AMERICAN UNIVERSITY OF BEIRUT

Characteristic Mode Solution of Complex-Coefficient Complex-Solution Differential Equations

by
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An Abstract of the Thesis of

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Computation of complex coefficient complex solution differential equations has been a problem that arises in various domains of science and engineering, and has thus received much attention in years, with marked recent progress. This thesis aims at applying the Theory of the Characteristic Mode (TCM) approach along with the Method of Moments (MoM) in solving these problems with emphasis on procedures for higher order differential equations. Several methods, known in literature, are available for solving the problem and their complexity differs based on the accuracy of the solution. The general method is presented and a simplified version of it is adopted to address this problem considering first, third, and forth orders complex-coefficients complex solution differential equations. These examples are considered to clarify the approach and present the simplicity of the method. As illustrated in this proposal, The approach can be introduced along with other methods, and can be considered as an attractive way to solve these special occurrences differential equations and other boundary value problems

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Chapter 1

Introduction

This thesis introduces the Theory of Characteristic Mode (TCM) along with the Method of Moment (MoM) for numerically computing the solution of a complex-coefficient complex-solution differential equation. Numerical methods for finding these solutions are of great importance and of practical values in various branches of applied mathematics, science, and engineering. Over the years, the subject has been an active research area, and many solutions have been proposed [1-6] for the solution of the problem.

The Characteristic Mode Theory (TCM) for either conducting body [7] or Apertures [8], are simple and structured methods to solve integro-differential complex equations and proven, as demonstrated by the authors, to have good convergence. The simplest, which is the characteristic mode theory for apertures, has been applied successfully to many engineering problems including slots in a conducting plane [9-13], slots in a conducting cylinder [14-17], waveguide problem [18], rectangular apertures [19], and others [20-26]. In short, TCM found a great characteristic as it lies in its ability to fully characterize the radiation and scattering properties of an arbitrary object based only on the objects geometry and material properties.

This property of TCM has also extended to other applications and was proven to provide valuable insights into an antennas behavior independent of the feeding arrangement as well as providing information about how desirable radiation modes can be excited [26]. This feature has led to its use to design integrated antennas in the High Frequency (HF) band for land vehicles, ships and aircraft. However, TCM had largely remained a specialist field until it was rediscovered for aiding the design of mobile handset antennas. In particular, TCM provides a powerful tool to understand and exploit excitation of the terminal chassis to enhance antenna performance. Another powerful feature of TCM is that multiple characteristic modes at a given frequency facilitate orthogonal radiation patterns, which provide effective Multiple-Input Multiple-Output (MIMO) antennas [27].

Chapter 2

Literature Review

The objective of this literature survey is to give a general overview about some important concepts related to the thesis work. We start by giving an overview about MoM, it is good to introduce them to new terms such as expansion and testing. This is because the MoM method starts by expanding the unknown quantity, which is to be solved for, into a set of known functions with unknown coefficients. The resulting equation is then converted into a linear system of equations by enforcing the boundary conditions at a number of points. This resulting linear system is then solved analytically for the unknown coefficients. It is here to note that such an approach is very simple and quite interesting when applied to differential equation of order less than 3, but it is applicable for equations of higher order.

As readers of this proposal may not be familiar with radiation and scattering problem, TCM can simply be described as a numerical procedure based on dividing the general solution of a problem into a set of known solutions with different weights. The orthogonality property of the set function, derived in the procedure, is used to determine the weighted sum and hence a general solution is obtained. As dealing with complex quantities, the solution of a complex-coefficient complex function is complex by nature and hence it can be divided into a real part and an imaginary part. This division, once implemented in the differential equation under solution, will result into two interconnected equations, whose solution using the Method of Moment (MoM) will lead to an eigenvalue problem subject to be solved under the domain of the problem. Once these eigenmodes and eigenvalues are obtained, a general solution of the problem is then found.

The Method of Moments (MoM) [28] is a computational method based on, first discretizing the equations into small segments, then employing known basis functions with unknown coefficients and using the Galerkins solution to produce a large matrix of equations. The Galerkin method employed in the MoM is defined by using the same functions for both expansion and testing. The most delicate

part of this method resides in the fact that the selection of the basis and testing functions is unlimited, however, it highly affects the accuracy of the solution. The idea is to use basis functions to expand the solutions into sets and solving for a null residual error with sets of weighting functions.

The MoM is used to find numerical solutions for applications related to the computational electromagnetic field [29]. Compared to other numerical techniques, the MoM is characterized by its accuracy and high speed. In electromagnetic problems, some metals could have complex and frequency dependent permittivity which increase the number of unknowns to the solver. It could require high computational time and large memory requirements because these are always increasing proportionally to the electrical size of the integral equation. The solution for that is to combine the MoM with other methods to get higher speed and better accuracy.

The MoM is joint with the gradient method to create a novel antenna design method based on the representation of an antenna structure by a combination of small segments. It is joint with the unscented transform in [30] to model the uncertainty that could occur in electromagnetic computations, with same accuracy but less simulation than the Monte Carlo approach. The authors of [31] also used the MoM to analyze effectively the uncertainties of the geometric shape of 3D objects presented by multiple random variables in the integral equations. In [31], an improved MoM is proposed to analyze the scattering properties of multi-object electromagnetic systems while adopting an iterative process to increase the accuracy of the method used. The authors of [32] proposed the use of the wavelet packet transform with the MoM in the purpose of reducing the computation time and the storage capacity when dealing with large array elements.

In [33], the MoM was used to find the modal characteristics such as the attenuation and phase constants of plasmonic transmission lines. The starting point of the algorithm proposed in [34] is the MoM along with the Sherman-Morrison-Woodbury identity; it performs an investigation of nearest neighbors on a Hamming graph. The MoM is also used in [35] to solve a time-domain integral equation applied to a wire-grid model of a conducting body.

This thesis will first outline the problem; derive a general solution where the unknown function is expanded into a sum of weighted basis functions, where the weight coefficients are to be found. The Galerkin method, which selects testing functions equal to the basis functions, is adopted. The problem then becomes a system of linear equations, which is solved analytically or numerically to find the needed weight coefficients. Simple examples using first, third, and forth orders complex-coefficients differential equations are then considered for illustration.

Chapter 3

Problem Formulation

For readers to be familiarized with MoM, it is good to introduce them to new terms such as expansion and testing. This is because the MoM method starts by expanding the unknown quantity, which is to be solved for, into a set of known functions with unknown coefficients. The resulting equation is then converted into a linear system of equations by enforcing the boundary conditions at a number of points. This resulting linear system is then solved analytically for the unknown coefficients. It is here to note that such an approach is very simple and quite interesting when applied to differential equation of order less than 3, but it is applicable for equations of higher order.

Accordingly, it is advisable to start with some basic mathematical techniques for reducing functional equations to matrix equations. A deterministic problem is considered, which will be solved by reducing it to a suitable matrix equation, and hence the solution could be found by matrix inversion. The examples that we will choose are simple and easily to illustrate the theory without any complicated mathematics. Linear spaces and operators will be used in our solution. At first, it is recommended to introduce TCM and define some terms related to first order non-homogeneous differential equation. The choice of this equation is important only for better understanding of the solution.

3.1 General Formulation

A general pth order complex-coefficients linear differential equation, defined over a domain D, has the form

$$C_p \frac{\mathrm{d}^p f(x)}{\mathrm{d}x^p} + C_{p-1} \frac{\mathrm{d}^{p-1} f(x)}{\mathrm{d}x^{p-1}} + \cdots + C_1 \frac{df(x)}{dx} f(x) + C_0 f(x) = g(x). \qquad (3.1)$$

In (3.1), the coefficients $C_p, C_{p-1}, \cdots, C_1, C_0$, $f(x)$, and $g(x)$ are all complex quantities.

In addition, all coefficients and g(x) are known functions and f(x) is the function whose solution is to be determined.
(3.1) is called the operator equation and can be written as:

$$L(f(x)) = g(x). \qquad (3.2)$$

In (3.2), L is the operator, operating on $f(x)$, and given by:

$$L = C_p \frac{\mathrm{d}^p f(x)}{\mathrm{d}x^p} + C_{p-1} \frac{\mathrm{d}^{p-1} f(x)}{\mathrm{d}x^{p-1}} + \cdots + C_0 f(x). \qquad (3.3)$$

In (3.3), p is the order of the differential equation.
As $f(x)$ is usually a complex-valued solution, and the coefficients $C_p, C_{p-1}, \cdots, C_1, C_0$ are complex, the operator in (3.3) can be defined, according to the TCM theory, as the sum of a real operator G and an imaginary operator B. Each operator is given by:

$$L(f(x)) = G(f(x)) + jB(f(x)). \qquad (3.4)$$

where

$$G(f(x)) = \text{The Real part of } g(x) = Re\{g(x)\}, \qquad (3.5)$$

and

$$B(f(x)) = \text{The Imaginary part of } g(x) = Im\{g(x)\}. \qquad (3.6)$$

These operators are determined from the given differential equation. In the next step, the domain of solution is divided into N segments of equal length, and the function f(x) is now defined over the n intervals of definition as follows:

$$f(x) = \sum_{n=1}^{N} i_n f_n(x), \qquad n = 1, 2, \cdots, N. \qquad (3.7)$$

In (3.7), f_n is the expansion function and it is a real-valued function defined over the nth interval and satisfy the boundary condition of the problem, i_n is its unknown coefficient to be determined and N is the total number of expansion

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functions. The method of Moments (MoM) with Glisson basis functions, used together with the model decomposition formulated by Harrington that implements the direct solution by MoM and the modal decomposition in terms of characteristic modes. Both methods work with the same discretization, and therefore with the same impedance matrix,Z.

The discretized impedance opertaror Z can be separated as: Z= R+jX

where R and X have to be real and symmetrical. Since both operators B and G are linear, the first equation (3.3) could be written as:

$$L\left(f(x)\right) = C_n \sum_{i=1}^{N} V_i \frac{\mathrm{d}^n(f_i(x))}{\mathrm{d}x^n} + C_{n-1} \sum_{i=1}^{N} V_{i-1} \frac{\mathrm{d}^{n-1}(f_{i-1}(x))}{\mathrm{d}x^{n-1}} + \cdots \qquad (3.8)$$

where C_n represents the complex coefficients that should be determined, V_i represents the multiplication of $[i_n * f_n]$, also the boundary conditions from 1 to N (N is chosen according to our need)

3.2 Method of Solution

Denoting $O(.)$ any of the two operators defined in (3.4) and g be the right side of equation (3.2), the problem now is to solve the equation

$$O(f) = g \Rrightarrow O(f) - g = 0. \tag{3.9}$$

In (3.9), $O(f)$ is a linear integro-differential operator, f is an unknown function to be solved and g is a known function. To find (f), it should be approximated by a sum of weighted basis or expansion functions as follows:

$$f \approx \sum_{n=1}^{N} i_n f_n, \qquad n = 1, 2, \cdots, N. \tag{3.10}$$

Where f_n is the expansion function, i_n is its unknown coefficients to be determined and N is the total number of expansion functions.Since the operator O is linear,the first equation could be written as:

$$O\left(\sum_{n=1}^{N} i_n f_n\right) \approx g. \tag{3.11}$$

Where the residual is given by:

$$R = g - O\left(\sum_{n=1}^{N} i_n f_n\right). \tag{3.12}$$

The residual or the error has to be equal to zero to get best and most accurate solution. Therefore, the next step is to make the inner product of the above equation with the testing or weighting functions called w_m which should be similar to f_n as imposed by the method of Galerkin. The inner product is defined as:

$$\langle C, D \rangle = \int_{\substack{\text{domain of} \\ \text{solution}}} C^*(x).D(x)\mathrm{d}x. \tag{3.13}$$

Applying (3.13), we obtain

$$< w_m, (O(f_n) - g) >= 0. \qquad m = 1, 2, \cdots, N. \tag{3.14}$$

Since i_n is a constant, it could be taken outside of the inner product and the inner product could be rewritten as:

$$\sum_{n=1}^{N} i_n < W_m, O(f_n) >=< w_m, g > . \qquad n = 1, 2 \cdots, N. \tag{3.15}$$

This could be written in matrix form as follows:

$$[Z][I] = [V]. \tag{3.16}$$

With matrix Z defined as:

$$\begin{bmatrix} \langle f_1, O(f_1) \rangle & \cdots & \langle f_1, O(f_N) \rangle \\ \vdots & \ddots & \vdots \\ \langle f_M, O(f_1) \rangle & \cdots & \langle f_M, O(f_N) \rangle \end{bmatrix} \tag{3.17}$$

And vectors I and V as follows:

$$[I] = \begin{bmatrix} i_1 i_2 \cdots i_N \end{bmatrix}^T. \tag{3.18}$$

$$[V] = [< g, w1 >< g, w2 > \cdots < g, w_M >]^T \tag{3.19}$$

It is then obvious that the vector of unknown coefficients. It could be solved as follows:

$$[I] = [Z]^{-1}[V]. \tag{3.20}$$

Chapter 4

Example of a Real Function

In this section, the presented solution will be applied to real coefficient real solution differential equation and to complex-coefficients complex-solution differential equations.

4.1 Solution to Real Coefficient Real Solution differential Equation

Considering this real differential equation:

$$-\frac{d^2 f(x)}{dx^2} = 3 + 2x^2. \tag{4.1}$$

subject to the boundary conditions $f(0) = f(1) = 0$. The operator used in the calculations below is $O = -\frac{d^2 \cdot}{dx^2}$ and the known function is $g(x) = 3 + 2x^2$. Using the Galerkins MoM, the solution for this equation is found by following the steps below:

1 - Choose the basis function as $f_n(x) = x - x^{n+1}$ for n=1,2,...,N to satisfy both boundary conditions $f(0) = f(1) = 0$. Where f_n should satisfy the order of differential equation.

2 -Write the approximated expansion of the unknown function as:

$$f(x) \approx \sum_{n=1}^{N} i_n f_n = i_1 f_1(x) + i_2 f_2(x) = i_1(x - x^2) + i_2(x - x^3). \tag{4.2}$$

Assuming N=2 is the total number of subsections on the interval [0,1]. For Galerkins MoM, choose the weighting functions as:

$$w_m(x) = x - x^{m+1} \qquad m = 1, 2, ..., M \tag{4.3}$$

For M=N=2, find the elements of the square matrix $[Z]$:

$$z_{11} = <w_1, O(f_1)> = \int_0^1 w_1(x)O(f_1(x))d_x = \int_0^1 (x - x^2)(2)dx = \frac{1}{3}. \tag{4.4}$$

$$z_{12} = <w_1, O(f_2)> = \int_0^1 w_1(x)O(f_2(x))d_x = \int_0^1 (x - x^2)(6x)dx = \frac{1}{2}. \tag{4.5}$$

$$z_{21} = <w_2, O(f_1)> = \int_0^1 w_2(x)O(f_1(x))d_x = \int_0^1 (x - x^3)(2)dx = \frac{1}{2}. \tag{4.6}$$

$$z_{22} = <w_2, O(f_2)> = \int_0^1 w_2(x)O(f_2(x))d_x = \int_0^1 (x - x^3)(6x)dx = \frac{4}{5}. \tag{4.7}$$

Find the elements of the vector $[V]$:

$$V_1 = <g, w_1> = \int_0^1 g(x)w_1(x)d_x = \int_0^1 (3 + 2x^2)(x - x^2)dx = \frac{3}{5}. \tag{4.8}$$

$$V_2 = <g, w_2> = \int_0^1 g(x)w_2(x)d_x = \int_0^1 (3 + 2x^2)(x - x^3)dx = \frac{11}{12}. \tag{4.9}$$

Replace the values found and solve for $[I]$

$$[Z][I] = [V] \Rightarrow \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{11}{12} \end{bmatrix}. \tag{4.10}$$

(4.10) is solved and the results are obtained.

$$[I] = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \frac{13}{10} \\ \frac{1}{3} \end{bmatrix} \tag{4.11}$$

Finally, find the unknown function f(x)

$$f(x) \approx i_1(x - x^2) + i_2(x - x^3) = \frac{13}{10}(x - x^2) + \frac{1}{3}(x - x^3). \tag{4.12}$$

All the above steps are done using a MatLab script and both, the analytical solution for this differential equation and the solution found by the MoM, are plotted. Figure 4.1 below represents a graph for analytical solution. Figure 4.2 represents the two plots which clearly show that f(x) found by the MoM is the same as

that found by MatLab. There are functions that MATLAB calls implicitly when you perform certain actions with objects You can change how user-defined objects behave by defining methods that control specific behaviors. To change a behavior, implement the appropriate method with the name and signature of the MATLAB function, so dsolve equation was the best command to define the differential equation with its condition which will tell us also the coefficients that we need to represents.

Some differential equations are not as well-behaved, and show singularities due to a failure to model the problem correctly, or a limitation of the model that was not apparent. Some DEs can be solved analytically in closed-form, but most have to be approximated by numerical procedures, which can be unstable.

Ordinary Differential Equations (ODEs), in which there is a single independent variable and one or more dependent variables, can be solved using DSolve.

Partial Differential Equations (PDEs) have two or more independent variables and one dependent variable. Finding exact symbolic solutions of PDEs is a difficult problem, but DSolve can solve most first-order PDEs and a limited number of the second-order PDEs.
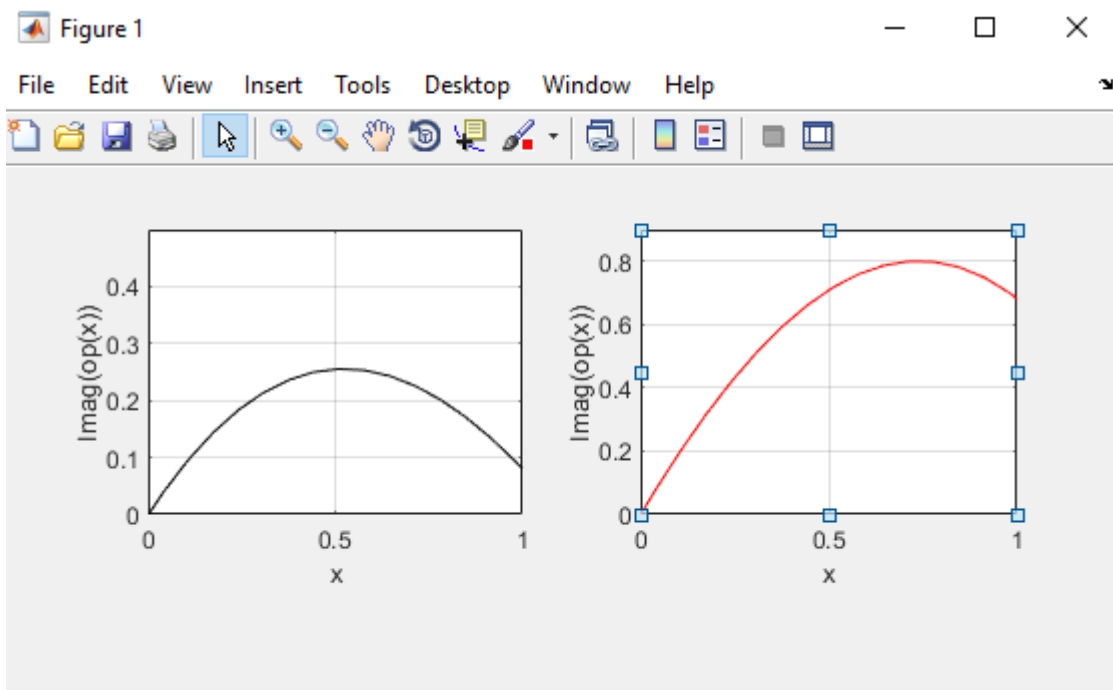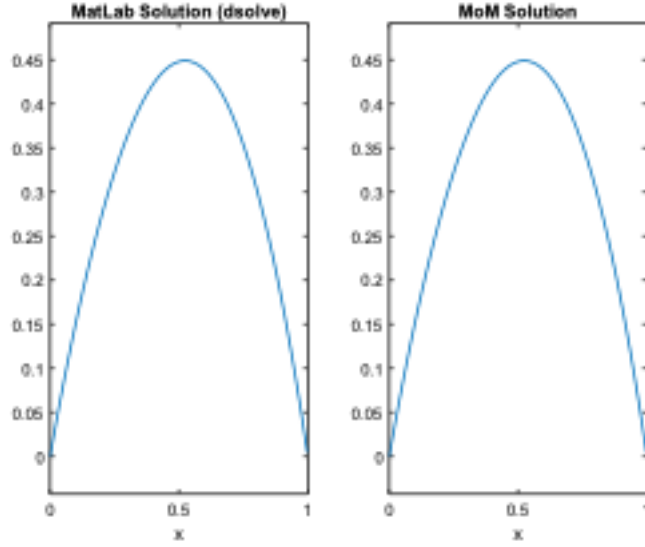

Figure 4.1: Analytical Solution

Figure 4.2: Real Function Solution

4.2 Solution to Complex Function Complex Coefficient Differential Equation

Similarly, consider the following complex-coefficient complex-solution differential equation:

$$O(f) = g \Rightarrow O(f) - g = 0. \tag{4.13}$$

Where O is a linear integro-differential operator, f is an unknown complex function to be solved and g is a known complex function. As dealing with complex quantities, the solution of a complex-coefficient complex-function is complex by nature and hence it can be divided into a real and an imaginary part. This division, and also the division of the operator into real and imaginary parts, once implemented in the differential equation under solution, will result in two interconnected equations, whose solution using the MoM will lead to a problem subject to be solved under the domain of definition. To find complex f, it should be approximated by a sum of weighted basis or expansion functions as follows:

$$f(x) \approx \sum_{n=1}^{N} i_n f_n = \sum_{n=1}^{N} a_n f_n + j \sum_{n=1}^{N} b_n f_n \qquad n = 1, 2, ..., N. \tag{4.14}$$

Where f_n is the expansion function, a_n are the unknown coefficients of the real part and b_n are the unknown coefficients of the imaginary part of f_n and N is the total number of expansion functions. a_n and b_n are to be determined. Op_{real} and Op_{imag} are the real and imaginary parts of the linear integro-differential

operator, respectively. Since the operator O is linear, the differential equation could be written as:

$$Op_{real}\left[\sum_{n=1}^{N} a_n f_n(x) + j\sum_{n=1}^{N} b_n f_n(x)\right] + jOp_{imag}\left[\sum_{n=1}^{N} a_n f_n(x) + j\sum_{n=1}^{N} b_n f_n(x)\right]$$
$$= g_R + jg_I \tag{4.15}$$

After expanding the equation, there exists two residual equations given by:

$$R(real) = g_R - Real(LHS). \tag{4.16}$$

$$R(imag) = g_I - Imag(LHS). \tag{4.17}$$

The next step is to make the inner product of the above equations with the testing function called w_m which should be equal to f_n as imposed by the method of Galerkin. In the examples below, the calculations are presented in details step by step.

4.3 First Order Example of a Complex Function Complex Coefficient Differential Equation

$$\frac{df(x)}{dx} + jf(x) = 2e^{jx} - jx^2 + x - 2. \tag{4.18}$$

The steps of the solution of this equation are presented below.

1- Identify the real and the imaginary parts of the operator

$$(f'_R + jf'_I) + j(f_R + jf_I). \tag{4.19}$$

where

$$Op_{real} = f'_R - f_I. \tag{4.20}$$

$$Op_{imag} = f'_I + f_R. \tag{4.21}$$

2- Identify the real and the imaginary parts of the right side function:

$$Real\{g(x)\} = 2\cos x + x - 2. \tag{4.22}$$

$$Imag\{g(x)\} = 2j\sin x - x^2. \tag{4.23}$$

3- Equate the real and the imaginary parts from both sides:

$$Op_{real} = Real\{g(x)\}. \tag{4.24}$$

$$Op_{imag} = Imag\{g(x)\}. \tag{4.25}$$

4- According to the Galerkins method, choose the same basis and weighting functions. It is to note that the chosen function must meet the boundary values of the problem. In this case, we select the following function:

$$f_n(x) = x^{n+1} \qquad\qquad n = 1, ..., N. \tag{4.26}$$

Then

$$f(x) = \sum_{n=1}^{N} a_n f_n(x) + j\sum_{n=1}^{N} b_n f_n(x). \tag{4.27}$$

and

$$w_m(x) = x^{m+1}. \tag{4.28}$$

5- Calculate the inner product of the previous equations with the weighting function chosen and apply the real and imaginary parts of the operator on the basis function chosen:

$$< (Op_{real}, w_m) >=< Real\{g(x)\}, w_m > . \tag{4.29}$$

$$< f'_R - f_I, w_m >=< 2\cos x + x - 2, w_m > . \tag{4.30}$$

Extending the real operator:

$$\sum_{n=1}^{N} a_n f_n'(x) - \sum_{n=1}^{N} b_n f_n(x). \tag{4.31}$$

Extending the real operator:

$$\sum_{n=1}^{N} a_n Op_1 - \sum_{n=1}^{N} b_n Op_2. \tag{4.32}$$

Taking the inner product of the above equation

$$\left\langle \left(\sum_{n=1}^{N} a_n Op_1 - \sum_{n=1}^{N} b_n Op_2 \right), w_m \right\rangle = \left\langle 2\cos x + x - 2, w_m \right\rangle. \tag{4.33}$$

Using the linearity property of the operator, we obtain:

$$\sum_{n=1}^{N} a_n \langle Op_1, w_m \rangle - \sum_{n=1}^{N} b_n \langle Op_2, w_m \rangle = \langle 2\cos x + x - 2, w_m \rangle. \tag{4.34}$$

This can be written in the following format

$$\sum_{n=1}^{N} a_n Z - \sum_{n=1}^{N} b_n A = V_R. \tag{4.35}$$

The same procedure is extended to the imaginary part of the operator. In this case,

$$\langle Op_{imag}, w_m \rangle = \langle Imag\{g(x)\}, w_m \rangle. \tag{4.36}$$

In this example,

$$\langle f_1' + f_R, w_m \rangle = \langle 2j\sin x - x^2, w_m \rangle. \tag{4.37}$$

Extending the imaginary operator, we obtain;

$$\sum_{n=1}^{N} b_n f_n'(x) + \sum_{n=1}^{N} a_n f_n(x) = \sum_{n=1}^{N} a_n Op_2 + \sum_{n=1}^{N} b_n Op_1. \tag{4.38}$$

Taking the inner product of this operator with the testing function, we obtain

$$\left\langle \left(\sum_{n=1}^{N} a_n Op_2 + \sum_{n=1}^{N} b_n Op_1 \right), w_m \right\rangle = < 2j\sin x - x^2, w_m > . \tag{4.39}$$

$$\sum_{n=1}^{N} a_n < Op_2, w_m > + \sum_{n=1}^{N} b_n < Op_1, w_m > = < 2j\sin x - x^2, w_m >, \tag{4.40}$$

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which can also be written in a matrix format

$$\sum_{n=1}^{N} a_n A + \sum_{n=1}^{N} b_n Z = V_I. \tag{4.41}$$

6 - Solve the system of the form A.x=b

$$\sum_{n=1}^{N} a_n Z - \sum_{n=1}^{N} b_n A = V_R. \tag{4.42}$$

$$\sum_{n=1}^{N} a_n A + \sum_{n=1}^{N} b_n Z = V_I. \tag{4.43}$$

Combining both equations of (4.42)and (4.43) into one matrix equation, we obtain

$$\begin{bmatrix} Z & A \\ A & Z \end{bmatrix} \begin{bmatrix} \sum_{n=1}^{N} a_n \\ \sum_{n=1}^{N} b_n \end{bmatrix} = \begin{bmatrix} V_R \\ V_I \end{bmatrix}. \tag{4.44}$$

$$\begin{bmatrix} Z & A \\ A & Z \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} = \begin{bmatrix} V_R \\ V_I \end{bmatrix}. \tag{4.45}$$

Such that:

$$\begin{bmatrix} Z \end{bmatrix} = \begin{bmatrix} \langle w_1, Op_1(f_1) \rangle & \cdots & \langle w_1, Op_1(f_N) \rangle \\ \vdots & \ddots & \vdots \\ \langle w_M, Op_1(f_1) \rangle & \cdots & \langle w_M, Op_1(f_N) \rangle \end{bmatrix} \tag{4.46}$$

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} \langle w_1, Op_2(f_1) \rangle & \cdots & \langle w_1, Op_2(f_N) \rangle \\ \vdots & \ddots & \vdots \\ \langle w_M, Op_2(f_1) \rangle & \cdots & \langle w_M, Op_2(f_N) \rangle \end{bmatrix} \tag{4.47}$$

$$\begin{bmatrix} V_R \end{bmatrix} = \begin{bmatrix} \langle g_R, w_1 \rangle \langle g_R, w_2 \rangle \cdots \langle g_R, w_M \rangle \end{bmatrix}^T. \tag{4.48}$$

$$\begin{bmatrix} V_I \end{bmatrix} = \begin{bmatrix} \langle g_I, w_1 \rangle \langle g_I, w_2 \rangle \cdots \langle g_I, w_M \rangle \end{bmatrix}^T. \tag{4.49}$$

7 - Finally, create the function $f(x) = \sum_{n=1}^{N} a_n f_n(x) + j \sum_{n=1}^{N} b_n f_n(x)$ and plot it. $M = N = 2$ is chosen because the plot does not present any major changes for more iterations. The elements of the square matrix $\begin{bmatrix} Z \end{bmatrix}$ are found to be:

$$Z_{11} = \langle w_1, Op_1(f_1) \rangle = \int_0^1 w_1(x) Op_1(f_1(x)) dx = \int_0^1 (x^2)(2x) dx = \frac{1}{2}. \tag{4.50}$$

$$Z_{12} = \langle w_1, Op_1(f_2) \rangle = \int_0^1 w_1(x) Op_1(f_2(x)) dx = \int_0^1 (x^2)(3x^2) dx = \frac{3}{5}. \tag{4.51}$$

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$$Z_{21} = \langle w_2, Op_1(f_1) \rangle = \int_0^1 w_2(x)Op_1(f_1(x))dx = \int_0^1 (x^3)(2x)dx = \frac{2}{5}. \quad (4.52)$$

$$Z_{22} = \langle w_2, Op_1(f_2) \rangle = \int_0^1 w_2(x)Op_1(f_2(x))dx = \int_0^1 (x^3)(3x^2)dx = \frac{1}{2}. \quad (4.53)$$

The elements of the square matrix $\left[A \right]$ are found to be:

$$A_{11} = \langle w_1, Op_2(f_1) \rangle = \int_0^1 w_1(x)Op_2(f_1(x))dx = \int_0^1 (x^2)(x^2)dx = \frac{1}{5}. \quad (4.54)$$

$$A_{12} = \langle w_1, Op_2(f_2) \rangle = \int_0^1 w_1(x)Op_2(f_2(x))dx = \int_0^1 (x^2)(x^3)dx = \frac{1}{6}. \quad (4.55)$$

$$A_{21} = \langle w_2, Op_2(f_1) \rangle = \int_0^1 w_2(x)Op_2(f_1(x))dx = \int_0^1 (x^3)(x^2)dx = \frac{1}{6}. \quad (4.56)$$

$$A_{22} = \langle w_2, Op_2(f_2) \rangle = \int_0^1 w_2(x)Op_2(f_2(x))dx = \int_0^1 (x^3)(x^3)dx = \frac{1}{7}. \quad (4.57)$$

The elements of the vectors $\left[V_R \right]$ and $\left[V_I \right]$ are:

$$V_{R1} = \langle g_R, w_1 \rangle = \int_0^1 g_R(x)w_1(x)dx = \int_0^1 (2\cos x + x - 2)(x^2)dx = 0.0616. \quad (4.58)$$

$$V_{R2} = \langle g_R, w_2 \rangle = \int_0^1 g_R(x)w_2(x)dx = \int_0^1 (2\cos x + x - 2)(x^3)dx = 0.0435. \quad (4.59)$$

$$V_{I1} = \langle g_I, w_1 \rangle = \int_0^1 g_I(x)w_1(x)dx = \int_0^1 (2j\sin x - x^2)(x^2)dx = 0.02465. \quad (4.60)$$

$$V_{I2} = \langle g_I, w_2 \rangle = \int_0^1 g_I(x)w_2(x)dx = \int_0^1 (2j\sin x - x^2)(x^3)dx = 0.1875. \quad (4.61)$$

After replacing the values found in the system of equations, the coefficients a_n and b_n are calculated by solving the system A.x=b.

$$a_n = \begin{bmatrix} 0.6092 \\ -0.2261 \end{bmatrix}, \quad b_n = \begin{bmatrix} 1.0188 \\ -0.5784 \end{bmatrix} \quad (4.62)$$

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Finally, the approximated expansion of the unknown function could be written:

$$
\begin{aligned}
f(x) &\approx \sum_{n=1}^{N} a_n f_n + j \sum_{n=1}^{N} b_n f_n \\
&= a_1 f_1(x) + a_2 f_2(x) + j b_1 f_1(x) + j b_2 f_2(x) \\
&= a_1 x^2 + a_2 x^3 + j b_1 x^2 + j b_2 x^3 \\
&= 0.6092 x^2 - 0.2261 x^3 + j 1.0188 x^2 - j 0.5784 x^3.
\end{aligned}
\tag{4.63}
$$

All the above steps are done using a MatLab script and both, the analytical solution and the solution found by the MoM for this complex-coefficients complex-solution differential equation, are plotted. Figure 4.2 represents the two plots which shows that $Real\{f(x)\}$ and $Imag\{f(x)\}$ found by the MoM is the same as those found by the MatLab function dsolve.
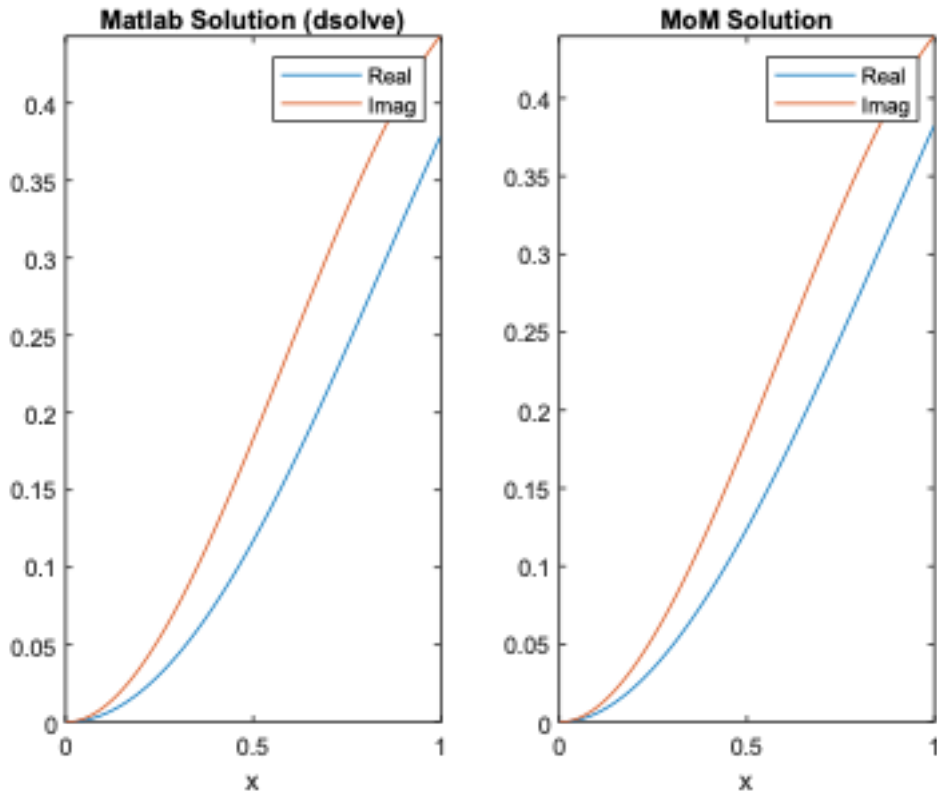

Figure 4.3: Complex Function First Order DE Solution

4.4 Third Order Example of a Complex Function Complex Coefficient Differential Equation

To prove that the method of moments could be applied to higher order differential equations, two more examples were added. Considering the following complex-function complex-solution third order differential equation:

$$j\frac{d^3f(x)}{dx^3} + (2+j)\frac{d^2f(x)}{dx^2} + (1+j)\frac{df(x)}{dx} + 2jf(x)$$
$$= -2jx^3 - (1+3j)x^2 - (12+6j)x + (2-8j). \qquad (4.64)$$

The steps of the solution of this equation are presented below:
1- Identify the real and the imaginary part of the operator:

$$j(f_R''' + jf_I''') + 2(f_R'' + jf_I'')$$
$$+ j(f_R'' + jf_I'') + (f_R' + jf_I') + j(f_R' + jf_I') + 2j(f_R + jf_I). \qquad (4.65)$$

$$Op_{Real} = -f_I''' + 2f_R'' - f_I'' + f_R' - f_I' - 2f_I. \qquad (4.66)$$

$$Op_{Imag} = f_R''' + 2f_I'' + f_R'' + f_I' + f_R' + 2f_R. \qquad (4.67)$$

2- Identify the real and the imaginary part of the right side function:

$$Real\{g(x)\} = -x^2 - 12x + 2. \qquad (4.68)$$

$$Imag\{g(x)\} = -2x^3 - 3x^2 - 6x - 8. \qquad (4.69)$$

3- Equate the real and the imaginary parts from both sides:

$$Op_{real} = Real\{g(x)\}. \qquad (4.70)$$

$$Op_{imag} = Imag\{g(x)\}. \qquad (4.71)$$

4- According to the Galerkins method, choose the same basis and weighting functions that meets the boundary conditions of the problem. In this case, we select:
$$f_n(x) = w_m(x) = x - x^{n+1}. \qquad (4.72)$$

5- Calculate the inner product of the previous equations with the weighting function chosen and apply the real and imaginary parts of the operator on the basis function chosen:

$$\langle Op_{real}, w_m \rangle = \langle Real\{g(x)\}, w_m \rangle$$
$$\langle -f_{nI}''' + 2f_{nR}'' - f_{nI}'' + f_{nR}' - f_{nI}' - 2f_{nI}, w_m \rangle = \langle -x^2 - 12x + 2, w_m \rangle \qquad (4.73)$$

Extending the real operator, we obtain:

$$
Op_{Real} = -\sum_{n=1}^{N} b_n f_n'''(x) + 2\sum_{n=1}^{N} a_n f_n''(x) - \sum_{n=1}^{N} b_n f_n''(x)
$$

$$
+ \sum_{n=1}^{N} a_n f_n'(x) - \sum_{n=1}^{N} b_n f_n'(x) - 2\sum_{n=1}^{N} b_n f_n(x)
$$

$$
= \sum_{n=1}^{N} a_n \left[2f_n''(x) + f_n{}'(x) \right] - \sum_{n=1}^{N} b_n \left[f_n'''(x) + f_n{}''(x) + f_n'(x) + 2f_n(x) \right]
$$

$$
= \sum_{n=1}^{N} a_n Op_1 - \sum_{n=1}^{N} b_n Op_2. \tag{4.74}
$$

Taking the inner product of (4.74) with the testing function, we obtain:

$$
\left\langle \sum_{n=1}^{N} a_n Op_1 - \sum_{n=1}^{N} b_n Op_2, w_m \right\rangle = < -x^2 - 12x + 2, w_m >
$$

$$
\sum_{n=1}^{N} a_n, < Op_1, w_m > - \sum_{n=1}^{N} b_n < Op_2, w_m > = < -x^2 - 12x + 2, w_m >, \tag{4.75}
$$

which can be written in a matrix format

$$
\sum_{n=1}^{N} a_n Z - \sum_{n=1}^{N} b_n A = V_R. \tag{4.76}
$$

Similar operation is done for the imaginary operator

$$
\langle Op_{imag}, w_m \rangle = \langle Imag\{g(x)\}, w_m \rangle \tag{4.77}
$$

$$
\langle f_{nR}''' + 2f_{nI}'' + f_{nR}'' + f_{nI}' + f_{nR}' + 2f_{nR}, w_m \rangle = \langle -2x^3 - 3x^2 - 6x - 8, w_m \rangle. \tag{4.78}
$$

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Extending the imaginary operator:

$$\sum_{n=1}^{N} a_n f_n'''(x) + 2\sum_{n=1}^{N} b_n f_n''(x) + \sum_{n=1}^{N} a_n f_n''(x) + \sum_{n=1}^{N} b_n f_n'(x)$$

$$+ \sum_{n=1}^{N} a_n f_n'(x) + 2\sum_{n=1}^{N} a_n f_n(x)$$

$$= \sum_{n=1}^{N} a_n \left[f_n'''(x) + f_n''(x) + f_n'(x) + 2f_n(x) \right] + \sum_{n=1}^{N} b_n \left[2f_n''(x) + f_n'(x) \right]$$

$$= \sum_{n=1}^{N} a_n Op_2 + \sum_{n=1}^{N} b_n Op_1$$

$$\left\langle \sum_{n=1}^{N} a_n Op_2 + \sum_{n=1}^{N} b_n Op_1, w_m \right\rangle = \left\langle -2x^3 - 3x^2 - 6x - 8, w_m \right\rangle$$

$$\sum_{n=1}^{N} a_n < Op_2, w_m > + \sum_{n=1}^{N} b_n < Op_1, w_m > = \left\langle -2x^3 - 3x^2 - 6x - 8, w_m \right\rangle \quad (4.79)$$

In a matrix format, 4.47 can be written as:

$$\sum_{n=1}^{N} a_n A + \sum_{n=1}^{N} b_n Z = V_I. \quad (4.80)$$

6- Solve the system (4.76) and (4.80) of the form $A.x = b$

$$\sum_{n=1}^{N} a_n Z - \sum_{n=1}^{N} b_n A = V_R. \quad (4.81)$$

$$\sum_{n=1}^{N} a_n A + \sum_{n=1}^{N} b_n Z = V_I. \quad (4.82)$$

$$\begin{bmatrix} Z & A \\ A & Z \end{bmatrix} \begin{bmatrix} \sum_{n=1}^{N} a_n \\ \sum_{n=1}^{N} b_n \end{bmatrix} = \begin{bmatrix} V_R \\ V_I \end{bmatrix}. \quad (4.83)$$

$$\begin{bmatrix} Z & A \\ A & Z \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} = \begin{bmatrix} V_R \\ V_I \end{bmatrix}. \quad (4.84)$$

Similarly, the solution for this equation in both ways, Matlab function solve and the method of moments, is presented in the figure below in which the first plot is the solution of the third order differential equation using solve and the second plot is the result found by using the MoM. The number of subsections on the interval $[-2, 2]$ is chosen to be $N = M = 12$ in this case also for the same reason

that, for higher values, no major change is observed on the curves.

7- finally, create the function $f(x) = \sum_{n=1}^{N} a_n f_n(x) + j \sum_{n=1}^{N} b_n f_n(x)$ and plot it.
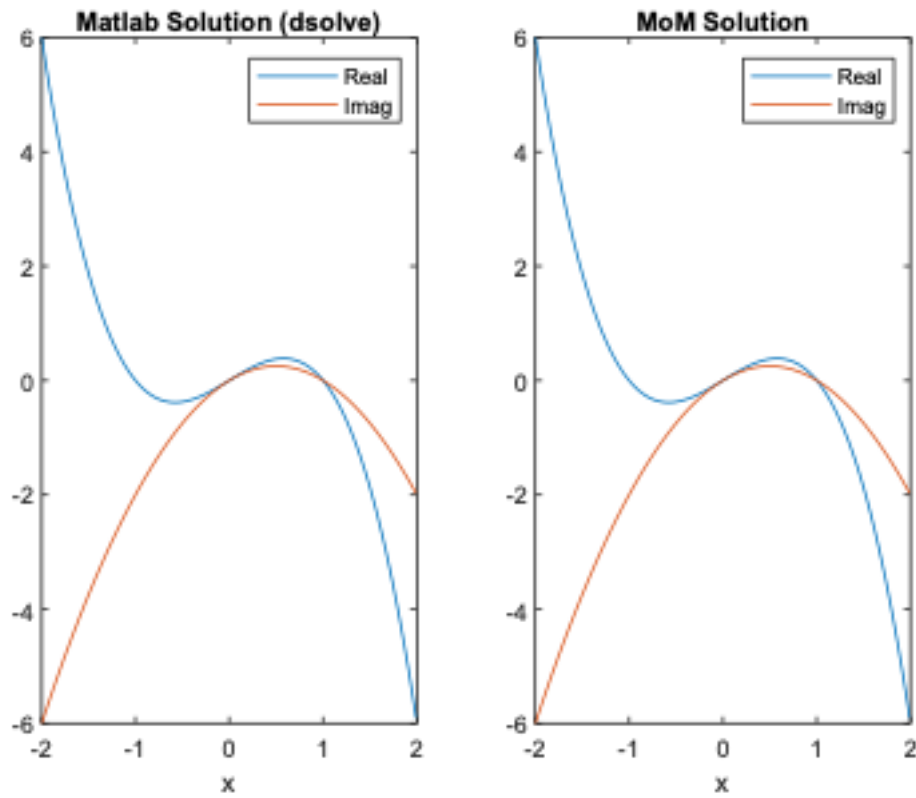

Figure 4.4: Complex Function Third Order DE Solution

4.5 Fourth Order Example of a Complex Function Complex Coefficient Differential Equation

The same steps as for the previous example are followed even for higher order differential equations. The details for solving the following fourth order differential equation are presented.

$$
\begin{aligned}
3j\frac{d^4 f(x)}{dx^4} &+ (2+j)\frac{d^3 f(x)}{dx^3} + j\frac{d^2 f(x)}{dx^2} - \frac{df(x)}{dx} - 2f(x) \\
&= -4jx^6 + (2-2j)x^5 - (55+2j)x^4 - (242-454)x^3 - (2295+55j)x^2 \\
&\quad - (32+298j)x - (59-18j).
\end{aligned}
$$

$$(4.85)$$

1- Identify the real and the imaginary part of the operator:

$$3j(f_R^{IV}+jf_I^{IV})+(2+j)(f_R'''+jf_I''')+j(f_R''+jf_I'')-(f_R'+jf_I')-2(f_R+jf_I). \quad (4.86)$$

$$
\begin{aligned}
Op_{real} &= -3f_I^{IV} + 2f_R''' - f_I''' - f_I'' - f_R' - 2f_R. \\
Op_{imag} &= 3f_R^{IV} + 2f_I''' + f_R''' + f_R'' - f_I' - 2f_I.
\end{aligned}
$$

$$(4.87)$$

2- Identify the real and the imaginary part of the right side function:

$$Real\Big\{g(x)\Big\} = 2x^5 - 55x^4 - 242x^3 - 2295x^2 - 32x - 59.$$

$$Imag\Big\{g(x)\Big\} = -4x^6 - 2x^5 - 2x^4 + 454x^3 - 55x^2 - 298x - 18. \quad (4.88)$$

3- Equate the real and the imaginary parts from both sides

$$Op_{real} = Real\Big\{g(x)\Big\}.$$

$$Op_{imag} = Imag\Big\{g(x)\Big\}. \quad (4.89)$$

4- According to the Galerkins method, choose the same basis and weighting functions

$$f_n(x) = w_m(x) = x - x^{m+1}.$$

$$f(x) = \sum_{n=1}^{N} a_n f_n(x) + j\sum_{n=1}^{N} b_n f_n(x). \quad (4.90)$$

5- Calculate the inner product of the previous equations with the weighting function chosen and apply the real and imaginary parts of the operator on the basis function chosen:

$$\langle Op_{real}, w_m \rangle = \langle Real\{g(x)\}, w_m \rangle$$
$$\langle -3f_{ni}^{IV} + 2f_{nr}''' - f_{ni}''' - f_{ni}'' - f_{nr}' - 2f_{nr}, w_m \rangle$$
$$= \langle 2x^5 - 55x^4 - 242x^3 - 2295x^2 - 32x - 59, w_m \rangle. \qquad (4.91)$$

Extending the real operator:

$$-3\sum_{n=1}^{N} b_n f_n^{IV}(x) + 2\sum_{n=1}^{N} a_n f_n'''(x) - \sum_{n=1}^{N} b_n f_n'''(x)$$

$$- \sum_{n=1}^{N} b_n f_n''(x) - \sum_{n=1}^{N} a_n f_n'(x) - 2\sum_{n=1}^{N} a_n f_n(x)$$

$$= \sum_{n=1}^{N} a_n \left[2f_n'''(x) - f_n'(x) - 2f_n(x) \right] - \sum_{n=1}^{N} b_n \left[3f_n^{IV}(x) + f_n'''(x) + f_n''(x) \right]$$

$$= \sum_{n=1}^{N} a_n Op_1 - \sum_{n=1}^{N} b_n Op_2. \qquad (4.92)$$

Taking the inner product of (4.92) with the testing function, we obtain:

$$\left\langle \left(\sum_{n=1}^{N} a_n Op_1 - \sum_{n=1}^{N} b_n Op_2 \right), w_m \right\rangle$$
$$= \langle 2x^5 - 55x^4 - 242x^3 - 2295x^2 - 32x - 59, w_m \rangle$$
$$\sum_{n=1}^{N} a_n \langle Op_1, w_m \rangle - \sum_{n=1}^{N} b_n \langle Op_2, w_m \rangle$$
$$= \langle 2x^5 - 55x^4 - 242x^3 - 2295x^2 - 32x - 59, w_m \rangle. \qquad (4.93)$$

In a matrix format,
$$\sum_{n=1}^{N} a_n Z - \sum_{n=1}^{N} b_n A = V_R \qquad (4.94)$$

The same procedure is used for the imaginary operator;

$$\langle Op_{imag}, w_m \rangle = \langle Imag\{g(x)\}, w_m \rangle$$
$$= \langle 3f_R^{IV} + 2f_I''' + f_R'' - f_I' - f_I' - 2f_I, w_m \rangle$$
$$= \langle -4x^6 - 2x^5 - 2x^4 + 454x^3 - 55x^2 - 298x - 18, w_m \rangle. \qquad (4.95)$$

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Extending the imaginary operator:

$$3\sum_{n=1}^{N} a_n f_n^{IV}(x) + 2\sum_{n=1}^{N} b_n f_n'''(x) + \sum_{n=1}^{N} a_n f_n'''(x)$$

$$+ \sum_{n=1}^{N} a_n f_n''(x) - \sum_{n=1}^{N} b_n f_n'(x) - 2\sum_{n=1}^{N} b_n f_n(x)$$

$$= \sum_{n=1}^{N} a_n \left[3f_n^{IV}(x) + f_n'''(x) + f_n''(x)\right] + \sum_{n=1}^{N} b_n \left[2f_n'''(x) - f_n'(x) - 2f_n(x)\right]$$

$$= \sum_{n=1}^{N} a_n Op_2 + \sum_{n=1}^{N} b_n Op_1. \tag{4.96}$$

Taking the inner product of (4.96) with the testing function, we obtain

$$\left\langle \left(\sum_{n=1}^{N} a_n Op_2 + \sum_{n=1}^{N} b_n Op_1\right), w_m \right\rangle$$

$$= \left\langle -4x^6 - 2x^5 - 2x^4 + 454x^3 - 55x^2 - 298x - 18, w_m \right\rangle$$

$$\sum_{n=1}^{N} a_n \langle Op_2, w_m \rangle + \sum_{n=1}^{N} b_n \langle Op_1, w_m \rangle$$

$$= \left\langle -4x^6 - 2x^5 - 2x^4 + 454x^3 - 55x^2 - 298x - 18, w_m \right\rangle. \tag{4.97}$$

In a matrix format, (4.97) is written as:

$$\sum_{n=1}^{N} a_n A + \sum_{n=1}^{N} b_n Z = V_I \tag{4.98}$$

6 - Solve the system of the form A.x=b

$$\sum_{n=1}^{N} a_n Z - \sum_{n=1}^{N} b_n A = V_R. \tag{4.99}$$

$$\sum_{n=1}^{N} a_n A + \sum_{n=1}^{N} b_n Z = V_I. \tag{4.100}$$

$$\begin{bmatrix} Z & A \\ A & Z \end{bmatrix} \begin{bmatrix} \sum_{n=1}^{N} a_n \\ \sum_{n=1}^{N} b_n \end{bmatrix} = \begin{bmatrix} V_R \\ V_I \end{bmatrix}. \tag{4.101}$$

$$\begin{bmatrix} Z & A \\ A & Z \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} = \begin{bmatrix} V_R \\ V_I \end{bmatrix}. \tag{4.102}$$

7- Finally, create the function $f(x) = \sum_{n=1}^{N} a_n f_n(x) + j \sum_{n=1}^{N} b_n f_n(x)$ and plot it.

Similarly, the solution for this equation in both ways, Matlab function dsolve and the method of moments, is presented in figure(4.4) below in which the first plot is the solution of the fourth order differential equation using dsolve and the second plot is the result found by using the MoM. The number of subsections on the interval [-2 2] is chosen to be N = M = 12 in this case also for the same reason that, for higher values, no major change is observed on the curves.
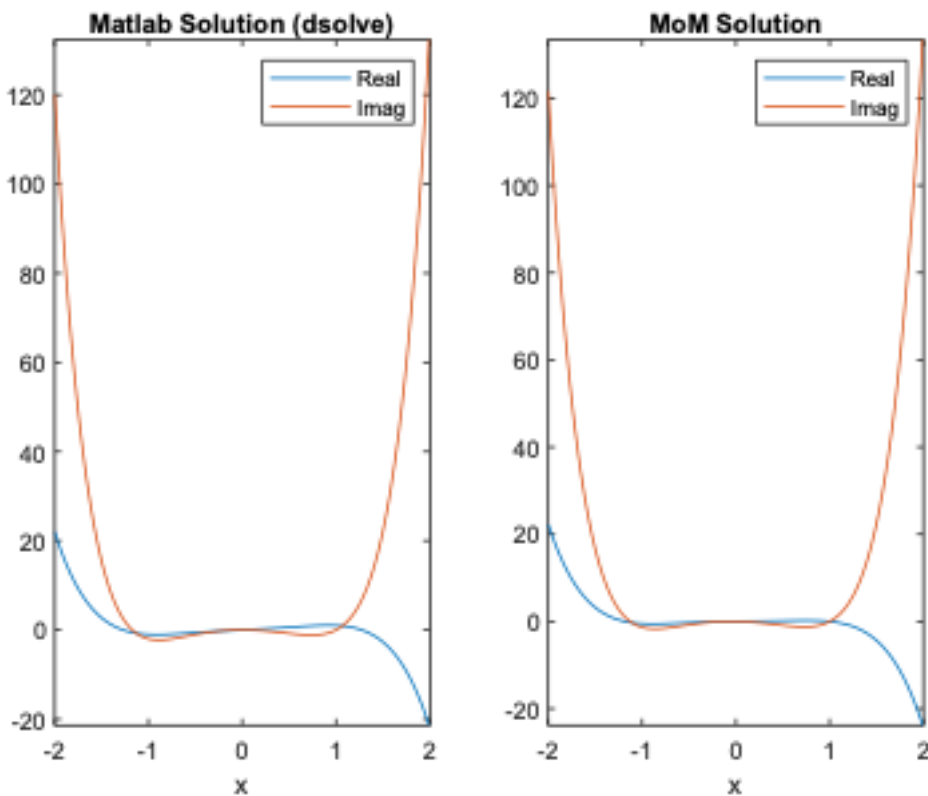

Figure 4.5: Complex Function Fourth Order DE Solution

Chapter 5

Conclusion

In this thesis, the method of moments along with the Characteristic Mode theory is used to solve real- and complex-solution differential equations. The functions to be found are expanded into a sum of unknown coefficients multiplied by a basis function. Then, the inner product of the differential equation with a weighting function, equal to the basis function as imposed by the method of Galerkin, is performed. Finally, the unknown coefficients are calculated by a simple matrix inversion. The sum of the basis function multiplied by the corresponding coefficients found will result in the solution of the unknown function.

Four different examples of differential equations are solved to prove the simplicity of the method: one real-solution second order differential equation and three complex-solution first-, third- and fourth-order differential equations.

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