AMERICAN UNIVERSITY OF BEIRUT

# ON THE GEOMETRY OF THE KORAS–RUSSELL CUBIC THREEFOLD

by

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# An Abstract of the Thesis of

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Major: Mathematics

Title: On the Geometry of the Koras–Russell Cubic Threefold

The Koras–Russel cubic threefold is a complex-affine manifold that is diffeomorphic to the three-dimensional complex-Euclidean space, but not algebraically isomorphic to the three dimensional complex-affine space as an affine variety. We analyze the topology of the Koras–Russel cubic threefold by means of Morse theory.

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# Chapter 1

# Introduction

The Koras–Russell cubic threefold, first introduced by Russell (1992), then studied by Koras and Russell later, is a smooth complex-affine manifold defined by

$$\{(x, y, z, w) \in \mathbb{C}^4 \colon x^2y + x + z^2 + w^3 = 0\}.$$

The Koras–Russell cubic threefold was studied in several papers, being a $\mathbb{C}^3$-like affine variety that is not $\mathbb{C}^3$, in the affine algebraic sense. Dimca (1992) showed that the Koras–Russell cubic threefold is a contractible hypersurface in $\mathbb{C}^4$, and proved that it is diffeomorphic to $\mathbb{C}^3$, see [Kraft, 1996]. Makar-Limanov (1994) proved that the Koras–Russell cubic threefold is not isomorphic to $\mathbb{C}^3$ as an affine variety, see [Kraft, 1996].

In this thesis we study by means of Morse theory that the topology of the Koras–Russell cubic threefold. We wish to make the thesis as much self-contained as possible in terms of basic definitions and main theorems. For that we present the preliminaries needed from topology and bilinear algebra in the second chapter.

The third chapter gives a tour on the theory of smooth manifolds, being the main objects under study in Morse theory, and extends to study complex manifolds highlighting some particular specializations that will be useful in the study of the Koras–Russell cubic threefold.

We define the notion of Morse functions on smooth manifolds in the forth chapter, prove the Lemma of Morse, and demonstrate the canceling of critical points theorem.

In the fifth chapter, we demonstrate that the Koras–Russell cubic threefold is not isomorphic to $\mathbb{C}^3$. For that we present basic definitions and properties of affine varieties and define isomorphisms between affine varieties in the suitable framework.

In the sixth chapter, we analyze the topology of the Koras–Russell cubic threefold using Morse theory. We define a convenient Morse function on the threefold, find its critical points, and study how the points are connected by the Morse flow.

# Chapter 2

# Background Theory

## 2.1 Topological Manifolds

**Definition 2.1.1.** *Let $X$ be a nonempty set. A collection $\tau$ of subsets of $X$ is said to be a topology on $X$ if :*

1. $\emptyset \in \tau$ *and* $X \in \tau$.
2. *for any family of sets* $\{A_i\}_{i\in I} \subseteq \tau$, $\bigcup\limits_{i\in I} A_i \in \tau$.
3. *for any finite family of sets* $\{A_1, ..., A_n\} \subseteq \tau$, $\bigcap\limits_{i=1}^{n} A_i \in \tau$.

*The pair $(X, \tau)$ is called a topological space. The elements of $\tau$ are called open sets. For $x \in X$, an open set containing $x$ is called a neighborhood of $x$.*

**Definition 2.1.2.** *A topological space $(X, \tau)$ is said to be a Hausdorff space if:*

$$\forall x, y \in X, (x \neq y \Rightarrow \exists U, V \in \tau \text{ such that } x \in U, y \in V, \text{ and } U \cap V = \emptyset).$$

**Definition 2.1.3.** *A collection of open sets $\mathfrak{B}$ is said to be a basis of the topology*

*$\tau$ of $X$ if:*

$$\forall x \in X, \forall U \in \tau, (x \in U \Rightarrow \exists B \in \mathfrak{B} \text{ such that } x \in B \subseteq U).$$

**Definition 2.1.4.** *A topological space is said to be second-countable if it has a countable basis for its topology.*

**Definition 2.1.5.** *Let $(X, \tau)$ and $(X', \tau')$ be two topological spaces. A map $f \colon X \to X'$ is said to be continuous if :*

$$\forall U \in \tau', f^{-1}(U) \in \tau$$

*Moreover if $f$ is bijective and $f^{-1}$ is continuous we say that $f$ is a homeomorphism and $X$ and $X'$ are homeomorphic*

**Definition 2.1.6.** *A topological space $(X, \tau)$ is said to be locally Euclidean of dimension $n$ if $\forall x \in X$, there exists:*

1. *an open set $U$ of $X$ containing $x$.*

2. *an open set $V$ of $\mathbb{R}^n$ (open with respect to the usual topology on $\mathbb{R}^n$).*

3. *a homeomorphism $\varphi \colon U \to V$.*

*Such a pair $(U, \varphi)$ is called a* ***coordinate chart****.*

**Definition 2.1.7.** *(Topological Manifold). A topological space $(X, \tau)$ is said to be a topological manifold of dimension $n$ if $(X, \tau)$ is a Hausdorff, second-countable, locally Euclidean space of dimension $n$.*

**Theorem 2.1.8.** *(Topological Invariance of Dimension). Let $M$ and $N$ be two topological manifolds of dimensions $n$ and $m$ respectively. If $M$ and $N$ are home-*

*omorphic, then $m = n$.*

*Proof.* See Theorem 17.26 of [Lee, 2013]. $\square$

## 2.2 Homotopy

The material of this section is from [Munkres, 2000].

**Definition 2.2.1.** (*Homotopy*) *Let $X$ and $Y$ be two topological spaces. A continuous map $F\colon X \times [0,1] \to Y$ is called a homotopy.*

**Definition 2.2.2.** *Let $X$ and $Y$ be two topological spaces, and $f, g\colon X \to Y$ be two continuous maps. $f$ and $g$ are said to be homotopic if there exists a homotopy $F\colon X \times [0,1] \to Y$ such that: $F(x,0) = f(x)$ and $F(x,1) = g(x)$, for all $x \in X$. We write $f \simeq g$.*

**Definition 2.2.3.** *Let $X$ be a topological space and $A \subseteq X$. A deformation retraction of $X$ onto $A$ is a homotopy $F\colon X \times [0,1] \to X$ with the following properties:*

1. *$F(x,0) = x$, for all $x \in X$,*
2. *$F(X,1) = A$,*
3. *$F(x,t) = x$, for all $t \in [0,1]$, and all $x \in A$.*

*If such a homotopy exists, we say that $X$ deformation retracts onto $A$.*

**Definition 2.2.4.** *A topological space $X$ is said to be contractible if $X$ deformation retracts onto a point.*

**Remark 2.2.5.** *Let $X$ be a topological space and $A \subseteq X$. A retraction of $X$ onto $A$ is a continuous map $r\colon X \to X$ such that $r(X) = A$ and $r(x) = x$ for all $x \in A$. It is clear that if $X$ deformation retracts onto $A$ then any retraction of $X$ onto $A$ is homotopic to the identity map of $X$.*

**Definition 2.2.6.** *Let $X$ and $Y$ be two topological spaces. A continuous map $f\colon X \to Y$ is said to be a homotopy equivalence if there a continuous map $g\colon Y \to X$ such that $g \circ f \simeq I_X$ and $f \circ g \simeq I_Y$, where $I$ is the identity map on each of the corresponding spaces. In this case we say that $X$ and $Y$ are homotopy equivalent or have the same homotopy type, and we write $X \simeq Y$. It is easy to check that this relation is in fact an equivalence relation on topological spaces.*

**Remark 2.2.7.** *Let $X$ and $Y$ be two topological spaces. If $X$ and $Y$ are homeomorphic, then it is follows easily that $X \simeq Y$, since every homeomorphism of topological spaces is in fact a homotopy equivalence.*

**Definition 2.2.8.** *Let $X$ be a topological space. Given a non-negative integer $k$, we define the $k$-cell, denoted by $e^k$, to be the closed unit ball in $\mathbb{R}^k$,*

$$e^k = \{x \in \mathbb{R}^k \colon \|x\| \leq 1\}$$

*where $\|\cdot\|$ is the standard Euclidean norm. The boundary of $e^k$ will be denoted by $\dot{e}^k$ or $S^{k-1}$,*

$$\dot{e}^k = \{x \in \mathbb{R}^k \colon \|x\| = 1\}$$

*with the convention that $e^0$ is a single point and $\dot{e}^0 = \emptyset$.*

**Definition 2.2.9.** *Let $X$ be a topological space, $k$ be a non-negative integer, and $g\colon S^{k-1} \to X$ be a continuous map. We obtain the space $X$ with a $k$-cell attached by $g$, denoted by $X \cup_g e^k$, by taking the disjoint union of $X$ and $k$-cell, $X \coprod e^k$ with* *the disjoint union topology, and then identifying each $x \in S^{k-1}$ with $g(x) \in X$.* $X \cup_g e^k = \left(X \coprod e^k\right) / \sim$, *equipped with the quotient topology, where $\sim$ is the equivalence relation defined on $X \coprod e^k$ by*

$$x \sim y \Leftrightarrow x = y \;\; or \;\; x = g(y) \;\; or \;\; y = g(x) \qquad \forall\, x, y \in X \coprod e^k$$

*$X$ with a 0-cell attached is just the disjoint union of $X$ and a point.*

*See Chapter 2 Section 22 in [Munkres, 2000] for the definition of the quotient topology.*

## 2.3 Bilinear Algebra

In this section, $K = \mathbb{R}$ or $K = \mathbb{C}$, will denote any of the two fields.

**Definition 2.3.1.** *(Bilinear Map) Let $V, W, S$ be vector spaces over $K$. A map $L\colon V \times W \to S$ is said to bilinear if $L$ is linear in each variable. That is for every $v_1, v_2 \in V$, $w_1, w_2 \in W$, $\alpha \in K$ we have:*

$$L(\alpha v_1 + v_2, w_1) = \alpha L(v_1, w_1) + L(v_2, w_1) \quad and \quad L(v_1, \alpha w_1 + w_2) = \alpha L(v_1, w_1) + L(v_1, w_2)$$

**Definition 2.3.2.** *(Tensor Product) Let $V, W, S$ be vector spaces over $K$, and $\varphi\colon V \times W \to S$ be a bilinear map. The pair $(S, \varphi)$ is called a tensor product of $V$ and $W$ if for every vector space $P$ over $K$, and every bilinear map $L\colon V \times W \to P$, there exists a unique linear map $\tilde{L}\colon S \to P$ that makes the following diagram commute:*

$$\begin{tikzcd} V \times W \arrow[r, "\varphi"] \arrow[dr, "L"'] & S \arrow[d, "\tilde{L}"] \\ & P \end{tikzcd}$$

**Proposition 2.3.3.** *Let $V, W$ be vector spaces over $K$. If $(S, \varphi)$ and $(R, \psi)$ are two tensor products of $V$ and $W$, then $S$ and $R$ are isomorphic.*

*Proof.* By the definition of tensor products there exist unique linear maps $\tilde{\psi}\colon S \to R$ and $\tilde{\varphi}\colon R \to S$ making the following diagram commute:

$$\begin{tikzcd}
V \times W \arrow[r, "\varphi"] \arrow[dr, "\psi"] \arrow[ddr, "\varphi"] \arrow[dddr, "\psi"] & S \arrow[d, "\tilde{\psi}"] \\
& R \arrow[d, "\tilde{\varphi}"] \\
& S \arrow[d, "\tilde{\psi}"] \\
& R
\end{tikzcd}$$

Now $\tilde{\varphi} \circ \tilde{\psi}\colon S \to S$ makes the below diagram commute:

$$\begin{tikzcd}
V \times W \arrow[r, "\varphi"] \arrow[dr, "\varphi"] & S \arrow[d, "\tilde{\varphi} \circ \tilde{\psi}"] \\
& S
\end{tikzcd}$$

Then by uniqueness of such map $\tilde{\varphi} \circ \tilde{\psi} = I_S$.

By a similar argument $\tilde{\psi} \circ \tilde{\varphi} = I_R$, so that $\tilde{\varphi}$ is a linear isomorphism. Therefore $S \cong R$. $\square$

**Proposition 2.3.4.** *Let $V$ and $W$ be vector spaces over $K$. Then there exists a unique tensor product of $V$ and $W$ (up to isomorphisms).*

*Proof.* For the construction of a tensor product see Proposition 12.10 of [Lee, 2013]. Proposition 2.3.3 guarantees the uniqueness of a tensor product (up to isomorphisms). $\square$

**Remark 2.3.5.** *Let $V, W$ be vector spaces over $K$. We will denote the tensor* *product of $V$ and $W$ by $V \otimes W$, with the corresponding bilinear map from $V \times W$ to $V \otimes W$ denoted by $(v, w) \mapsto v \otimes w$. Elements of the form $v \otimes w \in V \otimes W$ are called elementary tensors. The construction in Proposition 12.10 of [Lee, 2013] shows that every element in $V \otimes W$ can be expressed as a linear combination of elementary tensors. When more than one field is considered, we write $\otimes_K$ to denote the tensor product with respect to the field $K$.*

**Proposition 2.3.6.** *Let $V, W$ be finite-dimensional vector spaces of dimensions $m, n$, respectively. Let $(E_1, \ldots, E_m)$ be a basis for $V$, and $(F_1, \ldots, F_n)$ be a basis for $W$. Then the set:*

$$\{E_i \otimes F_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

*is a basis for $V \otimes W$, and so $V \otimes W$ has dimension $mn$.*

*Proof.* See Proposition 12.8 of [Lee, 2013]. □

**Remark 2.3.7.** *One particular application of tensor products, that we will refer to in Chapter 3, is the* ***complexification*** *of a real vector space. Let $V$ be a vector space over $\mathbb{R}$. Consider the tensor product $V^{\mathbb{C}} :\ = V \otimes_{\mathbb{R}} \mathbb{C}$ (where $\mathbb{C}$ is viewed as a vector space over $\mathbb{R}$). We can make $V^{\mathbb{C}}$ a vector space over $\mathbb{C}$, by defining a scalar multiplication:*

$$\mathbb{C} \times V^{\mathbb{C}} \to V^{\mathbb{C}}$$

*on elementary tensors of $V^{\mathbb{C}}$ by $\lambda(v \otimes \alpha) = v \otimes (\lambda\alpha)$, for all $v \in V$ and $\lambda, \alpha \in \mathbb{C}$, and extending it to elements of $V_{\mathbb{C}}$ by:*

$$\lambda \sum_{k=1}^{n} (v_k \otimes \alpha_k) = \sum_{k=1}^{n} (v_k \otimes \lambda\alpha_k)$$

*for* $v_1, \dots, v_n \in V$, $\lambda, \alpha_1, \dots, \alpha_n \in \mathbb{C}$. *The* $\mathbb{C}$*-vector space* $V^{\mathbb{C}}$ *is called the complexification of* $V$*.*

*If* $V$ *and* $W$ *are two vector spaces over* $\mathbb{R}$ *and* $L\colon V \to W$ *be a linear map.* $L$ *extends to a* $\mathbb{C}$*-linear map* $L\colon V^{\mathbb{C}} \to W^{\mathbb{C}}$ *defined on elementary tensors by*

$$L(v \otimes \alpha) = L(v) \otimes \alpha$$

*for all* $v \in V$ *and* $\alpha \in \mathbb{C}$*.*

**Definition 2.3.8.** (*Dual Space*) *Let* $V$ *be a vector space over* $K$*. The dual space of* $V$*, denoted by* $V^*$*, is the set of all linear maps from* $V$ *to* $K$*.*

$$V^* = \{f\colon V \to K : f \text{ is linear}\}$$

*Elements of* $V^*$ *are called covectors, or linear forms.*

**Proposition 2.3.9.** *Let* $V$ *be a finite-dimensional vector space over* $K$ *of dimension* $n$*, and let* $(v_1, \dots, v_n)$ *be a basis for* $V$*. For* $i \in \{1, \dots, n\}$*, consider the map:*

$$\lambda_i\colon V \to K$$

*defined on the given basis of* $V$ *by* $\lambda_i(v_j) = \delta_{ij}$*, for* $j \in \{1, \dots, n\}$ *(where* $\delta_{ij}$ *is the Kronecker delta symbol), and extended to a linear map on* $V$*. Then* $(\lambda_1, \dots, \lambda_n)$ *is a basis for* $V^*$*, and so* $\dim V^* = \dim V$*. This basis is called the dual basis of* $V^*$ *relative to the basis* $(v_1, \dots, v_n)$*.*

*Proof.* See Proposition 11.1 of [Lee, 2013]. □

**Definition 2.3.10.** *Let* $V$ *and* $W$ *be two vector spaces over* $K$*, and* $L\colon V \to W$ *be a linear map. We define the linear map* $L^*\colon W^* \to V^*$*, called the dual map or* *transpose of $L$, by*

$$(L^*\omega)v = \omega(Lv) \quad \textit{for } \omega \in W^*,\, v \in V.$$

**Definition 2.3.11.** *Let $V$ be a vector space over $K$. A bilinear map from $V \times V$ to $K$ is said to be a bilinear form on $V$. The set of all bilinear forms on $V$ is denoted by $\mathcal{B}(V)$.*

$$\mathcal{B}(V) = \{L\colon V \times V \to K \mid L \textit{ is bilinear}\}$$

*$\mathcal{B}(V)$ is in fact a vector space over $K$ (with the usual scalar multiplication).*

**Remark 2.3.12.** *Let $V$ be a vector space over $K$, and let $\alpha, \beta \in V^*$. Define the map:*

$$\alpha \otimes \beta\colon V \times V \to K$$

*by $\alpha \otimes \beta(v, w) = \alpha(v)\beta(w)$, $\forall v, w \in V$. $\alpha \otimes \beta$ is clearly a bilinear form. The choice of the tensor symbol will be justified in Remark 2.3.14.*

**Proposition 2.3.13.** *Let $V, W$ be a finite dimensional vector space over $K$ of dimensions $m$ and $n$ respectively, and let $(\alpha_1, \ldots, \alpha_m)$ be a basis of $V^*$ and $(\beta_1, \ldots, \beta_n)$ be a basis of $W^*$. Then then set:*

$$\{\alpha_i \otimes \beta_j\colon 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

*is a basis for the space $L(V, W; K) = \{f\colon V \times W \to K \mid f \text{ is bilinear}\}$.*

*Proof.* See Proposition 12.4 in [Lee, 2013]. □

**Remark 2.3.14.** *Let $V$ be a finite dimensional vector space over $K$ of dimension $n$. Consider the linear map $L\colon V^* \otimes V^* \to \mathcal{B}(V)$ defined by*

$$L(\alpha \otimes \beta) = \alpha \otimes \beta$$

*on elementary tensors of $V^* \otimes V^*$. $L$ is clearly an isomorphism since it maps the basis of $V^* \otimes V^*$ to the basis of $\mathcal{B}(V)$. In what follows we will identify $\mathcal{B}(V)$ with $V^* \otimes V^*$. Elements of $V^* \otimes V^*$ will be regarded as bilinear forms on $V$, by the action of elementary tensors on $V \times V$:*

$$\alpha \otimes \beta(v, w) = \alpha(v)\beta(w)$$

$\forall v, w \in V$.

**Remark 2.3.15.** *Let $V$ be a finite dimensional vector space over $K$ of dimension $n$. Let $(v_1, \ldots, v_n)$ be a basis of $V$ and $(\lambda_1, \ldots, \lambda_n)$ be the dual basis of $V^*$ relative to the basis $(v_1, \ldots, v_n)$. The set $\{\lambda_i \otimes \lambda_j\colon 1 \leq i \leq n,\ 1 \leq j \leq n\}$ is a basis for $V^* \otimes V^*$, then for every bilinear map $L \in V^* \otimes V^*$,*

$$L = \sum_{i,j=1}^{n} a_{ij} \lambda_i \otimes \lambda_j$$

*for some coefficients $a_{ij} \in K$. The matrix $(a_{ij})$ is called the representative matrix of the bilinear form $L$ in the coordinates $(v_1, \ldots, v_n)$ of $V$.*

**Definition 2.3.16.** *Let $V$ be a finite dimensional vector space over $K$. A bilinear form $L \in V^* \otimes V^*$ is said to be symmetric if*

$$L(v, w) = L(w, v) \qquad \forall v, w \in V$$

*We denote by $\Sigma^2(V)$ the set of all symmetric bilinear forms on $V$. $\Sigma^2(V)$ is in fact a subspace of $V^* \otimes V^*$, and the linear map $Sym\colon V^* \otimes V^* \to \Sigma^2(V)$ defined by*

$$Sym(L)(v,w) = \frac{1}{2}(L(v,w) + L(w,v)) \qquad \forall v, w \in V$$

*is a projection, i.e. $Sym(L) = L$, $\forall L \in \Sigma^2(V)$.*

**Definition 2.3.17.** *Let $V$ be a finite dimensional vector space over $K$. For $\alpha, \beta \in V^*$, we define the symmetric product of $\alpha$ and $\beta$, denoted by $\alpha\beta$, by*

$$\alpha\beta = Sym(\alpha \otimes \beta) = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha).$$

The above procedure could be done in general for $n$ variables by considering multilinear maps instead of bilinear maps, one can also define antisymmetric (or alternating) multilinear maps, and alternating product. However we are only interested in symmetric bilinear forms in the thesis. See Chapter 12 in [Lee, 2013] for a general approach.

**Definition 2.3.18.** *Let $V$ be a finite dimensional vector space over $K$, and $L \in \Sigma^2(V)$, then the map*

$$q\colon V \to K$$

*defined by $q(v) = L(v,v)$ for all $v \in V$ is called the quadratic form associated to the symmetric bilinear form $L$.*

**Definition 2.3.19.** *Let $V$ be a finite dimensional vector space over $\mathbb{R}$, and $L \in \Sigma^2(V)$. $L$ is said to be **positive-definite** (respectively **negative-definite**) if $L(v,v) > 0$ (respectively $L(v,v) < 0$) for every $v \in V \setminus \{0\}$.*

*A positive-definite symmetric bilinear form on $V$ is called an inner product on $V$.*

**Definition 2.3.20.** (*Nullity and Index*) *Let $V$ be a finite dimensional vector space over $\mathbb{R}$ of dimension $n$, and let $L \in \Sigma^2(V)$. We define the* ***null space of L****, denoted by* $\ker L$*, by*

$$\ker L = \{v \in V : L(v, w) = 0 \;\; \forall\, w \in V\}.$$

$\ker L$ *is clearly a subspace of $V$. The dimension of the null space of $L$ is called the* ***nullity of L****. We define the* ***index of L*** *to be the maximum dimension of a subspace of $V$ on which $L$ is negative-definite.*

**Theorem 2.3.21.** *(Sylvester's Law of Inertia) Let $V$ be a finite dimensional vector space over $\mathbb{R}$ of dimension $n$, and let $L \in \Sigma^2(V)$. Then there exists a basis of $V$ such that the representative matrix of the $L$ in this basis is a diagonal matrix with diagonal entries $1, 0, -1$. Moreover for any such basis of $V$, the number of each, $1$'s, $0$'s and $-1$'s on the diagonal entries, is invariant.*

*Proof.* See Section 9.3.1 in [Carrell, 2017]. □

**Remark 2.3.22.** *Let $V$ be a finite dimensional vector space over $\mathbb{R}$ of dimension $n$, and let $L \in \Sigma^2(V)$. Let $(v_1, \dots, v_n)$ be any basis of $V$ for which the representative matrix of the $L$ in $(v_1, \dots, v_n)$ is a diagonal matrix with diagonal entries $1, 0, -1$. It follows from this representation of $L$ that number of $0$'s in the diagonal entries of such representative matrix of $L$, is the nullity of $L$, and the number of $-1$'s is the the index of $L$.*

# Chapter 3

# Smooth Manifolds

In this chapter we establish the basic definitions and properties of smooth manifolds. The material of this chapter is from [Lee, 2013] and [Forstnerič, 2017].

## 3.1 Smooth Structure

In this section we will define an additional structure on topological manifolds, namely a smooth structure, that will enable us to define smooth maps between manifolds.

**Definition 3.1.1.** *Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ be two open sets, and $F\colon V \to W$ be a map. $F$ is said to be $C^\infty$ smooth if each of its component functions has continuous partial derivatives of all orders. If in addition $F$ is bijective and $F^{-1}$ is $C^\infty$ smooth, $F$ is called a diffeomorphism.*

Throughout the thesis the word smooth will be used to denote $C^\infty$ smoothness.

**Remark 3.1.2.** *Let $M$ be a topological manifold of dimension $n$ and let $(U, \varphi)$ and $(V, \psi)$ be two coordinate charts. Note that $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open* *subsets of* $\mathbb{R}^n$*, and whenever* $U \cap V \neq \emptyset$*, the map* $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ *is a homeomorphism. The map* $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ *is called the transition map from* $\varphi$ *to* $\psi$*.*

**Definition 3.1.3.** *Let* $M$ *be a topological manifold of dimension* $n$ *and let* $(U, \varphi)$ *and* $(V, \psi)$ *be two coordinate charts.* $(U, \varphi)$ *and* $(V, \psi)$ *are said to be smoothly compatible if either* $U \cap V = \emptyset$ *or the map* $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ *is smooth.*

**Definition 3.1.4.** *Let* $M$ *be a topological manifold of dimension* $n$*. A collection of coordinate charts* $\{(U_i, \varphi_i)\}_{i \in I}$ *is said to be an* ***atlas*** *for* $M$ *if* $\bigcup_{i \in I} U_i = M$*.*

*An atlas* $\mathcal{A}$ *is called a smooth atlas if any two charts in* $\mathcal{A}$ *are smoothly compatible with each other.*

**Remark 3.1.5.** *Let* $M$ *be a topological manifold of dimension* $n$ *and* $\mathcal{A}$ *be a smooth atlas for* $M$*. Let* $f \colon M \to \mathbb{R}$*. Suppose that there exists a chart* $(U, \varphi)$ *of* $\mathcal{A}$ *such that the map* $f \circ \varphi^{-1} \colon \varphi(U) \to \mathbb{R}$ *is smooth. Then for any coordinate chart* $(V, \psi)$ *of* $\mathcal{A}$ *with* $U \cap V \neq \emptyset$*,* $f \circ \psi^{-1} \colon \psi(U \cap V) \to \mathbb{R}$ *is smooth, as* $f \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1})$ *in* $\psi(U \cap V)$*.*

*Hence we would like to say that* $f$ *is smooth whenever the maps* $f \circ \varphi^{-1}$ *are smooth for* $(U, \varphi) \in \mathcal{A}$*. However in general, there will be many possible smooth atlases that determine the same collection of smooth functions on* $M$*. For this reason we will define a maximal atlas, according to which we will define a smooth structure on* $M$*.*

**Definition 3.1.6.** *Let* $M$ *be a topological manifold. A smooth atlas* $\mathcal{A}$ *is said to be maximal if for every smooth atlas* $\mathcal{B}$ *such that* $\mathcal{A} \subseteq \mathcal{B}$*,* $\mathcal{A} = \mathcal{B}$*. Equivalently,*

*$\mathcal{A}$ is maximal if for any chart $(U, \varphi)$ that is smoothly compatible with every chart in $\mathcal{A}$, $(U, \varphi) \in \mathcal{A}$. We call a maximal smooth atlas for $M$ a* ***smooth structure****.*

**Proposition 3.1.7.** *Let $M$ be a topological manifold. Every smooth atlas $\mathcal{A}$ for $M$ is contained in a unique maximal smooth atlas.*

*Proof.* Let $\mathcal{A}$ be a smooth atlas for $M$, and denote by $\tilde{\mathcal{A}}$ the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. It is clear that $\mathcal{A} \subseteq \tilde{\mathcal{A}}$. Let $(U, \varphi), (V, \psi) \in \tilde{\mathcal{A}}$ and consider the map $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$.
Let $x \in \varphi(U \cap V)$. $\varphi^{-1}(x) \in M$ then there exists a chart $(W, \nu)$ in $\mathcal{A}$ such that $\varphi^{-1}(x) \in W$. Since every chart in $\tilde{\mathcal{A}}$ is smoothly compatible with $(W, \nu)$, $\nu \circ \varphi^{-1}$ and $\psi \circ \nu^{-1}$ are smooth in $\varphi(U \cap W)$ and $\nu(V \cap W)$ respectively. $\varphi^{-1}(x) \in U \cap V \cap W$, then $(\psi \circ \nu^{-1}) \circ (\nu \circ \varphi^{-1})$ is smooth in a neighborhood of $x$, thus $\psi \circ \varphi^{-1}$ is smooth in a neighborhood of each point in $\varphi(U \cap V)$. Therefore $\tilde{\mathcal{A}}$ is a smooth atlas. Now to prove uniqueness, assume that $\mathcal{B}$ is a maximal smooth atlas containing $\mathcal{A}$, then each chart of $\mathcal{B}$ is compatible with each chart of $\mathcal{A}$, so $\mathcal{B} \subseteq \tilde{\mathcal{A}}$. By maximality of $\mathcal{B}$, $\mathcal{B} = \tilde{\mathcal{A}}$. $\square$

We call the unique maximal smooth atlas containing $\mathcal{A}$, the smooth structure determined by $\mathcal{A}$.

**Proposition 3.1.8.** *Let $M$ be a topological manifold. Two smooth atlases for $M$ determine the same smooth structure if and only if their union is a smooth atlas.*

*Proof.* $\boxed{\Rightarrow}$ Let $\mathcal{A}, \mathcal{B}$ be two smooth atlases. Assume that $\mathcal{A}$ and $\mathcal{B}$ determine the same smooth structure, namely $\tilde{\mathcal{A}}$. Then $\mathcal{A} \cup \mathcal{B} \subseteq \tilde{\mathcal{A}}$, so for any two charts $(U, \varphi), (V, \psi) \in \mathcal{A} \cup \mathcal{B}$, $(U, \varphi), (V, \psi) \in \tilde{\mathcal{A}}$ so $(U, \varphi)$ and $(V, \psi)$ are smoothly compatible. Therefore $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas.
$\boxed{\Leftarrow}$ Let $\mathcal{A}, \mathcal{B}$ be two smooth atlases. Now assume that $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas,

and let $\mathcal{C}$ be the smooth structure determined by $\mathcal{A} \cup \mathcal{B}$. $\mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B} \subseteq \mathcal{C}$, $\mathcal{B} \subseteq \mathcal{A} \cup \mathcal{B} \subseteq \mathcal{C}$, and $\mathcal{C}$ is a maximal smooth atlas, then $\mathcal{C}$ is the smooth structure determined by $\mathcal{A}$ and $\mathcal{B}$. $\square$

**Definition 3.1.9.** *(Smooth Manifold). Let $M$ be a topological manifold and $\mathcal{A}$ be a maximal smooth atlas for $M$. The pair $(M, \mathcal{A})$ is called a smooth manifold.*

**Definition 3.1.10.** *Let $(M, \mathcal{A})$ be a smooth manifold of dimension $n$. We say that $f\colon M \to \mathbb{R}^k$ is a smooth function if for every $p \in M$, there exists a smooth chart $(U, \varphi) \in \mathcal{A}$ with $p \in U$ such that $f \circ \varphi^{-1}$ is smooth in the open subset $\varphi(U) \subseteq \mathbb{R}^n$.*

**Remark 3.1.11.** *The set of all real-valued smooth functions $f\colon M \to \mathbb{R}$ is denoted by $C^\infty(M)$. Since sums and constant multiples of smooth functions are smooth, $C^\infty(M)$ is a vector space over $\mathbb{R}$. Moreover the point-wise product of two real-valued smooth functions is again a smooth function and hence $(C^\infty(M), +, \cdot)$ is a ring.*

**Remark 3.1.12.** *Let $M$ be a smooth manifold and $U \subseteq M$ be open. A map $f\colon U \to \mathbb{R}$ is said to be smooth if for every chart $(V, \psi)$ with $U \cap V \neq \emptyset$, the map $f \circ \psi^{-1}\colon \psi(U \cap V) \to \mathbb{R}$ is smooth. The space of all smooth function $f\colon U \to \mathbb{R}$ is denoted by $C^\infty(U)$.*

**Definition 3.1.13.** *Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ be smooth manifolds. We say that $F\colon M \to N$ is a smooth function if for every $p \in M$, there exist smooth charts $(U, \varphi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(U) \subseteq V$, and $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$.*

*Moreover if $F$ is bijective and $F^{-1}$ is smooth, we say that $F$ is a diffeomorphism and that $M$ and $N$ are diffeomorphic.*

## 3.2 The Tangent Space

In what follows we will define the tangent space to a smooth manifold at a given point. We wish to give a definition independent of the coordinate charts. The smooth manifold $(M, \mathcal{A})$ will be simply denoted by $M$.

**Definition 3.2.1.** *Let $M$ be a smooth manifold and $p \in M$. A linear map $V\colon C^\infty(M) \to \mathbb{R}$ is said to be a* ***derivation at p*** *if for every $f, g \in C^\infty(M)$, $V(fg) = f(p)V(g) + g(p)V(f)$.*

**Remark 3.2.2.** *For a general definition of derivations on maps between modules over the same ring, we also require that $V(c) = 0$ for any constant map $c$. However this condition follows from the above definition in case of real valued functions (see Proposition 3.2.7).*

**Remark 3.2.3.** *The sums and constant multiples of derivations at a point $p$ are also derivations at the point $p$, thus the set of all derivations at $p$ is a vector space over $\mathbb{R}$.*

**Definition 3.2.4.** *(Tangent Space). Let $M$ be a smooth manifold and $p \in M$. We define the tangent space to $M$ at point $p$, denoted by $T_pM$, to be the space of all derivations at $p$.*

$$T_pM = \{V\colon C^\infty(M) \to \mathbb{R} : V \textit{ is a derivation at } p\}.$$

**Remark 3.2.5.** *Note that for $M = \mathbb{R}^n$, $p \in \mathbb{R}^n$, the partial derivatives at $p$, $\{\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p\}$ are derivations at $p$ so that $\mathrm{Span}\{\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p\} \subseteq T_p\mathbb{R}^n$. Moreover the vectors $\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p$ are linearly independent. To see this let $a_1, ..., a_n \in \mathbb{R}$ such that $\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}|_p = 0$. Apply the latter sum on the (smooth)* *projection maps $x_j$, we get $a_j = 0, \forall j = 1, .., n$.*

*In what follows we will prove that $\{\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p\}$ is in fact a basis for $T_p\mathbb{R}^n$.*

**Remark 3.2.6.** *Let $M$ be a smooth manifold and $p \in M$. Define $m_p \coloneqq \{f \in C^\infty(M) : f(p) = 0\}$.*

*$m_p$ is clearly an ideal of the ring $C^\infty(M)$, and $m_p^2$ is the ideal formed by taking finite sums of elements the form $fg$, where $f, g \in m_p$.*

**Proposition 3.2.7.** *Let $M$ be a smooth manifold and $p \in M$. A linear map $V: C^\infty(M) \to \mathbb{R}$ is a derivation at $p$ if and only if:*

1. *$V(c) = 0$ for any constant function $c \in C^\infty(M)$*

2. *$V(h) = 0$ for all $h \in m_p^2$*

*Proof.* $\boxed{\Rightarrow}$ Let $V: C^\infty(M) \to \mathbb{R}$ be a derivation at $p$. $V(1) = V(1 \cdot 1) = 1(p)V(1) + 1(p)V(1) = 2V(1)$, and so $V(1) = 0$, thus $V(c) = cV(1) = 0$. Now for all $f, g \in m_p$, $V(fg) = f(p)V(g) + g(p)V(f) = 0$, therefore by linearity of $V$, $V(h) = 0, \forall h \in m_p^2$.

$\boxed{\Leftarrow}$ Let $V: C^\infty(M) \to \mathbb{R}$ be a linear map satisfying *1* and *2*. Let $f, g \in C^\infty(M)$, then $f - f(p), g - g(p) \in m_p$ and so $V((f - f(p))(g - g(p))) = 0$ (by *2*).

$0 = V((f - f(p))(g - g(p))) = V(fg - f(p)g - g(p)f + f(p)g(p)) = V(fg) - V(f(p)g) - V(g(p)f) + V(f(p)g(p))$.

Now $V(f(p)g(p)) = 0$ (by *1*), therefore $V(fg) = f(p)V(g) + g(p)V(f)$. $\square$

**Proposition 3.2.8.** *$\{\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p\}$ forms a basis for the vector space $T_p\mathbb{R}^n$.*

*Proof.* By Remark 3.2.5, $\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p$ are linearly independent, so it is enough to show that $\text{Span}\{\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p\} = T_p\mathbb{R}^n$.

Let $V \in T_p\mathbb{R}^n$. Let $f \in C^\infty(\mathbb{R}^n)$. By Taylor's theorem we can write

$$f(x) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - p_i) + \sum_{i,j=1}^{n} (x_i - p_i)(x_j - p_j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(p + t(x-p))dt$$

with $x = (x_1, ..., x_n), p = (p_1, ..., p_n)$.

Let $g(x) = \sum_{i,j=1}^{n} (x_i - p_i)(x_j - p_j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(p + t(x-p))dt$, then $g \in m_p^2$, thus by Proposition 3.2.7, $V(g) = 0$.

Applying $V$ to both sides of Equation (2.2.1) we get:

$V(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)V(x_i) = \sum_{i=1}^{n} V(x_i)\frac{\partial}{\partial x_i}|_p f$ for all $f \in C^\infty(\mathbb{R}^n)$,

thus $V = \sum_{i=1}^{n} V(x_i)\frac{\partial}{\partial x_i}|_p$, therefore $V \in \text{Span}\{\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p\}$. □

We will prove next that if $M$ is a smooth manifold of dimension $n$, $p \in M$, then $T_pM$ is a finite dimensional vector space of dimension $n$. To achieve that we will first prove that $T_pM$ could be determined locally by derivations of $C^\infty(U)$ at $p$, where $U$ is any neighborhood of $p$ in $M$.

**Lemma 3.2.9.** *Let $M$ be a smooth manifold, $p \in M$, and $U \subseteq M$ be an open neighborhood of $p$ in $M$. Then there exists a smooth function $\psi \in C^\infty(M)$ and an open neighborhood $\tilde{U}$ of $p$ compactly contained in $U$ ($\tilde{U} \Subset U$) such that $\psi = 1$ on $\tilde{U}$ and supp($\psi$) $\subset U$. Such a function is called a bump function.*

*Proof.* See Proposition 2.25 of [Lee, 2013]. □

**Proposition 3.2.10.** *Let $M$ be a smooth manifold, $p \in M$, and $f \in C^\infty(M)$. If $f$ is identically zero in a neighborhood of $p$, then $V(f) = 0$.*

*Proof.* Let $U$ be a neighborhood of $p$ such that $f = 0$ in $U$. Let $\psi$ be a bump function as in Lemma 3.2.9 and consider the function $(1 - \psi)f$ defined on M.

$(1-\psi)f \in C^\infty(M)$, $(1-\psi)(p) = f(p) = 0$ then $(1-\psi)f \in m_p^2$.

For $x \in U$, $f(x) = 0 = (1-\psi)f(x)$.

For $x \in M \setminus U$, $\psi(x) = 0$ so $(1-\psi)f(x) = f(x)$.

Thus $f = (1-\psi)f$ on $M$, and so $V(f) = V((1-\psi)f) = 0$ (by Proposition 3.2.7).

$\square$

**Corollary 3.2.11.** *Let $M$ be a smooth manifold, $p \in M$, and $V$ be a derivation at $p$. Then $\forall f, g \in C^\infty(M)$ such that $f = g$ in a neighborhood of $p$ in $M$, we have: $V(f) = V(g)$.*

*Proof.* Let $f, g \in C^\infty(M)$ such that $f = g$ in a neighborhood of $p$, then $f - g \in C^\infty(M)$ and $f - g = 0$ in a neighborhood of $p$, thus by Proposition 3.2.10, $V(f-g) = 0$. Therefore $V(f) = V(g)$. $\square$

**Remark 3.2.12.** *Let $M$ be a smooth manifold, $p \in M$, and $U$ an open neighborhood of $p$. Define the set $T_pU$ to be the set of all derivations on $C^\infty(U)$.*

$$T_pU = \{V : C^\infty(U) \to \mathbb{R} : V \text{ is a derivation at } p\}.$$

**Proposition 3.2.13.** *Let $M$ be a smooth manifold, $p \in M$, and $U$ an open neighborhood of $p$. Then $T_pM \cong T_pU$.*

*Proof.* Consider the map $i : T_pU \to T_pM$, for $V \in T_pU$ and $f \in C^\infty(M)$,

$$i(V)(f) = V(f|_U).$$

$\forall f \in C^\infty(M)$, $f|_U \in C^\infty(U)$, $i(V) : C^\infty(M) \to \mathbb{R}$ is a linear map, and for $f, g \in C^\infty(M)$, $i(V)(fg) = V(fg|_U) = V(f|_U \cdot g|_U) = f|_U(p)V(g|_U) + g|_U(p)V(f|_U) = f(p)i(V)(g) + g(p)i(V)(f)$, thus $i(V) \in T_pM$. This shows that $i$ is well-defined, and $i$ is clearly linear.

Now let $\psi$ be a bump function supported in $U$ such that $\psi = 1$ in a neighborhood of $p$. For $f \in C^\infty(U)$ define the map $\tilde{f}$ to be $\psi f$ in $U$ and identically zero elsewhere. It is clear that $\tilde{f} \in C^\infty(M)$. Now consider the map $r\colon T_pM \to T_pU$, for $W \in T_pM$ and $f \in C^\infty(U)$, let $r(W)f = W(\tilde{f})$.
For $f, g \in C^\infty(U)$, $r(W)(fg) = W(\widetilde{fg})$. Note that in a neighborhood of $p$, $\psi = 1$ so the maps $\tilde{f} \cdot \tilde{g}$ and $\widetilde{fg}$ are equal in a neighborhood of $p$, so by Corollary 3.2.11, $r(W)(fg) = W(\widetilde{fg}) = W(\tilde{f} \cdot \tilde{g}) = \tilde{f}(p)W(\tilde{g}) + \tilde{g}(p)W(\tilde{f}) = f(p)r(W)(g) + g(p)r(W)(f)$. So $r(W) \in T_pU$, for all $W \in T_pM$. So $W$ is well-defined, and $W$ is clearly linear.

Now for $V \in T_pU$ and $f \in C^\infty(U)$, $r \circ i(V)f = r(V(f|_U)) = V(\widetilde{f|_U}) = V(f)$ by Corollary 3.2.11 and the fact that $f$ and $\widetilde{f|_U}$ are equal in a neighborhood of $p$. Thus $r \circ i(V) = V$ for all $V \in T_pU$.
Similarly, $i \circ r(W) = W$ for all $W \in T_pM$. Therefore $i$ is an isomorphism of vector spaces, and so $T_pM \cong T_pU$. $\square$

*We will usually identify $T_pU$ with $i(T_pU) = T_pM$.*

**Proposition 3.2.14.** *Let $M$ be a smooth manifold of dimension $n$. Then for every $p \in M$, $T_pM$ is an $n$-dimensional vector space.*

*Proof.* Let $p \in M$ and let $(U, \varphi)$ be a chart such that $p \in U$. Define the map

$$\varphi_*\colon T_pU \to T_{\varphi(p)}\varphi(U)$$

by $\varphi_*(V)f = V(f \circ \varphi)$ for all $V \in T_pU$ and $f \in C^\infty(\varphi(U))$.

$\varphi_*$ is well defined and linear, and $\varphi_*$ is invertible, with

$$(\varphi_*)^{-1} = (\varphi^{-1})_*\colon T_{\varphi(p)}\varphi(U) \to T_pU$$

defined by $(\varphi^{-1})_*(W)f = W(f \circ \varphi^{-1})$ for all $W \in T_{\varphi(p)}\varphi(U)$ and $f \in C^\infty(U)$. Thus $\varphi_*$ is an isomorphism, and so $T_pU \cong T_{\varphi(p)}\varphi(U)$. Therefore $T_pM \cong T_{\varphi(p)}\mathbb{R}^n$ by Proposition 3.2.13. and so $T_pM$ is an $n$-dimensional vector space by Proposition 3.2.8. □

**Remark 3.2.15.** *Let $M$ be a smooth manifold of dimension $n$, $p \in M$, and $\varphi = (\tilde{x_1}, \ldots, \tilde{x_n})\colon U \to \mathbb{R}^n$ is a smooth coordinate chart in a neighborhood $U$ of $p$. The partial derivatives $\{\frac{\partial}{\partial x_1}|_{\varphi(p)}, \ldots, \frac{\partial}{\partial x_n}|_{\varphi(p)}\}$ is a basis for $T_{\varphi(p)}\mathbb{R}^n$. Now consider the linear isomorphism defined in the proof of Proposition 3.2.14:*

$$(\varphi^{-1})_*\colon T_{\varphi(p)}\varphi(U) \to T_pU$$

*by $(\varphi^{-1})_*(W)f = W(f\circ\varphi^{-1})$ for all $W \in T_{\varphi(p)}\varphi(U)$ and $f \in C^\infty(U)$. $\{\frac{\partial}{\partial x_1}|_{\varphi(p)}, \ldots, \frac{\partial}{\partial x_n}|_{\varphi(p)}\}$ is a basis for $T_{\varphi(p)}\varphi(U)$ (identified with $T_{\varphi(p)}\mathbb{R}^n$). For $i \in \{1, \ldots, n\}$, define*

$$\frac{\partial}{\partial \tilde{x}_i}|_p := (\varphi^{-1})_* \left(\frac{\partial}{\partial x_i}|_{\varphi(p)}\right).$$

*Then $\{\frac{\partial}{\partial \tilde{x}_1}|_p, \ldots, \frac{\partial}{\partial \tilde{x}_n}|_p\}$ is a basis for $T_pU$. For each $i \in \{1, \ldots, n\}$, we see that $\frac{\partial}{\partial \tilde{x}_i}|_p$ acts on a function $f \in C^\infty(U)$ by*

$$\frac{\partial}{\partial \tilde{x}_i}|_p f = (\varphi^{-1})_* \left(\frac{\partial}{\partial x_i}|_{\varphi(p)}\right) f = \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}(\varphi(p))$$

*so* $\frac{\partial}{\partial \tilde{x_i}}|_p$ *acts on* $\tilde{x_k} \in C^\infty(U)$ *by*

$$\frac{\partial}{\partial \tilde{x_i}}|_p \tilde{x_k} = \frac{\partial(\tilde{x_k} \circ \varphi^{-1})}{\partial x_i}(\varphi(p)) = \frac{\partial(x_k)}{\partial x_i}(\varphi(p)) = \delta_{ik}$$

*for all* $i, k \in \{1, \ldots, n\}$.

**Definition 3.2.16.** *Let* $M$ *and* $N$ *be two smooth manifolds,* $p \in M$*, and let* $F\colon M \to N$ *be a smooth map. We define the differential (or push-forward) of* $F$ *at* $p$*, denoted by* $F_*$ *or* $D_pF$*, by the linear map :* $F_*\colon T_pM \to T_pN$*,*

$$F_*(V)g = V(g \circ F)$$

*for all* $V \in T_pM$ *and* $g \in C^\infty(N)$*.*

*Throughout the thesis, we will use* $F_*$ *and* $D_pF$ *interchangeably to denote the same map.*

**Remark 3.2.17.** *(Calculating the Differential in Coordinates) Given* $M$ *and* $N$ *two smooth manifolds of dimensions* $m$ *and* $n$ *respectively,* $p \in M$*, and* $F\colon M \to N$ *a smooth map, there exist charts* $(U, \varphi)$ *and* $(W, \psi)$ *on* $M$ *and* $N$ *respectively, such that* $p \in U$*,* $F(U) \subseteq W$ *and* $\psi \circ F \circ \varphi^{-1}$ *is smooth. Set* $\varphi = (\tilde{x_1}, \ldots, \tilde{x_m})$ *and* $\psi = (\tilde{y_1}, \ldots, \tilde{y_n})$*. We call the map* $\tilde{F} = \psi \circ F \circ \varphi^{-1} = (\tilde{F}_1, \ldots, \tilde{F}_n)$ *the representative of* $F$ *in the given coordinate charts. We consider the differential of* $F$ *at* $p$*,* $D_pF\colon T_pM \to T_{F(p)}N$*.* $\{\frac{\partial}{\partial \tilde{x_1}}|_p, \ldots, \frac{\partial}{\partial \tilde{x_n}}|_p\}$ *is a basis for* $T_pM$ *so it is enough to find* $D_pF(\frac{\partial}{\partial \tilde{x_i}}|_p)$ *for* $i = 1, .., m$*.* $D_pF(\frac{\partial}{\partial \tilde{x_i}}|_p) \in T_{F(p)}N$*, so* $D_pF(\frac{\partial}{\partial \tilde{x_i}}|_p) = \sum_{j=1}^n a_{ij} \frac{\partial}{\partial \tilde{y_j}}|_{F(p)}$*.*
*Applying both sides to the maps* $\tilde{y_k} \in C^\infty(N)$*, for* $k = 1, ..., n$ *we get:* $a_{ik} = D_pF(\frac{\partial}{\partial \tilde{x_i}}|_p)(\tilde{y_k})$*.*

*Now by definition* $D_pF(\frac{\partial}{\partial \tilde{x_i}}|_p)(\tilde{y_k}) = \frac{\partial}{\partial \tilde{x_i}}|_p(\tilde{y_k} \circ F) = \frac{\partial}{\partial x_i}|_{\varphi(p)}(\tilde{y_k} \circ \tilde{F} \circ \varphi^{-1}) = \frac{\partial \tilde{F}_k}{\partial x_i}|_{\varphi(p)} = a_{ik}$*, and the matrix of* $D_pF$ *with respect to the coordinate bases is*

$$\begin{pmatrix} \frac{\partial \tilde{F}_1}{\partial x_1}|_x & \cdots & \frac{\partial \tilde{F}_1}{\partial x_m}|_x \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{F}_n}{\partial x_1}|_x & \cdots & \frac{\partial \tilde{F}_n}{\partial x_m}|_x \end{pmatrix}$$

*We call this matrix the Jacobian of* $F$ *in the given coordinate charts.*

## 3.3 The Tangent Bundle

**Definition 3.3.1.** *(Tangent Bundle). Let* $M$ *be a smooth manifold. We define the tangent bundle of* $M$*, denoted by* $TM$*, to be the disjoint union of the tangent spaces at all points of* $M$*:*

$$TM = \coprod_{p \in M} T_pM.$$

*The tangent bundle is equipped with the natural projection map* $\pi \colon TM \to M$ *which sends each vector in* $T_pM$ *to the point* $p$ *at which it is tangent.*

**Remark 3.3.2.** *Given a smooth manifold of dimension* $n$*, we wish to define a smooth structure on* $TM$ *to transform it into a smooth manifold. For that we start by defining a topology on* $TM$*. Consider the family* $\tau$*, of subsets of* $TM$*,* $\tau = \{\pi^{-1}(U) : U \text{ is open in } M\}$*. We will see that* $\tau$ *is a topology on* $TM$*,and* $(TM, \tau)$ *is a topological manifold. Moreover for every chart* $(U, \varphi)$ *of* $M$*,* $\varphi = (x_1, ..., x_n)$*, setting* $\tilde{U} = \pi^{-1}(U)$ *and defining* $\tilde{\varphi} \colon \tilde{U} \to \mathbb{R}^{2n}$ *by*

$$\tilde{\varphi}(p, \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}|_p) = (x_1(p), .., x_n(p), v_1, .., v_n)$$

*we get that* $\tilde{\varphi}(\tilde{U})$ *is open in* $\mathbb{R}^{2n}$ *and* $\tilde{\varphi} \colon \tilde{U} \to \tilde{\varphi}(\tilde{U})$ *is a homeomorphism.*

**Proposition 3.3.3.** *Let $(M, \mathcal{A})$ be a smooth manifold of dimension $n$. Then $(TM, \tau)$ is a topological manifold of dimension $2n$, and the collection of charts $\tilde{\mathcal{A}} = \{(\tilde{U}, \tilde{\varphi})\}_{(U,\varphi)\in\mathcal{A}}$ is a smooth atlas, hence defines a smooth structure for $TM$, where $\tau$ and $(\tilde{U}, \tilde{\varphi})$ are as defined in Remark 3.3.2.*

*Moreover the natural projection $\pi\colon TM \to M$ is smooth.*

*Proof.* See Proposition 3.18 of [Lee, 2013]. □

**Definition 3.3.4.** *(Global Differential). Let $M$ and $N$ be smooth manifolds, and let $F\colon M \to N$ be a smooth map. For every $p \in M$, consider the differential of $F$ at $p$, $D_pF\colon T_pM \to TN$ ($T_{F(p)}N$ viewed as a subset of $TN$). Then by the universal property of the coproduct, there exists a unique map $DF\colon TM \to TN$ whose restriction to each tangent space $T_pM$ is $D_pF$. The map $DF$ is called the global differential of $F$.*

**Proposition 3.3.5.** *Let $M$ and $N$ be smooth manifolds of dimensions $m$ and $n$ respectively, and let $F\colon M \to N$ be a smooth map. Then $DF\colon TM \to TN$ is smooth.*

*Proof.* Let $v \in TM$, we express $DF$ with respect to local coordinate charts of $TM$ in a neighborhood of $v$ and of $TN$ in a neighborhood of $DF(v)$.

$DF(x_1, .., x_m, v_1, .., v_m) = (F_1(x), .., F_n(x), \sum_{i=1}^m \frac{\partial F_1}{\partial x_i}(x)v_i, ..., \sum_{i=1}^m \frac{\partial F_n}{\partial x_i}(x)v_i)$.

The latter is smooth since $F$ is smooth. □

## 3.4 Smooth Vector Fields

**Definition 3.4.1.** *Let $M$ be a smooth manifold. A vector field is a continuous map $X\colon M \to TM$ with the property that $\pi \circ X = Id_M$.*

**Definition 3.4.2.** *(Smooth Vector Fields) Let $M$ be a smooth manifold and $X\colon M \to TM$ be a vector field. $X$ is said to be a smooth vector field if $X$ is smooth map (between the corresponding manifolds).*

**Proposition 3.4.3.** *Let $M$ be a smooth manifold, $p \in M$, and $v \in T_p(M)$, then there exists a smooth vector field $X$ on $M$, such that $X(p) = v$. We call $X$ an extension of $v$.*

*Proof.* See Proposition 8.7 of [Lee, 2013]. □

**Proposition 3.4.4.** *Let $M$ be a smooth manifold of dimension $m$ and let $X\colon M \to TM$ be a vector field. Let $(U, (x_1, ..., x_m))$ be a coordinate chart. Then the restriction of $X$ to $U$ is smooth if and only if its component functions with respect to this chart are smooth.*

*Proof.* Let $(x_1, .., x_m, v_1, .., v_m)$ be the natural coordinates on $\pi^{-1}(U)$ associated with the chart $(U, (x_1, ..., x_m))$.

$$X(x_1, .., x_m) = (x_1, .., x_m, X_1(x), .., X_m(x))$$

where $X_i$ denotes the $i$th component function of $X$ in $(x_1, ..., x_m)$ coordinates. It follows directly that smoothness of $X$ in $U$ is equivalent to smoothness of its component functions. □

We will use the notation $\mathfrak{X}(M)$ to denote the set of all smooth vector fields on $M$. It is a vector space under point-wise addition and scalar multiplication:

$$(aX + bY)_p = aX_p + bY_p.$$

In addition, smooth vector fields can be multiplied by smooth real-valued functions: if $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, we define $fX \colon M \to TM$ by:

$$(fX)_p = f(p)X_p.$$

By expressing $fX$ in some coordinate charts and by Proposition 3.4.4, it is clear that $fX$ is a smooth vector field whenever $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$. Thus the above operation makes $\mathfrak{X}(M)$ a module over the ring $C^\infty(M)$.

**Proposition 3.4.5.** *Let $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$. Consider the function $Xf \colon M \to \mathbb{R}$, defined by*

$$(Xf)(p) = X_p f.$$

*Then $Xf$ is smooth.*

*Proof.* For $p \in M$ we can choose smooth coordinates $(x_1, .., x_m)$ on a neighborhood $U$ of $p$, then for $x \in U$ we can write

$$Xf(x) = \left( \sum_{i=1}^{m} X_i(x) \frac{\partial}{\partial x_i}|_x \right) f = \sum_{i=1}^{m} X_i(x) \frac{\partial f}{\partial x_i}(x).$$

Since the component functions $X_i$ are smooth on $U$ by Proposition 3.4.4, it follows that $Xf$ is smooth in $U$. Since this is true for all $p \in M$ and $U$ a neighborhood of $p$, $Xf$ is smooth on $M$. □

As a consequence of the preceding proposition, every smooth vector field $X \in \mathfrak{X}(M)$ defines a map $X \colon C^\infty(M) \to C^\infty(M)$ by $f \longmapsto Xf$. This map is clearly linear and $X(fg) = fXg + gXf$ for all $f, g \in C^\infty(M)$.

**Definition 3.4.6.** *(Lie bracket). Let $M$ be a smooth manifold. For $X, Y \in \mathfrak{X}(M)$, we define the Lie bracket of $X$ and $Y$ to be the operator $[X, Y] \colon C^\infty(M) \to$*

$C^\infty(M)$

$$[X, Y]f = X(Yf) - Y(Xf)$$

**Proposition 3.4.7.** *Let $M$ be a smooth manifold, then the Lie bracket of two smooth vector fields is a smooth vector field.*

*Proof.* See Lemma 8.25 of [Lee, 2013]. □

**Proposition 3.4.8.** *Let $M$ be a smooth manifold, then the Lie bracket satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$ :*

1. *Bilinearity: For $a, b \in \mathbb{R}$,*

$$[aX + bY, Z] = a[X, Z] + b[Y, Z]$$

$$[Z, aX + bY] = a[Z, X] + b[Z, Y]$$

2. *Antisymmetry:*

$$[X, X] = 0$$

3. *Jacobi Identity:*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

4. *For all $f, g \in C^\infty(M)$,*

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X$$

*Proof.* See Proposition 8.28 of [Lee, 2013]. □

**Definition 3.4.9.** *Let $M$ be a smooth manifold, and $I \subseteq \mathbb{R}$ be an open interval. A smooth map $\gamma\colon I \to M$ is called a smooth curve.*

If $\gamma\colon I \to M$ is a smooth curve, it is seen that $D\gamma\colon TI \to TM$ is smooth. In what follows we will simply write $\gamma'(t)$ instead of $D_t\gamma$. Note that $\forall t \in I$, $\gamma'(t) \in T_{\gamma(t)}M$.

**Definition 3.4.10.** *Let $M$ be a smooth manifold, and $V \in \mathfrak{X}(M)$. A smooth curve $\gamma\colon I \to M$ is said to be an integral curve of $V$ if*

$$\gamma'(t) = V_{\gamma(t)}, \forall t \in I.$$

**Definition 3.4.11.** *(Flow). Let $M$ be a smooth manifold. A* ***flow domain*** *for $M$ is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$ the set $\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t, p) \in \mathcal{D}\}$ is an open interval containing $0$. A* ***flow*** *on $M$ is a continuous map $\theta\colon \mathcal{D} \to M$, where $\mathcal{D} \subseteq \mathbb{R} \times M$ is a flow domain, that satisfies the following group laws: $\forall p \in M$,*

$$\theta(0, p) = p$$

*and for all $s \in \mathcal{D}^{(p)}$ and $t \in \mathcal{D}^{\theta(s,p)}$ such that $s + t \in \mathcal{D}^{(p)}$,*

$$\theta(t, \theta(s, p)) = \theta(t + s, p).$$

*If $\theta$ is smooth, the* ***infinitesimal generator of $\boldsymbol{\theta}$*** *at $p$ is defined by*

$$V_p = \frac{\partial \theta}{\partial t}(0, p).$$

*If $\forall p \in M$, $\mathcal{D}^{(p)} = \mathbb{R}$, $\theta$ is said to be a* ***global flow*** *on $M$ (also called a*

***one-parameter group action***).

**Proposition 3.4.12.** *Let $M$ be a smooth manifold and $\theta\colon \mathcal{D} \to M$ be a smooth flow, then the infinitesimal generator $V$ of $\theta$ is a smooth vector field, and each curve $\theta(\cdot, p)\colon \mathcal{D}^{(p)} \to M$ is an integral curve of $V$.*

*Proof.* See Proposition 9.11 of [Lee, 2013]. □

**Definition 3.4.13.** *A **maximal integral curve** is an integral curve that cannot be extended to an integral curve on any larger open interval, and a **maximal flow** is a flow that admits no extension to a flow on any larger flow domain.*

**Theorem 3.4.14.** (*Fundamental Theorem on Flows*) *Let $M$ be a smooth manifold and $V$ be a smooth vector field. Then there exists a unique smooth maximal flow $\theta\colon \mathcal{D} \to M$ whose infinitesimal generator is $V$. This flow has the following properties:*

1. *For each $p \in M$, the curve $\theta(\cdot, p)\colon \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of $V$ satisfying $\gamma(0) = p$.*
2. *If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))}$ is the interval $\mathcal{D}^{(p)} - s$.*
3. *For each $t \in \mathbb{R}$, the set $M_t = \{p \in M : (t, p) \in \mathcal{D}\}$ is open and $\theta(t, \cdot)\colon M_t \to M_{-t}$ is a diffeomorphism with inverse $\theta(-t, \cdot)$.*

*Proof.* See Theorem 9.12 of [Lee, 2013]. □

**Remark 3.4.15.** *We call the unique smooth maximal flow whose infinitesimal generator is $V$, the flow generated by $V$. We say a vector field is **complete** if it generates a global flow.*

**Theorem 3.4.16.** *Let $M$ be a smooth manifold. Then every compactly supported smooth vector field on $M$ is complete.*

*Proof.* See Theorem 9.16 of [Lee, 2013]. $\square$

## 3.5 Embedded Submanifolds

**Definition 3.5.1.** *Let $M$ and $N$ be two smooth manifolds, $p \in M$ and $F\colon M \to N$ be a smooth map. We define the* ***rank of $F$*** *at $p$ to be the rank of the linear map $D_pF\colon T_pM \to T_{F(p)}N$. If $F$ has the same rank $r$ at every point, we say that it has constant rank and we write $\operatorname{rank} F = r$.*

**Remark 3.5.2.** *Note that a rank of a smooth map $F\colon M \to N$ is at each point is bounded above by the minimum of $\{\dim M, \dim N\}$. If the rank of $F$ is equal to this upper bound at some point $p \in M$, we say that $F$ has a full rank at $p$, and if $F$ has full rank everywhere, we say that $F$ has full rank.*

**Definition 3.5.3.** *(Submersions and Immersions). Let $M$ and $N$ be two smooth manifolds, $F\colon M \to N$ be a smooth map. $F$ is called a smooth submersion if its differential is surjective at each point, equivalently if $F$ has a constant rank and $\operatorname{rank} F = \dim N$. $F$ is called a smooth immersion if its differential is injective at each point, equivalently if $F$ has a constant rank and $\operatorname{rank} F = \dim M$.*

**Proposition 3.5.4.** *Let $M$ and $N$ be two smooth manifolds of dimensions $m$ and $n$ respectively, $p \in M$ and $F\colon M \to N$ be a smooth map. If $D_pF$ is surjective, then there exists a neighborhood $U$ of $p$ such that $F|_U$ is a submersion. If $D_pF$ is injective, then there exists a neighborhood $U$ of $p$ such that $F|_U$ is an immersion.*

*Proof.* Assume that $D_pF$ is surjective. Choose any smooth coordinates for $M$ near $p$ and $N$ near $F(p)$ and consider the Jacobian matrix $J$ of $F$ in these coordinates. The rank of the Jacobian matrix at $p$ is $n$, then $J$ has an $n \times n$ submatrix $A$ of nonzero determinant at $p$. $\det(A)$ is continuous near $p$, then there exists a neighborhood $U$ of $p$ such that $\det(A) \neq 0$ on $U$. Thus the rank of $J$ is $n$ in $U$, so that the rank of $F$ is $n$ in $U$. Therefore $F|_U$ is a submersion. We use the exact same argument to prove that if $D_pF$ is injective, then there exists a neighborhood $U$ of $p$ such that $F|_U$ is an immersion. $\square$

**Remark 3.5.5.** *Let $M$ be a smooth manifold and $p \in M$. A smooth chart $(U, \varphi)$ is said to be centered at $p$ if $p \in U$ and $\varphi(p) = 0$. It is always possible to choose a smooth chart centered at $p$ by starting from any chart $(U, \varphi)$ containing $p$, and defining the map $\tilde{\varphi}$ on $U$ by $\tilde{\varphi}(x) = \varphi(x) - \varphi(p)$ for all $x \in U$. $(U, \tilde{\varphi})$ is smoothly compatible with all the coordinate charts on $M$ and $(U, \tilde{\varphi})$ is centered at $p$.*

**Theorem 3.5.6.** *(Inverse Function Theorem for Smooth Manifolds). Let $M$ and $N$ be two smooth manifolds, and $F \colon M \to N$ be a smooth map. If for some $p \in M$, $D_pF$ is invertible, then there exist connected neighborhoods $U_0$ of $p$ and $V_0$ of $F(p)$ such that $F|_{U_0} \colon U_0 \to V_0$ is a diffeomorphism.*

*Proof.* Since $D_pF$ is invertible then $T_pM$ and $T_{F(p)}N$ have the same dimension, so $M$ and $N$ have the same dimension, say $n$. Choose smooth charts $(U, \varphi)$ centered at $p$ and $(V, \psi)$ centered at $F(p)$, with $F(U) \subseteq V$. Then $\tilde{F} = \psi \circ F \circ \varphi^{-1}$ is a smooth map from an open set $\tilde{U} = \varphi(U) \subseteq \mathbb{R}^n$ into $\tilde{V} = \varphi(V) \subseteq \mathbb{R}^n$, with $\tilde{F}(0) = 0$. Now $D_0\tilde{F} = D_{F(p)}\psi \circ D_pF \circ D_0(\varphi^{-1})$, so $D_0\tilde{F}$ is invertible, thus by the inverse function theorem there exist open connected subsets $\tilde{U_0} \subseteq \tilde{U}$ and $\tilde{V_0} \subseteq \tilde{V}$ containing 0 such that $\tilde{F}|_{\tilde{U_0}} \colon \tilde{U_0} \to \tilde{V_0}$ is a diffeomorphism. Then $U_0 = \varphi^{-1}(\tilde{U_0})$ and $V_0 = \varphi^{-1}(\tilde{V_0})$ are connected neighborhoods of $p$ and $F(p)$, respectively, and it follows from composition that $F|_{U_0}$ is a diffeomorphism from $U_0$ to $V_0$. $\square$

**Theorem 3.5.7.** *(Rank Theorem). Let $M$ and $N$ be smooth manifolds of dimensions $m$ and $n$ respectively, and $F \colon M \to N$ be a smooth map with constant rank $r$. For each $p \in M$ there exist smooth charts $(U, \varphi)$ for $M$ centered at $p$ and*

*$(V, \psi)$ for $N$ centered at $F(p)$ such that $F(U) \subseteq V$, in which $F$ has a coordinate representation of the form*

$$\tilde{F}(x_1, ..., x_r, x_{r+1}, ..., x_m) = (x_1, ..., x_r, 0, ..., 0)$$

*In particular, if $F$ is a smooth submersion this become*

$$\tilde{F}(x_1, ..., x_n, x_{n+1}, ..., x_m) = (x_1, ..., x_n)$$

*and if $F$ is a smooth immersion it becomes*

$$\tilde{F}(x_1, ..., x_m) = (x_1, ..., x_m, 0, ..., 0).$$

*Proof.* See Theorem 4.12 of [Lee, 2013]. □

**Theorem 3.5.8.** *(Global Rank Theorem). Let $M$ and $N$ be smooth manifolds, let $F: M \to N$ be a smooth map with constant rank.*

1. *If $F$ is surjective, then it is a smooth submersion.*
2. *If $F$ is injective, then it is a smooth immersion.*
3. *If $F$ is bijective, then it is a diffeomorphism.*

*Proof.* See Theorem 4.14 of [Lee, 2013]. □

**Definition 3.5.9.** *(Embeddings). Let $M$ and $N$ be two smooth manifolds. We say that a map $F: M \to N$ is a smooth embedding of $M$ into $N$ if $F$ is a smooth immersion and a topological embedding, i.e., a homeomorphism onto its image $F(M) \subseteq N$.*

**Definition 3.5.10.** *(Embedded Submanifold). Let $M$ be a smooth manifolds. A subset $S \subseteq M$ is said to be an embedded submanifold of $M$ if $S$ is a topological manifold in the subspace topology, endowed with a smooth structure that makes the inclusion map $\iota \colon S \hookrightarrow M$ a smooth embedding.*

If $S$ is an embedded submanifold of $M$, $\dim S < \dim M$ since the inclusion map is a smooth embedding. The difference $\dim M$-$\dim S$ is called the codimension of $S$ in $M$.

**Remark 3.5.11.** *Let $M$ be a smooth manifold, and $S \subseteq M$ be an embedded submanifold, and let $p \in S$. The inclusion $\iota \colon S \hookrightarrow M$ induces an injective linear map:*

$$D_p\iota \colon T_pS \hookrightarrow T_pM$$

*So that $D_p\iota(T_pS) \cong T_pS$.*
*We will identify $T_pS$ with the subspace $D_p\iota(T_pS) \subseteq T_pM$.*

**Theorem 3.5.12.** *Let $M$ be a smooth manifold of dimension $n$ and $S \subseteq M$. $S$ is an embedded submanifold of $M$ of dimension $k$ if and only if for all $p \in S$ there exists a neighborhood $U$ of $p$ in $M$ and a diffeomorphism $\varphi \colon U \to \varphi(U)$ satisfying:*

$$\varphi(U \cap S) = \varphi(U) \cap \{(x_1, ..., x_k, 0, ..., 0) : (x_1, ..., x_k) \in \mathbb{R}^k\}$$

*Proof.* See Theorem 5.8 of [Lee, 2013]. □

**Proposition 3.5.13.** *Let $M$ be a smooth manifold of dimension $n$ and $S \subseteq M$. $S$ is an embedded submanifold of $M$ of dimension $k$ if and only if for all $p \in S$ there exists a neighborhood $U$ of $p$ in $M$ and a smooth submersion $F \colon U \to \mathbb{R}^{n-k}$*

*such that $U \cap S = F^{-1}(\{0\})$. We call such submersions local defining maps for $S$.*

*Proof.* See Proposition 5.16 of [Lee, 2013]. $\square$

**Proposition 3.5.14.** *Let $M$ be a smooth manifold of dimension $n$, and $S \subseteq M$ be an embedded submanifold of dimension $k$, and let $p \in S$. If $F\colon U \to \mathbb{R}^{n-k}$ is a local defining map for $S$ in some neighborhood $U$ of $p$, then*

$$T_pS = \ker D_pF$$

*where $D_pF\colon T_pM \to T_{F(p)}\mathbb{R}^{n-k}$ is the differential of $F$ at $p$, and $T_pS$ is identified with the subspace $D_p\iota(T_pS) \subseteq T_pM$ by Remark 3.5.11.*

*Proof.* See Proposition 5.38 in [Lee, 2013]. $\square$

**Theorem 3.5.15.** *(Whitney Embedding Theorem) Every smooth manifold of dimension $n$ admits a smooth embedding into $\mathbb{R}^{2n}$.*

*Proof.* See Theorem 5 of [Whitney, 1944]. $\square$

## 3.6 The Cotangent Bundle

**Definition 3.6.1.** *(The Cotangent Space) Let $M$ be a smooth manifold and $p \in M$. We define the cotangent space at $p$, denoted by $T_p^*M$, to be the dual space to $T_pM$:*

$$T_p^*M = (T_pM)^*$$

*Elements of $T_p^*M$ are called tangent covectors at $p$.*

*If $(x_1, \ldots, x_n)$ is a smooth coordinate chart containing $p$, then $\left(\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\right)$*

*is a basis for $T_pM$. This coordinate basis gives rise to a dual basis for $T_p^*M$, denoted by $(dx_1|_p, \ldots, dx_n|_p)$, such that*

$$dx_i|_p \left( \frac{\partial}{\partial x_j}|_p \right) = \delta_{ij} \qquad \forall i, j \in \{1, \ldots, n\}$$

**Definition 3.6.2.** *(The Cotangent Bundle) Let $M$ be a smooth manifold. We define the cotangent bundle of $M$, denoted by $T^*M$, to be the coproduct:*

$$T^*M = \coprod_{p \in M} T_p^*M$$

*with the natural projection map $\pi \colon T^*M \to M$ that maps a covector $\omega \in T_p^*M$ to $p$.*

**Remark 3.6.3.** *Let $M$ be a smooth manifold of dimension $n$. Following the same construction of Remark 3.3.2, we can define a smooth structure on $T^*M$, which turns $T^*M$ into a smooth manifold, on which $\pi \colon T^*M \to M$ is a smooth map. Given a smooth chart $(U, \varphi)$ for $M$, with $\varphi = (x_1, \ldots, x_n)$, we obtain a smooth chart $(\tilde{U}, \tilde{\varphi})$ for $T^*M$, with $\tilde{U} = \pi^{-1}(U)$ and $\tilde{\varphi} \colon \tilde{U} \to \mathbb{R}^{2n}$ defined by:*

$$\varphi(p, \sum_{i=1}^{n} \xi_i dx_i|_p) = (x_1(p), \ldots, x_n(p), \xi_1, \ldots, \xi_n)$$

*See Proposition* 11.9 *of [Lee, 2013] for more details.*

**Definition 3.6.4.** *(Covector Field) Let $M$ be a smooth manifold. A covector field is a continuous map $\omega \colon M \to T^*M$ such that $\pi \circ \omega = Id_M$. If moreover $\omega \colon M \to T^*M$ is a smooth map (between the corresponding manifolds), we say that $\omega$ is a smooth covector field.*

**Proposition 3.6.5.** *Let $M$ be a smooth manifold and $\omega \colon M \to T^*M$ be a covector* *field, then the following are equivalent:*

1. *$\omega$ is smooth.*

2. *In every coordinate chart, the component functions of $\omega$ are smooth.*

3. *Each point of $M$ is contained in some coordinate chart in which $\omega$ has smooth component functions.*

4. *For every smooth vector field $X \in \mathfrak{X}(M)$, the function $\omega(X)\colon M \to \mathbb{R}$ is smooth on $M$.*

5. *For every open subset $U \subseteq M$ and every smooth vector field $X$ on $U$, the function $\omega(X)\colon U \to \mathbb{R}$ is smooth on $U$.*

*Proof.* See Proposition 11.11 of [Lee, 2013]. □

We will use the notation $\mathfrak{X}^*(M)$ to denote the set of all smooth covector fields on $M$. It is a vector space under point-wise addition and scalar multiplication. In addition, smooth covector fields can be multiplied by smooth real-valued functions: if $f \in C^\infty(M)$ and $\omega \in \mathfrak{X}^*(M)$, we define $f\omega\colon M \to T^*M$ by:

$$(f\omega)_p = f(p)\omega_p.$$

The above operation makes $\mathfrak{X}^*(M)$ a module over the ring $C^\infty(M)$.

**Definition 3.6.6.** *Let $M$ be a smooth manifold and $f \in C^\infty(M)$. We define the differential of $f$ at $p$, $df_p\colon T_pM \to \mathbb{R}$, by*

$$df_p(v) = vf \qquad \forall v \in T_pM$$

*and we define the differential map of f, $df\colon M \to T^*M$ to be the unique map such that*

$$df_p(v) = vf \qquad \forall p \in M \qquad and \;\; v \in T_pM$$

**Proposition 3.6.7.** *Let $M$ be a smooth manifold and $f \in C^\infty(M)$. Then $df\colon M \to T^*M$ is a smooth covector field.*

*Proof.* For every $p \in M$, $df_p \in T_p^*M$, $df$ is a covector field. Now for every smooth vector field $X \in \mathfrak{X}(M)$, $df(X) = Xf\colon M \to \mathbb{R}$ is smooth on $M$, hence $df$ is a smooth covector field by Proposition 3.6.5. □

**Remark 3.6.8.** *(Calculating the Differential in Coordinates) Let $M$ be a smooth manifold of dimension $n$, and $p \in M$, and let $(U, \varphi)$ be a smooth coordinate chart containing $p$, with $\varphi = (x_1, \ldots, x_n)$. For $f \in C^\infty(M)$, $df_p \in T_p^*M$, so expressing $df_p$ in the basis $(dx_1|_p, \ldots, dx_n|_p)$ of $T_p^*M$ we get*

$$df_p = \sum_{i=1}^{n} a_i(p) dx_i|_p$$

*for some maps $a_i\colon U \to \mathbb{R}$. Applying both side to the coordinate basis of $T_pM$, $\frac{\partial}{\partial x_k}$, for $k \in \{1, \ldots, n\}$ we get*

$$a_k = df_p\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial f}{\partial x_k}(p)$$

*so that $df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_k}(p) dx_i|_p$.*

## 3.7 Riemannian Metrics

**Definition 3.7.1.** *Let $M$ be a smooth manifold and $p \in M$. Consider the space of bilinear forms on $T_pM$, $T_p^*M \otimes T_p^*M$. We define the bundle of covariant 2-tensors* *on $M$ by*

$$T^2T^*M = \coprod_{p\in M}(T_p^*M \otimes T_p^*M)$$

*with the natural projection map $\pi\colon T^2T^*M \to M$ that maps a bilinear form $L \in T_p^*M \otimes T_p^*M$ to $p$.*

**Remark 3.7.2.** *Let $M$ be a smooth manifold of dimension $n$, $p \in M$, and $(U, \varphi)$ be a smooth coordinate chart containing $p$, with $\varphi = (x_1, \dots, x_n)$. A basis of $T_p^*M \otimes T_p^*M$ is given by*

$$\{dx_i|_p \otimes dx_j|_p \colon 1 \leq i \leq n,\ 1 \leq j \leq n\}$$

**Remark 3.7.3.** *Let $M$ be a smooth manifold of dimension $n$. Following the same construction of Remark 3.3.2, we can define a smooth structure on $T^*M$, which turns $T^2T^*M$ into a smooth manifold, on which $\pi\colon T^2T^*M \to M$ is a smooth map. Given a smooth chart $(U, \varphi)$ for $M$, with $\varphi = (x_1, \dots, x_n)$, we obtain a smooth chart $(\tilde{U}, \tilde{\varphi})$ for $T^2T^*M$, with $\tilde{U} = \pi^{-1}(U)$ and $\tilde{\varphi}\colon \tilde{U} \to \mathbb{R}^n \times \mathbb{R}^{n^2}$ defined by:*

$$\tilde{\varphi}(p, \sum_{i,j=1}^{n} \mu_{i,j} dx_i|_p \otimes dx_j|_p) = (x_1(p), \dots, x_n(p), \mu_{1,1}, \mu_{1,2}, \dots, \mu_{n,1}, \dots, \mu_{n,n})$$

*See the definition of (smooth) vector bundles, Chapter 10 in [Lee, 2013], for a general approach, and refer to the proof of Proposition 3.3.3.*

**Remark 3.7.4.** *Let $M$ be a smooth manifold. A* ***covariant 2-tensor field*** *is a continuous map $g\colon M \to T^2T^*M$ satisfying $\pi \circ g = Id_M$. Moreover if $g$ is a smooth map, we say that $g$ is a smooth covariant 2-tensor field.*

**Proposition 3.7.5.** *Let $M$ be a smooth manifold and let $g\colon M \to T^2T^*M$ be a* *covariant 2-tensor field. Then the following are equivalent.*

1. *$g$ is smooth.*

2. *In every smooth coordinate chart, the component functions of $g$ are smooth.*

3. *Each point of $M$ is contained in some coordinate chart in which $g$ has smooth component functions.*

4. *For every smooth vector fields $X, Y \in \mathfrak{X}(M)$, the map $g(X,Y)\colon M \to \mathbb{R}$ , defined by*
$$g(X,Y)(p) = g_p(X_p, Y_p)$$
*is smooth.*

5. *For every $U \subseteq M$ and $X, Y \in \mathfrak{X}(U)$, $g(X,Y)\colon U \to \mathbb{R}$ is smooth.*

*Proof.* See Proposition 12.19 of [Lee, 2013]. □

**Definition 3.7.6.** (*Riemannian Metric*) *Let $M$ be a smooth manifold. A Riemannian metric on $M$ is a smooth symmetric covariant* 2*-tensor field on $M$ that is positive definite at each point; i.e. a Riemannian metric is a smooth map $g\colon M \to T^2T^*M$ such that for every $p \in M$, $g_p \in \Sigma^2(T_pM)$, and $g_p(v,v) > 0$ $\forall v \in T_pM \setminus \{0\}$.*
*For each $p \in M$, $g_p$ is an inner product on $T_pM$, so we often use the notation $\langle v, w\rangle_g$ to denote the real number $g_p(v,w)$ for $v, w \in T_pM$.*

**Definition 3.7.7.** *(Riemannian Manifold) A Riemannian manifold is a pair $(M, g)$, where $M$ is a smooth manifold and $g$ is a Riemannian metric on $M$.*

**Remark 3.7.8.** *Let $M$ be a smooth manifold of dimension $n$, $g$ be a Riemannian metric on $M$, $p \in M$, and $(U, \varphi)$ be a smooth coordinate chart containing $p$, with*

*$\varphi = (x_1, \ldots, x_n)$. In the given coordinate chart, $g$ can be written as*

$$g = \sum_{i,j=1}^{n} g_{ij} dx_i \otimes dx_j$$

*where $g_{ij} \in C^\infty(U)$ for every $i, j \in \{1, \ldots, n\}$, and the matrix $(g_{ij}(x))$ is a symmetric positive definite matrix for every point $x \in U$. Since $g_{ij} = g_{ji}$ for all $i, j \in \{1, \ldots, n\}$, we can write $g$ in terms of the symmetric products:*

$$\begin{aligned}
g &= \sum_{i,j=1}^{n} g_{ij} dx_i \otimes dx_j \\
&= \frac{1}{2} \sum_{i,j=1}^{n} (g_{ij} dx_i \otimes dx_j + g_{ji} dx_i \otimes dx_j) \\
&= \frac{1}{2} \sum_{i,j=1}^{n} (g_{ij} dx_i \otimes dx_j + g_{ij} dx_j \otimes dx_i) \\
&= \sum_{i,j=1}^{n} g_{ij} \frac{1}{2} (dx_i \otimes dx_j + dx_j \otimes dx_i) \\
&= \sum_{i,j=1}^{n} g_{ij} dx_i dx_j
\end{aligned}$$

**Proposition 3.7.9.** *Every smooth manifold admits a Riemannian metric.*

*Proof.* See Proposition 13.3 of [Lee, 2013]. □

**Definition 3.7.10.** *Let $(M, g)$ be a Riemannian manifold, and $p \in M$. We define the norm of a tangent vector $v \in T_pM$ by $|v|_g = \langle v, v \rangle_g^{\frac{1}{2}}$.*

**Definition 3.7.11.** *Let $(M, g)$ be a Riemannian manifold, and $p \in M$. We say that two tangent vectors $v, w \in T_pM$ are orthogonal if $\langle v, w \rangle_g = 0$.*

**Definition 3.7.12.** *Let $(M, g)$ be a Riemannian manifold, and $\gamma \colon [a, b] \to M$ be* *a piecewise smooth curve. We define the length of $\gamma$ to be*

$$L_g(\gamma) = \int_a^b |\gamma'(t)|_g dt.$$

**Proposition 3.7.13.** *Let $(M, g)$ be a Riemannian manifold, and $\gamma\colon [a, b] \to M$ be a piecewise smooth curve. If $\tilde{\gamma}$ is a reparameterization of $\gamma$, then $L_g(\gamma) = L_g(\tilde{\gamma})$.*

*Proof.* See Proposition 13.25 in [Lee, 2013]. □

**Definition 3.7.14.** *Let $(M, g)$ be a connected Riemannian manifold. For $p, q \in M$, denote by $S(p, q)$ the set of all piecewise smooth curves on $M$ whose start point is $p$ and end point is $q$. We define the Riemannian distance function $d_g\colon M \times M \to \mathbb{R}$ by*

$$d_g(p, q) = \inf_{\gamma \in S(p,q)} L_g(\gamma)$$

*We will see in the following theorem that the above function is in fact a distance function on $M$.*

**Theorem 3.7.15.** *Let $(M, g)$ be a connected Riemannian manifold. Then $M$ is a metric space with the Riemannian distance function, and the metric topology is the same as the original manifold topology.*

*Proof.* See Theorem 13.29 in [Lee, 2013]. □

**Corollary 3.7.16.** *Every smooth manifold is metrizable.*

*Proof.* Let $M$ be a smooth manifold. By Proposition 3.7.9, $M$ admits a Riemannian metric, say $g$. If $M$ is connected then $M$ is metrizable by Theorem 3.7.15. For the general case, let $\{M_i\}$ be the connected components of $M$, and choose a point $p_i \in M_i$ for each $i$. For $x \in M_i$ and $y \in M_j$, we define the distance between $x$ and $y$, $d(x, y)$, as follows:

- If $i = j$, let $d(x, y) = d_g(x, y)$ be the Riemannian distance function defined on the connected component $M_i$.
- If $i \neq j$, let

$$d(x, y) = d_g(x, p_i) + 1 + d_g(p_j, y)$$

It follows directly that $d\colon M \times M \to \mathbb{R}$ is a distance function, and hence $M$ is metrizable. $\square$

**Remark 3.7.17.** *The above corollary can be proved alternatively by using the Whitney embedding theorem, see Theorem 3.5.15. Given $M$ a smooth manifold of dimension $n$, there exists a smooth embedding $i\colon M \to \mathbb{R}^{2n}$. By restricting the metric of $\mathbb{R}^{2n}$ to $i(M)$, we get that $i(M)$ is metrizable, thus so is $M$.*

**Proposition 3.7.18.** *Let $(M, g)$ be a Riemannian manifold. We define the map $\hat{g}\colon TM \to T^*M$ as follows. For each $p \in M$ and $v \in T_pM$ we let $\hat{g}(v) \in T_p^*M$ be the covector defined by*

$$\hat{g}(v)(w) = g_p(v, w) \quad \forall\, w \in T_pM.$$

*Then $\hat{g}$ is a diffeomorphism, and for all $p \in M$, $\hat{g}(v)\colon T_pM \to T_p^*M$ is an isomorphism.*

*Proof.* See Chapter 13 in [Lee, 2013]. In fact $\hat{g}$ is a smooth bundle isomorphism, see Chapter 10 of [Lee, 2013] for the definition. $\square$

**Remark 3.7.19.** *Let $(M, g)$ be a Riemannian manifold. Let $p \in M$, and let $(U, \varphi)$ be a smooth coordinate chart containing $p$, with $\varphi = (x_1, \dots, x_n)$. Then we can write $g = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j$ where $g_{ij} \in C^\infty(U)$ for every $i, j \in \{1, \dots, n\}$.*

*For* $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$, $Y = \sum_{i=1}^n Y_i \frac{\partial}{\partial x_i} \in \mathfrak{X}(U)$, *we have*

$$\hat{g}(X)(Y) = \sum_{i,j=1}^n g_{ij} X_i Y_j$$

*so the covector field* $\hat{g}(X)$ *has the coordinate expression*

$$\hat{g}(X) = \sum_{i,j=1}^n g_{ij} X_i dx_j$$

*So for every* $q \in U$, *the matrix of the linear isomorphism* $\hat{g}\colon T_qM \to T_q^*M$ *is the transpose of the matrix of* $g$, *but because* $(g_{ij}(q))$ *is symmetric, we conclude that* $(g_{ij}(q))$ *is the matrix of* $\hat{g}$ *relative to the basis* $(\frac{\partial}{\partial x_1}|_q, \ldots, \frac{\partial}{\partial x_n}|_q)$ *of* $T_qM$ *and* $(dx_1|_q, \ldots, dx_n|_q)$ *of* $T_q^*M$. *This shows that* $(g_{ij}(q))$ *is invertible. Letting* $(h_{ij}(q))$ *be the inverse of* $(g_{ij}(q))$ *at every point* $q \in M$ *(* $h_{ij}$ *are smooth maps defined on* $U$, *by the smoothness of* $\hat{g}^{-1}$ *) the map* $\hat{g}^{-1}\colon T_q^*M \to T_qM$ *has* $(h_{ij}(q))$ *as a matrix representation in the given basis, i.e. for every* $\omega = \sum_{i=1}^n \omega_i dx_i \in \mathfrak{X}^*(U)$,

$$\hat{g}^{-1}(\omega) = \sum_{i,j=1}^n h_{ij}\omega_j \frac{\partial}{\partial x_i}.$$

**Definition 3.7.20.** *(The Gradient) Let* $(M, g)$ *be a Riemannian manifold, and* $f \in C^\infty(M)$, *we define the* ***gradient of f*** *to be the vector field*

$$\operatorname{grad} f = \hat{g}^{-1}(df).$$

**Remark 3.7.21.** *Let* $(M, g)$ *be a Riemannian manifold, and* $f \in C^\infty(M)$. *For* $X \in \mathfrak{X}(M)$ *we have*

$$\langle \operatorname{grad} f, X\rangle_g = \hat{g}(\operatorname{grad} f)(X) = df(X) = Xf$$

*Now assume that $Y$ is a vector field satisfying $\langle Y, X\rangle_g = Xf$, $\forall X \in \mathfrak{X}(M)$, then $\hat{g}(Y)(X) = df(X)$ for all $X \in \mathfrak{X}(M)$, so the covector fields $df$ and $\hat{g}(Y)$ are equal, thus $Y = \hat{g}^{-1}(df)$, so $\operatorname{grad} f$ can be characterized as the unique vector field satisfying*

$$\langle \operatorname{grad} f, \cdot\rangle_g = df$$

*In a smooth coordinate chart $(x_1, \ldots, x_n)$, we have*

$$\operatorname{grad} f = \hat{g}^{-1}(df) = \sum_{i,j=1}^{n} h_{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}.$$

*In particular this shows that $\operatorname{grad} f$ is smooth.*

## 3.8 Complex Manifolds

In this section we will define complex manifolds and formulate some of their properties. The material of this section is taken from [Forstnerič, 2017]. We will begin by defining the notion of holomorphicity, or complex differentiability. Let $z = (z_1, \ldots, z_n)$ denote the complex coordinates on $\mathbb{C}^n$. For $j \in \{1, \ldots, n\}$, write $z_j = x_j + \mathrm{i}\, y_j$, where $x_j, y_j \in \mathbb{R}$. Then $x_j = \frac{1}{2}(z_j + \bar{z_j})$ and $y_j = \frac{1}{2\mathrm{i}}(z_j - \bar{z_j})$, where $\bar{z_j} = x_j - \mathrm{i}\, y_j$ is the complex conjugate of $z_j$. The coordinate map

$$\varphi \colon \mathbb{C}^n \to \mathbb{R}^{2n}$$

defined by $\varphi(z_1, \ldots, z_n) = \big(\frac{1}{2}(z_1 + \bar{z_1}), \frac{1}{2\mathrm{i}}(z_1 - \bar{z_1}), \ldots, \frac{1}{2}(z_n + \bar{z_n}), \frac{1}{2\mathrm{i}}(z_n - \bar{z_n})\big)$ is a homeomorphism, where $\mathbb{C}^n$ is equipped with the usual topology. Then $\mathbb{C}^n$ is a topological manifold of dimension $2n$, and the atlas $\mathcal{A} = \{(\mathbb{C}^n, \varphi)\}$ is smooth since it contains only one chart, so $\mathcal{A}$ determines a smooth structure on $\mathbb{C}^n$, making $\mathbb{C}^n$ a smooth manifold of dimension $2n$.

For $j \in \{1, \dots, n\}$ we define the following differential operators on a smooth function $f\colon \mathbb{C}^n \to \mathbb{C}$

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - \mathrm{i}\,\frac{\partial f}{\partial y_j}\right), \qquad \frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + \mathrm{i}\,\frac{\partial f}{\partial y_j}\right)$$

and the corresponding forms

$$dz_j = dx_j + \mathrm{i}\,dy_j, \qquad d\bar{z}_j = dx_j - \mathrm{i}\,dy_j$$

It is easy to verify that

$$dz_j\left(\frac{\partial}{\partial z_k}\right) = \delta_{jk}, \quad d\bar{z}_j\left(\frac{\partial}{\partial z_k}\right) = 0$$

$$dz_j\left(\frac{\partial}{\partial \bar{z}_k}\right) = 0, \quad d\bar{z}_j\left(\frac{\partial}{\partial \bar{z}_k}\right) = \delta_{jk}$$

for all $j, k \in \{1, \dots, n\}$, and

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$$

**Definition 3.8.1.** *Let $D \subseteq \mathbb{C}^n$ be a domain, and $f\colon D \to \mathbb{C}$ be a smooth function. We say that $f$ is holomorphic if the differential $df_z$ is $\mathbb{C}$-linear at every point $z \in D$. Equivalently, $f$ is holomorphic if and only if*

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \qquad for\ j \in \{1, \dots, n\}$$

**Remark 3.8.2.** *Let $D \subseteq \mathbb{C}^n$ be a domain, and $f\colon D \to \mathbb{C}$ be a smooth function.*

*Writing $f = f_1 + \mathrm{i}\, f_2$, where $f_1$ is the real part of $f$ and $f_2$ is the imaginary part of $f$, we see that $f$ is holomorphic if and only if*

$$\frac{\partial f_1}{\partial x_j} = \frac{\partial f_2}{\partial y_j} \quad \text{and} \quad \frac{\partial f_2}{\partial x_j} = -\frac{\partial f_1}{\partial y_j}$$

*for all $j \in \{1, \ldots, n\}$. The above system is called the Cauchy–Riemann equations.*

**Definition 3.8.3.** *Let $D \subseteq \mathbb{C}^n$ be a domain, and $f = (f_1, \ldots, f_m)\colon D \to \mathbb{C}^m$ be a smooth function. We say that $f$ is holomorphic if each component $f_j\colon D \to \mathbb{C}$ is holomorphic. When $n = m$, we say that $f$ is biholomorphic onto its image $D' = f(D)$ if $f\colon D \to D'$ is holomorphic, bijective, and its inverse $f^{-1}\colon D' \to D$ is holomorphic.*

**Remark 3.8.4.** *An injective holomorphic map of a domain $D \subseteq \mathbb{C}^n$ to $\mathbb{C}^n$ is always biholomorphic onto its image; see Chapter I, Theorem 2.14 of [Range, 1986].*

**Definition 3.8.5.** (*Complex Manifolds*) *Let $M$ be a topological manifold of dimension $2n$. A complex atlas on $M$ is a collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$, where $\{U_\alpha\}_{\alpha \in I}$ is an open cover of $M$ and $\varphi_\alpha$ is a homeomorphism onto an open set of $\mathbb{C}^n$ such that for every $\alpha, \beta \in I$, the transition map*

$$\varphi_\alpha \circ \varphi_\beta^{-1}\colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

*is biholomorphic.*

*Two complex atlases on $M$ are said to be holomorphically compatible if their union is also a complex atlas. Each complex atlas $\mathcal{A}$ is contained in a unique maximal complex atlas, called the complex structure determined by $\mathcal{A}$. Two complex atlases determine the same complex structure if and only if they are holomorphically* *compatible.*

*A complex manifold of complex dimension* $n$ *is a topological manifold of dimension* $2n$ *equipped with a complex structure.*

**Definition 3.8.6.** *Let* $M$ *be a complex manifold of dimension* $n$*. A function* $f\colon M \to \mathbb{C}$ *is said to be holomorphic if for any chart* $(U, \varphi)$ *of the complex structure on* $M$*, the function*

$$f \circ \varphi^{-1}\colon \varphi(U) \to \mathbb{C}$$

*is holomorphic on the open set* $\varphi(U) \subseteq \mathbb{C}^n$*.*

**Definition 3.8.7.** *Let* $M$ *and* $N$ *be complex manifolds of dimensions* $m$ *and* $n$*, respectively. A function* $f\colon M \to N$ *is said to be holomorphic if for any* $p \in M$ *there exist complex charts* $(U, \varphi)$ *on* $M$*,* $(V, \psi)$ *on* $N$ *such that* $p \in U$*,* $f(U) \subseteq V$*, and the map*

$$\psi \circ f \circ \varphi^{-1}\colon \varphi(U) \to \psi(V) \subseteq \mathbb{C}^n$$

*is holomorphic on the open set* $\varphi(U) \subseteq \mathbb{C}^m$*. The choice of the charts is unimportant in the definition since the charts in a complex atlas are holomorphically compatible. If* $f$ *is holomorphic, bijective, and* $f^{-1}\colon N \to M$ *is holomorphic, then we say that* $f$ *is biholomorphic. By Remark 3.8.4, every bijective holomorphic map between complex manifolds is biholomorphic. A biholomorphic map* $f\colon M \to M$ *is called a holomorphic automorphism of the complex manifold* $M$*, and we denote the collection of all automorphisms on* $M$ *by* $Aut(M)$*.* $(Aut(M), \circ)$ *is a clearly group, we call* $Aut(M)$ *the holomorphic automorphism group of* $M$*.*

**Definition 3.8.8.** *Let* $D \subseteq \mathbb{C}^n$ *be a domain, and* $f = (f_1, \ldots, f_m)\colon D \to \mathbb{C}^m$ *be a holomorphic map. We define the complex rank of* $f$ *at a point* $p \in D$*, denoted* *by* $\operatorname{rank}_p f$*, to be the rank of the* $m \times n$ *Jacobian matrix*

$$J_p f = \left(\frac{\partial f_j}{\partial z_k}\right)$$

*Clearly* $\operatorname{rank}_p f \leq \min\{n, m\}$. $f$ *is called a holomorphic immersion at* $p$ *if* $\operatorname{rank}_p f = n$*, and* $f$ *is called a holomorphic submersion at* $p$ *if* $\operatorname{rank}_p f = m$*. A holomorphic immersion that is a topological embedding is called a holomorphic embedding. These notions, being local, extend to holomorphic maps between complex manifolds.*

**Definition 3.8.9.** *Let* $M$ *be a complex manifold of dimension* $n$*. A subset* $S$ *of* $M$ *is said to be a complex submanifold of dimension* $k \in \{0, 1, \ldots, n\}$*, and codimension* $n-k$*, if every point* $p \in S$ *admits an open neighborhood* $U \subset M$ *and a holomorphic chart* $\varphi\colon U \to U' \subset \mathbb{C}^n$ *such that* $\varphi(U \cap S) = \varphi(U) \cap (\mathbb{C}^k \times \{0\}^{n-k})$.

**Definition 3.8.10.** *Let* $M$ *be a complex manifold of dimension* $n$*. A subset* $A$ *of* $M$ *is said to be a complex* (*analytic*) *subvariety of* $M$ *if for every point* $p \in A$ *there exist an open neighborhood* $U \subset M$ *of* $p$ *and holomorphic functions* $f_1, \ldots, f_d\colon U \to U' \subset \mathbb{C}^n$ *such that*

$$A \cap U = \{x \in U\colon f_1(x) = \cdots = f_d(x) = 0\}.$$

**Definition 3.8.11.** (*The Tangent Bundle*) *Let* $M$ *be a complex manifold of dimension* $n$*. The complex structure on* $M$ *defines in particular a* (*real*) *smooth structure making* $M$ *a* (*real*) *smooth manifold of dimension* $2n$*. Let* $TM$ *be the real tangent bundle of the smooth manifold* $M$*. The complexification of* $TM$*,* $\mathbb{C}TM = TM \otimes \mathbb{C} \cong \coprod_{p \in M} T_pM \otimes \mathbb{C}$*, is called the complexified tangent bundle of* $M$*, and a map from* $M$ *to* $\mathbb{C}TM = TM \otimes \mathbb{C}$ *that send every point in* $p \in M$ *to a* *vector in $T_pM \otimes \mathbb{C}$ are called a complex vector field on $M$.*

*There exists a unique real linear map $J\colon TM \to TM$, called the almost complex structure operator (see Section 1.6 in [Forstnerič, 2017] ), which is given in any local holomorphic coordinate chart $(z_1, \dots, z_n)$, with $z_j = x_j + \mathrm{i}\, y_j$, by*

$$J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}, \qquad J\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}$$

*$J$ extends to a $\mathbb{C}$-linear map on $\mathbb{C}TM$ by $J(v \otimes \alpha) = J(v) \otimes \alpha$ for $v \in TM$ and $\alpha \in \mathbb{C}$. Since $J^2 = -Id$, the eigenvalues of $J$ are $\mathrm{i}$ and $-\mathrm{i}$. Hence we have the decomposition*

$$\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M$$

*where $T^{1,0}M$ is the $\mathrm{i}$ eigenspace and $T^{0,1}M$ is the $-\mathrm{i}$ eigenspace of $J$. In any holomorphic coordinates $(z_1, \dots, z_n)$ on an open set $U \subset M$ we have*

$$T^{1,0}M = \mathrm{Span}_{\mathbb{C}}\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\}, \qquad T^{0,1}M = \mathrm{Span}_{\mathbb{C}}\{\frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n}\}$$

*where*

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \mathrm{i}\,\frac{\partial}{\partial y_j}\right) \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \mathrm{i}\,\frac{\partial}{\partial y_j}\right)$$

*We have an $\mathbb{R}$-linear isomorphism $\phi\colon TM \to T^{1,0}M$ given by*

$$\phi(v) = \frac{1}{2}(v - \mathrm{i}\, J(v))$$

*for every $v \in TM$. In any local holomorphic coordinates $\phi$ is given by*

$$\sum_{j=1}^{n}\left(a_j\frac{\partial}{\partial x_j} + b_j\frac{\partial}{\partial y_j}\right) \mapsto \sum_{j=1}^{n}(a_j + \mathrm{i}\, b_j)\frac{\partial}{\partial z_j}$$

**Remark 3.8.12.** *For $z \in \mathbb{C}^n$, using the isomorphism $T_z\mathbb{C}^n \cong T_z^{1,0}\mathbb{C}^n \cong \mathbb{C}^n$, we identify a tangent vector at $z$, $v = \sum_{i=j}^n v_j \frac{\partial}{\partial z_j} \in T_z^{1,0}\mathbb{C}^n$ with $(v_1, \ldots, v_n) \in \mathbb{C}^n$.*

**Definition 3.8.13.** *Let $M$ be a complex manifold. A real vector field $V \colon M \to TM$ is said to be holomorphic if $\phi(V)$ is a holomorphic section of $T^{1,0}M$. Equivalently $V$ is holomorphic iff in any local holomorphic coordinate chart $(z_1, \ldots, z_n)$, with $z_j = x_j + \mathrm{i}\, y_j$, and $V = \sum_{j=1}^n \left( a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j} \right)$, the functions $a_j + \mathrm{i}\, b_j \colon M \to \mathbb{C}$ are holomorphic.*

**Proposition 3.8.14.** *Let $M$ and $N$ be complex manifolds, and let $J_M$ and $J_N$ be the almost complex structure operators on $M$ and $N$ respectively. A smooth map $f \colon M \to N$ is holomorphic if and only if the differential of $f$ commutes with the almost complex structure operators on $M$ and $N$:*

$$df \circ J_M = J_N \circ df$$

*In this case $df$ respects the decomposition $\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M$ and $df_p \colon T_p^{1,0}M \to T_{f(p)}^{1,0}N$ is $\mathbb{C}$-linear for every $p \in M$.*

*Proof.* See Proposition 1.6.3 of [Forstnerič, 2017]. $\square$

**Definition 3.8.15.** *Let $M$ be a complex manifold of dimension $n$. $T^*M$ is the (real) cotangent bundle; i.e. the real dual of the tangent bundle $TM$. The $\mathbb{R}$-linear map $J$ of $\mathbb{C}TM$ induces the dual linear map $J^*$ of the complexified cotangent bundle $\mathbb{C}T^*M$ with $(J^*)^2 = -Id$ so we have the decomposition*

$$\mathbb{C}T^*M = T^{*1,0}M \oplus T^{*0,1}M$$

*into the $\mathrm{i}$ and $-\mathrm{i}$ eigenspaces of $J^*$. In any holomorphic coordinates $(z_1, \ldots, z_n)$ on an open subset $U \subset M$ and a point $p \in U$, the forms $dz_1, \ldots, dz_n$ at $p$ are* *complex a basis of $T_p^{*1,0}M$, while $d\bar{z}_1, \dots, d\bar{z}_n$ are a complex basis of $T_p^{*0,1}M$. Moreover $T^{*1,0}M$ and $T^{*0,1}M$ are the complex dual spaces of $T^{1,0}M$ and $T^{0,1}M$, respectively.*

**Definition 3.8.16.** *Let $M$ be a complex manifold and denote by $\mathbb{D}$ the open unit disc in $\mathbb{C}$; $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. An upper semicontinuous function $u \colon M \to [-\infty, +\infty]$ which is not identically $-\infty$ on any connected component of $M$ is said to be plurisubharmonic if for every continuous map $f \colon \overline{\mathbb{D}} \to M$ which is holomorphic in $\mathbb{D}$ we have the submeanvalue property*

$$u\left(f(0)\right) \leq \int_0^{2\pi} u\left(f(e^{\mathrm{i}\,\theta})\right) \frac{d\theta}{2\pi}.$$

**Definition 3.8.17.** *Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, and let $\rho \colon U \to \mathbb{C}$ be a $C^2$ function, where $U$ be an open neighborhood of $z$ in $\mathbb{C}^n$. We define the complex Hessian of $\rho$ at $z$ to be the Hermitian bilinear form $H_{\rho,z} \colon T_z\mathbb{C}^n \times T_z\mathbb{C}^n \to \mathbb{C}$ given by*

$$H_{\rho,z}(v,w) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) v_j \bar{w}_k$$

*for $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) \in T_z\mathbb{C}^n \cong \mathbb{C}^n$.*
*The associated Hermitian quadratic form is called the Levi form of $\rho$ at $z$:*

$$\mathcal{L}_{\rho,z}(v) = H_{\rho,z}(v,v)$$

**Remark 3.8.18.** *Given $z \in \mathbb{C}^n$, and $\rho$ as in the above definition, the Levi form of $\rho$ at $z$ is*

$$\mathcal{L}_{\rho,z}(v) = H_{\rho,z}(v,v) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) v_j \bar{v}_k = \left.\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}}\right|_{\zeta=0} \rho(z + \zeta v)$$

*the latter equals one quarter of the Laplacian* $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ *of the function* $\zeta = x + \mathrm{i}\, y \mapsto \rho_{z,v}(\zeta) = \rho(z + \zeta v)$ *at* $\zeta = 0$*:*

$$\mathcal{L}_{\rho,z}(v) = \frac{1}{4}\Delta\rho_{z,v}(0).$$

*Hence, for a domain* $\Omega \subset \mathbb{C}^n$*, and a* $C^2$ *function* $\rho\colon \Omega \to \mathbb{R}$*,* $\rho$ *is plurisubharmonic if and only if its Levi form is positive semi-definite for every point* $z \in \Omega$*; i.e.* $\mathcal{L}_{\rho,z}(v) \geq 0$ *for every* $z \in \Omega$ *and* $v \in T_z\mathbb{C}^n$*.*

**Definition 3.8.19.** *Let* $\Omega$ *be a domain in* $\mathbb{C}^n$*, and* $\rho\colon \Omega \to \mathbb{R}$ *be a* $C^2$ *function.* $\rho$ *is said to be strongly plurisubharmonic if its Levi form,* $\mathcal{L}_{\rho,z}$*, is positive definite for every* $z \in \Omega$*.*

# Chapter 4

# Morse Theory

The material of this chapter is from [Milnor, 1963].

## 4.1 Definitions and Lemmas

**Definition 4.1.1.** *Let $M$ be a smooth manifold, $p \in M$, and $f\colon M \to \mathbb{R}$ be a smooth map. $p$ is said to be a critical point of $f$ if the differential map of $f$ at $p$, $f_*\colon T_pM \to T_{f(p)}\mathbb{R}$ is the zero map. Equivalently if the differential of $f$ at $p$, $df_p\colon T_pM \to \mathbb{R}$ is the zero linear form. In that case, the real number $f(p)$ is said to be a critical value of $f$.*

**Remark 4.1.2.** *Let $M$ be a smooth manifold, $p \in M$, and $f\colon M \to \mathbb{R}$ be a smooth map. For every smooth coordinate chart $(U, \varphi)$ containing $p$, with $\varphi = (x_1, \ldots, x_n)$, $p$ is a critical point of $f$ if and only if $\frac{\partial f}{\partial x_k}(p) = 0$ for every $k \in \{1, \ldots, n\}$. In fact this follows directly from the coordinate representation of the differential map with respect to a given coordinate chart. See Remark 3.2.17.*

**Remark 4.1.3.** *Let $M$ be a smooth manifold, $p \in M$, and $f\colon M \to \mathbb{R}$ be a smooth map. Assume that $p$ is a critical point of $f$. By Proposition 3.4.3, every* *vector $v \in T_pM$ extends to a smooth vector field $\tilde{v} \in \mathfrak{X}(M)$, such that $\tilde{v}(p) = v$. We define the* ***Hessian of f at p****,*

$$f_{**} \colon T_pM \times T_pM \to \mathbb{R}$$

*by $f_{**}(v,w) = \tilde{v}_p(\tilde{w}(f))$ for any $v,w \in T_pM$ and any extensions $\tilde{v}$ and $\tilde{w}$ of $v$ and $w$ respectively. We will show that $f_{**}$ is a well defined map (independent of the choice of extensions of vectors) that is a symmetric bilinear form on $T_pM$.*
*For any $X,Y \in \mathfrak{X}(M)$, $X_p(Y(f)) - Y_p(X(f)) = [X,Y]_p(f)$, where $[X,Y]$ is the Lie bracket of $X$ and $Y$. By Proposition 3.4.7, $[X,Y]$ is again a smooth vector field. Expressing $[X,Y]$ with respect to some smooth coordinate charts $(x_1,\dots,x_n)$ in a neighborhood of $p$, $[X,Y] = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}$, we get*

$$[X,Y]_p(f) = \sum_{i=1}^n \xi_i(p)\frac{\partial f}{\partial x_i}(p)\frac{\partial}{\partial x_i}|_p$$

*the latter is zero by Remark 4.1.2, so $X_p(Y(f)) = Y_p(X(f))$. Then for any $v,w \in T_pM$, and any extensions $\tilde{v}$ and $\tilde{w}$ of $v$ and $w$ respectively, we have $\tilde{v}_p(\tilde{w}(f)) = \tilde{w}_p(\tilde{v}(f))$. This shows that $f_{**}$ is symmetric.*
*Now for $v,w \in T_pM$, $f_{**}(v,w) = \tilde{v}_p(\tilde{w}(f)) = v(\tilde{w}(f))$ is independent of the choice of $\tilde{v}$, and $f_{**}(v,w) = \tilde{w}_p(\tilde{v}(f)) = w(\tilde{v}(f))$ is independent of the choice of $\tilde{w}$. This shows that $f_{**}$ is well defined.*
*For $\alpha \in \mathbb{R}$, $v_1, v_2, w \in T_pM$, $f_{**}(\alpha v_1 + v_2, w) = (\alpha v_1 + v_2)(\tilde{w}(f)) = \alpha v_1(\tilde{w}(f)) + v_2(\tilde{w}(f)) = \alpha f_{**}(v_1,w) + f_{**}(v_2,w)$, so $f_{**}$ is linear in the first variable, so $f_{**}$ is bilinear since $f_{**}$ is symmetric.*

*In a smooth coordinate chart $(x_1,\dots,x_n)$ defined in some neighborhood of $p$, for*

$v = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}|_p, w = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}|_p \in T_pM$, *choosing* $\tilde{w} = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$ *with* $b_i$ *are constant functions in a neighborhood of* $p$, *we have*

$$f_{**}(v, w) = \sum_{i,j=1}^{n} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}(p)$$

*Thus the matrix representation of* $f_{**}$ *with respect to the coordinate basis* $(\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p)$ *is* $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$.

**Definition 4.1.4.** *Let* $M$ *be a smooth manifold,* $p \in M$, *and* $f \colon M \to \mathbb{R}$ *be a smooth map. Assume that* $p$ *is a critical point of* $f$. *We say that* $p$ *is a* ***non-degenerate critical point of f*** *if the nulity of the Hessian of* $f$ *at* $p$ *is zero. The* ***index of f at p*** *is the index of the Hessian of* $f$ *at* $p$.

**Definition 4.1.5.** *Let* $M$ *be a smooth manifold. A smooth function* $f \colon M \to \mathbb{R}$ *is said to be a Morse function if all the critical points of* $f$ *are non-degenerate.*

**Lemma 4.1.6.** *Let* $V$ *be a convex neighborhood of* $0$ *in* $\mathbb{R}^n$, *and* $f \colon V \to \mathbb{R}$ *be a smooth function such that* $f(0) = 0$. *Then there exist smooth functions* $g_1, \ldots, g_n \colon V \to \mathbb{R}$ *with*

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \ldots, x_n),$$

$g_i(0) = \frac{\partial f}{\partial x_i}(0)$, *and* $\frac{\partial g_i}{\partial x_j}(0) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right)$ *for every* $i, j \in \{1, \ldots, n\}$.

*Proof.* Let $(x_1, \ldots, x_n) \in V$, and consider the curve $\gamma \colon [0,1] \to \mathbb{R}^n$ defined by $\gamma(t) = (tx_1, \ldots, tx_n)$. Since $\gamma(0) = 0$, $\gamma(1) = (x_1, \ldots, x_n)$ and $V$ is convex, we have $\gamma([0,1]) \subseteq V$. Now $f \circ \gamma \colon [0,1] \to \mathbb{R}$ is a smooth map, hence by the fundamental theorem of calculus

$$f\circ\gamma(1)-f\circ\gamma(0)=\int_0^1\frac{d(f\circ\gamma)}{dt}dt=\int_0^1\frac{d(f(tx_1,\ldots,tx_n))}{dt}dt$$

so that

$$f(x_1,\ldots,x_n)=\int_0^1\sum_{i=1}^n x_i\frac{\partial f}{\partial x_i}(tx_1,\ldots,tx_n)dt$$

setting $g_i(x_1,\ldots,x_n)=\int_0^1\frac{\partial f}{\partial x_i}(tx_1,\ldots,tx_n)dt$ for $i\in\{1,\ldots,n\}$, we get that

$$f(x_1,\ldots,x_n)=\sum_{i=1}^n x_ig_i(x_1,\ldots,x_n),$$

and by the Leibniz Integral Rule it follows that for every $i,j\in\{1,\ldots,n\}$, $g_i\in C^\infty(V)$ and

$$\begin{aligned} g_i(0) &= \int_0^1\frac{\partial f}{\partial x_i}(0,\ldots,0)dt \\ &= \frac{\partial f}{\partial x_i}(0)\int_0^1 dt \\ &= \frac{\partial f}{\partial x_i}(0) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial g_i}{\partial x_j}(x_1,\ldots,x_n) &= \int_0^1\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}(tx_1,\ldots,tx_n)\right)dt \\ &= \int_0^1 t\left(\frac{\partial^2 f}{\partial x_i\partial x_j}(tx_1,\ldots,tx_n)\right)dt \end{aligned}$$

so that

$$\begin{aligned}
\frac{\partial g_i}{\partial x_j}(0) &= \int_0^1 t\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0,\dots,0)\right) dt \\
&= \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right)\int_0^1 t dt \\
&= \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right)
\end{aligned}$$

□

**Lemma 4.1.7.** *Let $V$ be an open neighborhood of $0$ in $\mathbb{R}^n$, $h_{ij}\colon V \to \mathbb{R}$ be smooth functions for $i, j \in \{1,\dots,n\}$, and $q\colon V \to \mathbb{R}$*

$$q(x_1,\dots,x_n) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x_1,\dots,x_n).$$

*If the matrix $(h_{ij}(0))$ is symmetric and non-singular, then there exists a smooth coordinate chart $(U,\varphi)$, around $0$ in $V$, $\varphi = (z_1,\dots,z_n)$, such that*

$$q(z_1,\dots,z_n) = \pm(z_1)^2 \pm \dots \pm (z_n)^2$$

*Proof.* We will show by induction that for all $r$, $1 \le r \le n+1$, there exist smooth coordinates $(u_1,\dots,u_n)$ (that depends on $r$), in a neighborhood $U_r$ of $0$ in $V$ such that

$$q = \pm(u_1)^2 \pm \dots \pm (u_{r-1})^2 + \sum_{i,j\ge r} u_i u_j H^r_{ij}(u_1,\dots,u_n)$$

on $U_r$, and the $(n-r+1)\times(n-r+1)$ matrix

$$\begin{pmatrix}
H^r_{rr}(u_1,\dots,u_n) & & H^r_{rn}(u_1,\dots,u_n) \\
& \ddots & \\
H^r_{nr}(u_1,\dots,u_n) & & H^r_{nn}(u_1,\dots,u_n)
\end{pmatrix}$$

is symmetric on $U_r$, and non-singular at 0, and the maps $H^r_{ij}\colon V \to \mathbb{R}$ are smooth. For $r = 1$, let $(H^1_{ij}) = \left(\frac{1}{2}(h_{ij} + h_{ji})\right)$, $(u_1, \ldots, u_n) = (x_1, \ldots, x_n)$, and $U_1 = V$. $(H^1_{ij})$ is symmetric on $U$, and $(H^1_{ij}(0)) = (h_{ij}(0))$ is non-singular, and

$$q(u_1, \ldots, u_n) = \sum_{i,j=1}^{n} u_i u_j H^1_{ij}(u_1, \ldots, u_n).$$

Now assume that the above holds for order $r$,

$$q = \pm(u_1)^2 \pm \cdots \pm (u_{r-1})^2 + \sum_{i,j \geq r} u_i u_j H^r_{ij}(u_1, \ldots, u_n). \qquad (*)$$

Since the matrix

$$\begin{pmatrix} H^r_{rr}(0) & & H^r_{rn}(0) \\ & \ddots & \\ H^r_{nr}(0) & & H^r_{nn}(0) \end{pmatrix}$$

is non-singular, we can assume after a linear change in the last $n - r + 1$ coordinates that $H^r_{rr}(0) \neq 0$, so by continuity, $H^r_{rr}(u_1, \ldots, u_n)$ is of constant sign on a neighborhood $V_{r+1} \subseteq U_r$ of 0. Let $g(u_1, \ldots, u_n)$ be the the square root of $|H^r_{rr}(u_1, \ldots, u_n)|$. $g$ is smooth and nonzero on $U_{r+1}$. Now we define the new coordinates $(v_1, \ldots, v_n)$ by

$$v_i = u_i \quad \text{for } i \neq r$$

$$v_r(u_1, \ldots, u_n) = g(u_1, \ldots, u_n) \left[ u_r + \sum_{i > r} u_i \frac{H^r_{ir}(u_1, \ldots, u_n)}{H^r_{rr}(u_1, \ldots, u_n)} \right].$$

The Jacobian matrix of $(v_1, \ldots, v_n)$ with respect to the coordinate chart $(u_1, \ldots, u_n)$ is non-singular at zero, so by the inverse function theorem, $(v_1, \ldots, v_n)$ is a diffeomorphism in a neighborhood $U_{r+1} \subseteq V_{r+1}$ of 0 onto its image, and by direct computation:

$$\begin{aligned}
\pm(v_r)^2 &= H^r_{rr}\left(u_r + \sum_{i>r} u_i \frac{H^r_{ir}}{H^r_{rr}}\right)^2 \\
&= (u_r)^2 H^r_{rr} + 2\sum_{i>r} u_r u_i H^r_{ir} + \sum_{i,j>r} u_i u_j \frac{\left(H^r_{ij}\right)^2}{H^r_{rr}} \\
&= \sum_{i,j\geq r} u_i u_j H^r_{ij} + \sum_{i,j>r} u_i u_j \frac{\left(H^r_{ij}\right)^2}{H^r_{rr}}.
\end{aligned}$$

Substituting for $\sum_{i,j\geq r} u_i u_j H^r_{ij}$ in $(*)$:

$$\begin{aligned}
q &= \pm(u_1)^2 \pm \cdots \pm (u_{r-1})^2 \pm (v_r)^2 - \sum_{i,j>r} u_i u_j \frac{\left(H^r_{ij}\right)^2}{H^r_{rr}} \\
&= \pm(v_1)^2 \pm \cdots \pm (v_{r-1})^2 \pm (v_r)^2 + \sum_{i,j>r} v_i v_j \left(-\frac{\left(H^r_{ij}\right)^2}{H^r_{rr}}\right) \\
&= \sum_{i\leq r} \pm(v_i)^2 + \sum_{i,j>r} v_i v_j H^{r+1}_{ij}(v_1, \ldots, v_n)
\end{aligned}$$

where $(H^{r+1}_{ij}) = -\dfrac{\left(H^r_{ij}\right)^2}{H^r_{rr}}$ for $i, j > r$ are smooth functions on $U_{r+1}$ satisfying the above assumptions.

Thus the above holds for all $1 \leq r \leq n+1$, in particular for $r = n+1$ we obtain the desired result. $\square$

**Lemma 4.1.8.** (*Lemma of Morse*) *Let $M$ be a smooth manifold of dimension $n$, and $f\colon M \to \mathbb{R}$ be a smooth function. Let $p \in M$ be a non-degenerate critical point of $f$. Then there exists a smooth coordinate chart $(U, \varphi)$ containing $p$, with $\varphi = (y_1, \ldots, y_n)$, such that $\varphi(p) = 0$ and*

$$f(x) = f(p) - (y_1(x))^2 - \cdots - (y_\lambda(x))^2 + (y_{\lambda+1}(x))^2 + \cdots + (y_n(x))^2$$

*for all $x \in U$, where $\lambda$ is the index of $f$ at $p$.*

*Proof.* Let $\lambda$ by the index of $f$ at $p$. We will prove first that if

$$f(x) = f(p) - (y_1(x))^2 - \cdots - (y_k(x))^2 + (y_{k+1}(x))^2 + \cdots + (y_n(x))^2$$

for a smooth coordinate chart $(y_1, \ldots, y_n)$ in a neighborhood of $p$, and some $k \in \{0, 1, \ldots, n\}$, then $k = \lambda$. In fact if $f$ can be represented as above in some smooth coordinate chart $(y_1, \ldots, y_n)$, then the matrix representation of the Hessian of $f$ at $p$ with respect to the coordinate basis $(\frac{\partial}{\partial y_1}|_p, \ldots, \frac{\partial}{\partial y_n}|_p)$ is the diagonal matrix

$$\begin{pmatrix} -2 & & & & & \\ & \ddots & & & & \\ & & -2 & & & \\ & & & 2 & & \\ & & & & \ddots & \\ & & & & & 2 \end{pmatrix}$$

where the number of $-2$'s on the diagonal is $k$ and the number of 2's is $n - k$. Hence the index of the Hessian of $f$ is $k$, so that $k = \lambda$.

It remains to prove that $f$ can be represented as above in some smooth coordinate chart. We first assume that $f(p) = 0$. Let $(W, \tilde{\psi})$ be a smooth coordinate chart containing $p$, and set $\psi \colon W \to \mathbb{R}^n$, defined by $\psi(x) = \tilde{\psi}(x) - \tilde{\psi}(p)$. The map $\psi$ is a diffeomorphism onto its image, and $\psi$ is compatible with the charts of $M$, then $(W, \psi)$ is a smooth coordinate chart and $\psi(p) = 0$. Let $V$ be an open ball in $\mathbb{R}^n$ centered at 0, with $V \subseteq \psi(W)$. Write $\psi = (x_1, \ldots, x_n)$. Let $\tilde{f} = f \circ \psi^{-1} \colon V \to \mathbb{R}$ be a representation of $f$ in the smooth coordinates $(x_1, \ldots, x_n)$. $\tilde{f}$ is smooth and

$\tilde{f}(0) = 0$, then by Lemma 4.1.6, there exist smooth functions $g_i \colon V \to \mathbb{R}$ such that

$$\tilde{f}(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \dots, x_n)$$

and $g_i(0) = \frac{\partial \tilde{f}}{\partial x_i}(0) = \frac{\partial f}{\partial x_i}(p) = 0$ for every $i \in \{1, \dots, n\}$. By applying again Lemma 4.1.6 on the smooth maps $g_i \colon V \to \mathbb{R}$, there exist smooth functions $h_{ij} \colon V \to \mathbb{R}$ such that for every $i \in \{1, \dots, n\}$,

$$g_i(x_1, \dots, x_n) = \sum_{j=1}^{n} x_j h_{ij}(x_1, \dots, x_n)$$

and $h_{ij}(0) = \frac{\partial g_i}{\partial x_j}(0) = \frac{1}{2}\frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(p)$ for every $i, j \in \{1, \dots, n\}$, so that

$$\tilde{f}(x_1, \dots, x_n) = \sum_{i,j=1}^{n} x_i x_j h_{ij}$$

on $V$, and the matrix $(h_{ij}(0))$ is $\left(\frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$, which is symmetric and non-singular since $p$ is a non-degenerate critical point of $f$. Then by Lemma 4.1.7, there exists a smooth coordinate chart $(\tilde{U}, \tilde{\varphi})$, with $0 \in \tilde{U} \subseteq V$, $\tilde{\varphi} = (z_1, \dots, z_n)$, and

$$\tilde{f}(z_1, \dots, z_n) = \pm(z_1)^2 \pm \cdots \pm (z_n)^2.$$

Setting $U = \psi^{-1}(\tilde{U})$ and $\varphi = \tilde{\varphi} \circ \psi \colon U \to \mathbb{R}^n$ is a diffeomorphism onto its image, so in the smooth chart $(U, \varphi)$, writing $\varphi = (y_1, \dots, y_n)$, we get that

$$f(x) = \pm(y_1(x))^2 \pm \cdots \pm (y_n(x))^2$$

so after a suitable rearrangement of the maps $(y_1, \dots, y_n)$,

$$f(x) = -(y_1(x))^2 - \cdots - (y_k(x))^2 + (y_{k+1}(x))^2 + \cdots + (y_n(x))^2$$

Now for the general case, apply the above result on the smooth function $f - f(p)$ so that

$$f(x) = f(p) - (y_1(x))^2 - \cdots - (y_k(x))^2 + (y_{k+1}(x))^2 + \cdots + (y_n(x))^2.$$

This ends the proof. $\square$

**Remark 4.1.9.** *Let $M$ be a complex manifold of dimension $n$, and $f\colon M \to \mathbb{R}$ be a smooth function. $M$ could be regarded as a smooth manifold of (real) dimension $2n$, so for a non-degenerate critical point $p$ of $f$, the index of $f$ at $p$ belongs to the set $\{0, 1, \dots, 2n\}$.*

**Theorem 4.1.10.** *Let $p \in \mathbb{C}^n$ and $\rho$ be a strongly plurisubharmonic function defined on an open neighborhood of $p$. Assume that $p$ is a non-degenerate critical point of $\rho$, then the index of $\rho$ at $p$ belongs to the set $\{0, 1, \dots, n\}$.*

*Proof.* See Lemma 3.10.1 of [Forstnerič, 2017]. $\square$

**Remark 4.1.11.** (*Local Extrema*) *Let $M$ be a smooth manifold of dimension $n$ and $f\colon M \to \mathbb{R}$. Assume $f$ has a local minimum (respectively local maximum) at a point $p \in M$. Then for every smooth coordinate $(U, \varphi)$ around $p$, $f \circ \varphi^{-1}\colon \mathbb{R}^n \to \mathbb{R}$ has a local minimum (respectively local maximum) at $\varphi(p)$, so $\varphi(p)$ is a critical point for $f \circ \varphi^{-1}$, and so $p$ is a critical point for $f$.*

*Conversely assume that $p$ is a non-degenerate critical point of $f$, then $f$ has* *a local minimum (respectively local maximum) at $p$ if and only if the index of $f$ at $p$ is $0$ (respectively $n$). In fact this follows from Lemma 4.1.8, since $p$ is non-degenerate, there exist smooth coordinates $(y_1, \dots, y_n)$ in an open neighborhood $U$ of $p$ such that*

$$f(x) = f(p) - (y_1(x))^2 - \cdots - (y_\lambda(x))^2 + (y_{\lambda+1}(x))^2 + \cdots + (y_n(x))^2$$

*in $U$, where $\lambda$ is the index of $p$ in $f$. Thus it follows directly from the above representation of $f$ that $f$ has a local minimum (respectively local maximum) at $p$ if and only if $\lambda = 0$ (respectively $\lambda = n$).*

**Corollary 4.1.12.** *Let $M$ be a smooth manifold, and $f\colon M \to \mathbb{R}$ be a Morse function. Then the critical points of $f$ are isolated.*

*Proof.* Let $p$ be a critical point of $f$, and let $\lambda$ be the index of $f$ at $p$. $p$ is non-degenerate, then by Lemma 4.1.8 there exist smooth coordinates $\varphi = (y_1, \dots, y_n)$ defined on a neighborhood $U$ of $p$, such that $\varphi(p) = 0$ and

$$f(x) = f(p) - (y_1(x))^2 - \cdots - (y_\lambda(x))^2 + (y_{\lambda+1}(x))^2 + \cdots + (y_n(x))^2$$

for all $x \in U$. Thus the differential of $f$ in the coordinate basis relative to $(y_1, \dots, y_n)$ is

$$Df = -2y_1\frac{\partial}{\partial y_1} - \cdots - 2y_\lambda\frac{\partial}{\partial y_\lambda} + 2y_{\lambda+1}\frac{\partial}{\partial y_{\lambda+1}} + \cdots + 2y_n\frac{\partial}{\partial y_n}$$

in $U$, so that $Df = 0 \Leftrightarrow y_i = 0$ for all $i$, hence $p$ is the only critical point of $f$ in $U$. $\square$

## 4.2 Morse Theory

**Definition 4.2.1.** *Let $M$ be a topological manifold and $f\colon M \to \mathbb{R}$ be a real valued function. For $a \in \mathbb{R}$ we define the set $M^a \subseteq M$ by*

$$M^a = f^{-1}(-\infty, a] = \{x \in M \colon f(x) \leq a\}$$

**Theorem 4.2.2.** *Let $M$ be a smooth manifold and $f\colon M \to \mathbb{R}$ be a smooth function. Let $a < b$ and suppose that $f^{-1}[a, b]$ is compact and contains no critical points of $f$. Then $M^a$ is diffeomorphic to $M^b$. Furthermore $M^a$ is a deformation retract of $M^b$, so that the inclusion map $M^a \hookrightarrow M^b$ is a homotopy equivalence.*

*Proof.* Let $\langle \cdot, \cdot \rangle$ be a Riemannian metric on $M$ (recall that by Proposition 3.7.9 every smooth manifold admits a Riemannian metric). Consider the gradient of $f$, characterized by the identity

$$\langle \operatorname{grad} f, X \rangle = df(X) = X(f) \qquad \text{for all } X \in \mathfrak{X}(M).$$

Note that the vector field $\operatorname{grad} f$ vanishes precisely at the critical points of $f$. Let $\psi\colon M \to \mathbb{R}$ be a non-negative (smooth) bump function that is identically 1 on the compact set $f^{-1}[a, b]$, and vanishes outside a compact neighborhood of this set, not containing any critical point of $f$. Define $\rho\colon M \to \mathbb{R}$ by

$$\rho = \psi / \langle \operatorname{grad} f, \operatorname{grad} f \rangle.$$

$\rho$ is well defined since $\langle \operatorname{grad} f, \operatorname{grad} f \rangle > 0$ in the support of $\psi$, and $\psi$ is smooth.

Define the vector field $X\colon M \to TM$ by

$$X_p = \rho(p)(\operatorname{grad} f)_p \qquad \forall p \in M,$$

then $X$ is smooth and compactly supported, and thus by Theorem 3.4.16, $X$ is complete. Let $\varphi\colon \mathbb{R} \times M \to M$ be the global flow generated by $X$. For $t \in \mathbb{R}$ and $x \in M$ we will write $\varphi_t(x)$ to denote $\varphi(t,x)$. For a fixed $q \in M$ consider the smooth function $t \mapsto f(\varphi_t(q))$ defined for every $t \in \mathbb{R}$.

$$\frac{d\, f(\varphi_t(q))}{dt} = df\left(\frac{d\varphi_t(q)}{dt}\right) = \left\langle \operatorname{grad} f, \frac{d\varphi_t(q)}{dt}\right\rangle = \langle \operatorname{grad} f, X\rangle = \psi(\varphi_t(q))$$

the latter is non-negative so $t \mapsto f(\varphi_t(q))$ is non-decreasing, and if $\varphi_t(q) \in f^{-1}(a,b)$, then $\dfrac{d\, f(\varphi_t(q))}{dt} = 1$ in an open neighborhood of $t$, so $f(\varphi(t,q)) = t + f(\varphi(0,q)) = t + f(q)$ as long as $a < f(\varphi(t,q)) < b$, and the equality extends, by continuity, for $a \leq f(\varphi(t,q)) \leq b$. We consider the diffeomorphism $\varphi_{b-a}\colon M \to M$. We will show that $\varphi_{b-a}(M^a) = M^b$, thus $\varphi_{b-a}$ carries $M^a$ diffeomorphically to $M^b$. Let $x \in M^a$, and suppose to the contrary that $\varphi_{b-a}(x) \notin M^b$ i.e. $f(\varphi_{b-a}(x)) > b$. Now $f(\varphi_0(x)) = f(x) \leq a$, $f(\varphi_{b-a}(x)) > b$, and $t \mapsto f(\varphi_t(x))$ is continuous and non-decreasing, then by Intermediate Value Theorem there exists $t_1, t_2 \in [0, b-a]$ such that $t_1 < t_2$, $f(\varphi_{t_1}(x)) = a$, $f(\varphi_{t_2}(x)) = b$, and $f(\varphi_t(x)) \in [a,b]$ for all $t \in [t_1, t_2]$. Thus for all $t \in [t_1,t_2]$, $f(\varphi_t(x)) = t + f(x)$, in particular

$$\begin{aligned} a &= f(\varphi_{t_1}(x)) = t_1 + f(x) \\ b &= f(\varphi_{t_2}(x)) = t_2 + f(x) \end{aligned}$$

hence $t_2 - t_1 = b - a$, and so $t_2 = b - a$ (since $[t_1, t_2] \subseteq [0, b-a]$). However $f(\varphi_{b-a}(x)) > b = f(\varphi_{t_2}(x))$, a contradiction. So $\varphi_{b-a}(x) \in M^b$ whenever $x \in M^a$.

Now for $y \in M^b$, let $x = \varphi_{a-b}(y)$, then $\varphi_{b-a}(x) = y$ and a similar argument as above shows that $x \in M^a$.

Now for the second part of the theorem, we define the homotopy $r_t\colon [0,1] \times M^b \to M^a$ by

$$r_t(x) = \begin{cases} x & \text{if } f(x) \leq a \\ \varphi_{t(a-f(x))}(x) & \text{if } a < f(x) \leq b \end{cases}$$

It is clear that $r_t$ is continuous, $r_0$ is the identity of $M^b$, and $r_1$ is a retraction from $M^b$ to $M^a$. Therefore $M^a$ is a deformation retract of $M^b$. □

**Theorem 4.2.3.** *Let $M$ be a smooth manifold and $f\colon M \to \mathbb{R}$ be a smooth function. Let $p$ be a non-degenerate critical point of $f$ with index $\lambda$. Set $f(p) = c$. Suppose that there exists $\delta > 0$ such that $f^{-1}[c-\delta, c+\delta]$ is compact and contains no critical point of $f$ other than $p$. Then for all sufficiently small $\epsilon > 0$, the set $M^{c+\epsilon}$ has the same homotopy type of $M^{c-\epsilon}$ with a $\lambda$-cell attached.*

*Proof.* See Theorem 3.2 of [Milnor, 1963]. □

**Theorem 4.2.4.** (*Reeb Sphere Theorem*) *Let $M$ be a compact manifold of dimension $n$ and $f\colon M \to \mathbb{R}$ be a smooth function with only 2 critical points, both of which are non-degenerate, then $M$ is homeomorphic to the $n$-sphere, $S^n$.*

*Proof.* $M$ is compact and $f$ is continuous then the maximum and minimum of $f$ on $M$ are attained. Hence by Remark 4.1.11 $f$ attains its minimum at one of the two critical points, say $p$, and its maximum at the other, say $q$. Set $a = f(p)$ and $b = f(q)$. $p$ is non-degenerate, and the index of $f$ at $p$ is 0, then by Lemma 4.1.8, there exist smooth coordinates $(y_1, \dots, y_n)$ in an open neighborhood $U$ of $p$ such that

$$f(x) = a + (y_1(x))^2 + \dots + (y_n(x))^2.$$

Since $M$ is compact, there exists $\delta > 0$ such that $f^{-1}[a, a + \delta] \subset U$, thus

$$x \in f^{-1}[a, a + \delta] \Leftrightarrow (y_1(x))^2 + \dots + (y_n(x))^2 < \delta$$

so the map $(y_1/\sqrt{\delta}, \dots, y_n/\sqrt{\delta})\colon f^{-1}[a, a + \delta] \to e^n$ defines a diffeomorphism, so $f^{-1}[a, a + \delta]$ is diffeomorphic to the $n$-cell, $e^n$. Similarly there exists $\epsilon > 0$ such that $f^{-1}[b - \epsilon, b]$ is diffeomorphic to the the $n$-cell, $e^n$. Now $f$ has no critical points in $f^{-1}[a + \delta, b - \epsilon]$ (assuming that $\delta$ and $\epsilon$ are taken small enough), and $f^{-1}[a + \delta, b - \epsilon]$ is compact, so by Theorem 4.2.2, $M^{a+\delta} = f^{-1}[a, a + \delta]$ and $M^{b-\epsilon}$ are diffeomorphic. In particular we get that $f^{-1}[a, b - \epsilon]$ is homeomorphic to $e^n$ and $f^{-1}[b - \epsilon, b]$ is homeomorphic to $e^n$, where $f^{-1}\{b - \epsilon\}$ is mapped to the $\dot{e}^n$ by both homeomorphisms. Therefore $M$ is homeomorphic to two $n$-cells glued at their boundaries which is homeomorphic to $S^n$. $\square$

**Remark 4.2.5.** *$M$ in the above theorem need not be diffeomorphic to $S^n$ with its usual smooth structure, see [Milnor, 1956] for a counterexample.*

**Definition 4.2.6.** (*Upper disk and lower disk*) *Let $M$ be a Riemannian manifold and $f\colon M \to \mathbb{R}$ be a Morse function. Assume that $f$ has $m$ critical points, $p_1, p_2, \dots, p_m$, with $f(p_i) = c_i$, such that $c_1 < c_2 < \dots < c_m$. For $\varepsilon > 0$, consider the set $M_{[c_{i-1}+\varepsilon, c_i+\varepsilon]} = \{x \in M\colon c_{i-1} + \varepsilon \le f(x) \le c_i + \varepsilon\} = f^{-1}[c_{i-1} + \varepsilon, c_i + \varepsilon]$. We define the lower disk (resp. upper disk) corresponding to the critical point $p_i$ to be the set of all points in $M_{[c_{i-1}+\varepsilon, c_i+\varepsilon]}$ that converge to the critical point $p_i$ along the integral curves of* $\operatorname{grad} f$ *as the parameter $t$ tends to $+\infty$, (resp. $-\infty$).*

*The lower and upper disks associated to $p_i$ will be denoted by $D_l(p_i)$ and $D_u(p_i)$ respectively.*

**Remark 4.2.7.** *Given a Riemannian manifold $M$ of dimension $n$ and a Morse function $f\colon M \to \mathbb{R}$. Assume that $p_i \in M$ is a critical point of $f$ (the critical points of $f$ are indexed as in the above definition), $c_i = f(p_i)$, and let $\lambda_i$ be the index of $f$ at $p_i$. Since $p_i$ is a critical point of $f$, then $\operatorname{grad} f(p_i) = 0$, thus $p_i$ is fixed by the integral curves of $\operatorname{grad} f$, so $p_i \in D_l(p_i) \cap D_u(p_i)$. By the Lemma of Morse (Lemma 4.1.8), we can choose local coordinates $(x_1, \dots, x_n)$ in a neighborhood $U$ of $p_i$ such that $f = c_i - x_1^2 - \dots - x_{\lambda_i}^2 + x_{\lambda_i+1}^2 + \dots + x_n^2$ in $U$, thus for $\varepsilon$ small enough we have:*

$$D_l(p_i) \cap M_{[c_i-\varepsilon, c_i+\varepsilon]} = \{(x_1, \dots, x_n)\colon x_1^2 + \dots + x_{\lambda_i}^2 \leq \varepsilon,\ x_{\lambda_i+1} = \dots = x_n = 0\}.$$

*And since $f$ has no critical points in $M_{[c_{i-1}+\varepsilon, c_i-\frac{\varepsilon}{2}]}$, then $D_l(p_i) \cap M_{[c_{i-1}+\varepsilon, c_i-\frac{\varepsilon}{2}]}$ is carried diffeomorphically along the integral curves of $\operatorname{grad} f$ to $D_l(p_i) \cap M_{[c_i-\varepsilon, c_i-\frac{\varepsilon}{2}]}$, see page 113 of [Matsumoto, 2002] and the proof of Theorem 4.2.2. Writing $D_l(p_i)$ as the union of the two sets, we see that $D_l(p_i)$ is diffeomorphic to the $\lambda_i$-cell, $e^{\lambda_i}$.*
*Similarly $D_u(p_i)$ is diffeomorphic to the $(n - \lambda_i)$-cell.*

**Definition 4.2.8.** *Let $M$ be a smooth manifold of dimension $n$, and $A$ and $B$ be submanifolds of $M$ of dimensions $k$ and $n - k$, respectively. We say that $A$ and $B$ intersect transversely at a point $p \in M$ if there exist an open neighborhood $U$ of $p$ in $M$, and local coordinates $(x_1, \dots, x_n)$ defined in $U$ such that:*

$$A \cap U = \{(x_1, \dots, x_n) \in U\colon x_{k+1} = \dots = x_n = 0\}$$

*and*

$$B \cap U = \{(x_1, \ldots, x_n) \in U : x_1 = \cdots = x_k = 0\}.$$

(*From here we see that* $x_1(p) = \cdots = x_k(p) = x_{k+1}(p) = \cdots = x_n(p) = 0$, *and that* $A \cap B \cap U = \{p\}$.)

**Theorem 4.2.9.** *Let $M$ be a smooth manifold and $f\colon M \to \mathbb{R}$ be a Morse function with at least two critical points. Arrange the critical points of $f$ as in Definition 4.2.6. For a subscript $i$, we consider the set $M_{[c_{i-2}+\varepsilon, c_i+\varepsilon]}$, containing the two consecutive critical points $p_{i-1}$ and $p_i$. Assume the following conditions hold:*

1. *The index of $p_i$ is one larger than the index of $p_{i-1}$.*
2. *For some $\varepsilon > 0$, $\partial D_l(p_i)$ and $\partial D_u(p_{i-1})$ intersect transversely at a single point in the level surface $f^{-1}(c_{i-1} + \varepsilon)$.*

*Then there exists a Morse function $g\colon M \to \mathbb{R}$ such that:*

1. *$g$ has no critical points in the interior of $M_{[c_{i-2}+\varepsilon, c_i+\varepsilon]}$.*
2. *$g$ coincides with $f$ near the boundary and outside of $M_{[c_{i-2}+\varepsilon, c_i+\varepsilon]}$.*

*Proof.* See Theorem 3.28 of [Matsumoto, 2002]. □

**Remark 4.2.10.** *Let $M$ be a smooth manifold of dimension $n$, $f\colon M \to \mathbb{R}$ be a Morse function, and $p_{i-1}, p_i$ be two consecutive critical points of $f$ of indices $\lambda - 1$ and $\lambda$ respectively. The dimension of $f^{-1}(c_{i-1} + \varepsilon)$ as a smooth manifold is $n - 1$. The dimension of $\partial D_l(p_i)$ is $\lambda - 1$, and that of $\partial D_u(p_{i-1})$ is $n - (\lambda - 1) - 1 = n - \lambda$, so that*

$$\dim f^{-1}(c_{i-1} + \varepsilon) = \dim \partial D_l(p_i) + \dim \partial D_u(p_{i-1})$$

*which makes it possible that $\partial D_l(p_i)$ and $\partial D_u(p_{i-1})$ intersect transversely, viewed as embedded submanifolds of $f^{-1}(c_{i-1} + \varepsilon)$. The above theorem is referred to as* *cancellation of critical points, since $g$ can be viewed as a perturbance of $f$, with 2 less critical points.*

# Chapter 5

# On The Koras–Russell Cubic Threefold

In this chapter we will give basic definitions from algebraic geometry and prove that the Koras–Russell cubic threefold is not isomorphic to the complex affine 3-space.

## 5.1 Algebraic Geometry

This section is from Chapter 1 of [Hartshorne, 1977]. In what follows $K$ denotes an algebraically closed field.

**Definition 5.1.1.** *We define the **affine n-space** over $K$, denoted by $\mathbb{A}^n_K$ or simply $\mathbb{A}^n$, to be the set of all $n$-tuples of elements of $K$. An element $P \in \mathbb{A}^n$ will be called a point, and if $P = (a_1, \ldots, a_n)$ with $a_i \in K$, then $a_i$ will be called the coordinates of $P$.*

*$K[x_1, \ldots, x_n]$ will denote the polynomial ring in $n$ variables over $K$. Elements of $K[x_1, \ldots, x_n]$ will be interpreted as functions from $\mathbb{A}^n$ to $K$, by defining $f(P) =$*

*$f(a_1, \dots, a_n)$, where $f \in K[x_1, \dots, x_n]$ and $P \in \mathbb{A}^n$. Thus for $f \in K[x_1, \dots, x_n]$, we can define the zeros of $f$ as a subset of the affine $n$-space, $Z(f) = \{P \in \mathbb{A}^n \colon f(P) = 0\}$. More generally, if $T$ is a subset of $K[x_1, \dots, x_n]$, we define the zero set of $T$ to be the common zeros of all elements of $T$,*

$$Z(T) = \{P \in \mathbb{A}^n \colon f(P) = 0 \ \textit{for all} \ f \in T\}$$

*Clearly if $I$ is the ideal of $K[x_1, \dots, x_n]$ generated by $T$, then $Z(I) = Z(T)$. Furthermore, since $K[x_1, \dots, x_n]$ is a noetherian ring, any ideal $I$ has a finite set of generators $f_1, \dots, f_r$. Thus $Z(T)$ can be expressed as the common zeros of the finite set of polynomials $\{f_1, \dots, f_r\}$.*

**Definition 5.1.2.** *A subset $Y$ of $\mathbb{A}^n$ is called an algebraic set if there exists a subset $T \subseteq K[x_1, \dots, x_n]$ such that $Y = Z(T)$.*

**Proposition 5.1.3.** *The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.*

*Proof.* See Proposition 1.1 of [Hartshorne, 1977]. □

**Definition 5.1.4.** *We define the* ***Zariski topology*** *on $\mathbb{A}^n$ by taking open sets to be the complements of the algebraic sets. This defines a topology on $\mathbb{A}^n$ due to Proposition 5.1.3.*

**Definition 5.1.5.** *A nonempty subset $Y$ of a topological space $X$ is said to be irreducible if it cannot be expressed as the union of two proper subsets each of which is closed in $Y$.*

**Definition 5.1.6.** *An affine algebraic variety (or simply affine variety) is an* *irreducible closed subset of $\mathbb{A}^n$ (with the Zariski topology). An open subset of an affine variety is called a quasi-affine variety.*

**Definition 5.1.7.** *Let $Y$ be a subset of $\mathbb{A}^n$. We define the ideal of $Y$ in $K[x_1, \ldots, x_n]$, denoted by $I(Y)$, by*

$$I(Y) = \{f \in K[x_1, \ldots, x_n] \colon f(P) = 0 \text{ for all } P \in Y\}$$

**Theorem 5.1.8.** (*Hilbert's Nullstellensatz*) *Let $K$ be an algebraically closed field, let $J$ be an ideal in $K[x_1, \ldots, x_n]$, and let $f \in K[x_1, \ldots, x_n]$ be a polynomial which vanishes at all points of $Z(J)$. Then $f^r \in J$ for some integer $r > 0$.*

*Proof.* See page 85 in [Atiyah and Macdonald, 1969]. □

**Definition 5.1.9.** *Let $R$ be a commutative ring. Let $J$ be an ideal in $R$. We define the radical of $J$, denoted by $\sqrt{J}$ as*

$$\sqrt{J} = \{f \in R \colon f^r \in J \text{ for some integer } r > 0\}$$

*If $J = \sqrt{J}$ we say that $J$ is a radical ideal.*

**Proposition 5.1.10.** *The following properties hold:*

*a) If $T_1 \subseteq T_2$ are two subsets of $K[x_1, \ldots, x_n]$, then $Z(T_1) \supseteq Z(T_2)$.*

*b) If $Y_1 \subseteq Y_2$ are two subsets of $\mathbb{A}^n$, then $I(Y_1) \supseteq I(Y_2)$.*

*c) For any two subsets $Y_1$ and $Y_2$ of $\mathbb{A}^n$ we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.*

*d) For any ideal $J$ of $K[x_1, \ldots, x_n]$, $I(Z(J)) = \sqrt{J}$.*

*e) For any subset $Y \subseteq \mathbb{A}^n$, $Z(I(Y)) = \overline{Y}$, the closure of $Y$ in $\mathbb{A}^n$ endowed with the Zariski topology.*

*Proof.* See Proposition 1.2 of [Hartshorne, 1977]. □

**Corollary 5.1.11.** *There is a one-to-one inclusion-reversing correspondence between algebraic sets in* $\mathbb{A}^n$ *and radical ideals in* $K[x_1, \ldots, x_n]$ *given by* $Y \mapsto I(Y)$ *and* $J \mapsto Z(J)$*. Furthermore, an algebraic set is irreducible if and only if its ideal is a prime ideal.*

*Proof.* See Corollary 1.4 of [Hartshorne, 1977]. □

**Remark 5.1.12.** *Since the correspondence in Corollary 5.1.11 is inclusion-reversing, every maximal ideal* $M$ *of* $K[x_1, \ldots, x_n]$ *corresponds to a minimal irreducible closed subset of* $\mathbb{A}^n$*, which must be a point. Thus an* $n$*-affine space can be identified by the set of maximal ideals of* $K[x_1, \ldots, x_n]$*. We call this set the maximal spectrum of the ring* $K[x_1, \ldots, x_n]$ *and denote it by* $\mathrm{Spec}_{max}(K[x_1, \ldots, x_n])$*. For an ideal* $J$ *in* $K[x_1, \ldots, x_n]$*, letting* $V_J$ *be the set of all maximal ideals containing* $J$*, one can check that the collection of subsets*

$$\{S \subseteq \mathrm{Spec}_{max}(K[x_1, \ldots, x_n]) \colon S^c = V_J \text{ for some ideal } J \text{ of } K[x_1, \ldots, x_n]\}$$

*defines a topology on* $\mathrm{Spec}_{max}(K[x_1, \ldots, x_n])$*, called the Zariski topology as well. In fact* $\mathrm{Spec}_{max}(K[x_1, \ldots, x_n])$ *is homeomorphic to the affine* $n$*-space, and thus* $\mathbb{A}^n$ *could be identified with* $\mathrm{Spec}_{max}(K[x_1, \ldots, x_n])$ *as a topological space.*

**Definition 5.1.13.** *Let* $Y \subseteq \mathbb{A}^n$ *be an affine algebraic set. We define the affine coordinate ring* $A(Y)$ *of* $Y$ *to be* $K[x_1, \ldots, x_n]/I(Y)$*.*

**Definition 5.1.14.** *A topological space* $X$ *is said to be noetherian if it satisfies the descending chain condition for closed subsets: for any sequence* $Y_1 \supseteq Y_2 \supseteq \ldots$ *of closed subsets, there exists a positive integer* $r$ *such that* $Y_r = Y_{r+1} = \ldots$

**Remark 5.1.15.** *$\mathbb{A}^n$ is a noetherian topological space. Indeed, if $Y_1 \supseteq Y_2 \supseteq \ldots$ is a descending chain of closed subsets, then $I(Y_1) \subseteq I(Y_2) \subseteq \ldots$ is an ascending chain of ideals in $K[x_1, \ldots, x_n]$, which is a noetherian ring, then the chain of ideals is eventually stationary. Note that for all $i$, $Y_i$ is closed, so $Y_i = \overline{Y_i} = Z(I(Y_i))$, so the chain $Y_i$ is also stationary.*

**Proposition 5.1.16.** *Let $X$ be a noetherian topological space, then every nonempty closed subset $Y$ of $X$ can be expressed as a finite union of irreducible closed subsets $Y_i$. Moreover if we require that no one of the subsets $Y_i$ is contained in the other, then the $Y_i$ are uniquely determined.*

*Proof.* See Proposition 1.5 of [Hartshorne, 1977]. □

**Definition 5.1.17.** *Let $X$ be a topological space, we define the dimension of $X$ to be the supremum of all integers $n$ such that there exists a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of distinct irreducible closed subsets of $X$. We define the dimension of an affine or quasi-affine variety to be its dimension as a topological space.*

**Definition 5.1.18.** *Given a ring $R$, we define the height of a prime ideal $\mathfrak{p}$ to be the supremum of all integers $n$ such that there exists a chain $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{p}_n = \mathfrak{p}$ of distinct prime ideals. We define the dimension (or Krull dimension) of $R$ to be the supremum of the heights of all prime ideals.*

**Proposition 5.1.19.** *If $Y$ is an affine algebraic set, then the dimension of $Y$ is equal to the dimension of its affine coordinate ring $A(Y)$.*

*Proof.* See Proposition 1.7 of [Hartshorne, 1977]. □

**Definition 5.1.20.** *Let $Y$ be a quasi-affine variety in $\mathbb{A}^n$. We say that a function $f\colon Y \to K$ is regular at a point $P \in Y$ if there exist an open neighborhood $U$ of*

*$P$ in $Y$, and polynomials $g, h \in K[x_1, \ldots, x_n]$ such that $h$ is nowhere zero on $U$, and $f = g/h$ on $U$. We say that $f$ is regular on $Y$ if it is regular at every point of $Y$.*

**Definition 5.1.21.** *Let $X$ and $Y$ be two affine varieties. A **morphism** $\varphi\colon X \to Y$ is a continuous map such that for every open set $V \subseteq Y$, and for every regular function $f\colon V \to K$, the function $f \circ \varphi\colon \varphi^{-1}(V) \to K$ is regular.*

**Remark 5.1.22.** *Clearly the composition of two morphisms is a morphism, and $Id_X\colon X \to X$ is a morphism when $X$ is an affine variety, thus we have a category whose objects are affine varieties over a fixed algebraically closed field $K$. In particular, we have the notion of isomorphisms. Given two affine varieties $X$ and $Y$ over $K$, an isomorphism $\varphi\colon X \to Y$ is a morphism which admits an inverse morphism $\psi\colon Y \to X$ with $\psi \circ \varphi = Id_X$ and $\varphi \circ \psi = Id_Y$. An automorphism of an affine variety $X$ is an isomorphism from $X$ to $X$.*

**Definition 5.1.23.** *A subset of a topological space is locally closed if it is an open subset of its closure. If $X$ is an affine variety and $Y$ is an irreducible locally closed subset of $X$, then $Y$ is also an affine variety. We call $Y$ a subvariety of $X$.*

**Definition 5.1.24.** *Let $X$ and $Z$ be two affine varieties. An embedding $\varphi\colon X \to Z$ is an injective morphism whose image is closed, and which induces an isomorphism between $X$ and $\varphi(X)$. Two embeddings $\varphi_1, \varphi_2\colon X \to Z$ are said to be equivalent if there exists an automorphism $\phi$ of $Z$ such that $\phi \circ \varphi_1 = \varphi_2$.*

**Definition 5.1.25.** *Two subvarieties $X_1$ and $X_2$ of an affine variety $Z$ are said to be equivalent if there exists an automorphism $\phi$ of $Z$ such that $\phi(X_1) = X_2$.*

## 5.2 The Koras–Russell Cubic Threefold

In this section we will define the Koras–Russell cubic threefold, and prove that it is not isomorphic to the affine 3-space over $\mathbb{C}$.

Let $\mathbb{A}^4$ be the affine 4-space over $\mathbb{C}$ and consider the polynomial $h \in \mathbb{C}[x, y, z, w]$

$$h = x^2y + x + z^2 + w^3$$

and consider the algebraic affine variety of $\mathbb{A}^4$, $Z(h) = \{P \in \mathbb{A}^4 \colon h(P) = 0\}$, of dimension 3. When $\mathbb{A}^4$ is identified with $\mathbb{C}^4$, $h \colon \mathbb{C}^4 \to \mathbb{C}$ is a holomorphic map, whose (complex) Jacobian is never zero, thus $h$ is a complex submersion and so $h^{-1}(\{0\})$ is a complex submanifold of $\mathbb{C}^4$ of codimension 1, i.e. of complex dimension 3. In particular $h^{-1}(\{0\})$ can be viewed as a six dimensional smooth (real) manifold. We will denote by the Koras–Russell cubic threefold the set $X = \{(x, y, z, w) \in \mathbb{C}^4 \colon x^2y + x + z^2 + w^3 = 0\}$ regarded as an object in any of the above categories accordingly.

The Koras–Russell cubic threefold belongs to the family $X_{d,k,l}$ of affine hypersurfaces in $\mathbb{A}^4_{\mathbb{C}}$, where $d \geq 2$, $2 \leq k < l$ with $k$ and $l$ relatively prime, and $X_{d,k,l}$ is the zero set of the polynomial $P_{d,k,l} = x^dy + z^k + w^l + x$. Those varieties were first introduced by Koras and Russell, see [Kraft, 1996], when they were proving that all algebraic actions of $\mathbb{C}^*$ on the affine 3-space are linearizable; i.e. given a group action $\varphi \colon \mathbb{C}^* \times \mathbb{A}^3 \to \mathbb{A}^3$ of the multiplicative group $(\mathbb{C}^*, \cdot)$ on the set $\mathbb{A}^3$, that is also a morphism between the corresponding affine varieties, then in a suitably chosen coordinates for $\mathbb{A}^3$, $\varphi$ is linear. For this purpose it was necessary to show that none of the varieties $X_{d,k,l}$ are isomorphic to $\mathbb{A}^3$. For some of these varieties, this could be done with geometric invariants. However the problem was difficult for the Koras–Russell cubic threefold. In 1994, Makar-Limanov proved the result using the Makar-Limanov invariant,

$$ML(Y) = \bigcap_{\delta \in LND(Y)} \ker \delta$$

where $Y$ is an affine variety and $LND(Y)$ is the set of all locally nilpotent derivations on the ring of regular functions on $Y$, see [Kraft, 1996]. Makar-Limanov showed that $ML(X) \neq \mathbb{C}$ whereas $ML(\mathbb{A}^3) = \mathbb{C}$. It has been shown later by Makar-Limanov and Kaliman in [Kaliman and Makar-Limanov, 1997] that all members of the family of varieties $X_{d,k,l}$ defined above have non-trivial Makar-Limanov invariant, and therefore not isomorphic to $\mathbb{A}^3$.

In what follows we will demonstrate a simpler proof for the fact that the Koras–Russell cubic threefold is not isomorphic to $\mathbb{A}^3$. The proof is from [Dubouloz et al., 2010]. $X$ will denote the Koras–Russell cubic threefold.

**Lemma 5.2.1.** *If $\phi$ is an algebraic automorphism of $\mathbb{A}^4$ which restricts to an automorphism of $X$, then $\phi$ fixes the point $(0, 0, 0, 0)$.*

*Proof.* See Proposition 2.5 in [Dubouloz et al., 2010]. □

**Lemma 5.2.2.** *Every automorphism of $X$ extends to an automorphism of $\mathbb{A}^4$.*

*Proof.* See Theorem 4.3 in [Dubouloz et al., 2010]. □

**Corollary 5.2.3.** *Every automorphism of $X$ fixes the point $(0, 0, 0, 0)$.*

*Proof.* Let $\psi$ be an automorphism of $X$, then by Lemma 6.2 $\psi$ extends to an an automorphism of $\mathbb{A}^4$, namely $\phi$. $\phi|_X = \psi$, then by Lemma 6.1 $\phi(0, 0, 0, 0) = (0, 0, 0, 0) = \psi(0, 0, 0, 0)$. □

**Theorem 5.2.4.** *$X$ is not isomorphic to $\mathbb{A}^3$.*

*Proof.* $\mathbb{A}^3$ will be identified with the affine variety $\{(x, y, z, w) \in \mathbb{A}^4 \colon w = 0\}$. Suppose to the contrary that $X$ is isomorphic to $\mathbb{A}^3$. Let $\psi\colon X \to \mathbb{A}^3$ be an isomorphism. The mapping $\phi\colon \mathbb{A}^3 \to \mathbb{A}^3$ given by $(x, y, z, 0) \mapsto (x, y, z + 1, 0)$ is an isomorphism, then the map

$$\psi^{-1} \circ \phi \circ \psi\colon X \to X$$

is an automorphism of $X$, thus by Corollary 5.2.3, $\psi^{-1} \circ \phi \circ \psi(0, 0, 0, 0) = (0, 0, 0, 0)$, so that $\phi \circ \psi(0, 0, 0, 0) = \psi(0, 0, 0, 0)$. Setting $\psi(0, 0, 0, 0) = (a, b, c, 0)$, we get that $(a, b, c + 1, 0) = \phi(a, b, c, 0) = (a, b, c, 0)$. A contradiction. This ends the proof. $\square$

# Chapter 6

# Main Result

In this chapter we will denote by $X$ the Koras–Russell Cubic Threefold, and by $h$ the defining function of $X$; i.e. $h\colon \mathbb{C}^4 \to \mathbb{C}$, defined by $h(x, y, z, w) = x^2y + x + z^2 + w^3$ and $X = \{(x, y, z, w) \in \mathbb{C}^4 \colon h(x, y, z, w) = 0\}$. It was proven by Dimca (1992) that $X$ is diffeomorphic to $\mathbb{R}^6$, see [Kraft, 1996]. The proof was based on a theorem by Dimca that states that every smooth contractible affine variety (over $\mathbb{C}$) of dimension $d \geq 3$ is diffeomorphic to $\mathbb{C}^d$. In what follows we will study the Koras–Russell Cubic Threefold using Morse Theory. For that we will define a suitable smooth function, $f\colon \mathbb{C}^4 \to \mathbb{R}$, whose restriction to $X$ is a Morse function; i.e. the critical points of $f$ restricted to $X$ are nondegenerate. Rigorously, $\tilde{f} = f \circ \iota\colon X \to \mathbb{R}$ is a Morse function, where $\iota$ is the inclusion map $X \hookrightarrow \mathbb{C}^4$. It is clear that $p \in X$ is a critical point of $\tilde{f}$ if and only if the complex differential, $d\tilde{f}|_p$ of $\tilde{f}$ at $p$ is zero, which is equivalent to say that there exist $\mu, \lambda \in \mathbb{C}$ (Lagrange multipliers) such that $df\,|_p = \lambda\, dh\,|_p + \mu\, d\bar{h}\,|_p$. We will prove that in this case, $\mu$ and $\lambda$ should be complex conjugates.

**Lemma 6.1.** *Let $f\colon \mathbb{C}^4 \to \mathbb{R}$ be a smooth function, $p \in X$, and $\mu, \lambda \in \mathbb{C}$ such that $df\,|_p = \lambda\, dh\,|_p + \mu\, d\bar{h}\,|_p$, then $\lambda = \bar{\mu}$.*

*Proof.* $f$ is a real valued function, then $f = \bar{f}$, hence $df = d\bar{f} = \overline{df}$. So we have

$$df\,|_p = \overline{df}\,|_p$$

$$\lambda\, dh\,|_p + \mu\, d\bar{h}\,|_p = \bar{\lambda}\, d\bar{h}\,|_p + \bar{\mu}\, dh\,|_p$$

$$(\lambda - \bar{\mu})\, dh\,|_p = (\bar{\lambda} - \mu)\, d\bar{h}\,|_p$$

Now suppose to the contrary that $\lambda \neq \bar{\mu}$, then $dh\,|_p = \dfrac{\bar{\lambda} - \mu}{\lambda - \bar{\mu}}\; d\bar{h}\,|_p$. By Proposition 3.8.14, and since $h$ is holomorphic, $dh\,|_p = 0$ on $T_p^{0,1}\mathbb{C}^4$ and $d\bar{h}\,|_p = 0$ on $T_p^{1,0}\mathbb{C}^4$. But $dh\,|_p = \dfrac{\bar{\lambda} - \mu}{\lambda - \bar{\mu}}\; d\bar{h}\,|_p$, so $dh\,|_p = 0$ on $T_p^{1,0}\mathbb{C}^4$, hence $h$ is singular at $p$, a contradiction, since $dh = (2xy + 1)\, dx + x^2\, dy + 2z\, dz + 3w^2\, dw$ never vanishes. Therefore $\lambda = \bar{\mu}$. $\square$

**Lemma 6.2.** *Let $a \in \mathbb{R}$, and let $f\colon \mathbb{C}^4 \to \mathbb{R}$ be defined as $f(x, y, z, w) = |x - a|^2 + |y|^2 + |z|^2 + |w|^2$, and $\tilde{f}\colon X \to \mathbb{R}$ be the restriction of $f$ on $X$. Then for $a = 0$, $\tilde{f}$ is not a Morse function, and for any other value of $a$, $\tilde{f}$ has at least 9 critical points.*

*Proof.* Let $\zeta = (x, y, z, w) \in \mathbb{C}^4$. $\zeta$ is a critical point of $\tilde{f}$ if and only if $\zeta \in X$, and $df\,|_\zeta = \lambda\, dh\,|_\zeta + \mu\, d\bar{h}\,|_\zeta$, for some $\mu, \lambda \in \mathbb{C}$. This reduces to solving the following system:

$$\begin{cases} h(\zeta) = 0 \\ df\,|_\zeta = \lambda\, dh\,|_\zeta + \mu\, d\bar{h}\,|_\zeta \end{cases}$$

If $\zeta = (x, y, z, w)$ is a solution for the above system, then by Corollary 6.1, $\lambda = \bar{\mu}$, so the system becomes:

$$\begin{cases} h(\zeta) = 0 \\ df\,|_\zeta = \lambda\, dh\,|_\zeta + \bar{\lambda}\, d\bar{h}\,|_\zeta \end{cases}$$

Writing $f$ as $f = (x-a)(\bar{x}-a) + y\bar{y} + z\bar{z} + w\bar{w}$, we get that

$$df = (x-a)\,d\bar{x} + y\,d\bar{y} + z\,d\bar{z} + w\,d\bar{w} + (\bar{x}-a)\,dx + \bar{y}\,dy + \bar{z}\,dz + \bar{w}\,dw$$

so the system becomes:

$$\begin{cases} x^2y + x + z^2 + w^3 = 0 \\ \bar{x} - a = \lambda(2xy+1) \\ \bar{y} = \lambda x^2 \\ \bar{z} = 2\lambda z \\ \bar{w} = 3\lambda w^2 \end{cases}$$

For $a = 0$, we note that for every $\lambda \in \partial\mathbb{D} = \{t \in \mathbb{C} \colon |t| = 1\}$, the point $(\frac{-1}{\lambda}, \lambda, 0, 0)$ solves the above system, and hence $\tilde{f}$ is not a Morse function in this case, since the critical points of a Morse function are isolated.

For $a \neq 0$, we can deduce directly that $\lambda \neq 0$ (if $\lambda = 0$ then $x = a$ and $y = z = w = 0$, so it follows from the defining equation that $a = 0$). Multiplying the third equation by $y$, the forth by $z$, the fifth by $w$, and substituting in the defining equation we get

$$\begin{aligned} |y|^2 + \lambda x + \frac{|z|^2}{2} + \frac{|w|^2}{3} &= 0 \\ |y|^2 + \frac{|z|^2}{2} + \frac{|w|^2}{3} &= -\lambda x \end{aligned} \tag{$*$}$$

so $\lambda x$ is a (nonpositive) real number, hence either $x = 0$, or $\lambda = -r\bar{x}$, for some $r > 0$. If $x = 0$, then by the above equation $|y|^2 + \frac{|z|^2}{2} + \frac{|w|^2}{3} = 0$,so $y = z = w = 0$, and $\lambda = -a$, thus $(0,0,0,0)$ is a critical point. Now if $x \neq 0$, then $\lambda = -r\bar{x}$ for some $r > 0$, we substitute for $\lambda$ in the initial system to get:

$$\begin{cases} x^2y + x + z^2 + w^3 = 0 \\ \bar{x} - a = -r\bar{x}(2xy + 1) \\ \bar{y} = -r\bar{x}x^2 \\ \bar{z} = -2r\bar{x}z \\ \bar{w} = -3r\bar{x}w^2 \end{cases}$$

from the second and third equation we get

$$\bar{x}(1 - 2r^2|x|^4 + r) = a$$

then $x \in \mathbb{R}$, and so by the third equation, $y \in \mathbb{R}$ as well. Knowing that $x$ and $y$ are real numbers the system becomes:

$$\begin{cases} x^2y + x + z^2 + w^3 = 0 \\ x - a = -rx(2xy + 1) \\ y = -rx^3 \\ \bar{z} = -2rxz \\ \bar{w} = -3rxw^2 \end{cases}$$

from the forth equation it follows that $z(1-4r^2x^2) = 0$, hence $z = 0$ or $2rx = \pm 1$. Suppose that $z \neq 0$, then $2rx = \pm 1$, with the third and forth equation, we get a system of equations in $x$ and $r$:

$$\begin{cases} \frac{-1}{2}x^3 + x + rx = a \\ rx = \pm\frac{1}{2} \end{cases}$$

with the constraint that $\frac{|z|^2}{2} + \frac{|w|^2}{3} > 0$, so substituting for $\lambda$ and $y$ in $(*)$ we get that $rx^4 < 1$. For a wise choice of $a$, the above system has no real solution in $x$ and $r$.

Now for $z = 0$, we study the equation in $w$. $\bar{w} = -3rxw^2$ implies that $w(1 + (3rxw)^3) = 0$, so either $w = 0$ or $w^3 = \dfrac{-1}{27r^3x^3}$. If $w = 0$, the system reduces to 3 equations in $x$,$y$,$r$:

$$\begin{cases} x^2y + x = 0 \\ x - a = -rx(2xy + 1) \\ y = -rx^3 \end{cases}$$

since $x \neq 0$, then $xy = -1$, substituting for $y$ we get 2 equations in $r$ and $x$:

$$\begin{cases} rx^4 = 1 \\ x - a = rx \end{cases}$$

which reduces to a forth degree equation in $x$,

$$x^4 + ax - 1 = 0$$

the above equation admits 2 real nonzero roots for all values of $a$, every solution $x$, determines one critical point $(x, \frac{-1}{x}, 0, 0)$, thus this case yields two critical points. Finally if $w \neq 0 (z = 0$ here), then $w^3 = \dfrac{-1}{27r^3x^3}$, substituting for $w^3$ in the defining equation, we obtain 3 equations in $x$, $y$ and $r$:

$$\begin{cases} x^2y + x - \frac{1}{27r^3x^3} = 0 \\ x - a = -rx(2xy+1) \\ y = -rx^3 \end{cases}$$

substituting for $y$ we get 2 equations in $x$ and $r$:

$$\begin{cases} x - a = 2r^2x^5 - rx \\ 27r^3x^4 - 27r^4x^8 - 1 = 0 \end{cases}$$

for all values of a the above system has at least 2 solutions in $\mathbb{R}^{*2}$, each solution $(r, x)$ gives 3 critical points $(x, -rx^3, 0, \dfrac{-1}{3rx}u_i)$, with $i = 1, 2, 3$, where $u1, u2, u3$ are the cubic roots of the unity, thus this case yields at least 6 critical points. Therefore for any choice of $a \neq 0$, $\tilde{f}$ has at least 9 critical points. In fact 9 critical points could be attained for a suitable choice of $a$; e.g. $a = 3$. □

**Lemma 6.3.** *Let $b \in \mathbb{R}^*$, and let $f\colon \mathbb{C}^4 \to \mathbb{R}$ be defined as $f(x, y, z, w) = |x|^2 + |y|^2 + |z|^2 + |w - b|^2$, and $\tilde{f}\colon X \to \mathbb{R}$ be the restriction of $f$ to $X$. Then $\tilde{f}$ has at least 3 critical points, and for $b = 2$, $\tilde{f}$ has exactly 3 critical points.*

*Proof.* As in the proof of Lemma 6.2, $\zeta$ is a critical point of $\tilde{f}$ if and only if $\zeta = (x, y, z, w)$ solve the following system:

$$\begin{cases} h(\zeta) = 0 \\ df\,|_\zeta = \lambda\, dh\,|_\zeta + \bar{\lambda}\, d\bar{h}\,|_\zeta \end{cases}$$

equivalently

$$\begin{cases} x^2y + x + z^2 + w^3 = 0 \\ \bar{x} = \lambda(2xy + 1) \\ \bar{y} = \lambda x^2 \\ \bar{z} = 2\lambda z \\ \bar{w} - b = 3\lambda w^2 \end{cases}$$

Note that $\lambda \neq 0$, otherwise we get that $x = y = z = 0$ and $w = b$, substituting in the defining equation we get that $b = 0$. Multiplying the first equation by $\lambda$, the second equation by $x$, the third by $y$, the forth by $z$, and the fifth by $w$ we get

$$\begin{cases} \lambda x^2y + \lambda x + \lambda z^2 + \lambda w^3 = 0 \\ |x|^2 = 2\lambda x^2y + \lambda x \\ |y|^2 = \lambda x^2 y \\ |z|^2 = 2\lambda z^2 \\ |w|^2 - bw = 3\lambda w^3 \end{cases}$$

note that $|x|^2 - |y|^2 = \lambda x^2y + \lambda x$, so substituting in the first equation we get

$$|x|^2 - |y|^2 + \frac{|z|^2}{2} + \frac{|w|^2}{3} - \frac{bw}{3} = 0$$

so that

$$\frac{bw}{3} = |x|^2 - |y|^2 + \frac{|z|^2}{2} + \frac{|w|^2}{3}$$

$b \in \mathbb{R}^*$, hence $w \in \mathbb{R}$, and $w \neq 0$, since otherwise the equation $\bar{w} - b = 3\lambda w^3$

would imply that $b = 0$. Solving for $\lambda$ we get that

$$\lambda = \frac{\bar{w} - b}{3w^2} = \frac{w - b}{3w^2}$$

so $\lambda$ is real. Note that $|x|^2 - 2|y|^2 = \lambda x$, and $\lambda \in \mathbb{R}^*$, so $x \in \mathbb{R}$, and thus $y \in \mathbb{R}$, since $\bar{y} = \lambda x^2$. Now the equation $\bar{z} = 2\lambda z$ implies that $z^2 = \dfrac{|z|^2}{2\lambda}$, so $z^2$ is real and thus one of the following holds:

1. $z \in \mathbb{R}^*$ and $\lambda = \frac{1}{2}$

2. $z \in \mathrm{i}\,\mathbb{R}^*$ and $\lambda = -\frac{1}{2}$

3. $z = 0$

If (1) holds: Substituting for $\lambda = \frac{1}{2}$ in $w - b = 3\lambda w^2$, we get the following quadratic equation:

$$\frac{3}{2}w^2 - w + b = 0$$

whose discriminant is $\Delta = -6b + 1$ so that

$$b > \frac{1}{6} \Leftrightarrow \Delta < 0 \Leftrightarrow \text{ the above equation has no real solutions}$$

If (2) holds: Substituting for $\lambda = -\frac{1}{2}$ in $w - b = 3\lambda w^2$, we get the following quadratic equation:

$$\frac{3}{2}w^2 + w - b = 0$$

whose discriminant is $\Delta = 6b + 1$ so that

$$b < -\frac{1}{6} \Leftrightarrow \Delta < 0 \Leftrightarrow \text{ the above equation has no real solutions}$$

We will show that for $b = 2$, there are no critical points with $z \neq 0$. Set $b = 2$

and assume $z \neq 0$, thus (1) is an impossible case, and so case (2) holds. $\lambda = -\frac{1}{2}$, so substituting for $\lambda$ in the equations with $x$, $y$ and $\lambda$ we get that

$$\begin{cases} y = -\frac{1}{2}x^2 \\ x = -\frac{1}{2}(2xy + 1) \end{cases}$$

which reduces to a third degree polynomial in $x$:

$$\frac{x^3}{2} - x - \frac{1}{2} = 0$$

which factors into

$$\frac{1}{2}(x+1)\left(x - \frac{\sqrt{5}+1}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right) = 0$$

for each value of $x$ we obtain one value of $y$, using the equation $y = -\frac{1}{2}x^2$,

$$\begin{cases} x = -1, y = -1/2 \\ x = \frac{\sqrt{5}+1}{2}, y = \frac{-\sqrt{5}-3}{4} \\ x = \frac{1-\sqrt{5}}{2}, y = \frac{\sqrt{5}-3}{4} \end{cases}$$

For each couple $(x, y)$ of the solutions above, the value of $(-x^2y - x)^{\frac{1}{3}}$ is strictly greater than $\frac{22}{25}$. Now from the defining equation, $w^3 + x^2y + x + z^2 = 0$, we have that $z \in \mathrm{i}\,\mathbb{R}^*$, so $z^2 < 0$, so that

$$\begin{aligned} w^3 + x^2y + x &> 0 \\ w^3 + &> -x^2y - x \\ w &> (-x^2y - x)^{\frac{1}{3}} \\ w &> \frac{22}{25} \end{aligned}$$

Now solving the quadratic equation in $w$, for $b = 2$, we get that $w = \dfrac{\sqrt{13} - 1}{3}$ or $w = \dfrac{-\sqrt{13} - 1}{3}$, both of which are less than $\frac{22}{25}$ . Thus for $b = 2$, there are no critical points with $z \neq 0$, and so the least possible number of critical points is 0 so far.

<u>If (3) holds:</u> In this case $z = 0$, then the system becomes:

$$\begin{cases} x^2y + x + w^3 = 0 \\ x = \lambda(2xy + 1) \\ y = \lambda x^2 \\ w - b = 3\lambda w^2 \end{cases}$$

Substituting $\lambda x^2$ for $y$ and $\dfrac{w - b}{3w^2}$ for $\lambda$, we get:

$$\begin{cases} x = \dfrac{w - b}{3w^3}\left(2\left(\dfrac{w - b}{3w^3}\right)x^3 + 1\right) \\ \left(\dfrac{w - b}{3w^3}\right)x^4 + x + w^3 = 0 \end{cases}$$

This system has at least 3 solutions, and for $b = 2$, the system has exactly 3 solutions $(x_i, w_i)$, each of which determines exactly one critical point. This shows that $\tilde{f}$ has at least 3 critical points, and for $b = 2$, we obtain the optimal case. $\square$

**Remark 6.4.** *The functions* $|x|^2+|y-a|^2+|z|^2+|w|^2$ *and* $|x|^2+|y|^2+|z-b|^2+|w|^2$ *have more than* 3 *critical points when restricted to* $X$ *for any* $a, b \in \mathbb{R}^*$. *In none of the* 2 *cases one can conclude that* $w$ *should be real.*

**Remark 6.5.** *We will fix* $b = 2$ *and consider the function defined in Lemma 6.3. Throughout the chapter,* $f$ *will denote the smooth map* $f\colon \mathbb{C}^4 \to \mathbb{R}$ *defined by* $f(x, y, z, w) = |x|^2 + |y|^2 + |z|^2 + |w - 2|^2$, *and by* $\tilde{f}$ *its restriction to* $X$. *In fact* $\tilde{f}$ *is a strongly plurisubharmonic function, so by Theorem 4.1.10, the index of* $\tilde{f}$ *at any non-degenerate critical point is at most* 3. *We will call the three critical points of* $\tilde{f}$, $p_1$, $p_2$ *and* $p_3$, *such that* $f(p_1) < f(p_2) < f(p_3)$. *Numerically, the approximate solutions to the last system of equations in Lemma 6.3 are* $(1.2320, 0.7570)$,$(-0.5995, 0.8721)$,$(0.6539, -0.6161)$, *hence the three critical points are numerically approximated by*

$$p_1 \approx (-0.5995, -0.1776, 0, 0.8721) \text{ and } f(p_1) \approx 1.6630$$

$$p_2 \approx (1.2320, -1.0974, 0, 0.7570) \text{ and } f(p_2) \approx 4.2699$$

$$p_3 \approx (0.6539, -0.9822, 0, -0.6161) \text{ and } f(p_3) \approx 8.2365$$

**Lemma 6.6.** *The function* $\tilde{f}\colon X \to \mathbb{R}$ *is a Morse function, and the index of* $\tilde{f}$ *at each of* $p_1, p_2, p_3$ *is* 0, 2 *and* 3 *respectively.*

*Proof.* We will define local coordinate charts, and calculate the Hessian of $\tilde{f}$ in those charts at $p_1$,$p_2$ and $p_3$. For $\zeta = (x, y, z, w) \in X$ such that $x \neq 0$, let $U$ be an open neighborhood of $\zeta$ in $X$ away from $x = 0$. For $(x, y, z, w) \in U$ write $y = \dfrac{x + z^2 + w^3}{-x^2}$. In real coordinates, $x = x_1 + \mathrm{i}\, x_2$, $y = y_1 + \mathrm{i}\, y_2$, $z = z_1 + \mathrm{i}\, z_2$,

$w = w_1 + \mathrm{i}\, w_2$, we have

$$\begin{aligned}
y =& \frac{\bar{x}}{-x\bar{x}} + \frac{z^2\bar{x}^2}{-x^2\bar{x}^2} + \frac{w^3\bar{x}^2}{-x^2\bar{x}^2} \\
=& \frac{x_1 - \mathrm{i}\, x_2}{-x_1^2 - x_2^2} + \frac{(z_1 + \mathrm{i}\, z_2)^2(x_1 - \mathrm{i}\, x_2)^2}{-x_1^4 - 2x_1^2x_2^2 - x_2^4} + \frac{(w_1 + \mathrm{i}\, w_2)^3(x_1 - \mathrm{i}\, x_2)^2}{-x_1^4 - 2x_1^2x_2^2 - x_2^4} \\
=& \frac{x_1}{-x_1^2 - x_2^2} + \frac{-\mathrm{i}\, x_2}{-x_1^2 - x_2^2} + \frac{z_1^2x_1^2 - z_1^2x_2^2 + 4z_1z_2x_1x_2 - z_2^2x_1^2 + z_2^2x_2^2}{-x_1^4 - 2x_1^2x_2^2 - x_2^4} \\
&+ \frac{-2\,\mathrm{i}\, z_1^2x_1x_2 + 2\,\mathrm{i}\, z_1z_2x_1^2 - 2\,\mathrm{i}\, z_1z_2x_2^2 + 2\,\mathrm{i}\, z_2^2x_1x_2}{-x_1^4 - 2x_1^2x_2^2 - x_2^4} \\
&+ \frac{w_1^3x_1^2 - w_1^3x_2^2 + 6w_1^2w_2x_1x_2 - 3w_1w_2^2x_1^2 + 3w_1w_2^2x_2^2 - 2w_2^3x_1x_2}{-x_1^4 - 2x_1^2x_2^2 - x_2^4} \\
&+ \frac{-2\,\mathrm{i}\, w_1^3x_1x_2 + 3\,\mathrm{i}\, w_1^2w_2x_1^2 - 3\,\mathrm{i}\, w_1^2w_2x_2^2 + 6\,\mathrm{i}\, w_1w_2^2x_1x_2 - \mathrm{i}\, w_2^3x_1^2 + \mathrm{i}\, w_2^3x_2^2}{-x_1^4 - 2x_1^2x_2^2 - x_2^4}
\end{aligned}$$

So we have

$$\begin{aligned}
y_1 =& \frac{x_1}{-x_1^2 - x_2^2} + \frac{z_1^2x_1^2 - z_1^2x_2^2 + 4z_1z_2x_1x_2 - z_2^2x_1^2 + z_2^2x_2^2}{-x_1^4 - 2x_1^2x_2^2 - x_2^4} \\
&+ \frac{w_1^3x_1^2 - w_1^3x_2^2 + 6w_1^2w_2x_1x_2 - 3w_1w_2^2x_1^2 + 3w_1w_2^2x_2^2 - 2w_2^3x_1x_2}{-x_1^4 - 2x_1^2x_2^2 - x_2^4}
\end{aligned}$$

and

$$\begin{aligned}
y_2 =& + \frac{-x_2}{-x_1^2 - x_2^2} + \frac{-2z_1^2x_1x_2 + 2z_1z_2x_1^2 - 2z_1z_2x_2^2 + 2z_2^2x_1x_2}{-x_1^4 - 2x_1^2x_2^2 - x_2^4} \\
&+ \frac{-2w_1^3x_1x_2 + 3w_1^2w_2x_1^2 - 3w_1^2w_2x_2^2 + 6w_1w_2^2x_1x_2 - w_2^3x_1^2 + w_2^3x_2^2}{-x_1^4 - 2x_1^2x_2^2 - x_2^4}
\end{aligned}$$

So the representative of $\tilde{f}$ in the above chart is

$$\tilde{f}(x_1, x_2, z_1, z_2, w_1, w_2) = x_1^2 + x_2^2 + y_1^2 + y_2^2 + z_1^2 + z_2^2 + (w_1 - 2)^2 + w_2^2$$

where $y_1 = y_1(x_1, x_2, z_1, z_2, w_1, w_2)$ and $y_2 = y_2(x_1, x_2, z_1, z_2, w_1, w_2)$ are the smooth functions defined above. Letting $u, v \in \{x_1, x_2, z_1, z_2, w_1 - 2, w_2\}$ be any of the coordinate maps, the first order partial derivatives of $\tilde{f}$ are:

$$\frac{\partial \tilde{f}}{\partial v} = 2v + 2y_1\frac{\partial y_1}{\partial v} + 2y_2\frac{\partial y_2}{\partial v}$$

and the second order partial derivatives of $\tilde{f}$ are:

$$\frac{\partial^2 \tilde{f}}{\partial v \partial u} = \begin{cases} 2 + 2\left(\frac{\partial y_1}{\partial v}\right)^2 + 2y_1\frac{\partial^2 y_1}{\partial v^2} + 2\left(\frac{\partial y_2}{\partial v}\right)^2 + 2y_2\frac{\partial^2 y_2}{\partial v^2} & \text{if } u = v \\ 2\frac{\partial y_1}{\partial v}\frac{\partial y_1}{\partial u} + 2y_1\frac{\partial^2 y_1}{\partial v \partial u} + 2\frac{\partial y_2}{\partial v}\frac{\partial y_2}{\partial u} + 2y_2\frac{\partial^2 y_2}{\partial v \partial u} & \text{if } u \neq v \end{cases}$$

Note that the three critical points of $\tilde{f}$ have real $x, y, w$ coordinates, and $z = 0$, so they correspond to the points $(x_1^i, 0, 0, 0, w_1^i, 0)$, for $i = 1, 2, 3$. Calculating the Hessian of $\tilde{f}$ in the given coordinate charts and evaluating the matrix at the a point of the form $(x_1, 0, 0, 0, w_1, 0)$, we obtain the following matrix:

$$\begin{pmatrix} m_{11}(x_1, w_1) & 0 & 0 & 0 & m_{15}(x_1, w_1) & 0 \\ 0 & m_{22}(x_1, w_1) & 0 & 0 & 0 & m_{26}(x_1, w_1) \\ 0 & 0 & m_{33}(x_1, w_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{44}(x_1, w_1) & 0 & 0 \\ m_{15}(x_1, w_1) & 0 & 0 & 0 & m_{55}(x_1, w_1) & 0 \\ 0 & m_{26}(x_1, w_1) & 0 & 0 & 0 & m_{66}(x_1, w_1) \end{pmatrix}$$

where

$$
\begin{aligned}
m_{11}(x_1, w_1) &= \frac{2(10w_1^6 + x_1^6 + 12w_1^3 x_1 + 3x_1^2)}{x_1^6} \\
m_{22}(x_1, w_1) &= -\frac{2(2w_1^6 - x_1^6 + 4w_1^3 x_1 + x_1^2)}{x_1^6} \\
m_{33}(x_1, w_1) &= \frac{2(x_1^4 + 2w_1^3 + 2x_1)}{x_1^4} \\
m_{44}(x_1, w_1) &= -\frac{2(-x_1^4 + 2w_1^3 + 2x_1)}{x_1^4} \\
m_{55}(x_1, w_1) &= \frac{2(15w_1^4 + x_1^4 + 6w_1 x_1)}{x_1^4} \\
m_{66}(x_1, w_1) &= \frac{2(3w_1^4 + x_1^4 - 6w_1 x_1)}{x_1^4} \\
m_{15}(x_1, w_1) &= -\frac{6w_1^2(4w_1^3 + 3x_1)}{x_1^5} \\
m_{26}(x_1, w_1) &= \frac{2w_1^2}{x_1^4}
\end{aligned}
$$

It remains to evaluate the above matrix at each of the critical points, and find the index of the real symmetric matrix obtained in each case. The three obtained matrices are nonsingular, this shows that the three critical points are nondegenerate, and hence $\tilde{f}$ is a Morse function. Moreover for $p_1$ the index is 0, for $p_2$ the index is 2, and for $p_3$ the index is 3. $\square$

**Proposition 6.7.** *$X$ has the same homotopy type as $e^6 \cup_\varphi e^2 \cup_\psi e^3$ where $\varphi\colon S^1 \to e^6$ and $\psi\colon S^2 \to e^6 \cup_\varphi e^2$ are the attaching maps.*

*Proof.* $\tilde{f}$ has a global minimum at $p_1$, $\tilde{f}^{-1}[c_1, c_1+1]$ is a compact subset of $X$, then by the Lemma of Morse and the proof of Theorem 4.2.4, there exists $0 < \delta < 1$ such that $\tilde{f}^{-1}[c_1, c_1+\delta]$ is diffeomorphic to a 6-cell. Now $\tilde{f}^{-1}[c_2-1, c_2+1]$ is compact, and contains no critical points of $\tilde{f}$ other than $p_2$, then by Theorem 4.2.3, for all sufficiently small $\varepsilon > 0$, $\tilde{f}^{-1}[c_1, c_2+\varepsilon]$ has the same homotopy type as

$\tilde{f}^{-1}[c_1, c_2 - \varepsilon]$ with a 2-cell attached (the index of $p_2$ is 2). $\tilde{f}^{-1}[c_1 + \delta, c_2 - \varepsilon]$ is compact and contains no critical points of $\tilde{f}$, thus by Theorem 4.2.2, $\tilde{f}^{-1}[c_1, c_2 - \varepsilon]$ is diffeomorphic to $\tilde{f}^{-1}[c_1, c_1 + \delta]$, which is diffeomorphic to a 6-cell, so that $\tilde{f}^{-1}[c_1, c_2 + \varepsilon]$ has the same homotopy type as a 6-cell with a 2-cell attached. Now applying again Theorem 4.2.3 and Theorem 4.2.2, in a similar manner as before, $\tilde{f}^{-1}[c_1, c_3 + \varepsilon]$ has the same homotopy type as a 6-cell with a 2-cell attached, with a 3-cell attached. It remains to note that $X$ deformation retracts to $\tilde{f}^{-1}[c_1, c_3 + \varepsilon]$ along the integral curves of $-\operatorname{grad} \tilde{f}$. Rigorously, define a homotopy $H\colon [0,1] \times X \to \tilde{f}^{-1}[c_1, c_3 + \varepsilon]$ that is the identity on $\tilde{f}^{-1}[c_1, c_3 + \varepsilon]$ for any time $t$, and at any point $x \in f^{-1}(c_3 + \varepsilon, +\infty)$, $H(t, x)$ is the integral curve of $-\operatorname{grad} \tilde{f}$ starting at $x$ and flowing to a point in $f^{-1}(\{c_3 + \varepsilon\})$. $\square$

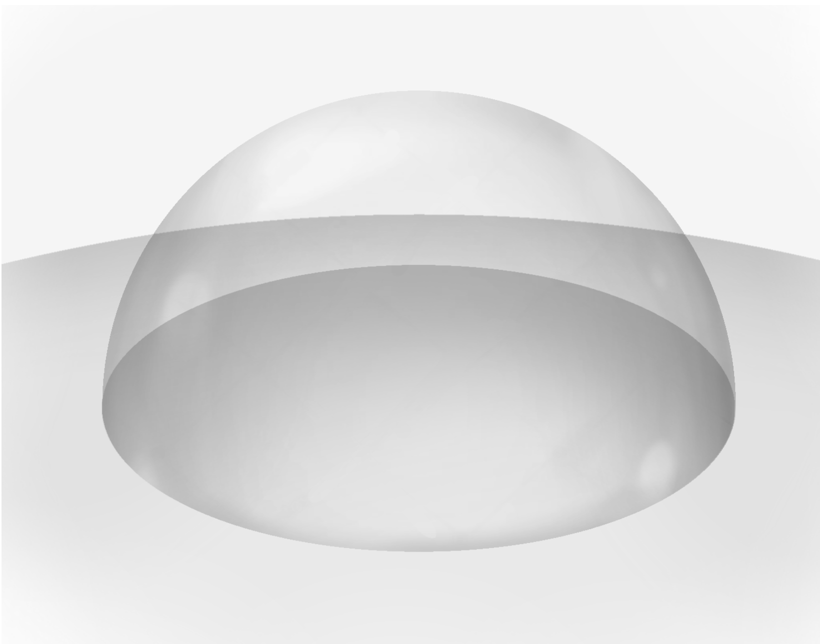

Figure 6.1: $e^6$ with a 2-cell attached

**Remark 6.8.** *Figure 6.1 shows a possible way to attach a 2-cell to $e^6$, where $e^6$ is presented abstractly. Since $X$ is contractible, see [Kaliman, 1993], the only way* *to attach a* 3*-cell to the above figure to obtain* $X$*, is by mapping the boundary of* $e^3$ *in such a way that* $e^3$ *fills the hole. Note that the theorems used in the proof of Proposition 6.7 are not enough to predict how the cells are attached. For that we have to study the behavior of the integral curves of* $\operatorname{grad}\tilde{f}$ *between the critical points.*

**Remark 6.9.** *Writing the defining function of* $X$ *as* $h = h_1 + \mathrm{i}\, h_2$*, were* $h_1$ *is the real part of* $h$*, and* $h_2$ *is the imaginary part of* $h$*, we get that*

$$h_1(x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2) = w_1^3 - 3w_1w_2^2 + x_1^2y_1 - 2x_1x_2y_2 - x_2^2y_1 + z_1^2 - z_2^2 + x_1$$

*and*

$$h_2(x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2) = 3w_1^2w_2 - w_2^3 + x_1^2y_2 + 2x_1x_2y_1 - x_2^2y_2 + 2z_1z_2 + x_2$$

*so* $X$ *can be described in real coordinates as*

$$\{\zeta = (x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2) \in \mathbb{R}^8 \colon h_1(\zeta) = h_2(\zeta) = 0\}$$

*We will consider the usual Riemannian metric on* $\mathbb{R}^8$*, defined on* $T\mathbb{R}^8$ *by:*

$$\langle V, W\rangle = \sum_{i=1}^{8} v_i w_i$$

*where* $V, W \in T\mathbb{R}^8$ *and* $(v_i), (w_i)$ *are smooth functions such that* $(v_i(p))$ *and* $(w_i(p))$ *coordinates of* $V(p)$ *and* $W(p)$*, respectively, in the basis*

$$\left(\frac{\partial}{\partial x_1}|_p, \frac{\partial}{\partial x_2}|_p, \frac{\partial}{\partial y_1}|_p, \frac{\partial}{\partial y_2}|_p, \frac{\partial}{\partial z_1}|_p, \frac{\partial}{\partial z_2}|_p, \frac{\partial}{\partial w_1}|_p, \frac{\partial}{\partial w_2}|_p\right)$$

*of* $T_p\mathbb{R}^8$*. Note by Remark 3.7.21 that for any smooth map* $f\colon \mathbb{R}^8 \to \mathbb{R}$*, the coordinates of the vector field* $\operatorname{grad} f$ *(with respect to the usual Riemannian metric) are* $\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial w_1}, \frac{\partial f}{\partial w_2}\right)$*, so* $\operatorname{grad} f$ *will be identified by the differential map,* $df$*, of* $f$*.*

We now wish to study the flow of $\operatorname{grad} \tilde{f}$. The local coordinate charts introduced in Lemma 6.6 are only defined when $x = x_1 + \mathrm{i}\,x_2 \neq 0$, so we would not be able to study the flow of $\operatorname{grad} \tilde{f}$ globally in those coordinates. Instead we will work with the coordinates of $\mathbb{R}^8$. Since $h_1$ and $h_2$ are nonsingular, $X$ is an embedded submanifold of $\mathbb{R}^8$, and at every point $p \in X$, $T_pX$ can be viewed as the 6-dimensional subspace of $T_p\mathbb{R}^8$:

$$\ker(dh_1|_p) \cap \ker(dh_2|_p) = \{V \in T_p\mathbb{R}^8 \colon dh_1(V) = dh_2(V) = 0\}$$

or in terms of the usual Riemannian metric:

$$\{V \in T_p\mathbb{R}^8 \colon \langle dh_1|_p, V\rangle = \langle dh_2|_p, V\rangle = 0\}$$

where $dh_1$ and $dh_2$ are identified with $\operatorname{grad} h_1$ and $\operatorname{grad} h_2$, respectively.

By the Cauchy–Riemann equations, since $h$ is holomorphic, $dh_1$ and $dh_2$ are orthogonal. $d\tilde{f}|_p$ is the projection of $df|_p$ on $T_pX$, so $d\tilde{f}|_p = df|_p - \alpha(p)dh_1|_p - \beta(p)dh_2|_p$, with

$$\alpha = \frac{\langle df, dh_1\rangle}{\langle dh_1, dh_1\rangle} \quad \text{and} \quad \beta = \frac{\langle df, dh_2\rangle}{\langle dh_2, dh_2\rangle}$$

Letting $\psi$ be the flow of $d\tilde{f}$, $\frac{d\psi}{dt} = d\tilde{f}|_{\psi(t)}$, and expressing $\psi$ in the coordinates of $\mathbb{R}^8$, $\psi = (x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2)$ we get a system of ordinary differential equations, with $\psi(0)$ to be precised.

In what follows we will calculate and visualize the flow by solving numerically the above system using the Runge–Kutta method, for particular choices of $\psi(0)$. The index of $p_3$ is 3, so the Hessian of $\tilde{f}$ at $p_3$ has 3 negative eigenvalues. We will consider the eigenvectors corresponding to those eigenvalues, namely $v_1, v_2$ and $v_3$. From the proof of the Morse Lemma, we see that $f$ decreases its value when flowing from $p_3$ along these directions. For that we take $\varepsilon > 0$, and we set each of the six points $p_3 \pm \varepsilon v_i$, $i = 1, 2, 3$, as a starting points for the flow. For $\varepsilon$ small enough, the flow must converge to either $p_1$ or $p_2$. The figures below represent the variation of the $x_1$,$y_1$, and $w_1$ coordinates of the flow given numerically by the the Runge–Kutta method, with time steps $\tau = 0.0002$ and $\varepsilon = 0.1$, in each of the six cases:

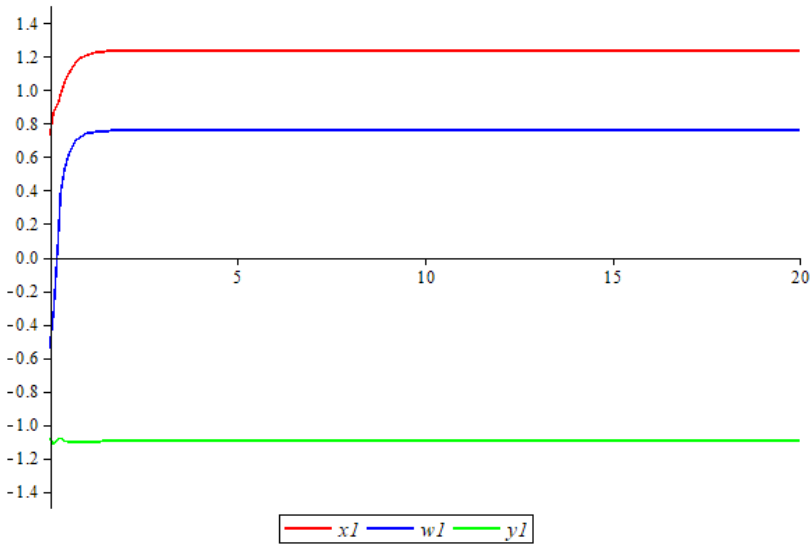


Figure 6.2: flowing from $p_3 + \varepsilon v_1$

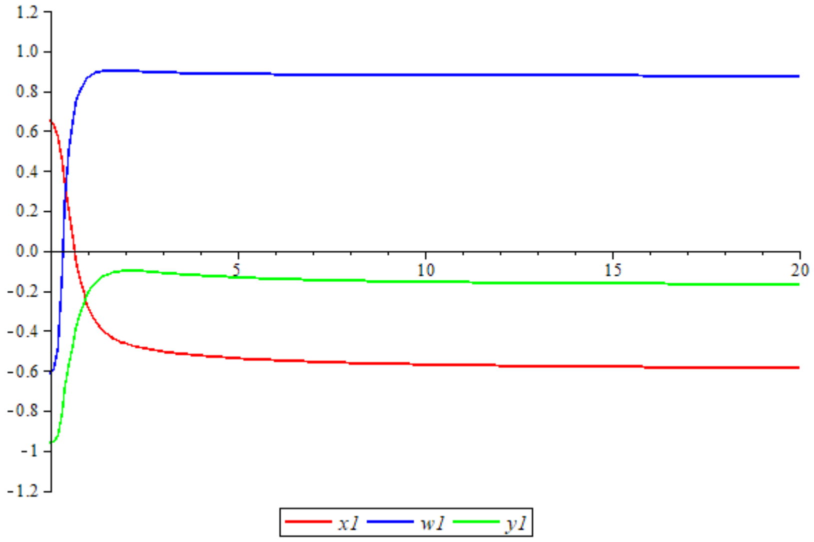


Figure 6.3: flowing from $p_3 - \varepsilon v_1$

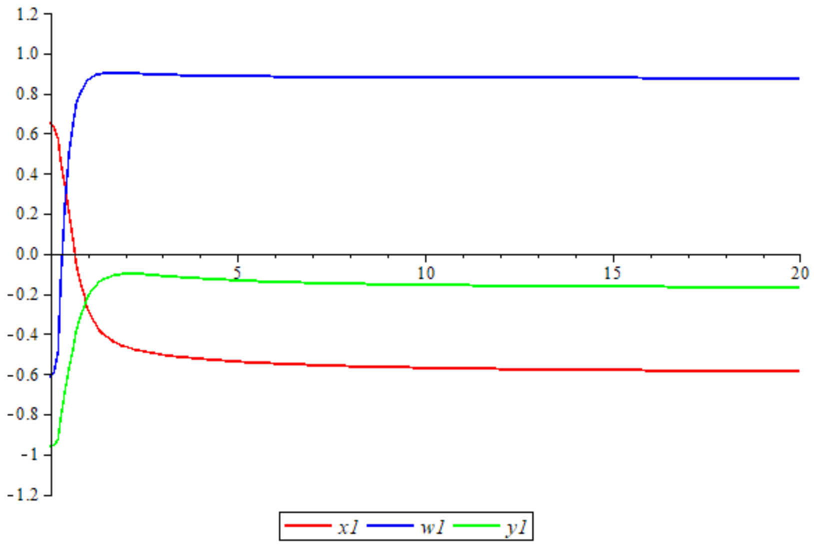


Figure 6.4: flowing from $p_3 + \varepsilon v_2$

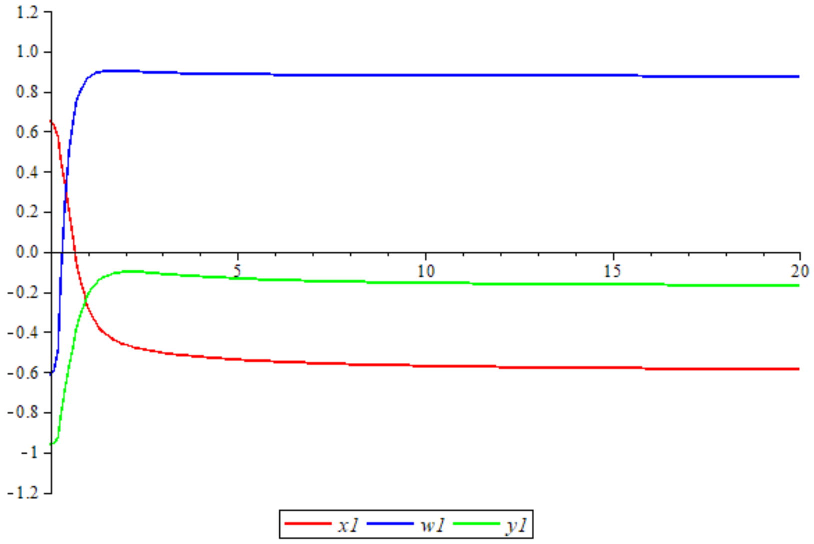


Figure 6.5: flowing from $p_3 - \varepsilon v_2$

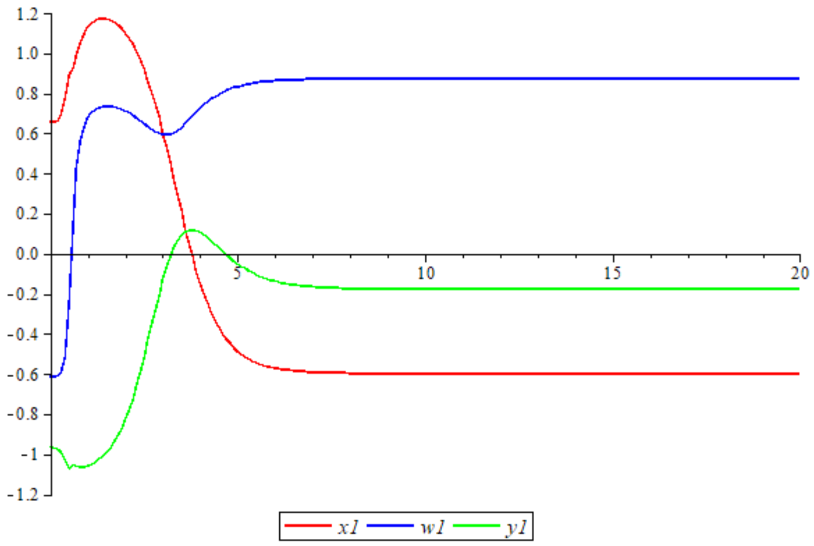


Figure 6.6: flowing from $p_3 + \varepsilon v_3$

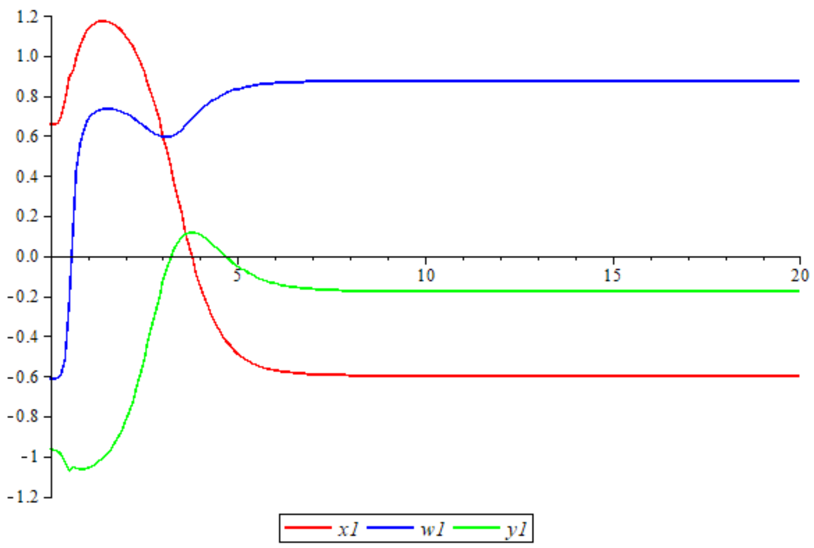


Figure 6.7: flowing from $p_3 - \varepsilon v_3$

Figure 6.2 shows that the point $p_3 + \varepsilon v_1$ flows directely to $p_2$. Figures 6.3, 6.4 and 6.5 show that the points $p_3 - \varepsilon v_1, p_3 + \varepsilon v_2, p_3 - \varepsilon v_2$ flow to $p_1$. For the points $p_3 \pm \varepsilon v_3$, Figures 6.3 and 6.4 show that the flow approaches $p_2$, then converges to $p_1$. We can not be certain in this case that the flow should converge to $p_1$. It is possible due to a numerical error that the numerical solution misses $p_2$, and converges along some other flow to $p_1$.

**Remark 6.10.** *If with further numerical assertions one can show that the flow starting from each of the points $p_3 \pm \varepsilon v_3$ converges to $p_1$, then we would have good numerical evidence that for some $\varepsilon > 0$, $\partial D_l(p_3)$ and $\partial D_u(p_2)$ intersect transversely at a single point in the level surface $\tilde{f}^{-1}(c_2 + \varepsilon)$, so that $p_2$ and $p_3$ satisfy the assumptions of Theorem 4.2.9, and thus obtain a Morse function $g\colon X \to \mathbb{R}$ that coincides with $\tilde{f}$ near the boundary and outside of $\tilde{f}^{-1}[c_1+\varepsilon, c_3+\varepsilon]$ and has no critical points in the interior of $\tilde{f}^{-1}[c_1+\varepsilon, c_3+\varepsilon]$. Hence $g$ has a global minimum at $p_1$, and has no other critical points. So any point in $X$, other than $p_1$, flows to $\infty$ along the integral curves of* $\operatorname{grad} g$, *as the parameter $t$ tends to $+\infty$,* *and flows to $p_1$ as $t$ tends to $-\infty$. This allows us to construct a diffeomorphism between $X$ and $\mathbb{R}^6$, and thus reprove that the Koras–Russell cubic threefold is diffeomorphic to $\mathbb{R}^6$.*

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