AMERICAN UNIVERSITY OF BEIRUT

PRICING AND INVENTORY DECISIONS FOR AN ASSORTMENT UNDER EOQ SUPPLY AND LOGIT DEMAND

by
REEM NABIL ALAMEDDINE

A thesis
submitted in partial fulfillment of the requirements
for the degree of Master of Science
to the Department of Computational Science
of the Faculty of Arts and Sciences
at the American University of Beirut

Beirut, Lebanon
August 2021

AMERICAN UNIVERSITY OF BEIRUT

PRICING AND INVENTORY DECISIONS FOR AN ASSORTMENT UNDER EOQ SUPPLY AND LOGIT DEMAND

by
REEM NABIL ALAMEDDINE

Approved by:

	Signature
Dr. BacelMaddah, Professor Industrial Engineering and Management	Advisor
	Signature
Dr. Hussein Tarhini, Assistant Professor Industrial Engineering and Management	Member of Committee
	Signature
Dr. MoueenSalameh, Adjunct Professor Industrial Engineering and Management	Member of Committee

Date of thesis defense: August 25, 2021

AMERICAN UNIVERSITY OF BEIRUT

THESIS RELEASE FORM

Student Name: Alameddine______________Reem________________Nabil____________
Last First Middle

I authorize the American University of Beirut, to: (a) reproduce hard or electronic copies of my thesis; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes:

☒ As of the date of submission

☐ One year from the date of submission of my thesis.

☐ Two years from the date of submission of my thesis.

☐ Three years from the date of submission of my thesis.

25 August 2021

Signature Date

ACKNOWLEDGEMENTS

I would like to express my sincere appreciation to my advisor, Dr. Bacel Maddah, whose constant support and mentorship helped in bringing this thesis paper to life. I can’t thank you enough for your encouragement and understanding which helped enrich my scholarly experience.
I would also like to thank my thesis committee members, Dr. Hussein Tarhini and Dr. Moueen Salemeh, whose valuable input helped in enhancing and shaping this paper.

ABSTRACT
OF THE THESIS OF

Reem Nabil Alameddiine for Master of Science
Major: Computational Science

Title: Pricing and Inventory Decisions for an Assortment Under EOQ Supply and Logit Demand

The topic of pricing and revenue optimization is receiving increasing interest as time evolves, especially in the retailing field. With the advance in technology comes an increase of sophistication in the algorithms used for pricing and related decisions in retail operations.
The increasingly quantitative approaches adopted by researchers of the retailing field aims to incorporate and integrate retail data with emerging technologies. The retailer's profit is tightly tied to three main decisions: pricing, inventory (shelf) and assortment. Determining prices and assortment is typically the main concern of the marketing department of a firm, while the operations department handles ordering from suppliers and stocking decisions. This paper takes an integrative view, in-line with the modern paradigm in the literature, and jointly analyzes pricing and inventory decisions for a given assortment of substitutable products. The demand is based on a multinomial logit consumer choice which is highly effective in capturing real-world consumer's behavior. The supply framework adopted is that of the EOQ (Economic Ordering Quantity) model which exploits the balance between economies of scales in ordering and inventory financing cost.

TABLE OF CONTENTS

ILLUSTRATIONS

Figure

TABLES

Table

CHAPTER I

INTRODUCTION

The retail sector has been growing steadily. This growth is coupled with fierce competition between online and brick and mortar retailing along with technology-empowered customers who are very demanding. With enormous amount of data that retailers now have on hand, a crucial factor for retail management in this day and age is the adoption of sophisticated analytical approaches resting on mathematical and statistical analysis. A recent report by McKinsey&Company's depicts how the world of supermarkets is being managed in a highly scientific way, specifically Fast Moving Consumer Goods (FMCG) which are at the heart of this evolution. According to that report, the winners in the retail and FMCG world are those that "harness the power of digitization and analytics, implement agile methodologies, and put talent at the top of the CEO agenda" (Kelly, 2019). The Wall Street Journal, on the other hand, stated in a recent article that pricing can be an effective tool for retailers to drive profitability and increase market share (Chawla, 2021). Nevertheless, there is limited literature on models for pricing a product line of substitutable FMCG. This paper provides a contribution in that area.

The aforementioned evolution in retail management has been labeled by Fisher et al. (2000) as the emergence of "Rocket Science Retailing." Retailers are in constant pursuit to increase their revenue while minimizing their costs and boosting their bottom line, i.e. their profit. Both revenue and costs, and hence profit, depend on three main tactical decisions, assortment (i.e. what products to offer in store), inventory levels and selling prices of products in the offered assortment.

In this paper, we adopt the Multinomial Logit (MNL) choice model which is one of the most suitable methods to capture a consumer's behavior when presented with a set of substitutable products that differ in characteristics such as color or style. The customer makes the decision of buying based on a random utility that measures the overall perception of a certain product. The logit choice method is the most frequently used when it comes to modeling a consumer's behavior (e.g. Maddah et al. 2011).

On the supply side, the Economic Order Quantity (EOQ) model is used for inventory decisions, which is one of the well-established models in inventory management. The EOQ model applies when the demand is steady and can be estimated with certainty, which applies mainly to fast-moving consumer goods (FMCG, i.e. grocery items). Having in mind the analysis of pricing and inventory decisions for an assortment of FMCG products (e.g. different brands of coffee or sugar), we assume that demand is deterministic but depends on the prices of the products in the assortment of the retailer via the logit choice model. As detailed in Chapter II, to the best of our knowledge, no work in the literature considers joint pricing and inventory decisions for an assortment under logit demand and EOQ-type supply, as we propose here. Hence our work addresses an important gap in the literature.

The paper includes, in Chapter II, a literature review where previous research done on the topic of pricing, inventory and assortment decisions is discussed. In Chapter III, we begin by introducing our model formulation and the assumptions made, specifically, Section A includes a brief explanation of the assumptions and notations used, Section B describes our demand model and Section 3.3 depicts our supply mode. We assume equal profit margins in our analysis and we compare this approach with that of the heterogeneous pricing approach. In Chapter IV, we present analytical results that prove the unimodularity of the profit function (Section A), and we investigate the effect of inventory considerations on pricing (Section B). In Chapter V, we present numerical results and insights. Particularly, an illustrative example is given in Section A, a sensitivity analysis on a base case is presented in Section B, and lastly a numerical comparison between the optimal profit margin and the classical one ignoring inventory cost is revealed in section C. We end this paper in Chapter VI by concluding the work done and presenting ideas for future work.

CHAPTER II

LITERATURE REVIEW

The main concern of any retailer is to maximize profit. To do that, the optimization of three interdependent measures (pricing, inventory and assortment) is essential. Optimizing two of these decisions (or even all three) jointly, serves the purpose of attracting more customers, satisfying their demand and, ultimately, maximizing profit subject to various constraints (Katsifou al. 2014, Maddah et al. 2011). Consumer purchasing patterns imply how customer service level is defined, how retail assortments are selected, and how the target inventory levels for individual items are set (Agrawal and Smith, 2003). Aydin and Porteus (2008) state that when presented with an assortment of substitutable products (e.g. choosing a certain ice cream flavor), a consumer must make a decision, with the price playing an important role in this decision. Therefore, the price of products in an assortment affects the demand in general and how it would be allocated among the products. This in turn influences the inventory decisions making and makes it important to jointly decide on inventory and prices within a given assortment (Aydin and Porteus, 2008).

A retailer must decide on inventory levels and order quantities that minimize cost. This report adopts the Economic Order Quantity (EOQ) framework to determine the optimal order quantity. There is a significant amount of work on pricing within the single product EOQ model. Examples of these works include Whitin (1955), Wagner and Whitin (1957), Cohen (1977), Ladany and Sternlieb (1974), Porteus (1985), Chakravarty and Martin (1989), Cheng (1990), Chen and Min (1994), and Federgruen (1999). The demand function in this literature can be classified into two main categories, additive, with $D(p) = \alpha p + \beta$, where p is the price, $\alpha > 0, \beta > 0$, and multiplicative, $D(p) = e^{-\alpha p+\beta}$. Some of the works that include the additive demand model include Whitin (1955), Federgruen (1999) and Avinadav et al. (2014). Examples of multiplicative demand models in an EOQ framework include Arcelus and Srinivasan (1987), Avinadav et al. (2014, 2017). Most of the previous work done involves optimizing over the price then over the inventory level. Our work is based on the logit or attraction demand model, where $D(p) = \frac{e^{\alpha-bp}}{1+e^{\alpha-bp}}$ for the single product case , and we optimize over then inventory level then over the price. Not a lot of work has been done around the latter except for the work of Avinadav et al. (2014, 2017) and Tarhini et al. (2020), which we adopted and extend to include multiple products.

The work done by Avinadav et al. (2014, 2017) is highly relevant to our proposed research. Avinadav et al. (2014) used the line search method to obtain an optimal pricing and inventory policy for a single-product under EOQ supply and logit demand. Avinadav et al. (2017) extend the work to include promotion expenditures associated with perishable products. However, the logit model Avinadav et al. (2014, 2017) is less general than that of Tarhini et al. (2020) who consider a similar problem and establish useful concavity and monotinicity results. In our research, we propose to extend this work to pricing an assortment of multiple products under EOQ and logit.

Recent work has also been done on pricing and inventory decisions for a given assortment. Examples of those include Aydin and Porteus (2008), Maddah and Bish (2007) and Maddah et al. (2014). Maddah and Bish (2007, 2014) consider joint pricing, inventory and assortment decisions under a mixed multiplicative/additive demand. They show that for horizontally differentiated products with homogeneous costs, the optimal assortment has a popular set structure. They also argued that the optimal assortment has products with approximately equal profit margins and proposed a heuristic that builds on this argument. Maddah et al. (2014) consider joint pricing and inventory decisions for a given assortment under a demand model similar to Maddah and Bish (2014). They assumed homogeneous prices and costs and showed that the expected profit at optimal inventory levels is unimodal in the price. Maddah et al. (2014) also analyzed the effect of inventory on pricing and found that the optimal “risky" price can be above or below the optimal riskless price which ignores inventory costs. Aydin and Porteus (2008) on the other hand, seek optimal inventory levels and prices of multiple products and prove that the optimal prices are the unique solutions to the first order optimality conditions, under a multiplicative model. The previously mentioned papers adopt the newsvendor model whereas in our thesis we use the EOQ model, which to the post of our knowledge is the first work in its league.

Table 1 below positions our work with respect to the literature on pricing and inventory decisions.

	Demand Function			**Number of Products**		**Supply Side**	
Paper	**Additive**	**Multiplicative**	**Logit**	**Single**	**Multiple**	**EOQ**	**Newsvendor**
This Paper	×	×	✓	×	✓	✓	×
Whitin (1955)	✓	×	×	✓	×	✓	×
Federgruen (1999)	✓	×	×	✓	×	✓	×
Avinadav et al. (2014, 2017)	✓	✓	×	✓	×	✓	×
Arcelus & Srinivasan (1987)	×	✓	×	✓	×	✓	×
Tarhini et al. (2020)	×	×	✓	✓	×	✓	×
Maddah & Bish (2007)	✓	✓	×	×	✓	×	✓
Maddah et al. (2014)	✓	✓	×	×	✓	×	✓
Aydin & Porteus (2008)	×	✓	×	×	✓	×	✓

Table 1 Summary of works on pricing and inventory decisions

CHAPTER III

MODEL FORMULATION AND ASSUMPTIONS

In this chapter we introduce our model formulation and the assumptions made. Section A includes a brief explanation of the assumptions and notations used, Section B describes our demand model and Section C depicts our supply mode.

A. Assumptions and Notations

We aim to find the optimal prices for Products 1,2,…,*n* that would maximize our profit. In the absence of inventory costs, this pricing rule using equal profit margins has shown to be optimal (e.g. Aydin & Ryan (2002), Anderson and de Palma (1992)). Therefore we adopt this approach as an approximation and assume equal profit margin for all products. i.e. $p_1 - c_1 = p_2 - c_2 = \cdots = m$.

For simplicity of the presentation, we first define our notations,

α_i Customer mean reservation price for Product *i* which is the maximum price a customer is willing to pay for a certain product, on average

b Price elasticity metric

p_i Retail price or offered price of Product *i*

m Profit margin, same for all products

ε_i Random Gumbel variable capturing randomness in Product *i* utility

$D_i(\mathbf{p})$ Demand for Product *i* is a function of the prices of products in the assortment given by $\mathbf{p} = (p_1, \dots, p_n)$

$D_i(m)$ Demand for Product *i* is a function of the profit margin of products in the assortment

M	Total market size
h_i	Unit holding cost for Product *i*
c_i	Variable ordering cost for Product *i*
K_i	Fixed ordering cost for Product *i*
y_i	Order size for Product *i*
S	Assortment $S = \{1, 2, \ldots, n\}$ of *n* products
U_i	Product utility which a customer assigns to a certain product I, U_0 is the no-purchase utility
$q_i(\boldsymbol{p}), q_0(\boldsymbol{p})$	Purchasing/non-purchasing probabilities, and $q_i(\boldsymbol{p})$ is the probability that the customer has the maximum utility for product *i*
$q_i(m)$	Purchasing probabilities as a function of the profit margin
$T_i(m)$	Order cycle duration
$\Pi_i(m, y_i)$	Profit from selling item i per ordering cycle
$\Pi_{ui}(m, y_i)$	Profit per unit time
$\Pi_{ui}(m)$	Profit per unit time for Product i at optimal inventory level
$\Pi_u(m)$	Total profit of the offered assortment at optimal inventory levels

B. Demand Model

On the demand side, the logit model was used, which is accurate in capturing the consumer choice process. We describe our demand model as follows: the retailer offers an assortment $S = \{1,2, \ldots, n\}$ of products. A customer enters a store and assigns a certain utility, U_i, to each Product *i*, $i = 1,\ldots,n$, in the offered assortment, and a utility $U_0 = \varepsilon_0$ for the no-purchase option. The product utility is $U_i = \alpha_i - bp_i + \varepsilon_i$ where α_i is the mean reservation price for Product *i*. To estimate this mean reservation price α_i, a combination of panel data (people are surveyed) and scanner data (historical data on consumer purchases) is used, for example see Guadagni and Little (1983). Part of the panel data is from conjoint analysis where a customer is presented with variants of a certain product (e.g. Green et al. 2004) that differ in attributes like size or color. The customer is then asked to give a score to each of those variants and then one can estimate the mean reservation price by calculating the total weighted average of the product attributes.

The variable p_i is the price of the Product i, b is a price elasticity metric, and ε_i, $i = 0, \ldots, n$, are independent and identically distributed Gumbel random variables following a Gumbel distribution. Part of the reason the Gumbel distribution is used because it is closed under maximization; the maximum of many Gumbel random variables is a Gumbel random variable.

The fact that the Gumbel random variable is closed under maximization implies the following closed form for probability of purchasing Product i $q_i(\mathbf{p}) = \mathrm{P}\{U_i = \max_{j \in S \cup \{0\}\}} U_j\}$ and not purchasing $q_{\mathbf{0}}(\mathbf{p})$, (e.g. Anderson and de Palma (1992)),

$$q_i(\mathbf{p}) = \frac{e^{\alpha_i - bp_i}}{1+\sum_{i=1}^{n} e^{\alpha_i - bp_i}}, \tag{1}$$

$$q_0(\mathbf{p}) = \frac{1}{1+\sum_{i=1}^{n} e^{\alpha_i - bp_i}}. \tag{2}$$

This implies that in a market with M customers, the demand of a Product i is a function of the price of all products the assortment, and is given by

$$D_i(\mathbf{p}) = M \frac{e^{\alpha_i - bp_i}}{1+\sum_{j \epsilon S} e^{\alpha_j - bp_j}}\,. \tag{3}$$

Assuming equal profit margins m for all products, where $m = p_1 - c_1 = p_2 - c_2 = \cdots = p_n - c_n$, the demand function can be written as

$$D_i(m) = M\,\frac{e^{\alpha_i - b e^{\alpha_i - b(c_i+m)}}}{1+\sum_{j\epsilon S} e^{\alpha_j - b(c_j+m)}}\,. \quad \textbf{(4)}$$

C. Supply Model

On the supply side, the Economic Ordering Quantity (EOQ) framework is used, fixed setup cost per order, an inventory holding cost, and no shortages are allowed (Zipkin 2000). When an order of size y_i for a Product i is placed, the ordering cost would be $K_i+c_iy_i$, where K_i and c_i are the fixed and variable order costs for Product $\boldsymbol{i}$ respectively. Holding inventory inflicts a unit cost h_i (\$/unit/unit of time) and is proportional to the average inventory level. This leads to a holding cost of $h_iy_i^2/D_i(m)$ per ordering cycle. The ordering cycle duration is given by

$$T_i(m) = \frac{y_i}{D_i(m)}\,. \quad \textbf{(5)}$$

The profit from selling Product i per ordering cycle is

$$\Pi_i(m, y_i) = (m + c_i)y_i - [K_i + c_iy_i + h_i\,\frac{y_i^2}{2D_i(m)}]\,. \quad \textbf{(6)}$$

The profit per unit time is then given as

$$\Pi_{ui}(m, y_i) = \frac{\Pi_i(m,y_i)}{Ti(m)} = mD_i(m) - \frac{K_iD_i(m)}{y_i} + \frac{h_iy_i}{2}\,. \quad \textbf{(7)}$$

It can be easily shown that for a given profit margin m, $\Pi_{ui}(m, y)$ is concave in y_i since

$$\frac{\partial \Pi_{ui}(m,y_i)}{\partial y_i} = \frac{K_iD_i(m)}{y_i^2} - \frac{h_i}{2}\,, \quad \textbf{(8)}$$

$$\frac{\partial^2 \Pi_{ui}(m,y_i)}{\partial y_i^2} = \frac{-2K_i D_i(m)}{y_i^3} \leq 0 \ . \quad \textbf{(9)}$$

The optimal order quantity is obtained from the first-order optimality conditions (setting the right-hand of **(8)** equal to zero), and is given by

$$y_i^*(m) = \sqrt{\frac{2K_i D_i(m)}{h_i}} \ . \quad \textbf{(10)}$$

Substituting **(10)** in **(7)** gives the profit per unit time at optimal inventory level for Product *i*,

$$\Pi_{ui}(m) = mD_i(m) - \sqrt{2K_i h_i D_i(m)} \ . \quad \textbf{(11)}$$

Note that the profit function in **(11)** is divided into two parts, gross profit from sales $mD_i(m)$ and the operational cost $\sqrt{2K_i h_i D_i(m)}$.

Finally, the total profit of the offered assortment at optimal inventory levels is given by

$$\Pi_u(m) = \sum_{i=1}^{n} \Pi_{ui}(m), \quad \textbf{(12)}$$

where $\Pi_{ui}(m)$ is given in **(11).** Equivalently,

$$\Pi_u(m) = \sum_{i=1}^{n} \left[mM \frac{e^{\alpha_i - b(c_i+m)}}{1+\sum_{j=1}^{n} e^{\alpha_i - b(c_i+m)}} - \sqrt{2K_i h_i M \frac{e^{\alpha_i - b(c_i+m)}}{1+\sum_{j=1}^{n} e^{\alpha_i - b(c_i+m)}}} \right]. \quad \textbf{(13)}$$

CHAPTER IV

ANALYTICAL RESULTS

We begin by stating an assumption which guarantees a positive optimal profit function.

Assumption A1 $A^2 < \Phi(\widetilde{m})$

Let $A = \sqrt{\sum_{i=1}^{n} 2K_i \mathrm{h}_i M e^{\alpha_i - bc_i}}$

and $\widetilde{m} = \frac{2+W[2\gamma e^{-2}]}{b}$ is the unique maximizer of

$$\Phi(m) = M^2 m^2 \frac{[\sum_{i=1}^{n} e^{\alpha_i - bc_i}]^2}{e^{bm} + \sum_{j=1}^{n} e^{\alpha_i - bc_i}}$$

It can easily be shown that Assumption (A1) implies that $\Pi_u(m) > 0$ over some range of *m*.

A. Concavity Results

In this chapter, we perform some analysis on the resulting profit function and prove its pseudo-concavity. The following lemma defines a range where the profit function is positive.

Lemma 1 *The expected profit,* $\Pi_u(m)$*, has two roots* $\underline{m}$ *and* $\overline{m}$ *> 0, and* $\Pi_u(m) > 0$ *if and only if* $m \in (\underline{m}, \overline{m})$*, where* $\underline{m}$ *and* $\overline{m}$ *are the unique solutions to the equation* $\Phi(m) = A^2$

Proof *See Appendix.*

Lemma 1 helps in searching for the optimal price since we limit our search between two roots, $\underline{m}$ and $\overline{m}$.

The following theorem presents our main result on the structure of the profit function $\Pi_u(m)$.

Theorem 1 *$\Pi_u(m)$ is unimodal for $m \in (\underline{m}, \overline{m})$ which are defined in Lemma 1*

Proof *See Appendix.*

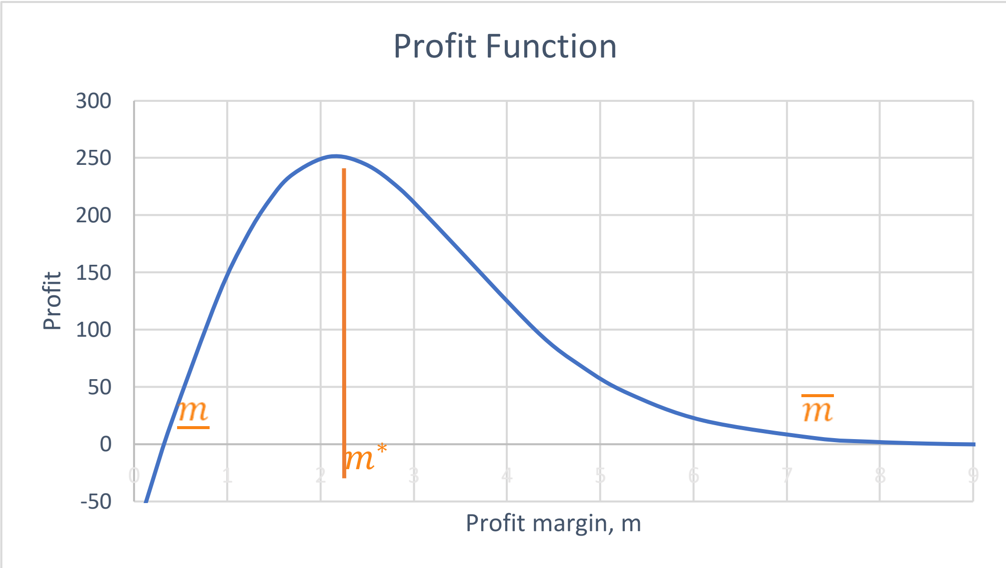

Figure 1 Profit Function as a Function of *m*

Figure 1 illustrates the behavior of the expected profit based on Theorem 1. The expected profit $\Pi_u(m)$ starts negative at $m = 0$ and increases till $\underline{m}$ where it reaches zero. Then, between $\underline{m}$ and m^*, it keeps increasing reaching its maximum at m^*. It then decreases till it reaches zero again at $\overline{m}$. For $m > \overline{m}$, the expected profit function remains negative. When $m < \overline{m}$, the profit margin is too low to cover operation costs, even though the assortment demand is high. When $m > \overline{m}$, the opposite happens, the profit margin is too high but demand is too low, again failing to cover costs. The range $(\underline{m}, \overline{m})$ is the sweet spot where profit margin and assortment demand are balanced, and

m^* is the ultimate pricing level where this balance achieves its climax yielding peak profit.

B. Comparing Risky and Riskless Profit Margins

Note that in ample inventory case, where no inventory costs are incurred, the expected profit is given by

$$\Pi_u^0(m) = \; Mm \frac{\sum_{i=1}^{n} e^{\alpha_i - bc_i}}{e^{bm} + \sum_{j=1}^{n} e^{\alpha_i - bc_i}} \tag{14}$$

Moreover, the optimal riskless profit margin that optimizes $\Pi_u^0(m)$, is given by (e.g. Li and Huh, 2011)

$$m^0 = 1 + W\left(\sum_{j=1}^{n} e^{\alpha_j - c_j - 1}\right) \tag{15}$$

The following Lemma compares the optimal (risky) and riskless profit margins.

Lemma 2 *The optimal profit margin is greater than or equal to the riskless profit margin,* $m^* > m^0$

Proof *See Appendix.*

Previous work comparing risky prices, p^*, with the riskless price, p^0 in the context of the newsvendor model, showed that for the additive demand function $p^* \leq p^0$ (e.g. Mills, 1959). On the other hand, for the multiplicative demand case, $p^* \geq p^0$ (e.g. Karlin and Carr 1962). It is interesting that we observe a similar result with multiplicative demand and EOQ supply. Young (1978) and Maddah et al. (2007, 2014) find that $p^* \leq p^0$ and $p^* \geq p^0$ may both hold under mixed multiplicative additive demand in a newsvendor setting.

Lemma 2 allows narrowing the range where the search for the optimal profit margin takes place as shown in the following corollary.

Corollary 1 The optimal profit margin satisfies $m^* \in (m^0, \overline{m})$ where $\overline{m}$ is as defined in lemma 1 and m^0 is the riskless margin given in **(15).**

Proof *Follows from Lemmas 1 and 2.*

CHAPTER V

NUMERICAL RESULTS AND INSIGHTS

In the following chapter, we use numerical results to prove the previously stated analytical results. In Section A, we describe our base case and in Section B we show some sensitivity analysis.

A. Base Case Analysis

For our illustrative case, the following values were used to depict the behavior of the profit function for a scenario were three products are involved:

Market size: $M = 300$,

Costs: $c_1 = 5,\ c_2 = 4, c_3 = 3$,

Reservation prices: $\alpha_1 = 6, \alpha_2 = 5, \alpha_3 = 4$,

Holding costs $\left(h_j = rc_j\right)$: $h_1 = 1, h_2 = 0.8, h_3 = 0.6$, with $r = 0.2$,

Fixed costs $\left(K_j = \mu h_j\right)$: $h_1 = 7, h_2 = 5.6, h_3 = 4.2$, with $\mu = 7$,

Price elasticity metric: $b = 1$

To find the optimal profit margin, Lemma 1 provides with bounds on the optimal margin. For the base case at hand, setting $\Pi_u(m) = 0$ or $\Phi(m) - A^2 = 0$ resulted in $\underline{m} = 0.32$ and $\overline{m} = 8.88$ and $m^0 = 2.05$.

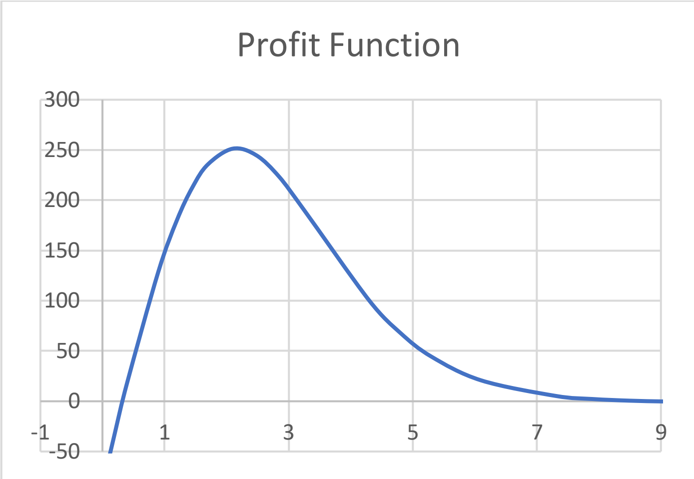

Figure 2 Profit function as a function of *m*

The optimal profit margin for this base case was found to be $m^* = 2.158$ via a numerical search in Excel and MATLAB. *Figure 2* depicts the behavior of the expected profit function at optimal inventory levels which is in-line with *Theorem 1*.

To check the effectiveness of the equal profit margin policy, we compared the results against the case of having three heterogeneous profit margins where the expected profit at optimal inventory level is given by

$$\Pi^*(\mathbf{p}) = \sum_{i=1}^{n} D_i(\boldsymbol{p})(p_i - c_i) - \sum_{i=1}^{n} \sqrt{2K_i h_i D_i(\mathbf{p})}, \tag{16}$$

where $\mathbf{p} = (p_1, p_2, p_3)$. For the latter case, the three optimal prices obtained were $p_1^* = 7.22$, $p_2^* = 6.158$, and $p_3^* = 5.099$. Note that $p_1^* - c_1 = 2.22$, $p_2^* - c_2 = 2.158$, and $p_3^* - c_3 = 2.099$. These margins are close in value to the optimal profit margin which supports the profit margin method used. Next, we estimate the regret of choosing equal profit margins over the original heterogeneous case. Note that the optimal under equal profit margins is $\Pi_u(m^*) = 251.54$, and under heterogeneous margin $\Pi_u(p_1^*, p_2^*, p_3^*) = 251.70$

Then,

$$Regret = \frac{\Pi_u(p_1^*, p_2^*, p_3^*) - \Pi_u(m^*)}{\Pi_u(p_1^*, p_2^*, p_3^*)} = 0.063\%, \qquad \textbf{(17)}$$

which indicates an excellent performance of the equal profit margin pricing approach.

B. Sensitivity Analysis

For our sensitivity analysis, we change the values of the cost *c* of product *i* and the mean reservation price α of product *i*. The new values now are

Market size *M* = 300,

Unit Cost of product *i*: $c_i = c_1 - 0.5 * (i - 1)$, where $i = 1,2,3$ and $c_1 = 3$,

Mean reservation price α of product *i*: $\alpha_i = [\gamma + (k - i + 1)\delta]c_i$, where $i = 1,2,3$ and the intercept is $\gamma = 1$ and the slope is $\delta = \frac{1}{8}$.

Holding cost *h* of product *i*: $h_i = r * c_i$, where $r = 0.2$.

Fixed cost *K* of product *i*: $K_i = \mu * h_i$, where $\mu = 7$.

We calculate the optimal profit to be, $m^* = 1.991$. We also find the optimal ordering quantities, y_i^*, where $i = 1,2,3$, as well as the the riskless profit margin, to be $m_0 = 1.917 < m^*$, $y_1^* = 28.02$, $y_2^* = 21.82$, and $y_3^* = 18.09$.

Above, we perform some sensitivity analysis on the base case presented in Section A, while focusing on estimating the regret from using equal profit margins over the optimal unrestricted prices. We vary each model parameter from its base values, one at a time, while keeping the other parameters at their base values.

1. Varying the market size M

Starting with a market size of 300 and decreasing the latter, the regret increases. It must be noted that for market size less than 5, assumption A1 does not hold *Figure 3* displays the regret as the market size varies from $M = 10$ to $M = 300$. Obviously this indicates a superior performance of the equal profit margin policy with a regret below 0.1%.

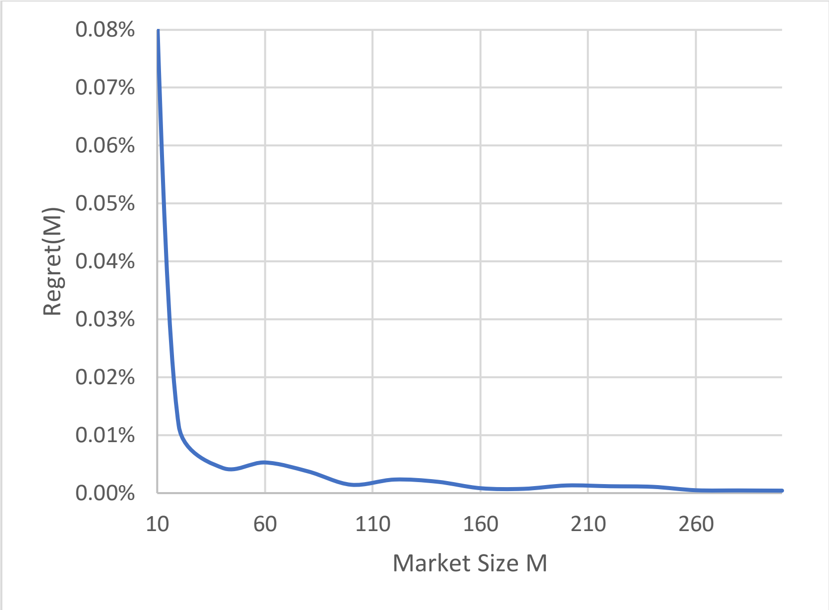

Figure 3 Regret as Market Size M increases

2. ***Varying the intercept γ ($0 < \gamma < 2$) and the slope δ ($-\frac{1}{3} \leq \delta \leq \frac{1}{2}$) of the mean reservation prices, α_i***

Recall that we assume a linear structure of the reservation prices which is proportional to the unit costs, $\alpha_i = [\gamma + (k - i + 1)\delta] * c_i, i = 1,2,3$. As the intercept γ increases from 0 to 2, the regret function decreases to reach its minimum at the value $\gamma = 1$ and increases for greater values of γ (demonstrated in *Figure 4*). For all these values of γ, the regret remains negligible at below 0.25%.

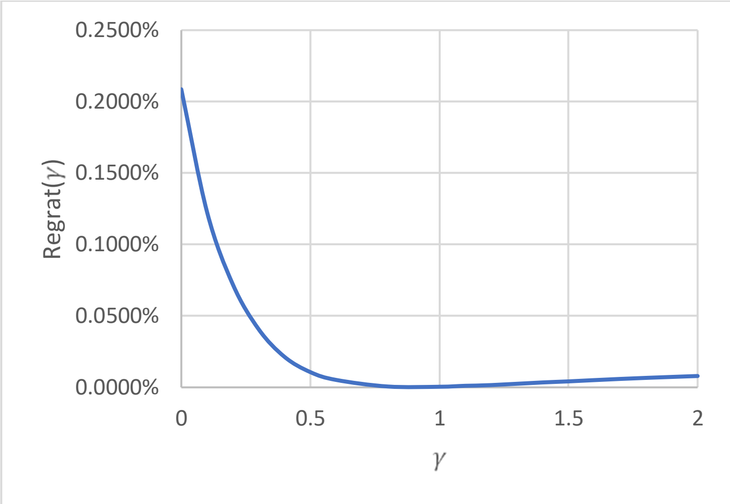

Figure 4 Regret as a intercept γ increases

As for the slope δ, we started our regret analysis for the value $\delta = -\frac{1}{3}$. The regret function decreases to reach its minimum at $\delta = \frac{1}{8}$ and increases again after that (shown in *Figure 5*).

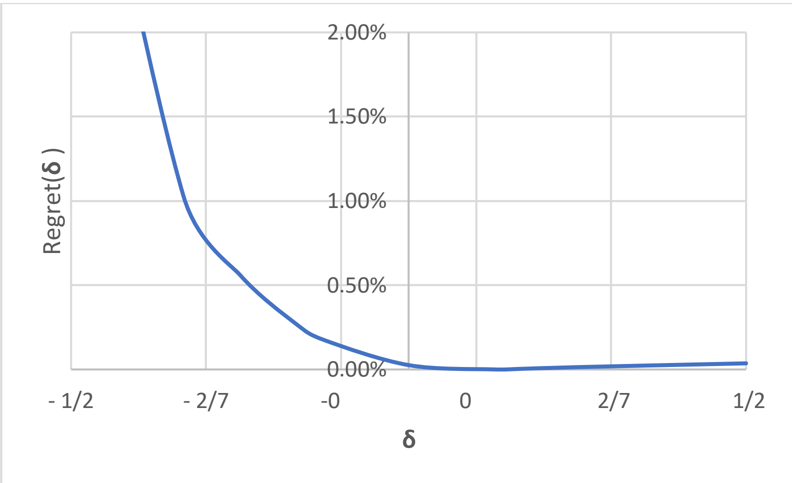

Figure 5 Regret as slope δ increases

Again, the regret is quite small with values below 2% in all the tested cases.

3. *Varying cost coefficients r* $(0.1 < i < 0.4)$ *and* μ $(4 < \mu < 12)$

We varied the value of the inventory financing cost according to its boundaries. The resulting graph did not show a consistency in behavior, but did demonstrate a slight increase in regret as the coefficient r increases in value as shown in *Figure 6*, and more importantly negligible regret values below 0.003%.

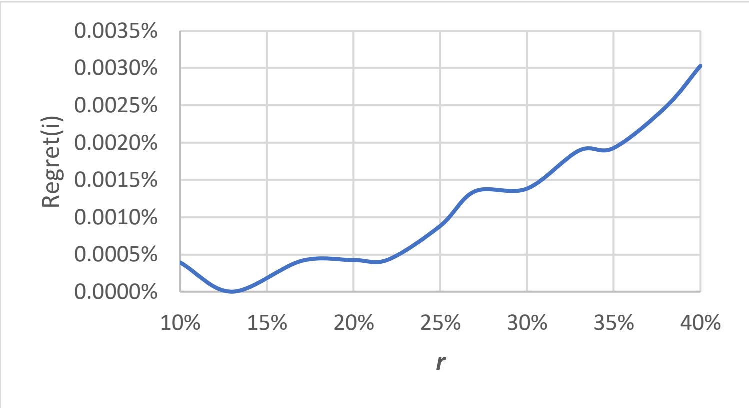

Figure 6 Regret as coefficient r increases

We also varied the value of coefficient μ according to its boundaries. The graph did not show a consistent increase rather an overall increase in regret as the coefficient μ increases in value as shown in *Figure 7*. The regret is again insignificant being below 0.001% for all values of the multiplier μ.

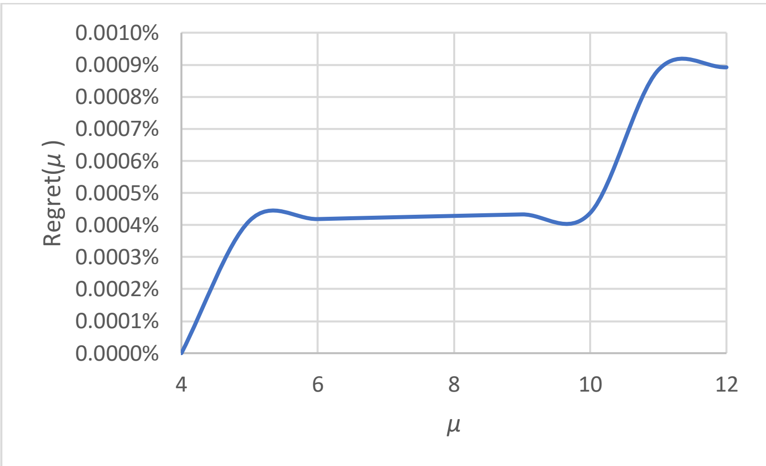

Figure 7 Regret as coefficient μ increases

4. *Varying base value of cost of Product 1, c_1*

As the cost of Product 1 changes, costs of the other products of the assortment change since the their costs follow the equation: $c_i = c_1 - 0.5 * (i - 1)$. We started our analysis with $c_1 = 0.5$ and increased the value till 5. *Figure 8* shows the graph of the regret resulting from this variation in the value of c_1.

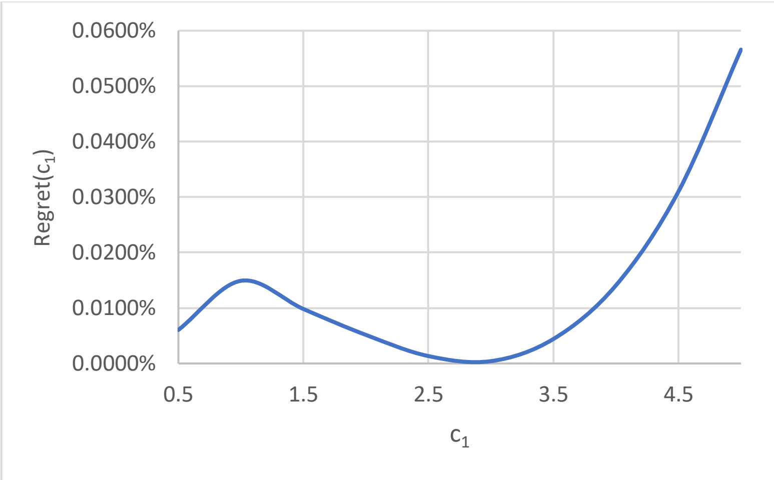

Figure 8 Regret as base value of cost of product 1, c_1 increases

It is evident from the graph that the smallest regret is for the cost of Product 1 $c_1 = 3$. However, the regret, while displaying an increasing behavior, remains insignificant (below 0.06%).

5. *Varying assortment size*

Our base case includes an assortment of three products. We tested the regret resulting from including a different number of items in our assortment. Specifically, we performed our analysis on 2, 4, 5 and 6 products. *Figure 9* shows that the regret is smallest for an assortment of three products. The regret increases as the assortment size increases, while remaining negligible at values below 0.04%.

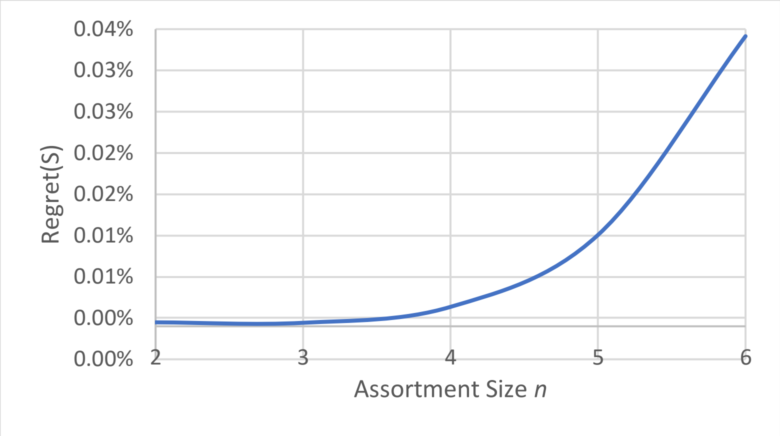

Figure 9 Regret as assortment size *n* increases

C. Sensitivity Analysis

We performed some numerical analysis to characterize the effect of inventory on the price (reflected by the profit margin *m*) and the profit. The numerical findings obtained are demonstrated in *Tables 2* to *6.* Tables 2-6 confirm the result in Lemma 2 that the risky profit margin, m^*, is always larger than the riskless one, m^0. The maximum deviation between m^* and m^0 occurs when

(i) The market size M is small,

(ii) The slope and intercept of the mean reservation price are small,

(iii) The unit cost is large,

(iv) The assortment size is large.

M	m^0	m^*
300	1.917	1.991
280	1.917	1.994
260	1.917	1.997
240	1.917	2
220	1.917	2.004
200	1.917	2.008
180	1.917	2.014
160	1.917	2.02

140	1.917	2.027
120	1.917	2.037
100	1.917	2.049
80	1.917	2.066

Table 2 Comparing the riskless, m^0, and risky profit margin, m^*, as the market size M varies

γ	m_0	m^*
0	1.152	1.405
0.1	1.188	1.409
0.4	1.343	1.493
0.7	1.583	1.687
1	1.917	1.991
1.4	2.496	2.546
1.7	3.015	3.054
1.9	3.394	3.427
2	3.592	3.623

Table 3 Comparing the riskless, m^0, and risky profit margin, m^*, as the intercept γ varies

δ	m_0	m^*
-1/2	1.147	1.338
- 1/3	1.222	1.388
- 1/4	1.278	1.43
- 1/6	1.355	1.49
- 1/8	1.403	1.53
0	1.604	1.704
1/8	1.917	1.991
1/6	2.051	2.118
1/4	2.369	2.422
1/2	3.671	3.698

Table 4 Comparing the riskless, m^0, and risky profit margin, m^*,, as the slope δ varies

c_1	m_0	m^*
1.5	1.729	1.775
2	1.788	1.853
2.5	1.85	1.933
3	1.917	2.015
3.5	1.987	2.099

4	2.061	2.185
4.5	2.139	2.272
5	2.221	2.363

Table 5 Comparing the riskless, m^0, and risky profit margin, m^*, as the unit cost $\boldsymbol{c_1}$ varies

$\boldsymbol{n}$	$\boldsymbol{m_0}$	$\boldsymbol{m}^*$
2	1.81	1.882
4	1.989	2.063
5	2.047	2.119
6	2.101	2.169

Table 6 Comparing the riskless, m^0, and risky profit margin, m^*, as the assortment size n varies

CHAPTER VI

CONCLUSION AND FUTURE RESEARCH

In this thesis, we show that pricing an assortment under logit demand and EOQ supply can be done with ease under a special equal profit margin pricing rule, due to the concavity results (unimodularity of the profit function). We also derive useful bounds that simplify the search for the optimal profit margin. Moreover, we numerically demonstrate that the equal profit margin approach we obtained is an excellent approximation that gives near optimal results. The regret of using equal profit margins instead of heterogeneous prices is found to be below 2%. We also investigate the effect of inventory costs on the pricing, and find that prices increase when limited inventory exists.

This thesis can be further extended to include future work in several aspects. One feature to explore is adding another decision lever which is the assortment to carry. Assortment decisions can either be made under exogenous prices or joint assortment, pricing and inventory decisions can be studied. Another extended work can include pricing and inventory decisions under Nested Multinomial Logit (NMNL) demand model which helps eliminate the limitation of the MNL model, and allow modeling an assortment with more product differentiation. Supply chain integration is another aspect that can be further explored, where a model other than the EOQ from the supply side can be studied. One example would be the Integrated Procurement Production model which deals with the effective management of goods from raw materials to finished products (Maddah et al., 2015). Another addition to our work would be to include inflation and study if the pricing methods we use would be effective in responding to the fast cost changes.

APPENDIX

Proof of Lemma 1. Referring to equation **(13)**,

$$\Pi_u(m) = Mm\frac{\sum_{i=1}^{n} e^{\alpha_i - bc_i}}{e^{bm} + \sum_{j=1}^{n} e^{\alpha_i - bc_i}} - \sqrt{\frac{1}{e^{bm} + \sum_{j=1}^{n} e^{\alpha_i - bc_i}}}\left(\sum_{i=1}^{n}\sqrt{2K_i h_i M e^{\alpha_i - bc_i}}\right)$$

Setting $\Pi_u(m) = 0$, gives

$$Mm\frac{\sum_{i=1}^{n} e^{\alpha_i - bc_i}}{e^{bm} + \sum_{j=1}^{n} e^{\alpha_i - bc_i}} = \sqrt{\frac{1}{e^{bm} + \sum_{j=1}^{n} e^{\alpha_i - bc_i}}}\left(\sum_{i=1}^{n}\sqrt{2K_i h_i M e^{\alpha_i - bc_i}}\right)$$

which is equivalent to

$$\left(\sum_{i=1}^{n}\sqrt{2K_i h_i M e^{\alpha_i - bc_i}}\right) = Mm\frac{\sum_{i=1}^{n} e^{\alpha_i - bc_i}}{\sqrt{e^{bm} + \sum_{j=1}^{n} e^{\alpha_i - bc_i}}}$$

Let $A = \sum_{i=1}^{n}\sqrt{2K_i h_i M e^{\alpha_i - bc_i}}$. Then,

$$A^2 = M^2 m^2 \frac{(\sum_{i=1}^{n} e^{\alpha_i - bc_i})^2}{e^{bm} + \sum_{j=1}^{n} e^{\alpha_i - bc_i}} = \frac{h_1(m)}{h_2(m)} = \Phi(m)$$

Let $\beta = M^2(\sum_{i=1}^{n} e^{\alpha_i - bc_i})^2$ and $\theta = \sum_{j=1}^{n} e^{\alpha_i - bc_i}$

$$\Phi(m) = M^2 m^2 \frac{(\sum_{i=1}^{n} e^{\alpha_i - bc_i})^2}{e^{bm} + \sum_{j=1}^{n} e^{\alpha_i - bc_i}} = \frac{\beta m^2}{e^{bm} + \theta}$$

$$\frac{\partial \Phi(m)}{\partial m} = \frac{2\beta m e^{bm} + 2\beta m\gamma - \beta b m^2 e^{bm}}{(e^{bm} + \gamma)^2} = \frac{\beta m(2e^{bm} + 2\gamma - bme^{bm})}{(e^{bm} + \gamma)^2}$$

Setting $\frac{\partial \Phi(m)}{\partial m} = 0$ gives

$2\beta m e^{bm} + 2\beta m\gamma - \beta b m^2 e^{bm} = 0$

$\beta m(2e^{bm} + 2\gamma - bme^{bm}) = 0$

$2e^{bm} + 2\gamma - bme^{bm} = 0$

$e^{bm}(bm - 2) = 2\gamma$

$(bm - 2) = 2\gamma e^{-bm}$

Multiply both sides by $e^{(bm-2)}$

$(bm-2)e^{(bm-2)} = 2\gamma e^{-2}$

Using the Lambert function on both sides we get:

$W\left[(bm-2)e^{(bm-2)}\right] = W[2\gamma e^{-2}]$

$$\widetilde{m} = \frac{2 + W[2\gamma e^{-2}]}{b}$$

The maximum number of intersection points between A^2 and $\Phi(m)$ is 2 (Refer to *Figure 10)*. The first derivative of $\Phi(m)$ has a unique solution, therefore this proves that $\Phi(m)$ is unimodal (pseudo-concave) for $m \in (0, \infty)$. Thus, the profit function has exactly 2 roots.

For $m = 0$, $\Phi(m) = 0$ and as $m \to \infty$ $\Phi(m) \to 0$. Otherwise, $\Phi(m) > 0$. Therefore, $\Phi(m)$ is is increasing after $m = 0$ reaching a maximum at $\widetilde{m} = \frac{2+W[2\gamma e^{-2}]}{b}$, then decreases to 0 as $m \to \infty$.

For $m = 0$,

$$\Pi_u(m) = -\sqrt{\frac{1}{e^{bm} + \sum_{i=1}^{n} e^{\alpha_i - bc_i}}}\left(\sum_{i=1}^{n}\sqrt{2K_i h_i M e^{\alpha_i - bc_i}}\right) < 0$$

As $m \to \infty$,

$$\lim_{m\to\infty}\Pi_u(m) = \lim_{m\to\infty}\left[Mm\frac{\sum_{i=1}^{n} e^{\alpha_i - bc_i}}{e^{bm}+\sum_{j=1}^{n} e^{\alpha_i - bc_i}} - \sqrt{\frac{1}{e^{bm}+\sum_{i=1}^{n} e^{\alpha_i - bc_i}}}\left(\sum_{i=1}^{n}\sqrt{2K_i h_i M e^{\alpha_i - bc_i}}\right)\right] =$$

$$\lim_{m\to\infty}\left[\frac{m*Mr}{e^{bm}+c} - \sqrt{\frac{1}{e^{bm}+c}}s\right]$$

where $r = \sum_{i=1}^{n} e^{\alpha_i - bc_i}$ and $s = \left(\sum_{i=1}^{n}\sqrt{2K_i h_i M e^{\alpha_i - bc_i}}\right)$

Therefore,

$$\lim_{m\to\infty}\Pi_u(m) = \lim_{m\to\infty}\left[\sqrt{\frac{1}{e^{bm}+c}}\left(\frac{m*Mr}{\sqrt{e^{bm}+c}} - s\right)\right] = 0^-$$

Note that $\Pi_u(0) < 0$, and $\lim_{m\to\infty} \Pi_u(m) \to 0^-$. Therefore, $\Pi_u(m)$ starts negative at m = 0, increases to $\underline{m}$ and becomes positive and then decreases to $\bar{m}$, and then becomes negative and remains negative for $m > \bar{m}$.

Figures 10 and *11* show that the intersection points between Φ(m) and A^2 is are equivalent to the intersection between the profit function and the x-axis.

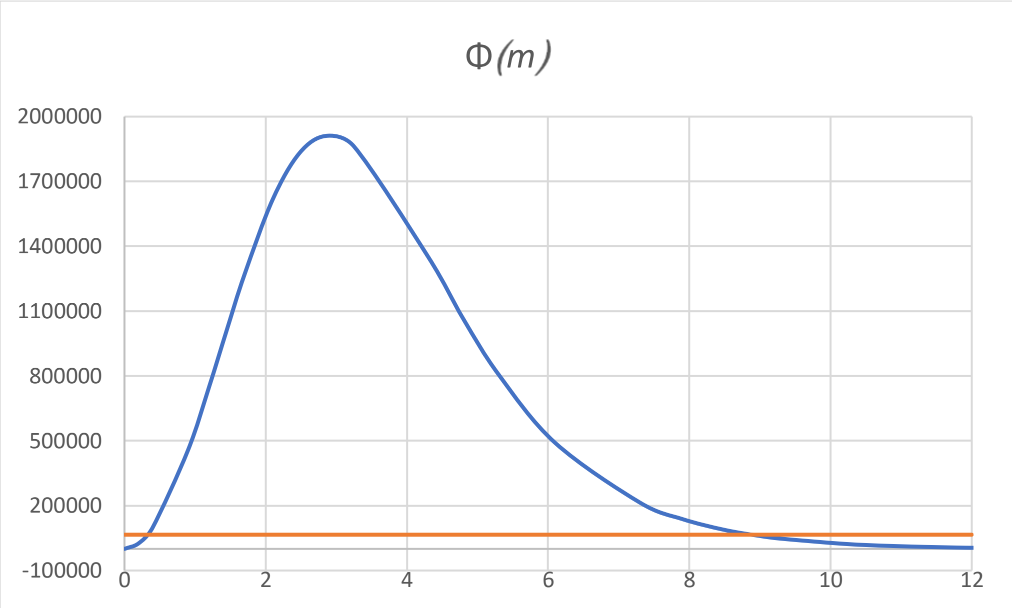

Figure 10 Intersection between A^2 and $\Phi(m)$

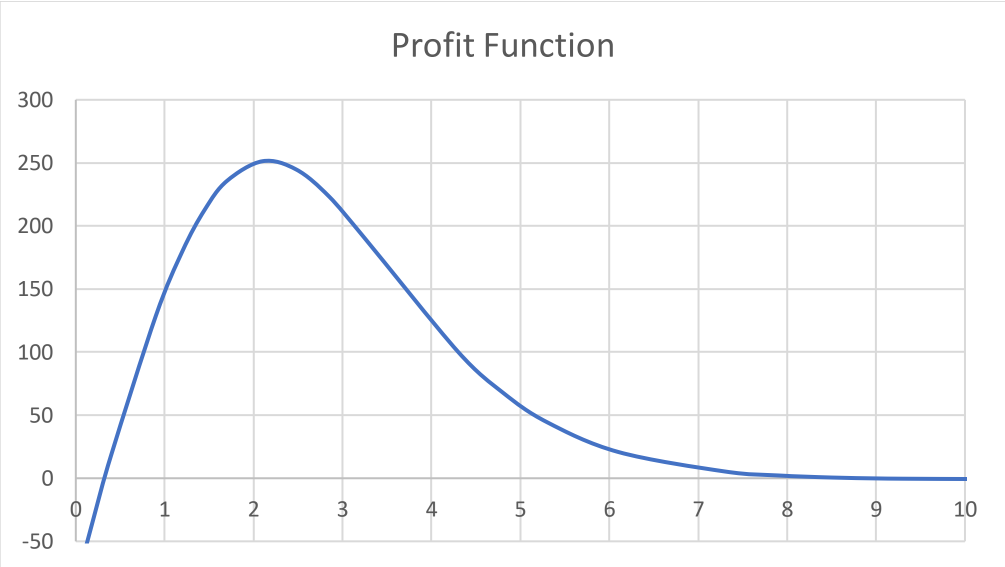

Figure 11 Intersection between $\Pi_u(m)$ and x-axis

$\blacksquare$

Proof of Theorem 1. Recall from the proof of Lemma 1 that the expected profit can be written as

$$\Pi_u(m) = Mm\frac{\sum_{i=1}^{n} e^{\alpha_i - bc_i}}{e^{bm} + \sum_{i=1}^{n} e^{\alpha_i - bc_i}} - \sqrt{\frac{1}{e^{bm} + \sum_{i=1}^{n} e^{\alpha_i - bc_i}}}\left(\sum_{i=1}^{n} \sqrt{2K_i h_i M e^{\alpha_i - bc_i}}\right)$$

$$= \frac{Mm\sum_{i=1}^{n} e^{\alpha_i - bc_i} - \sqrt{e^{bm} + \sum_{i=1}^{n} e^{\alpha_i - bc_i}}\left(\sum_{i=1}^{n} \sqrt{2K_i h_i M e^{\alpha_i - bc_i}}\right)}{e^{bm} + \sum_{i=1}^{n} e^{\alpha_i - bc_i}}$$

$$Let\ \psi = \sum_{i=1}^{n} e^{\alpha_i - bc_i}\ and\ \beta = \left(\sum_{i=1}^{n} \sqrt{2K_i h_i M e^{\alpha_i - bc_i}}\right)$$

then,

$$\Pi_u(m) = \frac{\psi Mm - \sqrt{e^{bm} + \psi} * \beta}{e^{bm} + \psi} = \frac{f(m)}{g(m)}$$

$$-\Pi_u(m) = \frac{\sqrt{e^{bm} + \psi} * \beta - \psi Mm}{e^{bm} + \psi} = \frac{-f(m)}{g(m)}$$

$$f(m) = \sqrt{e^{bm} + \psi} * \beta - \gamma Mm$$

$\sqrt{\mathrm{e}^{\mathrm{bm}} + \psi} * \beta$ is a positive convex function and $-\psi Mm$ is linear. The sum of two convex functions is convex. Therefore, $-f(m)$ is a negative convex function. Note that $\Pi_u(m)$ is positive and therefore $f(m)$ is also positive.

$$g(m) = e^{bm} + \psi$$

$g(m)$ is a positive convex function.

Theorem 6.9 in Avriel (2003) shows that $-\Pi_u(m)$ *i*s pseudo-convex. Therefore, $\Pi_u(m)$ *i*s pseudo-concave.

$\blacksquare$

Proof of Lemma 2.

We define the inventory cost as $CI(m) = \sqrt{2K_i h_i M \frac{e^{\alpha_i - b(c_i+m)}}{1+\sum_{j=1}^{n} e^{\alpha_i - b(c_i+m)}}}$ which is decreasing with m.

$$\pi(m^*) = \pi^0(m^*) - CI(m^*)$$

$$\pi(m^0) = \pi^0(m^0) - CI(m^0)$$

$\pi(m^*) > \pi(m^0)$ and $\pi^0(m^*) < \pi^0(m^0)$ since π^* is optimal for m^* which means $\pi^*(m^*) > \pi^*(m); \forall\, m$

and π^0 is optimal for m^0 which means that $\pi^0(m^0) > \pi^0(m);\ \forall\, m$.

$$\pi(m^*) > \pi(m^0)$$

$$\pi^0(m^*) - CI(m^*) > \pi^0(m^0) - CI(m^0)$$

$$\pi^0(m^*) - \pi^0(m^0) - CI(m^*) > -CI(m^0)$$

$$\text{but } \pi^0(m^*) - \pi^0(m^0) < 0$$

$$-CI(m^*) > -CI(m^0)$$

which implies that

$$CI(m^*) < CI(m^0)$$

Therefore, $m^* > m^0$. ■

REFERENCES

Agrawal, N., & Smith, S. A. (2003). Optimal retail assortments for substitutable items purchased in sets. *Naval Research Logistics*, *50*(7), 793–822.

Anderson, S. P., & Palma, A. D. (1992). The Logit As A Model Of Product Differentiation. *Oxford Economic Papers*, *44*(1), 51–67.

Arcelus, F. J., & Srinivasan, G. (1987). Inventory Policies Under Various Optimizing Criteria and Variable Markup Rates. *Management Science*, *33*(6), 756–762.

Avinadav, T., Chernonog, T., Lahav, Y., & Spiegel, U. (2017). Dynamic pricing and promotion expenditures in an EOQ model of perishable products. *Annals of Operations Research*, *248*(1-2), 75–91.

Avinadav, T., Herbon, A., & Spiegel, U. (2014). Optimal ordering and pricing policy for demand functions that are separable into price and inventory age. *International Journal of Production Economics*, *155*, 406–417.

Aydin, G., & Porteus, E. L. (2008). Joint Inventory and Pricing Decisions for an Assortment. *Operations Research*, *56*(5), 1247–1255.

Aydin, G. & Ryan, J. (2002), Product Line Selection and Pricing under the Multinomial Logit Choice Model. Working paper, Purdue University.

Chakravarty, A. K., & Martin, G. E. (1989). Discount pricing policies for inventories subject to declining demand. *Naval Research Logistics*, *36*(1), 89–102.

Chawla, Gagan. (2021). *Precision Pricing Drives Growth in Margin*, Wall Street Journal.

Chen, C. K., & Min, K. J. (1994). An analysis of optimal inventory and pricing policies under linear demand. *Asia-Pacific Journal of Operational Research*, *11*(2): 117-129.

Cheng, T. (1990). An EOQ model with pricing consideration. *Computers & Industrial Engineering*, *18*(4), 529–534.

Cohen, M. A. (1977). Joint pricing and ordering policy for exponentially decaying inventory with known demand. *Naval Research Logistics Quarterly*, *24*(2), 257–268.

Corless, R.M., Gonnet, G.H., Hare, D.E.G., Jeffrey, D.J. and Knuth, D.E. (1996) On the LambertWfunction. *Advances in Computational Mathematics*, 5, 329–359.

Federgruen, A., & Heching, A. (1999). Combined Pricing and Inventory Control Under Uncertainty. *Operations Research, 47*(3), 454–475.

Fisher ML, Raman A, McClelland A. (2000). Rocket-science retailing is almost here: are you ready? *Harvard Business Review*, 78:115–24.

Green, P. E., Krieger, A. M., Wind, Y. (2004). Thirty Years of Conjoint Analysis: Reflections and Prospects. *International Series in Quantitative Marketing Marketing Research and Modeling: Progress and Prospects*, 117–139.

Guadagni, P. M., Little, J. D. (1983). A Logit Model of Brand Choice Calibrated on Scanner Data. *Marketing Science*, *2*(3), 203–238.

Karlin, S. and Carr, C.R. (1962) Prices and optimal inventory policy. *Studies inApplied Probability and Management* Science, Arrow,K.J., Karlin, S. and Scarf, H. (eds), Stanford University Press, Stanford, CA, pp. 159-172

Katsifou, A., Seifert, R. W., & Tancrez, J.-S. (2014). Joint product assortment, inventory and price optimization to attract loyal and non-loyal customers. *Omega*, *46*, 36–50.

Kelly, G. (2019). Forward to Perspectives on retail and consumer goods. Available online at https://www.mckinsey.com/.

Ladany, S., & Sternlieb, A. (1974). The Interaction of Economic Ordering Quantities and Marketing Policies. *A I I E Transactions*, *6*(1), 35–40.

Li, H., & Huh, W. T. (2011). Pricing Multiple Products with the Multinomial Logit and Nested Logit Models: Concavity and Implications. *Manufacturing & Service Operations Management*, *13*(4), 549–563.

Maddah, B., Bish, E. K. (2007). Joint Pricing, Assortment, and Inventory Decisions for a Retailer's Product Line. *Naval Research Logistics*, 54, 315-330.

Maddah, B., Bish, E. K., & Munroe, B. (2011). Pricing, Variety, and Inventory Decisions for Product Lines of Substitutable Items. *International Series in Operations Research & Management Science Planning Production and Inventories in the Extended Enterprise*, 367–391.

Maddah, B., Bish, E. K., & Tarhini, H. (2014). Newsvendor pricing and assortment under Poisson decomposition. *IIE Transactions*, *46*(6), 567–584.

Maddah, B., Ghoneim, A., & Bou Youness, M. (2015), *Product Line Optimization under Quantity Discount Incentives and Logistical Considerations*. Working paper, American University of Beirut, 2015.

Mills, E. S. (1959), Uncertainty and Price Theory. *Quarterly J. of Economics*, 73:117-130, 1959.

Porteus, E. L. (1985). Investing in reduced setups in the EOQ model. *Management Science*, *31*(8), 998–1010.

Tarhini, H., Maddah, B., & Jaafar, M. (2020), Pricing and Inventory Decisions in an EOQ setting Under Logit Demand. Working paper, American University of Beirut, 2020.

Wagner, H. M., & Whitin, T. M. (1957). *Dynamic problems in the theory of the firm*. Defense Technical Information Center.

Whitin, T. M. (1955). Inventory Control and Price Theory. *Management Science*, *2*(1), 61–68.

Young, L. (1978) Price, inventory and the structure of uncertain demand. *New Zealand Operations Research*, 6, 157–177.

Zipkin, P.H. (2000). *Foundations of Inventory Management*. McGraw-Hill New York.