)^{n-2}a\\ &= r^2(ra)^{n-2}a^2\\ &= r^n a^n\\ &= 0 \text{ since } a^n = 0.\end{aligned}$$

And,

$$\begin{aligned}(ar)^n &= (ar)^{n-1}(ar)\\ &= a(ar)^{n-1}r\\ &= a(ar)^{n-2}(ar)r\\ &= a(ar)^{n-2}ar^2\\ &= a^n r^n = 0.\end{aligned}$$

Case 2: If $(ar)^k \in$ C for all k $\geq$ K, for some K positive integer.

$(ar)^K \in C \Rightarrow (ar)^K a = a(ar)^K = a^2 b$, where b $\in$ R.
Also $(ar)^{K+1} = (ar)^K(ar) = ((ar)^K a)r = a^2br = a^2d \in C$.
But $(a^2d)^n = a^{2n}d^n = 0$ since $a^n = 0$.
Hence $a^2d \in$ N. This implies that $(ar)^{K+1} \in$ N.

So ar $\in$ N and ra $\in$ N.

Thus ar and ra both commute with a since N is commutative, and again as in the previous case we get,
$(ra)^n = r^n a^n = 0$ and $(ar)^n = a^n r^n = 0$.

Thus N is an ideal. Therefore C(R) $\subseteq$ N.

Definition 2.3: An element g is said to be *a torsion element of a group* if it has finite order, i.e., if there is a positive integer m such that $g^m = e$, where e denotes the identity element of the group, and g^m denotes the product of m copies of g.
An element x of a ring R is called *a torsion element* if there exists a regular element r of the ring (an element that is neither a left nor a right zero divisor) that annihilates x, i.e., r.x = 0.
A ring R is said to be *torsion-free* if zero is the only torsion element.

Theorem 2.4: If R is a torsion-free c*-ring with 1, then R is commutative.
Proof: Let x be an element of R, such that x $\in$ P,

- If 2x $\in$ P, then there exist distinct positive integers m, n such that $x^m = x^n$ and $(2x)^m = (2x)^n$, so that $(2^m - 2^n)x^m = 0$ and hence $x^m = 0$ by torsion-freeness.
- If 2x $\notin$ P, then there exists M such that $(2x)^m \in C\ \forall\ m \geq M$, and by torsion-freeness we get $x^m \in C$ for all m $\geq$ M.

Notice that R is a c*-ring, so x is either periodic or $x^k \in C\ \forall\ k \geq K$ for some K positive integer. Hence in all cases, for every x $\in$ R,
$x^k \in C$ for all k $\geq$ K for some K positive integer. And by Lemma 2.4, $x^K[x,y] = 0$ for all y $\in$ R.

Also x + 1 $\in$ R, so $(x + 1)^k \in C$ for all k $\geq$ K' for some K' positive integer.
So take k" = max {K, K'}, then $x^k \in C$ and $(x + 1)^k \in C\ \forall\ k \geq$ k". And by Lemma 2.4, $x^{k''}[x,y] = (x + 1)^{k''}[x + 1,y] = 0$ for all y$\in$ R.
But $(x + 1)^{k''}[x + 1,y] = (x + 1)^{k''}[x,y] = 0$ for all y $\in$ R.
Now by Lemma 2.3, [x,y] = 0 for all y $\in$ R. Hence R is commutative.

Lemma 2.7: If R is a ring and C(R) $\subseteq$ C, then the idempotents are central.

Proof: Let e be an idempotent element in R, and let x $\in$ R
Then [e,x] = ex - xe is central, $\Rightarrow$ e(ex - xe) = (ex - xe)e
$\Rightarrow$ ex - exe = exe - xe
Multiplying by e both sides from the right,
exe - exe = exe - xe $\Rightarrow$ exe = xe $\forall$ e idempotent and $\forall$ x $\in$ R,

Similarly, exe = ex.
So ex = xe $\forall$ x $\in$ R. Hence idempotents in R are central.

Lemma 2.8 Let R be a ring with identity 1. If a $\in$ N, then 1 - a and 1 + a are invertible.

Proof: a $\in$ N, then $a^n = 0$ for some positive integer n.
$1 = 1 - a^n = (1 - a)(1 + a + ... + a^{n-1}))$. So 1 - a is invertible.
Also -a is nilpotent with $(-a)^n = 0$, so 1 + a = 1 - (-a) is invertible.

Definition 2.4: A group is said to be a *torsion group* if every element has a finite order.

Theorem 2.5: If R is a c*-ring with identity 1 and N $\subseteq$ C, then R is commutative.

Proof: Since N $\subseteq$ C, by Theorem 2.3 commutators are nilpotent and hence central.

First, we will consider the case of (R,+) a torsion group.
We see that if 1 + x $\in$ P, then the subring generated by x is finite, and hence x is periodic.

Let $\bar{R}$ be a subdirectly irreducible homomorphic image of R.
Then, by Lemma 2.6, $\bar{R}$ is a c*-ring and satisfies the property that commutators are central, which implies by Lemma 2.7 that the idempotents in $\bar{R}$ are central. Then by Lemma 1.4, the idempotents in $\bar{R}$ are either 0 or 1.

For each x $\in$ R, we denote by $\bar{x}$ its image in $\bar{R}$.

Since R is a c*-ring, x $\in$ P or for some integer K, $x^k \in C \; \forall \; k \geq K$.

For x $\in$ P, by Lemma 2.1(c), we write x = a + u, where u $\in$ N and a $\in P_0$. So $a^n = a$ for some integer n > 1.
Then $\bar{x} = \bar{a} + \bar{u}$ where $\bar{a}^n = \bar{a}$ and $\bar{u}$ is a central nilpotent element of $\bar{R}$.

Observe that, for n > 2

$$\begin{aligned}(\bar{a}^{n-1})^2 &= \bar{a}^{n-1}.\bar{a}^{n-1}\\ &= \bar{a}^{n+n-2}\\ &= \bar{a}^{n}.\bar{a}^{n-2}\\ &= \bar{a}.\bar{a}^{n-2}\\ &= \bar{a}^{n-1}\end{aligned}$$

And for n=2, clearly $\bar{a}^{2-1} = \bar{a}$.
So we proved that $\bar{a}^{n-1}$ is idempotent $\forall$ n $\geq$ 2.

Hence either $\bar{a}^{n-1} = \bar{0}$ or $\bar{a}^{n-1} = \bar{1}$.
But $\bar{a}^{n-1} = \bar{0}$ implies that $\bar{a}^n = \bar{0}$ (by multiplying by $\bar{a}$ both sides), then $\bar{a} = \bar{0}$.
Thus either $\bar{a} = \bar{0}$ or $\bar{a}^{n-1} = \bar{1}$.

- If $\bar{a} \neq 0$, so $\bar{a}^{n-1} = \bar{1}$ then $(\bar{a} + \bar{u})^{n-1} = \bar{1}$ + v, where v is nilpotent in $\bar{R}$ (since N $\subseteq$ C).
  So $(\bar{a} + \bar{u})^{n-1}$ is invertible by Lemma 2.8.

Hence $\bar{x} = \bar{a} + \bar{u}$ is periodic and invertible, which implies that $\bar{x}$ is potent.

- If $\bar{a} = 0$, then $\bar{x} = \bar{u}$. So $\bar{x}$ is nilpotent, hence central (N $\subseteq$ C).

We have proved so far that if x $\in$ P, then $\bar{x}$ is either potent or central in $\bar{R}$.

Now suppose x $\notin$ P, then as noted before, 1 + x $\notin$ P. Hence there exists K such that x^k, $(1 + x)^k \in C \ \forall \ k \geq K$. Then by Lemma 2.4 applied to x and 1 + x, we get $x^k[x,y] = 0$ and $(1+x)^k[x+1,y] = 0$ for all y $\in$ R.

But $(1+x)^k[x+1,y] = (1+x)^k[x,y]$.
So $x^k[x,y] = (1+x)^k[x,y] = 0$, which implies by Lemma 2.3 that $[x,y] = 0 \ \forall$ y $\in$ R. Hence x $\in$ C, and therefore $\bar{x}$ is central in $\bar{R}$.

We see that in both cases, $\bar{x}$ is either potent or central in $\bar{R}$.
Using Lemma 2.5, we get $\bar{R}$ is commutative. But R is a subdirect product of commutative subdirectly irreducible homomorphic images, hence R must be commutative.

So far we proved that, if (R,+) is a torsion group, then R is commutative.

Now we proceed to the general case.
Consider a potent element a, with $a^n = a$.
Assume there exists K such that $(2a)^k \in C$ and $(3a)^k \in C \ \forall \ k \geq K$.
Since $a^n = a$ for some positive integer n > 1, then

- If n $\geq$ K, then $2^n a = (2a)^n \in C$.
- If n < K, then $a^n a^{n-1} = a^n = a \Rightarrow a^{2n-1} = a$.
  Now if 2n-1 $\geq$ K, we get $2^{2n-1}a \in C$, and if 2n-1 < K, then $a^{3n-2} = a$.
  We continue in this fashion until we get an m $\geq$ K, such that $2^m a \in C$ and $3^m a \in C$.

Now since 2^m and 3^m are coprime, we have by Bezout that $1 = 2^m\alpha + 3^m\beta$ where α, β are integers. Then a is a linear combination of $2^m a$ and $3^m a$. So a $\in$ C.

Thus if a $\notin$ C, we may assume that 2a is periodic. So there exists an integer m such that $a^m = a^n$ and $(2a)^m = (2a)^n$. Hence $(2^m - 2^n)a^n = 0 = (2^m - 2^n)a$. Thus a is a torsion element.

Therefore we proved that every potent element is either central or torsion.

Now let $a^n = a$ where n > 1 and a be a nonzero torsion element. Then e = a $^{n-1}$ is idempotent, hence central since N $\subseteq$ C and using lemma 2.,; and consider eR the c*-ring with 1.

e is a torsion element since a is torsion, and hence (eR,+) is a torsion group.

Thus eR is commutative, and [ea,eb] = eaeb - ebea = ebea - ebea = 0
Then e[a,b] = 0 (since e central) $\Rightarrow a^{n-1}[a,b] = 0$
$\Rightarrow a^{n-1}(ab - ba) = 0$

$\Rightarrow$ $a^n b - a^{n-1} ba = a^n b - ba^n = 0$
$\Rightarrow$ ab = ba $\forall$ b $\in$ R.

Hence a is a central element.

We have shown that every potent element of R is a central element; and by Lemma 2.1(c), we get P $\subseteq$ C.

Now suppose R has a noncentral element x. Then 1 + x $\notin$ C; otherwise x $\in$ C. Hence x and 1 + x are not periodic, so there exists K > 0 such that $x^k \in C$ and $(1 + x)^k \in C$ $\forall$ k $\geq$ K.
So by Lemma 2.4, we get $x^K[x,y] = 0 = (1 + x)^K[x,y]$ $\forall$ y $\in$ R.
And by Lemma 2.3, we get [x,y] = 0 for all y, which contradicts the fact that x $\notin$ C.

We conclude that every element of R is central. Hence R is commutative as required.

Theorem 2.6 Let R be a ring with each element is either periodic or central. If N $\subseteq$ C, then R is commutative.

Proof: Let R be a ring with each element is either periodic or central.
Let a $\in$ C, then ay = ya for all y in R.
Multiplying by a both sides from the left,
$a^2y = aya = ya^2$ for all y in R.
Again, multiplying by a both sides from the left,
$a^3y = aya^2 = ya^3$ for all y in R.
Continuing in this fashion, we prove that $a^n \in C$ $\forall$ n $\geq$ 1.

Then R is a c*-ring with N $\subseteq$ C, and by Theorem 2.5, R is commutative.

Chapter 3

Rings in which every Zero Divisor is either Nilpotent or Idempotent

We consider rings in which N is commutative, and satisfying the property " every zero divisor is uniquely represented in the form
$x = e + a$ where $e^2 = e$ and $a \in N$ ". And we get that N is an ideal of R and R/N is a subdirect sum of fields. Also if we consider rings with identity such that every zero divisor is either nilpotent or idempotent, then N is still an ideal of R and R/N is either a Boolean ring or a field.

Theorem 3.1: Let R be a periodic ring. Suppose that,
(a) N is commutative
(b) Every x in R is uniquely written in the form $x = e + a$ where $e^2 = e$ and $a \in N$
Then N is an ideal of R, and R/N is Boolean. In fact, R is commutative.

Proof: Let e be an idempotent element of R; $e^2 = e$, $x \in R$, and let
$f = e + ex - exe$. Then
$$\begin{aligned} f^2 &= (e + ex - exe)^2 \\ &= (e + ex - exe)(e + ex - exe) \\ &= e^2 + eex - eexe + exe + exex - exexe - exee - exeex + exeexe \\ &= e + ex - exe + exe + exex - exexe - exe - exex + exexe \\ &= e + ex - exe \\ &= f. \end{aligned}$$

Now on one hand, we have that $f = f + 0$ where $f^2 = f$ and $0 \in N$.

On the other hand, we have that $f = e + ex - exe$ where $e^2 = e$ and $ex - exe \in N$ since $(ex - exe)^2 = 0$.
It follows from (b) that $ex - exe = 0$, so $ex = exe$.
Similaly $xe = exe$. Therefore, $ex = xe$ for every x in R and every idempotent e

in R. Thus the idempotents are central.

Now, let x, y $\in$ R and we show that xy = yx.
x = $x_1 + x_2$, where $x_1^2 = x_1$, $x_2 \in$ N and y = $y_1 + y_2$, where $y_1^2 = y_1$, $y_2 \in$ N.
xy = $(x_1 + x_2)(y_1 + y_2)$
= $x_1(y_1 + y_2) + x_2(y_1 + y_2)$
= $(y_1 + y_2)x_1 + x_2y_1 + x_2y_2$ (since x_1 is central)
= $(y_1 + y_2)x_1 + y_1x_2 + y_2x_2$ (since y_1 is central and N is commutative)
= $(y_1 + y_2)x_1 + (y_1 + y_2)x_2$
= $(y_1 + y_2)(x_1 + x_2)$= yx.
Therefore R is commutative, and hence N is an ideal of R.

Consider now the quotient ring R/N,
let x + N $\in$ R/N, where x + N $\neq$ N. So x $\notin$ N.
But x is uniquely written in the form x = e + a where e^2 = e and a $\in$ N.
So x + N = e + a + N = e + N (since a $\in$ N).

Observe that $(e + N)^2 = (e + N)(e + N) = e^2 + N = e + N$ with e + N $\neq$ N; otherwise x $\in$ N, which is a contradiction.
Hence x + N is idempotent. Thus R/N is Boolean.

Lemma 3.1: If the commutator ideal C(R) is nil, then the set N of nilpotent element forms an ideal.

Proof: Let C(R) be the commutator ideal of R which is the ideal generated by all [x,y] where x, y $\in$ R.
Consider the quotient ring R/C(R), it is a commutative ring since for any x + C(R), y + C(R) $\in$ R/C(R),

$$(x + C(R))(y + C(R)) = xy + C(R)$$
and
$$(y + C(R))(x + C(R)) = yx + C(R)$$

But xy - yx = [x,y] $\in$ C(R).
So xy - yx + C(R) = C(R), hence xy + C(R) = yx + C(R).

To show that N is an ideal of R,
Let x $\in$ R, a, b $\in$ N where $a^n = 0 = b^m$ for some positive integers m and n.
We have x + C(R) $\in$ R/C(R), $(ax + C(R))^n = (a^n + C(R))(x^n + C(R))$= C(R).
So $(ax)^n \in$ C(R) where C(R) $\subseteq$ N since C(R) is nil. Therefore ax $\in$ N.
Similarly xa $\in$ N.

Furthermore, $((a + C(R)) - (b + C(R)))^{n+m}$ = C(R) since R/C(R) is commutative.
So $(a - b)^{n+m} \in C(R) \subseteq N$. Therefore a - b $\in$ N.

Hence N is an ideal.

Theorem 3.2: Let R be a periodic ring with identity 1. Suupose that,

(a) N is commutative.

(b) Every zero divisor x can be uniquely written in the form

$x = e + a$ where $e^2 = e$ and $a \in N$.

Then N is an ideal of R and R/N is isomorphic to a subdirect sum of fields.

Proof: Let $e \in R$ such that e is idempotent, $x \in R$, and let $f = e + ex - exe$.

- If $f = 1$, then $ef = e$. So $e^2 + e^2x - e^2xe = e$ which implies that $ex = exe$.
- If $f \neq 1$, then $f^2 = f$, $f \neq 1$. Hence $f \in D$ since $f(f - 1) = 0$.

Since $f = f + 0$, it follows from (b) that $ex - exe = 0$ and thus $ex = exe$.

Similarly $xe = exe$. Hence all idempotents are central.

Now let $x \in R$. Since R is periodic, so there exist distinct positive integers m and n such that $x^m = x^n$ with $n > m \geq 1$. Hence $x^{m(n-m)}$ is idempotent by Lemma 2.1(a).

Therefore, $x^{m(n-m)}y - yx^{m(n-m)} = 0$ for every $y \in R$.

A well known theorem of Herstein [10] asserts that the commutator ideal C(R) is nil hence N is an ideal of R by Lemma 3.1.

Also,

$$\begin{aligned}(x^{n-m+1} - x)^m &= (x^{n-m+1} - x)(x^{n-m+1} - x)^{m-1} \\ &= x^{n-m+1} - x)(x(x^{n-m} - 1))^{m-1} \\ &= x^{n-m+1} - x)x^{m-1}(x^{n-m} - 1)^{m-1} \\ &= x^{n-m+1} - x)x^{m-1}g(x) \\ &= (x^n - x^m)g(x) \\ &= 0.\end{aligned}$$

Hence $(x^{n-m+1} - x) \in N$. Thus, $x^{n-m+1} + N = x + N$, $n-m+1 > 1$.

Therefore the quotient ring R/N is a commutative ring isomorphic to a subdirect product of fields by Jacobson's theorem [13].

Lemma 3.2: Let R be a ring, e idempotent element and a an element in N such that $ea^2e = (eae)^2$. Then $eae \in N$.

Proof: Let e be an idempotent element, $a \in N$ such that $ea^2e = (eae)^2$.

By induction, $(eae)^{2^2} = (eae)^{2\times 2} = ((eae)^2)^2 = (ea^2e)^2 = ea^{2^2}e$.

Also $(eae)^{2^3} = ((eae)^{2^2})^2 = (ea^{2^2}e)^2 = ea^{2^3}e$.

Suppose this is true for $k = n - 1$ we get, $(eae)^{2^{n-1}} = ea^{2^{n-1}}e$ and show it for $k = n$.

$$\begin{aligned}(eae)^{2^n} &= (eae)^{2^{n-1}\times 2} \\ &= ((eae)^{2^{n-1}})^2 \\ &= (ea^{2^{n-1}}e)^2 \\ &= ea^{2^n}e.\end{aligned}$$

The induction shows that $(eae)^{2^n} = ea^{2^n}e$ for all positive integers n.

But $a \in N$, so $a^k = 0$ for some positive integer k which implies that $a^j = 0$

$\forall$ j $\geq$ k.
Moreover $(eae)^{2^k} = ea^{2^k}e = e.0.e = 0$ since $2^k > k$. Hence $(eae)^{2^k} = 0$, and therefore eae $\in$ N.

Lemma 3.3: Let R be a ring such that every element is either idempotent or nilpotent or a unit and suppose N is commutative. Then for every element a in N and every unit x in R, ax and xa $\in$ N.

Proof: Let a $\in$ N, and x a unit. Then $a^n = 0$ for some positive integer n. Suppose ax $\notin$ N. Then,

(1) $$ax \neq xa.$$

Because otherwise, if ax = xa, so $(ax)^n = axaxa...ax = x^n a^n = 0$ which is a contradiction to our assumption.

Also ax is not a unit in R. Because otherwise, there exists y $\in$ R such that y(ax) = (ax)y = 1.
This implies that $ya = x^{-1}$ and hence xya = 1. So a is invertible, which is a contradiction since a $\in$ N.
Therefore ax is idempotent, and hence $axax = (ax)^2 = ax$.
So

(2) $$axa = a \text{ since x is invertible.}$$

Now (1 + x) $\notin$ N, since N is commutative and a(1 + x) $\neq$ (1 + x)a.
Also if $(1 + x)^2 = 1 + x$, then $x^2 = -x$. So x = -1 since x is invertible, which implies that ax = xa but this contradicts (1).
Hence

(3) $$1 + x \text{ is a unit in R.}$$

Now ax $\notin$ N, it follows that a(1 + x) = a + ax $\notin$ N. Also a(1 + x) is not a unit in R, since otherwise a will be a unit.
So a(1 + x) is idempotent. Thus $(a + ax)^2 = a + ax$.
So $a^2 + axax + a^2x + axa = a + ax$
Using (2) and $(ax)^2 = ax$, we get $a^2 + a^2x = 0$, that is ;
$a^2(1 + x) = 0$. Then (3) implies that

(4) $$a^2 = 0.$$

On the other hand, $a^n = 0$ and
$$\begin{aligned}(x^{-1}ax)^n &= x^{-1}ax.x^{-1}ax...x^{-1}ax \\ &= x^{-1}a^n x = 0.\end{aligned}$$
So $x^{-1}ax \in$ N. Therefore,
$$\begin{aligned}a(x^{-1}ax) &= (x^{-1}ax)a \text{ since N is commutative} \\ &= x^{-1}(axa) \\ &= x^{-1}a \text{ using (2).}\end{aligned}$$
Hence $ax^{-1}ax = x^{-1}a$.
Multiplying by a from the left, and using (4) we get $ax^{-1}a = 0$.

Then $x^{-1}a = ax^{-1}ax = 0.x = 0$. Thus a = 0, which contradicts (1).
Therefore, ax$\in$ N .
Similarly, xa $\in$ N.

Theorem 3.3: Let R be a periodic ring with identity 1. Suppose that
(a) N is commutative
(b) Every x in D is either idempotent or nilpotent
Then N is an ideal of R, and R/N is either Boolean or a field.

Proof: Suppose x $\in$ R, x $\notin$ D. Since R is periodic, let $x^m = x^n$, m>n$\geq$1.
Then $x^n(x^{m-n} - 1) = 0$ implies $xx^{n-1}(x^{m-n} - 1) = 0$ and since x $\notin$ D, $x^{n-1}(x^{m-n} - 1) = 0$. Similary $x^{n-1}(x^{m-n} - 1) = xx^{n-2}(x^{m-n} - 1) = 0$, then $x^{n-2}(x^{m-n} - 1) = 0$.
Continuing the same way we obtain $x(x^{m-n} - 1) = 0$. But x $\notin$ D, so $x^{m-n} - 1 = 0$. Hence $x^{m-n} = 1$. Therefore x is a unit with inverse x^{m-n-1}.
Hence by (b) ,for every x in R, x is nilpotent or idempotent or a unit.

It is easy to see that N is a subring of R, since the sum and the product of two nilpotent elements is nilpotent (N is commutative). In order to prove N is an ideal of R, it remains to show that for every x $\in$ R and a$\in$ N, xa $\in$ N and ax $\in$ N.

Case 1: If x is nilpotent,

Then xa = ax $\in$ N.

Case 2: If x is idempotent,

Then $(xa - xax)^2 = 0$. Hence xa - xax $\in$ N.
But N is commutative, so a(xa - xax) = (xa - xax)a.
$\Rightarrow$ axa - axax = xa^2 - xaxa
Multilplying both sides by x from the right and the left, we get,
$\Rightarrow$ xaxax - xaxax = xa^2x - xaxax
$\Rightarrow$ xaxax = x^2x
$\Rightarrow$ $(xax)^2$ = xaxxax = xaxa = xa^2x
Hence by Lemma 3.2, xax $\in$ N
Then xa = xa - xax + xax $\in$ N since N is a subring. Similarly for ax $\in$ N.
Case 3: If x is a unit element and a $\in$ N,

Then ax $\in$ N and xa $\in$ N by Lemma 3.3.

Therefore N is an ideal of R.

Consider now the quotient ring R/N.
Let x + N be any nonzero right zero divisor in R/N.
So (y + N)(x + N) = N, with x $\notin$ N and y $\notin$ N. Thus yx + N = N, and hence yx $\in$ N.

Note that x is not a unit; otherwise, $y = (yx)x^{-1} \in N$ since N is an ideal.
Thus x is an idempotent element of R, hence

$$(x + N)^2 = x^2 + N = x + N.$$

This shows that every right zero divisor of R/N is idempotent.
Moreover, we see that every $x + N \in R/N$ is idempotent or a unit of R/N.

Claim:
If R/N has an idempotent different from N and 1 + N, then R/N is Boolean.

Proof: Let $(f + N) \in R/N$ such that $(f + N)^2 = f + N$ with $f \notin N$ and $f - 1 \notin N$.
Suppose there exists an element $u + N \in R/N$ that is not idempotent.
Then u + N is a unit in R/N.
Note that $(f + N)(u + N)$ is not a unit in R/N; otherwise, f + N would be a unit in R/N. Hence,

(5) $(f + N)(u + N)$ is idempotent.

Note that R/N has no nonzero nilpotents; because if x + N is a nilpotent element,then $(x + N)^k = N$ so $x^k = 0$, which implies that $x \in N$.

Now, since R/N is periodic and has no nonzero nilpotents, by a well known theorem of Herstein [9], R/N is commutative. Combining this with (5),

We see that

$$(f + N)(u + N) = ((f + N)(u + N))^2 = fu^2 + N.$$

And hence $f(u - u^2) + N = N$. But u + N is a unit and hence $f(1 - u) + N = N$.

Thus, $(1 - u) + N$ is a right zero divisor (since $f \notin N$), and hence $(1 - u) + N$ is idempotent.

Now $(1 - u)^2 + N = (1 - u) + N$ implies that $u^2 + N = u + N$ and hence $u + N = 1 + N$ since u + N is a unit.
But then u + N is idempotent since 1 + N is idempotent, which is a contradiction. This contradiction proves the Claim.

On the other hand, if R/N has no idempotent other than 1 + N and N, then all nonzero elements in R/N are unit elements (since every x+N is either idempotent or a unit in R/N). Therefore, R/N is a division ring and commutative.

So R/N is a field.

Chapter 4

Structure of Certain Von Neumann π-Regular Rings with Prime Centers

In this chapter we introduce the notion of Prime Center and Semiprime Center, and study the structure of certain classes of these rings. We also study structure of certain Von Neumann π-regular rings having a Prime center and other constraints. For example, we prove that a π-regular ring with identity and a prime center is strongly π-regular commutative ring.
We also give an example of a noncommutative ring with prime center.
Hazar Abu-Khuzam and Adel Yaqub introduced the notions of Prime center and Semi-prime center in [2].

Definition 4.1: The center C of a ring R is said to be a *Prime center* if:
ab $\in$ C $\Rightarrow$ a $\in$ C or b $\in$ C $\forall$ a, b $\in$ R.
In this case, R is said to have a *Prime Center*.

Definition 4.2: The center C of a ring R is said to be a *Semiprime center* if:
$a^n \in C \Rightarrow a \in C$ $\forall$ a $\in$ R.
In this case, R is said to have a *Semiprime Center*.

Remark 4.1:

1. Every commutative ring has a prime center (R = C).
2. If the center of R is prime, then it is also semi-prime.

Lemma 4.1: Let R be a ring having a semiprime center C. Then the nilpotent and the idempotent elements of R belong to the center.

Proof: Let a be a nilpotent element, then $a^n = 0$ for some positive integer n.
But $0 \in C$, so $a^n \in C$. Hence $a \in C$ (since C is semi-prime).

Let e be an idempotent element, then $e^2 = e$.
So $(ex - exe)^2 = 0 \ \forall\ x \in R$, which implies that (ex - exe) is nilpotent. Hence (ex - exe) belongs to the center, and

$$e(ex - exe) = (ex - exe)e \ \forall\ x \in R.$$

This implies that $ex = exe \ \forall\ x \in R$.
Similarly $xe = exe \ \forall\ x \in R$ and hence e belongs to the center of R.

Lemma 4.2: Let R be a ring with identity 1 and having a prime center C. Let U be the set of units of R. Then $U \subseteq C$.

Proof: Let u be a unit in R, then $u.u^{-1} = 1$.
But $1 \in C$, so $u \in C$ or $u^{-1} \in C$ (since C is a prime center).

Note that, if $u^{-1}x = xu^{-1}$ for all x in R, then by multiplying by u from the left, we get $x = uxu^{-1}$. And again, by multiplying by u from the right, we get
$xu = ux$ for all x in R.
Hence $U \subseteq C$.

Lemma 4.3: Let R be a ring with identity 1 and having a prime center C. Let J be the Jacobson radical of R. Then $J \subseteq C$.

Proof: Let $x \in J$, then x belongs to all the regular maximal ideals of R, and hence
1 + x doesn't belong to any maximal ideal of R.
Claim: 1 + x is a unit in R.

Proof:
Assume 1 + x is not a unit of R, then the ideal $(1 + x) \subsetneq R$.
So there must exists a maximal ideal M with $(1 + x) \subseteq M$.
Hence $1 = 1 - x + x \in M$, which implies that R = M and this is a contradiction. This completes the proof of the claim.

Therefore 1 + x is a unit of R, and hence belongs to the center by Lemma 4.2.
Write x = x + 1 - 1, then x belongs to the center. Thus $J \subseteq C$.

Lemma 4.4: Let R be a prime ring with identity 1 and a semiprime center C. If e is an idempotent in R, then e = 0 or e = 1.

Proof: Let e be an idempotent element in R, then e belongs to the center by Lemma 4.1. Hence

$$eaex = ae^2x = aex = eax \ \forall\ a, x \in R$$

Thus

$$eR(ex - x) = 0 \ \forall\ x \in R$$

Since R is a prime ring, then e = 0 or $ex = x \ \forall\ x \in R$.
If $e \neq 0$, then ex = x for all $x \in R$. But $e \in C$, so xe = x for all $x \in R$.
This means that xe = ex =x. Therefore e = 1.

Theorem 4.1: Let R be a periodic ring. R is commutative if and only if R has a semi-prime center.

Proof:
($\Rightarrow$) Clearly if R is commutative, then R has a semi-prime center.

($\Leftarrow$) Assume R has a semi-prime center.
Let $x \in R$, then x is a periodic element of R (since R is periodic). And by Lemma 2.1 (a), x^k is idempotent for some positive integer k, and hence x^k is a central idempotent by Lemma 4.1. But R has a semi-prime center, so $x \in C$. Therefore R commutative.

Theorem 4.2: If R is a prime ring with a prime center C. Then R is a domain.

Proof: Assume R is a prime ring having a prime center C, hence R has a semi-prime center C. Then by Lemma 4.1, $N \subseteq C$.

Claim: C has no nonzero zero divisors.

Proof:
Let $0 \neq x \in C$ such that $yx = xy = 0$ for some $y \neq 0$. Hence

$$xry = rxy = 0 \ \forall \ r \in R.$$

Thus

$$xRy = 0 \Rightarrow x = 0 \text{ or } y = 0 \text{ as R is prime.}$$

This is a contradiction. So C has no nonzero zero divisors, and hence R has no nonzero nilpotent elements. So $N = \{0\}$.

Now we have R a prime ring with $N = \{0\}$, and we let $a, b \in R$. Then

$$\begin{aligned} ab = 0 \Rightarrow (bra)^2 &= (bra)(bra) = (br).0.(ra) = 0 \ \forall \ r \in R \\ &\Rightarrow bra \in N = \{0\} \ \forall \ r \in R \\ &\Rightarrow bra = 0 \ \forall \ r \in R \\ &\Rightarrow b = 0 \text{ or } a = 0 \end{aligned}$$

Therefore R is a domain.

Lemma 4.5: Let R be a ring with no nonzero nilpotent ideals. Let $I \neq 0$ be a minimal right ideal of R. Then $I = eR$ for some idempotent element e of R.

Proof: As R has no nonzero nilpotent ideals, then $I^2 \neq 0$. Hence there exists $x \in I$ such that $xI \neq 0$.

But xI is an ideal of R and $xI \subseteq I$, so by minimality of I we get $xI = I$.

So there exists $e \in I$ such that $xe = x$.
Observe that $x(e^2 - e) = 0$. So we let $I_0 = \{a \in I; xa = 0\}$. Then I_0 is a right ideal of R, contained in I and is not I (since $xI \neq 0$).
Hence by minimality of I, we get $I_0 = 0$.
But $e^2 - e \in I_0$, so $e^2 = e$.

Theorefore e idempotent element of R and I = eR.

Lemma 4.6: Let R be a ring and suppose that for some a $\in$ R, a^2 - a is nilpotent. Then either a is nilpotent, or for some polynomial q(x) with integer coefficients,
e = a.q(a) is a nonzero idempotent.

Proof: Suppose that $(a^2 - a)^k = 0$ for some positive integer k; expanding this we get $a^k = a^{k+1}p(a)$ where p(x) is a polynomial with integer coefficients.

$$\begin{aligned} a^k &= a^{k+1}p(a) \\ &= a^k ap(a) \\ &= a^{k+2}p(a)^2 \end{aligned}$$

Continuing in this fashion, we get $a^k = a^{2k}p(a)^k$.

Assume a is not nilpotent, so $a^k \neq 0$. Then

$$e = a^k p(a)^k \neq 0,$$

and

$$e^2 = a^{2k}p(a)^{2k} = a^{2k}p(a)^k p(a)^k = a^k p(a)^k = e.$$ This completes our proof.
This proof is done in Lemma 1.3.2 in [11].

Lemma 4.7: If R is an Artinian ring and I $\neq$ 0 is a nonnilpotent ideal of R, then I contains a nonzero idempotent element.

Proof: Since I is not nilpotent, then by Theorem 1.5.5 I is not contained in J. Consider $\bar{R}$ = R/J which is a semisimple ring by Theorem 1.4.3, then $\bar{R}$ has no nonzero nilpotent ideals.

Let π(I) be the image of I in $\bar{R}$ under the canonical epimorphism π: R $\longrightarrow \bar{R}$, and we denote π(I) by $\bar{I}$.

Since $\bar{I} \neq 0$, then $\bar{I}$ contains a minimal ideal $\bar{I}_0$ of $\bar{R}$ (since $\bar{R}$ is artinian by Theorem 1.5.3).

Now by Lemma 4.5, $\bar{I}_0 = \bar{R}\bar{e}$, with $\bar{e}$ idempotent element in $\bar{I}_0$.

Since π is surjective, then there exists a $\in$ I such that $\pi(a) = \bar{e}$. Hence $\pi(a^2 - a) = 0$, which implies that a^2 - a $\in$ J, and thus it is nilpotent by Theorem 1.5.5.

If a is a nilpotent element, then $a^k = 0$, so $(\pi(a))^k = 0$.
But $(\pi(a))^k = \bar{e}^k = \bar{e} \neq 0$. Hence a is not nilpotent, and by Lemma 4.6 there exists a polynomial q(x) with integer coefficients such that e = aq(a) a nonzero idempotent.
Since a $\in$ I, then e $\in$ I. And this completes our proof.

Theorem 4.3: Let R be an Artinian prime ring with identity 1. If R has a prime center then R is a field.

Proof: We will first show that R is simple, and that it is isomorphic to a division ring.

Let I be a nonzero ideal of R, so there exists a nonzero element x in I.
If $x^n = 0$ for some positive integer n, then $x \in C$ by Lemma 4.1. So $x^{n-1} = 0$ or $x = 0$ (since R prime ring).
If $x \neq 0$, then $x^{n-1} = 0$ which implies that $x^{n-2} = 0$. If we continue in this fashion, we get $x = 0$ which contradicts our assumption.
Hence I is a non-nilpotent ideal.

By Lemma 4.7, I contains a nonzero idempotent element. And since every prime center is a semiprime center then by Lemma 4.4, $1 \in I$. Therefore $I = R$. We have proved so far that R has no nonzero proper two sided ideals and thus R is simple.

Now, R is simple and artinian, so R is isomorphic to a complete matrix ring $Mat_n D$ over a division ring D by Wedderburn-Artin.

This implies that the matrix ring over the division ring D has a prime center, which is impossible for $n > 1$.
($E_{11}E_{22} = 0 \in C$ but $E_{11} \notin C$ and $E_{22} \notin C$, with E_{ii} be the matrix only its (i,i) entry is 1 and the others are equal to zero).
Then $n = 1$ and thus R is a division ring.

We now have that every nonzero element of R is a central unit by Lemma 4.2. Hence R is a commutative division ring, and therefore R is a field.

Example of a noncommutative ring with a prime center

Let F be an infinite field, let σ be an automorphism of F with infinite order. Let $F[x,\sigma]$ be the ring of all polynomials p(x) over F such that $x^n a = \sigma^n(a) x^n$ $\forall$ $a \in F$ and for every n positive integer.

Claim: $F[x,\sigma]$ is a domain.

Proof:
Assume $F[x,\sigma]$ is not a domain, then it has zero divisors; that is,
$\exists$ p(x), $g(x) \in F[x,\sigma]$ such that $p(x).g(x) = 0$ with $p(x) \neq 0$ and $g(x) \neq 0$.

$p(x) = a_n x^n + a_{n-1} x_{n-1} + ... + a_0$.
$g(x) = b_n x^n + b_{n-1} x_{n-1} + ... + b_0$.

$p(x).g(x) = a_n x^n b_n x^n + ... = a_n \sigma^n(b_n) x^{2n} + ...$

But $p(x).g(x) = 0$, then all coefficients are zero; that is $a_n \sigma^n(b_n) = 0$ $\forall$ n.
However $a_n \sigma^n(b_n) \in F$ and F is a field, so it has no zero divisors, thus either $a_n = 0$ or $\sigma^n(b_n) = 0$.
Repeating this for all coefficients we obtain a possibility where a_i=0 $\forall$ i, thus p(x)=0. Contradiction.

Now let R = F[x,σ]x. We will prove that this R is a noncommutative ring with a prime center.
Let C denote the center of R, and let P(x) = $a_1x + a_2x^2 + ... + a_nx^n$ be a nonzero element in C.

$$\begin{aligned} x.P(x) &= xa_1x + xa_2x^2 + ...+ xa_nx^n \\ &= \sigma(a_1)x.x + \sigma(a_2)x.x^2 + ... + \sigma(a_n)x.x^n \\ &= \sigma(a_1)x^2 + \sigma(a_2)x^3 + ... + \sigma(a_n)x^{n+1} \end{aligned}$$

$$\begin{aligned} P(x).x &= a_1x.x + a_2x^2.x + a_nx^n.x \\ &= a_1x^2 + a_2x^3 + a_nx^{n+1} \end{aligned}$$

But P(x) $\in$ C, then x.P(x) = P(x).x, therefore

(1) $$\sigma(a_1) = a_1,\ \sigma(a_2) = a_2,\ ...,\ \sigma(a_n) = a_n$$

Note that σ has an infinite order, so there exists a $\in$ F such that $\sigma^n(a) \neq \sigma(a)$ = a for all n, then

(2) $$\sigma^n(a) \neq a\ \forall\ n.$$

Now,

$$\begin{aligned} axP(x) &= axa_1x + axa_2x^2 + ...+ axa_nx^n \\ &= a\sigma(a_1)x^2 + a\sigma(a_2)x^3 + ... + a\sigma(a_n)x^{n+1} \\ &= aa_1x^2 + aa_2x^3 + ... + aa_nx^{n+1} \qquad \text{by (1)} \end{aligned}$$

Also,

$$\begin{aligned} P(x)ax &= a_1xax + a_2x^2ax + ...\ a_nxnax \\ &= a_1\sigma(a)x^2 + a_2\sigma^2(a)x^3 + ... + a_n\sigma^n(a)x^{n+1}. \end{aligned}$$

But P(x) $\in$ C, then a.x.P(x) = P(x).a.x, therefore

(3) $$aa_1 = a_1\sigma(a),\ aa_2 = a_2\sigma^2(a),\ ...,\ aa_n = a_n\sigma^n(a)$$

We assume P(x) $\neq$ 0, then $a_i \neq 0$ for some positive integer i. Since F is a field, $aa_i = a_ia$ with a, a_i are units. Hence using (3) we get

$$\sigma^i(a) = a \text{ for some positive integer i}$$

But this contradicts (2), thus P(x) = 0 and it follows that C = 0.
So we proved that R is a domain with center {0}, this means that R has a prime center.
Also R is noncommutative, since xa = σ(a)x $\neq$ ax.
Therefore R is a noncommutative ring with prime center.

Definition 4.3: A ring R is said to be *Von Neumann regular* if for any a $\in$ R, there exists x $\in$ R such that a = axa.

Definition 4.4: A ring R is said to be Von Neumann π-*regular* if for every x $\in$ R, there exist y $\in$ R and a positive integer n such that $x^n = x^nyx^n$.

Remark 4.1: A Von Neumann regular ring is π-regular.

Definition 4.5: A ring R is said to be *strongly π-regular* if for every $x \in R$, there exist $y \in R$ and a positive integer n such that $x^n = x^{n+1}y$.

Theorem 4.4: Let R be a Von Neumann π-regular ring with identity. If R has a prime center, then R is strongly π-regular commutative ring.

Proof: Let x be an element in R, then there exist y in R and a positive integer n such that $x^n = x^n y x^n$.
Observe that $x^n y = x^n y x^n y = (x^n y)^2$ and $y x^n = (y x^n)^2$.
Then $x^n y$ and $y x^n$ are idempotent elements of R, and hence $x^n y$ and $y x^n$ are central elements of R by Lemma 4.1.

$x^n y \in C \Rightarrow x^n \in C$ or $y \in C$ since R has a prime center.

- If $x^n \in C$, then $x \in C$. So

$$x^n = x^n y x^n = x^{n+1} x^{n-1} y,$$

which implies that R is strongly π-regular commutative ring.

- If $x^n \notin C$, then $y \in C$ and

$$\begin{aligned} x^n r &= x^n y x^n r \ \forall\, r \in R \\ &= x^n r x^n y \text{ since } x^n y \in C \ \forall\, r \in R \\ &= x^n r y x^n \text{ since } y \in C \\ &= x^n y r x^n \text{ since } y \in C \\ &= r x^n x^n y \text{ since } x^n y \in C \\ &= r x^n y x^n \text{ since } y \in C \\ &= r x^n \ \forall\, r \in R \end{aligned}$$

So $x^n \in C$, this contradicts our assumption.
And our proof is done.

References

1. Abu Khuzam H. and Yaqub A, *Structure of certain periodic rings*, Canad. Math. Bull. 28,1 (1985), 120-123.

2. Abu Khuzam H. and Yaqub A, *On rings with prime centers*, International J. Math & Math Sci 17, No.4 (1994), 667-670.

3. Abu Khuzam H. and Bell H, *On rings with condition on Nonperiodic element*, Result. Math. 43 (2003), 198-201.

4. Abu Khuzam H, *the structure of certain von Neumann regular rings and π-regular rings with certain constraints such as having prime/semi-prime centers* (Submitted for publication), (2020).

5. Anderson W. and Fuller K.R, *Rings and categories of modules , Graduate texts in Mathematics* 13, Springer-Verlag (1992).

6. Bell H, *A commutativity study for periodic rings*, Pacific J. Math. 70 (1977), 29-36.

7. Bell H, *A near-commutativity property for rings*, Result. Math. 42 (2002), 28-31.

8. Guo X, *On the periodic radical of a ring*, Canad. Math. 42 (2002), 28-31.

9. Herstein I.N, *A note on rings with central nilpotent elements*, Proc. Amer. Math. Soc, 5 (1954), 620.

10. Herstein I.N, *A commutativity theorem*, Journal of Algebra, 38 (1976), 238-241.

11. Herstein I.N, *Noncommutative Rings*, The Carus Math. Monographs, John Wiley and Sons, Inc. (1968).

12. Hungerford T, *Algebra, Graduate Texts in Mathematics* 73, Springer Verlag (1974).

13. Jacobson N, *Structure theory for algebraic algebras of bounded degrees*, Ann. of Math 46 (1945), 695-707.

14. Lam T.Y, *A first course in noncommutative rings, Graduate texts in Mathematics* 131, Springer-Verlag (1991).