AMERICAN UNIVERSITY OF BEIRUT

SEMI-DEFINITE PROGRAMMING (SDP) APPROACH FOR ROBUST STATE PD CONTROL DESIGN FOR DESCRIPTOR SYSTEMS

by

ISRAA YEHYA AZZAM

A thesis
submitted in partial fulfillment of the requirements
for the degree of Master of Engineering
to the Department of Mechanical Engineering
of the Maroun Semaan Faculty of Engineering and Architecture
at the American University of Beirut

Beirut, Lebanon
August 2021

AMERICAN UNIVERSITY OF BEIRUT

Semi-Definite Programming (SDP) Approach for Robust State PD Control Design for Descriptor Systems

by

Israa Yehya Azzam

Approved by:

Dr. Dany Abou Jaoude, Assistant Professor — Advisor
Mechanical Engineering

Dr. Elie Shammas, Associate Professor — Co-Advisor
Mechanical Engineering

Dr. Naseem Daher, Assistant Professor — Member of Committee
Electrical and Computer Engineering

Date of thesis defense: 8/20/2021

AMERICAN UNIVERSITY OF BEIRUT

THESIS, DISSERTATION, PROJECT RELEASE FORM

Student Name: Azzam Israa Yehya

Last First Middle

☑ Master's Thesis ◯ Master's Project ◯ Doctoral Dissertation

☑ I authorize the American University of Beirut to: (a) reproduce hard or electronic copies of my thesis, dissertation, or project; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes.

☐ I authorize the American University of Beirut, to: (a) reproduce hard or electronic copies of it; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes after:
One ___ year from the date of submission of my thesis, dissertation or project.
Two ___ years from the date of submission of my thesis , dissertation or project.
Three ___ years from the date of submission of my thesis , dissertation or project.

[signature] 9/17/2021

Signature Date

This form is signed when submitting the thesis, dissertation, or project to the University Libraries

Acknowledgements

I like to express my sincere gratitude to my advisors *Dr. Dany Abou Jaoude* and *Dr. Elie Shammas*, for their tolerance, expert advice, and skillful leadership that supported me throughout this Thesis.

Special thanks to my parents *Mr. Yehya Azzam and Mrs. Souad Azzam*, who never let me down and always provided me with inestimable encouragement and continued love. Greetings to my sisters *Sereen Azzam* and *Sara Azzam* for their shore and support. My success in completing this Thesis would not have been possible without my family's succor and nurturing that prolonged me with a profound belief in my work and abilities.

I like to express my gratitude to *Dr. Naseem Daher* who encouraged this work and provided me with insightful comments and valuable questions that helped me improve and reinforce this work.

My sincere thanks to *Dr. Issam Lakkis* for permitting me to work in his lab during my two years at AUB.

Last but not least, I can not forget to thank my colleague *Farah Fadel* who supported me through these two years at AUB and was such a great ennobling friend.

This thesis is dedicated to my beloved parents, advisors, professors, friends, and everyone who supported me during my academic track.

This thesis was supported by the University Research Board (URB) at the American University of Beirut.

An Abstract of the Thesis of

Israa Yehya Azzam for Master of Engineering
Major: Mechanical Engineering

Title: SDP Approach for Robust State PD Control Design for Descriptor Systems

Various practical systems, such as robotic, aerospace, and electronic systems, appear to be singular by nature and cannot be represented by the explicit state-space formulation. Consequently, the descriptor state-space formulation, also known as the implicit state-space formulation, is used for representing such singular systems.

This thesis addresses the stabilization problem for linear time-invariant (LTI) descriptor systems. Techniques are proposed for synthesizing state Proportional-Derivative (PD) controllers that ensure the stability of the resulting closed-loop system and further guarantee desired decay rates.
The results are then extended to handle the robust stabilization problem for uncertain polytopic descriptor systems that have an affine dependence on the parametric uncertainties.

Using Schur complement-based lemmas, the synthesis problems are cast as semi-definite programs (SDPs) to be solved via linear matrix inequality (LMI) techniques or as nonlinear problems to be solved using the bisection method along with LMI techniques. Furthermore, the proposed formulation of the synthesis problems is augmented to account for energy optimization by minimizing the sought control gains.

Illustrative examples are provided to demonstrate the efficacy of the proposed control strategies. The results show how the synthesized state PD controllers may allow for asymptotic stabilization, decay rate maximization, gain minimization, or a combination of the latter two objectives. The simulation results also demonstrate that the robust controller is able to handle system failures modeled as parametric uncertainties, unlike a controller designed for nominal parameter values.

Contents

List of Figures

List of Tables

List of Notation

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	Set of real vectors of dimension n
$\mathbb{R}^n_+$	Set of non-negative real vectors of dimension n
$\mathbb{R}^{n\times m}$	Set of real matrices of dimension $n \times m$
$\mathbb{C}$	Set of complex numbers
$\mathbb{S}^n$	Set of symmetric matrices of dimension $n \times n$
$\mathbb{S}^n_{++}$	Set of positive definite matrices of dimension $n \times n$
$\succ$	Generalized inequality on $\mathbb{S}^n_{++}$
$\in$	Set membership
$A \subseteq B$	A is subset of B
I	Identity matrix with appropriately determined dimension
X^T	Transpose of matrix X
X^{-1}	Inverse of matrix X
$X^{-T}=(X^{-1})^T$	Inverse transpose of matrix X
$\det(X)$	Determinant of matrix X
$\mathrm{tr}(X)$	Trace of matrix X
$\mathrm{spec}(X)$	Spectrum of matrix X
$\mathrm{rank}[X]$	Rank of matrix X
$\deg(p(x))$	Degree of polynomial $p(x)$
$\|x\|$	Euclidean norm of the vector x
$\overline{\rho}$, $\underline{\rho}$	Upper and lower bounds on scalar parameter ρ

Chapter 1

Introduction

Proportional-Derivative (PD) feedback possesses a strong motivation in descriptor systems theory. For explicit systems, a proportional (P) feedback controller may be enough to stabilize the closed-loop system and achieve the desired mode requirements. In such cases, adding a derivative (D) term in the feedback law may not contribute any additional improvements. However, for descriptor systems, which are also known as implicit systems, synthesizing a D controller could alter the dynamics of the singular system. Namely, this allows transforming the implicit open-loop system into an explicit closed-loop system. But, D feedback alone can not handle descriptor systems with a singular system matrix. In such cases, using P feedback is required to modify the dynamics of the system matrix.

This thesis is a novel formulation of the state PD feedback control synthesis problem for linear time-invariant (LTI) descriptor systems. It deals with the problem of finding state PD controller gains for LTI descriptor systems such that the resulting closed-loop system is asymptotically stable or exponentially stable with a guaranteed decay rate. Furthermore, the results are extended to tackle the robust stabilization problem for uncertain descriptor systems of polytopic type.

Using Schur complement-based Lemmas, the synthesis problems are cast as semi-definite programs (SDPs) to be solved via linear matrix inequality (LMI) techniques or as nonlinear problems to be solved by the combination of the bisection method with LMI techniques. Furthermore, it is shown how to augment the proposed formulation of the synthesis problems to ensure the minimization of the controller gains.

The thesis is outlined as follows. Chapter 2 presents a review of the relevant literature. Chapter 3 presents the contribution of the proposed work, defines the problem, and introduces preliminary results and other required material. Chapter 4 treats the state PD control synthesis problems for LTI descriptor systems, discusses how to minimize the controller gains, and gives an illustrative example. Chapter 5 extends on the work done in Chapter 4. It treats the robust state PD control problem for uncertain descriptor systems and gives an illustrative example. Finally, Chapter 6 concludes the thesis.

Chapter 2

Literature Review

Descriptor systems, also known as singular systems [2] have a particular mathematical structure that makes them appealing for modeling and control design in applications such as constrained mechanical systems, chemical systems, and power electronic systems [2, 3, 4].

The importance of descriptor systems stems from the fact that they help resolve some intricacies associated with standard state-space systems. For instance, “standard”, uncertain, state-space systems in which the system matrices exhibit a rational dependence on the uncertainties can be reformulated as uncertain descriptor systems wherein the dependence of the system matrices on the uncertainties is affine; see, for instance, [5].

Several works in the literature address the stabilization problem for descriptor systems using state P, D, and PD feedback controllers. The works of [6, 7, 8, 9], address the robust stabilization problem for descriptor systems using P feedback controllers. In [6], a P feedback controller is designed to handle the robust stabilization problem for uncertain descriptor systems using a parameter-independent Lyapunov function (PILF) approach. The considered class of uncertain descriptor systems is that of affine parametric systems of polytopic type. The formulated synthesis problem is solved using LMI techniques. The LMI constraints are derived using the singular value decomposition, the inversion lemma, and other complicated manipulations.

In [7], a robust controller is designed to stabilize uncertain descriptor systems with norm-bounded perturbations in the derivative matrix. It is assumed therein that the perturbations do not change the rank of the derivative matrix.

The work in [8] derives sufficient conditions for addressing the Schur admissibilization problem for discrete-time descriptor systems, i.e., regularization and stabilization, using the Lyapunov LMI-based results for implicit systems. The derived conditions are extended to address the robust admissibilization problem for polytopic uncertain systems in which the derivative and the system matrices are subjected to parametric uncertainties.

The work in [9] proposes algorithms for strong stabilization of impulsive descriptor systems via pole assignment technique. Two approaches are considered: the first approach depends on direct stabilization techniques, and the second approach relies on iterative algorithms for pole assignment.

A number of works like the ones in [10, 11, 12, 13] address the robust stabilization problem of uncertain descriptor systems using state D feedback only. For example, in [10], asymptotic and exponential stabilization conditions are derived for uncertain polytopic descriptor systems with non-singular system matrices. The conditions are derived using a PILF approach and then formulated into LMIs to be solved using convex optimization solvers. The theorems derived in [10] are considered as the generalized versions of the theorems derived in [14], which address the robust stabilization problem for explicit systems.

The works in [15, 16, 17, 18] treat the stabilization problem for descriptor systems in the "certain" LTI system setting using state PD controllers. In [15], a controller is designed via the transfer matrix approach. The proposed controller works on shifting the controllable finite and infinite modes of the open-loop descriptor system to the desired finite location in the complex plane while fulfilling regularity conditions. In [16], the problem of stabilizing impulsive descriptor systems is addressed. The proposed method consists of replacing the infinite frequency modes with finite frequency modes by altering the dynamic order of the singular system through a derivative gain. In [18], a new Sylvester matrix equation is derived to give parametric solutions to the descriptor eigen-structure assignment problem. Namely, the work deals with determining suitable parametric representations for the proportional and derivative controller gains such that the closed-loop system is regular, impulse-free, stable, and its closed-loop derivative and system matrices satisfy a desired parametric formulation.

Adding to the prior works, the works in [19, 20, 21, 22, 23] address the robust stabilization problem for specific classes of uncertain descriptor systems using state PD controllers. The work in [19] tackles the robust stabilization problem for uncertain polytopic descriptor systems. In this work, the stability analysis condition is reformulated by dividing it into two coupled matrix inequality constraints, and the latter are used to derive the synthesis conditions. However, the derived constraints can not be directly solved using LMI solvers. The adopted methodology uses a PILF, and the proposed algorithms rely on the choice of a non-singular matrix and other variables to satisfy the derived conditions.

In [20], a PD controller is derived to tackle the robust stabilization problem for uncertain descriptor fractional-order systems involving norm-bounded uncertainties in the system matrix. The fractional-order system is normalized using a derivative controller. Then, proportional state feedback is applied to achieve the robust asymptotic stabilization of the obtained normalized system.

Similar to the work in [20], the work of [21] derives state PD controllers for descriptor systems with appropriately structured uncertainties in the system matrices. The S-procedure and dilation techniques for LMIs are leveraged to show how to compute the P gain for an appropriately chosen D gain.

In [22], a robust H_∞ control approach is adopted to design a PD controller for descriptor systems with norm-bounded uncertainties. Starting with Lyapunov theorems for explicit systems and utilizing Young's relation, the PD gains are computed such that the closed-loop system is quadratically normal quadratically stable and the H_∞ norm bound constraints are satisfied. The proposed method can be extended to tackle descriptor systems with time-varying parameteric uncertainties.

In [23], starting with positive realness and passivity conditions, a robust PD controller for a class of linear descriptor systems with linear fractional parameter uncertainties is derived. The parametric uncertainties appearing in system matrices are norm-bounded and have a known structure.

In addition to the above works on control synthesis for uncertain descriptor systems, the following works deal with the system analysis problem. Namely, the works of [24, 25, 26] give sufficient conditions for the robust stability of regular uncertain descriptor systems. For example, the work of [24] deals with uncertain systems of polytopic type. In this work, the stability conditions are derived using a PILF approach. However, to obtain less conservative results, new conditions are formulated using a parameter-dependent Lyapunov function (PDLF) approach. Similarly, in [25], robust stability conditions are formulated for uncertain descriptor systems subjected to system perturbations using a PDLF approach. Finally, in [26], the stability conditions are formulated for uncertain polytopic descriptor systems. The stability conditions therein are formulated using a copositive matrix approach.

Chapter 3

Problem Setup and Preliminary Results

This chapter presents the contributions of this thesis, describes the problem setup, and gives some preliminary results.

3.1 Contributions

The contribution of this thesis is a novel formulation of the state PD control synthesis problem for LTI descriptor systems using Schur complement-based Lemmas. Namely, the contribution consists of using the Schur complement lemma and its reverse to derive Theorems 4.1 and 4.2 that allow for computing the PD gains to asymptotically and exponentially stabilize LTI descriptor systems.

The adopted strategy relies on starting from the Lyapunov stability conditions for explicit systems and using Schur complement-based lemmas to formulate the control synthesis problems as SDPs to be solved via LMI techniques or as nonlinear problems to be solved by the combination of the bisection method with LMI techniques. Furthermore, the proposed formulation of the synthesis problems is augmented to ensure the minimization of the controller gains.

The methods derived for the state PD control design for LTI descriptor systems are extended to allow for designing robust state PD controllers for polytopic descriptor systems with affine parametric uncertainties. Thus, expanding on Theorems 4.1 and 4.2, Theorems 5.1 and 5.2 are derived to allow for computing state PD controller gains that robustly stabilize the uncertain closed-loop system. Throughout this thesis, the SDPs are modeled using Yalmip [27] and solved using SDPT3 [28].

3.2 Convex Analysis

This section presents the necessary background from convex analysis. For more details on the standard material presented here, the reader is referred to the reference books in [29] and [30].

3.2.1 Affine and Convex Sets

Consider the two distinct points p_1 and $p_2 \in \mathbb{R}^n$. The line passing through p_1 and p_2 is generated by the affine combination of the points p_1 and p_2, i.e., the points of the form $q = \sigma p_1 + (1-\sigma)p_2$ with $\sigma \in \mathbb{R}$. Then, every point p_3 defined as $p_3 = \sigma p_1 + (1-\sigma)p_2$ with $\sigma \in \mathbb{R}$, lies on the line p_1p_2. For $0 \leq \sigma \leq 1$, the point p_3 lies on the line segment between p_1 and p_2. However, for $\sigma > 1$, p_3 lies on the line p_1p_2 beyond p_1. On the other hand, for $\sigma < 0$, p_3 lies on the line p_1p_2 beyond p_2.

A set $\mathcal{P}$ is said to be an affine set if for any p_1, $p_2 \in \mathcal{P}$ and any $\sigma \in \mathbb{R}$, the point $q = \sigma p_1 + (1-\sigma)p_2$ lies in $\mathcal{P}$. The set of all the affine combinations of the points in $\mathcal{P}$ is called the affine hull of $\mathcal{P}$. The set $\mathcal{P}$ is said to be a convex set if for any p_1, $p_2 \in \mathcal{P}$ and any σ between 0 and 1 , the point $q = \sigma p_1 + (1-\sigma)p_2$ lies in $\mathcal{P}$. For $0 \leq \sigma \leq 1$, the point $q = \sigma p_1 + (1-\sigma)p_2$ is called a convex combination of the points p_1 and p_2. The set of all the convex combinations of the points in $\mathcal{P}$ is called the convex hull of $\mathcal{P}$.

These concepts can be generalized to combinations of more than two points. Consider a set $\mathcal{P}$ and points $p_1, p_2, .., p_k$ in $\mathcal{P}$. The set $\mathcal{P}$ is an affine set if and only if all the points of the form $\sum_{i=1}^{k} \sigma_i p_i$, where $\sum_{i=1}^{k} \sigma_i = 1$, lie in $\mathcal{P}$.
The set $\mathcal{P}$ is a convex set if and only if all the points of the form $\sum_{i=1}^{k} \sigma_i p_i$ with $\sum_{i=1}^{k} \sigma_i = 1$ and $\sigma_i \geq 0$ up to 1 lie in $\mathcal{P}$.

Let $\mathbf{C_o}\{\mathcal{P}\}$ and $\mathbf{A_o}\{\mathcal{P}\}$, respectively, denote the convex hull and the affine hull of the set $\mathcal{P}$. Then, $\mathbf{C_o}\{\mathcal{P}\}$ and $\mathbf{A_o}\{\mathcal{P}\}$ are defined as

$$\mathbf{C_o}\{\mathcal{P}\} := \left\{ \sum_{i=1}^{k} \sigma_i p_i \,\middle|\, p_i \in \mathcal{P}, \sigma_i \geq 0, \sum_{i=1}^{k} \sigma_i = 1 \right\},$$

$$\mathbf{A_o}\{\mathcal{P}\} := \left\{ \sum_{i=1}^{k} \sigma_i p_i \,\middle|\, p_i \in \mathcal{P}, \sum_{i=1}^{k} \sigma_i = 1 \right\}.$$

The convex hull of $\mathcal{P}$ is convex, and the affine hull of $\mathcal{P}$ is affine. Finally, every affine set is a convex set, but not every convex set is an affine set.

A convex polytope is defined as the convex hull of a finite set of points, called vertices, including the set of extreme points of the polytope.

Let $\mathcal{V} := \{v_1, v_2, ..., v_N\}$ denote the set of vertices. Then, the convex polytope is defined as

$$\daleth_N := \mathbf{C_o}\{\mathcal{V}\} := \left\{ \sum_{i=1}^{N} \chi_i v_i \,\middle|\, \chi_i \geq 0, \sum_{i=1}^{N} \chi_i = 1 \right\}.$$

3.3 Explicit State-Space System Analysis

This section introduces the preliminary results for the explicit state-space system analysis. It presents the initial findings for LTI explicit systems and uncertain explicit systems.

3.3.1 LTI Explicit Systems

Consider the continuous-time LTI autonomous explicit system

$$\dot{x}(t) = Ax(t), \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $t \geq 0$ denotes the continuous time, and $A \in \mathbb{R}^{n \times n}$ is the system matrix.

The system in (3.1) is quadratically stable if there exists a quadratic Lyapunov function $V(x) = x(t)^T P x(t)$ with $P \in \mathbb{S}^n_{++}$ that decreases over all nonzero trajectories $x(t)$ of (3.1).
The system in (3.1) is asymptotically stable if all the eigenvalues of the matrix A lie in the open left-half of the complex plane, i.e., $\lim_{t \to \infty} \|x(t)\| = 0$ for all $x(0) \in \mathbb{R}^n$. Since the system in (3.1) is LTI, quadratic stability is equivalent to asymptotic stability.
The system in (3.1) is further said to be exponentially stable with a decay rate α if $\lim_{t \to \infty} e^{\alpha t} \|x(t)\| = 0$.

This section is concluded with two important lemmas to be used in proving the results in this thesis.

***Lemma* 3.1** *The system in* (3.1) *is asymptotically stable if and only if there exists $P \in \mathbb{S}^n_{++}$ such that $A^T P + PA \prec 0$.*

***Lemma* 3.2** *The system in* (3.1) *is exponentially stable with a decay rate $\alpha > 0$ if and only if there exists $P \in \mathbb{S}^n_{++}$ such that $A^T P + PA \prec -2\alpha P$.*

***Remark* 3.1** *Lemmas 3.1 and 3.2 are standard LMI-based Lyapunov results and can be found in multiple references, see e.g., [31, 32].*

3.3.2 Uncertain Explicit Systems

Consider the continuous-time uncertain system

$$\dot{x}(t) = A(\varphi)x(t), \tag{3.2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $t \geq 0$ denotes continuous time, $A(\varphi) \in \mathbb{R}^{n\times n}$ is the uncertain system matrix, and φ is the uncertain parameter vector.
Let $\Delta_\varphi \subseteq \mathbb{R}^N$ be the compact set of allowable values of φ, where N is the total number of parameters.

The uncertain system is said to be robustly asymptotically stable if $\mathrm{spec}(A(\varphi))$ lies in the open left-half of the complex plane for all $\varphi \in \Delta_\varphi$ [33]. This means that all the trajectories $x(t)$ of (3.2) converge to 0 as $t \to \infty$ for all $\varphi \in \Delta_\varphi$. The system is further said to be robustly exponentially stable with a decay rate α if $\lim_{t\to\infty} e^{\alpha t}\|x(t)\| = 0$ for all trajectories $x(t)$ of (3.2) for all $\varphi \in \Delta_\varphi$.

The uncertain system in (3.2) is said to be robustly quadratically stable if there exists a quadratic Lyapunov function $V(x) = x(t)^T Px(t)$ with $P \in \mathbb{S}^n_{++}$ that decreases over all nonzero trajectories $x(t)$ of (3.2). Robust quadratic stability is only sufficient for robust asymptotic stability [33], and one can significantly reduce conservatism by searching for a parameter-dependent Lyapunov function. Nonetheless, in this thesis, we use the following sufficient characterization of robust asymptotic stability (this characterization is necessary and sufficient for robust quadratic stability) to allow for a preliminary extension of our PD control design techniques for LTI descriptor systems to the context of polytopic descriptor systems. Future work will look at reducing the associated conservatism by using better characterizations of robust asymptotic stability.

***Lemma* 3.3** *The system in* (3.2) *is robustly asymptotically stable if there exists* $P \in \mathbb{S}^n_{++}$ *such that* $A(\varphi)^T P + PA(\varphi) \prec 0$ *for all* $\varphi \in \Delta_\varphi$.

***Lemma* 3.4** *The system in* (3.2) *is robustly exponentially stable with a decay rate* $\alpha > 0$ *if there exists* $P \in \mathbb{S}^n_{++}$ *such that* $A(\varphi)^T P + PA(\varphi) \prec -2\alpha P$ *for all* $\varphi \in \Delta_\varphi$.

3.4 Descriptor State-Space System Analysis

Given the matrices M and N, consider the matrix pencil $\lambda M - N$. The matrix pencil $\lambda M - N$ is said to be regular if there exists $\lambda_0 \in \mathbb{C}$ such that $\det(\lambda_0 M - N) \neq 0$. $\tilde{\lambda} \in \mathbb{C}$ is a (finite) eigenvalue of $\lambda M - N$ if $\det(\tilde{\lambda}M - N) = 0$. $\tilde{\lambda}$ is an infinite eigenvalue of $\lambda M - N$ if 0 is an eigenvalue of $\lambda N - M$.

Consider the continuous-time LTI *descriptor* system

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{3.3}$$

where $x(t) \in \mathbb{R}^n$ is the generalized state vector and $u(t) \in \mathbb{R}^m$ is the input vector. $E \in \mathbb{R}^{n\times n}$ is the derivative matrix of rank $r \leq n$, $A \in \mathbb{R}^{n\times n}$ is the system matrix, and $B \in \mathbb{R}^{n\times m}$ is the input matrix.

3.4.1 Definitions

Having defined the system equation, the following definitions are introduced [34]:

***Definition* 3.1** *The descriptor system in* (3.3) *is said to be regular if the matrix pencil $\lambda E - A$ is regular. A regular system is further said to be impulse-free, i.e, non-impulsive, if* $\deg(\det(\lambda E - A)) = \mathit{rank}[E]$.

***Definition* 3.2** *The descriptor system in* (3.3) *is said to be stable if it is regular and all eigenvalues of $\lambda E - A$ lie in the open left-half of the complex plane. The system is said to be admissible if it is regular, impulse-free, and stable.*

***Definition* 3.3** *Consider the descriptor system in* (3.3) *and assume it to be regular. The system is said to be completely controllable (C-controllable) if for any initial condition $x(0) = x_0 \in \mathbb{R}^n$, it is possible to find a sufficiently smooth control input $u(t)$ that will drive the state response from x_0 to any final state $x_f \in \mathbb{R}^n$ in any specified period of time $t_1 > 0$, i.e., $x(t_1) = x_f$.*

3.4.2 Controllability Conditions

Consider the open-loop descriptor system in (3.3). As per [35], the following assumptions are introduced:

***Assumption* 3.1** *The input matrix B is full column rank, i.e., rank $[B] = m$.*

***Assumption* 3.2** *The system in* (3.3) *is a regular system.*

Consider the following conditions:

C.1. rank $\begin{bmatrix}\gamma E - \theta A & B\end{bmatrix} = n$ for all $(\gamma, \theta) \in \mathbb{C} \times \mathbb{C} \setminus \{0, 0\}$.

C.2. rank $\begin{bmatrix}\lambda E - A & B\end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$.

Condition C.1. implies condition C.2. This can be shown by letting $\lambda = \gamma/\theta$ and $\theta \neq 0$. Furthermore, condition C.1. is satisfied if and only if condition C.2. and the condition rank$[E\ B] = n$ are satisfied.

If the descriptor system in (3.3) is regular, then conditions C.1. and C.2. can be used to characterize the controllability of the system. In the rest of this section, it is supposed that Assumption 3.2 is satisfied. Then, the system in (3.3) is C-controllable if and only if condition C.1. is satisfied. Moreover, it follows that rank$[E\ B] = n$ is a necessary condition for the system to be C-controllable.

For more details, the reader is referred to [36, 37, 38, 39].

3.4.3 Polytopic Uncertain Descriptor Systems

Consider the uncertain polytopic *descriptor* system

$$E(\rho)\dot{x}(t) = A(\delta)x(t) + B(\beta)u(t), \tag{3.4}$$

where $x(t) \in \mathbb{R}^n$ is the generalized state vector, $u(t) \in \mathbb{R}^m$ is the input vector, and $t \geq 0$ denotes continuous time. The derivative matrix $E(\rho) \in \mathbb{R}^{n\times n}$, the system matrix $A(\delta) \in \mathbb{R}^{n\times n}$, and the input matrix $B(\beta) \in \mathbb{R}^{n\times m}$ are subjected to parametric uncertainties and belong to the following convex polytopes:

$$\mathcal{E} := \left\{ \sum_{k=1}^{v_e} \rho_k E_k \;\middle|\; \rho_k \geq 0, \sum_{k=1}^{v_e} \rho_k = 1 \right\}, \tag{3.5}$$

$$\mathcal{A} := \left\{ \sum_{i=1}^{v_a} \delta_i A_i \;\middle|\; \delta_i \geq 0, \sum_{i=1}^{v_a} \delta_i = 1 \right\}, \tag{3.6}$$

$$\mathcal{B} := \left\{ \sum_{j=1}^{v_b} \beta_j B_j \;\middle|\; \beta_j \geq 0, \sum_{j=1}^{v_b} \beta_j = 1 \right\}, \tag{3.7}$$

respectively, where E_k, A_i, and B_j are given real constant matrices at the k^{th}, i^{th}, and j^{th} vertices of the sets $\mathcal{E}$, $\mathcal{A}$, and $\mathcal{B}$, respectively, and v_e, v_a, and v_b denote the total number of the polytope vertices in $\mathcal{E}$, $\mathcal{A}$, and $\mathcal{B}$, respectively. ρ_k with $k = 1, 2.....v_e$, δ_i with $i = 1, 2.....v_a$, and β_j with $j = 1, 2.....v_b$ are non-negative unknown real constants that satisfy $\sum_{k=1}^{v_e} \rho_k = 1$, $\sum_{i=1}^{v_a} \delta_i = 1$, and $\sum_{j=1}^{v_b} \beta_j = 1$, respectively.

To illustrate, let e, a, and b be the total numbers of uncertain parameters in $E(\rho)$, $A(\delta)$, and $B(\beta)$, respectively, and let $\Upsilon_E = \begin{bmatrix}\Upsilon_{E_1} & \Upsilon_{E_2} & ... & \Upsilon_{E_e}\end{bmatrix}^T \in \mathbb{R}^e$, $\Upsilon_A = \begin{bmatrix}\Upsilon_{A_1} & \Upsilon_{A_2} & ... & \Upsilon_{A_a}\end{bmatrix}^T \in \mathbb{R}^a$, and $\Upsilon_B = \begin{bmatrix}\Upsilon_{B_1} & \Upsilon_{B_2} & ... & \Upsilon_{B_b}\end{bmatrix}^T \in \mathbb{R}^b$ be the vectors of the uncertain parameters. If each uncertain parameter $\Upsilon_{E_{h_e}}$ for $h_e = 1, 2, ..., e$, $\Upsilon_{A_{h_a}}$ for $h_a = 1, 2, ..., a$, and $\Upsilon_{B_{h_b}}$ for $h_b = 1, 2, ..., b$ is bounded by a minimum and a maximum value, namely, $\Upsilon_{E_{h_e}} \in \begin{bmatrix}\underline{\Upsilon_{E_{h_e}}}, & \overline{\Upsilon_{E_{h_e}}}\end{bmatrix}$, $\Upsilon_{A_{h_a}} \in \begin{bmatrix}\underline{\Upsilon_{A_{h_a}}}, & \overline{\Upsilon_{A_{h_a}}}\end{bmatrix}$, and $\Upsilon_{B_{h_b}} \in \begin{bmatrix}\underline{\Upsilon_{B_{h_b}}}, & \overline{\Upsilon_{B_{h_b}}}\end{bmatrix}$ then all the possible combinations of the minimum and the maximum values of the parameters define a convex polytope having the total number of vertices $v_T = v_e \times v_a \times v_b$ with $v_e = 2^e$, $v_a = 2^a$, and $v_b = 2^b$.

3.5 Schur Complement-Based Lemmas

This section introduces the Schur complement-based Lemmas to be called upon in later chapters of this thesis.

***Lemma* 3.5** *[40] Consider $A \in \mathbb{S}^n$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{S}^m$, and $X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \in \mathbb{S}^{n+m}$. Let $X/A = C - B^T A^{-1} B$ and $X/C = A - BC^{-1}B^T$ be the Schur complements of A in X and the Schur complement of C in X, respectively. The following statements are equivalent:*

- $X \succ 0$.
- $X/A \succ 0\,,\;\; A \succ 0$.
- $X/C \succ 0\,,\;\; C \succ 0$.

Lemma 3.5 is referred to as the Schur complement lemma [40].

***Lemma* 3.6** *[41] Consider $P \in \mathbb{S}^n_{++}$, and H, Z, and G in $\mathbb{R}^{n\times n}$, where G is invertible. Then, $\begin{pmatrix} H+H^T-Z & G^T \\ G & P \end{pmatrix} \succ 0$ implies that $H^T G^{-1} P G^{-T} H \succ Z$. When $G = P$, the inequalities become equivalent.*

This chapter is concluded with the following standard result, see e.g., [31, 42]:

***Lemma* 3.7** *If the matrix $G \in \mathbb{R}^{n\times n}$ satisfies $G + G^T \prec 0$, then G is a nonsingular matrix.*

Chapter 4

PD State Feedback Control Synthesis

This chapter addresses the problem of synthesizing state PD feedback controllers such that the resulting closed-loop system is asymptotically stable or exponentially stable with a guaranteed decay rate α. Also, it discusses how to minimize the controller gains. Furthermore, this chapter provides an example to illustrate the proposed control strategies. The example deals with stabilizing a two degrees-of-freedom (2-DOF) ballbot system around the unstable equilibrium position by applying the state PD controllers derived throughout this chapter.

Throughout this chapter, it is assumed that the descriptor system in (3.3) is regular, impulse-free, and C-controllable. Also, it is assumed that Assumption 3.1 holds.

4.1 Control Synthesis

Consider the state PD feedback control law of the form

$$u(t) = -K_o x(t) - F_o \dot{x}(t), \tag{4.1}$$

where K_o and F_o in $\mathbb{R}^{m \times n}$ are the proportional and the derivative state feedback gains, respectively. The control synthesis problem addressed here is to find feedback gains K_o and F_o that stabilize the descriptor system in (3.3).

To do so, the control law in (4.1) is applied to the system in (3.3) to obtain the following closed-loop system:

$$(E + BF_o)\dot{x}(t) = (A - BK_o)x(t).$$

Assuming that $(E + BF_o)$ is non-singular, the closed-loop system is equivalently written as

$$\dot{x}(t) = (E + BF_o)^{-1}(A - BK_o)x(t). \tag{4.2}$$

It is shown throughout how to compute an F_o that makes $(E+BF_o)$ non-singular. It is required to stabilize the descriptor system in (3.3) using the state PD control law (4.1) by rendering the closed-loop system in (4.2) asymptotically stable. Also, it is desired to achieve the best convergence rates and minimize the PD control gains.

4.1.1 Stabilization

The control synthesis objective is to make the closed-loop system in (4.2) asymptotically stable. Also, it is desirable to achieve the best decay rates for exponential stability. As such, two stabilization problems are considered, namely, the asymptotic and exponential stabilization problems. The synthesis conditions in the asymptotic stabilization problem are expressed as LMIs, and so the problem is formulated as an SDP. In the exponential stabilization problem, the synthesis conditions are expressed as LMIs for a given decay rate. Thus, to maximize the decay rate of the closed-loop system, the bisection method [43] is used, wherein one SDP is solved at each iteration.

***Theorem* 4.1** *Consider the descriptor system in* (3.3) *and the control law in* (4.1). *If there exist $Q \in \mathbb{S}^n_{++}$, $X_1 \in \mathbb{S}^n$, and Y_1 and $Y_2 \in \mathbb{R}^{m\times n}$ that satisfy the following LMIs:*

$$\begin{pmatrix} Q & 0 & (BY_1)^T \\ 0 & Q & (BY_2)^T \\ (BY_1) & (BY_2) & X_1 \end{pmatrix} \succ 0, \tag{4.3}$$

$$\begin{pmatrix} B(Y_1+Y_2)+(B(Y_1+Y_2))^T - cvx - X_1 & Q \\ Q & Q \end{pmatrix} \succ 0, \tag{4.4}$$

where $cvx = \Lambda + \Lambda^T$ with $\Lambda = EQA^T - EY_1^T B^T + BY_2 A^T$, then the choice $K_o = Y_1 Q^{-1}$ and $F_o = Y_2 Q^{-1}$ renders the closed-loop system in (4.2) *asymptotically stable.*

***Proof* 4.1** *By Lemma 3.5, the LMI in* (4.3) *is equivalent to the following inequality:*

$$X_1 \succ \begin{pmatrix} (BY_1)^T \\ (BY_2)^T \end{pmatrix}^T \begin{pmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{pmatrix} \begin{pmatrix} (BY_1)^T \\ (BY_2)^T \end{pmatrix}, \text{with} \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \succ 0. \tag{4.5}$$

Similarly, using Lemma 3.6, the LMI in (4.4) *is equivalent to the following inequality:*

$$cvx - B(Y_1+Y_2)Q^{-1}(B(Y_1+Y_2))^T + X_1 \prec 0. \tag{4.6}$$

Then, for X_1 satisfying the inequality in (4.5)*, the inequality in* (4.6) *implies the following:*

$$cvx - B(Y_1 + Y_2)Q^{-1}(B(Y_1 + Y_2))^T + \begin{pmatrix}(BY_1)^T \\ (BY_2)^T\end{pmatrix}^T \begin{pmatrix}Q^{-1} & 0 \\ 0 & Q^{-1}\end{pmatrix} \begin{pmatrix}(BY_1)^T \\ (BY_2)^T\end{pmatrix} \prec 0. \quad (4.7)$$

Next, replace Y_1 and Y_2 by K_oQ and F_oQ, respectively, and do the needed multiplications to get the following result:

$$cvx - B(K_oQF_o^T + F_oQK_o^T)B^T \prec 0.$$

Now, substitute $cvx = \Lambda + \Lambda^T$, where $\Lambda = EQA^T - E(K_oQ)^T B^T + B(F_oQ)A^T$, and expand to get the following:

$$EQA^T + AQE^T - EQK_o^T B^T + BF_oQA^T + AQF_o^T B^T - BK_oQE^T - BF_oQK_o^T B^T - BK_oQF_o^T B^T \prec 0.$$

Then, factorize to get

$$(E + BF_o)Q(A - BK_o)^T + (A - BK_o)Q(E + BF_o)^T \prec 0.$$

By Lemma 3.7, it follows that $(E + BF_o)$ is a non-singular matrix because the matrices $(E+BF_o)$, Q, and $(A-BK_o)$ are square matrices. Thus, we have found an F_o such that $(E + BF_o)$ is non-singular.

To proceed with the proof, pre and post-multiply the previous inequality by $Q^{-1}(E+BF_o)^{-1}$ and its transpose, respectively, and replace the positive definite matrix Q^{-1} by $P \succ 0$ to obtain the following equivalent inequality:

$$(A-BK_o)^T(E+BF_o)^{-T}P+P(E+BF_o)^{-1}(A-BK_o)\prec 0. \quad (4.8)$$

By Lemma 3.1, the closed-loop system in (4.2) *is asymptotically stable, which concludes the proof.*

By Theorem 4.1, one can find a state PD controller that achieves closed-loop asymptotic stability by solving the LMIs (4.3) and (4.4). Expanding on Theorem 4.1, Theorem 4.2 shows how to design a state PD controller that further guarantees closed-loop exponential stability with a decay rate α.

Theorem 4.2 *Consider the descriptor system in* (3.3) *and the control law in* (4.1)*. If there exist $Q \in \mathbb{S}^n_{++}$, Y_1 and $Y_2 \in \mathbb{R}^{m\times n}$, and X_1 and $X_2 \in \mathbb{S}^n$ satisfying* (4.3)*,*

$$\begin{pmatrix}-(cvx + X_1 - X_2) & (EQ + BY_2) \\ (EQ + BY_2)^T & Q/2\alpha\end{pmatrix} \succ 0, \quad (4.9)$$

$$\begin{pmatrix} B(Y_1+Y_2)+(B(Y_1+Y_2))^T - X_2 & Q \\ Q & Q \end{pmatrix} \succ 0, \tag{4.10}$$

where $cvx = \Lambda + \Lambda^T$ *with* $\Lambda = EQA^T - EY_1^T B^T + BY_2 A^T$, *then the choice* $K_o = Y_1 Q^{-1}$ *and* $F_o = Y_2 Q^{-1}$ *makes the closed-loop system in* (4.2) *exponentially stable with a guaranteed decay rate* α.

***Proof* 4.2** *Referring to Lemma 3.5, the LMI in* (4.3) *is equivalent to the inequality in* (4.5)*, and* (4.9) *is equivalent to the following inequality:*

$$cvx + X_1 - X_2 \prec -2\alpha(EQ + BY_2)Q^{-1}(EQ + BY_2)^T. \tag{4.11}$$

Similarly, by Lemma 3.6, because $Q \succ 0$ *and so non-singular, the LMI in* (4.10) *is equivalent to the following:*

$$X_2 \prec B(Y_1 + Y_2)Q^{-1}(B(Y_1 + Y_2))^T. \tag{4.12}$$

For X_1 *satisfying the inequality in* (4.5) *and* X_2 *satisfying the inequality in* (4.12)*, the inequality in* (4.11) *implies the following:*

$$cvx - B(Y_1+Y_2)Q^{-1}(B(Y_1+Y_2))^T + \begin{pmatrix} (BY_1)^T \\ (BY_2)^T \end{pmatrix}^T \begin{pmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{pmatrix} \begin{pmatrix} (BY_1)^T \\ (BY_2)^T \end{pmatrix} \prec -2\alpha(EQ + BY_2)Q^{-1}(EQ + BY_2)^T.$$

Following a procedure similar to the one in Proof 4.1, replace Y_1 *and* Y_2 *by* K_oQ *and* F_oQ*, respectively, and perform the needed matrix multiplications to get the following inequality:*

$$cvx - B(K_oQF_o^T + F_oQK_o^T)B^T \prec -2\alpha(E + BF_o)QQ^{-1}Q(E + BF_o)^T,$$

where $\Lambda = EQA^T - E(K_oQ)^T B^T + B(F_oQ)A^T$. *Then, replace* $Q \succ 0$ *by* P^{-1} *and factorize to get*

$$(E + BF_o)P^{-1}(A - BK_o)^T + (A - BK_o)P^{-1}(E + BF_o)^T \prec -2\alpha(E + BF_o)P^{-1}(E + BF_o)^T. \tag{4.13}$$

Now, to prove that $(E + BF_o)$ *is non-singular, rearrange the equation to get the following equivalent inequality:*

$$(E + BF_o)P^{-1}((A - BK_o) + \alpha(E + BF_o))^T + ((A - BK_o) + \alpha(E + BF_o))P^{-1}(E + BF_o)^T \prec 0.$$

Then, using Lemma 3.7, because $Q = P^{-1} \succ 0$ *and the matrices* $(E + BF_o)$ *and* $((A - BK_o) + \alpha(E + BF_o))$ *are square matrices, it follows that* $(E + BF_o)$ *is a* *non-singular matrix. Therefore, we have found an* F_o *that renders* $(E + BF_o)$ *non-singular.*

To complete the proof, pre and post-multiply the inequality in (4.13) *by* $P(E + BF_o)^{-1}$ *and its transpose, respectively, to get the following equivalent inequality:*

$$(A - BK_o)^T(E + BF_o)^{-T}P + P(E + BF_o)^{-1}(A - BK_o) \prec -2\alpha P.$$

Thus, by Lemma 3.2, the closed-loop system in (4.2) *is exponentially stable with a guaranteed decay rate* α*.*

Based on Theorem 4.2, for a given decay rate α, one can solve for Q, X_1, X_2, Y_1, and Y_2 that satisfy the LMIs in (4.3), (4.9), and (4.10). This allows for certifying the feasibility of the given α as well as computing the corresponding gains. However, solving for the gains that achieve the maximum feasible decay rate is a non-convex problem, which is handled by resorting to the bisection method.

4.1.2 Control Gain Minimization

In addition to stabilizing the system, it might be required to minimize the controller gains to reduce the input energy consumption. This section discusses how to minimize the PD control gains. The discussion builds on the heuristic for the gain minimization presented in [44].

Consider the asymptotic stabilization problem using the state PD feedback control law in (4.1). In addition to the LMIs (4.3) and (4.4) in Theorem 4.1, it is shown that imposing the additional LMI constraints $Q \succ \zeta I$, $\begin{pmatrix} \varrho_1 I & Y_1 \\ Y_1^T & I \end{pmatrix} \succ 0$, and $\begin{pmatrix} \varrho_2 I & Y_2 \\ Y_2^T & I \end{pmatrix} \succ 0$ ensures that the gains K_o and F_o that asymptotically stabilize the system satisfy the inequalities $K_o K_o^T \prec \varrho_1 I/\zeta^2$ and $F_o F_o^T \prec \varrho_2 I/\zeta^2$, where ϱ_1 and ϱ_2 are variables to be minimized and $0 < \zeta < 1$ is a given scaling factor. Namely, it is desired to prove that the constraints $Q \succ \zeta I$ and $\begin{pmatrix} \varrho_1 I & Y_1 \\ Y_1^T & I \end{pmatrix} \succ 0$ imply that $K_o K_o^T \prec \varrho_1 I/\zeta^2$. The inequality $F_o F_o^T \prec \varrho_2 I/\zeta^2$ can then be proved similarly.

By Lemma 3.5, $\begin{pmatrix} \varrho_1 I & Y_1 \\ Y_1^T & I \end{pmatrix} \succ 0$ is equivalent to $Y_1 Y_1^T \prec \varrho_1 I$. Also, $Q \succ \zeta I$ can be equivalently rewritten as $QQ \succ \zeta Q$ by pre and post-multiplying it by $Q^{1/2} \succ 0$ from both sides, where $Q^{1/2}Q^{1/2} = Q$. Pre and post-multiplying the previous inequality by K_o and its transpose, and dividing its both sides by $\zeta > 0$ result in $(1/\zeta)K_o QQK_o^T \succeq K_o QK_o^T$. Then, substituting $Y_1 = K_o Q$ gives $K_o QK_o^T \preceq (1/\zeta)Y_1 Y_1^T \prec (\varrho_1/\zeta)I$. Moreover, $Q \succ \zeta I$ implies that $K_o QK_o^T \succeq \zeta K_o K_o^T$. Hence,

it follows that $\zeta K_o K_o^T \preceq K_o Q K_o^T \prec (\varrho_1/\zeta)I$, i.e., $K_o K_o^T \prec (\varrho_1/\zeta^2)I$.
Thus, it is desired to minimize the vector-valued objective function $\varrho = (\varrho_1, \varrho_2)$ subject to the original synthesis conditions as well as the added LMI constraints for gain minimization. In practice, this objective function is scalarized by introducing the vector $\mu = (\mu_1, \mu_2)$, where $\mu_1, \mu_2 > 0$, $\mu_1 + \mu_2 = 1$, and μ_j is the weight associated with the j-th component of the objective function for $j = 1, 2$.
In conclusion, to obtain PD gains that render the closed-loop system asymptotically stable and are small with respect to the scalarized objective function (for given μ_1 and μ_2), the following SDP is solved:

$$\begin{aligned} &\text{minimize} && \mu_1\varrho_1 + (1-\mu_1)\varrho_2 \\ &\text{subject to} && Q \succ \zeta I,\ \begin{pmatrix} \varrho_1 I & Y_1 \\ Y_1^T & I \end{pmatrix} \succ 0,\ \begin{pmatrix} \varrho_2 I & Y_2 \\ Y_2^T & I \end{pmatrix} \succ 0, \\ &&& \text{LMIs in (4.3) and (4.4).} \end{aligned} \tag{4.14}$$

To obtain a PD control law with small gains that renders the closed-loop system exponentially stable with a given decay rate α, a problem similar to the SDP in (4.14) is solved, wherein the LMIs (4.3) and (4.4) of Theorem 4.1 are replaced by the LMIs (4.3), (4.9), and (4.10) of Theorem 4.2.

4.1.3 Algorithmic Implementation

This section summarizes the control synthesis problems defined throughout. It gives algorithms to compute the state PD gains that asymptotically stabilize the closed-loop system in (4.2) while achieving the best decay rates and allowing for energy optimization. Doing so requires defining the following SDPs:

***SDP* 4.1** *Solve for Q, X_1, X_2, Y_1, and Y_2 such that the LMIs* (4.3) *and* (4.4) *in Theorem 4.1 are satisfied.*

***SDP* 4.2** *For a given decay rate α, solve for Q, X_1, X_2, Y_1, and Y_2 that satisfy the LMIs in* (4.3), (4.9), *and* (4.10) *in Theorem 4.2.*

***SDP* 4.3** *For a given μ_1, solve the SDP defined in* (4.14).

***SDP* 4.4** *For a given decay rate α and a given μ_1, solve the SDP defined in* (4.14), *with the LMIs* (4.3), (4.9), *and* (4.10) *defined in Theorem 4.2 instead of the LMIs* (4.3) *and* (4.4) *defined in Theorem 4.1.*

Solving SDP 4.2 allows to certify the feasibility of a given decay rate α and to find the corresponding state PD gains. Let α^* be the maximum feasible decay rate. α^* is located using the bisection method according to Algorithm 4.1.

Denote by α_{low} and α_{up} a prior known feasible lower bound and infeasible upper bound on α^*, respectively. As shown in Algorithm 4.1, the feasibility of the midpoint $\alpha_{mid} = (\alpha_{low} + \alpha_{up})/2$ is determined by solving SDP 4.2 . If α_{mid} is a feasible decay rate, α_{low} is updated to α_{mid}. Otherwise, if α_{mid} is an infeasible decay rate, α_{up} is updated to α_{mid}. The process repeats until convergence, i.e., $\alpha_{up} - \alpha_{low} \leq \tau$, where $\tau > 0$ is a pre-specified tolerance value. Upon convergence, α^* is set equal to the final value of α_{low}. Thus, Algorithm 4.1 shows how to find the state PD controller gains that maximize α to attain a good performance for the resulting closed-loop system.

Algorithm 4.1

Input: E, A, B, α_{up}, α_{low}, and τ
Output: α^*, K_o, and F_o
Step 1: **while** $\alpha_{up} - \alpha_{low} > \tau$
  Assign $\alpha_{mid} = (\alpha_{up} + \alpha_{low})/2$
  Solve SDP 4.2 at α_{mid}
  if SDP 4.2 is feasible
  $\alpha_{low} = \alpha_{mid}$
  else
  $\alpha_{up} = \alpha_{mid}$
  end
 end
Step 2: **Solve** SDP 4.2 at α_{low}
 $\alpha^* = \alpha_{low}, \quad K_o = Y_1 Q^{-1}, \quad F_o = Y_2 Q^{-1}.$

Furthermore, it is possible to obtain an optimal performance with minimum input energy consumption by computing smaller controller gains with maximum allowable decay rates. If it is required to maximize the decay rate α while minimizing the state PD controller gains, then Algorithm 4.2 can be applied. To trace the optimal front exhibiting the trade-off between the two sub-objectives, i.e., ϱ_1 and ϱ_2, the weight μ_1 is varied between 0 and 1.

Algorithm 4.2

Input: E, A, B, α_{up}, α_{low}, τ, μ_1, and ζ
Output: α^*, K_o, F_o, ϱ_1, and ϱ_2
Step 1: **while** $\alpha_{up} - \alpha_{low} > \tau$
  Assign $\alpha_{mid} = (\alpha_{up} + \alpha_{low})/2$
  Solve SDP 4.4 at α_{mid}
  if SDP 4.4 is feasible
  $\alpha_{low} = \alpha_{mid}$
  else
  $\alpha_{up} = \alpha_{mid}$
  end
 end
Step 2: **Solve** SDP 4.4 at α_{low}
 $\alpha^* = \alpha_{low}, \quad K_o = Y_1 Q^{-1}, \quad F_o = Y_2 Q^{-1}.$

4.2 A 2-DOF BallBot System Example

Consider a 2-DOF mechanical system describing an under-actuated ball-balancing robot [45], hereafter referred to as ballbot. This robot has encoders to measure the angular positions and the angular velocities for each of the generalized coordinates. The ball of the ballbot system is actuated using the inverse ball mouse driving mechanism.

It is required to stabilize the ballbot in the upright position while forcing the angular position of the ball to track a certain square signal by applying the proposed state PD controllers derived in Section 4.1. To do so, it is needed to establish the descriptor state-space representation for this mechanical system.

It is important to note that the ballbot system under consideration is not descriptor by nature. Namely, it can be readily transformed into an explicit system by inverting its non-singular derivative matrix. Clearly, the advantage of the descriptor system representation is that it eliminates the need for the inversion of the derivative matrix. This can prove particularly helpful for larger scale examples. Moreover, as will be shown in Chapter 5 for the quarter car active suspension system considered therein, the descriptor system representation becomes natural for such mechanical systems when the system parameters are uncertain.

4.2.1 Mathematical Model

As in [46], it is assumed that the motion of the ball in the XZ plane is identical to and decoupled from that in the YZ plane. Then, it is possible to look at the ballbot as two separate, similar, planar wheeled inverted pendula. Based on this assumption, only one plane of motion is analyzed. A scheme for the planar wheeled inverted pendulum in the XZ plane is shown in Figure 4.1

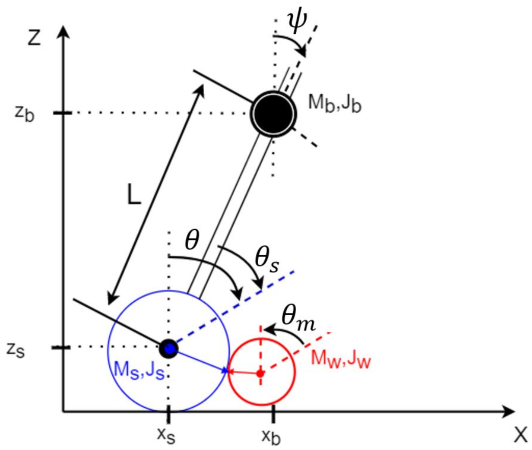


Figure 4.1: Schematic Diagram for a Planar, Wheeled, Inverted Pendulum Representing the Ballbot Motion in the XZ Plane.

The physical parameters of the pendulum and the motor specifications are summarized in Table 4.1 and are obtained from [1]. The plastic ball is driven by a wheel supported by a servomotor. Let θ_s be the angular position of the ball with respect to the body axis, and θ_m be the drive wheel angle. It is assumed that there is no slippage between the drive wheel and the plastic ball. As such, $R_w\theta_m = R_s\theta_s$.

Table 4.1: Physical Parameters and Motor Specifications from [1].

Parameters	**Description**	**Value**
M_s	Ball mass	$0.013\ kg$
R_s	Ball radius	$0.026\ m$
J_s	Ball moment of inertia	$(2M_sR_s^2)/3\ kg.m^2$
M_b	Body mass	$0.682\ kg$
L	Ball to body center of mass distance	$0.17\ m$
J_b	Body moment of inertia	$(M_bL^2)/3\ kg.m^2$
M_w	Motor wheel mass	$0.015\ kg$
R_w	Motor wheel radius	$0.021\ m$
J_w	Motor wheel moment of inertia	$(M_wR_w^2)/2\ kg.m^2$
g	Gravity acceleration	$9.8\ m/s^2$
R_m	DC motor resistance	$6.69\,\Omega$
J_m	DC motor moment of inertia	$1{\times}10^{-5}\ kg.m^2$
f_m	Body-DC motor friction factor	0.0022
K_b	DC motor back EMF	$0.468\ V.s/rad$
K_t	DC motor torque constant	$0.317\ N.m/A$
$x_b,\ z_b$	Body center of mass coordinates	-
$x_s,\ z_s$	Ball center of mass coordinates	-

As shown in Figure 4.1, θ is the angular position of the ball with respect to the vertical axis, while ψ is the angular position of the pendulum with respect to the same axis. Let θ and ψ be the generalized coordinates within the Euler–Lagrange formulation. Then, the equations of motion of the wheeled inverted pendulum are given by

$$((M_s + M_b)R_s^2 + q_1 + J_s)\ddot{\theta} + (q_2 \cos\psi - q_1)\ddot{\psi} - q_2\dot{\psi}^2 \sin\psi = F_\theta,$$
$$(\, q_2 \cos\psi - q_1)\,\ddot{\theta} + (\, M_bL^2 + J_b + q_1)\,\ddot{\psi} - M_bgL \sin\psi = F_\psi,$$

where $q_1{=}(Rs/Rw)^2(J_m{+}J_w)$, $q_2{=}M_bR_sL$, and the applied forces are defined by

$$F_\theta = \gamma v - (\eta + f_s)\dot{\theta} + \eta\dot{\psi},\ F_\psi = -\gamma v + \eta\dot{\theta} - \eta\dot{\psi},$$

in which $\gamma = \frac{K_t}{R_m}$, $\eta = \frac{R_s}{R_w}(\frac{K_tK_b}{R_m} + f_m)$, v is the applied input voltage, and f_s is the friction between the ball and the ground. As in [46], it is assumed that $f_s = 0$.

Linearizing the non-linear equations of motion around the upright position yields the following system of equations expressed in constrained mechanical form:

$$M_o \begin{pmatrix} \ddot{\theta} \\ \ddot{\psi} \end{pmatrix} + C_r \begin{pmatrix} \dot{\theta} \\ \dot{\psi} \end{pmatrix} + G \begin{pmatrix} \theta \\ \psi \end{pmatrix} = Hv, \tag{4.15}$$

where the inertia matrix M_o, the coriolis matrix C_r, the potential matrix G, and the input matrix H are defined by

$$M_o = \begin{pmatrix} (M_b + M_s)R_s^2 + Js + q_1 & q_2 - q_1 \\ q_2 - q_1 & M_b L^2 + J_b + q_1 \end{pmatrix},$$

$$C_r = \begin{pmatrix} \eta + f_s & -\eta \\ -\eta & \eta \end{pmatrix}, G = \begin{pmatrix} 0 & 0 \\ 0 & -M_b g L \end{pmatrix}, H = \begin{pmatrix} \gamma \\ -\gamma \end{pmatrix}.$$

Then, the equations in (4.15) are written in the descriptor state-space form, $E\dot{x}(t) = Ax(t) + Bu(t)$, by defining $u(t) = v$, $x(t) = \begin{pmatrix} \theta & \psi & \dot{\theta} & \dot{\psi} \end{pmatrix}^T$,

$$E = \begin{pmatrix} I & 0 \\ 0 & M_o \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ -G & -C_r \end{pmatrix}, \quad \text{and } B = \begin{pmatrix} 0 \\ H \end{pmatrix}.$$

This descriptor system can be confirmed to be regular, impulse-free, and C-controllable by readily verifying that $\det(\lambda E - A)$ is not identically equal to zero, $\deg(\det(\lambda E - A)) = \text{rank}[E]$, $\text{rank}\begin{bmatrix} E & B \end{bmatrix} = 4$, and $\text{rank}\begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$.

4.2.2 Control Design and Simulation Results

As mentioned earlier, the goal is to stabilize the body of the pendulum around the unstable equilibrium position while forcing the angular position of the ball, θ, to track a given square signal. This motion corresponds to forward-backward movement of the ballbot, with the pendulum maintained in an upright position. The state PD control strategies derived in Section 4.1 are applied to achieve the desired tasks. To quantify the controller performance, two quantities are introduced: the tracking error $(T.E)$ and the input effort $(I.E)$ defined as

$$T.E = \int_0^{t_f} (x_d - x)^T (x_d - x) \, dt, \quad I.E = \int_0^{t_f} u^T u \, dt,$$

where x_d denotes the desired state trajectory and $t_f = 80sec$. Table 4.2 summaries the control design results.

Three different state PD controllers are designed by solving the SDPs and applying the algorithms derived in Section 4.1.3: Controller 1 to asymptotically stabilize the system by solving SDP 4.1, Controller 2 to exponentially stabilize the system by applying Algorithm 4.1, and Controller 3 to exponentially stabilize the system while minimizing the gains by applying Algorithm 4.2. The results are outlined in Table 4.2.

Table 4.2: Reported Results from Applying Different State PD Control Strategies to the Ballbot System

Controller	**SDP / Algorithm**	**PD Gains and Optimized Decay Rate**	***T.E*** *($rad^2.sec$)* ***I.E*** *($V^2.sec$)*
1	**SDP 4.1**	$K_{o_1}=\begin{pmatrix}26 & -44 & 50 & 186\end{pmatrix}$ $F_{o_1}=\begin{pmatrix}10 & 162 & 30 & 188\end{pmatrix}$	$T.E = 18$ $I.E = 2.5$
2	**Algorithm 4.1** $\alpha_{up}=100$ $\alpha_{low}=0.01$	$K_{o_2}=\begin{pmatrix}60 & 178 & 50 & 186\end{pmatrix}$ $F_{o_2}=\begin{pmatrix}-17 & -5.5 & 2.9 & 17\end{pmatrix}$ $\alpha^* = 1.6$	$T.E$=13.5 $I.E$=4.3
3	**Algorithm 4.2** α_{up}=100, $\alpha_{low}=0.01$ ζ=0.01, $\mu_1 = 0.2$	$K_{o_3}=\begin{pmatrix}24 & -62 & 38 & 122\end{pmatrix}$ $F_{o_3}=\begin{pmatrix}0.28 & 69 & 11.2 & 68\end{pmatrix}$ $\alpha^* = 0.9$	$T.E$=14.7 $I.E$=2.75

Asymptotically stabilizing the system by solving SDP 4.1 results in the PD gains K_{o_1} and F_{o_1}. Using these gains, the control law in (4.1) is applied to the non-linear system to perform the desired task. The simulation results of the input signal and the corresponding outputs are plotted in Figures 4.3 and 4.2, respectively. The system starts from a nonzero initial condition $\psi(0) = 9\,\text{deg}$. The ball angle tracks the desired square input and the pendulum is maintained close to the upright position. This results in $T.E= 18\,rad^2.sec$ and $I.E= 2.5\,V^2.sec$ as shown in Figure 4.4.

On the other hand, applying Algorithm 4.1 to exponentially stabilize the system yields a maximum decay rate $\alpha^* = 1.6$ in addition to the PD gains K_{o_2} and F_{o_2}. Simulation results starting from the same initial condition as before are also shown in Figures 4.2 , 4.3, and 4.4. As expected from optimizing the decay rate, in this case, $T.E$ reduces to $13.5\,rad^2.sec$ (reduction of 25%), while $I.E$ increases to $4.3\,V^2.sec$ (increase of 72%). This is clearly demonstrated in Figure 4.4.

To get a small input effort $I.E$ with an acceptable tracking error $T.E$, Algorithm 4.2 is applied. For $\zeta = 0.01$ and $\mu_1 = 0.2$, we get $\alpha^* = 0.9$ with K_{o_3} and F_{o_3}. This gives the acceptable trade-off values of $T.E= 14.7\,rad^2.sec$ and $I.E= 2.75\,V^2.sec$ as captured in Figures 4.2 , 4.3, and 4.4.

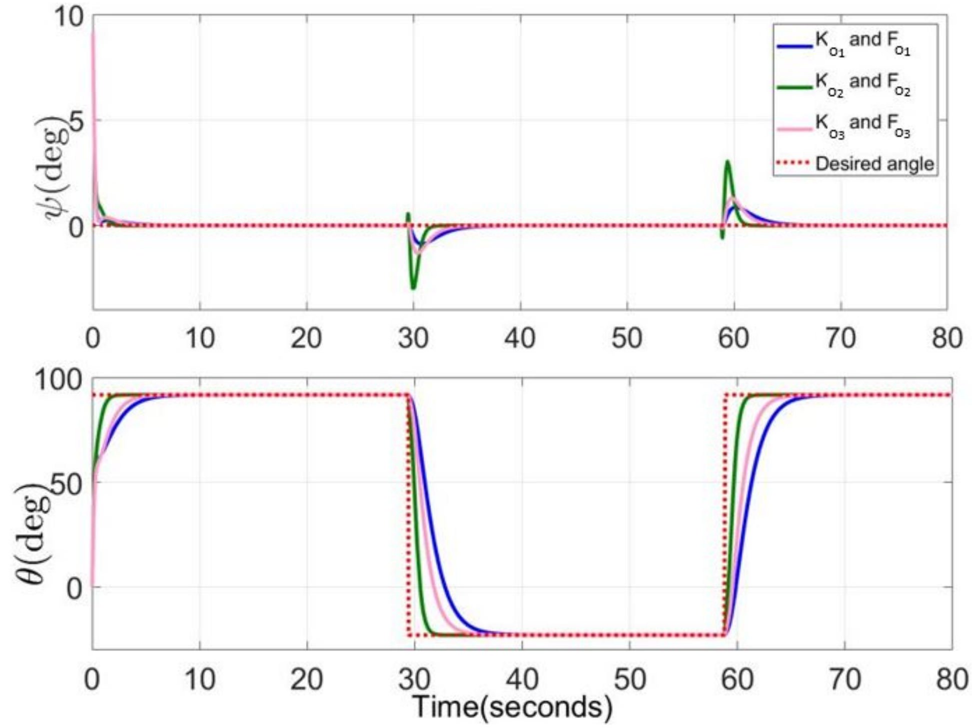


Figure 4.2: Simulation Results from Applying Different State PD Controllers to the Nonlinear Equations of the Ballbot System.

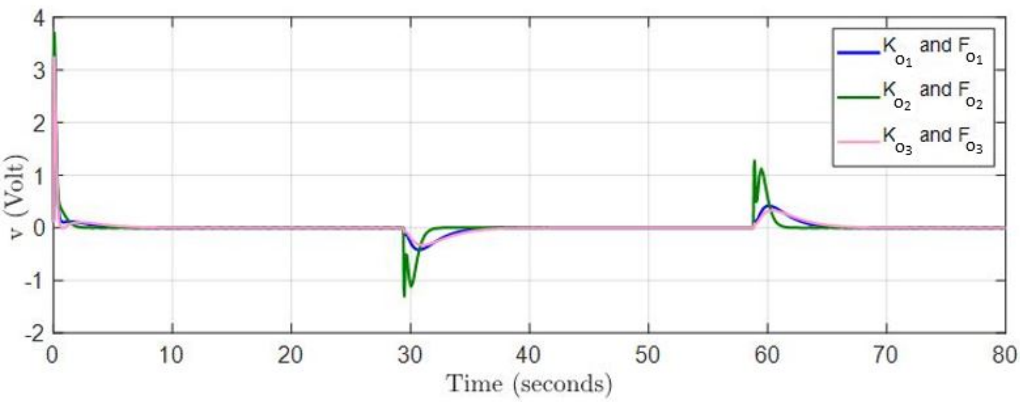


Figure 4.3: Simulation Results of the Input Signals of the State PD Controllers.

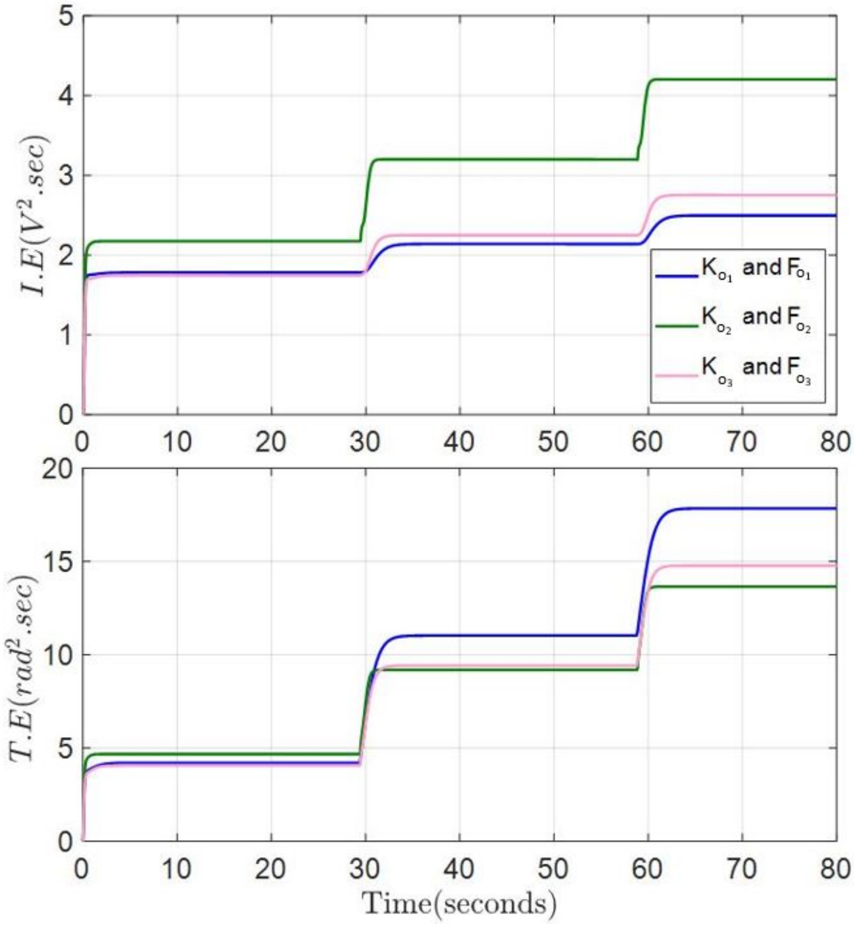


Figure 4.4: Simulation Results Capturing the Input Effort and the Tracking Error after Applying the State PD Controllers to the Nonlinear Equations of the Ballbot System

In brief, the ballbot system example demonstrates how the proposed state PD controllers are capable of stabilizing the non-linear system while achieving other design criteria.

Chapter 5

Robust PD State Feedback Control Synthesis

After treating the problem of synthesizing state PD feedback controllers to stabilize the LTI descriptor system defined in (3.3), in this chapter, the work is extended to tackle the robust stabilization problem for uncertain polytopic descriptor systems with affine parametric uncertainties.
Robust state PD feedback controllers are synthesized to address the robust stabilization problem of the polytopic descriptor system in (3.4). The results in this chapter are based on Lemmas 3.3 and 3.4, which give sufficient conditions for robust asymptotic and exponential stability of the explicit uncertain system defined in (3.2). An illustrative example is provided to demonstrate the robustness of the proposed controllers. The example deals with stabilizing a quarter car suspension system subjected to damper failures.

5.1 Control Synthesis

Consider the robust state PD feedback control law of the form

$$u(t) = -K_R x(t) - F_R \dot{x}(t), \tag{5.1}$$

where K_R and F_R in $\mathbb{R}^{m\times n}$ are the robust proportional and derivative state feedback gains, respectively.
Extending on the state PD control problem in Chapter 4, in this chapter, the main goal is to find feedback gains K_R and F_R that robustly asymptotically stabilize the uncertain descriptor system defined in (3.4). Namely, it is required to find robust gains K_R and F_R such that the resulting closed-loop system is robustly asymptotically stable for all possible uncertain parameter values.
Accordingly, the control law in (5.1) is applied to the uncertain descriptor system in (3.4) to get the following uncertain closed-loop system:

$$(E(\rho) + B(\beta)F_R)\dot{x}(t) = (A(\delta) - B(\beta)K_R)x(t).$$

Assuming that the robust derivative gain F_R renders $(E(\rho) + B(\beta)F_R)$ non-singular for all ρ such that $E(\rho) \in \mathcal{E}$ defined in (3.5) and for all β such that $B(\beta) \in \mathcal{B}$ defined in (3.7), the closed-loop system can be written as

$$\dot{x}(t) = (E(\rho) + B(\beta)F_R)^{-1}(A(\delta) - B(\beta)K_R)x(t). \tag{5.2}$$

It is shown throughout how to compute a gain F_R that makes $(E(\rho) + B(\beta)F_R)$ non-singular for all allowable ρ and β. Thus, it is the purpose of this chapter to find controller gains F_R and K_R such that the closed-loop system in (5.2) is robustly asymptotically stable. As mentioned before, we will use Lemmas 3.3 and 3.4, which only give sufficient conditions for robust asymptotic stability. Indeed, the conditions in Lemma 3.3, which assume a parameter-independent Lyapunov function, equivalently characterize robust quadratic stability of the uncertain explicit system. Using Lemmas 3.3 and 3.4 introduces some conservatism into our proposed approach, however, it allows for a transparent extension of the results of Chapter 4 to the uncertain descriptor system setting.

5.1.1 Robust Stabilization

The control synthesis objective is to make the closed-loop system in (5.2) robustly asymptotically stable by finding appropriate PD gains K_R and F_R. Furthermore, it is desired to achieve the best decay rates.

Expanding on Theorems 4.1 and 4.2, Theorems 5.1 and 5.2 allow for finding robust state PD controller gains to robustly asymptotically or exponentially stabilize the polytopic system in (3.4) for all permissible parameter values.

***Theorem* 5.1** *Consider the uncertain polytopic descriptor system in* (3.4) *and the control law in* (5.1)*. If there exist $\hat{Q} \in \mathbb{S}^n_{++}$, $\hat{X}_1$ in $\mathbb{S}^n$, and $\hat{Y}_1$ and $\hat{Y}_2$ in $\mathbb{R}^{m\times n}$ that satisfy the following two sets of LMIs:*

$$\begin{pmatrix} \hat{Q} & 0 & (B_j\hat{Y}_1)^T \\ 0 & \hat{Q} & (B_j\hat{Y}_2)^T \\ (B_j\hat{Y}_1) & (B_j\hat{Y}_2) & \hat{X}_1 \end{pmatrix} \succ 0, \tag{5.3}$$

$$\begin{pmatrix} B_j(\hat{Y}_1 + \hat{Y}_2) + (B_j(\hat{Y}_1 + \hat{Y}_2))^T - cvx_{ijk} - \hat{X}_1 & \hat{Q} \\ \hat{Q} & \hat{Q} \end{pmatrix} \succ 0, \tag{5.4}$$

where $cvx_{ijk}=\Lambda_{ijk}+\Lambda_{ijk}^T$ with $\Lambda_{ijk}=E_k\hat{Q}A_i^T - E_k\hat{Y}_1^T B_j^T + B_j\hat{Y}_2 A_i^T$, for $i = 1, 2, ..., v_a$, $j = 1, 2, ..., v_b$, and $k = 1, 2, ..., v_e$, then the choice $K_R = \hat{Y}_1\hat{Q}^{-1}$ and $F_R = \hat{Y}_2\hat{Q}^{-1}$ renders the closed-loop system in (5.2) *asymptotically stable for all permissible parameter values ρ, δ, and β.*

***Proof* 5.1** *For each* $j = 1, 2, ..., v_b$*, multiply the inequalities* (5.3) *and* (5.4) *by* $\beta_j \geq 0$ *such that* $\sum_{j=1}^{v_b} \beta_j = 1$ *and sum the corresponding resulting inequalities to get*

$$\begin{pmatrix} \hat{Q} & 0 & (B(\beta)\hat{Y}_1)^T \\ 0 & \hat{Q} & (B(\beta)\hat{Y}_2)^T \\ (B(\beta)\hat{Y}_1) & (B(\beta)\hat{Y}_2) & \hat{X}_1 \end{pmatrix} \succ 0, \tag{5.5}$$

$$\begin{pmatrix} (B(\beta)(\hat{Y}_1 + \hat{Y}_2)) + (B(\beta)(\hat{Y}_1 + \hat{Y}_2))^T - cvx_{ik} - \hat{X}_1 & \hat{Q} \\ \hat{Q} & \hat{Q} \end{pmatrix} \succ 0, \tag{5.6}$$

where $cvx_{ik}{=}\Lambda_{ik}{+}\Lambda_{ik}^T$ *with* $\Lambda_{ik}{=}E_k\hat{Q}A_i^T{-}E_k\hat{Y}_1^T B(\beta)^T{+}B(\beta)\hat{Y}_2A_i^T$ *and* $B(\beta) = \sum_{j=1}^{v_b} \beta_j B_j$ *is in the set* $\mathcal{B}$ *defined in* (3.7)*.*

Similarly, for each $i = 1, 2, ..., v_a$*, multiply the inequality in* (5.6) *by* $\delta_i \geq 0$ *such that* $\sum_{i=1}^{v_a} \delta_i = 1$ *and sum all the resulting inequalities to get*

$$\begin{pmatrix} (B(\beta)(\hat{Y}_1{+}\hat{Y}_2)){+}(B(\beta)(\hat{Y}_1{+}\hat{Y}_2))^T - cvx_k - \hat{X}_1 & \hat{Q} \\ \hat{Q} & \hat{Q} \end{pmatrix} \succ 0, \tag{5.7}$$

with $cvx_k = \Lambda_k{+}\Lambda_k^T$ *with* $\Lambda_k = E_k\hat{Q}A(\delta)^T{-}E_k\hat{Y}_1^T B(\beta)^T{+}B(\beta)\hat{Y}_2A(\delta)^T$ *and* $A(\delta) = \sum_{i=1}^{v_b} \delta_i A_i$ *is in the set* $\mathcal{A}$ *defined in* (3.6)*.*

Finally, for each $k = 1, 2, ..., v_e$*, multiply the inequality in* (5.7) *by* $\rho_k \geq 0$ *such that* $\sum_{k=1}^{v_e} \rho_k = 1$ *and sum all the resulting inequalities to get*

$$\begin{pmatrix} (B(\beta)(\hat{Y}_1 + \hat{Y}_2)) + (B(\beta)(\hat{Y}_1 + \hat{Y}_2))^T - cvx(\rho, \delta, \beta) - \hat{X}_1 & \hat{Q} \\ \hat{Q} & \hat{Q} \end{pmatrix} \succ 0, \tag{5.8}$$

where $cvx(\rho, \delta, \beta) = \Lambda(\rho, \delta, \beta) + \Lambda(\rho, \delta, \beta)^T$ *with* $\Lambda(\rho, \delta, \beta) = E(\rho)\hat{Q}A(\delta)^T - E(\rho)\hat{Y}_1^T B(\beta)^T + B(\beta)\hat{Y}_2A(\delta)^T$ *and* $E(\rho) = \sum_{k=1}^{v_e} \rho_k E_k$ *is in the set* $\mathcal{E}$ *defined in* (3.5)*.*

By Lemma 3.5, the inequality in (5.5) *is equivalent to the following inequality:*

$$\hat{X}_1 \succ \begin{pmatrix} (B(\beta)\hat{Y}_1)^T \\ (B(\beta)\hat{Y}_2)^T \end{pmatrix}^T \begin{pmatrix} \hat{Q}^{-1} & 0 \\ 0 & \hat{Q}^{-1} \end{pmatrix} \begin{pmatrix} (B(\beta)\hat{Y}_1)^T \\ (B(\beta)\hat{Y}_2)^T \end{pmatrix}, with \begin{pmatrix} \hat{Q} & 0 \\ 0 & \hat{Q} \end{pmatrix} \succ 0. \tag{5.9}$$

Similarly, using Lemma 3.6, the inequality in (5.8) *is equivalent to*

$$cvx(\rho, \delta, \beta) - B(\beta)((\hat{Y}_1 + \hat{Y}_2)\hat{Q}^{-1}(B(\beta)(\hat{Y}_1 + \hat{Y}_2))^T + \hat{X}_1 \prec 0. \tag{5.10}$$

Then, for $\hat{X}_1$ satisfying the inequality in (5.9)*, the inequality in* (5.10) *implies*

$$cvx(\rho,\delta,\beta) - B(\beta)(\hat{Y}_1+\hat{Y}_2)\hat{Q}^{-1}(B(\beta)(\hat{Y}_1+\hat{Y}_2))^T + \begin{pmatrix}(B(\beta)\hat{Y}_1)^T\\(B(\beta)\hat{Y}_2)^T\end{pmatrix}^T \begin{pmatrix}\hat{Q}^{-1} & 0\\ 0 & \hat{Q}^{-1}\end{pmatrix}\begin{pmatrix}(B(\beta)\hat{Y}_1)^T\\(B(\beta)\hat{Y}_2)^T\end{pmatrix} \prec 0.$$

Next, replace $\hat{Y}_1$ and $\hat{Y}_2$ by $K_R\hat{Q}$ and $F_R\hat{Q}$, respectively, and do the needed multiplications to get the following result:

$$cvx(\rho,\delta,\beta) - B(\beta)(K_R\hat{Q}F_R^T + F_R\hat{Q}K_R^T)B(\beta)^T \prec 0.$$

Now, substitute $cvx(\rho,\delta,\beta) = \Lambda(\rho,\delta,\beta) + \Lambda(\rho,\delta,\beta)^T$, where

$$\Lambda(\rho,\delta,\beta) = E(\rho)\hat{Q}A(\delta)^T - E(\rho)(K_R\hat{Q})^T B(\beta)^T + B(\beta)(F_R\hat{Q})A(\delta)^T,$$

and expand to get the following:

$$E(\rho)\hat{Q}A(\delta)^T + A(\delta)\hat{Q}E(\rho)^T - E(\rho)\hat{Q}K_R^TB(\beta)^T + B(\beta)F_R\hat{Q}A(\delta)^T + A(\delta)\hat{Q}F_R^TB(\beta)^T - B(\beta)K_R\hat{Q}E(\rho)^T - B(\beta)F_R\hat{Q}K_R^TB(\beta)^T - B(\beta)K_R\hat{Q}F_R^TB(\beta)^T \prec 0.$$

Then, factorize to get

$$(E(\rho)+B(\beta)F_R)\hat{Q}(A(\delta)-B(\beta)K_R)^T + (A(\delta)-B(\beta)K_R)\hat{Q}(E(\rho)+B(\beta)F_R)^T \prec 0.$$

By Lemma 3.7, because the matrices $(E(\rho)+B(\beta)F_R)$, $\hat{Q}$, and $(A(\delta)-B(\beta)K_R)$ are square matrices, it follows that $(E(\rho)+B(\beta)F_R)$ is a non-singular matrix for all β such that $\beta_j \geq 0$ and $\sum_{j=1}^{v_b}\beta_j = 1$ and all ρ such that $\rho_k \geq 0$ and $\sum_{k=1}^{v_e}\rho_k = 1$.

Then, pre and post-multiply the previous inequality by $\hat{Q}^{-1}(E(\rho)+B(\beta)F_R)^{-1}$ and its transpose, respectively, and replace the positive definite matrix $\hat{Q}^{-1}$ by $\hat{P} \succ 0$ to obtain the following equivalent inequality:

$$(A(\delta)-B(\beta)K_R)^T(E(\rho)+B(\beta)F_R)^{-T}\hat{P} + \hat{P}(E(\rho)+B(\beta)F_R)^{-1}(A(\delta)-B(\beta)K_R) \prec 0. \tag{5.11}$$

Since inequality (5.11) *holds for all $A(\delta) \in \mathcal{A}$, $B(\beta) \in \mathcal{B}$, and $E(\rho) \in \mathcal{E}$, then by Lemma 3.3, the closed-loop system in* (5.2) *is robustly asymptotically stable for all permissible parameter values δ, β, and ρ.*

Theorem 5.1 allows for finding robust state PD controller gains that guarantee the robust asymptotic stability of the uncertain closed-loop system in (5.2) by solving the LMIs in (5.3) and (5.4) at all the $v_T = v_e \times v_a \times v_b$ vertices of the uncertain polytopic descriptor system. Expanding on Theorem 5.1, Theorem 5.2 allows for finding state PD controller gains that further guarantee the robust exponential stability of the closed-loop system with a decay rate α.

***Theorem* 5.2** *Consider the uncertain polytopic descriptor system in* (3.4) *and the control law in* (5.1)*. If there exist* $\hat{Q} \in \mathbb{S}^n_{++}$*,* $\hat{Y}_1$ *and* $\hat{Y}_2$ *in* $\mathbb{R}^{m\times n}$*, and* $\hat{X}_1$ *and* $\hat{X}_2$ *in* $\mathbb{S}^n$ *satisfying* (5.3)*,*

$$\begin{pmatrix} -(cvx_{ijk} + \hat{X}_1 - \hat{X}_2) & (E_k\hat{Q} + B_j\hat{Y}_2) \\ (E_k\hat{Q} + B_j\hat{Y}_2)^T & \hat{Q}/2\alpha \end{pmatrix} \succ 0, \tag{5.12}$$

$$\begin{pmatrix} B_j(\hat{Y}_1 + \hat{Y}_2) + (B_j(\hat{Y}_1 + \hat{Y}_2))^T - \hat{X}_2 & \hat{Q} \\ \hat{Q} & \hat{Q} \end{pmatrix} \succ 0, \tag{5.13}$$

where $cvx_{ijk}{=}\Lambda_{ijk}{+}\Lambda_{ijk}^T$ *with* $\Lambda_{ijk}{=}E_k\hat{Q}A_i^T{-}E_k\hat{Y}_1^TB_j^T{+}B_j\hat{Y}_2A_i^T$*, for* $i = 1, 2, ..., v_a$*,* $j = 1, 2, ..., v_b$*, and* $k = 1, 2, ..., v_e$*, then the choice* $K_R = \hat{Y}_1\hat{Q}^{-1}$ *and* $F_R = \hat{Y}_2\hat{Q}^{-1}$ *renders the closed-loop system in* (5.2) *exponentially stable with a guaranteed decay rate* α*.*

***Proof* 5.2** *For each* $j = 1, 2, ..., v_b$*, multiply the inequalities* (5.3)*,* (5.12)*, and* (5.13) *by* $\beta_j \geq 0$ *such that* $\sum_{j=1}^{v_b} \beta_j = 1$ *and sum the corresponding resulting inequalities to get the inequality in* (5.5) *in addition to the following inequalities:*

$$\begin{pmatrix} -(cvx_{ik} + \hat{X}_1 - \hat{X}_2) & (E_k\hat{Q} + B(\beta)\hat{Y}_2) \\ (E_k\hat{Q} + B(\beta)\hat{Y}_2)^T & \hat{Q}/2\alpha \end{pmatrix} \succ 0, \tag{5.14}$$

$$\begin{pmatrix} B(\beta)(\hat{Y}_1 + \hat{Y}_2) + (B(\beta)(\hat{Y}_1 + \hat{Y}_2))^T - \hat{X}_2 & \hat{Q} \\ \hat{Q} & \hat{Q} \end{pmatrix} \succ 0, \tag{5.15}$$

where $cvx_{ik} = \Lambda_{ik} + \Lambda_{ik}^T$ *with* $\Lambda_{ik} = E_k\hat{Q}A_i^T - E_k\hat{Y}_1^TB(\beta)^T + B(\beta)\hat{Y}_2A_i^T$ *and* $B(\beta) = \sum_{j=1}^{v_b} \beta_j$ *is in the set* $\mathcal{B}$ *defined in* (3.7)*.*

Similarly, for each $k = 1, 2, ..., v_e$*, multiply the inequality in* (5.14) *by* $\rho_k \geq 0$ *such that* $\sum_{k=1}^{v_e} \rho_k = 1$ *and sum all the resulting inequalities to get*

$$\begin{pmatrix} -(cvx_i + \hat{X}_1 - \hat{X}_2) & (E(\rho)\hat{Q} + B(\beta)\hat{Y}_2) \\ (E(\rho)\hat{Q} + B(\beta)\hat{Y}_2)^T & \hat{Q}/2\alpha \end{pmatrix} \succ 0,$$

with $cvx_i = \Lambda_i{+}\Lambda_i^T$ *with* $\Lambda_i = E(\rho)\hat{Q}A_i^T - E(\rho)\hat{Y}_1^TB(\beta)^T + B(\beta)\hat{Y}_2A_i^T$ *and* $E(\rho) = \sum_{k=1}^{v_e} \rho_kE_k$ *is in the set* $\mathcal{E}$ *defined in* (3.5)*.*

Finally, for each $i = 1, 2, ..., v_a$*, multiply the previous inequality by* $\delta_i \geq 0$ *such that* $\sum_{i=1}^{v_a} \delta_i = 1$ *and sum all the resulting inequalities to get*

$$\begin{pmatrix} -(cvx(\rho,\delta,\beta) + \hat{X}_1 - \hat{X}_2) & (E(\rho)\hat{Q} + B(\beta)\hat{Y}_2) \\ (E(\rho)\hat{Q} + B(\beta)\hat{Y}_2)^T & \hat{Q}/2\alpha \end{pmatrix} \succ 0, \tag{5.16}$$

where $cvx(\rho,\delta,\beta) = \Lambda(\rho,\delta,\beta) + \Lambda(\rho,\delta,\beta)^T$ *with* $\Lambda(\rho,\delta,\beta) = E(\rho)\hat{Q}A(\delta)^T - E(\rho)\hat{Y}_1^TB(\beta)^T + B(\beta)\hat{Y}_2A(\delta)^T$ *and* $A(\delta) = \sum_{i=1}^{v_a} \delta_iA_i$ *is in the set* $\mathcal{A}$ *defined* *in* (3.6).

By Lemma 3.5, the inequality in (5.5) *is equivalent to the inequality in* (5.9)*, and the inequality in* (5.16) *is equivalent to the following inequality:*

$$cvx(\rho,\delta,\beta) + \hat{X}_1 - \hat{X}_2 \prec -2\alpha(E(\rho)\hat{Q} + B(\beta)\hat{Y}_2)\hat{Q}^{-1}(E(\rho)\hat{Q} + B(\beta)\hat{Y}_2)^T. \quad (5.17)$$

Also, by Lemma 3.6, the inequality in (5.15) *is equivalent to the following inequality:*

$$\hat{X}_2 \prec (B(\beta)(\hat{Y}_1 + \hat{Y}_2))\hat{Q}^{-1}(B(\beta)(\hat{Y}_1 + \hat{Y}_2))^T. \quad (5.18)$$

Then, for $\hat{X}_1$ *satisfying the inequality in* (5.9) *and for* $\hat{X}_2$ *satisfying the inequality in* (5.18)*, the inequality in* (5.17) *implies*

$$\begin{aligned} &cvx(\rho,\delta,\beta) - (B(\beta)(\hat{Y}_1 + \hat{Y}_2))\hat{Q}^{-1}(B(\beta)(\hat{Y}_1 + \hat{Y}_2))^T \\ &\quad + \begin{pmatrix} (B(\beta)\hat{Y}_1)^T \\ (B(\beta)\hat{Y}_2)^T \end{pmatrix}^T \begin{pmatrix} \hat{Q}^{-1} & 0 \\ 0 & \hat{Q}^{-1} \end{pmatrix} \begin{pmatrix} (B(\beta)\hat{Y}_1)^T \\ (B(\beta)\hat{Y}_2)^T \end{pmatrix} \prec \\ &\quad - 2\alpha(E(\rho)\hat{Q} + B(\beta)\hat{Y}_2)\hat{Q}^{-1}(E(\rho)\hat{Q} + B(\beta)\hat{Y}_2)^T. \end{aligned}$$

Now, replace $\hat{Y}_1$ *and* $\hat{Y}_2$ *by* $K_R\hat{Q}$ *and* $F_R\hat{Q}$*, respectively, and do the needed multiplications to get the following result:*

$$\begin{aligned} &cvx(\rho,\delta,\beta) - B(\beta)(K_R\hat{Q}F_R^T + F_R\hat{Q}K_R^T)B(\beta)^T \prec \\ &\quad - 2\alpha(E(\rho) + B(\beta)F_R)\hat{Q}\hat{Q}^{-1}\hat{Q}(E(\rho) + B(\beta)F_R)^T. \end{aligned}$$

To proceed, substitute $cvx(\rho,\delta,\beta) = \Lambda(\rho,\delta,\beta) + \Lambda(\rho,\delta,\beta)^T$ *with* $\Lambda(\rho,\delta,\beta) = E(\rho)\hat{Q}A(\delta)^T - E(\rho)(K_R\hat{Q})^T B(\beta)^T + B(\beta)(F_R\hat{Q})A(\delta)^T$*, expand, perform the required factorization, and replace* $\hat{Q} \succ 0$ *by* $\hat{P}^{-1}$ *to get*

$$\begin{aligned} &(E(\rho)+B(\beta)F_R)\hat{P}^{-1}(A(\delta)-B(\beta)K_R)^T + (A(\delta)-B(\beta)K_R)\hat{P}^{-1}(E(\rho)+B(\beta)F_R)^T \\ &\quad \prec -2\alpha(E(\rho) + B(\beta)F_R)\hat{P}^{-1}(E(\rho) + B(\beta)F_R)^T. \end{aligned}$$

Now, to prove that $(E(\rho) + B(\beta)F_R)$ *is non-singular for all* ρ *such that* $E(\rho) \in \mathcal{E}$ *defined in* (3.5) *and for all* β *such that* $B(\beta) \in \mathcal{B}$ *defined in* (3.7)*, rearrange the above inequality to get the following equivalent inequality:*

$$\begin{aligned} &\Big(E(\rho) + B(\beta)F_R\Big)\hat{P}^{-1}\Big((A(\delta) - B(\beta)K_R) + \alpha(E(\rho) + B(\beta)F_R)\Big)^T + \\ &\quad \Big((A(\delta) - B(\beta)K_R) + \alpha(E(\rho) + B(\beta)F_R)\Big)\hat{P}^{-1}\Big(E(\rho) + B(\beta)F_R\Big)^T \prec 0. \end{aligned}$$

Then, using Lemma 3.7, because $\hat{Q} = \hat{P}^{-1} \succ 0$ *and the matrices* $(E(\rho)+B(\beta)F_R)$ *and* $((A(\delta) - B(\beta)K_R) + \alpha(E(\rho) + B(\beta)F_R))$ *are square matrices, it follows that*

$(E(\rho) + B(\beta)F_R)$ is a non-singular matrix. Therefore, we have found an F_R that renders $(E(\rho) + B(\beta)F_R)$ non-singular for all permissible ρ and β.

To complete the proof, pre and post-multiply the previous inequality by $\hat{P}(E(\rho) + B(\beta)F_R)^{-1}$ and its transpose, respectively, to obtain the following equivalent inequality:

$$(A(\delta)-B(\beta)K_R)^T(E(\rho)+B(\beta)F_R)^{-T}\hat{P}+\hat{P}(E(\rho)+B(\beta)F_R)^{-1}(A(\delta)-B(\beta)K_R) \prec -2\alpha\hat{P}. \quad (5.19)$$

Since inequality (5.19) holds for all $A(\delta) \in \mathcal{A}$, $B(\beta) \in \mathcal{B}$, and $E(\rho) \in \mathcal{E}$, then by Lemma 3.4, the closed-loop system in (5.2) is exponentially stable with guaranteed decay rate α for all permissible parameter values of δ, β, and ρ.

Theorem 5.2 allows for finding robust state PD controller gains that guarantee the robust exponential stability of the uncertain closed-loop system in (5.2) by solving the LMIs in (5.3), (5.12), and (5.13) at all the $v_T = v_e \times v_a \times v_b$ vertices of the uncertain polytopic descriptor system.

5.1.2 Control Gain Minimization

If it is desired to minimize the robust PD controller gains, a method similar to the one in Section 4.1.2 can be followed. In the case of uncertain polytopic descriptor systems, the inequalities (4.3) and (4.4) of Theorem 4.1 are replaced by the sets of inequalities in (5.3) and (5.4) of Theorem 5.1 and the inequalities (4.3), (4.9), and (4.10) of Theorem 4.2 are replaced by the sets of inequalities in (5.3), (5.12), and (5.13) of Theorem 5.2.

5.2 Quarter Car Active Suspension System Example

Consider the physical model in Figure 5.1 for the quarter car active suspension system studied in [47]. The model consists of the vehicle's mass denoted by M_c, the driver's mass including the seat's mass denoted by m_s, and two shock absorbers used to attenuate the vertical vibrations while driving the car on non-smooth roads. The first shock absorber has a spring with a stiffness coefficient k_1 and a damper with a viscous damping coefficient b_1, and it is connected between the wheel and the vehicle's mass M_c. The second absorber has a spring with a stiffness coefficient k_2 and a damper with a viscous damping coefficient b_2, and it is connected between M_c and m_s. Furthermore, the system has two control input signals $u_1(t)$ and $u_2(t)$. The first control signal $u_1(t)$ is applied to M_c and the second control signal $u_2(t)$ is applied to both the vehicle's mass M_c and the driver's mass m_s.

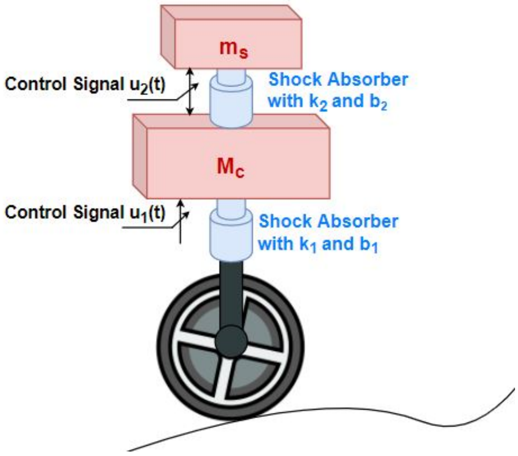


Figure 5.1: Schematic Showing the Physical Model of a Quarter Car Suspension System.

The masses M_c and m_s are considered to be constant real parameters but are not precisely known such that $70 \leq m_s \leq 120kg$ and $1400 \leq M_c \leq 1600kg$. The viscous damping coefficients b_1 and b_2 are also considered to be unknown real constant coefficients with $0 \leq b_1 \leq 2000Ns/m$ and $0 \leq b_2 \leq 1400Ns/m$. Namely, the dampers in the installed shock absorbers may be subjected to structural failures which can cause inadmissible vibrations in the system. At $b_1 = 2000Ns/m$ and $b_2 = 1400Ns/m$ the dampers work effectively, but at $b_1 = 0$ and $b_2 = 0$ a consequent failure occurs in the system. One way to address this problem is to design robust PD controllers to control the vertical displacement of the masses M_c and m_s by applying Theorems 5.1 and 5.2 derived in Section 5.1.1. The stiffness coefficients of the springs are given by $k_1 = 40000N/m$ and $k_2 = 5000N/m$.

5.2.1 Mathematical Model

We first establish the polytopic descriptor formulation of the system in which the uncertain parameters M_c, m_s, b_1, and b_2 appear affinely in the system matrices. The equations of motion of the quarter car suspension system are given by

$$M_c\ddot{x}_1 + (b_1 + b_2)\dot{x}_1 + (k_1 + k_2)x_1 - b_2\dot{x}_2 - k_2x_2 = u_1 - u_2,$$

$$m_s\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 - k_2x_1 - b_2\dot{x}_1 = u_2,$$

where $x_1(t)$ and $x_2(t)$ denote the vertical displacements of M_c and m_s, respectively.

Next, the equations of motion are written in explicit state-space form, $\dot{x}(t) = A_e x(t) + B_e u(t)$, by defining $u(t) = \begin{pmatrix} u_1 & u_2 \end{pmatrix}^T$, $x(t) = \begin{pmatrix} x_1 & x_2 & \dot{x}_1 & \dot{x}_2 \end{pmatrix}^T$,

$$A_e = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2)/M_c & k_2/M_c & -(b_1+b_2)/M_c & b_2/M_c \\ k_2/m_s & -k_2/m_s & b_2/m_s & -b_2/m_s \end{pmatrix},$$

$$B_e = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1/M_c & -1/M_c \\ 0 & 1/m_s \end{pmatrix}.$$

The uncertain parameters M_c, m_s, b_1, and b_2 appear in A_e and B_e having a rational dependence. Hence, the uncertain explicit state-space system is reformulated as the uncertain polytopic descriptor system in (3.4) to make the uncertain parameters appear linearly. This results in the following uncertain matrices:

$$A(\delta) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2) & k_2 & -(b_1+b_2) & b_2 \\ k_2 & -k_2 & b_2 & -b_2 \end{pmatrix},$$

$$E(\rho) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & M_c & 0 \\ 0 & 0 & 0 & m_s \end{pmatrix}, \; B(\beta) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Thus, the descriptor state-space system representation allows for treating the complexities associated with the standard state-space system representation.

The system has four uncertain parameters, namely, M_c and m_s in $E(\rho)$, and b_1 and b_2 in $A(\delta)$. $B(\beta)$ is constant, and we let $B = B(\beta)$. Then, $\Upsilon_A = \begin{bmatrix} b_1 & b_2 \end{bmatrix}^T$, $\Upsilon_E = \begin{bmatrix} M_c & m_s \end{bmatrix}^T$, $v_e = 2^2 = 4$, $v_a = 2^2 = 4$, and $v_b = 2^0 = 1$. Accordingly, the sets $\mathcal{E}$ and $\mathcal{A}$ have four vertices each defined as follows:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \overline{M_c} & 0 \\ 0 & 0 & 0 & \overline{m_s} \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \underline{M_c} & 0 \\ 0 & 0 & 0 & \underline{m_s} \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \overline{M_c} & 0 \\ 0 & 0 & 0 & \underline{m_s} \end{pmatrix}, E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \underline{M_c} & 0 \\ 0 & 0 & 0 & \overline{m_s} \end{pmatrix},$$

$$A_1=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2) & k_2 & -(\overline{b_1}+\overline{b_2}) & \overline{b_2} \\ k_2 & -k_2 & \overline{b_2} & -\overline{b_2} \end{pmatrix}, A_2=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2) & k_2 & -(\underline{b_1}+\underline{b_2}) & \underline{b_2} \\ k_2 & -k_2 & \underline{b_2} & -\underline{b_2} \end{pmatrix},$$

$$A_3=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2) & k_2 & -(\overline{b_1}+\underline{b_2}) & \underline{b_2} \\ k_2 & -k_2 & \underline{b_2} & -\underline{b_2} \end{pmatrix}, A_4=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2) & k_2 & -(\underline{b_1}+\overline{b_2}) & \overline{b_2} \\ k_2 & -k_2 & \overline{b_2} & -\overline{b_2} \end{pmatrix}.$$

Therefore, the total number of the polytope vertices in the system is $v_T = v_a v_e v_b = 16$. The vertices are listed in Table 5.1.

Table 5.1: Polytope Vertices of the Quater Car Suspension System.

All the Polytope Vertices in the System	
Vertex 1: E_1, A_1, B	Vertex 9: E_2, A_3, B
Vertex 2: E_2, A_2, B	Vertex 10: E_2, A_4, B
Vertex 3: E_3, A_3, B	Vertex 11: E_3, A_1, B
Vertex 4: E_4, A_4, B	Vertex 12: E_3, A_2, B
Vertex 5: E_1, A_2, B	Vertex 13: E_3, A_4, B
Vertex 6: E_1, A_3, B	Vertex 14: E_4, A_1, B
Vertex 7: E_1, A_4, B	Vertex 15: E_4, A_2, B
Vertex 8: E_2, A_1, B	Vertex 16: E_4, A_3, B

To illustrate the robustness of the proposed control synthesis approach to parameter variations, the performance of the robust PD controller designed using the techniques of Chapter 5 is compared to that of a PD controller designed using the techniques of Chapter 4 for nominal parameter values. The nominal system matrices are given by $B_n = B$,

$$A_n = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2) & k_2 & -(b_{1_n}+b_{2_n}) & b_{2_n} \\ k_2 & -k_2 & b_{2_n} & -b_{2_n} \end{pmatrix}, E_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & M_{c_n} & 0 \\ 0 & 0 & 0 & m_{s_n} \end{pmatrix}, \tag{5.20}$$

where $M_{c_n} = (\overline{M_c} + \underline{M_c})/2$, $m_{s_n} = (\overline{m_s} + \underline{m_s})/2$, $b_{1_n} = (\overline{b_1} + \underline{b_1})/2$, and $b_{2_n} = (\overline{b_2} + \underline{b_2})/2$ are the nominal parameters of the system. Namely, the nominal parameters are taken as the average of their extreme values.

5.2.2 Control Design

Robust state PD controllers are designed by applying Theorems 5.1 and 5.2. Furthermore, nominal state PD controllers are designed for the nominal system defined in (5.20) by applying Theorems 4.1 and 4.2. Then, the resulting PD control laws are applied to the nominal system and the subsystems formed at the vertices of the uncertain system.

In this example, K_{R_i} and F_{R_i} are the computed robust controller gains, and K_{o_i} and F_{o_i} are the computed nominal controller gains. $i = 1$ stands for the (robust) asymptotic stability problem solved by applying Theorem 5.1 for the uncertain system and Theorem 4.1 for the nominal system. $i = 2$ stands for the (robust) exponential stability problem solved by applying Theorem 5.2 for the uncertain system and Theorem 4.2 for the nominal system. Furthermore, throughout this example, we let α_R^* be the maximum decay rate obtained by applying the bisection method along with Theorem 5.2. Similarly, we let α_o^* be the maximum decay rate obtained by applying the bisection method along with Theorem 4.2.

Table 5.2: Nominal and Robust State PD Control Gains for the Quarter Car Suspension System.

Control Design Problem	**State PD Controller Gains**
Problem 1 **Asymptotic Stabilization**	***Robust PD by Applying Theorem 5.1*** $K_{R_1}=\begin{pmatrix} -538 & -577 & 206 & 20.8 \\ 315 & -630 & 50.2 & 8.4 \end{pmatrix},$ $F_{R_1}=\begin{pmatrix} 2398 & 419.9 & -269.6 & -26.8 \\ -124.4 & 647.3 & 157.9 & -33.8 \end{pmatrix}.$
	Nominal PD by Applying Theorem 4.1 $K_{o_1}=\begin{pmatrix} 0.38 & -0.037 & -0.017 & 0.008 \\ 0.29 & -0.048 & -0.016 & 0.005 \end{pmatrix},$ $F_{o_1}=\begin{pmatrix} 0.068 & -0.02 & 0.01 & -0.0001 \\ 0.064 & -0.02 & 0.01 & -0.0005 \end{pmatrix}.$
Problem 2 **Exponential Stabilization** $\alpha_{up} = 100$ $\alpha_{low} = 0.01$	***Robust PD by Applying Theorem 5.2*** $K_{R_2}=\begin{pmatrix} -2815.4 & 277.9 & 817.7 & 105 \\ -653.6 & 30.34 & 88.5 & 86.1 \end{pmatrix},$ $F_{R_2}=\begin{pmatrix} 4209.5 & 310.9 & -1046.6 & -69.1 \\ -128.1 & 680.8 & 0.7 & -30.9 \end{pmatrix}.$
	Nominal PD by Applying Theorem 4.2 $K_{o_2}=\begin{pmatrix} -1568.8 & 119 & 1013.7 & 68.6 \\ -916.7 & 34.5 & 515.8 & 37.4 \end{pmatrix},$ $F_{o_2}=\begin{pmatrix} 2762.5 & -34.4 & -1238.3 & -88.9 \\ 1455.6 & 11.1 & 64.5 & -83.4 \end{pmatrix}.$

The results are reported in Table 5.2. Solving the robust asymptotic stabilization problem by applying Theorem 5.1 results in the robust PD controller gains K_{R_1} and F_{R_1}. Clearly, the robust gains stabilize the system at the nominal parameters. To demonstrate the advantage of the robust controller over the nominal controller, Theorem 4.1 is applied to compute the nominal PD gains K_{o_1} and F_{o_1}.

To maximize the decay rate α of the closed-loop system, the robust exponential stabilization problem is solved by applying the bisection method with initial bounds $\alpha_{low} = 0.01$ and $\alpha_{up} = 100$ along with Theorem 5.1 for the uncertain system and Theorem 4.2 for the nominal system. This results in the robust PD gains K_{R_2} and F_{R_2}, and $\alpha_R^* = 2.3$, and the nominal PD controller gains K_{o_2} and F_{o_2}, and $\alpha_o^* = 5.1$.

Now, the PD control laws with the robust and nominal gains is applied to the active suspension system starting from a certain nonzero initial condition. The parameters are instantiated at their values at all the vertices of the uncertain system as well as at their nominal values. The simulation results are shown in Figures 5.2, 5.4, 5.3, 5.5, and 5.6.

5.2.3 Simulation Results

This section presents the simulation results from applying the robust and the nominal PD controllers obtained by solving Problems 1 and 2 to the equations of the active suspension system.

Simulation Results for Asymptotic Stabilization

This section analyzes the simulation outcomes resulting from applying the PD control law with the controller gains obtained by solving Problem 1. The PD control law is applied to the nominal system in (5.20) with the nominal gains K_{o_1} and F_{o_1} and then with the robust gains K_{R_1} and F_{R_1}. Figure 5.2 shows the simulation results of the vertical displacements of M_c and m_s, i.e., $x_1(t)$ and $x_2(t)$, respectively, starting from the initial conditions $x_1(0) = 0.1m$ and $x_2(0) = 0.3m$. To compare between the response in orange, i.e., the response resulting from applying the control law with the nominal PD controller gains to the nominal system, and the response in blue, i.e., the response resulting from applying the control law with the robust PD controller gains to the nominal system, it is required to compute the error energy $\int_0^{T_s} ||x(t)||_2^2 \, dt$, where $T_s = 5$sec is the simulation time. This permits quantifying the performance of the controllers. As shown in Figure 5.2, both controllers asymptotically stabilize the system at its nominal parameters. However, the error energy of the response in orange is less than that of the response in blue. Hence, both controllers are good enough in that they stabilize the nominal system and eliminate any inadmissible vibrations. Yet, the nominal controller performs better when applied to the nominal system. This result is expected because the controller with the gains K_{o_1} and F_{o_1} is designed at the nominal parameters of the system.

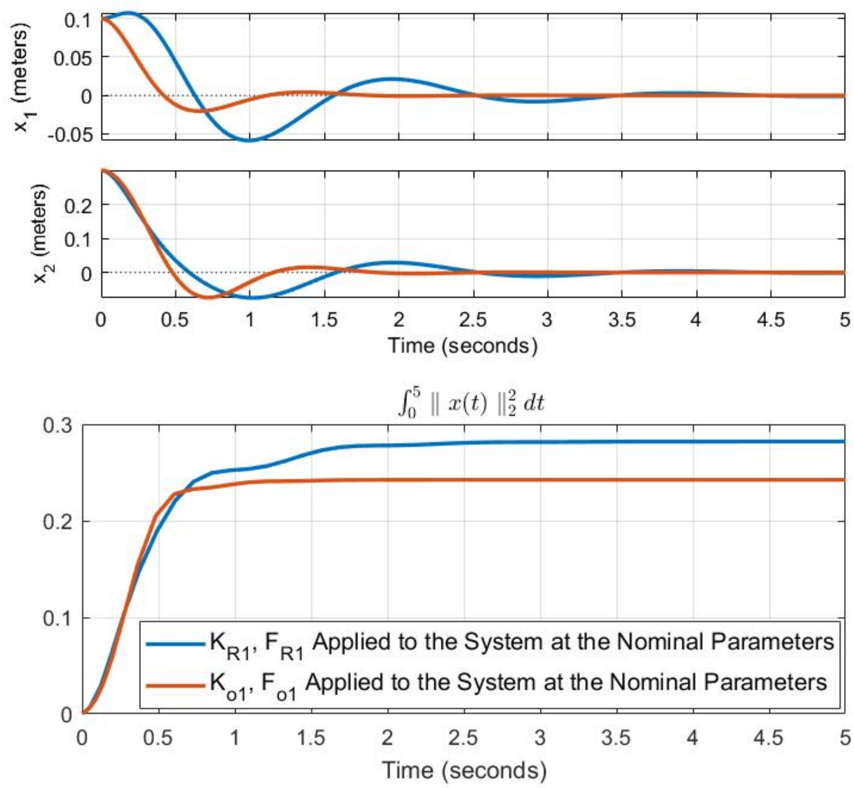
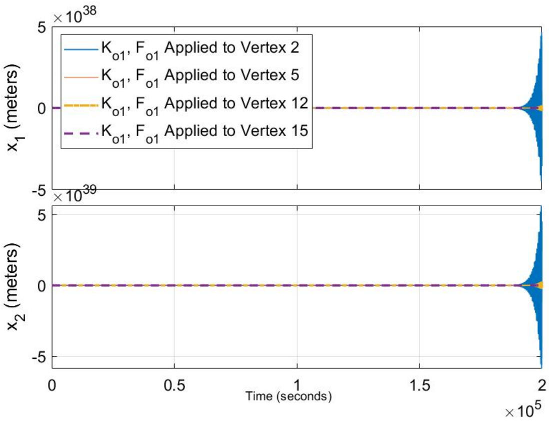
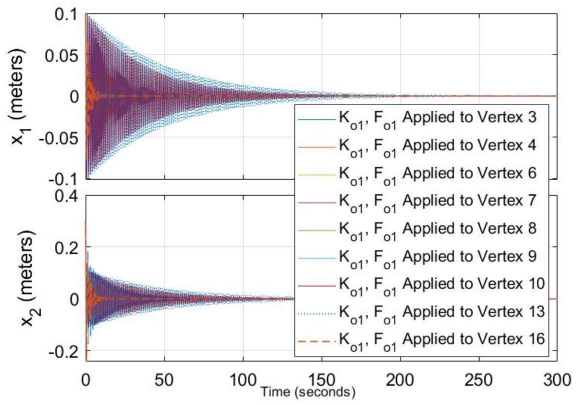


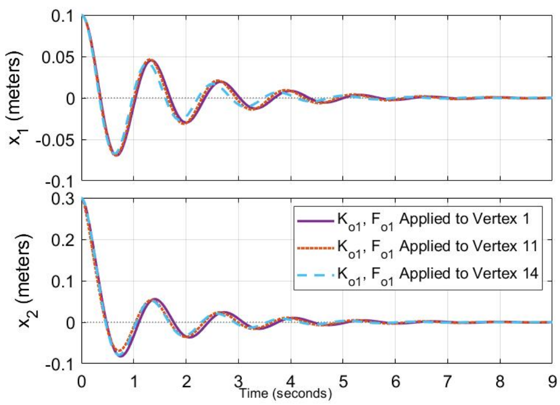
Figure 5.2: Simulation Results from Applying the Robust and the Nominal PD Controllers Obtained by Solving the Asymptotic Stabilization Problem, i.e., Problem 1, to the Equations of the Active Suspension System at the Nominal Parameters Values.



(a) K_{o_1} and F_{o_1} Applied to the System at Vertices 2, 5, 12, and 15.

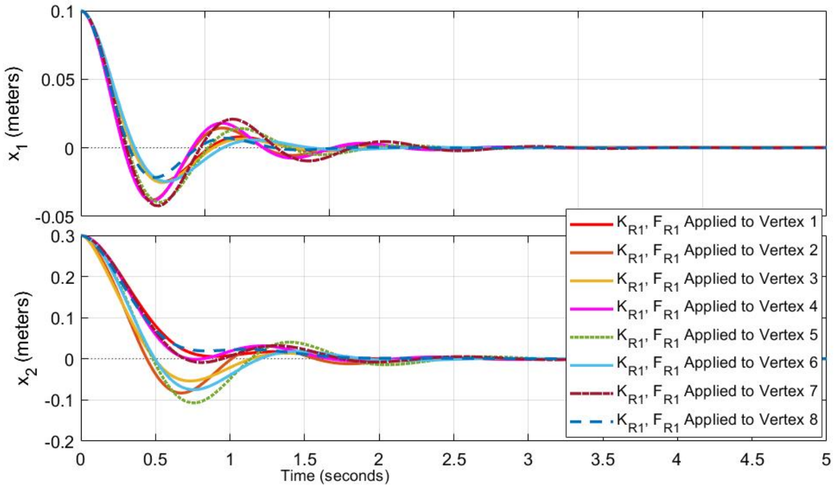


(b) K_{o_1} and F_{o_1} Applied to the System at Vertices 3, 4, 6, 7, 8, 9, 10, 13, and 16.

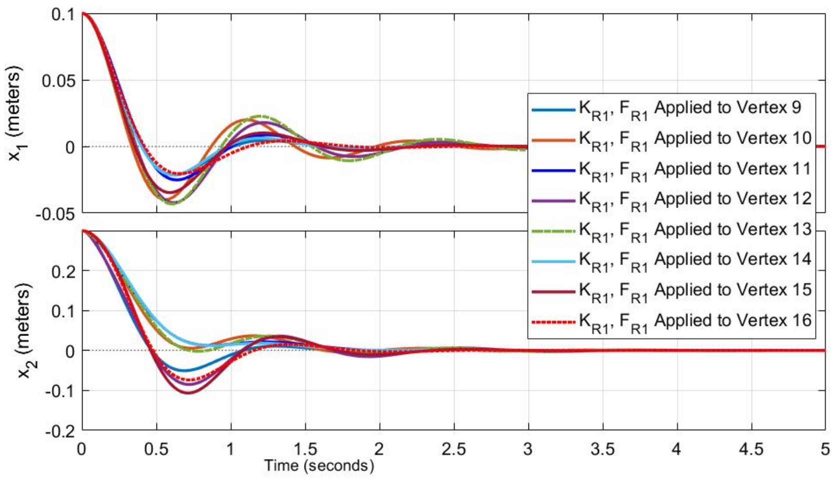


(c) K_{o_1} and F_{o_1} Applied to the System at Vertices 1, 11, and 14.

Figure 5.3: Simulation Results from Applying the Nominal PD Controller Obtained by Solving the Asymptotic Stabilization Problem, i.e., Problem 1, to the Equations of the Active Suspension System at the Vertex Parameter Values.



(a) K_{R_1} and F_{R_1} Applied to the System at its First Eight Vertices.



(b) K_{R_1} and F_{R_1} Applied to the System at its Second Eight Vertices.

Figure 5.4: Simulation Results from Applying the Robust PD Controller Obtained by Solving the Asymptotic Stabilization Problem, i.e., Problem 1, to the Equations of the Active Suspension System at the Vertex Parameter Values.

Figures 5.3 and 5.4, respectively, show the simulation outcomes resulting from applying the nominal PD controller, i.e., the control law with the gains K_{o_1} and F_{o_1}, and the robust PD controller, i.e., the control law with the gains K_{R_1} and F_{R_1}, to the uncertain system in (3.4) at all its 16 vertices.

As shown in Figure 5.3, the nominal PD controller stabilizes the uncertain system at the vertices 1, 11, and 14 with a settling time $T_{s_{o_1}} = 9s$. However, the nominal controller fails to stabilize the uncertain system at the vertices 2, 5, 12, and 15. Namely, the nominal PD controller fails to stabilize the system when it is subjected to damper failure, i.e., at $b_1 = b_2 = 0$. Also, the nominal controller malfunctions in controlling the vibrations when applied to vertices 3, 4, 6, 7, 8, 9, 10, 13, and 16. On the other hand, the robust PD controller stabilizes the uncertain system at all its 16 vertices with a settling time $T_{s_{R_1}} = 5s$ as shown in Figure 5.4.

Simulation Results for Exponential Stabilization

This section analyzes the simulation outcomes resulting from applying the PD control law with the controller gains obtained by solving Problem 2. First, the PD control law is applied to the nominal system with the nominal PD gains K_{o_2} and F_{o_2}. Then, the control law is applied to the nominal system with the robust gains K_{R_2} and F_{R_2}. Similar to the results shown above, both controllers exponentially stabilize the system at its nominal parameters. However, the nominal controller performs better when applied to the system at the nominal parameter values. This is shown in Figure 5.5, where the error energy $\int_0^{T_s} ||x(t)||_2^2 \, dt$ with $T_s = 2$sec of the response resulting from applying the nominal controller is less than that of the response resulting from applying the robust controller.

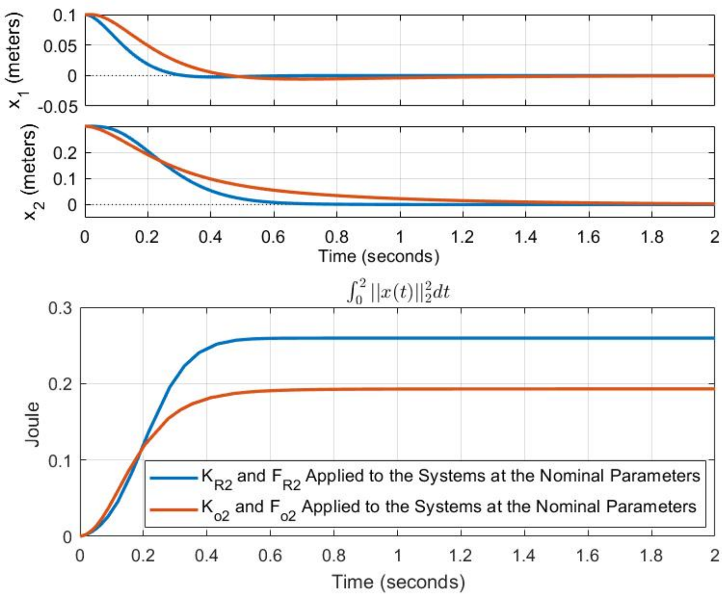
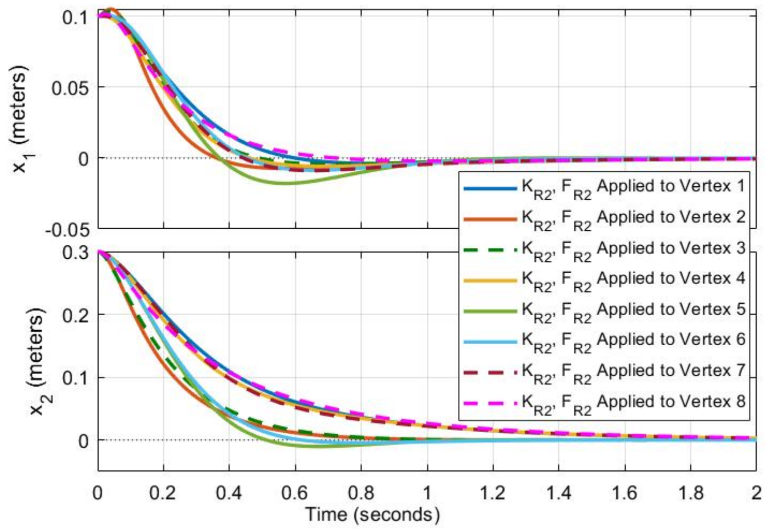
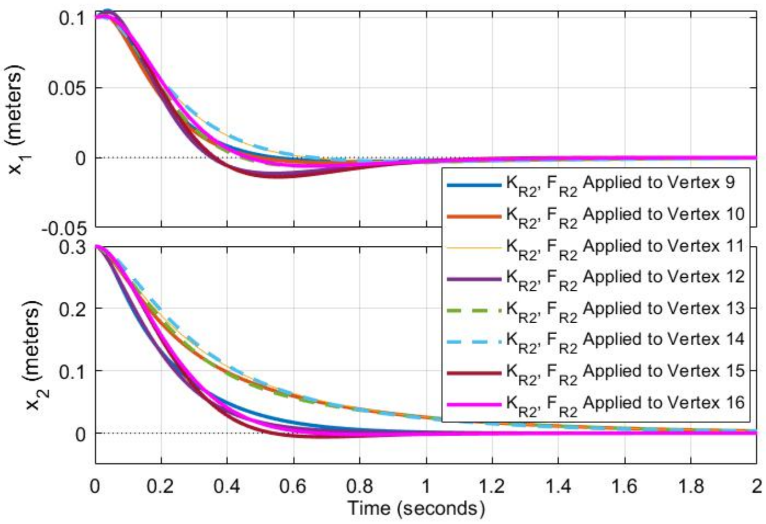


Figure 5.5: Simulation Results from Applying the Robust and the Nominal PD Controllers Obtained by Solving the Exponential Stabilization Problem, i.e., Problem 2, to the Equations of the Active Suspension System at the Nominal Parameters Values.

Figure 5.6 shows the simulation results obtained from applying the PD control law with the robust controller gains K_{R_2} and F_{R_2} to the uncertain system in (3.4) at all its 16 vertices. As shown in the figure, the robust controller stabilizes the the uncertain closed-loop system at all its 16 vertices with a convergence time $T_{s_{R_2}} = 2s < T_{s_{R_1}} = 5s$.



(a) K_{R_2} and F_{R_2} Applied to the System at the First 8 Vertices.



(b) K_{R_2} and F_{R_2} Applied to the System at the Second 8 Vertices.

Figure 5.6: Simulation Results from Applying the Robust PD Controller Obtained by Solving the Exponential Stabilization Problem, i.e., Problem 2, to the Equations of the Active Suspension System at the Vertex Parameter Values.

All in all, applying Theorems 4.1 and 4.2 to compute the nominal state PD controller gains allows for stabilizing the uncertain closed-loop system at its nominal parameters, i.e., the nominal closed-loop system. However, the nominal state PD controller fails to stabilize the uncertain system when it is subjected to component failures. On the other hand, applying the extended theorems, i.e., Theorems 5.1 and 5.2, allows for robustly stabilizing the uncertain polytopic system. Namely, the robust state PD controller stabilizes the subsystems located at the vertices of the polytopic system.

Chapter 6

Conclusions

This thesis proposes methods and techniques for designing state PD control laws to stabilize LTI descriptor systems. The derived control strategies guarantee the asymptotic stability and the exponential stability of the resulting closed-loop system. Using Schur complement-based lemmas, the control synthesis problems are formulated either as SDPs to be solved via LMI techniques or as nonlinear problems to be solved via the bisection method with LMI techniques. Also, this thesis discusses how to minimize the controller gains by appropriately augmenting the derived theorems.

Furthermore, this thesis presents a preliminary extension of the results to the uncertain polytopic descriptor system setting. The class of polytopic uncertain descriptor systems is of interest since, standard, uncertain, state-space systems in which the system matrices have a rational dependence on the uncertain parameters can be reformulated as uncertain polytopic descriptor systems. Thus, the results in this thesis derived for LTI descriptor systems are extended to address the robust stabilization problem for uncertain polytopic descriptor systems.

Future work will look at addressing the state P and output PD control synthesis problems for LTI and polytopic uncertain descriptor systems. It is also of interest, in the uncertain descriptor system setting, to investigate the benefit of starting from analysis results for uncertain explicit systems based on a parameter-dependent Lyapunov function approach, instead of the more conservative analysis results adopted in this thesis that are based on a parameter-independent Lyapunov function approach.

Appendix A

Abbreviations

LTI	Linear Time-Invariant
LMI	Linear Matrix Inequality
PILF	Parameter-Independent Lyapunov Function
PDLF	Parameter-Dependent Lyapunov Function
SDP	Semi-Definite Program
P	Proportional
D	Derivative
PD	Proportional-Derivative

Bibliography

[1] Y. Yamamoto, "NXT Ballbot Model Based Design: Control of Self-Balancing on a Ball Built with Lego Mindstorm NXT," *Cybernet Systems*, vol. 1, pp. 1–68, 2009.

[2] L. Dai, *Singular Control Systems*, vol. 118. Berlin: Springer-Verlag, 1989.

[3] P. C. Müller, "Linear Mechanical Descriptor Systems: Identification, Analysis and Design," *IFAC Proceedings Volumes*, vol. 30, no. 6, pp. 375–380, 1997.

[4] F. L. Lewis, "A Survey of Linear Singular Systems," *Circuits, Systems and Signal Processing*, vol. 5, no. 1, pp. 3–36, 1986.

[5] R. Sakuwa and Y. Fujisaki, "Robust Stability Analysis of Single-Parameter Dependent Descriptor Systems," in *Proceedings of the 44th IEEE Conference on Decision and Control*, pp. 2933–2938, 2005.

[6] E. Uezato and M. Ikeda, "Strict LMI Conditions for Stability, Robust Stabilization, and H_∞ Control of Descriptor Systems," in *Proceedings of the 38th IEEE Conference on Decision and Control*, vol. 4, pp. 4092–4097, IEEE, 1999.

[7] C. Lin, J. L. Wang, G.-H. Yang, and J. Lam, "Robust Stabilization via State Feedback for Descriptor Systems with Uncertainties in the Derivative Matrix," *International Journal of Control*, vol. 73, no. 5, pp. 407–415, 2000.

[8] B. Sari, O. Bachelier, and D. Mehdi, "Robust State Feedback Admissibilisation of Discrete Linear Polytopic Descriptor Systems: A Strict Linear Matrix Inequality Approach," *IET Control Theory & Applications*, vol. 6, no. 8, pp. 1097–1108, 2012.

[9] A. Varga, "On Stabilization Methods of Descriptor systems," *Systems & Control Letters*, vol. 24, no. 2, pp. 133–138, 1995.

[10] F. A. Faria, E. Assunção, M. Teixeira, and R. Cardim, "Robust State-Derivative Feedback LMI-Based Designs for Linear Descriptor Systems," *Mathematical Problems in Engineering*, vol. 2010, 2010.

[11] G.-R. Duan and X. Zhang, "Dynamical Order Assignment in Linear Descriptor Systems via State Derivative Feedback," in *Proceedings of the 41st IEEE Conference on Decision and Control*, pp. 4533–4538, 2002.

[12] R. Zaghdoud, S. Salhi, and M. Ksouri, "Optimal State Derivative Feedback Control for Singular Systems," in *Proceedings of the 14th International Conference on Sciences and Techniques of Automatic Control & Computer Engineering*, pp. 110–113, 2013.

[13] J. Bai and H. Su, "Robust D Stabilization of Singular Systems with Polytopic Uncertainties," in *Proceedings of the 2009 American Control Conference*, pp. 3706–3709, 2009.

[14] F. A. Faria, E. Assunção, M. C. Teixeira, R. Cardim, and N. Da Silva, "Robust State-Derivative Pole Placement LMI-Based Designs for Linear Systems," *International Journal of Control*, vol. 82, no. 1, pp. 1–12, 2009.

[15] V. X. Le, "Synthesis of Proportional-Plus-Derivative Feedbacks for Descriptor Systems," *IEEE Transactions on Automatic Control*, vol. 37, no. 5, pp. 672–675, 1992.

[16] R. Mukundan and W. Dayawansa, "Feedback Control of Singular Systems-Proportional and Derivative Feedback of the State," *International Journal of Systems Science*, vol. 14, no. 6, pp. 615–632, 1983.

[17] H.-H. Yu, "Parametric Stabilization for Descriptor Linear Systems via State-Proportional and-Derivative Feedback," *Journal of the Franklin Institute*, vol. 353, no. 5, pp. 992–1004, 2016.

[18] G.-R. Duan and H.-H. Yu, "Robust Pole Assignment in High-Order Descriptor Linear Systems via Proportional Plus Derivative State Feedback," *IET Control Theory & Applications*, vol. 2, no. 4, pp. 277–287, 2008.

[19] G. Duan, G. Irwin, and G. Liu, "Robust Stabilization of Descriptor Linear Systems via Proportional-Plus-Derivative State Feedback," in *Proceedings of the 1999 American Control Conference*, vol. 2, pp. 1304–1308, 1999.

[20] I. N'Doye, M. Darouach, M. Zasadzinski, and N.-E. Radhy, "Robust Stabilization of Uncertain Descriptor Fractional-Order Systems," *Automatica*, vol. 49, no. 6, pp. 1907–1913, 2013.

[21] H. Tian and Y. Hou, "Robust Stabilization for a Class of Uncertain Descriptor Systems via Both Proportional and Derivative State Feedback," in *2009 Second International Conference on Intelligent Computation Technology and Automation*, vol. 1, pp. 874–877, 2009.

[22] J. Ren and Q. Zhang, "Robust H_∞ Control for Uncertain Descriptor Systems by Proportional-Derivative State Feedback," *International Journal of Control*, vol. 83, no. 1, pp. 89–96, 2010.

[23] Y. Xun and X. Quan, "Robust Passivity Control for Descriptor Systems with Linear Fractional Parameter Uncertainties," in *Proceedings of the 2009 Second International Conference on Intelligent Computation Technology and Automation*, vol. 1, pp. 866–869, 2009.

[24] C.-H. Fang, "Stability Robustness Analysis of Uncertain Descriptor Systems-an LMI Approach," in *Proceedings of the 41st IEEE Conference on Decision and Control, 2002.*, vol. 2, pp. 1459–1460, IEEE, 2002.

[25] C. Yang, Q. Zhang, F. Zhang, and Z. Zhou, "Robustness Analysis of Descriptor Systems with Parameter Uncertainties," *International Journal of Control, Automation and Systems*, vol. 8, no. 2, pp. 204–209, 2010.

[26] L. H. Yen and V. N. Phat, "Stability Analysis of Linear Polytopic Descriptor Systems Using a Novel Copositive Matrix Approach," *IEEE Transactions on Automatic Control*, vol. 64, no. 11, pp. 4684–4690, 2019.

[27] J. Lofberg, "Yalmip: A Toolbox for Modeling and Optimization in MATLAB," in *Proceedings of the 2004 IEEE International Conference on Robotics and Automation*, pp. 284–289, 2004.

[28] K.-C. Toh, M. J. Todd, and R. H. Tütüncü, "SDPT3 - A MATLAB Software Package for Semidefinite Programming, Version 1.3," *Optimization Methods and Software*, vol. 11, no. 1-4, pp. 545–581, 1999.

[29] L. Boyd Stephen, Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[30] A. Brondsted, *An Introduction to Convex Polytopes*, vol. 90. Springer Science & Business Media, 2012.

[31] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15. Philadelphia: Society for Industrial and Applied Mathematics, 1994.

[32] J.-J. E. Slotine, W. Li, *et al.*, *Applied Nonlinear Control*, vol. 199. Prentice Hall Englewood Cliffs, NJ, 1991.

[33] C. Briat, *Linear Parameter-Varying and Time-Delay Systems*, vol. 3. Springer, 2014.

[34] G.-R. Duan, *Analysis and Design of Descriptor Linear Systems*, vol. 23. Springer Science & Business Media, 2010.

[35] G.-R. Duan and X. Zhang, "Regularizability of Linear Descriptor Systems via Output Plus Partial State Derivative Feedback," *Asian Journal of Control*, vol. 5, no. 3, pp. 334–340, 2003.

[36] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nichols, "Feedback Design for Regularizing Descriptor Systems," *Linear Algebra and its Applications*, vol. 299, no. 1-3, pp. 119–151, 1999.

[37] E. Yip and R. Sincovec, "Solvability, Controllability, and Observability of Continuous Descriptor Systems," *IEEE Transactions on Automatic Control*, vol. 26, no. 3, pp. 702–707, 1981.

[38] D. Chu and D. W. Ho, "Necessary and Sufficient Conditions for the Output Feedback Regularization of Descriptor Systems," *IEEE Transactions on Automatic Control*, vol. 44, no. 2, pp. 405–412, 1999.

[39] V. Lovass-Nagy, D. Powers, and R. Schilling, "On Regularizing Descriptor Systems by Output Feedback," *IEEE Transactions on Automatic Control*, vol. 39, no. 7, pp. 1507–1509, 1994.

[40] F. Zhang, *The Schur Complement and its Applications*, vol. 4. Springer Science & Business Media, 2006.

[41] R. J. Caverly and J. R. Forbes, "LMI Properties and Applications in Systems, Stability, and Control Theory," *arXiv preprint arXiv:1903.08599*, 2019.

[42] B. Sereni, M. Beteto, E. Assunção, and M. Teixeira, "Pole Placement LMI Constraints for Stability and Transient Performance of LPV systems with Incomplete State Measurement," *Journal of the Franklin Institute*, 2021.

[43] G. R. Wood, "The Bisection Method in Higher Dimensions," *Mathematical Programming*, vol. 55, no. 1, pp. 319–337, 1992.

[44] E. Assunção, M. C. Teixeira, F. A. Faria, N. D. Silva, and R. Cardim, "Robust State-Derivative Feedback LMI-Based Designs for Multivariable Linear Systems," *International Journal of Control*, vol. 80, no. 8, pp. 1260–1270, 2007.

[45] U. Nagarajan, G. Kantor, and R. Hollis, "The Ballbot: An Omnidirectional Balancing Mobile Robot," *The International Journal of Robotics Research*, vol. 33, no. 6, pp. 917–930, 2014.

[46] R. A. Garcia-Garcia and M. Arias-Montiel, "Linear Controllers for the NXT Ballbot with Parameter Variations Using Linear Matrix Inequalities," *IEEE Control Systems Magazine*, vol. 36, no. 3, pp. 121–136, 2016.

[47] E. Reithmeier and G. Leitmann, “Robust Vibration Control of Dynamical Systems Based on the Derivative of the State,” *Archive of Applied Mechanics*, vol. 72, no. 11, pp. 856–864, 2003.