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MINIMAL FREE RESOLUTIONS AND PROJECTIVE DIMENSION ≤ 1

by

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An Abstract of the Thesis of

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Let $R = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables and I an ideal in R. We first define the notions of minimal free resolutions of algebras R/I and multigraded minimal resolutions of monomial ideals I. We then discuss the following established result in [6]:

$$\text{projdim } (I) \leq 1 \iff \text{a graph tree supports the minimal free resolution of } R/I$$

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Chapter 1

Introduction

Let $R = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables with maximal ideal $\mathfrak{m}$, and let I be an ideal of R. A free resolution of R/I is an exact sequence of free modules that describes relations on the generators of the ideal. The resolution is minimal whenever the matrices representing the maps in the exact sequence have entries in the maximal ideal $\mathfrak{m}$. Constructing minimal free resolutions of algebras R/I has been of interests to many authors.

Suppose I is a monomial ideal i.e generated by monomials. Finding the minimal free resolution of R/I known as the minimal monomial resolution, can be quite complex despite the combinatorial structure that monomial ideals have. An important tool in studying monomial resolutions is to find topological objects whose chain maps can be homogenized to obtain free resolutions of these ideals. This approach began with Diana Taylor in her thesis [11] in 1966. It consists of labeling the faces of the simplex by the lcm of monomial generators of the ideal. Many mathematicians tried to generalize Taylor's approach by considering smaller topological objects with the hope of obtaining minimal free resolutions.

In this thesis, we first define the notions of minimal free resolutions of algebras R/I of a general ideal I and multigraded minimal resolutions of monomial ideals I. We then discuss the following established result by Hersey and Faridi in [6] where they prove:

$$\text{projdim}(I) \leq 1 \iff \text{a graph tree supports the minimal free resolution of } R/I$$

Chapter 2

Preliminaries

2.1 Notions on Commutative Rings

Let R be a commutative unitary ring. Here are some useful definitions on elements of the ring R.

Definition 2.1. A **zero divisor** in R is an element x for which $\exists\ y \neq 0$ such that $xy = 0$.

Example 2.2. *In $M_2(\mathbb{R})$, consider A and B to be the following matrices*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

their product AB is the zero matrix while A and B are not, so A and B are two zero divisors.

A ring with no zero divisors $\neq 0$ (and in which $1 \neq 0$) is called an integral domain, just like $\mathbb{Z}$, $k[x_1, x_2, \ldots, x_n]$, where k is a field and $n \in \mathbb{N}$, are integral domains.

Definition 2.3. A **unit** in R is an element x which "divides 1", i.e an element x such that $xy = 1$ for some y in R.

Note that the element y is then uniquely determined by x and is written x^{-1}, and the units in R form a (multiplicative) abelian group. For all $r \in R$, the multiples rx of an element $x \in R$ form a principal ideal, denoted by (x) or Rx . Note that, $(x) = R = (1) \iff x$ is a unit , because $rx = 1 \; \forall \; r \in R$. Note that the zero ideal is usually denoted by 0.

Now, we can introduce **Regular Sequences**, that will be used as an application afterwards.

Definition 2.4. Let R be a ring. Let M be an R-module. A sequence of elements $r_1, r_2, \dots, r_n \in R$ is called a **regular sequence on M (or M-sequence)** if

1. $(r_1, r_2, \dots, r_n)M \neq M$ and

2. for $i = 1, \dots, n$, r_i is a non zero divisor on $M/(r_1, r_2, \dots, r_{i-1})M$.

Example 2.5. *Let $M = R = k[x, y]$, $r_1 = xy, r_2 = x^2$ is not a regular sequence since x^2 is a zero divisor on $R/(xy)$ as $x^2y = x(xy) = 0$ in $R/(xy)$ and $y \neq 0$.*

Definition 2.6. A **field** is a ring R in which $1 \neq 0$ and every non-zero element is a unit.

Example 2.7. *$\mathbb{R}$, $\mathbb{C}$, $GL_n(\mathbb{R})$.*

We note that every field is an integral domain but not conversely.

Proposition 2.8. *Let R be a ring $\neq 0$, then the following are equivalent:*

1. *R is a field.*

2. *The only ideals in R are 0 and (1)*

3. *Every non zero homomorphism Φ: $R \longrightarrow R'$ is injective for all rings $R' \neq 0$.*

Proof. $1 \longrightarrow 2$) Let $\mathcal{A} \neq 0$ be an ideal in R, then $\mathcal{A}$ contains a non-zero element x, then x is a unit, hence $\mathcal{A} \supseteq (x)$ =(1) so $\mathcal{A} = 1$.

$2 \longrightarrow 3$) Let $\Phi : R \longrightarrow R'$ be a ring homomorphism , then ker(Φ) is an ideal $\neq (1)$ $\subseteq R$, so ker(Φ)= 0 hence Φ is injective.

$3 \longrightarrow 1$) Let $x \in R$ be a non-unit then $(x) \neq (1)$ hence $R' := R/(x)$ is not a zero ring . Let Φ: $R \longrightarrow R'$ be the natural homomorphism of R onto R', its kernel is (x). By hypothesis, Φ is injective, hence (x)= 0 so $x = 0$. □

Now we pass on to a discussion of the following significant ideals,

Definition 2.9. An ideal P in R is **prime** if $P \neq (1)$ and if $xy \in P$, then $x \in P$ or $y \in P$.

Example 2.10. *$p\mathbb{Z}$, where p is a prime number.*

Proof. Let $x, y \in p\ \mathbb{Z}$ such that $xy \in p\mathbb{Z} \implies p \mid xy$, and by Euclid's Lemma since p is prime then $p \mid x$ or $p \mid y \implies x \in p\mathbb{Z}$ or $y \in p\mathbb{Z}$. □

Definition 2.11. An ideal M in R is **maximal** if $M \neq (1)$ and if there is no ideal $\mathcal{A}$ in R such that $M \subsetneq \mathcal{A} \subsetneq (1)$.

Example 2.12. *$2\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$.*

Proof. Suppose $\exists\ J$ ideal of $\mathbb{Z}$ so $J = a\mathbb{Z}$ such that $2\mathbb{Z} \subseteq a\mathbb{Z} \subseteq \mathbb{Z}$,then $a \mid 2$ and $1 \mid a \implies a = 2$ or $a = 1 \implies a\mathbb{Z} = 2\mathbb{Z}$ or $a\mathbb{Z}$= $\mathbb{Z}$ and thus $2\mathbb{Z}$ is maximal. □

Example 2.13. *R= $k[x_1, x_2, \ldots, x_n]$, where k is a field and $n \in \mathbb{N}$. Let $f \in R$ be an irreducible polynomial. By unique factorization, the ideal (f) is prime.*

A **principal ideal domain** is an integral domain in which every ideal is principal. In such a ring, every non-zero prime ideal is maximal. For if $(x) \neq 0$ is a prime ideal and $(y) \supset (x)$, we have $x \in (y)$, say $x = yz$, so that $yz \in (x)$ and $y \notin (x)$, hence $z \in (x)$, say $z = tx$. Then $x = yz = ytx$ so that $yt = 1$ and then $(y)=$ (1).

Proposition 2.14. *Let P, M be ideals of R.*

1. *P is prime $\iff$ R/P is an integral domain.*

2. *M is maximal $\iff$ R/M is a field.*

   Hence a maximal ideal is prime but the converse is not true.

Proof. 1. ($\Longrightarrow$) P is a prime ideal of $R \implies P \neq (1)$ and if $xy \in P$, then $x \in P$ or $y \in P \implies R/P \neq 0$ and if $\overline{x}$, $\overline{y} \in R/P$ st $\overline{x}\,\overline{y} = \overline{0}$ then $\overline{x} = \overline{0}$ or $\overline{y}$ $= \overline{0} \implies R/P$ has no zero divisors $\implies R/P$ is an integral domain.

($\Longleftarrow$) Let $\overline{a}$,$\overline{b} \in R/P$ st $\overline{a}\overline{b}= \overline{0}$, but R/P is an integral domain, so it has no zero divisors, then $\overline{a} = \overline{0}$, or $\overline{b} = \overline{0} \implies a, b \in R$ with $ab \in P$ st $a \in P$ or $b \in P \implies P$ is a prime ideal of R.

2. ($\Longrightarrow$) M is a maximal ideal of $R \implies$ if J is an ideal in R such that $M \subseteq J \subseteq R$ then $J = M$ or $J = R \implies$ if J/M is an ideal in R/M such that $0 \subseteq J/M \subseteq R/M$ then $J/M = 0$ or $J/M = R/M \implies$ the only ideals of R/M are 0 and $R/M \implies R/M$ is a field.

   ($\Longleftarrow$) R/M is a field $\implies$ the only ideals of R/M are 0 and (1) $\implies$ if J/M is an ideal of R/M ($J \subseteq R$ and $M \subseteq J$) then $J/M = 0$ or $J/M = R/M$

$\Longrightarrow$ if J is an ideal of R such that $M \subseteq J$ then $J = M$ or $J = R \Longrightarrow M$ is a maximal ideal of R.

□

Example 2.15. *$R=\mathbb{Z}$, every ideal in $\mathbb{Z}$ is of the form (m) for some $m \geq 0$. The ideal (m) is prime $\Longleftrightarrow m = 0$ or a prime number. All the ideals (p) where p is a prime number are maximal and $\mathbb{Z}/(p)$ is the field of p elements.*

Theorem 2.16. *Every ring $R \neq 0$ has at least one maximal ideal.*

Proof. Let Σ be the set of all ideals $\neq (1)$ in R. Order Σ by inclusion, Σ is non-empty since $0 \in \Sigma$, to apply Zorn's Lemma we have to show that every chain in Σ has an upper bound in Σ , let ($\mathcal{A}_i$) be a chain of ideals in Σ , so that for each pair of indices i, j we have either $\mathcal{A}_i \subseteq \mathcal{A}_j$ or $\mathcal{A}_j \subseteq \mathcal{A}_i$. Let $\mathcal{A} = \cup_i \mathcal{A}_i$, then $\mathcal{A}$ is an ideal such that $1 \notin \mathcal{A}$ because $1 \notin \mathcal{A}_i$ $\forall i$. Hence, $\mathcal{A} \in \Sigma$, and $\mathcal{A}$ is an upper bound of the chain, by Zorn's Lemma Σ has a maximal element . □

Corollary 2.17. *If $\mathcal{A} \neq (1)$ is an ideal of R, then there exists a maximal ideal of R containing $\mathcal{A}$.*

Proof. Apply theorem 2.16 to $R/\mathcal{A}$. we have that $\mathcal{A} \neq R$, then $R/\mathcal{A} \neq 0$, so $R/\mathcal{A}$ has at least one maximal ideal, so $\exists$ $M/\mathcal{A}$, M ideal of R containing $\mathcal{A}$ such that $M/\mathcal{A}$ is maximal in $R/\mathcal{A}$. We still have to prove that M maximal in R containing $\mathcal{A}$. Let J be an ideal of R such that $\mathcal{A} \subseteq J$ and $M \subseteq J \subseteq R$, then $M/\mathcal{A} \subseteq J/\mathcal{A} \subseteq R/\mathcal{A}$. But, $M/\mathcal{A}$ is maximal in $R/\mathcal{A}$, so $J/\mathcal{A} = M/\mathcal{A}$ or $J/\mathcal{A} = R/\mathcal{A}$, therefore $J = M$ or $J = R$ and then M is maximal in R containing

$\mathcal{A}$. □

Proposition 2.18. *Every non-unit of R is contained in a maximal ideal.*

Proof. Let m be a non-unit element of R, then $(m) \neq R$ (otherwise $(m) = R = (1)$ and m is a unit, contradiction!) , then by corollary 2.17 , $\exists$ M maximal ideal such that $(m) \subseteq M \subseteq R$, so $(m) \subseteq M$ and $1 \in R$ so $m \in (m) \subseteq M$, then $m \in M$. □

Definition 2.19. The **Jacobson Radical** $\mathcal{J}$ of R is the intersection of all maximal ideals of R.

Proposition 2.20. $x \in \mathcal{J} \iff 1 - xy$ *is a unit in R for all $y \in R$.*

Proof. $\Longrightarrow$) Suppose $1 - xy$ is a non-unit, by Proposition 2.18 it belongs to some maximal ideal m of R, but $x \in \mathcal{J} \subseteq M$, hence $xy \in M$ and therefore $1 \in M$, which is absurd.

$\Longleftarrow$) Suppose $x \notin M$, for some maximal ideal M. Then, M and x generate the unit ideal (1), so that we have $u + xy = 1$, for some $u \in M$ and some $y \in R$. Hence, $1 - xy \in M$ and is therefore not a unit . □

Definition 2.21. R is a **local ring** iff R has a unique maximal ideal.

Example 2.22. *For every field k, its 0 is a maximal ideal, because by proposition 2.8, the only ideals of a field are 0 and (1),i.e.and there is no ideals in between, so the zero ideal is a maximal ideal and it's unique as (1) can't be a maximal ideal by definition.*

2.2 Modules

Let R be a commutative ring.

Definition 2.23. An R**-module** is an abelian group M (written additively) on which R acts linearly, i.e. it's a pair (M,μ), where M is an abelian group and μ is a mapping of $R \times M$ into M such that if we write rx for $\mu(r, x)$, with $r \in R$ and $x \in M$, then the following axioms are true:

1. $r(x + y) = rx + ry$

2. $(r + r')x = rx + r'x$

3. $(rr')x = r(r'x)$

4. $1x = x$

Example 2.24. *An ideal $\mathcal{A}$ of R is an R-module . In particular, R is an R-module.*

Example 2.25. *R is a field "k", then any R-module is a k-vector space .*

Example 2.26. *If $R = \mathbb{Z}$, then $\mathbb{Z}$-module = abelian group (where nx defined to be $x + x + \ldots + x, n$-times).*

Example 2.27. *$R = k[x]$, the polynomial ring with one variable is a k-module.*

Definition 2.28. Let M, N be R-modules. A mapping f: $M \longrightarrow N$ is an R-module homomorphism if :

1. $f(x + y) = f(x) + f(y)$

2. $f(rx) = rf(x)$ $\forall r \in R$, $\forall x, y \in M.$

If R is a field , then an R-module homomorphism is the same as a linear transformation of vector spaces.

Definition 2.29. A **submodule** M' of M is a subgroup of M which is closed under multiplication by elements of R .

Definition 2.30. The abelian group M/M' inherits an R-module structure from M defined by $r(x+M) = rx + M$. The **quotient module** of M by M' is the R-module M/M' with the above multiplication. The natural map of M onto M/M' is an R-module homomorphism. There is a 1-1 order-preserving correspondence between the submodules of M which contain M', and submodules of M/M'.

Definition 2.31. If $f : M \longrightarrow N$ is an R-module homomorphism , then the **kernel of** f is the set $\ker(f) = \{x \in M : f(x) = 0\}$ and is a submodule of M. The **image of** f is the set Im(f)= $f(M)$ and is a submodule of N.

The **cokernel of** f is coker(f)= $N/$Im(f) which is a quotient module of N.

If M' is a submodule of M such that $M' \subseteq \ker(f)$, then f gives rise to a homomorphism $\overline{f} : M/M' \longrightarrow N$, defined as follows ; if $\overline{x} \in M/M'$ is the image of $x \in M$, then $\overline{f}\ (\overline{x}) = f(x)$, and the kernel of $\overline{f}$ is $\ker(f)/\ M'$. The homomorphism $\overline{f}$ is said to be **induced** by f. In particular, taking M'= $\ker(f)$, we have an isomorphism of R-modules $M/\ker(f) \cong$ Im(f).

2.2.1 Direct Sum and Product

Definition 2.32. If M and N are R-modules, then their direct sum $M \oplus N$ is the set of all pairs (x, y) such that $x \in M, y \in N$.

It's an R-module as we define addition and scalar multiplication as follows:

- $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

- $r(x,y) = (rx, ry)$

More generally, if $(M_i)_{i\in I}$ is any family of R-modules, we can define then direct sum $\underset{i\in I}{\oplus} M_i$, its elements are families $(x_i)_{i\in I}$ such that $x_i \in M_i$ $\forall i \in I$ and almost all x_i's are zeros. If we drop the restriction on the number of non-zero x_i's we have the direct product $\prod\limits_{i\in I} M_i$. Direct sum and direct product are then the same if I is finite, but not otherwise in general.

Suppose that the ring R is a direct product $\prod\limits_{i=1}^{n} R_i$, then the set of all elements of the form $(0,\ldots,0,r_i,0,\ldots,0)$; $r_i \in R_i, \forall i \in I$ is an ideal $\mathcal{A}_i$ of R. It's not a subring of R except in trivial cases, because it does not contain the identity element of R. The ring R, considered as an R-module, is the direct sum of ideals $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$. Conversely, given a module decomposition $R= \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots \mathcal{A}_n$ of R as a direct sum of ideals, we have R $\cong \prod\limits_{i=1}^{n} R/b_i$; where $b_i = \underset{j\neq i}{\oplus} \mathcal{A}_j$. Each ideal $\mathcal{A}_i$ is a ring isomorphic to R/b_i. The identity element e_i of $\mathcal{A}_i$ is an idempotent in R and $\mathcal{A}_i$=e_i; i.e. e_i e_i= e_i.

2.2.2 Finitely Generated Modules

Let M be an R-module.

Definition 2.33. If x is an element of M, the set of all multiples rx; $r \in R$, is a submodule of M denoted by Rx or (x). If $M= \sum\limits_{i\in I} Rx_i$, the x_i's are said to be a set of generators of M; this means that every element of M can be expressed (not necessarily uniquely) as a finite linear combination of the x_i's with coefficients in R.

Definition 2.34. An R-module M is said to be **finitely generated** if it has a finite set of generators.

Example 2.35. $\mathbb{C} = (1, i)$ *is a finitely generated* $\mathbb{R}$*-module.*

Definition 2.36. A **free** R-module is one which is isomorphic to an R-module of the form $\bigoplus\limits_{i \in I} M_i$, where each $M_i \cong R$ as an R-module. The notation R^I is some times used. A finitely generated free R-module is therefore isomorphic to $R \oplus R \oplus \ldots \oplus R$ (n summands); which is denoted by R^n .

Conventionally, R^0 is the zero module, denoted by 0.

Proposition 2.37. *M is a finitely generated R-module $\iff$ M is isomorphic to a quotient of R^n for some integer $n > 0$.*

Proof. $\Longrightarrow$) Let $x_1, \ldots, x_n$ generate M. Define Φ: $\mathrm{R}^n \longrightarrow M$ such that $\Phi(r_1, \ldots, r_n) = r_1 x_1 + \ldots + r_n x_n$. Then, Φ is an R-module homomorphism onto M, therefore $M \cong R^n/ker(\Phi)$.

$\Longleftarrow$) we have an R-module homomorphism Φ of R^n onto M. If $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, then $\{e_i\}$ for $i = 1, \ldots, n$ generate R^n hence $\{\Phi(e_i)\}$ for $i = 1, \ldots, n$ generate M. □

Proposition 2.38. *Let M be a finitely generated R-module , let $\mathcal{A}$ be an ideal of R and let Φ be an R-module endomorphism of M such that $\Phi(M) \subseteq \mathcal{A}M$. The Φ satisfies an equation of the form: $\Phi^n + a_1\Phi^{n-1} + \ldots + a_n = 0$; where $a_i \in \mathcal{A}, \forall i = 1, \ldots, n$.*

Proof. Let $x_1, x_2, \ldots, x_n$ be a set of generators of M , then each $\Phi(x_i) \in \mathcal{A}M$ so $\Phi(x_i) = \sum\limits_{j=1}^{n} a_{ij} x_j$ $(1 \leq i \leq n; a_{ij} \in \mathcal{A})$ i.e. $\sum\limits_{j=1}^{n} (\delta_{ij}\Phi - a_{ij})x_j = 0$ where δ_{ij} is the kronecker delta.

By multiplying on the left by the adjoint of the matrix $(\delta_{ij}\Phi - a_{ij})$ it follows that

$det(\delta_{ij}\Phi - a_{ij})$ annihilates each x_i , hence is the zero endomorphism of M.

Expanding out the determinant, we will have the above equation. $\square$

Corollary 2.39. *Let M be a finitely generated R-module , $\mathcal{A}$ be an ideal of R such that $\mathcal{A}M = M$, then $\exists\ x \equiv 1$ (mod $\mathcal{A}$) such that $xM = 0$.*

Proof. take $\Phi =$ Identity , $x = 1 + a_1 + \ldots + a_n$ in Proposition 2.38. $\square$

Proposition 2.40. *(Nakayama's Lemma)*

Let M be a finitely generated R-module and $\mathcal{A}$ an ideal of R contained in the Jacobson Radical $\mathcal{J}$ of R , then $\mathcal{A}M = M \implies M = 0$.

Proof. First way:

By corollary 2.39 we have $xM = 0$ for some $x \equiv 1\ (mod \mathcal{J})$. By Proposition 2.38 x is a unit in R, hence $M = x^{-1}xM = 0$.

Second way:

Suppose $M \neq 0$, and let $u_1, .., u_n$ be a minimal set of generators of M, then $u_n \in \mathcal{A}M$ as $\mathcal{A}M = M$, hence we have an equation of the form $u_n = a_1u_1 + \ldots + a_nu_n$; $a_i \in \mathcal{A}$ hence $(1 - a_n)u_n = a_1u_1 + \ldots + a_{n-1}u_{n-1}$, since $a_n \in \mathcal{J}$, it follows that $1 - a_n$ is a unit in R. Hence, u_n belongs to the submodule of M generated by $u_1, \ldots, u_{n-1}$, contradiction! $\square$

Corollary 2.41. *Let M be a finitely generated R-module, N a submodule of M, $\mathcal{A} \subseteq \mathcal{J}$ an ideal. Then, $M = \mathcal{A}M + N \implies M = N$.*

Proof. Apply Proposition 2.40 to M/N, observing that $\mathcal{A}(M/N)=$ $(\mathcal{A}M + N)/N$. $\square$

Remark 2.42. *Let R be a local ring, $\mathfrak{m}$ its maximal ideal, $k = R/\mathfrak{m}$ its* ***residue field****. Let M be a finitely generated R-module. $M/\mathfrak{m}M$ is annihilated by $\mathfrak{m}$, hence is naturally an $R/\mathfrak{m}$-module, i.e. a k-vector space, and as such is finite-dimensional.*

Proposition 2.43. *Let x_i for $i = 1, .., n$ be elements of M whose images in $M/\mathfrak{m}M$ form a basis of this vector space. Then, the x_i generate M.*

Proof. Let N be the submodule of M generated by the x_i . Then the composite map $N \longrightarrow M \longrightarrow M/\mathfrak{m}M$ maps N onto $M/\mathfrak{m}M$,then $N/\mathfrak{m}M \cong M/\mathfrak{m}M$, so $N + \mathfrak{m}M = M$, hence $M = N$ (by Corollary 2.41). □

2.2.3 Algebras

Let $f : R \longrightarrow R'$ be a ring homomorphism. If $r \in R$, $r' \in R'$, define a product $rr' = f(r)r'$.

This definition of scalar multiplication makes the ring R' into an R-module (it's a particular example of restriction of scalars).Thus R' has an R-module structure as well as a ring structure, and these two structures are compatible in a sense which the reader will be able to formulate for himself.

Definition 2.44. The ring R', equipped with this R-module structure is said to be an R**-algebra**. Thus, an R**-algebra** is by definition a **ring** R' **with a ring homomorphism** $f : R \longrightarrow R'$.

Example 2.45. *Let $R = k(x, y, z)$,and $I = (x^2, yz)$ an ideal of R, then R/I is an R-algebra.*

Definition 2.46. Let R', R'' be two rings. An R**-algebra homomorphism**

$h : R' \longrightarrow R''$ is a ring homomorphism which is also an R-module homomorphism.

2.3 Complexes

Let R be a commutative ring.

Definition 2.47.

A **finite complex E** is a sequence of homomorphisms of R-modules of the form:

$$0 \longrightarrow E^0 \xrightarrow{d^0} \ldots \xrightarrow{d^n} E^{n+1} \longrightarrow 0$$

where $d^i : E^i \longrightarrow E^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all i. Thus, $\mathrm{Im}(d^i) \subseteq \ker(d^{i+1})$.

Definition 2.48. The **Homology** H^i of the complex is defined to be $H^i = \ker (d^{i+1})/\mathrm{Im}(d^i)$. By definition, $H^0 = E^0$ and $H^n = E^n/\mathrm{Im}(d^n)$.

Definition 2.49. Let E and F be two complexes. A homomorphism $f : E \longrightarrow F$, is a sequence of homomorphisms $d^i : E^i \longrightarrow F^i$ making the diagram commutative for every i.

$$\begin{tikzcd} \ldots \arrow[r] & E^i \arrow[r, "d_E^i"] \arrow[d, "f^i"'] & E^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \ldots \\ \ldots \arrow[r] & F^i \arrow[r, "d_F^i"] & F^{i+1} \arrow[r] & \ldots \end{tikzcd} .$$

2.3.1 Exact Sequences

Most important kind of a complex is the exact sequence.

Definition 2.50. A sequence of R-modules and R-homomorphisms

$$\ldots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \ldots$$

is said to be **exact** at M_i if $\operatorname{Im}(f_i) = \ker(f_{i+1})$. The sequence is **exact** if it's exact at each M_i.

In particular,

1. $0 \longrightarrow M' \xrightarrow{f} M$. is exact $\iff$ f is injective.

2. $M \xrightarrow{g} M'' \longrightarrow 0$ is exact $\iff$ g is surjective.

3. $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is exact $\iff$ f is injective and g is surjective.

And g induces an isomorphism of coker(f)= $M/f(M') = M/ker(g)$ onto M''. A sequence of last type is called **short exact sequence**.

2.4 Chain Conditions

Let Σ be a set partially ordered by a relation $\leq$. (i.e. $\leq$ is reflexive , transitive, and is such that $x \leq y$ and $y \leq x \implies x = y$.

Proposition 2.51. *The following conditions on Σ are equivalent:*

1. *Every increasing sequence $x_1 \leq x_2 \leq \ldots$ in Σ is stationary.*

2. *Every non-empty subset of Σ has a maximal element*

Proof. 1 $\implies$ 2) Suppose (2) is false, then there is a non-empty subset T of Σ with no maximal element, and we can construct inductively a non-terminating strictly increasing sequence in T which contradicts 1.

2 $\implies$ 1) The set $\{x_i\}_{i>0}$ has a maximal element , say x_n as it is a non-empty subset of Σ, then this increasing sequence is stationary. □

Definition 2.52. If Σ is the set of submodules of a module M, ordered by the relation($\subseteq$), then the first is called the **ascending chain condition** (acc), and the second is called the **maximal condition**.

Definition 2.53. A module M satisfying either of these equivalent conditions is said to be **Noetherian**.

Definition 2.54. If Σ is ordered by ($\supseteq$), then the first is called the **descending chain condition** (dcc) and the module is called **artinian**.

Example 2.55. *A finite abelian group (as $\mathbb{Z}$-module) satisfies both acc and dcc.*

Example 2.56. *The ring $\mathbb{Z}$ satisfies acc but not dcc, because if $a \in \mathbb{Z}$, $a \neq 0$, we have $(a) \supset (a^2) \supset \ldots \supset (a^n) \supset \ldots$ (these are strict inclusions).*

Proposition 2.57. *M is a Noetherian R-module $\Longleftrightarrow$ every submodule of M is finitely generated.*

Proof. $\Longrightarrow$) Let N be a submodule of M, and let Σ be the set of all finitely generated submodules of N, then Σ is non-empty ($0 \in \Sigma$) and therefore has a maximal element, say N_0. If $N_0 \neq N$, consider the submodule $N_0 + Rx$ where $x \in N$ and $x \notin N_0$, now this is finitely generated and strictly contains N_0, contradiction! Hence, $N = N_0$ and so N is finitely generated.

$\Longleftarrow$) Let $M_1 \subseteq M_2 \subseteq \ldots$ be an ascending chain of submodules of M. Then, $N = \cup_{n=1}^{\infty} \mathrm{M}_n$ is a submodule of M, hence is finitely generated, say by $x_1, \ldots, x_r$, say $x_i \in M_{n_i}$ and Let $n = \max\limits_{i=1,..,r} n_i$, then each $x_i \in M_n$, hence $M_n = N$, so the chain is stationary. □

Proposition 2.58. *Let $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$, be an exact sequence of R-modules. Then, M is Noetherian $\Longleftrightarrow$ M' and M'' are Noetherian.*

Proof. $\Longrightarrow$) An ascending chain of submodules of M' (or M'') gives rise to a chain in M, hence is stationary.

$\Longleftarrow$) Let $(L_n)_{n\geq 1}$ be an ascending chain of submodules of M, then$(\alpha^{-1}(L_n))$ is a chain in M', and $(\beta(L_n))$ is a chain in M'', for a large enough n both these chains are stationary, and it follows that the chain (L_n) is stationary. □

Corollary 2.59. *If M_i $(1 \leq i \leq n)$ are Noetherian R-modules so is $\bigoplus\limits_{i=1}^{n} M_i$.*

Proof. Apply induction and (Proposition 2.58) to the exact sequence:

$$0 \longrightarrow M_n \longrightarrow \bigoplus_{i=1}^{n} M_i \longrightarrow \bigoplus_{i=1}^{n-1} M_i \longrightarrow 0$$

□

Definition 2.60. A **ring** is said to be **Noetherian** if it is Noetherian as R-module, that is it satisfies the following three equivalent conditions:

1. Every non-empty set of ideals in R has a maximal element.
2. Every ascending chain of ideals in R is stationary.
3. Every ideal in R is finitely generated.

Proof. Equivalence follows from (Propositions 2.51 and 2.57) □

Example 2.61. *Any field is Noetherian, so is the ring $\mathbb{Z}/(n), n \neq 0$.*

Example 2.62. *The ring $\mathbb{Z}$ is Noetherian.*

Example 2.63. *Any principal ideal domain is Noetherian (by proposition 2.57),as every ideal is finitely generated.*

Proposition 2.64. *Let R be a Noetherian ring. If M is a finitely generated R-module, then M is Noetherian.*

Proof. M is a quotient of R^n for some n, then apply (Propositions 2.58 and corollary 2.59) □

Proposition 2.65. *Let R be Noetherian, and $\mathcal{A}$ be an ideal of R , then $R/\mathcal{A}$ is a Noetherian ring.*

Proof. By (Proposition 2.58) $R/\mathcal{A}$ is Noetherian as an R-module , hence also an $R/\mathcal{A}$-module. □

Proposition 2.66. *If R is a Noetherian ring and Φ is a homomorphism of R onto a ring R', then R' is Noetherian.*

Proof. This follows from (Proposition 2.65) since $R' \cong R/\mathcal{A}$, where $\mathcal{A}= \ker(\Phi)$. □

Proposition 2.67. *Let R be a subring of R', suppose that R is Noetherian and R' is finitely generated as an R-module. Then, R' is a Noetherian ring.*

Proof. By (Proposition 2.64) R' is Noetherian as an R-module, hence also as R'-module. □

Theorem 2.68. *"Hilbert Basis Theorem". If R is Noetherian , then the polynomial ring $R[x]$ is so.*

Proof. Let $\mathcal{A}$ be an ideal in $R[x]$. The leading coefficients of the polynomials in $\mathcal{A}$ form an ideal I in R. Since R is Noetherian, then I is finitely generated, say by $a_1, a_2, \ldots, a_n$. For all $i \in 1, \ldots, n$, $\exists f_i \in R[x]$ of the form $f_i = a_i x^{r_i}$ + (lower terms). Let $r = \max\limits_{1 \le i \le n} r_i$. The $\{f_i\}$'s generate an ideal $\mathcal{A}' \subseteq \mathcal{A}$ in $R[x]$.

Let $f = ax^m +$ (lower terms) be any element of $\mathcal{A}$, we have $a \in I$. If $m \geq r$, we write $a = \sum\limits_{i=1}^{n} u_i a_i, u_i \in R$; then $f - \Sigma u_i f_i x^{m-r_i}$ is in $\mathcal{A}$ and has a degree $<$ m. Proceeding this way, we can go on subtracting elements of $\mathcal{A}'$ from f until we get a polynomial g, say of degree $<$ r, that is we have $f = g + h, h \in \mathcal{A}'$.

Let M be the R-module generated by $1, x, \ldots, x^{r-1}$, then what we have proved is that $\mathcal{A}$= $(\mathcal{A} \cap M) + \mathcal{A}'$. Now, M is finitely generated R-module, hence is Noetherian by Proposition 2.64, and $\mathcal{A} \cap M$ is finitely generated as an R-module by Proposition 2.57. If $g_1, \ldots, g_m$ generate $\mathcal{A} \cap M$, it is clear that the f_i and the g_i generate $\mathcal{A}$. Hence, $\mathcal{A}$ is finitely generated and so $R[x]$ is Noetherian. □

Corollary 2.69. *If R is Noetherian, so is $R[x_1, \ldots, x_n]$.*

Proof. By induction on n. □

Example 2.70. $R = k[x, y, z]$ *is Noetherian.*

Corollary 2.71. *Let R' be a finitely generated R-algebra. If R is Noetherian, then so is R'. In particular, every finitely generated ring and every finitely generated algebra over a field, is Noetherian.*

Proof. R' is a homomorphic image of a polynomial ring $R[x_1, \ldots, x_n]$ which is Noetherian by (Corollary 2.69). □

2.5 Tensor Product

Definition 2.72. Let M, N, and P be R-modules, define a **bilinear map** from $M \times N$ to P to be a map of sets $\psi : M \times N \longrightarrow P$ satisfying the condition of bilinearity:

$\psi((am+a'm')\times(bn+b'n')) = ab\psi(m\times n)+a'b\psi(m'\times n)+ab'\psi(m\times n')+a'b'\psi(m'\times n')$

Definition 2.73. Define the **tensor product** $M \otimes_R N$ to be the module with generators $\{m \otimes n \mid m \in M, n \in N\}$ and relations

$(am + a'm') \otimes (bn + b'n') = ab(m \otimes n) + a'b(m' \otimes n) + ab'(m \otimes n') + a'b'(m' \otimes n')$

Remark 2.74. *In particular, we have $r(m \otimes n) = (rm) \otimes n = m \otimes (rn)$. When the ring R is clear from context, we often write $M \otimes N$ for $M \otimes_R N$.*

Note that the map $m \times n \longrightarrow m \otimes n$ is a bilinear map from $M \times N$ to $M \otimes_R N$. Thus, if $\phi : M \otimes_R N \longrightarrow P$ is a homomorphism, then the map $\psi : M \times N \longrightarrow P$ defined by $\psi(m \times n) = \phi(m \otimes n)$ is bilinear. Conversely, since no relations other than the bilinear relations were imposed on $M \otimes_R N$, if $\psi : M \times N \longrightarrow P$ is bilinear then there is a unique homomorphism $\phi : M \otimes_R N \longrightarrow P$ satisfying $\psi(m \times n) = \phi(m \otimes n)$.

One point about this construction requires some care: Not every element of $M \otimes_R N$ may be written in the form $m \otimes n$. $m_i \in M$ and $n_i \in N$.

Rather, every element is expressible as a finite sum $\Sigma m_i \otimes n_i$. For any R-module M we have $M \otimes_R R = R \otimes_R M = M$ by isomorphisms sending $1 \otimes m$ and $m \otimes 1$ to m. Also, $M \otimes_R N \cong N \otimes_R M$ by a map sending $m \otimes n$ to $n \otimes m$.

Proposition 2.75. *The tensor product is **functorial** in the sense that if $\alpha : M' \longrightarrow M$ and $\beta : N' \longrightarrow N$ are homomorphisms, then there is an induced homomorphism called $\alpha \otimes \beta : M' \otimes_R N' \longrightarrow M \otimes_R N$ that sends $m' \otimes n'$ to $\alpha(m') \otimes \beta(n')$.*

Proposition 2.76. *The tensor product preserves direct sums in the sense that if*

$M = \bigoplus_i M_i$, *then* $M \otimes_R N = \bigoplus_i M_i \otimes_R N$.

Proposition 2.77. *The tensor product preserves cokernels in the sense that if* $\alpha : M' \longrightarrow M$ *is a map with cokernel coker(*α*)* $= M''$, *then for any module* N *the cokernel of the induced map* $\alpha \otimes 1 : M' \otimes_R N \longrightarrow M \otimes_R N$ *is* $M'' \otimes_R N$.

2.6 Graded Rings and Modules

Definition 2.78. A **graded ring** is a ring R together with a family $(R_n)_{n\geq 0}$ of subgroups of the additive group R, such that $R= \overset{\infty}{\underset{n=0}{\oplus}} R_n$ and $R_m R_n \subseteq R_{m+n} \forall m, n \geq 0$. Thus, R_0 is a subring of R, and each R_n is an R_0-module.

Definition 2.79. Let R is a graded ring, a **graded** R**-module** is an R-module M together with a family $(M_n)_{n\geq 0}$ of subgroups of M such that $M= \overset{\infty}{\underset{n=0}{\oplus}} M_n$ and $R_m M_n \subseteq M_{m+n} \forall m, n \geq 0$. Thus each M_n is an R_0-module.

Definition 2.80. An element x of M is **homogeneous** if $x \in M_n$ for some n that is said to be the degree of x. Any element $y \in M$ can be written uniquely as a finite sum $\underset{n}{\Sigma} y_n$, where $y_n \in M_n, \forall n \geq 0$, and all but a finite number of the y_n are 0.

Example 2.81. *Let* $R = k[x_1, x_2, \ldots, x_n]$. R *is a graded ring, because* $R_0 = K$, R_1 *is the set of all linear forms,* R_2 *is the* k*-space of all quadrics, etc.*

In $k[x, y]$, *the polynomial* $x^3y^2 - 2xy^4$ *is homogeneous because all of its terms have the same degree* 5.

Definition 2.82. If M,N are graded R-modules, a **homomorphism of graded** R**-modules** is an R-module homomorphism $f : M \longrightarrow N$ such that $f(M_n) \subseteq N_n, \forall n \geq 0$.

Definition 2.83. An ideal I in R is called **graded** or **homogeneous** if I has a system of homogeneous generators.

Remark 2.84. *We have seen in example 2.81, that the polynomial ring is a graded ring. This polynomial ring can be considered as a local ring, and ideals as homogeneous ideals.*

Maximal ideals of $R = k[x_1, x_2, \ldots, x_n]$ are of the form $(x_1 - a_1, \ldots, x_n - a_n) = \mathfrak{m}$. Hence, $R/k[x_1, x_2, \ldots, x_n] \cong k$. Since $\mathfrak{m}$ is homogeneous, then $a_1, \ldots, a_n$ are all zeros. Therefore $\mathfrak{m} = (x_1, \ldots, x_n)$ is considered to be the homogeneous maximal ideal of R.

Definition 2.85. Let $M = \bigoplus\limits_{d \in \mathbb{Z}} M_d$ be a finitely generated graded R-module with d-th graded component M_d. Denote by $M(a)$ the module M **shifted (or twisted** by a: $M(a)_d = M_{a+d}$.

Example 2.86. *If x has degree 1 in $R = \mathbb{R}[x]$ then x has degree $1 + m$ in $R(-m)$, because when $x \in R(-m)_{1+m}$ then $x \in R_1$.*

Definition 2.87. Let $\Phi : R \longrightarrow R$ be an R-homomorphism that assigns $\Phi(x)$ to every $x \in R$. It is said to **homogeneous homomorphism** if $\deg(x) =$ degree$(\Phi(x))$ $\forall x \in R$. These maps are also called **degree-0 maps**.

Chapter 3

Graded Free Resolutions

Let $R = k[x_0, \ldots, x_n]$ and $\mathfrak{m}$ denote the homogeneous maximal ideal in R.

Definition 3.1. A **free resolution** of a finitely generated R-module M is a sequence of homomorphisms of R-modules

$$F : \ldots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\delta_1} F_0$$

such that

1. F is a complex of finitely generated free R-modules F_i
2. F is exact
3. $M \cong F_0/Im(\delta_1)$.

Sometimes, for convenience, we write

$$F : \ldots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \longrightarrow 0.$$

In the literature the map δ_0 is called an augmentation map.

Example 3.2. *Let $R = k[x, y, z]$, and $M = (xy, yz)$. A resolution of R/M is:*

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -z \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} xy & yz \end{pmatrix}} R \longrightarrow R/M \longrightarrow 0$$

The resolution can continue to different steps like:

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -y \\ 1 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} -z & -zy \\ x & xy \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} xy & yz \end{pmatrix}} R \longrightarrow R/M \longrightarrow 0$$

But we study the minimal ones in the next section. Sometimes resolutions are also presented as follows

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -z \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} xy & yz \end{pmatrix}} M \longrightarrow 0$$

A resolution is **graded** if M is graded, F is a graded complex, and the isomorphism $F_0/Im(\delta_1) \cong U$ has degree 0. Fix a homogeneous basis of each free module F_i. Then the differential δ_i is given by a matrix D_i, whose entries are homogeneous elements in R. These matrices are called **differential matrices** (note that they depend on the chosen basis).

Construction 3.3. *Given homogeneous elements $m_i \in M$ of degree a_i that* *generate M as an R-module, we will construct a graded free resolution of M by induction on homological degree. First, set $M_0 = M$. Choose homogeneous generators $m_1, ..., m_r$ of M_0. Let $a_1, \dots, a_r$ be their degrees, respectively. Now set $F_0 = \underset{1\leq i\leq r}{\oplus} R(-a_i)$ We may define a map from the graded free module F_0 onto M by sending the i-th generator f_i of $R(-a_i)$ to m_i. (In this text a map of graded modules means a degree-preserving map, and we need the shifts a_i to make this true). Next, let $M_1 \subset F_0$ be the kernel of this map $F_0 \longrightarrow M$. By the Hilbert Basis Theorem, M_1 is also a finitely generated module. The elements of M_1 are called* ***syzygies*** *on the generators m_i, or simply* ***syzygies of*** *M.*

Choosing finitely many homogeneous syzygies that generate M_1, we may define a map from a graded free module F_1 to F_0 with image M_1. Continuing in this way we construct a sequence of maps of graded free modules, to obtain a ***graded free resolution*** *of M:*

$$\dots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{\delta_1} F_0$$

But each module F_i is a free finitely generated graded R-module, then we can write it as $\underset{p\in\mathbb{Z}}{\oplus} R(-p)^{c_{i,p}}$ Therefore, a graded complex of free finitely generated modules has the form

$$\dots \longrightarrow \underset{p\in\mathbb{Z}}{\oplus} R(-p)^{c_{i,p}} \xrightarrow{\delta_i} \underset{p\in\mathbb{Z}}{\oplus} R(-p)^{c_{i-1,p}} \longrightarrow \dots \longrightarrow R$$

It is an exact sequence of degree-0 maps between graded free modules such that the cokernel of δ_1 is M. Note that the numbers $c_{i,p}$ are the ***graded Betti numbers*** *of* *the complex.*

Example 3.4. *As first example, we take one of the simplest family of graded free resolutions that are called Koszul complexes. They resolve an ideal generated by a regular sequence. Take the following ideal* $I = (x_0, x_1, x_2) \in k[x_0, x_1, x_2]$

$$0 \longrightarrow R(-3) \xrightarrow{\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}} R^3(-2) \xrightarrow{\begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix}} R^3(-1) \xrightarrow{\begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix}} R$$

Theorem 3.5. *Hilbert Syzygy Theorem*

Any finitely generated graded R*-module* M *has a finite graded free resolution:* $0 \longrightarrow F_m \xrightarrow{\delta_m} F_{m-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\delta_1} 0.$

Moreover, we have $m \leq r+1$*, the number of variables in* R*.*

3.1 Minimal Graded free resolutions

Let $R = k[x_1, \ldots, x_n]$, let M be a graded R-module. Here we define minimal graded free resolutions.

Definition 3.6. A complex of graded R-modules

$$\ldots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \ldots$$

is called minimal if for each i the image of δ_i is contained in $\mathfrak{m}F_{i-1}$.

The above definition implies that the entries of the matrices representing the differential maps are elements of the maximal ideal $\mathfrak{m}$.

Construction 3.7. ***Minimal graded free resolutions*** *can be described as follows:*

Given a finitely generated graded module M, choose a ***minimal*** *set of homogeneous generators m_i. Map a graded free module F_0 onto M by sending a basis for F_0 to the set of m_i. Let M_0 be the kernel of the map $F_0 \longrightarrow M$, and repeat the procedure, starting with a* ***minimal*** *system of homogeneous generators of M_0.*

Example 3.8. *Given the polynomial ring $R= k[x, y, z, w]$, and the ideal $M = (xy, yz, zw)$ of a regular sequence, and we want to construct the* ***minimal*** *graded free resolution of R/M. We start by mapping F_0 onto M.*

- *Step 1: Set $F_0 = R$ our graded free k-module, and $\delta_0 : R \longrightarrow R/M$.*

- *Step 2: The elements xy, yz, zw are homogeneous generators of Ker(δ_0), each of degree 2. So, set $F_1 = \mathbb{R}^3(-2)$, and denote by f_i the 1-generator of each $R(-2)$ with $i \in \{1, 2, 3\}$. And we construct $\delta_1 : F_1 \longrightarrow F_0$ by having $Im(\delta_1) = ker(\delta_0) = M$, and we obtain the beginning of the resolution $\mathbb{R}^3(-2) \xrightarrow{\delta_1} R \longrightarrow R/M$.*

- *Step 3: First, we need to find homogeneous generators of Ker(δ_1) that requires some computation:*

$$\begin{pmatrix} xy & yz & zw \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

with $c_1, c_2, c_3 \in R$ being the unknowns. Then we can easily see three generators: $R_1 = \begin{pmatrix} -z & x & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & -w & y \end{pmatrix}, R_3 = \begin{pmatrix} -wz & 0 & xy \end{pmatrix}$.
But, $xR_2 + R_3 = wR_1$*, so the minimal set of generators of the solution* (c_1, c_2, c_3) *is* $\{R_1, R_2\}$*. Therefore,* $-zf_1 + xf_2$ *and* $-wf_2 + yf_3$ *are homogeneous generators of ker(* δ_1*). Their degrees are* $3 = deg(-z) + deg(f_1)$ *and* $3 = deg(-w) + deg(f_2)$*. Set* $F_2 = R^2(-3)$*, and denote by* g_1, g_2 *the* 1*-generators of* $R(-3)$ *and* $R(-3)$*. Hence* $deg(g_1) = 3$ *and* $deg(g_2) = 3$*. Defining* δ_2 *by* $g_1 \mapsto -zf_1 + xf_2, g_2 \mapsto -wf_2 + yf_3$ *we obtain the next step in the resolution :* δ_2*:* $\mathbb{R}^2(-3) \xrightarrow{\begin{pmatrix} -z & 0 \\ x & -w \\ 0 & y \end{pmatrix}} \mathbb{R}^3(-2)$*, satisfying Im(* δ_2*)= ker(* δ_1*).*

- *Step* 4*: Now* $Im(\delta_3) = ker(\delta_2)$ *has no non-trivial solutions, hence* $F_3 = 0$ *and* δ_3 *: 0* $\longrightarrow \mathbb{R}^2(-3)$*.*

Therefore, we get the following minimal graded free resolution:

$$0 \xrightarrow{0} \mathbb{R}^2(-3) \xrightarrow{\begin{pmatrix} -z & 0 \\ x & -w \\ 0 & y \end{pmatrix}} \mathbb{R}^3(-2) \xrightarrow{\begin{pmatrix} xy & yz & zw \end{pmatrix}} R \longrightarrow R/M \longrightarrow 0$$

Next we show that two minimal free resolutions of the same module are isomorphic. In order to do that, we prove the below results.

Lemma 3.9. *(Nakayama) Suppose M is a finitely generated graded R-module and $m_1, \ldots, m_n \in M$ generate $M/\mathfrak{m}M$. Then $m_1, \ldots, m_n$ generate M.*

Proof. Let $\bar{M} = M/\Sigma R m_i$. If the m_i generate $M/\mathfrak{m}M$ then $\bar{M}/\mathfrak{m}\bar{M} = 0$ so $\mathfrak{m}\bar{M} = \bar{M}$. If $\bar{M} \neq 0$, since $\bar{M}$ is finitely generated, there would be a nonzero element of least degree in $\bar{M}$; this element could not be in $\mathfrak{m}\ \bar{M}$. Thus, $\bar{M} = 0$, so M is generated by the m_i. □

Corollary 3.10. *A graded free resolution*

$$F : \ldots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \ldots$$

is minimal as a complex if and only if for each i the map δ_i takes a basis of F_i to a minimal set of generators of the image of δ_i.

Proof. Consider the right exact sequence $F_{i+1} \longrightarrow F_i \longrightarrow Im(\delta_i) \longrightarrow 0$. The above graded free resolution is minimal $\Longleftrightarrow \delta_{i+1}(F_{i+1}) \subset \mathfrak{m}F_i$ for each i
$\Longleftrightarrow F_{i+1} \longrightarrow F_i/\mathfrak{m}F_i$ is the zero map $\Longleftrightarrow$ the induced map
$\overline{\delta_{i+1}} : F_{i+1}/\mathfrak{m}F_{i+1} \longrightarrow F_i/\mathfrak{m}F_i$ is the zero map. This holds if and only if the induced map $F_i/\mathfrak{m}F_i \xrightarrow{\bar{\delta}_i} Im(\delta_i)/\mathfrak{m}\mathrm{Im}(\delta_i)$ is an isomorphism. If $\{f_1, \ldots, f_n\}$ is a basis (a minimal set of generators) of F_i, then $\bar{f}_1, \ldots, \bar{f}_n$ is a set of generators of $F_i/\mathfrak{m}F_i$ and it is minimal by Nakayama's Lemma. Therefore, $\bar{\delta}(\bar{f}_i) = \bar{m}_i$ is a minimal set of generators of $\mathrm{Im}(\delta_i)/\mathfrak{m}\ \mathrm{Im}(\delta_i)$ and again by Nakayama's Lemma m_i is a minimal set of generators of $\mathrm{Im}(\delta_i)$.

On the other hand, suppose δ_i takes a basis of F_i to a minimal set of generators of $Im(\delta_i)$. By Nakayama's Lemma, we have $\{\bar{f}_1, \ldots, \bar{f}_n\}$ is a minimal set of generators of $F_i/\mathfrak{m}F_i$ (a basis of that vector space) as well as $\{m_i\}$ a basis of $Im(\delta_i)/\mathfrak{m}\mathrm{Im}(\delta_i)$ of the same dimension as $F_i/\mathfrak{m}F_i$. Therefore,there is an isomorphism between F_i/mF_i and $\mathrm{Im}(\delta_i)/\mathfrak{m}\mathrm{Im}(\delta_i)$. By Nakayama's Lemma, this occurs if and only if a basis of F_i maps to a minimal set of generators of $\mathrm{Im}(\delta_i)$. $\square$

Considering all the choices made in the construction, it is perhaps surprising that minimal graded free resolutions are unique up to isomorphism:

Theorem 3.11. *Let M be a finitely generated graded R-module. If F and G are minimal graded free resolutions of M, then there is a graded isomorphism of complexes $F \longrightarrow G$ inducing the identity map on M.*

Proof.

$$F : \ldots F_1 \longrightarrow F_0 \xrightarrow{d_0} \underset{\downarrow id_M}{M} \longrightarrow 0.$$

$$G : \ldots G_1 \longrightarrow G_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

We first start by constructing the identity map on M. We have that $id_M \circ d_0$ maps F_0 to M, then since δ_0 is surjective, F_0 is free and every free module is a projective module i.e there exists a map $f_0 : F_0 \longrightarrow G_0$ such that the diagram commutes, that is $id_M \circ d_0 = \delta_0 \circ f_0$.

Now, we need to show that f_0 is an isomorphism. To do so, we tensor both F and G with $k= R/\mathfrak{m}$ and we show that $f_0\otimes\ id$ is an isomorphism.

$$F : \ \ldots F_1 \otimes k \longrightarrow F_0 \otimes k \xrightarrow{d_0 \otimes id} \underset{\downarrow id_M \otimes k}{M \otimes k} \longrightarrow 0.$$

$$G : \ \ldots G_1 \otimes k \longrightarrow G_0 \otimes k \xrightarrow{\delta_0 \otimes id} M \otimes k \longrightarrow 0.$$

Since F and G are minimal, $F_0 \otimes k \cong F_0/\mathfrak{m}F_0$ and $G_0 \otimes k \cong G_0/\mathfrak{m}G_0$ which are k-vector spaces then $d_0 \otimes id$ and $\delta_0 \otimes id$ are isomorphisms, then so is $f_0 \otimes id$. We will show that f_0 is an isomorphism. Let $f_0 = (a_{ij})$, then $f_0 \otimes id=$ $(a_{ij} \otimes 1)$ $=(a'_{ij})$ is invertible. Hence, $\det(a'_{ij})$ is a unit in k and $\det(a_{ij})$ is not in M, which implies that $\det(a_{ij})$ is a unit in R and the matrix is invertible. So, f_0 is an isomorphism. Now, to construct f_1 we proceed the same way. f_0 induces an isomorphism between $\ker(d_0)$ and $\ker(\delta_0)$. $\square$

As we have seen earlier in the construction of the a minimal graded free resolution, we map F_1 onto $\ker(d_0)$, so we obtain a surjective map : $F_1 \longrightarrow ker(d_0)$. Similarly, with G_1 and $\ker(\delta_0)$. We then follow the same procedure as above.

Definition 3.12. If M is a finitely generated graded R-module then the **projective dimension of** M is the minimal length of a projective resolution of M, that is equal to the length of the minimal graded free resolution, and is denoted by $pd_R(M)$.

Example 3.13. *Let $R = k[x, y, z]$, and following example 3.2 taking $I = (xy, yz)$, the projective dimension $\mathrm{pd}_R(I)$ is equal to 1 in the minimal resolution of I:*

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -z \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} xy & yz \end{pmatrix}} I \longrightarrow 0$$

Also, the projective dimension $\mathrm{pd}_R(R/I)$ is equal to 2 in the minimal resolution of R/I:

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -z \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} xy & yz \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

Chapter 4

Monomial Resolutions

Let M be a monomial ideal that is by definition an ideal that can be generated by monomials. In this chapter we discuss free resolutions of monomial ideals; we call them monomial resolutions. Describing the minimal free resolution of a monomial ideal is quite complex despite the helpful combinatorial structure of monomial ideals. But, here we will introduce beautiful and easy proofs.

4.1 Multigrading

Along with the above standard grading, R can also be **multigraded** mainly $\mathbb{N}^n$-graded by the multidegree of x_i being mdeg(x_i)= the i'th standard vector in $\mathbb{N}^n$. Now for any vector in $\mathbb{N}^n$; $a = (a_1, a_2, \ldots, a_n)$, it is an exponent vector for some monomial x in R such that $x^a = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$, and we say **multidegree** x^a. Here, R has a direct sum decomposition over monomials, before it was a summation over elements of same degree. In this case, every monomial has a unique degree. Therefore, $R = \bigoplus_m R_m$ as a k-vector space and $R_m R'_m = R_{m+m'} \forall$

m, m' monomials in R, the equality can be easily seen since the product of monomials is a monomial of mdeg= mdeg(m)+mdeg(m'). An R-module T is called multigraded, if it has a direct sum decomposition $T = \bigoplus_m R_m$, m is a monomial, as a k-vector space and $R_m T_m \subseteq T_{mm'}$ $\forall$ monomials m, m'. Denote by $R(x^a)$ the free R-module with one generator in multidegree x^a.

4.2 Multigraded Free Resolutions

Note that every monomial ideal is homogeneous with respect to the multigrading, so the construction in 3.3 works in the multigraded case. There exists a minimal free resolution F_M of R/M over R which is multigraded. We denote by δ the differential in F_M. Similar to the 3.3 construction, the resolution can be written as $\ldots \longrightarrow \bigoplus_m R^{c_{i,m}} \xrightarrow{\delta_i} \bigoplus_m R^{c_{i-1,p}} \longrightarrow \ldots \longrightarrow R$, where every sum runs over all monomials.

Example 4.1. *let $R = k[x, y]$ and M be generated by the monomials xy and y^2, a multigraded free resolution of R/M is:*

$$0 \longrightarrow R(xy^2) \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R(xy) \oplus R(y^2) \xrightarrow{\begin{pmatrix} xy & y^2 \end{pmatrix}} R$$

4.3 Homogenization

From now on, denote by M the monomial ideal in R minimally generated by monomials $m_1, \ldots, m_r$, and by L_M the set of the least common multiples of subsets of $\{m_1, \ldots, m_r\}$. By convention, $1 \in L_M$ considered as the *lcm* of the empty set. Note that M is homogeneous with respect to the standard grading on R and with respect to the multigrading. We are going to form a monomial free resolution from a complex of vector spaces through homogenization by referring to Peeva [8].

Definition 4.2. Let U be a complex of finite k-vector spaces $\{U_i\}$ such that:

1. $U_i = 0$, $\forall i \leq -1$ and $U_i= 0$ for a large i.

2. $U_0= k$.

3. $U_1= k^r$ for a given r.

4. $\forall w_i$ a basis vector in U_1, $\delta(w_i)= 1$.

then, U is said to be an r-**frame** having δ as the differential map.

Example 4.3. $0 \longrightarrow k \xrightarrow{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} k^3 \xrightarrow{\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}} k^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} k$, *is the 3-frame.*

Definition 4.4. Let G be a multigraded complex of finitely generated free multigraded R-modules $\{G_i\}$ such that:

1. $G_i = 0$, $\forall i \leq -1$ and $G_i = 0$ for a large i.

2. $G_0= R$.

3. $G_1= R(m_1) \oplus R(m_2) \oplus \ldots \oplus R(m_r)$.

4. $\forall w_i$ a basis element of G_1, $d(w_i) = m_i$.

Then G is said to be an M-**complex** with differential d and a fixed homogeneous basis with multidegrees in L_M.

Definition 4.5. Let U be an r-frame. The M-**homogenization** of U is sequence of free R-modules constructed by induction as follows: $G_0 = R$ and $G_1 = R(m_1) \oplus \ldots \oplus R(m_r)$. Let $u_1, \ldots, u_q$ be the basis of $G_{i-1} = R^q$ chosen on the previous step of the induction. Denote by $\overline{v_1}, \ldots, \overline{v_p}$ the given bases of U_i, and by $\overline{u_1}, \ldots, \overline{u_q}$ the given bases of U_{i-1}, and we are going to find $v_1, \ldots, v_p$ that will be a basis of $G_i = R^p$. If $\delta(\overline{v_j}) = \sum_{1 \leq s \leq q} \alpha_{s,j} \overline{u_s}$, with coefficients $\alpha_{s,j} \in k$, then set

- mdeg(v_j)= $lcm\{$ mdeg(u_s) $\mid \alpha_{s,j} \neq 0 \}$, note that $lcm(\phi)$= 1.
- $G_i = \bigoplus_{1 \leq j \leq p} R(mdeg(v_j))$
- $d(v_j) = \displaystyle\sum_{1 \leq s \leq q} \alpha_{s,j} \frac{mdeg(v_j)}{mdeg(u_s)} u_s$

is homogeneous by construction. Note that, Coker(d_1) $= R/M$ from the 4^{th} condition. We will show that the G is an M-complex of free R-modules with differential d, and say that the complex G is obtained from U by M-homogenization.

Example 4.6. *Let R= $k[x, y]$, $M = (x^3, xy, y^2)$, and consider the 3-frame U, then the M-homogenization of U is:*

$$G : 0 \longrightarrow R(x^3y^2) \xrightarrow{\begin{pmatrix} y \\ x^2 \\ 1 \end{pmatrix}} R(x^3y) \oplus R(xy^2) \oplus R(x^3y^2) \xrightarrow{\begin{pmatrix} -y & 0 & y^2 \\ x^2 & -y & 0 \\ 0 & x & -x^3 \end{pmatrix}} R(x^3) \oplus R(xy) \oplus R(y^2) \xrightarrow{\begin{pmatrix} x^3 & xy & y^2 \end{pmatrix}} R$$

Proposition 4.7. *If G is the M-homogenization of a frame U, then G is an*

M-complex.

Proof. Let

- $\overline{v_1}, \ldots, \overline{v_p}$ be given basis of U_i
- $\overline{u_1}, \ldots, \overline{u_q}$be given basis of U_{i-1}
- $\overline{w_1}, \ldots, \overline{w_t}$ be given basis of U_{i-2}

and let

- $v_1, \ldots, v_p$ be given basis of G_i
- $u_1, \ldots, u_q$ be given basis of G_{i-1}
- $w_1, \ldots, w_t$ be given basis of G_{i-2}

Fix $1 \leq j \leq p$. Since U is a complex, we have that

$$0 = \delta^2(\overline{v_j}) = \delta(\sum_{1\leq s\leq q} \alpha_{s,j}\overline{u_s}) = \sum_{1\leq s\leq q} \alpha_{s,j}(\sum_{1\leq l\leq t} \beta_{l,s}\overline{w_l}) = \sum_{1\leq l\leq t}(\sum_{1\leq s\leq q} \alpha_{s,j}\beta_{l,s})\overline{w_l}$$

with $\alpha_{s,j}, \beta_{l,s} \in k$.Hence, $\sum\limits_{1\leq s\leq q} \alpha_{s,j}\beta_{l,s} = 0 \forall 1 \leq l \leq t$. Now, the term under consideration is

$$\begin{aligned}
& d^2(v_j) = d(\sum_{1\leq s\leq q} \alpha_{s,j}\tfrac{mdeg(v_j)}{mdeg(u_s)}u_s) \\
& = \sum_{1\leq s\leq q} \alpha_{s,j}\tfrac{mdeg(v_j)}{mdeg(u_s)}(\sum_{1\leq l\leq t} \beta_{l,s}\tfrac{mdeg(u_s)}{mdeg(w_l)}w_l) \\
& = \sum_{1\leq l\leq t}(\sum_{1\leq s\leq q} \alpha_{s,j}\beta_{l,s}\tfrac{mdeg(v_j)mdeg(u_s)}{mdeg(u_s)mdeg(w_l)})w_l \\
& = \sum_{1\leq l\leq t}(\sum_{1\leq s\leq q} \alpha_{s,j}\beta_{l,s})\tfrac{mdeg(v_j)}{mdeg(w_l)}w_l \\
& = 0.
\end{aligned}$$

□

Remark 4.8. *We note that we can* ***dehomogenize*** *by setting $U = G \otimes R/(x_1 - 1, \ldots, x_n - 1)$, being the frame of G. And, U is a finite complex* *of finite k-vector spaces with fixed basis and its differential matrices are obtained by setting $x_1 = 1, \ldots, x_n = 1$ in the differential matrices of G. But we only care about homogenization.*

A fruitful approach for constructing minimal monomial resolutions is based on the fact that the minimal free resolution of any monomial ideal can be encoded in any of its frames; this was proved in [9][Theorem 4.14]:

Theorem 4.9. *The M-homogenization of any frame of the minimal multigraded free resolution F of R/M is F.*

4.4 Subresolutions

Here we provide a helpful criterion.

Definition 4.10. Let G be an M-complex, and $m \in M$ be a monomial. We denote by $G(\leq m)$ **the subcomplex of** G that is generated by the homogeneous basis elements of multidegrees dividing m.

Example 4.11. *Following the example 4.6, let $m = x^2y^2$, so the monomials generating M that divide $m = x^2y^2$, are xy and y^2, then*

$$G(\leq x^2y^2) : 0 \longrightarrow R(xy^2) \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R(xy) \oplus R(y^2) \xrightarrow{\begin{pmatrix} xy & y^2 \end{pmatrix}} R$$

Proposition 4.12. *Let $m \in M$ be a monomial. Set $m' = lcm\ \{m_i \mid m_i$ divides $m\}$. Then, $G(\leq m) = G(\leq m')$*

Proof. By 4.4, all the basis elements of G have multidegrees in L_M, so none of the $m_i's$ excluded in $G(\leq m)$ will be considered in $G(\leq m')$, otherwise some m_j will divide m' the lcm and won't divide m that is supposed to be divisible by m', contradiction. □

Definition 4.13. Let F be a graded complex. Since each F_i is graded we write $F_i = \bigoplus_j F_{i,j}$. The differential has degree 0, therefore $d(F_{i,j}) \subseteq F_{i-1,j}$ for each i, j. Thus, the complex can be written as the following where the first is the (j)'th row, and the second is the $(j-1)$'st row:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \oplus & & \oplus & & \oplus & & \\ \ldots & \longrightarrow & F_{i+1,j} & \longrightarrow & F_{i,j} & \longrightarrow & F_{i-1,j} & \longrightarrow & \ldots. \\ & & \oplus & & \oplus & & \oplus & & \\ \ldots & \longrightarrow & F_{i+1,j-1} & \rightarrow & F_{i,j-1} & \rightarrow & F_{i-1,j-1} & \longrightarrow & \ldots. \\ & & \oplus & & \oplus & & \oplus & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

The (j)'th row is called the **(j)'th graded component** of F. It is the sequence of k-vector spaces $\ldots \longrightarrow F_{i+1,j} \longrightarrow F_{i,j} \longrightarrow F_{i-1,j} \longrightarrow \ldots$. The complex is the direct sum of its components. Often, it is very useful to study a complex by studying its graded components.

Theorem 4.14. *Let G be an M-complex and $m \in M$ be a monomial. The component of G of multidegree m is isomorphic to the frame of the complex $G(\leq m)$.*

Proof. Note that G_m has basis of the form $\{ \frac{m}{mdeg(g)} g \mid g$ is in the fixed basis of G, and mdeg(g) divides m $\}$. Therefore the component of G of multidegree m is isomorphic to the frame of the complex $G(\leq m)$. □

Now consider the following theorem which represents a very useful criterion for exactness.

Theorem 4.15. *An M-complex G is a free multigraded resolution of R/M if and only if for all monomials $m \neq 1 \in L_M$ the frame of the complex $G(\leq m)$ is exact.*

Proof. Note that $G_0/d(G_1) = R/M$. Since the complex G is multigraded, it suffices to check exactness in each multidegree, because a graded complex F is exact if and only if each of its graded components is an exact sequence of k-vector spaces such as $(G_i)_m = 0$ for $i > 0$ and $m \notin M$. It suffices to check exactness in each multidegree $m \in M$. By 4.14, it suffices to check exactness of the frames $G(\leq m)$ for all monomials $m \in M$. Fix a monomial $m \in M$, and set $m' = lcm$ $\{m_i \mid m_i \text{ divides } m\}$ and apply 4.12. Hence, $G(\leq m) = G(\leq m')$. Therefore, it suffices to consider only the multidegrees in L_M. □

Now we will show that the minimal free resolution of R/M contains as subcomplexes the minimal free resolutions of certain smaller monomial ideals.

Proposition 4.16. *Let $u \in M$ be a monomial, and consider the monomial ideal $(M_{\leq u})$ generated by the monomials $\{m_i \mid m_i$ divides $u\}$. Fix a multi-homogeneous basis of a multigraded free resolution F_M of R/M.*

1. *The subcomplex $F_M(\leq u)$ is a multigraded free resolution of $R/(M_{\leq u})$.*

2. *If F_M is a minimal multigraded free resolution of R/M, then $F_M(\leq u)$ is independent of the choice of basis.*

3. *If F_M is a minimal multigraded free resolution of R/M, then the resolution $F_M(\leq u)$ is minimal as well.*

Proof. 1. Set $v = \mathrm{lcm}\{m_i \mid m_i \text{ divides } u\}$ and apply 4.12. Hence, $F_M(\leq u) = F_M(\leq v)$. Clearly, $(M_{\leq u}) = (M_{\leq v})$. Therefore, we can replace u by v. By 4.15, we see that we have to show that for every monomial $m \neq 1 \in L_{(M_{\leq v})}$ the frame of the complex $(F_M(\leq v))(\leq m)$ is exact. The frame of

$(F_M(\leq v))(\leq m)$ is equal to the frame of $F_M(\leq w)$, where w is the maximal monomial that divides both v and m, and is in the set L_M. Since F_M is exact, by 4.15 it follows that the frame of $F_M(\leq w)$ is exact.

2. Note that the multidegrees of the basis elements in F_M are determined by the multigraded Betti numbers. Therefore, they are independent of the choice of basis.

3. holds by construction.

□

4.5 Taylor's Resolution

One important construction is the **Taylor′s resolution** T_M that resolves all R/M for any monomial ideal M. However, it is usually highly non-minimal, but very useful because of its simple structure.

Definition 4.17. Let $f_1, \ldots, f_q$ be elements in R. Let E be the exterior algebra over k on basis elements $e_1, \ldots, e_q$; this means that E is the following quotient of a free algebra $E = k\langle e_1, \ldots, e_q\rangle/(\{{e_i}^2 \mid 1 \leq i \leq q\}, \{e_i e_j + e_j e_i \mid 1 \leq i < j \leq q\})$.

Denote by T_M the R-module $R \otimes E$ graded homologically by $\mathrm{hdeg}(e_{j_1} \wedge \ldots \wedge e_{j_i}) = i$ and equipped with the differential:

$$d(e_{j_1} \wedge \ldots \wedge e_{j_i}) = \sum_{1 \leq p \leq i} (-1)^{p-1} \frac{lcm\{m_{j_1}, \ldots, m_{j_i}\}}{lcm\{m_{j_1}, \ldots, \hat{m}_{j_p}, \ldots m_{j_i}\}} e_{j_1} \wedge \ldots \wedge \hat{e}_{j_p} \wedge \ldots \wedge e_{j_i}$$

where $\hat{e}_{j_p}$ and $\hat{m}_{j_p}$ mean that e_{j_p} and m_{j_p} are omitted respectively.

The standard grading of T_M is given by $\deg(e_{j_1} \wedge \ldots \wedge e_{j_i}) = \deg(\mathrm{lcm}(m_{j_1}, \ldots, m_{j_i}))$, and the multigrading is given by $\mathrm{mdeg}(e_{j_1} \wedge \ldots \wedge e_{j_i}) =$

$\mathrm{lcm}(m_{j_1}, \ldots, m_{j_i})$.

Example 4.18. *Let $R = k[x,y]$. The Taylor's resolution of $R/(x^3, xy, y^2)$ is:*

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ x^2 \\ 1 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -y & 0 & y^2 \\ x^2 & -y & 0 \\ 0 & x & -x^3 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x^3 & xy & y^2 \end{pmatrix}} R$$

Chapter 5

Simplicial Complexes

Definition 5.1. A **finite simplicial complex** $\triangle$ is a finite set of $\mathbb{N}$, called the set of vertices $V = v_1, \ldots, v_p$ (or nodes) of $\triangle$, and a collection F of subsets of V, called the faces of $\triangle$, such that if $A \in F$ is a face and $B \subset A$ then B is also in F. Maximal faces are called **facets**.

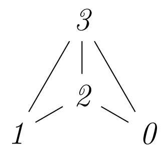
Definition 5.2. A **simplex** is a simplicial complex in which every subset of N is a face, that is it have only one facet: $v_1, \ldots, v_p$. For any vertex set V we may form the **void simplicial complex**, which has no faces at all. But if $\triangle$ has any faces at all, then the empty set ϕ is necessarily a face of $\triangle$. By contrast, we call the simplicial complex whose only face is ϕ the **irrelevant simplicial complex** on N.

Definition 5.3. The dimension of a face σ is $\mid \sigma \mid -1$. The dimension of $\triangle$ is the maximum of the dimensions of its faces, or $-\infty$ if $\triangle$ is the void complex. By convention, ϕ irrelevant simplicial complex has dimension -1. Throughout this section, $\triangle$ stands for a finite simplicial complex.

Example 5.4. *The simplicial complex on the set of vertices* $\{v_1, v_2, v_3\}$ *is* $\triangle = \{$

$\phi, \{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}\}$.

Example 5.5. *Let* $\{\{a, b, c\}$ *be the set of nodes of* $\triangle$ *and the sets of faces be* $\{\{a, b, c\}, \{a, b\}, \{a, c\}, \phi\}$*,is a non-simplicial complex as* $\triangle$ *doesn't contain the face* $\{c, b\}$ *which is a subset of the face* $\{a, b, c\}$*.*

Example 5.6. *Also,this is not a simplicial complex:* 

$\triangle =$
$\{\{0, 1, 2, 3\}, \{1, 2, 3\}, \{2, 3, 0\}, \{1, 2, 0\}, \{1, 2\}, \{2, 0\}, \{2, 3\}, \{1, 3\}, \{0, 3\}, \{0\}, \{1\}, \{2\}, \{3\}, \phi\}$ *misses* $\{1, 0\}$*.*

5.1 Simplicial Resolutions

Simplicial resolutions of $M = (m_1, \ldots, m_r)$ are free resolutions that are supported on simplicial complexes $\triangle$. Before we define them, we introduce the augmented chain complexes on $\triangle$.

Let $\triangle$ be a simplicial complex on vertices the monomials $\{m_1, \ldots, m_r\}$, and denote by τ the face of $\triangle$ in homological degree $\mid \tau \mid -1$. In order to construct the chain complex we have to define an incidence function.

Definition 5.7. Let τ' be a facet of τ, define an **incidence (orientation) function** $[\tau, \tau'] := (-1)^i$ if $\tau \setminus \tau'$ is the $(i + 1)$'st element in the sequence of the vertices of τ written in increasing order.

Example 5.8. *Let* $\triangle$ *be the simplicial complex on the vertices* m_1, m_2, m_3*. Take* τ *to be* $\{m_1, m_2, m_3\}$*, and take as a facet* τ' *to be the edge* $\{m_1, m_3\}$*, then* $[\tau, \tau'] = (-1)^1 = -1$ *because* $\tau \setminus \tau' = m_2$ *which is the second vertex so our* i *is equal to* 1*.*

Definition 5.9. The **augmented oriented simplicial chain complex of $\triangle$ over** k is $\tilde{C}(\triangle;k) = \underset{\tau\in\triangle}{\oplus} ke_\tau$, where e_τ denotes the basis element corresponding to the face τ, and the differential δ acts as $\delta(e_\tau) = \sum_{\tau' \textit{ is a facet of } \tau} [\tau,\tau']e'_\tau$

Example 5.10. *Consider the simplicial complex* $\triangle = \{\ \phi, \{x^2\}, \{xy\}, \{x^2, xy\}\}$. *We have that* $\tilde{C}(\triangle;k) = \underset{\tau\in\triangle}{\oplus} ke_\tau$, *so as a first step,* $\phi \in \triangle$ *has dimension* -1 *and we get* $\tilde{C} = 0$. *Next, for the vertices* $m_1 = x^2$, *and* $m_2 = xy$ *we get* k^2, *and finally for the edge we get* k. *Note that the first map* δ_1 *is zero and the second*
$\delta_2 = \begin{pmatrix} -1 \\ \\ 1 \end{pmatrix}$, *finally,* $\tilde{C}(\triangle;k)$: $k \xrightarrow{\delta_2} k^2 \xrightarrow{\delta_1} 0$.

Definition 5.11. After shifting $\tilde{C}(\triangle;k)$ in homological degree, we get that $\tilde{C}(\triangle;k)[-1]$ is a frame. Denote by $F_\triangle$ the M-homogenization of $\tilde{C}(\triangle;k)[-1]$, and we say that $F_\triangle$ is **supported on** $\triangle$, or $\triangle$ **supports** $F_\triangle$. The complex $F_\triangle$ is a **simplicial resolution** if it is exact.

For each vertex m_i of $\triangle$, we set that m_i has multidegree mdeg(m_i)= m_i. We define that a face τ has multidegree mdeg(τ)= lcm($m_i \mid m_i \in \tau$). By convention, mdeg(ϕ) = 1. And think of $\triangle$ as a simplicial complex with labeled faces: each face is labeled by its multidegree.

Theorem 5.12. *For each face* τ *of dimension* i *the complex* $F_\triangle$ *has the generator* e_τ *in homological degree* $i+1$. *We have*

1. *mdeg(e_τ) = mdeg(τ).*

2. *The differential in* $F_\triangle$ *is* $\delta(e_\tau) = \sum_{\tau' \textit{is a facet of} \tau} [\tau,\tau'] \frac{mdeg(\tau)}{mdeg(\tau')} e_{\tau'}$
$= \sum_{\tau' \textit{is a facet of} \tau} [\tau,\tau'] \frac{lcm(m_i|m_i\in\tau)}{lcm(m_i|m_i\in\tau')} e_{\tau'}$

Proof. 1. The first is done by induction on homological degree.

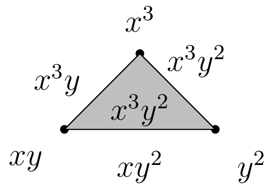


Figure 5.1: The labeled simplicial complex on the vertices x^3, xy, y^2

Note that $\mathrm{mdeg}(e_{m_i}) = m_i$ holds for each vertex m_i of $\triangle$.

Since $\delta(e_\tau)= \displaystyle\sum_{\tau' is\ a\ facet\ of \tau} [\tau, \tau']e_{\tau'}$, by definition 4.5 it follows that:

$\mathrm{mdeg}(e_\tau)=$ lcm$\{$ mdeg$(e_{\tau'})|$ τ' is a facet of $\tau\}$

$=$ lcm$\{$ mdeg$(\tau')|$ τ' is a facet of $\tau\}$

$=$ lcm$\{$lcm$\{m_i \mid m_i \in \tau'\} \mid \tau'$ is a facet of $\tau\}$

$=$ lcm$\{m_i \mid m_i \in \tau\}=$ mdeg(τ).

2. The second follows from the first and the fact that the differential is multihomogeneous.

□

Example 5.13. *As an example, we take the Taylor comlplex that is supported on the whole simplex. Consider the triangle $\triangle$ with vertices x^3, xy, y^2 that are the monomials generating M. We label each edge by the least common multiple of its vertices, so we get labels x^3y, xy^2, x^3y^2 on the edges. We label the simplicial complex by x^3y^2 the least common multiple of its vertices.*

The augmented oriented chain complex of this simplicial complex is the 3*-frame intoduced in example 4.3, and the corresponding M-homogenized complex is*

$$T_M\colon 0 \longrightarrow R(x^3y^2) \xrightarrow{\begin{pmatrix} y \\ x^2 \\ 1 \end{pmatrix}} R(x^3y) \oplus R(xy^2) \oplus R(x^3y^2) \xrightarrow{\begin{pmatrix} -y & 0 & y^2 \\ x^2 & -y & 0 \\ 0 & x & -x^3 \end{pmatrix}}$$

$$R(x^3) \oplus R(xy) \oplus R(y^2) \xrightarrow{\begin{pmatrix} x^3 & xy & y^2 \end{pmatrix}} R.$$

And T_M is a simplicial resolution, which is non-minimal, and in fact it is the taylor resolution.

Chapter 6

Monomial Ideals of Projective Dimension ≤ 1

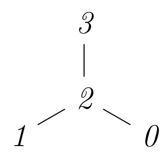
Recall the definition 5.1. It is easy to see that a simplicial complex $\triangle$ can be described completely by its facets, since every face is a subset of a facet and every subset of every facet is in a simplicial complex. So, if $\triangle$ has facets $F_0, \ldots, F_q$, we use the notation $\langle F_0, \ldots, F_q \rangle$ to describe $\triangle$. In this section, our main theorem is theorem 6.13. In order to do so, we consider the following definitions.

Definition 6.1. If $W \subseteq V$, we define the **induced subcomplex of $\triangle$ on W**, denoted $\triangle_W$, to be the simplicial complex on W given by
$\triangle_W = \{F \in \triangle \mid F \subseteq W\}$. A **subcollection of** $\triangle$ is a simplicial complex whose facets are also facets of $\triangle$.

Definition 6.2. The **dimension of a simplicial complex** $\triangle$ is $\dim(\triangle) = \max\{\dim(F) \mid F \in \triangle\}$, where the 0-dimensional faces are the vertices of $\triangle$ and the face ϕ has dimension -1.

Definition 6.3. A **leaf of** $\triangle$ is either the only facet of $\triangle$ or the facet F for which there is another facet τ of $\triangle$, called a **joint** such that $(F \cap H) \subseteq \tau$ for every facet $H \neq F$.

Example 6.4.



(6.1)

The facets are $F_1 = \{1, 2\}, F_2 = \{2, 3\}, F_3 = \{0, 2\}$. *Here every facet is a leaf with any other facet can be a joint, because the intersection is the vertex* 2 *which is common in all facets.*

Definition 6.5. A **free vertex of a simplicial complex** $\triangle$ is a vertex belonging to exactly one facet of $\triangle$. If F is a leaf of a simplicial complex, then F necessarily has a free vertex. For the sake of clarity, follow example 6.4 where the vertices $1, 3$ and 0 are free vertices.

Definition 6.6. A simplicial complex $\triangle$ is a **simplicial forest** if every nonempty subcollection of $\triangle$ has a leaf. We say $\triangle$ is **connected** if $\forall v_i, v_j \in V, \exists$ a sequence of faces $F_0, \ldots, F_k$ such that $v_i \in F_0, v_j \in F_k$ and $F_i \cap F_{i+1} \neq \phi$ for $i = 0, \ldots, k-1$. A connected simplicial forest is called a **simplicial tree**.

Remark 6.7. *One of the properties of simplicial trees that we will make particular use of is that whenever* $\triangle$ *is a simplicial tree we can always order the facets* $F_1, \ldots, F_q$ *of* $\triangle$ *so that* F_i *is a leaf of the induced subcollection* $\langle F_1, \ldots F_i \rangle$. *Such an ordering on the facets is called a* ***leaf order*** *and it is used to make the following definition.*

Definition 6.8. A **quasi-forest** is a simplicial complex $\triangle$ who has a leaf order. A connected quasi-forest is called a **quasi-tree**.

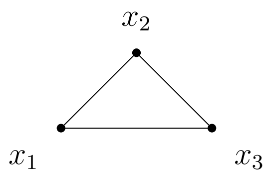
Example 6.9. *Consider the simplicial complex in example 6.4, it is a quasi-tree, because F_2 is a leaf of $\{F_1, F_2\}$, and F_3 is a leaf of $\{F_1, F_2, F_3\}$.*

Definition 6.10. If $\triangle = \langle F_1, \ldots, F_q \rangle$ is a simplicial complex on vertex set V , then the **complement of** $\triangle$ is the simplicial complex $\triangle^c = \langle {F_1}^c, \ldots {F_q}^c \rangle$, where ${F_i}^c = V \setminus F_i$.

Now we can construct square-free monomial ideals by means of simplicial complexes.

Definition 6.11. Let $\triangle$ be a simplicial complex whose vertices are labeled with the variables $x_1, \ldots, x_n$ in the ring R. Then the square-free monomial ideal $I = (x_{i_1}, \ldots, x_{i_r} \mid \{x_{i_1}, \ldots, x_{i_r}\}$ is a facet of $\triangle)$ is called the **facet ideal** of $\triangle$, denoted by $\mathfrak{F}(\triangle)$, and $\triangle$ is called the facet complex of I, denoted by $\mathfrak{F}(I)$.

Example 6.12.

$\triangle =$ 

(6.2)

let $I = (x_1, x_2, x_3)$, then $\triangle = \mathfrak{F}(I)$

We get to our main theorem.

Theorem 6.13. *A monomial ideal M has pd$(M) \leq 1$ if and only if R/M has a minimal resolution supported on a (graph) tree.*

Proof. The sufficient condition is easy, because geometrically the complex will only contain vertices and edges. So following the construction of simplicial resolutions, the homological degree of the free modules will not exceed 1, so the length of the free resolution that is the projective dimension of M will be atmost 1.

Conversely, suppose the $pd(M) = 0$, then $M = (m)$ is a principal ideal, hence the minimal free resolution of R/M is supported on the graph with a single vertex and no edges.

Now, assume that $pd(M) = 1$. Then R/M has a minimal resolution of the form:

$$0 \longrightarrow R^t \xrightarrow{\psi} R^r \xrightarrow{\phi} R \longrightarrow 0$$

where $\phi(e_i) = m_i$ for the basis elements e_i of R^r, and $\psi(g_j) = f_j$ where the g_j form a basis of R^t and the f_j form a minimal generating set of $\ker(\phi)$.

But (see [5], Corollary 4.13), $\ker(\phi)$ can be generated (though not necessarily minimally) by the elements: $\frac{lcm(m_i,m_j)}{m_i}e_i$ -$\frac{lcm(m_i,m_j)}{m_j}e_j$. Now let $f_1, \ldots, f_t$ be a minimal generating set of $\ker(\phi)$ which have this form. This gives us a complete description of the map ψ as a matrix with exactly two non-zero monomial entries in each column with coefficients corresponding to those appearing in the f_i (i.e one column entry has coefficient 1 and the other has coefficient -1). Dehomogenizing this resolution, gives us the sequence of vector spaces:

$$0 \longrightarrow k^t \xrightarrow{A} k^r \xrightarrow{\begin{pmatrix} 1 & 1 & \ldots & 1 \end{pmatrix}} k \longrightarrow 0 \tag{6.3}$$

which is exact and where A is a matrix in which every column has exactly one entry which is 1, one entry which is -1, and the rest equal to zero. If we consider each basis element of k^r as a vertex and each basis element of k^t as an edge between the two vertices determined by the basis elements of k^r, we may construct a graph G for which $C(G;k)$ is the chain complex in (6.3). Since this chain complex is exact the graph G is acyclic (graph with no holes). Hence, a tree (this would also imply that $t = r-1$). We show that the homogenization of $C(G;k)$ is minimal in the next proposition. $\square$

Proposition 6.14. *If M is a monomial ideal such that R/M has a resolution supported on a tree T, then that resolution is minimal.*

Proof. If $m_1, \ldots, m_r$ are the minimal generators of M then T would have to have r vertices and $r-1$ edges. When we regard T as a simplicial complex we get the simplicial chain complex:

$$C(T,k) : 0 \longrightarrow k^{r-1} \xrightarrow{\delta_2} k^r \overset{\begin{pmatrix} 1 & 1 & \ldots & 1 \end{pmatrix}}{\longrightarrow} k \longrightarrow 0$$

where δ_2 is a matrix in which every column has one entry equal to 1, one entry equal to -1, and the rest equal to zero, because the boundary map of an edge will have 2 non-zero entries corresponding to the vertices making this edge.

Following definition 4.5 about homogenization, fix a basis to $C(T,k)$ given by $\overline{m_1}, \ldots, \overline{m_r}$ as a basis of k^r, and $\overline{v_1}, \ldots, \overline{v_p}$ that of k^{r-1}, note that we had the following $\delta(\overline{v_j}) = \sum_{1 \leq s \leq r} \alpha_{s,j} \overline{m_s}$, with coefficients $\alpha_{s,j} \in k$. The M-homogenization would then give a resolution of M of the form

$$0 \longrightarrow \bigoplus_{j=1}^{r-1} R(-\alpha_j) \xrightarrow{d_2} \bigoplus_{j=1}^{r} R(-\beta_j) \xrightarrow{d_1} R \longrightarrow 0$$

with $\beta_j = \mathrm{mdeg}(m_j)$, and $\alpha_j = \mathrm{mdeg}(v_j)$. We call $v_1, \ldots, v_p$ the basis of R^{r-1}, where v_j= lcm$\{$ mdeg$(m_s) \mid \alpha_{s,j} \neq 0$ $\}$ for the $\alpha' s$ occuring in the boundary map δ

where for each j, exactly 2 of the $\alpha_{s,j} \neq 0$. So the multidegrees of the v_j's are actually of the form $\text{mdeg}(v_j) = \text{mdeg}(\text{lcm}(m_{i_1}, m_{i_2}))$ where m_{i_1},and m_{i_2} are minimal generators of M. Considering the boundary map of the M-complex, $d_2(v_j) = \sum_{1 \leq s \leq r} \alpha_{s,j} \frac{mdeg(v_j)}{mdeg(m_s)} m_s$, the matrix representation of d_2 has entries: $[d_2]_{s,j} = \alpha_{s,j} \frac{mdeg(v_j)}{mdeg(m_s)}$.

If $\alpha_{s,j} = 0$ then $[d_2]_{s,j} = 0$. If $\alpha_{s_1,j}, \alpha_{s_2,j} \neq 0$ then we have that $\text{mdeg}(v_j\ ) = \text{lcm}(m_{s_1}, m_{s_2})$. Since m_{s_1}, and m_{s_2} are minimal generators of M we know that m_{s_1}, and m_{s_2} strictly divide $\text{mdeg}(v_j\ ) = \text{lcm}(m_{s_1}, m_{s_2})$, so that $[d_2]_{s,j} \in \mathfrak{m}$ for all s, j. By construction, all entries of d_1 are in $\mathfrak{m}$ and we can conclude that this resolution is minimal. □

We next construct the tree. WLOG, we may consider square-free monomial ideals, since the polarization of M gives a square-free monomial ideal. It was also shown that the minimal free resolution of R/M, and that of R_{pol}/M_{pol} are homogenizations of the same frame see [9]. We will not discuss polarization process in this thesis.

Construction 6.15. *Consider M a square-free monomial ideal of R such that its projective dimension $pd_R(M) \leq 1$. To construct a tree starting from the minimal generating set $\{m_i\}_{i \leq q}$ of M, one has to follow the steps below:*

1. *Consider the facet complex of M, then take its complement and call it $\triangle$. Now order the facets of $\triangle$ by $F_1, F_2, \ldots, F_q$ such that F_i is a leaf of $\triangle_i = \langle F_1, \ldots, F_i \rangle$.*

2. *Start with one vertex v_1 equivalently it's a vertex tree $T_1 = (V_1, E_1)$, where $V_1 = \{v_1\}$ and $E_1 = \phi$.*

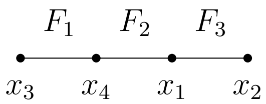
3. For each $i > 1$, let $\tau(i)$ be such that $F_{\tau(i)}$ is the joint of F_i in $\triangle_i$. Initially set $\tau(1) = 1$, and for $i = 2, \ldots, q$:

    - Pick $u < i$, such that F_u is a joint of F_i in $\triangle_i$. Now set $\tau(i) = u$;
    - Set $V_i = V_{i-1} \cup \{v_i\}$;
    - Set $E_i = E_{i-1} \cup \{(v_i, v_u)\}$;

4. That results in a tree $T = (V_q, E_q)$ with q vertices. Now label the vertex v_i of T with the monomial $m_i = \prod\limits_{x_t \notin F_i} x_t$.
   And note that the monomials $m_1, \ldots, m_q$ form a minimal generating set of M ordered as step one in this construction.

Remark 6.16. *In the final step of construction 6.15, note that F_i is a leaf of $\triangle_i$, so the free vertex $x \in F_i$ doesn't belong to any F_j such that $j < i$. Symbolically, to see it easily consider the complement of the facets, then $x \notin {F_i}^c$ and $x \in {F_j}^c$ for all $j < i$. Therefore, $\forall x \in \{1, \ldots, q\}$ there is a variable $x \in \{x_1, \ldots, x_n\}$ such that $x \nmid m_i$ and $x \mid m_j$ for all $j < i$.*

Example 6.17. 1. *Let $M = (x_1x_2, x_2x_3, x_3x_4)$, consider the facet complex $F(M)$ of facets $\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}$, then construct its complement $\triangle = F(M)^c$, its facets are $\{x_3, x_4\}, \{x_1, x_4\}, \{x_1, x_2\}$. Faridi and Hersey proved that we can order the facets of this simplicial complex as its a quasi-tree, then:*



Such that F_i is a leaf of $\triangle_i = \langle F_1, \ldots, F_i \rangle$ with $i = 2, 3$.

2. *Following the same procedure of the construction 6.15, we have* $\tau(2) = 1$, *and* $\tau(3) = 2$.

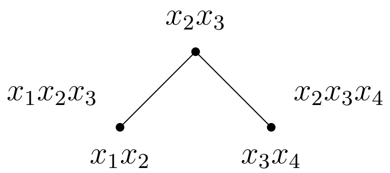
3. *Starting with* F_1, *construct inductively a graph with vertices labelled* F_1, F_2, F_3, *and with edges* $\{F_{\tau(i)}, F_i\}$ *for* $i = 2, 3$.

4. *Label each vertex of* F_i *with the monomial* $m_i =$ *the product of all variables that are not in* F_i.

   $m_1 = x_1x_2, m_2 = x_2x_3, m_3 = x_3x_4$.

5. *Label each edge with the lcm of the vertex labels.*

The labelled tree G *for this example is the graph supporting the minimal free resolution of* $M = (x_1x_2, x_2x_3, x_3x_4)$ *:*



The minimal free resoluion of $M = (x_1x_2, x_2x_3, x_3x_4)$ *is then :*

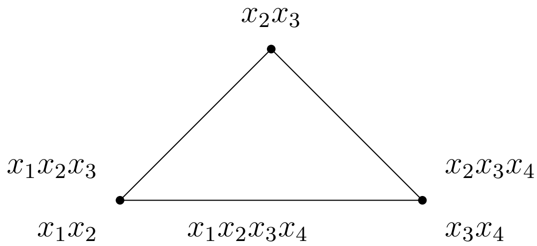
$0 \longrightarrow R(x_1x_2x_3) \oplus R(x_2x_3x_4) \xrightarrow{\delta} R(x_1x_2) \oplus R(x_2x_3) \oplus R(x_3x_4)$.

Considering the bases e_1, e_2, e_3 *corresponding to the three vertices, and* e_{12}, e_{23} *corresponding to the two edges joining, the map is:*

$\delta(e_{12}) = \frac{x_1x_2x_3}{x_2x_3}e_2 - \frac{x_1x_2x_3}{x_1x_2}e_1 = x_1e_2 - x_3e_1$

$\delta(e_{23}) = \frac{x_2x_3x_4}{x_3x_4}e_3 - \frac{x_2x_3x_4}{x_2x_3}e_2 = x_2e_3 - x_4e_2$.

Example 6.18. *We compare the Taylor resoluion of* $M = (x_1x_2, x_2x_3, x_3x_4)$ *to that in example 6.17 , we have to add the basis element* e_{13} *corresponding to the* 3^rd *edge that joins* x_1x_2 *with* x_3x_4, *along with* e_{123} *the facet, because the Taylor resolution considers the simplex that contains all possible faces, so starting from* *the figure it is :*



$$0 \longrightarrow R(x_1x_2x_3x_4) \xrightarrow{\delta_2} R(x_1x_2x_4) \oplus R(x_2x_3x_4) \oplus R(x_1x_2x_3x_4) \xrightarrow{\delta_1} R(x_1x_2) \oplus R(x_2x_3) \oplus R(x_3x_4)$$

where

$$\delta_1(e_{12}) = \tfrac{x_1x_2x_3}{x_2x_3}e_2 - \tfrac{x_1x_2x_3}{x_1x_2}e_1 = x_1e_2 - x_3e_1$$

$$\delta_1(e_{23}) = \tfrac{x_2x_3x_4}{x_3x_4}e_3 - \tfrac{x_2x_3x_4}{x_2x_3}e_2 = x_2e_3 - x_4e_2$$

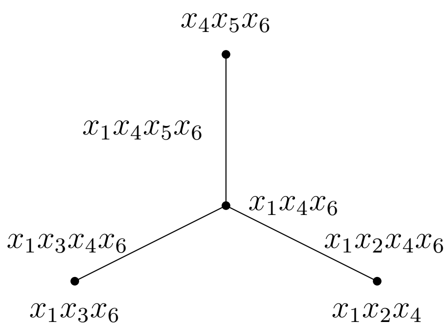
$$\delta_1(e_{13}) = \tfrac{x_1x_2x_3x_4}{x_3x_4}e_3 - \tfrac{x_1x_2x_3x_4}{x_1x_2}e_1 = x_1x_2e_3 - x_3x_4e_1$$

$$\delta_2(e_{123}) = \tfrac{x_1x_2x_3x_4}{x_1x_2x_4}e_{12} + \tfrac{x_1x_2x_3x_4}{x_2x_3x_4}e_{23} - \tfrac{x_1x_2x_3x_4}{x_1x_2x_3x_4}e_{13} = x_3e_{12} + x_1e_{23} - e_{13}.$$

See that in the last equation, the -1 *preceding* e_{13} *is an entry of the matrix corresponding to* δ_2*, therefore we can see that* $-1 \notin \mathfrak{m}$*, so our Taylor resolution is very non-minimal.*

Example 6.19. *Consider the ideal*

$M = (x_1x_3x_6, x_1x_4x_6, x_1x_2x_4, x_4x_5x_6) \subset k[x_1, \ldots, x_6]$*, and the labeled simplicial complex:*



From this labeled simplicial complex we construct the complex of R-modules:

$$0 \longrightarrow \begin{matrix} R(-x_1x_2x_4x_6) \\ \oplus \\ R(-x_1x_3x_4x_6) \\ \oplus \\ R(-x_1x_4x_5x_6) \end{matrix} \xrightarrow{\begin{pmatrix} x_6 & 0 & 0 \\ 0 & x_4 & 0 \\ -x_2 & -x_3 & x_5 \\ 0 & 0 & -x_1 \end{pmatrix}} \begin{matrix} R(-x_1x_2x_4) \\ \oplus \\ R(-x_1x_3x_6) \\ \oplus \\ R(-x_1x_4x_6) \\ \oplus \\ R(-x_4x_5x_6) \end{matrix} \xrightarrow{\begin{pmatrix} x_1x_2x_4 & x_1x_3x_6 & x_1x_4x_6 & x_4x_5x_6 \end{pmatrix}} R \to 0$$

which is the minimal multigraded free resolution of R/M.

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