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CONTINUOUS REVIEW INVENTORY MODEL WITH BUFFER STOCK AND RUSH ORDERS

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ABSTRACT
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The continuous review model is a widely used inventory management system in supply chains. It helps retailers and managers decide on the optimal quantity to order from the supplier(s), and the timing of the order. However, sometimes stock-outs occur and lead to loss of customer demand, which causes considerable financial losses. Ways to reduce shortages is to have a reserve stock or to utilize rush orders.
In our research, we extend the traditional continuous review system to account for an additional buffer stock, at an external location, that aims to reduce shortages. This first model works on obtaining the optimal buffer stock level, in addition to the reordering point and the order size, in order to minimize losses and reduce the total cost. We also develop another model that relies mainly on the use of rush orders to reduce shortages whenever the regular inventory is depleted. The extended models are effective in determining the optimal policy for inventory management, and they both present a promising decrease in total cost compared to the classical model.

Keywords: Inventory management; Continuous review model; Shortages; Buffer stock; Rush orders

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CHAPTER I

INTRODUCTION AND MOTIVATION

In this chapter, we introduce our problem statement and present some motivations. In Section 1.1, we discuss the importance of inventory management and reserve stocks, and present the main characteristics of the continuous review inventory model. In Section 1.2, we explain the main purpose of our research, and the study of the new extended models.

1.1 Introduction

Inventory management is the process of ordering, receiving and storing the physical products a company uses, and it applies to all types of industries. An effective management for the inventory allows the firm to maintain a satisfactory service for its customers and meet their demands, while a poor management can lead to loss of demand due to shortages, or to inventory damage and piling-up, which incurs significant costs.

1.1.1. Importance of Inventory Management

The importance of inventory management is usually most noticeable in manufacturing companies In 2007, 29% of the total investment in inventories in the United States was dedicated to manufacturing, with a value of $454.9 billion (Nahmias & Olsen, 2015, p.200). However, inventory management is not restricted to manufacturing companies only, but it also extends to services companies that usually have a faster production process and smaller order quantities. Dadic, Ribaric & Vlahov

(2020) conducted a study on bars and restaurants to understand how their employees and managers perceive the importance of inventory management. They found that most employees in these organizations are aware that the continuous and timely inventory monitoring has a great effect on the efficiency of the production and service processes, especially that these companies have a quick turnover for stock and lots of their products need careful management to avoid perishability.

1.1.2. Role of Reserve Stocks

Reserve stocks are one of the most adapted strategies used to mitigate the risk of supply stock-outs, which can occur due to internal (demand fluctuation, production interruption) or external (civil unrest, wars, natural catastrophes) reasons, and can sometimes lead to huge financial losses for the company. Reserve stocks are being used in many fields other than manufacturing and retail, including the oil industry, in response to the frequent oil price fluctuations (Xie, Yan, Zhou & Huang, 2017) and national security areas, such as the management of pharmaceuticals used for emergency preparedness to set up the Strategic National Stockpile (SNS) (Lee, Mu, Shen & Dessouky, 2014).

Some of the examples we can look at to show how reserve stocks could have helped in avoiding large stock-outs and supply chain crises include:

- The infamous toilet paper crisis that happened in the US with the beginning of the Covid19 pandemic in 2020, when a huge demand surge occurred due to people panicking and rushing to buy toilet papers prior to the lockdown. Reserve stocks and inventories were all depleted, and toilet paper was missing in more of

70% of all supermarkets across the US in few days only. The pandemic was believed to expose US supply chain flaws, especially the low inventory levels.

- With the pandemic lockdowns, people tended to rely more on home electronics which created an unusual demand that chip maker producing companies could not meet, and caused severe slowing down for the production of cars, smartphones and other sectors.
- With the launch of its twin-engined 787 in 2007, Boeing intended to set unprecedented production times, but the mission failed dramatically within few months only. Many factors caused this failure, but the main issue was in their supply chain and inventories, especially when they ran out of fasteners.
- Due to the pandemic and the lockdowns, Amazon was hit by a surge of online orders, and shopping from home hit an unusual record. However, the international and local lockdowns made it hard for Amazon to source products for its warehouses, causing large inventory stock-outs and delivery delays for customers.

1.1.3. Continuous Review Inventory Model

The continuous review inventory is a system where the inventory level is checked continuously, i.e. each time a product moves in or out of the system. Whenever the inventory level drops below a specified value called the reordering point, an order of a constant size (same number of units) is triggered for more stock. The order frequency is variable in this system, unlike the periodic review system where orders of different sizes are initiated at the same time in each cycle or period.

1.2 Motivation

In our research, we work on examining whether adding an additional buffer stock to an inventory management system helps in reducing the total cost. We seek to determine the optimal buffer stock level that balances the holding and shortage costs. We also investigate the inclusion of the concept of rush orders for limitation of shortages and how this affects the total cost. We base our work on the continuous review inventory system that is characterized by a continuous check for the inventory level in an uncertain demand environment. We extend it first to include an external buffer stock held at a different location that gets tapped into when the "regular" stock at the primary location is depleted, and then to include a policy of rush orders that are assumed to be zero lead-time orders also initiated when the regular inventory is completely used up.

The remainder of this thesis is organized as follows. In Chapter 2, we present a brief review of the related literature. In Chapter 3, we present the assumptions and the models including the classical model and the two extended ones. In Chapter 4, we present the numerical analysis for the models including finding the optimal solutions, performing a sensitivity analysis on the cost parameters, and comparing the costs of the models for different parameter values. In Chapter 5, we analytically approximate a way to find the optimal solution. Finally, in Chapter 6, we present a brief conclusion and some recommendations for future work.

CHAPTER II

LITERATURE REVIEW

Our literature review is divided into three related sections, reserve stocks in Section 2.1, rush orders in Section 2.2, and continuous review model extensions in Section 2.3.

2.1 Reserve Stocks

Having a reserve or safety stock means having an additional amount of product in inventory, and it aims to reduce the risk of shortage of this specific item. It is considered as a buffer stock that is used up when actual demand is greater than the planned or forecasted, or when there is a given disruption at the supplier.

Bhonsle, Rossetti & Robinson (2005) argue that setting the same level of safety stock for all items in inventory is cost inefficient, and that these levels should be determined by the degree of uncertainty or risk related to each item in terms of demand and sourcing, which reduces inventory holding costs. By experimental analysis, they found a 28.9% reduction in cost for a 95% service level, and an 8.8% reduction for a service level of 90%. This shows how the appropriate management of reserve stocks reduces costs considerably.

A good customer service and efficient operations in a company require a well-structured and coordinated supply chain. Graves & Willems (2000) develop an optimization algorithm to find the optimal position of "strategic" safety stock for a supply chain that is subject to demand uncertainty. The developed model aims to reduce inventory and increase the service performance, and finds the service times that are optimal for minimizing the holding costs for the safety stock.

Mekel, Anantadjaya & Lahindah (2014) conduct an empirical study to forecast the demand and determine the reordering point and the needed safety stock level for a pharmaceutical company in Indonesia by forecasting demand using double exponential smoothing. They check the error in forecasting demand to deduce if the company will face stock-outs or excessive settled inventory.

An important factor to consider while stocking for reserve is deterioration of inventory. Maddah, Yassine, Salameh & Chatila (2013) develop optimal stocking policies for exponentially deteriorating reserve stocks. The presented policies aim to balance the traditional costs with the additional costs of deterioration. They also propose a preventive replenishment policy with periodical orders, to keep the stock at a specified base level. The numerical analysis conducted shows an improvement of about 40% in offsetting additional costs.

Managing perishable items is also of great importance for humanitarian work. Logistic managers face a great challenge in the donations of perishable goods, including medicine and food, since distributing deteriorated items imposes a health danger to the population, and their disposal policies imply huge costs for the organizations (Ferreira, Arruda & Marujo, 2018). Ferreira et. al (2018) attempt to minimize the expected inventory cost, while avoiding shortage and deterioration of supplies as much as possible. They use a Markov Decision Process, considering both the demand and the supply as stochastic variables, with deterministic deterioration rates. They find that keeping rapidly perishable items in stock increases the need for disposal that is difficult and costly.

The above papers link the reserve stock to the regular safety stock, which differs from the buffer stock we are considering in our study. Safety stocks are usually maintained to mitigate the risk of disruptions or delays in delivery of raw materials from the supplier and can also hold finished goods, while buffer stocks act as a buffer between the actual and the forecasted demands, and is used up in lead-times between the placement and the receipt of the replenishment order.

2.2 Rush Orders

The mechanism of our secondary reserve stock system resembles that of rush orders which are placed when stock level is not enough to meet demand. As such, we review works on rush orders next. Rush orders are orders requested by customers to be supplied and delivered very quickly. These orders can be thought of as exceptional cases or some type of disturbance that affects the performance of a structured supply chain. However, rush orders including special demands for customers, replacement orders or prototypes, have become nowadays a regular component in the daily business of most companies.

Engelseth & White (2020) study the perception of rush orders from both the suppliers and customers' sides. The suppliers consider rush orders as a service provided to customers, having only a conception about the supply timing. Customers or dealers usually understand the complexity of rush orders but believe that it is impossible to plan or predict them.

Mahfouz & Arisha (2010) develop models to assess the influence of rush orders on cycle time and cost. They observe a negative impact results from the high priority given for rush orders which cause longer waiting times, frequent process interruptions and resources unavailability for regular orders. In order to mitigate this risk, they suggest applying a separate route at design, planning, engineering, production, purchasing and distribution centers.

Trzyna, Kuyumcy & Lodding (2012) present three main factors that affect rush orders' throughput times. First, having a high system utilization causes a deceleration in rush orders. Second, having a higher number of parallel machines shortens the waiting time for a rush order. The last and most influential factor is the work content of the standard orders; higher work contents and standard deviations of standard orders lead to longer interoperation times for the rush orders.

For a periodic review assemble-to-order system, with multiple components and multiple finished goods, Benbitour, Sahin & Dallery (2018) develop approximate expressions for determining the optimal safety stock for components, while minimizing the sum of holding and rush order costs. Orders for components are delivered in multiple shipments, and rush orders are offered at a higher cost when there is a possibility of shortage. Benbitour et al. (2018) study single shipments, and then generalize to multiple shipments. They assume that during the review time, there is only one possible rush order. For the multiple shipments case, they add a second assumption that a rush order will be more probably requested after receiving the last shipment in the order. The performance of the developed expressions is tested by a discrete event simulation. The developed model leads to a cost reduction of up to 66% due to the trade-off between holding and rush order costs.

Axsäter (2005) studies a continuous review inventory system with rush orders. He develops a heuristic decision assuming a single opportunity for a rush order He also assumes that a rush and normal order cannot be triggered at the same time. An emergency order is initiated when there are expected savings, i.e. when the difference between the expected cost of normal (Q, R) policy and with emergency ordering is significant.

2.3 Continuous Review Model: Applications and Extensions

The continuous review model is a well-known inventory management model, and it is applicable in many fields and industries. It has been subject to many extensions in the literature.

One of these aspects that are commonly considered in inventory management is perishability. Many researchers have applied this concept to the continuous review system. Chiu (1994) worked on determining the optimal (Q, r) policy while having a positive lead-time and perishable inventory. Furthermore, Lian & Liu (2000) proposed heuristics to derive optimal cost functions with positive lead times, derived from a generated Markovian process applicable in case of zero lead time. Similarly, Baron, Berman & Perry (2010) focus on perishable items that face spoilage due to disasters or to expirations. They develop heuristics to find the optimal solution and its cost, and are applicable in case of short lead times.

Another important factor to consider nowadays in warehousing management is the space restriction in warehouses, due to the high cost of acquiring lands in most countries. Receiving an order with no sufficient space to hold it might cause additional costs as supplier penalty or material handling. Hariga (2010) notes that since the lead-time demand is random, the amount of available space upon placing an order is usually unknown. He uses the continuous review model and adjusts it to develop a model that accounts for the space restriction. Other works on the continuous review model with limited space include the optimization model formulated by Zhao, Fan & Liu (2006) for a multi-item (Q, R) system, and Gholami-Qadikolaei, Mirzazadeh & Kajizad (2011).

Nevertheless, Moinzadeh & Nahmias (1988) extend the continuous review model to account for two supply options, with different reordering points ($R_1 > R_2$), order sizes (Q_1 and Q_2) and different lead times ($T_1 > T_2$). An order of size Q_1 is placed when the inventory level reaches R_1, and the same applies for Q_2 and R_2. However, the main assumption is that an order of Type 2 (emergency order) would not be triggered unless if it will arrive before the first order of Type 1. The procedure consists of first solving the simple probabilitized EOQ model to get a reordering value considered as R_1 and use it to find the value of R_2, and finally get Q_2 and Q_1. They find that their proposed model is most beneficial in cost saving when there are large stock-outs. This suggests that considering emergency ordering is most economical when the cost of stock-out is considerably high compared to other cost parameters.

Few authors study the effect of reducing the lead time in the continuous review inventory model. Moon & Cha (2004) extend the basic continuous review model by setting a relation connecting the lead-time, the production rate and the lot size. An additional cost is added to the general total cost, and is induced by the difference in values of the regular production and the new desired production rate. In addition, Gerchak & Parlar (1990) apply a mean-preserving transformation to the traditional continuous review model, in order to reduce the lead-time variability and turn it into a decision variable. This reduction results also in a decrease in the variability of lead-time demand, which might help in reducing the safety stock levels.

Finally, in their paper, Salameh, Abboud, El-Kassar & Ghattas (2003) examine a continuous review model under allowances for delays in payments. It means that either the retailer can pay immediately for the order upon receiving it, or he/she can delay the payment until the receipt of the next order with an additional interest charge. They study the effect of credit facilities on the inventory policy. Other works on the continuous review models with delay in payment include the model are developed by Mahata, Gupta & Mahata (2014) that also accounts for items perishability, and Wu (2001) who considers different demand distributions.

In our research, we focus mainly in our first model on inspecting the effect of adding an external buffer stock on the continuous review model. Such an extension does not seem to have been considered in the literature to our knowledge.

CHAPTER III

MODELS AND ASSUMPTIONS

In our study, we are considering the continuous review model. This model is characterized by, as the name states, a continuous check (review) for the inventory level. A constant reordering point, *R*, is specified, and whenever the inventory level is found to drop below this value, an order of size *y* is placed. Since it is not expected for the order to be received instantly, it is important to study the inventory level and shortages (if they exist) during the lead-time *T*. Depending on the lead-time demand *X*, inventory can be totally or partially consumed. In order to avoid (as much as possible) the occurrence of a shortage during this time, a buffer stock *B* or a rush order policy will be available to meet the demand.

In this chapter, we present first the basic classical continuous review model (without buffer stock nor rush orders) in Section 3.1, then we introduce the model with buffer stock first in Section 3.2, followed by the second model with rush orders, in Section 3.3.

3.1 Classical Model

In the classical model, we do not have an external buffer available, and we rely on the regular on-hand inventory only; therefore, the occurrence of a shortage depends on the size of the lead-time demand *X*. We assume that shortages are lost sales and cannot be backordered. When *X* is less than or equal to the available stock, i.e. the reordering point *R*, the inventory will be partially or fully depleted with no shortages. On the other hand, if *X* is greater than *R*, then the inventory will be completely consumed and shortages will occur before the arrival of the outstanding order. Note that we assume that shortages are lost sales and cannot be backordered.

The following graph in Figure 1 shows the inventory profile of the classical model for the two mentioned cases:

$$\begin{cases} X \leq R \rightarrow \text{no shortages} \\ X > R \rightarrow \text{shortages occur} \end{cases}$$

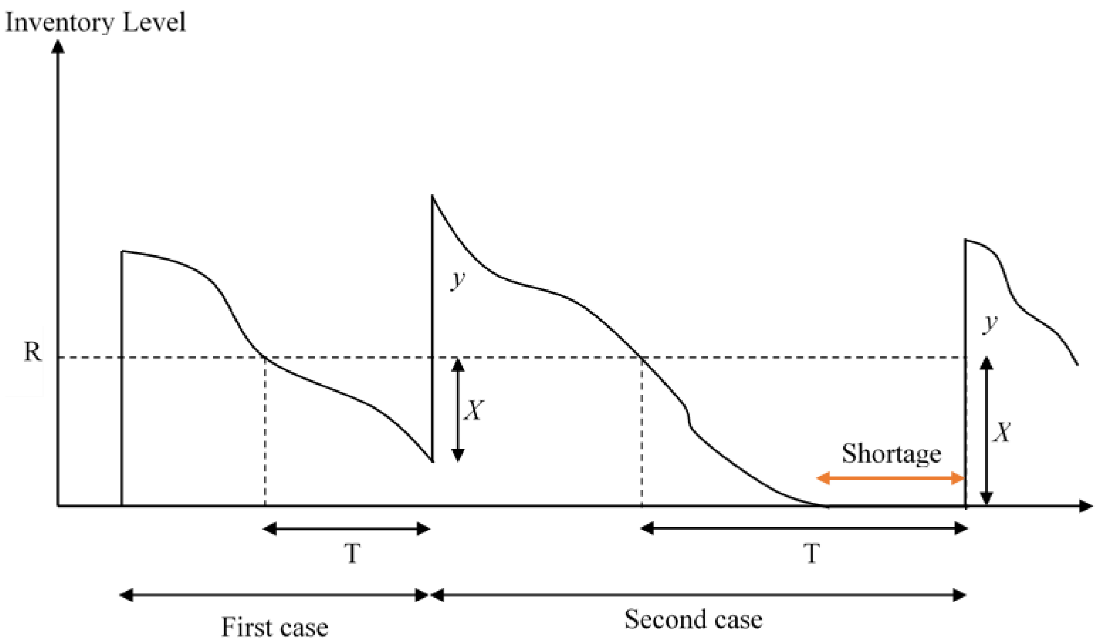

Figure 1: Inventory Profile for the Classical Model

The total cost of the classical model is composed of the fixed ordering cost, the holding cost and the shortage cost, each of which will be separately explained in the following sub-sections.

For every order we place in every cycle, there is a fixed ordering cost that we have to pay. The average fixed ordering cost per year is then the fixed cost per order K times the overall yearly number of orders, which is equal to the average demand per year D divided by the order size y. Hence, the annual fixed ordering cost is given by

$$K\frac{D}{y}$$

In order to get the holding cost, we should first calculate the average inventory, which is the average of the end and beginning of cycle inventories.

The end of cycle in-process inventory *EIPI* is what we have left at the end of each cycle before receiving a new order. *EIPI* is given by

$$EIPI = R - E\left[X\right] = R - \mu_X$$

This means that what we have left in inventory at the end of each cycle is the stock quantity we had i.e. *R* from which we remove what we consumed during the lead-time i.e. *X*, before receiving the outstanding order of size *y*.

The beginning of cycle in-process inventory is what we have after receiving the order. It is obtained by adding the order size *y* to the end of cycle inventory and is given by

$$BIPI = y + R - E\left[X\right] = y + R - \mu_X$$

The average on-hand inventory per cycle is then

$$AI = \frac{R - \mu_X + y + R - \mu_X}{2} = \frac{y}{2} + R - \mu_X$$

To calculate the annual inventory holding cost, we should multiply the average inventory by the unit holding cost h_0 to get

$$h_0\left(\frac{y}{2} + R - E[X]\right)$$

To calculate the shortage cost, we should first calculate the average number of units short per cycle $\overline{S}$, which is related to the probability of having a lead-time demand

X greater than the reordering level R, which is why we should use $f(x)$, the pdf of X.

Hence $\bar{S}$ is given by

$$\bar{S} = \int_R^{\infty} (x - R) f(x) dx \ ,$$

The shortage cost per year is then $\bar{S}$ times the unit shortage cost p for the given number of cycles per year as follows

$$\frac{pD}{y} \int_R^{\infty} (x - R) f(x) dx$$

By adding the three costs above, we get the total expected cost per year. The total cost formula for the classical model is then

$$TCU_0(y, R) = K\frac{D}{y} + h_0\left(\frac{y}{2} + R - E[X]\right) + p\frac{D}{y}\int_R^{\infty} (x - R) f(x) dx$$

If X is normally distributed with mean E[X] = μ_X and standard deviation σ_X, then it can be shown that

$$\int_R^{\infty} (x - R) f(x) dx = \sigma_X \varphi\left(\frac{R - \mu_X}{\sigma_X}\right) + \left(\mu_X - R\right)\left[1 - \phi\left(\frac{R - \mu_X}{\sigma_X}\right)\right]$$

where $\varphi(.)$ and $\phi(.)$ are the pdf and cdf of the standard Normal distribution. This is useful in analyzing the total expected cost when the lead-time demand is normally distributed.

3.2 Model with Buffer Stock

It is true that by adding the buffer stock to the model, the holding cost of the on-hand and the buffer inventory will be greater than that of the on-hand inventory alone, but on the other hand, the shortage cost (which is usually high) will be considerably less.

It is our aim to determine the optimal buffer stock level, in a way that balances the holding and shortage costs, and to integrate it into the optimal inventory policy.

Depending on the size of the lead-time demand X, three scenarios exist:

1. $X < R$
2. $R < X < R+B$
3. $X > R+B$

The following graph in Figure 2 shows the inventory profile of the buffer model for the three cases mentioned above.

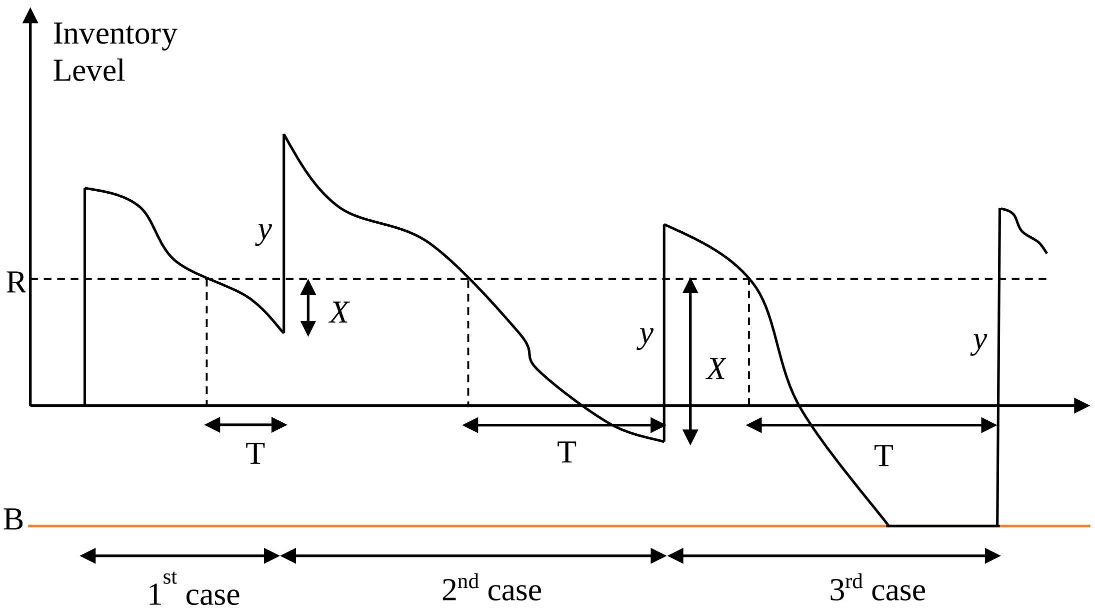

Figure 2: Buffer Model Cases

As shown in the previous graph of Figure 2, in the first case, the on-hand inventory is not completely used, so the buffer stock remains intact. In the second and third cases, the lead-time demand X is greater than the reordering point R, so the buffer stock is used, partially in case 2 where $R < X < R+B$ and fully in case 3 where $X > R+B$ and that is the case where shortages occur.

The graph in Figure 3 shows the buffer stock profile and how it acts in the three scenarios.

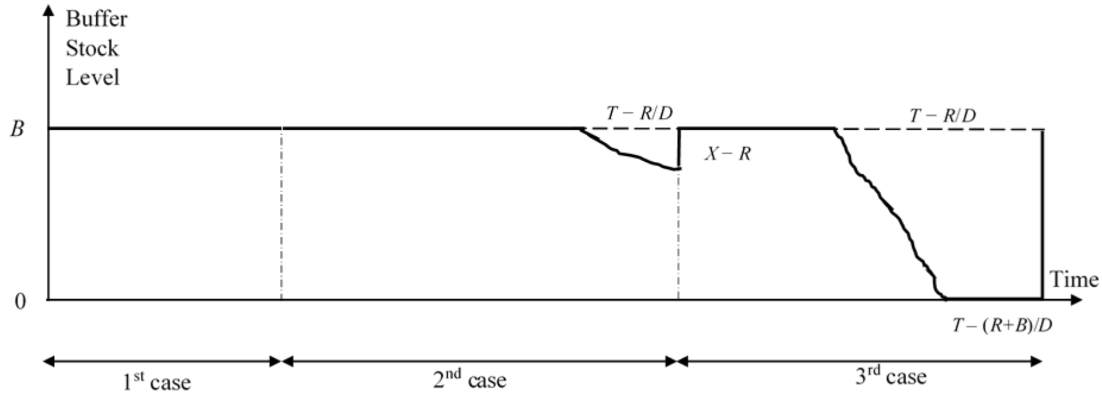

Figure 3: Buffer Inventory Profile

To estimate the end of cycle buffer inventory, we have to estimate the drop in buffer in each case as follows:

- If $X < R$, then there is no drop in buffer level.
- If $R < X < R+B$, then the drop in buffer does not start at the beginning of the cycle, but it starts approximately $\frac{R}{D}$ units of time after beginning of cycle.

  The average drop during remaining of cycle is $\frac{X-R}{2}$ over a period of $\left(T-\frac{R}{D}\right)$ units of time.

  Hence, the average drop in buffer per cycle in this case is

$$\frac{\left(T-\frac{R}{D}\right)\left(\frac{X-R}{2}\right)}{y/D} = \frac{(TD-R)(X-R)}{2y}$$

Since *TD* is approximately equal to *X*, the average drop in buffer per cycle is

$$\frac{(X-R)^2}{2y}$$

- If $X > R+B$, the drop in buffer starts after $\frac{R}{D}$ units of time from beginning of the cycle and lasts for $\left(T-\frac{R}{D}\right)$ units of time.

The average drop in buffer over the period $\left(T-\frac{R}{D}\right)$ is

$$\frac{B\left(T-\frac{R}{D}+T-\frac{R+B}{D}\right)}{2}=B\left(T-\frac{B}{2D}-\frac{R}{D}\right)$$

Therefore, the average drop in buffer level in this cycle is

$$\frac{B\left(T-\frac{B}{2D}-\frac{R}{D}\right)}{\frac{y}{D}}=\frac{B(2X-B-2R)}{2y}$$

The graph in Figure 4 shows the consumption of the buffer stock in the three cases.

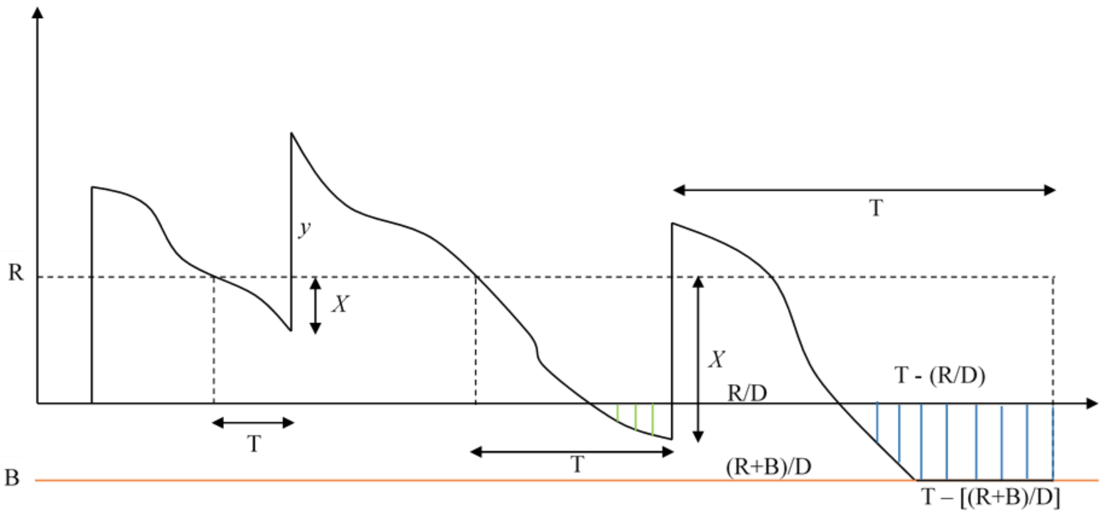

Figure 4: Buffer Stock Consumption

The buffer level *ABL*, depending on the three cases, is

$$ABL = \begin{cases} B & \text{if } X < R \\ B - \dfrac{\left(X - R\right)^2}{2y} & \text{if } R < X < R + B \\ B - \dfrac{B\left(2X - B - 2R\right)}{2y} & \text{if } X > R + B \end{cases}$$

The average buffer inventory *ABI* is then

$$ABI = E\left[ABL\right] = \int_0^R Bf(x)dx + \int_R^{R+B} \left[B - \frac{\left(x-R\right)^2}{2y}\right] f(x)dx + \int_{R+B}^{\infty} \left[B - \frac{B(2x-B-2R)}{2y}\right] f(x)dx$$

For simplification reasons, the average buffer inventory can be written as

$$ABI = \int_0^R Bf(x)dx + \int_R^{R+B} \left[B - \frac{\left(x-R\right)^2}{2y}\right] f(x)dx + \int_{R+B}^{\infty} \left[B - \frac{B(2x-B-2R)}{2y}\right] f(x)dx$$

$$ABI = \int_0^{\infty} Bf(x)dx - \int_R^{R+B} \frac{\left(x-R\right)^2}{2y} f(x)dx - \int_{R+B}^{\infty} \frac{B(2x-B-2R)}{2y} f(x)dx$$

$$ABI = B - \frac{1}{2y}\int_R^{R+B} \left(x-R\right)^2 f(x)dx - \frac{B}{2y}\int_{R+B}^{\infty} (2x-B-2R) f(x)dx$$

The average on-hand inventory is the same as in the classical model and the on-hand inventory holding cost is

$$h_0\left(\frac{y}{2} + R - E\left[X\right]\right)$$

The holding cost for the buffer inventory is equal to the average buffer inventory multiplied by the buffer inventory unit holding cost h_1. Here, we should note that h_1 is less than h_0, the regular inventory unit holding cost, and this is because in our model, we consider that the buffer stock is held at an external location, that is most probably distant from the main warehouse, and has a cheaper rent than the latter.

The buffer inventory holding cost is then

$$h_1\left(B-\frac{1}{2y}\int_R^{R+B}\left(x-R\right)^2 f(x)dx-\frac{B}{2y}\int_{R+B}^{\infty}(2x-B-2R)f(x)dx\right)$$

The total holding cost for both the regular and stock inventories is now

$$h_0\left(\frac{y}{2}+R-E\left[X\right]\right)+h_1\left(B-\frac{1}{2y}\int_R^{R+B}\left(x-R\right)^2 f(x)dx-\frac{B}{2y}\int_{R+B}^{\infty}(2x-B-2R)f(x)dx\right)$$

To order from the retailer, the fixed cost we have to pay for each order is the same as in the classical model. However, since the buffer stock is held at an external location, each time we want to use it, we have to transport or ship the stock to the main warehouse, and that incurs an additional cost, the fixed ordering cost from the buffer, which is equal to the fixed ordering cost per order times the probability of ordering from the buffer. Ordering from the buffer happens whenever the lead-time demand *X* is greater than the reordering point *R*, but not after we deplete the whole buffer (*X* > *R+B).* Therefore, the annual fixed ordering cost, from both the retailer and the buffer stock is

$$\left[K+K_1\int_R^{R+B} f(x)dx\right]\frac{D}{y}$$

Following a similar logic to the base model the shortage cost per cycle is

$$p\frac{D}{y}\int_{R+B}^{\infty}[x-(R+B)]f(x)dx$$

This means that shortages occur whenever the lead-time demand *X* is greater than the regular and buffer stocks combined.

The buffer inventory is replenished at the beginning of each cycle to the optimal level *B*. Hence, when the buffer inventory is partially or completely used-up during a given cycle, a replenishment cost is incurred to reset the buffer level to *B*.

The replenishment cost is equal to the unit replenishment cost c multiplied by the number of units used from the buffer and that have to be replenished. In the case where $R < X < R+B$, we replenish the units we used equal to $(X - R)$, but in the case where $X > R+B$, we replenish the entire buffer with size *B*.

The buffer replenishment cost is then equal to

$$c\left[\int_{R}^{R+B}(x-R)f(x)dx+\int_{R+B}^{\infty}Bf(x)dx\right]$$

By adding the above cost components, we get the formula for the total cost of the buffer model to be

$$TCU_B(y,R,B)=\left[K+K_1\int_{R}^{R+B}f(x)dx\right]\frac{D}{y}+h_0\left(\frac{y}{2}+R-E[X]\right)$$
$$+h_1\left(B-\frac{1}{2y}\int_{R}^{R+B}(x-R)^2\ f(x)dx-\frac{B}{2y}\int_{R+B}^{\infty}(2x-B-2R)f(x)dx\right)$$
$$+p\frac{D}{y}\int_{R+B}^{\infty}[x-(R+B)]f(x)dx+c\left[\int_{R}^{R+B}(x-R)f(x)dx+\int_{R+B}^{\infty}Bf(x)dx\right]$$

If $B = 0$, the buffer model should converge to the classical model, i.e. the total cost of the buffer model TCU_B should be equal to the total cost of the classical model TCU_0.

In order to verify this statement, we will assign a null value for *B* in the expression of TCU_B as follows

$$TCU_B = \left[K + 0K_1 \int_R^R f(x)dx \right] \frac{D}{y} + h_0 \left(\frac{y}{2} + R - E[X] \right)$$
$$+ h_1 \left(0 - \frac{1}{2y} \int_R^R (x-R)^2 f(x)dx - \frac{0}{2y} \int_{R+B}^{\infty} (2x - 2R) f(x)dx \right)$$
$$+ p \frac{D}{y} \int_R^{\infty} [x-R] f(x)dx + c_B \left[\int_R^R (x-R) f(x)dx + 0 \int_R^{\infty} f(x)dx \right]$$
$$TCU_B = [K + (0)] \frac{D}{y} + h_0 \left(\frac{y}{2} + R - E[X] \right) + h_1 \left(0 - \frac{1}{2y}(0) - 0 \right) + p \frac{D}{y} \int_R^{\infty} (x-R) f(x)dx + c_B [0+0]$$
$$TCU_B = K \frac{D}{y} + h_0 \left(\frac{y}{2} + R - E[X] \right) + p \frac{D}{y} \int_R^{\infty} (x-R) f(x)dx = TCU_0$$

The total cost formula for the buffer model is the same as that of the classical model when we set *B* to zero, which validates the accuracy of the buffer model.

3.3 Model with Rush Orders

As discussed in the literature review section, rush orders are requested to be delivered very quickly, so they are usually considered as zero lead-time orders (as assumed in our case). Since these orders are quickly received, they normally have higher cost than usual orders with lead times.

In the rush order alternative model, no buffer inventory will be held, and a rush order is used to limit shortages whenever the lead-time demand *X* exceeds the reordering point *R*. However, one main assumption that holds in our model, is that in each cycle, only one rush order can be initiated, even if its quantity is depleted before the arrival of the outstanding regular order.

A new decision variable is introduced, *W*, which is the quantity of units to be included in the rush order if initiated. A variable cost, c_R is incurred per unit of rush order, noting that this is only an additional cost for the rush order, given that the ordering cost for these units are included in the variable ordering cost for all demand cycles.

As in the buffer model, three scenarios exist depending on the size of the lead-time demand *X*.

1. $X < R$
2. $R < X < R+W$
3. $X > R+W$

The graph in Figure 5 aims to explain the mechanism of the model with rush orders, in the three cases mentioned above.

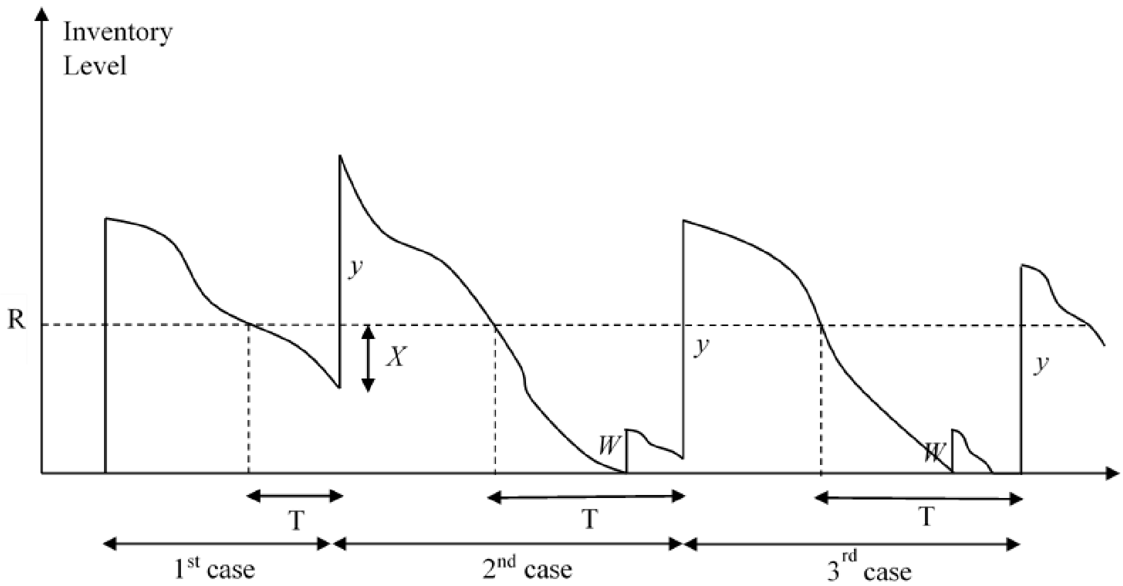

Figure 5: Rush Orders Model Cases

In the first case where $X < R$, no rush orders are initiated. In the second case, the regular inventory is depleted, so we initiate a rush order with *W* units but we do not entirely consume them so we have few left in inventory for the next cycle, and this is the case where $R < X < R+W$. In the last case, we initiate a rush order and we use all of the units included, and since we are assuming that only one rush order can be initiated per cycle, shortages occur before the receipt of the outstanding order, and this is the case where $X > R+W$.

Depending on the lead-time demand X, three cases exist for the end and beginning of cycle inventories presented below.

The end of cycle in-process inventory "EIPI" is

$$EIPI = \begin{cases} R - X & \text{if } X < R \\ R + W - X & \text{if } R < X < R + W \\ 0 & \text{otherwise} \end{cases}$$

The beginning of cycle in-process inventory is

$$BIPI = \begin{cases} y + R - X & \text{if } X < R \\ y + R + W - X & \text{if } R < X < R + W \\ y & \text{otherwise} \end{cases}$$

In order to account for the three different scenarios, based on the lead-time demand X, we will calculate the inventory level AI given by

$$AI = \begin{cases} \frac{y}{2} + R - X & \text{if } X < R \\ \frac{y}{2} + R + W - X & \text{if } R < X < R + W \\ \frac{y}{2} & \text{otherwise} \end{cases}$$

The expected value of AI is

$$E[AI] = \int_0^R \left(\frac{y}{2} + R - x \right) f(x)dx + \int_R^{R+W} \left(\frac{y}{2} + R + W - x \right) f(x)dx + \int_{R+W}^{\infty} \left(\frac{y}{2} \right) f(x)dx$$

$$E[AI] = \frac{y}{2}\left[\int_0^R f(x)dx + \int_R^{R+W} f(x)dx + \int_{R+W}^{\infty} f(x)dx \right] - \int_0^R (x - R) f(x)dx - \int_R^{R+W} (x - R - W) f(x)dx$$

$$E[AI] = \frac{y}{2} \int_0^{\infty} f(x)dx - \int_0^R (x - R) f(x)dx - \int_R^{R+W} (x - R - W) f(x)dx$$

$$E[AI] = \frac{y}{2} - \int_0^R (x - R) f(x)dx - \int_R^{R+W} (x - R - W) f(x)dx$$

Therefore the annual holding cost is

$$h_0 * \left[\frac{y}{2} * \int_0^{\infty} f(x)dx - \int_0^R (x-R) f(x)dx - \int_R^{R+W} (x-R-W) f(x)dx \right]$$

The fixed ordering cost in the model with rush orders is the same as in the classical model, with no additional costs as in the case of the buffer model.

$$K \frac{D}{y}$$

Following a similar logic to the base model the shortage cost per cycle is

$$p \frac{D}{y} \int_{R+W}^{\infty} [x-(R+W)] f(x)dx$$

This means that shortages occur whenever the lead-time demand *X* is greater than the regular stock in addition to the units we order as rush.

The ordering cost for the rush order units is included in the total ordering cost in the model; however, since these units arrive quickly, they have a higher cost than normal orders, so we have an additional cost for the rush order units. This cost is equal to the unit rush order cost c_R times the number of units ordered as rush. Rush orders are initiated only when the regular inventory is depleted, so we multiply these terms also by the pdf of *X* for *X* being greater than *R*.

$$c_R W \int_R^{\infty} f(x)dx \left(\frac{D}{y} \right)$$

After adding the four cost components, we get the total cost formula for the model with rush orders to be

$$TCU_R(y,R,W) = K \frac{D}{y} + h_0 \left[\frac{y}{2} - \int_0^R (x-R) f(x)dx - \int_R^{R+W} (x-R-W) f(x)dx \right]$$
$$+ p \int_{R+W}^{\infty} (x-R-W) f(x)dx \left(\frac{D}{y} \right) + c_R W \int_R^{\infty} f(x)dx \left(\frac{D}{y} \right)$$

If W is zero, the rush order model should converge to the classical model. i.e. the total cost of the rush order model TCU_R should be equal to the total cost of the classical model TCU_0.

In order to verify this statement, we will assign a null value for W in the expression of TCU_R as follows

$$TCU_R = K\frac{D}{y} + h_0\left[\frac{y}{2} - \int_0^R (x-R) f(x)dx - \int_R^R (x-R) f(x)dx\right] + p\int_R^\infty (x-R) f(x)dx\left(\frac{D}{y}\right) + c_R(0)\int_R^\infty f(x)dx\left(\frac{D}{y}\right)$$

$$TCU_R = K\frac{D}{y} + h_0\left[\frac{y}{2} - \int_0^R (x-R) f(x)dx - 0\right] + p\int_R^\infty (x-R) f(x)dx\left(\frac{D}{y}\right) + 0$$

$$TCU_R = K\frac{D}{y} + h_0\left[\frac{y}{2} - \int_0^R (x-R) f(x)dx\right] + p\int_R^\infty (x-R) f(x)dx\left(\frac{D}{y}\right)$$

The fixed ordering cost and the shortage cost are the same as in the classical model; however, the expression of the holding cost is different, and this is most probably due to the application of an approximation to the holding cost in the classical continuous review model, that ignores the case where $R - E[X]$ may be negative. In order to validate our hypothesis that the rush order model converges to the classical model in the case of W is zero, we proved it numerically using Excel, by setting W is zero and solving for the optimal y, R and TCU. We got the exact same values as in the classical model, which validates the accuracy of our developed rush order model.

CHAPTER IV

NUMERICAL ANALYSIS

This chapter is divided into three sections. In Section 5.1, we present a base example for the three models, after determining values for the cost parameters to solve for the optimal solution in each model. Then in Section 5.2, we perform a sensitivity analysis on the cost parameters for both the buffer and rush orders models, Finally, in Section 5.3, we present graphical and analytical comparisons for the costs of the models when varying cost parameters.

4.1. Base Example

In order to numerically test the effectiveness of our model in cost saving, we used Excel and Excel solver for the three models presented above. The three decision variables were the order size *y*, the reordering point *R* and the buffer stock *B*, and the objective was to minimize the cost function. We assumed a Normal distribution for the demand since it simplifies the calculations and it is widely used and observed in industry (Bhonsle, Rossetti, & Robinson, 2005).

The probability density function of the random normal variable X is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

For simplification reasons, we expressed the integrals as function of *z*, where *z* is a standard normal random variable and is given by $z = \frac{x-\mu}{\sigma}$.

We assigned the following values for the variable parameters:

- Average yearly demand D = 10000 units/year

  This is the average demand expected over a period of a year. It could vary depending on the supply and demand of each company.

- Fixed order cost K = \$100/order

  This is the cost to order from the retailer, and it is fixed for all orders. It could vary depending on the pricing of the supplier, the fragility and process of the shipment, or the distance between the retailer and the warehouse.

- Fixed order cost from buffer stock K_1 = \$20/order

  This is the cost to transport the stock from the buffer inventory to the main warehouse. It is less than K since there is no external party to pay for, and it represents only the transportation or shipment cost.

- On-hand inventory holding cost h_0 = \$10/unit

  This is the price we pay for each unit stored in inventory in the main warehouse.

- Buffer inventory holding cost h_1 = \$6/unit

  This is the price we pay for each unit stored in the buffer inventory. We can notice how its value is less than the value of the on-hand unit holding cost h_0 as we already stated.

- Shortage cost p = \$80/unit

  This is the cost we pay for each unit short during the lead-time. Shortage costs are usually high and this is why we always search for ways to reduce or avoid them.

- Buffer replenishment cost c = \$30/unit

  This is the fee we pay for replenishing each unit we use from the buffer at the beginning of each cycle. It is related to the buffer unit holding cost, where h_0 = 0.2c, since the storage cost accounts for only about 20% of the total inventory holding cost.

- Rush order unit cost c_R = \$50/unit

  This is the additional fee we pay for every unit we order as rush. Similarly, it is related to the on-hand inventory holding cost where h_0 = 0.2c_R.

- Mean of lead-time demand μ_X = 400 units

  This is the expected number of units to be ordered during the order lead-time and it is related to the average demand.

- Standard deviation of lead-time demand σ_X = 30 units

The objective was to minimize the cost function, and the decision variables were y and R in the classical model, y, R and B in the buffer model and y, R and W in the model with rush orders.

The following table shows the optimal values for the total cost and the decision variables for the models.

Table 1

Optimal Values for the Decision Variables and Total Costs for the Three Models

	y	R	B	W	TCU	% Improvement in TCU w.r.t TCU_0
Classical Model	**456.92**	**475.9**	**-**	**-**	**5328.05**	**-**
Buffer Model	**455.91**	**444.5**	**36.21**	**-**	**5247.8**	**1.5%**
Rush Order Model	**456.95**	**474.97**	**-**	**4.83**	**5319.86**	**0.15%**

The results in Table 1 can be analyzed as follows:

- For the order size *y*, the values are almost the same in the three models, and are slightly close the order size in the EOQ model which is $\sqrt{\frac{2KD}{h_0}}$ = 447.21.
- The reordering point *R* didn't change much in the rush order model compared to the classical one, but we notice that in the buffer model it is lower, and this is because of the role the buffer plays in acting as a safety stock that can help in reducing shortages considerably during lead-time. Therefore, it is safe to lower the reordering point without causing more shortages.
- What is most important is the decrease in the total cost in both extended models compared to the classical model. The decrease in the cost in the buffer model is greater than in the rush order model, but the latter can be a good choice when there is no possibility of holding a buffer at an external location with a convenient location and a cheaper rent.

5.2 Sensitivity Analysis

In order to study the effect of the variation of some parameters on the total cost of the models, and how the decision variables react to this change, we present a sensitivity analysis performed using Excel on the different cost parameters for both the buffer and the rush order models.

5.2.1. Sensitivity for the Buffer Model

Following is the sensitivity analysis for the buffer model performed on the following variables: fixed ordering cost from the retailer *K*, fixed ordering cost from the buffer K_1, on-hand inventory holding cost h_0, buffer holding cost h_1, unit replenishment cost c and shortage cost p.

<u>5.2.1.1. Sensitivity on the Fixed Ordering Cost from the Retailer K</u>

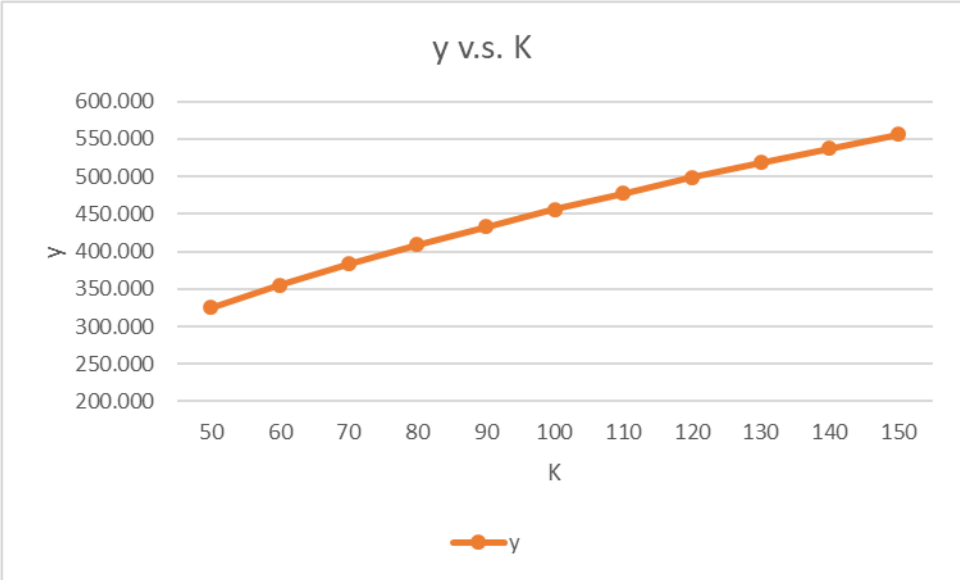

Figure 6: Variation of y with K

When the ordering cost increases, it obviously becomes more expensive to place an order; this is why it is more cost-effective to decrease the number of orders, and this can be achieved by increasing the fixed order size y each time, as we can see in the graph in Figure 6.

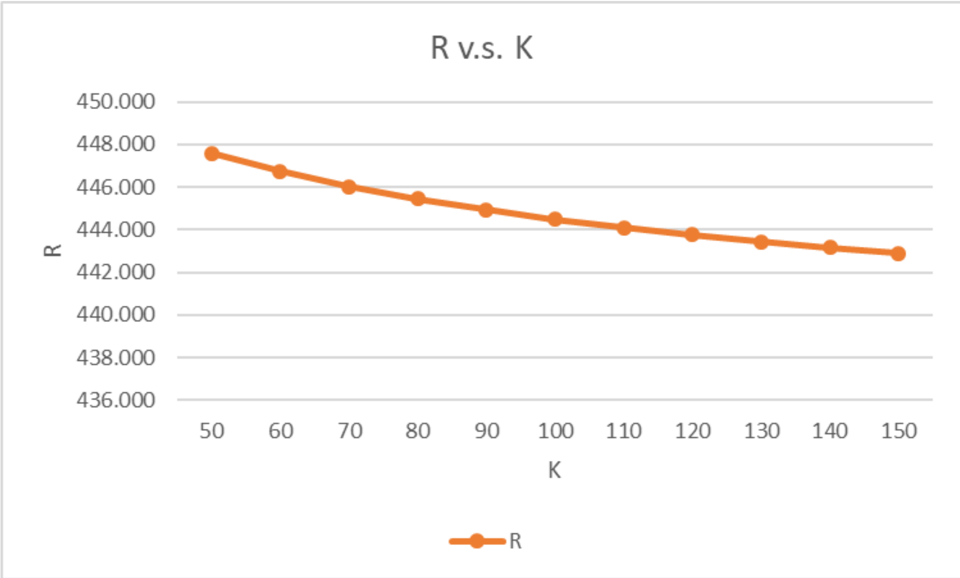

Figure 7: Variation of R with K

Since the order size increases with the increase of *K*, the reordering point *R* decreases as shown in the graph in Figure 7, and this is because we will have more on-hand inventory due to bigger order size, which allows us to have a lower reordering point without risking having more shortages.

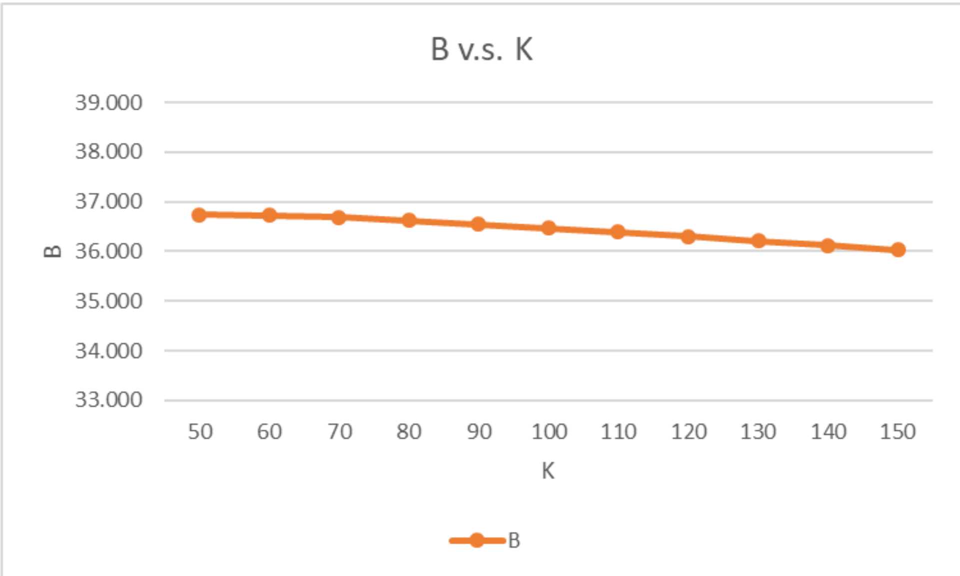

Figure 8: Variation of B with K

The buffer size *B* is almost insensitive to the variation of *K*, but it slightly decreases since we are having more on-hand inventory with the increase of *y*.

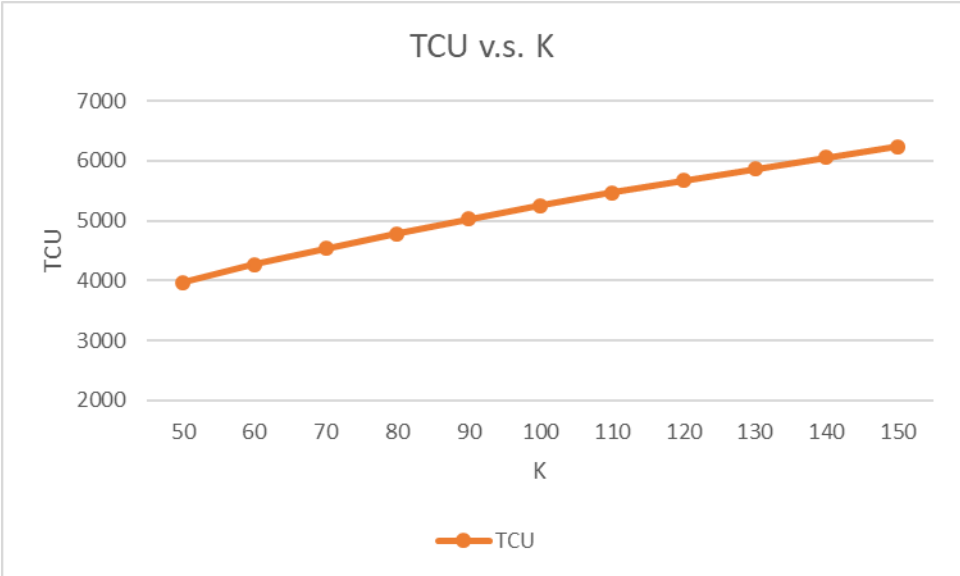

Figure 9: Variation of TCU_B with K

The total cost normally increases with the increase of any cost parameter, and in this case, the placement of an order becomes more expensive which results in a higher cost for the model as shown in Figure 9.

<u>5.2.1.2. Sensitivity on the Fixed Ordering Cost from the Buffer K_1</u>

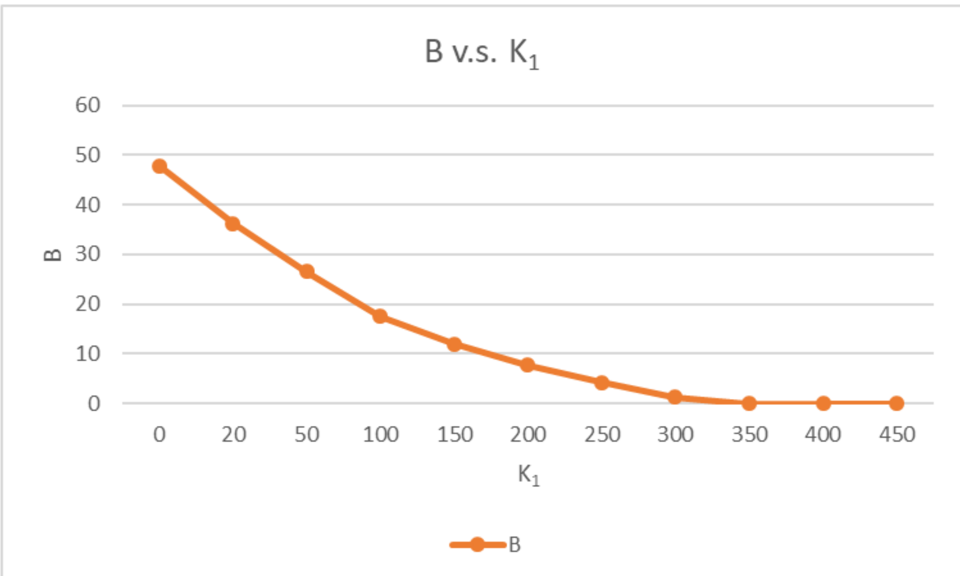

Figure 10: Variation of B with K_1

As we can observe in the graph in Figure 10, the buffer size decreases as K_1 increases, because for high values of K_1, it becomes expensive to rely and order from the buffer stock, until we reach a null buffer value starting K_1 = $350. In this case, it would be more cost effective to rely on the on-hand inventory instead of holding a buffer stock, and the model converges to the classical model.

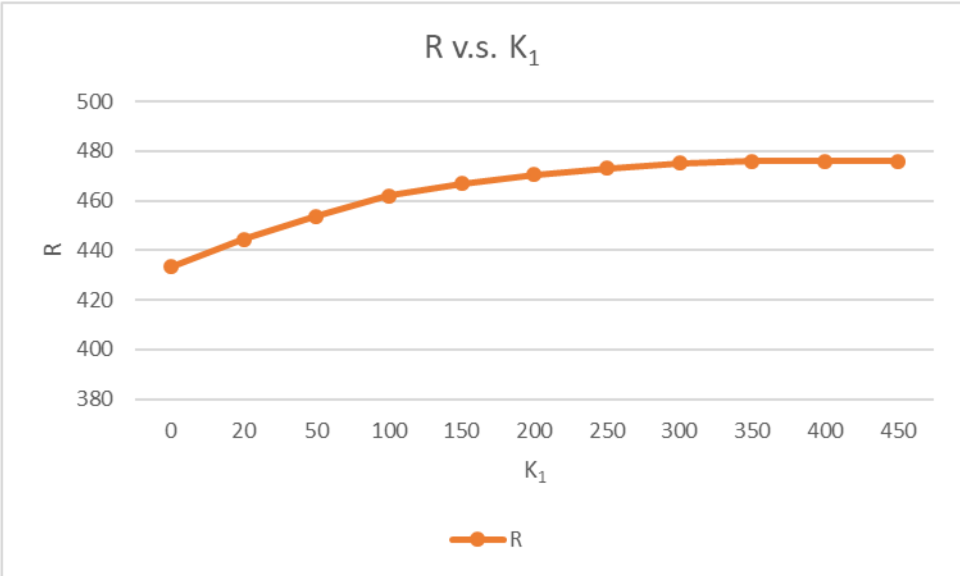

Figure 11: Variation of R with K_1

Since there will be less reliance on the buffer stock with the increase of K_1, the reordering point R increases to have more on-hand inventory and try to reduce shortages as much as possible with the partial or complete absence of the buffer stock.

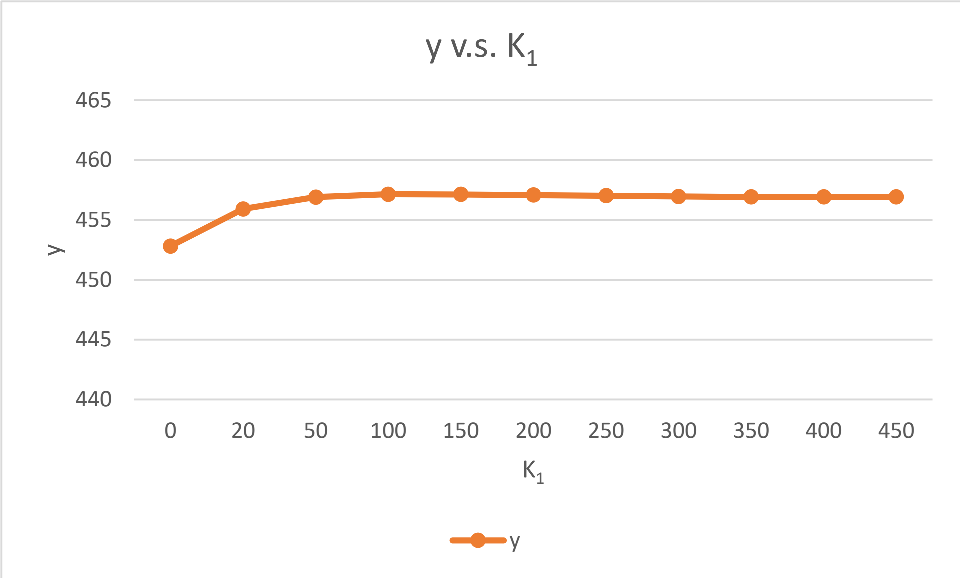

Figure 12: Variation of y with K_1

The graph in Figure 12 shows that the order size *y* is almost insensitive to the variation of K_1, since the increase of the reordering point ensures a sufficient on-hand inventory level to make up for the decrease of the buffer stock.

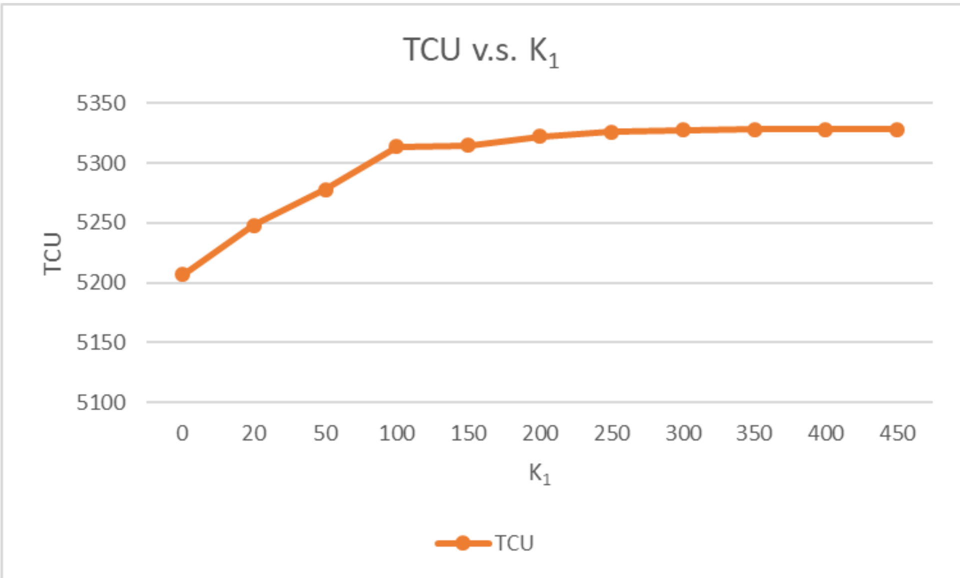

Figure 13: Variation of TCU_B with K_1

The total cost increases with the increase of K_1 until reaching a constant value equal to that of the classical model (5328) when the buffer reaches zero and the model converges to the classical one.

5.2.1.3. Sensitivity on the On-hand Inventory Holding Cost h_0

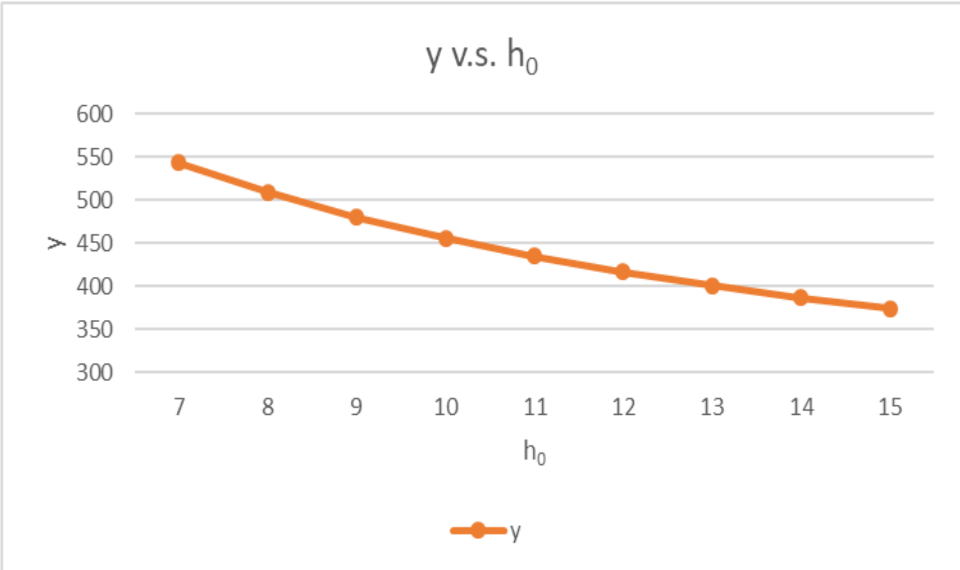

Figure 14: Variation of y with h_0

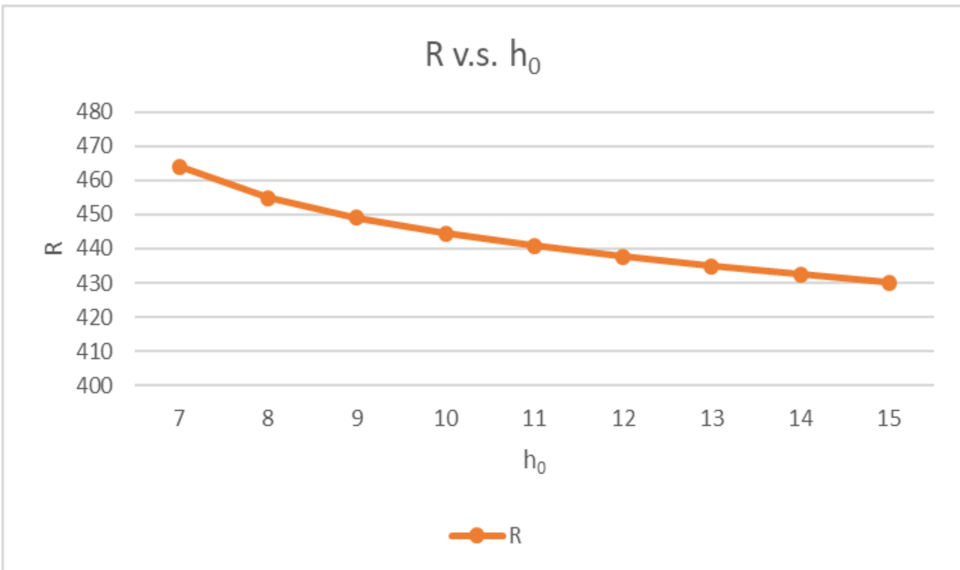

Figure 15: Variation of R with h_0

As long as h_0 increases, holding bigger amounts of inventory becomes more expensive, hence, it would be better to decrease the amount of on-hand inventory and rely more on the buffer stock. This is why in the graphs in Figures 14 and 15 respectively, we can notice that y and R are decreasing with the increase of h_0.

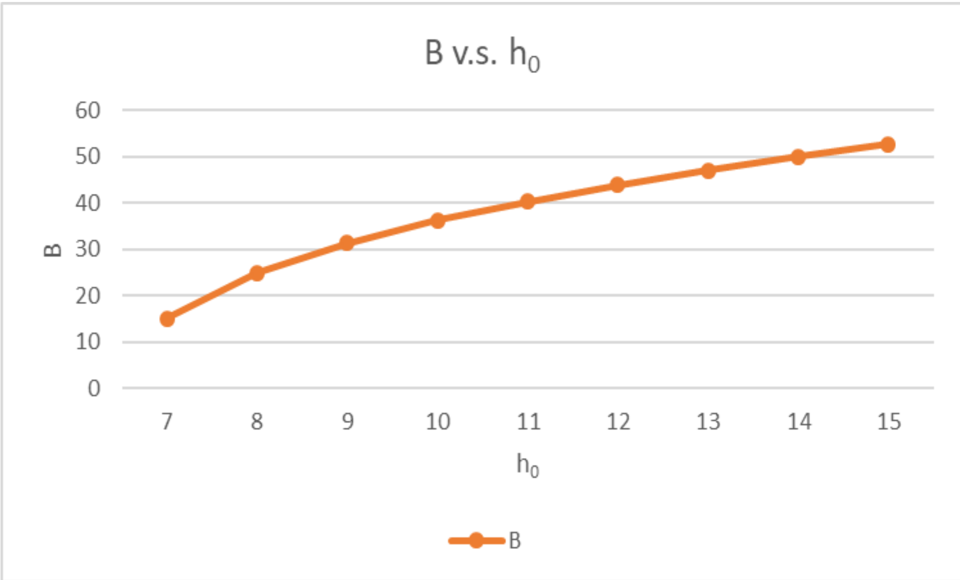

Figure 16: Variation of B with h_0

The decrease in the on-hand inventory signaled by the decrease of y and R is normally accompanied by an increase in the buffer size as shown in the graph in Figure 16, in order to increase the buffer inventory and limit the amount of shortage.

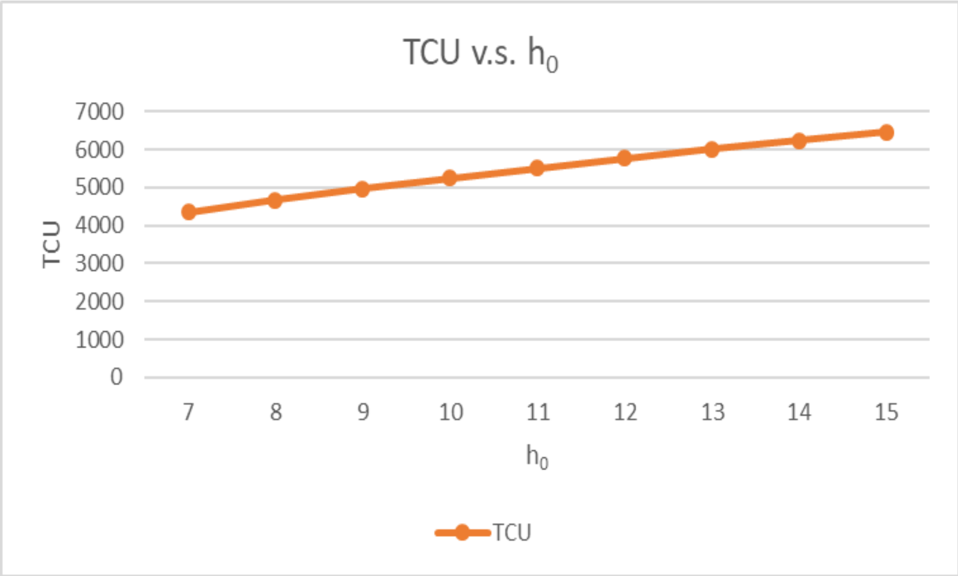

Figure 17: Variation of TCU_B with h_0

The total cost increases with the increase of h_0 since it the cost of holding the regular inventory is increasing.

5.2.1.4. Sensitivity on the Buffer Holding Cost h_1

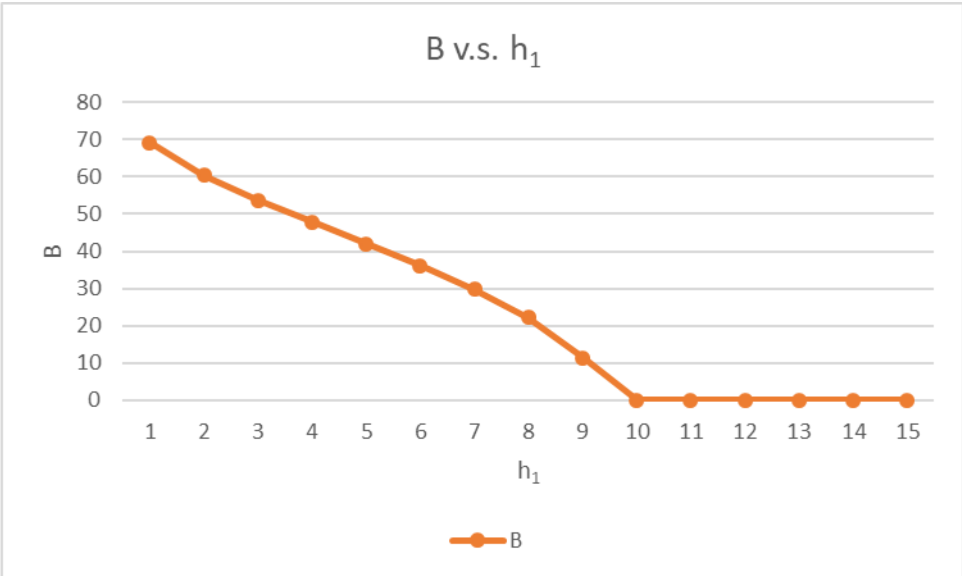

Figure 18: Variation of B with h_1

As we can see in Figure 18, the buffer size decreases linearly with the increase of h_1 until it reaches zero at $h_1 = 10$ (which is equal to the inventory holding cost in our model), and this is why it would be of no interest to hold a buffer stock at the same unit holding cost.

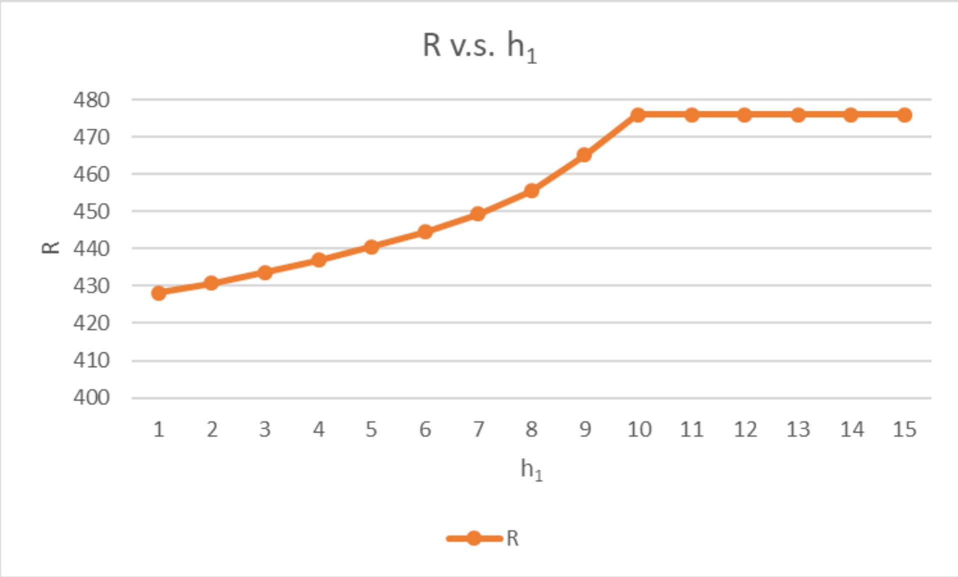

Figure 19: Variation of R with h_1

When the buffer stock decreases, the reordering point increases to hold more on-hold inventory and avoid shortages. When the buffer reaches zero at $h_1 = 10$, *R* reaches a constant value equal to its value in the classical model.

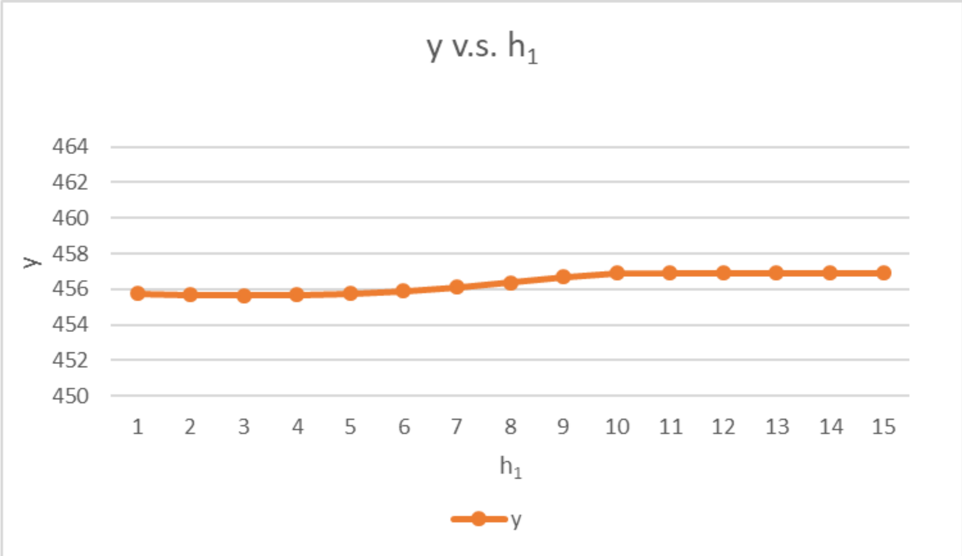

Figure 20: Variation of y with h_1

The order size is almost insensitive to the variation of h_1 as we can see in the graph of Figure 20. It also reaches a constant value starting h_1 = 10.

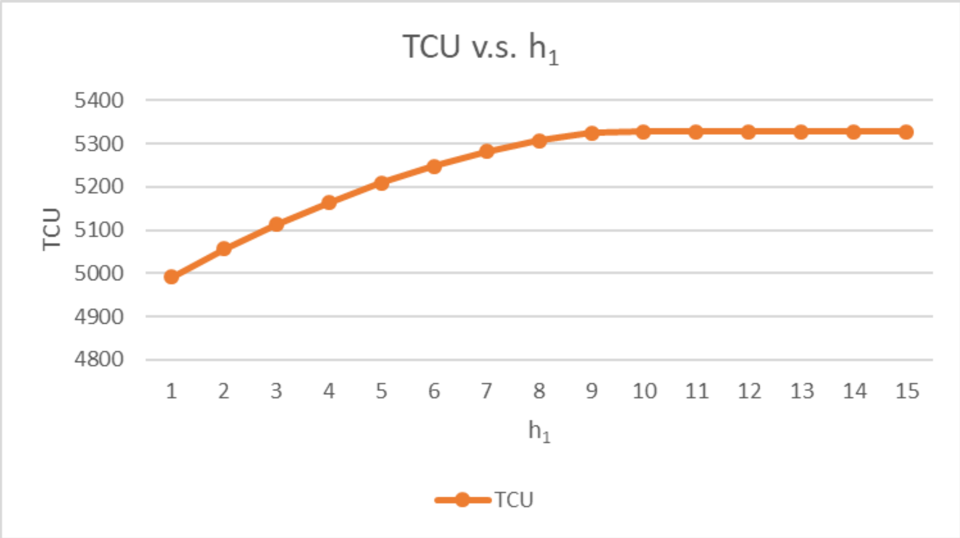

Figure 21: Variation of TCU_B with h_1

As in the case of K_1, the total cost increases linearly with the increase of h_1 until reaching a constant value of 5328 when the buffer stock hits zero, because this is when the model converges to the classical model with the same values of *y*, *R*, and *TCU*.

<u>5.2.1.5. Sensitivity on the Buffer Replenishment Cost c</u>

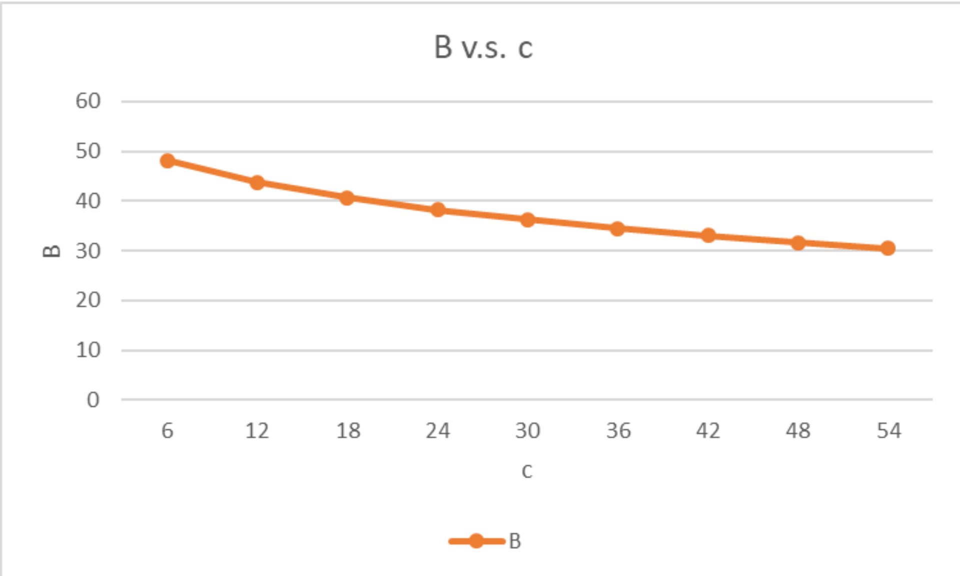

Figure 22: Variation of B with c

The buffer size decreases linearly with the increase of *c* as shown in Figure 22, and this is normal since the increasing cost of replenishing the buffer would outweigh the lower cost of holding inventory in it, so the buffer size will keep increasing until eventually reaching zero.

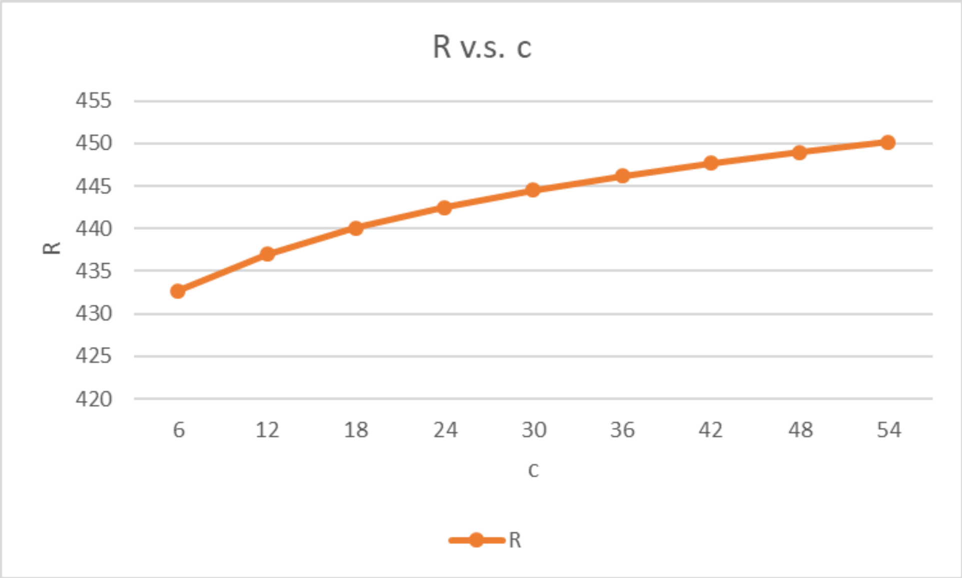

Figure 23: Variation of R with c

When the buffer stock decreases, the reordering point increases to hold more on-hold inventory and avoid shortages, and this is why we notice in Figure 23 a linear increase of *R* with the increase of *c*.

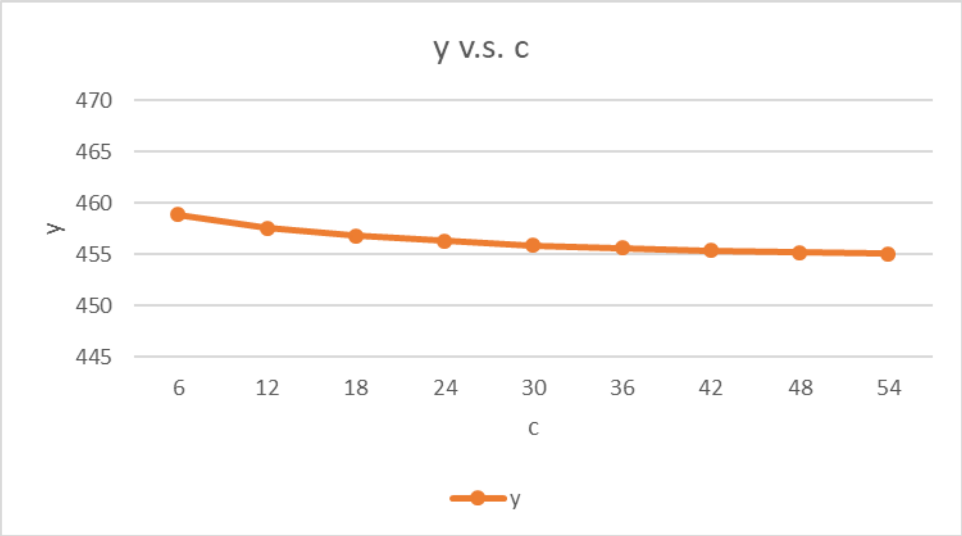

Figure 24: Variation of y with c

The increase of *R* results in having more on-hand inventory, so the order size *y* decreases slightly as seen in Figure 24.

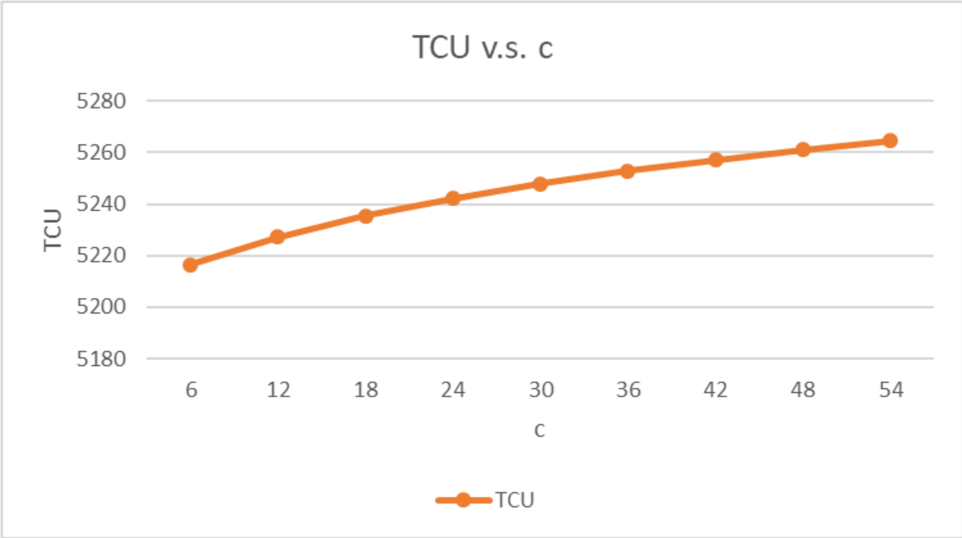

Figure 25: Variation of TCU_B with c

It is normal that an increase in any cost parameter in the model will cause an increase in the total cost, and this is shown in the graph of Figure 24 that signals a linear increase in TCU_B with the increase of *c*, until eventually reaching the value of the total cost in the classical model when *B* reaches zero.

5.2.1.6. Sensitivity on the Shortage Cost p

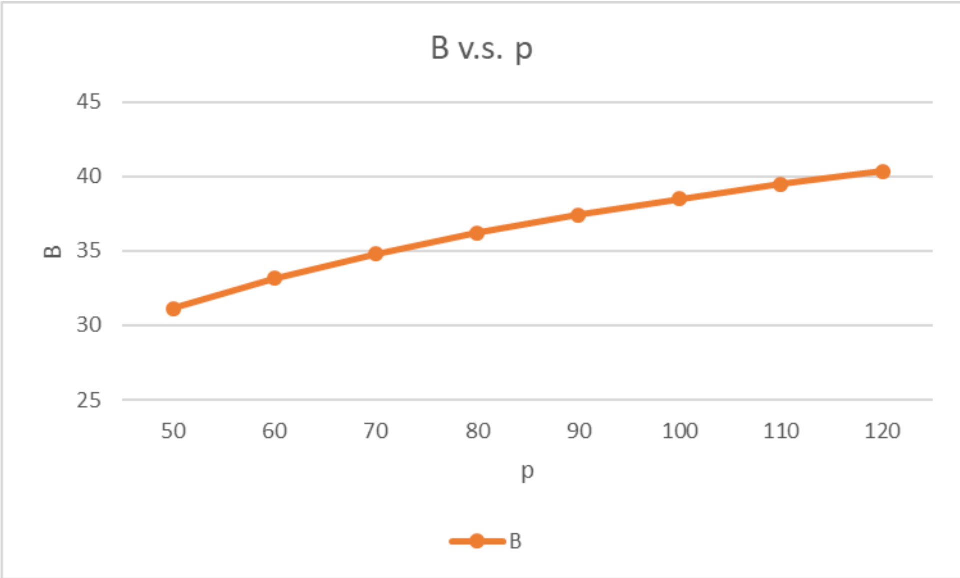

Figure 26: Variation of B with p

When the unit shortage cost increases, the model should work as much as possible on reducing shortages to reduce losses, and this can be achieved by holding a greater buffer inventory, hence *B* increases with the increase of *p* as shown in the graph of Figure 26.

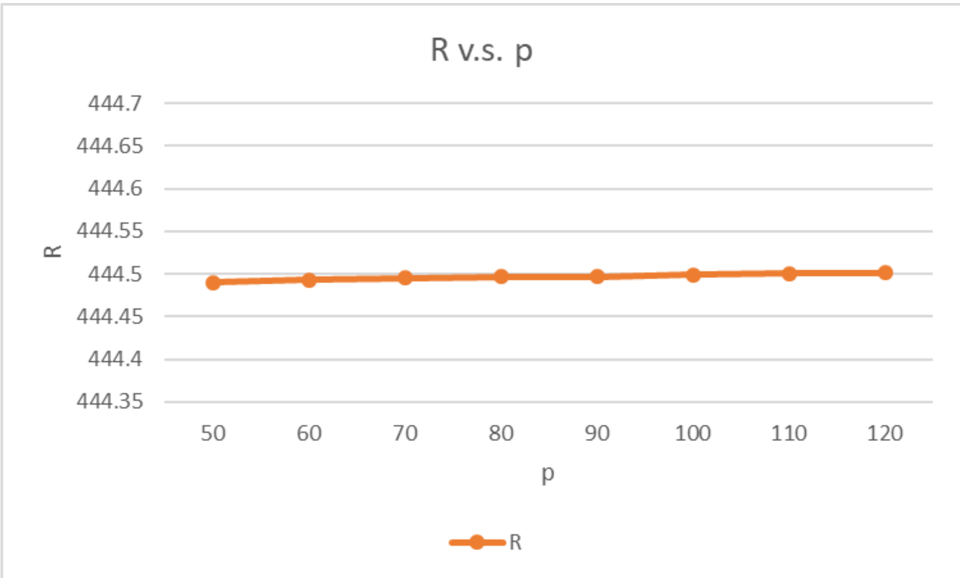

Figure 27: Variation of R with p

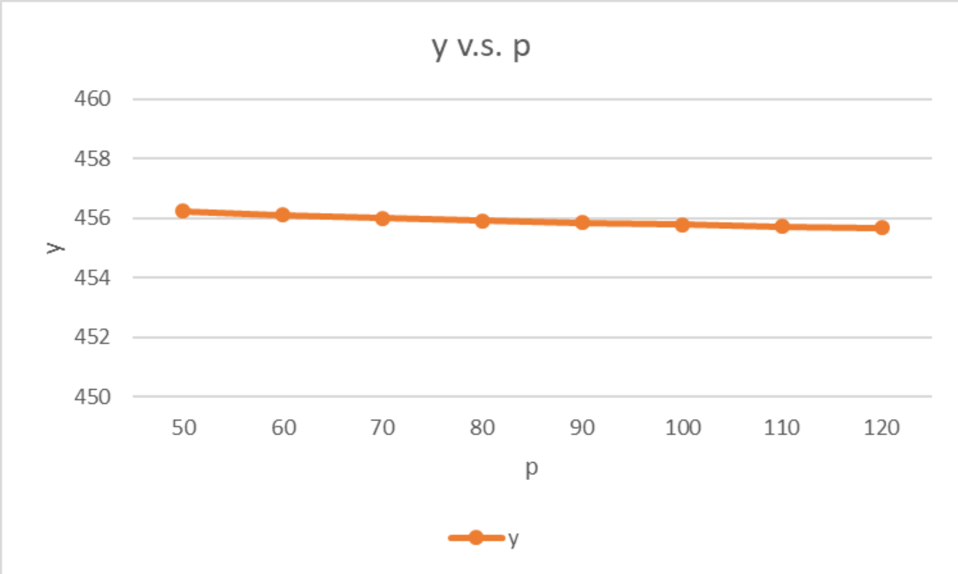

Figure 28: Variation of y with p

R and *y* (Figures 27 and 28 respectively) are almost constant with the increase of *p*, which means that *R* and *y* are nearly insensitive to the increase of the shortage cost. This is because the main increase in inventory is achieved in the buffer stock for its lower holding cost, which ensures a sufficient and safe inventory level without increasing much the cost.

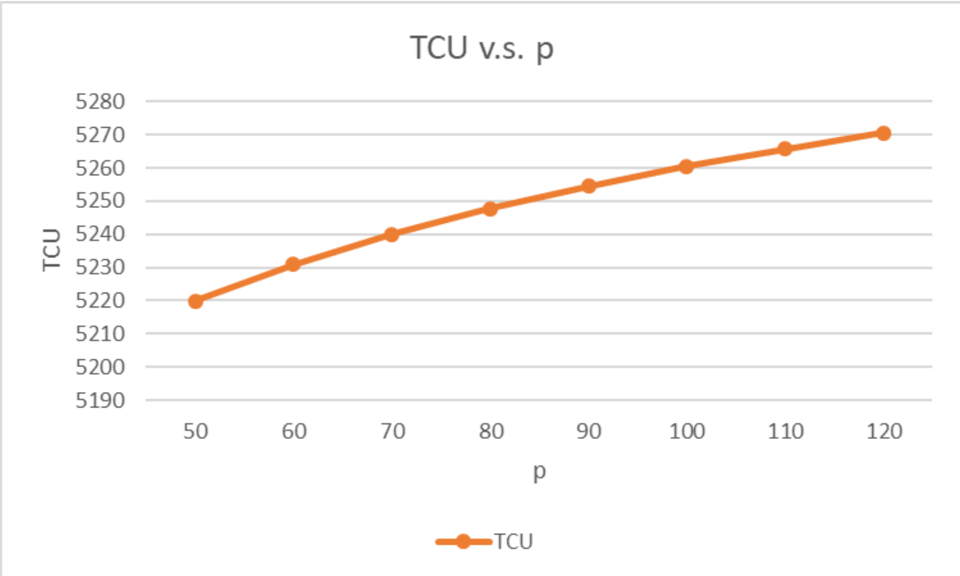

Figure 29: Variation of TCU_B with p

The increase in the unit shortage cost *p* will normally lead to an increase in the total cost as shown in Figure 29, since every unit short will cost more.

5.2.2. Sensitivity for the Rush Orders Model

Following is the sensitivity analysis for the model with rush orders performed on the following variables: fixed ordering cost from the retailer *K*, , on-hand inventory holding cost h_0, unit rush order cost *c* and shortage cost *p*.

<u>5.2.2.1. Sensitivity on the Fixed Ordering Cost from the Retailer K</u>

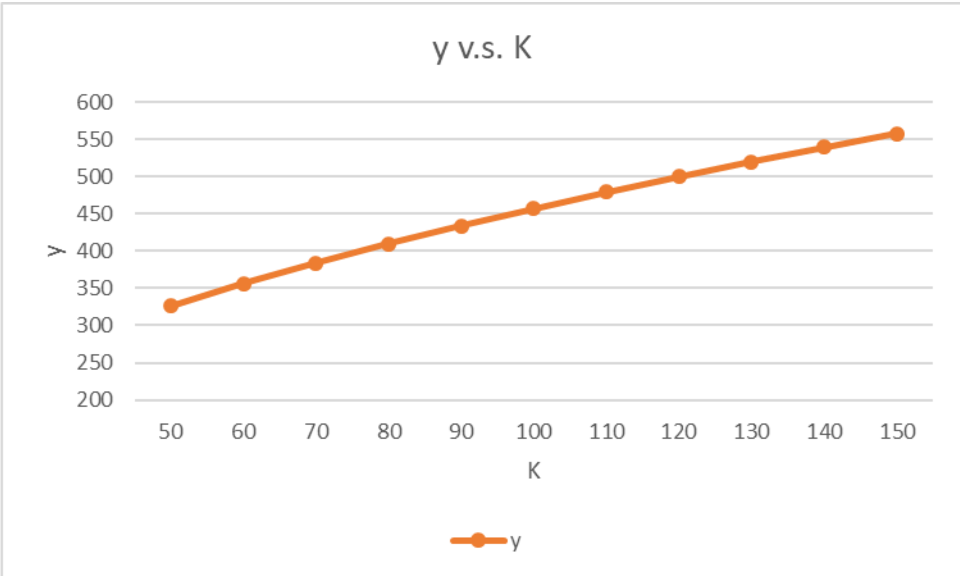

Figure 30: Variation of y with K

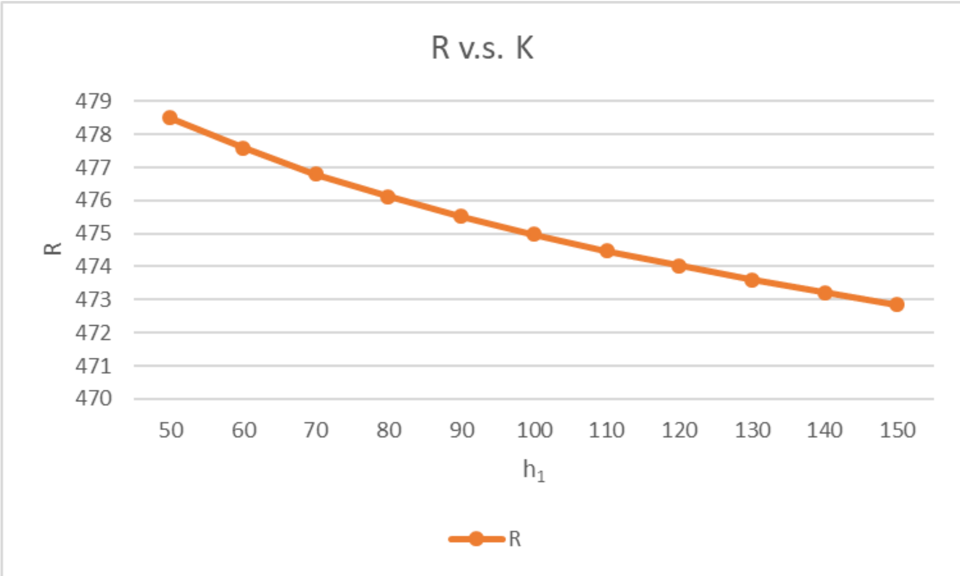

Figure 31: Variation of R with K

As seen in the buffer model, when the fixed ordering cost increases, the order size also increases (Figure 30) in order to reduce the number of outstanding orders needed, and hence cut the total ordering cost. This increase in the order size leads to a decrease in the reordering point as shown in the graph of Figure 31, since there will be more available inventory, so we can have a lower reordering point without causing more shortages.

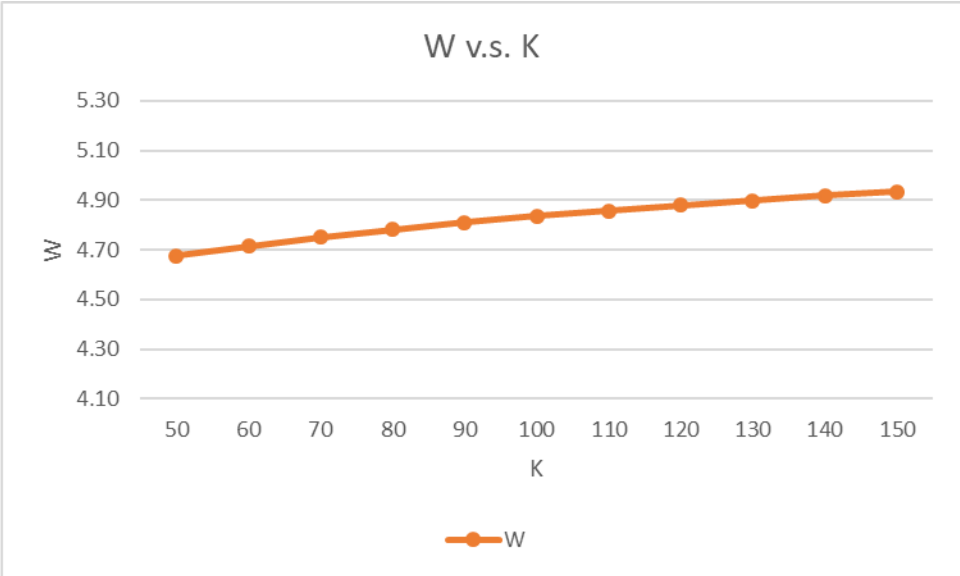

Figure 32: Variation of W with K

The rush order size is almost insensitive to the fixed ordering cost, with only a slight increase to account for the decrease of the reordering point, as shown in the graph of Figure 32.

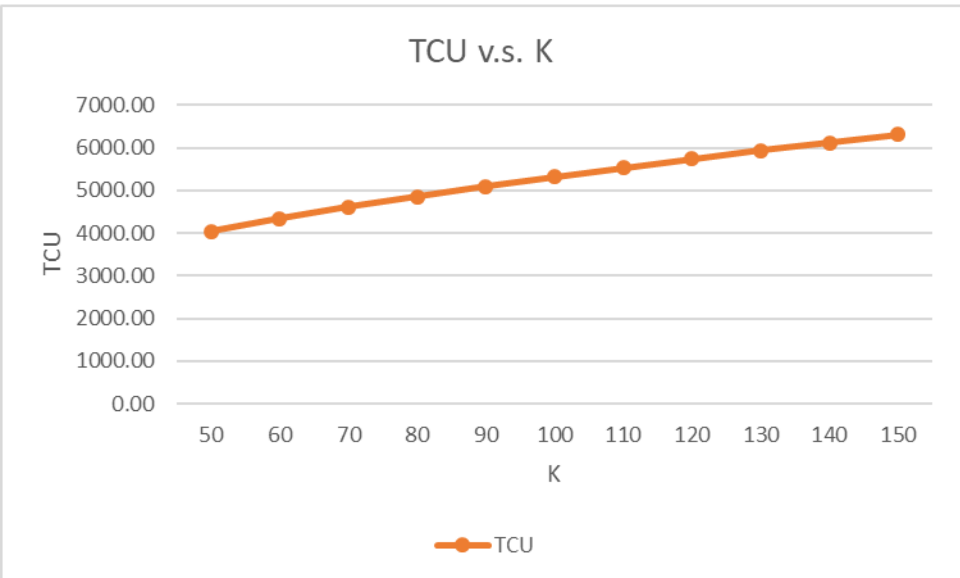

Figure 33: Variation of TCU_R with K

The total cost normally increases with the increase of the order cost as seen in Figure 33, since every order placed will be more expensive.

5.2.2.2. Sensitivity on the On-hand Inventory Holding Cost h_0

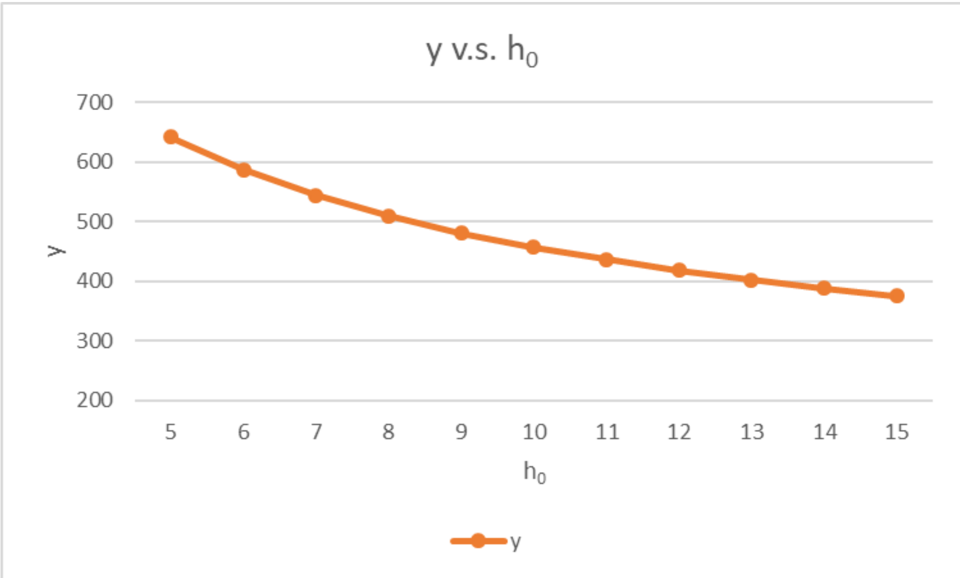

Figure 34: Variation of y with h_0

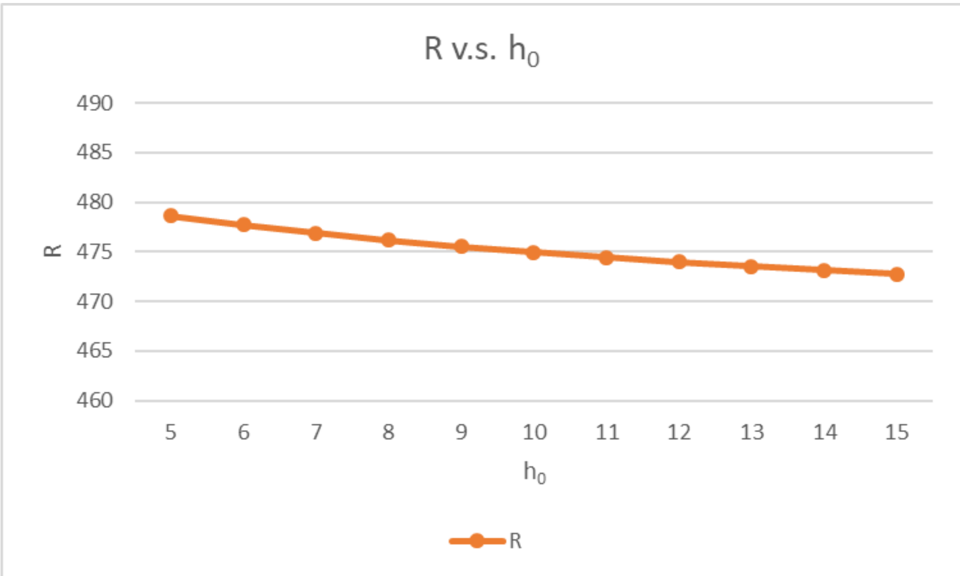

Figure 35: Variation of R with h_0

As shown in the buffer model previously, when the inventory holding cost increases, the order size and the reordering point decrease (Figures 34 and 35 respectively) in order to hold less on-hand inventory and decrease the overall holding cost.

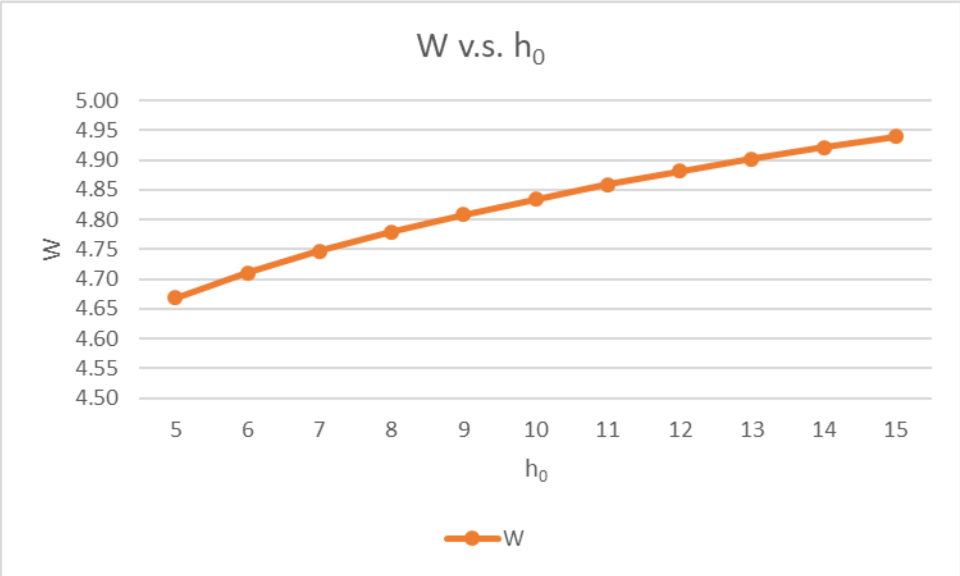

Figure 36: Variation of W with h_0

In order to make up for the smaller inventory available on-hand, the model relies more on the rush order alternative in order to limit shortages, and that's why the rush order size increases continuously as the holding cost increases as shown in Figure 36.

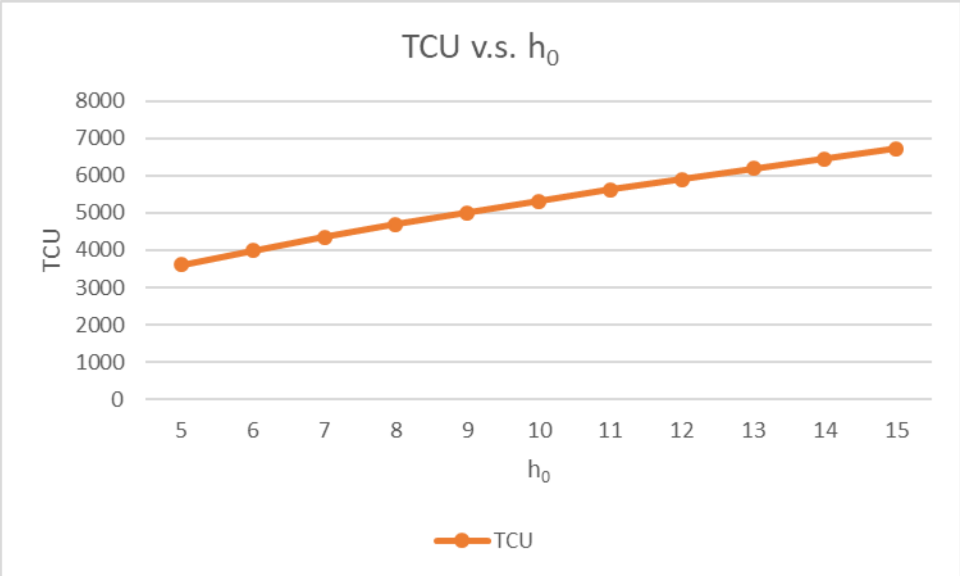

Figure 37: Variation of TCU_R with h_0

The total cost increases with the increases of h_0 since the cost of holding inventory is higher.

5.2.2.3. Sensitivity on Rush Order Unit Cost c_R

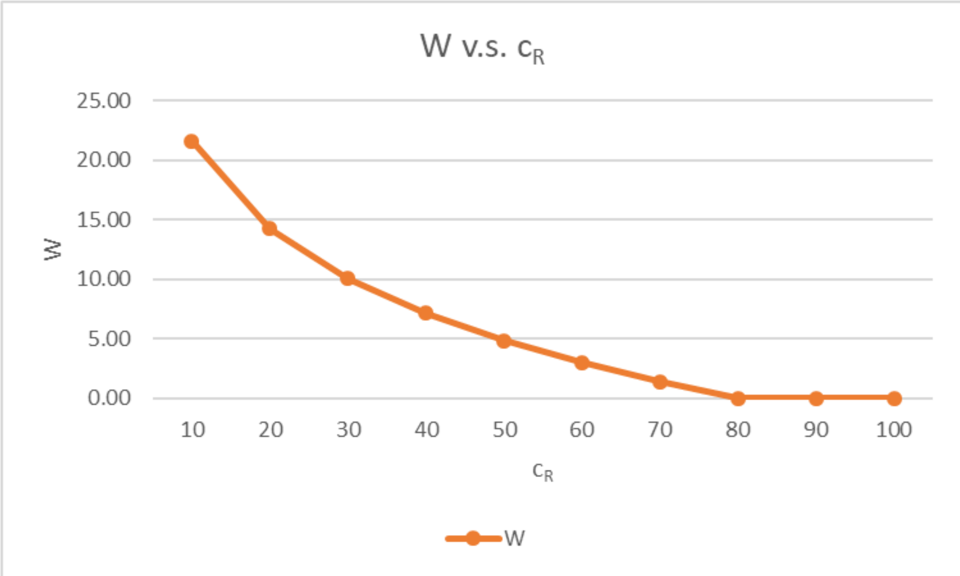

Figure 38: Variation of W with c_R

The main cost component related to the rush order model is the rush order unit cost c_R, so when this parameter increases, the high cost of the rush order will not be justified and therefore there will be less reliance on this option. This can be seen by the continuous decline of the graph of the rush order size W in Figure 38, until reaching a zero value for $c_R = 80$ when the rush order model is no longer used, and the model converges to the classical one.

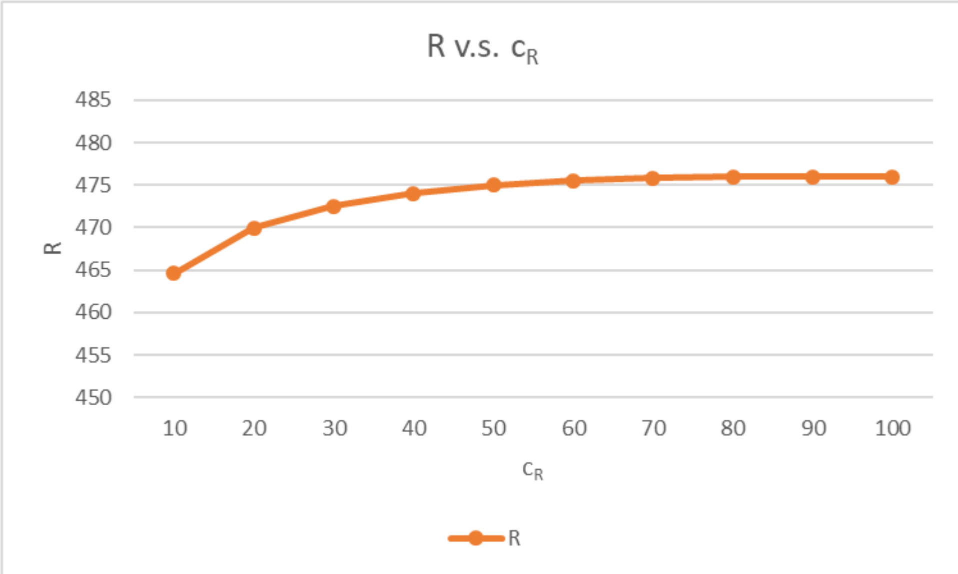

Figure 39: Variation of R with c_R

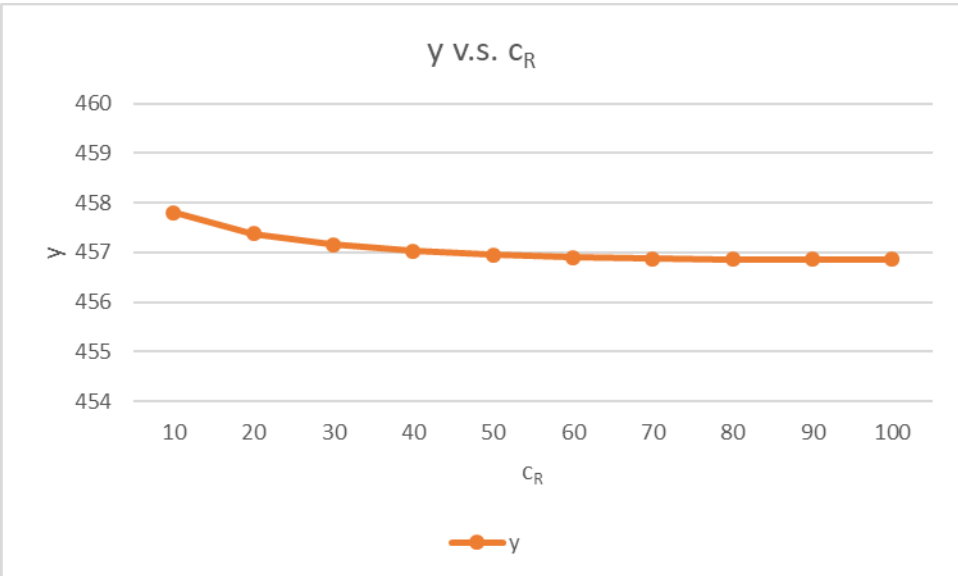

Figure 40: Variation of y with c_R

The reordering level increases with the increase of c_R (Figure 39) since we will need more on-hand inventory to account for the decrease of the rush order size W and reduce shortages, whereas the order size y is almost insensible to this variation as shown in Figure 40. Both R and y reach a constant value starting $c_R = 80$ where the rush order size W reaches zero and the model converges to the classical one.

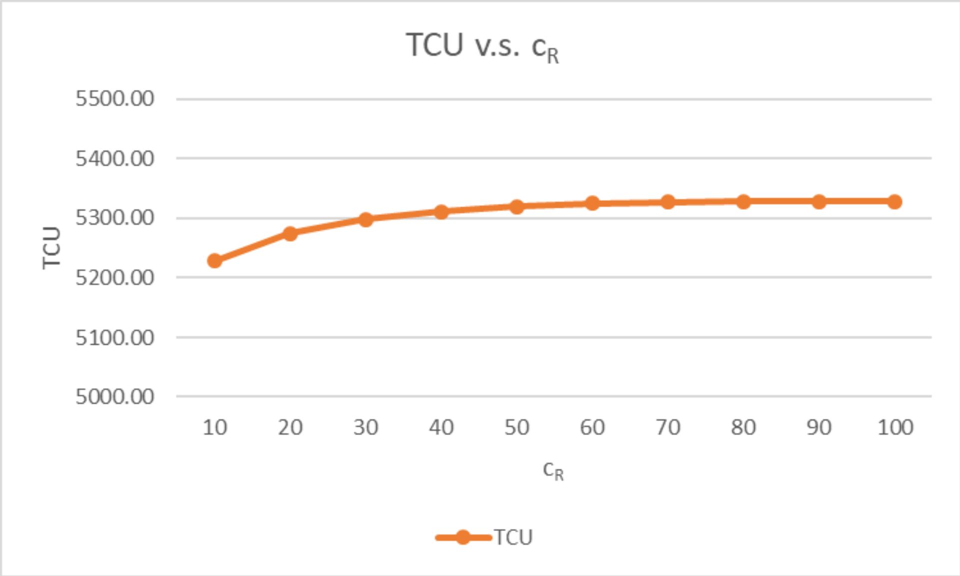

Figure 41: Variation of TCU_R with c_R

As shown in the graph of Figure 41, the total cost slightly increases with the increase of c_R until reaching a constant value of 5328 at $c_R = 80$ and hereafter, which is equal to the cost of the classical model in our case.

5.2.2.4. Sensitivity on Shortage Cost p

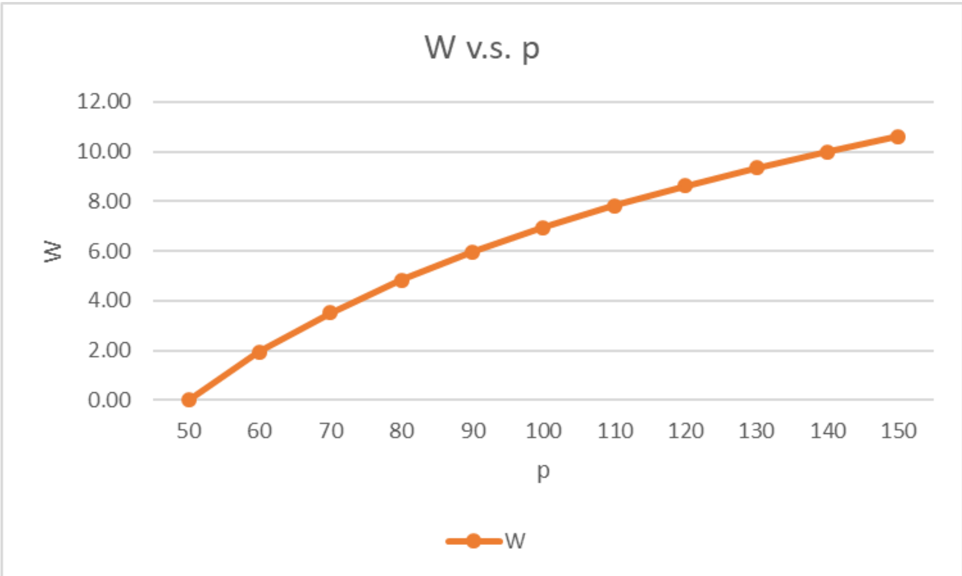

Figure 42: Variation of W with p

The main role of the rush order is to act fast to bring inventory and reduce shortages, and this is why when the shortage cost increases, the rush order size increases considerably to avoid any shortage occurrence as much as possible, as shown in Figure 42.

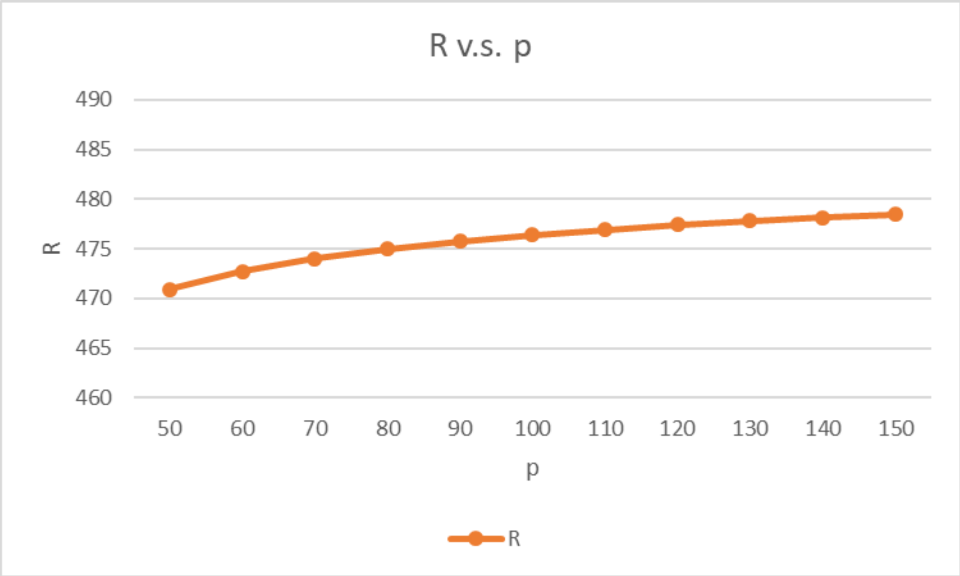

Figure 43: Variation of R with p

The increase of the shortage cost leads to a slight increase in the reordering point as seen in Figure 43, in order to reduce the amounts of units short.

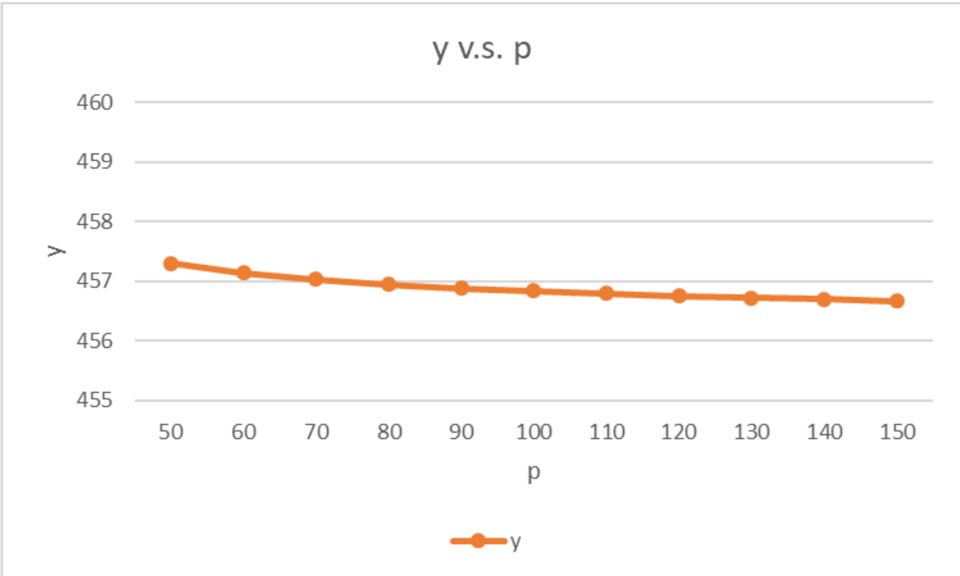

Figure 44: Variation of y with p

The graph in Figure 44 shows that the order size *y* is almost insensitive to the variation of *p*, it just slightly decreases since the reordering point is increasing.

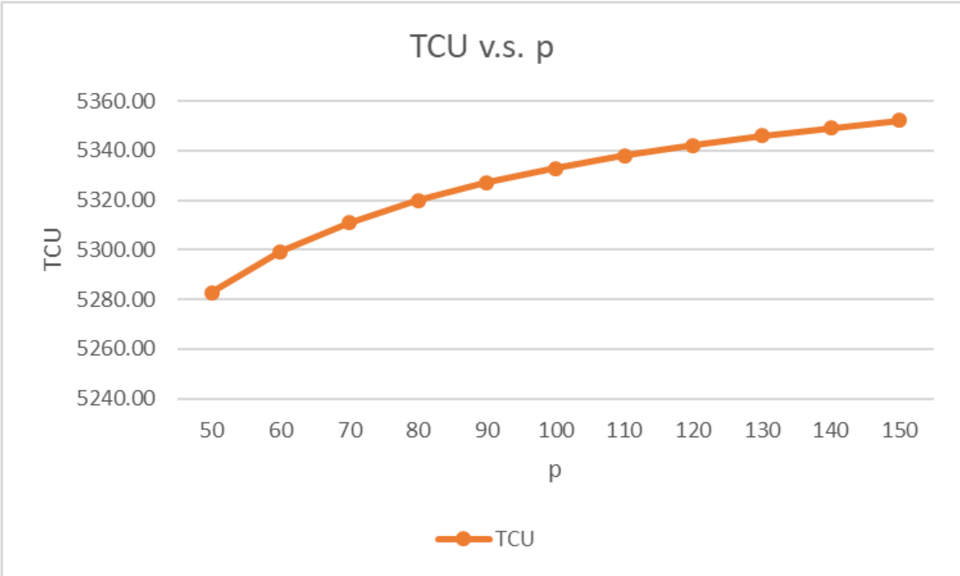

Figure 45: Variation of TCU_R with p

As in the case of any cost parameter, the total cost increases with the increase of the shortage cost, since any lost order will cost more money.

5.3 Cost Comparison

The main goal of extending the continuous review model was to try to reduce the total cost, that's why we present next a cost comparison between the models upon varying the cost parameter, to determine which model hold more cost savings in each case. We start by comparing the classical with the buffer model, and then the classical with the rush orders model, both graphically and analytically.

5.3.1. Graphical Analysis

<u>5.3.1.1. Classical vs. Buffer Models</u>

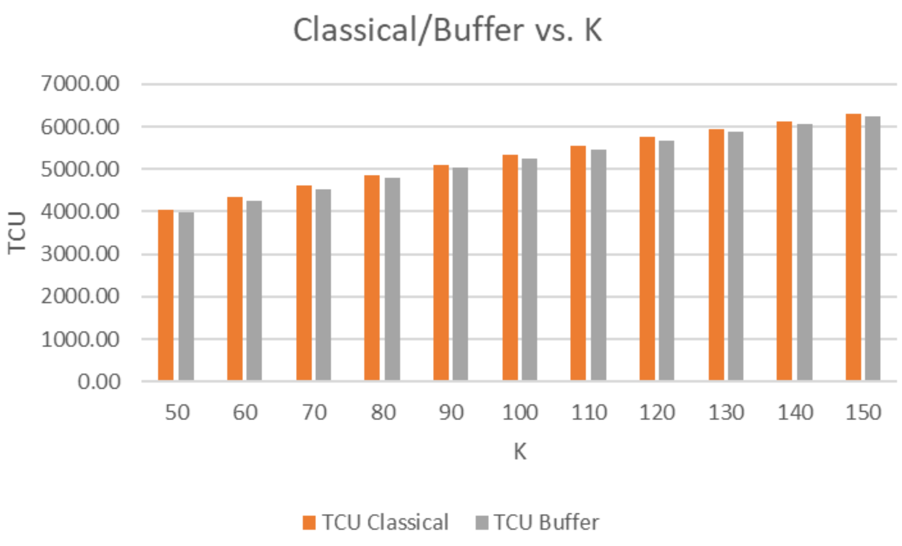

Figure 46: Classical/Buffer vs. K

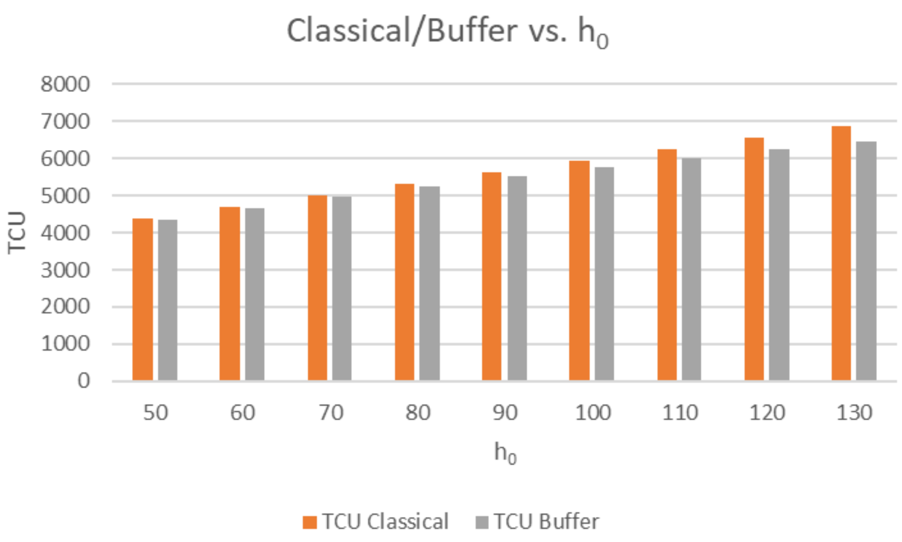

Figure 47: Classical/Buffer vs. h_0

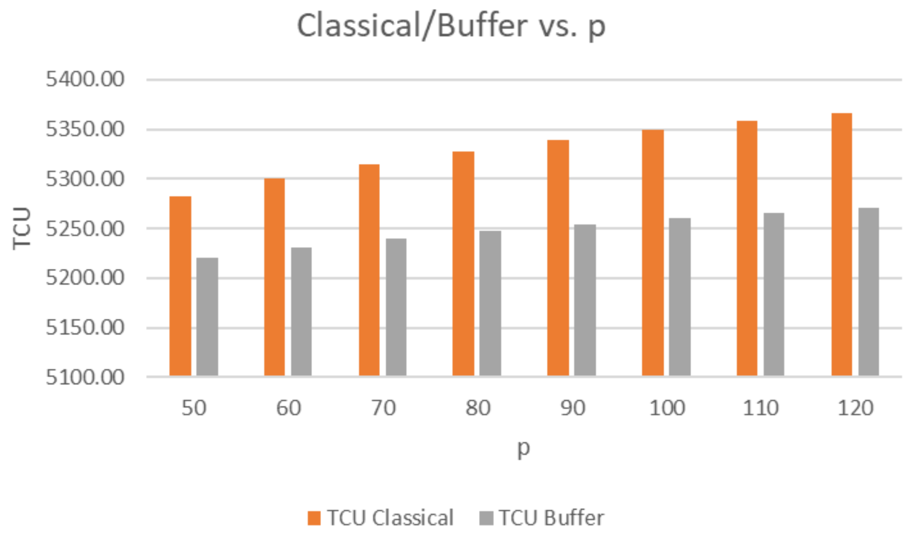

Figure 48: Classical/Buffer vs. p

Figures 46, 47 and 48 show the comparison of the buffer and classical models' costs when changing the parameters that are not directly related to the buffer stock, i.e. *K*, h_0 and *p*. What we can notice from these graphs is that the buffer model behaves better than the classical model in terms of cost for all increasing values of the designed parameters.

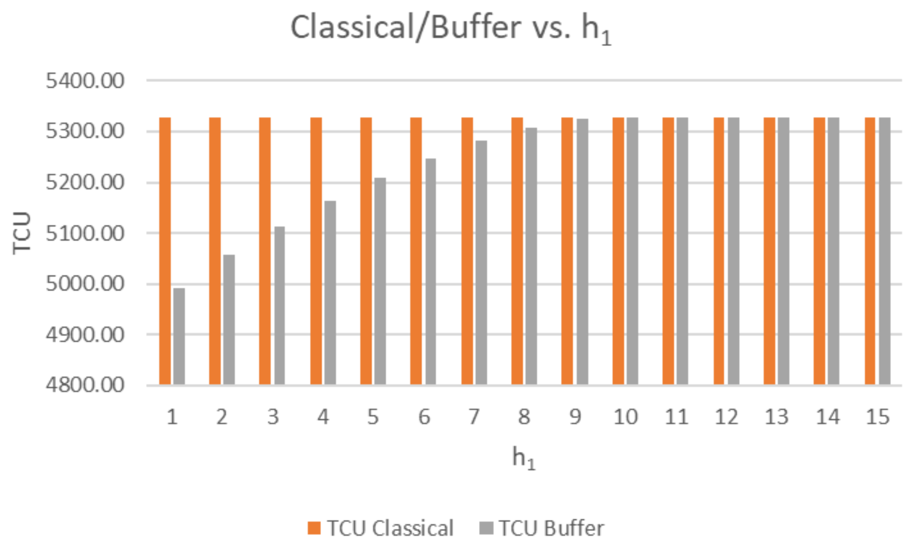

Figure 49: Classical/Buffer vs. h_1

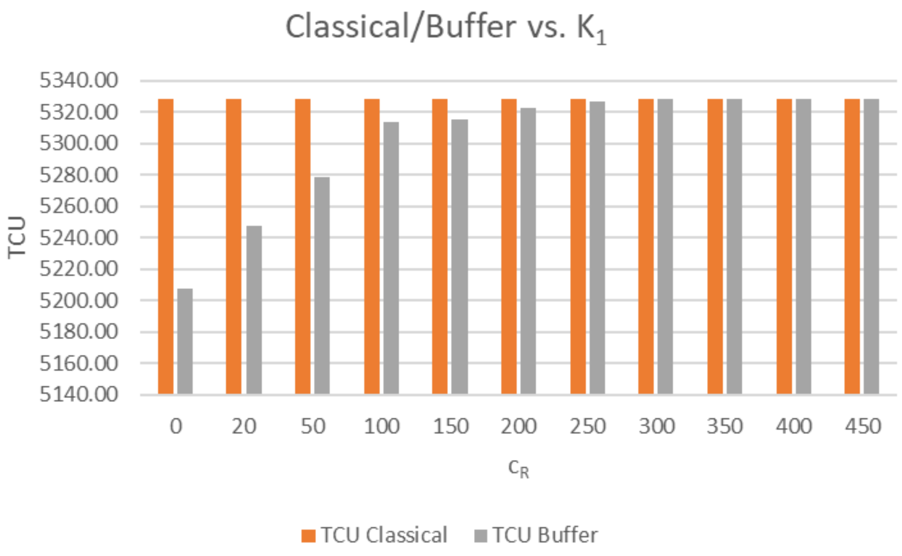

Figure 50: Classical/Buffer vs. K_1

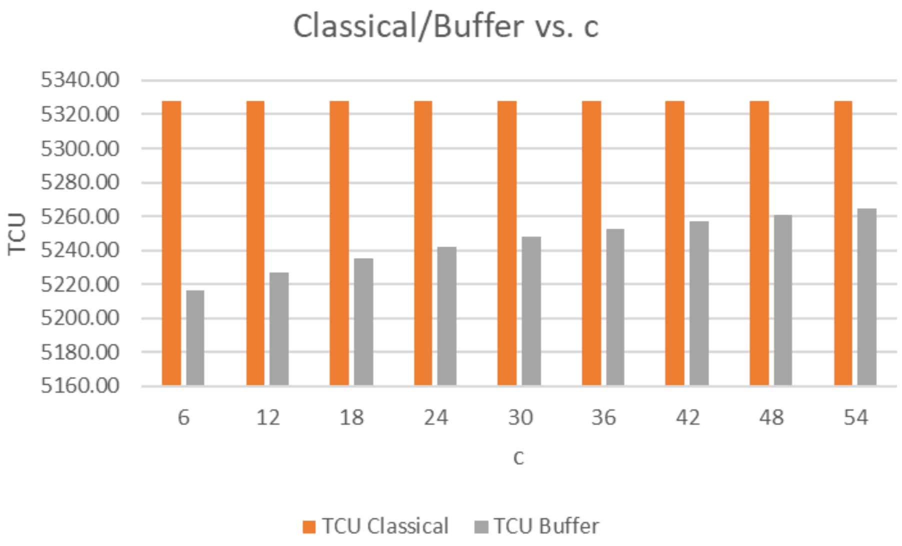

Figure 51: Classical/Buffer vs. c

In Figures 49, 50 and 51, we compare the costs of the classical and buffer models with respect to the cost parameters that are related to the buffer only, i.e. h_1, K_1 and c. As we can notice, when changing the buffer parameters, the total cost of the buffer model is always less than that of the classical, until they reach the same value when the buffer size reaches zero as we saw in the sensitivity analysis. For the case of the buffer replenishment cost *c* in Figure 51, the buffer model cost is continuously increasing and will also eventually reach the same value of the classical model's cost for a large *c* approximately equal to $630/unit (value obtained from experimenting with the optimal solution using Excel solver).

5.3.1.2. Classical vs. Rush Models

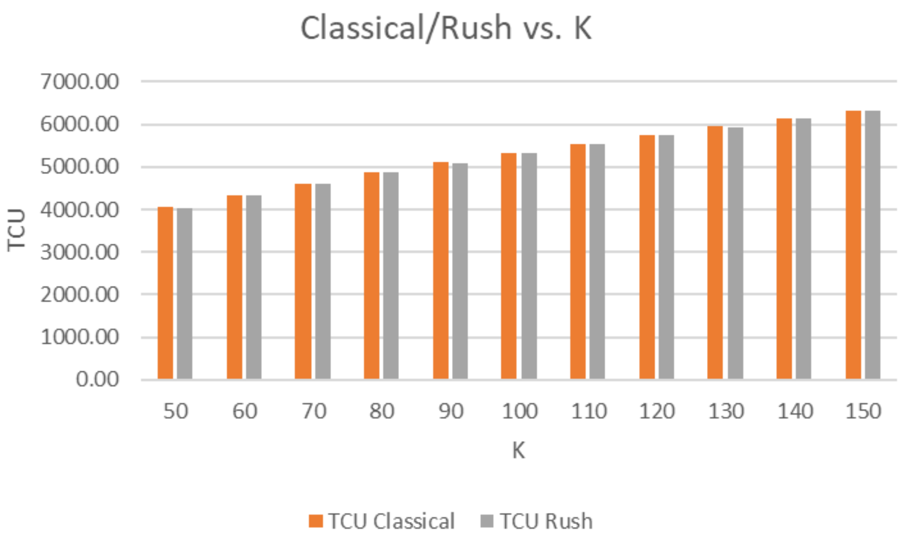

Figure 52: Classical/Rush vs. K

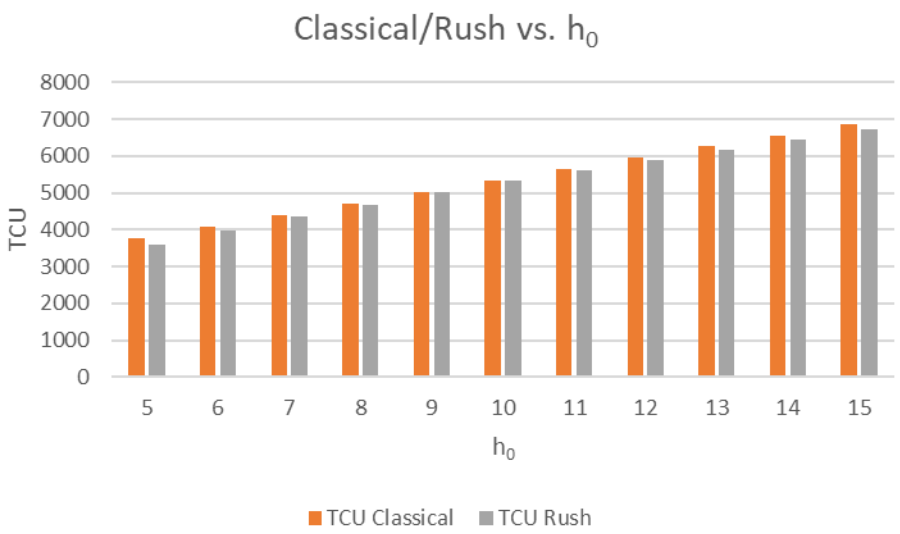

Figure 53: Classical/Rush vs. h_0

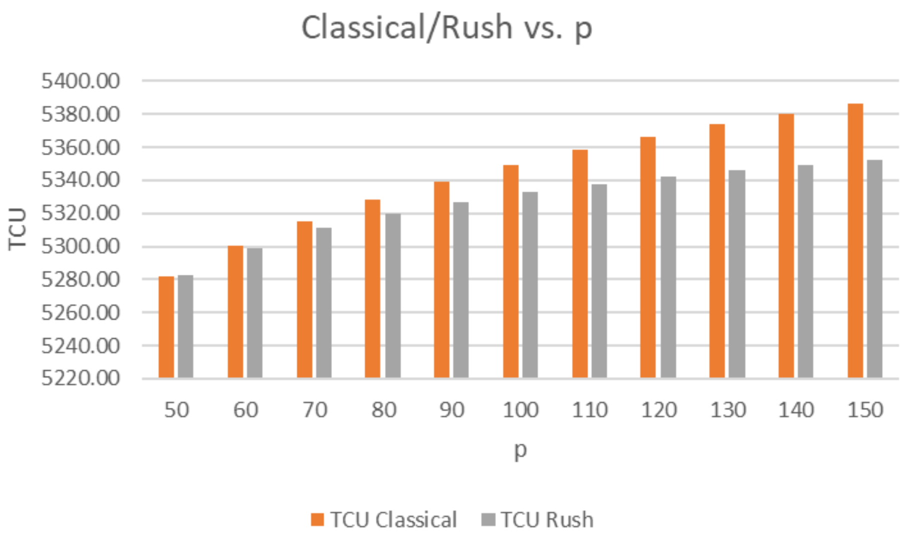

Figure 54: Classical/Rush vs. p

As in the buffer model, the cost of the rush orders model is less than that of the classical model in all cases presented above when varying the cost parameters h_0, K and p in Figures 52, 53 and 54.

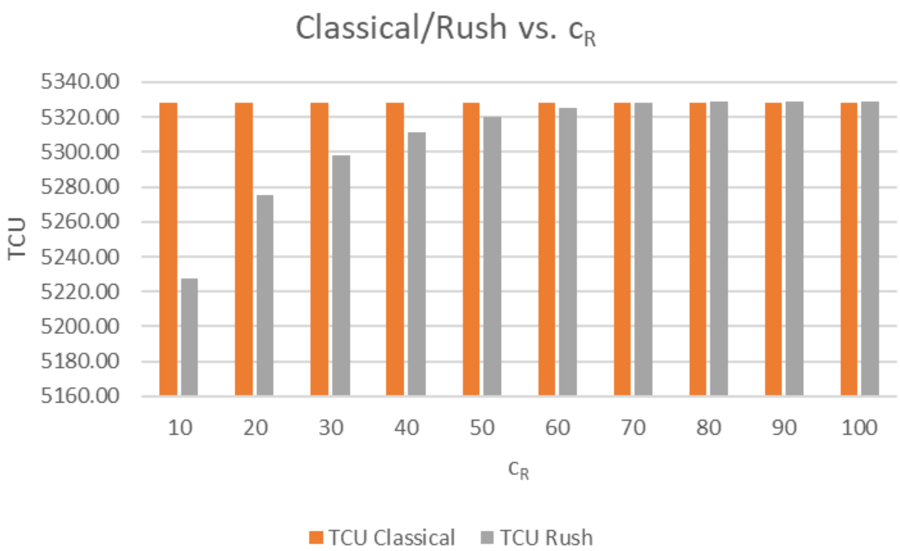

Figure 55: Classical/Rush vs. c_R

Figure 55 shows that the rush orders model has always a lower cost than the classical model, until they both reach the same cost value when the model converges to the classical one at $c_R = 80$.

In conclusion, the graphical analysis showed that the rush and buffer models always have a cost lower than or equal to that of the classical model, and never higher, since whenever the cost of these models becomes high, they converge to the classical model.

5.3.2. Analytical Analysis

5.3.2.1. Classical vs. Buffer Models

We recall the total cost formula of the classical model to be

$$TCU_0(y,R) = K\frac{D}{y} + h_0\left(\frac{y}{2} + R - E[X]\right) + \left(p\frac{D}{y}\right)\int_R^{\infty}(x-R)f(x)dx$$

The total cost formula for the buffer model is

$$\begin{aligned} TCU_B(y,R,B) = &\left[K + K_1\int_R^{R+B} f(x)dx\right]\frac{D}{y} + h_0\left(\frac{y}{2} + R - E[X]\right) \\ &+ h_1\left(B - \frac{1}{2y}\int_R^{R+B}(x-R)^2 f(x)dx - \frac{B}{2y}\int_{R+B}^{\infty}(2x - B - 2R)f(x)dx\right) \\ &+ p\frac{D}{y}\int_{R+B}^{\infty}[x-(R+B)]f(x)dx + c\left[\int_R^{R+B}(x-R)f(x)dx + \int_{R+B}^{\infty}Bf(x)dx\right] \end{aligned}$$

For simplification reasons, we will ignore the drop in buffer (which does not affect the result much as tested numerically in Excel). The total cost formula becomes

$$\begin{aligned} TCU_B = &\left[K + K_1\int_R^{R+B} f(x)dx\right]\frac{D}{y} + h_0\left(\frac{y}{2} + R - E[X]\right) + h_1 B \\ &+ p\frac{D}{y}\int_{R+B}^{\infty}[x-(R+B)]f(x)dx + c_B\left[\int_R^{R+B}(x-R)f(x)dx + \int_{R+B}^{\infty}Bf(x)dx\right] \end{aligned}$$

To compare the costs analytically, we will subtract the formulas from each other as follows.

$$TCU_0 - TCU_B = K\frac{D}{y} + h_0\left(\frac{y}{2} + R - E[X]\right) + \left(p\frac{D}{y}\right)\int_R^\infty (x-R)f(x)dx$$

$$-\left\{\begin{aligned}&\left[K + K_1\int_R^{R+B} f(x)dx\right]\frac{D}{y} + h_0\left(\frac{y}{2} + R - E\left[X\right]\right) + h_1B \\ &+ p\frac{D}{y}\int_{R+B}^\infty [x-(R+B)]f(x)dx + c_B\left[\int_R^{R+B}(x-R)f(x)dx + \int_{R+B}^\infty Bf(x)dx\right]\end{aligned}\right\}$$

$$TCU_0 - TCU_B = p\frac{D}{y}\left[\int_R^\infty (x-R)f(x)dx - \int_{R+B}^\infty [x-(R+B)]f(x)dx\right]$$

$$-\left\{\frac{K_1D}{y}\int_R^{R+B} f(x)dx + h_1B + c_B\left[\int_R^{R+B}(x-R)f(x)dx + \int_{R+B}^\infty Bf(x)dx\right]\right\}$$

The above expression signals that whenever the shortage cost is greater than the costs of the buffer model combined (ordering + holding + replenishment), the expression $TCU_0 - TCU_B$ will be positive, meaning that the total cost of the classical model will be greater than that of the buffer.

This result is logical, since the role of the buffer stock is to reduce shortages while minimizing the total cost, so whenever the cost of adding a buffer will exceed the cost of shortages, the use of the buffer stock would not be justified anymore and we will simply abide by the classical model. This is the analytical results conform with the graphical results that the cost of the buffer model is always less than or equal to that of the classical model.

5.3.2.2. Classical vs. Rush Models

We recall the total cost formula of the model with rush orders to be

$$TCU_R(y,R,W) = K\frac{D}{y} + h_0\left[\frac{y}{2} - \int_0^R (x-R)f(x)dx - \int_R^{R+W} (x-R-W)f(x)dx\right]$$
$$+p\int_{R+w}^{\infty}(x-R-w)f(x)dx\left(\frac{D}{y}\right) + c_R W\int_R^{\infty} f(x)dx\left(\frac{D}{y}\right)$$

For simplification reasons, we will replace the expression of the holding cost in the rush order model by the expression in the classical model, that gives the same result numerically as discussed previously, but the expression differs due to the approximation applied in the classical model.

The formula for the rush order model will now be

$$TCU_R(y,R,W) = K\frac{D}{y} + h_0\left(\frac{y}{2} + R - E[X]\right) + p\int_{R+W}^{\infty}(x-R-W)f(x)dx\left(\frac{D}{y}\right) + c_R W\int_R^{\infty} f(x)dx\left(\frac{D}{y}\right)$$

By subtracting the total cost of the rush orders model from the total cost of the classical model we get

$$TCU_0 - TCU_R = K\frac{D}{y} + h_0\left(\frac{y}{2} + R - E[X]\right) + \left(p\frac{D}{y}\right)\int_R^{\infty}(x-R)f(x)dx$$
$$-\left\{\begin{array}{l} K\frac{D}{y} + h_0\left[\frac{y}{2} - \int_0^R (x-R)f(x)dx - \int_R^{R+W}(x-R-W)f(x)dx\right] \\ +p\int_{R+w}^{\infty}(x-R-w)f(x)dx\left(\frac{D}{y}\right) + c_R W\int_R^{\infty} f(x)dx\left(\frac{D}{y}\right) \end{array}\right\}$$

$$TCU_0 - TCU_R = p\frac{D}{y}\left[\int_R^{\infty}(x-R)f(x)dx - \int_{R+w}^{\infty}(x-R-w)f(x)dx\right] - c_R W\int_R^{\infty} f(x)dx\left(\frac{D}{y}\right)$$

The same analysis applies here as in the case of the buffer model. Whenever the shortage cost is greater than the cost of the rush orders, the expression $TCU_0 - TCU_R$ will be positive, meaning that the total cost of the classical model will be greater than that of the rush orders model. When the cost of placing a rush order will be greater than the cost of experiencing shortages, the rush order alternative will not be used anymore and the model and its cost will converge to the classical model. Therefore, we can also say that the cost of the rush orders model is also always less than or equal to that of the classical model.

CHAPTER VI

ANALYTICAL APPROXIMATION FOR DERIVING THE OPTIMAL SOLUTION

One way to solve for the decision variables is by using certain approximations and solving for the optimal values. One approximation we can base our work on is the "probabilitized" EOQ model developed by Taha, H. (2006) in his book "Operations Research: An Introduction". The developed expressions for the order size *y* and the reordering point *R* based on the standard EOQ model are presented below.

$$y = \sqrt{\frac{2KD}{h}} \quad \text{and} \quad \int_{R}^{\infty} f(x) = \frac{hy}{pD}$$

The main assumption in the model is that the lead-time demand is normally distributed.

In our case, we can approximate the *y* and *R* values from the above expressions, and by setting the first-order derivatives of the total cost formula for the buffer and rush models to zero, we get values for *B* and *W*.

The first-order derivatives for the buffer model with respect to the three decision variables are

$$\frac{\partial TCU_B}{\partial y} = -\left[K + K_1 \int_{R}^{R+B} f(x)dx\right]\frac{D}{y^2} + \frac{h_0}{2} - p\frac{D}{y^2}\int_{R+B}^{\infty}[x-(R+B)]f(x)dx$$

$$\frac{\partial TCU_B}{\partial R} = \frac{K_1 D}{y}\left[f\left(R+B\right) - f(R)\right] + h_0 - p\frac{D}{y}\int_{R+B}^{\infty} f(x)dx - c\int_{R}^{R+B} f(x)dx$$

$$\frac{\partial TCU_B}{\partial B} = \frac{K_1 D}{y} f\left(R+B\right) + h_1 + \left(-\frac{pD}{y} + c\right)\int_{R+B}^{\infty} f(x)dx$$

The first-order derivatives for the rush orders model with respect to the three decision variables are

$$\frac{\partial TCU_R}{\partial y} = -K\frac{D}{y^2} + \frac{h_0}{2} - \frac{pD}{y^2}\int_{R+W}^{\infty}(x-R-W)f(x)dx - \frac{c_R WD}{y^2}\int_R^{\infty} f(x)dx$$

$$\frac{\partial TCU_R}{\partial R} = h_0\left[\int_0^R f(x)dx - Wf(R) + \int_R^{R+W} f(x)dx\right] - \frac{pD}{y}\int_{R+W}^{\infty} f(x)dx - c_R W\frac{D}{y}f(R)$$

$$\frac{\partial TCU_R}{\partial W} = h_0\left[\int_R^{R+W} f(x)dx - Wf(R)\right] - \frac{pD}{y}\int_{R+W}^{\infty} f(x)dx + \frac{c_R D}{y}\int_R^{\infty} f(x)dx$$

It is worth mentioning that in our models, we would get a more accurate value for the rush order size *W* than the buffer size *B*, since the reordering point value in the rush orders model is closer to the approximated value than in the buffer model.

To solve for *W* analytically, using the probabilitized expressions, we first set *y* and *R* to their approximated values, which are respectively 447.21 and 476.11. Then we consider one of the derivatives for the total cost formula of the rush orders model, let's say the derivative with respect to *W*, and we use the "Goal Seek" function in Excel to get the *W* value that sets that derivative to zero. By solving it, we get *W* = 4.82 which is almost equal to the value we got from the solver and which is 4.83. The total cost of the rush orders model using these values is 5321.28, which is very close to the value obtained using Solver equal to 5319.86.

To solve for *B*¸ we follow the same procedure as above. Table 2 below compares the results obtained for the two models using the Solver and using the analytical approximation.

Table 2

Comparison between Solver and Analytical Values for Both Models

	B/W		*TCU*		% change in *TCU*
	Solver	Analytical	Solver	Analytical	
Buffer Model	**36.214**	**4.794**	**5247.79**	**5321.07**	**1.39%**
Rush Orders Model	**4.83**	**4.82**	**5319.86**	**5321.28**	**0.027%**

As we can see, the approximated analytical solution can be beneficial especially in the rush orders model, where it presented very accurate results with only a slight 0.027% difference in total cost. However, for the buffer model, there was a larger difference of 1.39% in the total cost values, due to the difference in the reordering point values as already stated. If we want to use it for the buffer model, it is recommended to decrease the value of the reordering point by about 10% to get a more accurate value for *B*.

We can also notice how the total cost values for the buffer and rush orders models are almost the same in the analytical solution, probably resulting from having the same ordering policy in terms of *y* and *R*.

CHAPTER VIII

CONCLUSION AND RECOMMENDATIONS

In this study, we presented two extended models, one with a buffer stock and another with a rush orders option, for the continuous review inventory system that is widely used in inventory management. Our main goal was to present models that are effective in reducing shortages and losses, while also minimizing the total cost of the system. Our approach was robust and provided us with interesting and promising results, where both models were proven to be effective in finding the optimal policy for inventory management, while presenting cost reductions compared to the classical model.

While both models are effective and cost-saving, the choice of one or another depends on the preference of the decision makers, especially concerning the availability of an external warehouse for buffer storage or not. The percent reduction in cost would vary depending on the values of the cost parameters, which we logically estimated in our model based on industry standards and previous references.

The developed models could be further extended as future work to account for several aspects, from which we can mention the following:

- In some cases, the classical continuous review model does not have a feasible solution, depending on the parameter values; it is worth studying whether this applies also for the extended models and if there is cases where they do not give optimal solutions.

- Though the normal distribution is the most frequently used for the lead-time demand, it is worth applying other types of distributions to get the optimal solution and cost.
- The model with rush orders can be modified to account for a short lead-time instead of zero lead-time for the rush orders. Moreover, it can be stated that more than one rush order can be initiated per cycle, as opposed to our assumptions.
- The analytical approach for finding the optimal solution can be further developed to come up with accurate results especially for the buffer model.

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