AMERICAN UNIVERSITY OF BEIRUT

THE IMPACT OF GEOGEBRA WITH CONCEPTUAL CHANGE METHODS ON STUDENTS' CONCEPTUAL ENRICHMENT AND CONCEPTUAL CHANGE IN TRIGONOMETRY

by

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The purpose of this research study is to examine the effect of GeoGebra and conceptual change methods in enhancing conceptual change and conceptual enrichment in trigonometry of grade 10 students of different achievement levels in two schools of Lebanon. A mixed-method of quantitative and qualitative approaches is used in a quasi-experimental research design. The research study examines the differences of the impact of the three instructional designs (GeoGebra based explicit conceptual change (Geo + CC), GeoGebra based (Geo) and explicit conceptual change (CC)) on the conceptual enrichment, the overall conceptual change and the conceptual change in each of the concepts of periodicity, boundedness, non-monotonicity and non-linearity of trigonometric functions of students of different achievement levels. The findings of this research study show that the impact of combining the Geo and CC instructional designs on students' overall conceptual change is significantly more than each of Geo and CC instructional designs alone. However, the three instructional designs do not impact the students' conceptual enrichment differently nor the overall conceptual change or conceptual enrichment of students with different achievement levels. The impact of the instructional design differed for the four targeted concepts. For boundedness, the impact is mainly due to GeoGebra. For periodicity and non-linearity, the impact is mainly due to the explicit conceptual change methods. Instructional designs do not impact the concept of non-monotonicity differently. Four themes emerge from the qualitative analysis of students' self-reported conceptions of their conceptual change in each of the four targeted concepts. The four themes that emerged from the revised Bloom's taxonomy are non-recognition, recognition, explaining/exemplifying, and differentiation of conceptual change. The frequencies of the emerging themes in the three instructional designs are different for each of the four concepts. In general, the students of the instructional designs that had better conceptual change had cognitive processes that are higher on Bloom's revised taxonomy.

CONTENTS

ILLUSTRATIONS

TABLES

CHAPTER 1

INTRODUCTION

Constructivist theorists believe that learners need to construct their acquired knowledge actively. They discover information, compare new knowledge to the old, and revise rules when they do not apply (Bada, 2015). Constructivism's main idea is that learners construct new understandings using what they already know (Bada, 2015). Learners' prior knowledge influences what new or modified knowledge they will construct (Phillips, 1995). In mathematics education, the students' previous knowledge influences new concepts without significant discrepancies. Students can generalize their prior knowledge to new situations easily. Nevertheless, recently, many domains of mathematics are identified where the acquired knowledge is discrepant from learners' prior knowledge. In these domains, learners could not apply generalizations, and conceptual change is needed.

Kuhn (1962) defined the idea of the paradigm shift in science as a radical change of the basic concepts of science. Kuhn's notion of paradigm shift and Piaget's notion of accommodation are the bases of conceptual change in science education (Ozdemir & Clark, 2007). Conceptual change in science consists of the discrepancy between students' commonsense conceptions and scientific theory (Nersessian, 1989). In mathematics education, the conceptual change characterizes situations where learners' prior knowledge is incompatible with new concepts to be acquired (Merenluoto & Lehtinen, 2004).

Teachers use different strategies to enhance students' conceptual change in science and mathematics. Some of those strategies use an explicit comparison of students' prior knowledge and the new knowledge to be acquired. The computer-assisted instruction (CAI) is another strategy that would promote students' conceptual change.

The utilization of GeoGebra software is an example of CAI in mathematics education. GeoGebra is an interactive software where students can visualize and manipulate many abstract concepts of mathematics. It enhances teaching and learning in geometry, calculus, and trigonometry. Trigonometry is one of the fields of mathematics where students may face difficulties and develop many misconceptions. Trigonometric functions are defined for acute angles in a right triangle. Their definition is then generalized to larger angles on unit circles. Trigonometric functions have many properties that are discrepant from students' prior knowledge and cannot be generalized from it. They are periodic and non-monotonic. Other discrepant properties of trigonometric functions are their non-linearity with multiplication and addition and the boundedness of their image.

Purpose and Research Questions

The purpose of this research study is to examine the effect of GeoGebra and conceptual change methods in enhancing conceptual change and conceptual enrichment in trigonometry of grade 10 students of different achievement levels in two schools of Lebanon.

This research study aims to answer the following research questions:

1. Do the three instructional designs (GeoGebra based explicit conceptual change, GeoGebra based, and explicit conceptual change) impact students' overall conceptual change and their conceptual change on each of the concepts of

boundedness, periodicity, non-linearity and non-monotonicity of trigonometric functions differently?

2. Do the three instructional designs impact students' conceptual enrichment in Trigonometry differently?
3. Is there a significant interaction between instructional design and student level of achievement on the overall conceptual change and the conceptual change in each of the concepts of boundedness, periodicity, non-linearity, and non-monotonicity of trigonometric functions?
4. Is there a significant interaction between instructional design and student level of achievement on the conceptual enrichment in Trigonometry?
5. What themes emerge from students' self-reported conceptions of their conceptual change on each of the concepts of boundedness, periodicity, non-linearity, and non-monotonicity of trigonometric functions?
6. How does the saliency of the emerging themes differ for each of the four concepts in the three instructional designs?

Rationale of the Research Study

Tirosh and Tsamir (2004) question whether a conceptual change is applicable in mathematics education. They think that it is possible to predict cases in mathematics where the prior knowledge of learners is not sufficient for the development of the understanding of a mathematical notion and that radical reorganization of existing knowledge structures is likely to be required. Such cases appear in numbers (Lehtinen, Merenluoto & Kasanen, 1997; Christou, Vosniadou, & Vamvakoussi, 2007), Geometry (Van Dooren, De Bock, Hessels, Janssens & Verschaffel, 2004), problem-solving (Sarar & Al-Migdady, 2014), probability (Castro, 1998), and Calculus (Liang, 2016). The literature lacks research studies that address the application of the conceptual change in trigonometry even though it is a domain where many concepts do not flow continuously from students' prior knowledge. Trigonometric functions have many properties that are discrepant from students' prior knowledge and require conceptual change. Their image is bounded; they are periodic, non-linear, and non-monotonic.

Boundedness

The functions that students studied till grade 10, such as squares and radicals, have unbounded values, unlike the sine and cosine functions that are bounded in [-1;1]. Thus the need for conceptual change from unbounded functions to bounded functions in trigonometry.

Periodicity

Students' prior experiences were limited to visualizing numbers on a one-dimensional number line where the mapping from numbers to their representations is one-to-one. In trigonometry, angles are represented on a unit circle instead of the number line; this creates a mapping that is not one-to-one. Trigonometric functions are periodic, and many numbers have the same representation on the unit circle. Thus there is need for conceptual change from the one-to-one number line to the periodic trigonometric functions.

Non-linearity

At the intermediate level, learners study many linear transformations such as linear expansion with literal symbols. For example, they know that $a(x + y) = ax + ay$. On the other hand, they are introduced to many non-linear transformations too. When students are introduced to squares and radicals in grade 8, they do not apply the linearity with the addition. Instead, they use identities of squares such as $(x + y)^2 = x^2 + 2xy + y^2$. When students use inverse numbers in grade 6, they do not apply the linearity; rather, they use common denominators. In those cases, students use procedural methods to deal with non-linear transformations; they do not understand them conceptually. When students deal with trigonometric functions, they tend to apply the linearity rules, especially because of the written format of trigonometric functions that is very similar to literal expressions. They think that for example sin x + sin y =sin (x + y) and sin (ax)=a sinx. Conceptual change is needed to transfer the knowledge of linear transformations to non-linear trigonometric functions.

Non-monotonicity

Until grade 10, teachers introduce students to the conservation of order (i.e., if x<y then f(x) < f(y)) or the non-conservation of order (i.e., if x < y the f(x) > f(y)) without the concept of variations of a function or even without the idea of function instead by applying them as rules and properties. In general, students experience monotonic functions. Some of those functions are increasing, such as the square root that conserves the order. Others are decreasing, such as the inverse function that does not preserve the order. Students also study the square function that is increasing in some intervals and decreasing in others. Trigonometric functions are non-monotonic. They are increasing in some quadrants and decreasing in others. They change their monotonicity even in one period. Trigonometric functions conserve the order when they are increasing (i.e., if x<y, then sin x <sin y) but do not preserve the order when they are decreasing (i.e., if x<y, then sin x > sin y). Therefore the conceptual change from monotonic functions to non-monotonic trigonometric functions is needed, especially when comparing trigonometric functions.

Teachers need to apply the conceptual change theory in teaching trigonometry because trigonometric functions have many properties that are discrepant from students' prior knowledge. The conceptual change theory helps to detect and correct misconceptions that learners may have in trigonometry. Also, it promotes a conceptual understanding of trigonometric functions and their basic properties. No research discusses the application of the conceptual change theory in trigonometry. This research study extends the literature of conceptual change in mathematics education, applying the conceptual change theory in trigonometry. The application of the conceptual change theory in the trigonometry is a new and promising field of research in mathematics education.

The use of GeoGebra and CAI, in general, may facilitate the conceptual change in trigonometry. Many researchers have emphasized the use of GeoGebra in Mathematics education (Albaladejo, García and Codina, 2015; Kutluca, 2013; Akkayaa, Tatar, and Kagizmanli, 2011; Zengin, Furkan, Kutluca, 2012). Some researchers showed that CAI has positive effects on the overall performance of students in trigonometry (Naidoo & Govender, 2014; Blackett & Tall, 1991). The use of the Geometer's Sketchpad, the graphing calculators, and visualizations software impacts positively students' understanding of trigonometry (Johari, Chan, Ramli & Ahmat, 2010; Choi-Koh, 2003; Steckroth, 2007). No research has analyzed the influence of GeoGebra in enhancing conceptual change in Trigonometry. This research extends the literature to study the impact of GeoGebra on conceptual change in Trigonometry. Visualization through GeoGebra may promote conceptual change while teaching trigonometry and other concepts of mathematics.

Trundle and Bell (2009) used CAI as a tool to enhance conceptual change in science. Sander and Heib (2014) combined the CAI and the cognitive conflict methods in teaching trigonometry and found better results, especially for high achievers. Other researchers have combined CAI and conceptual change methods in science and found positive results (Ozmen, Demircioglu, & Demircioglu, 2009; Ozmen, 2011; Tas, Gulen, Oner, & Ozyurek, 2015; Yumusak, Maras & Sahin, 2015; Zietsman & Hewson, 1986; Yavuz, 2005; Wiser & Amin, 2001; Hameed, Hackling & Garnett, 1993; Talib, Matthew & Secombe, 2005). Sanger and Greenbowe (2000) combined CAI and the conceptual change methods in chemistry but could not reach definitive conclusions and suggested for further research. The literature lacks researches about the combination of the CAI and the conceptual change methods in mathematics education. This research extends the research of this combination to mathematics education in general and trigonometry in particular.

Significance of the Research Study

This research study examines the impact of three instructional designs (GeoGebra based explicit conceptual change (Geo + CC), GeoGebra based (Geo) and explicit conceptual change (CC)) on the conceptual enrichment, the overall conceptual change and the conceptual change in each of the concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions of students of different achievement levels.

This research study has three significant implications for research. First, it adds to the research about the influence of GeoGebra in teaching mathematics. This research study emphasizes on the role of GeoGebra to promote conceptual understanding and conceptual change in trigonometry.

Second, it contributes to the research about the conceptual change theory. It adds up to the literature of mathematics education a research study that applies the conceptual change theory to the domain of trigonometry particularly. The literature lacks research studies about conceptual change in trigonometry. The application of the conceptual change theory in trigonometry is a new field of research in mathematics education where positive results are expected.

Third, it combines the CAI and conceptual change methods as suggested by many other researchers especially in science education (Ozmen, Demircioglu, & Demircioglu, 2009; Ozmen, 2011; Tas, Gulen, Oner & Ozyurek, 2015; Yumusak, Maras & Sahin, 2015; Sander & Heib, 2004). This research study applies this combination in the context of mathematics education, which lacks researches about this combination. The combination of the conceptual change methods and the CAI is a promising research field in mathematics education where other researchers may expect similar results in trigonometry at different levels as well as in many other domains of mathematics.

This research study has many implications for practitioners, such as curriculum designers and teachers. It suggests the curriculum designers focus on the significant discrepancies that exist between the new properties of trigonometry and learners' prior knowledge to promote conceptual understanding in trigonometry. Curriculum designers may apply this focus to other mathematical concepts where there are significant discrepancies between the new properties to be acquired and students' prior knowledge.

This research study provides teachers a method that makes trigonometry teaching more interactive. It suggests a new and more holistic approach of teaching trigonometry where learners visualize the basic properties of trigonometry and contrast them to their prior knowledge. Combining GeoGebra and conceptual change methods may help students to avoid many misconceptions that they would encounter if they would generalize their prior knowledge to trigonometry. It allows teachers to detect and correct learners' misconceptions in trigonometry. Teachers may use this combination while teaching trigonometry or any other mathematical concept where major discrepancies exist between the new knowledge to be acquired and learners' prior knowledge.

This research study promotes the use of technology in the classroom to enhance a better conceptual understanding of students. It would help teachers to visualize abstract mathematical concepts concretely. Finally, it provides a constructivist approach where knowledge is constructed actively by learners who contrast the new knowledge to their prior knowledge explicitly and use technology in their classrooms.

CHAPTER 2

LITTERATURE REVIEW

This research study aims to examine the impact of GeoGebra and the conceptual change strategies in enhancing conceptual change and conceptual enrichment in trigonometry of students of different achievement levels.

In this chapter, first, the theory of conceptual change is presented. Then the need for a conceptual change in mathematics education and the need for a conceptual change in trigonometry are discussed. Finally, the computer-assisted instruction and GeoGebra and the relation of computer-assisted instruction and conceptual change in Mathematics and Sciences are presented.

Theory of Conceptual Change

Learning may be achieved through the process of enrichment, which suggests the continuous development of concepts. New concepts are introduced through the improvement of an existing knowledge structure (Merenluoto & Lehtinen, 2004). The constructivist approach suggests that students use their prior knowledge to construct the acquired knowledge. In some cases, they try to generalize the concepts that they already know to new situations. Discontinuity of learning occurs in situations where the prior knowledge of students is incompatible with the new information to be acquired. When the process of enrichment is not sufficient to construct the new knowledge, a significant restructuring or reorganization of the prior knowledge is needed (Merenluoto & Lehtinen, 2004). The conceptual change is this process of restructuring of the prior knowledge.

Posner, Strike, Hewson, and Gertzog (1982) developed one of the most prominent conceptual change theories. Their theory was based on Kuhn's notion of a paradigm shift or Piaget's notion of accommodation. They proposed that if learners' current conception is functional, and if they can solve problems within the existing conceptual schema, they do not feel a need to change their current conception. Even when the current conception does not successfully solve some problems, learners may make only moderate changes to their conceptions. In this case, they use assimilation. Posner et al. (1982) believe that learners must be dissatisfied with an initial conception to abandon it and accept a scientific conception for successful conceptual change. This more radical change is called accommodation or conceptual change. The process of conceptual change can be achieved through three methods: replacement, differentiation, and coalescence (Carey, 1991). In replacement, a new concept replaces another concept. During differentiation, the initial concept splits into two or more new concepts. Coalescence is the opposite process of differentiation; it involves two or more original concepts merging into a single concept (Carey, 1991).

The process of conceptual change is influenced by students' conceptual ecology (Talib, Matthew & Secombe, 2005). Posner et al. (1982) defined learners' conceptual ecology as the conceptions and ideas rooted in their epistemological beliefs. The constituents of a learner's conceptual ecology such as anomalies which result from the learner's existing knowledge; analogies and metaphors that help to suggest new ideas and make them intelligible, epistemological commitments and metaphysical beliefs about science determine the direction of the process of accommodation of new ideas (Posner et al., 1982).

Posner et al. (1982) described four conditions that must be fulfilled before accommodation takes place in a student's mind. First learners should be dissatisfied with his or her current conception. Also, the alternative conception should be intelligible, plausible, and fruitful. Other conditions for the process of conceptual change are the sensitivity to novel features and the process of the tolerance of ambiguity resulting from the conflict (Merenluoto & Lehtinen, 2004). The students have to tolerate the ambiguity that comes from newly learned concepts while they do not yet fully understand the concepts (Merenluoto & Lehtinen, 2004). According to Van Dooren, De Bock, Hessels, Janssens and Verschaffel (2004) the principles for the occurrence of the conceptual change are the necessity of being informed about students' prior knowledge, the need to explicitly address students' preconceptions during instruction, the facilitation of students' meta conceptual awareness, the motivation of students for obtaining conceptual change and the supply of appropriate mathematical models and related external representations. Van Dooren et al. (2004) showed that the illusion of linearity in calculating areas and volumes was de-constructed with students who experienced those conceptual change principles.

Metaconceptual awareness is defined to be learners' awareness of their own beliefs, their presuppositions, and the possible inconsistencies in them (Van Dooren at al, 2004). It is fundamental for the process of conceptual change (Van Dooren et al., 2004; Vosniadou, 2001). The lack of metaconceptual awareness prevents students from questioning their prior knowledge and from assimilating the new information into existing conceptual structures. Thus, this may result in the development of misconceptions (Vosniadou, 1994). The increase of the students' metaconceptual awareness is achieved by creating learning environments that help the students to express their internal representations of the phenomena, to share those representations with their peers, to defend criticism and compare those representations to the explanations of experts (Vosniadou, 2001). Such activities may be time-consuming, but they are essential for ensuring that students become aware of what they know and what they need to learn. Conceptual change cannot be achieved without the development of intentional learners who have the required metaconceptual awareness to understand the difference between their naïve beliefs and scientific concepts. Intentional learners are capable of producing intentional mechanisms that scientists use for hypothesis testing and conscious belief-revision (Vosniadou, 2001).

The learning environment facilitates the process of conceptual change. Vamvakoussi, Kargiotakis, Kollias, Nektarios, Mamalougos, and Vosniadou (2004) analyzed the effect of the software SYNERGEIA on students' understanding of natural numbers. SYNERGEIA is a database where students can upload and share files and participate in structured discussions. This environment gives to students the opportunity to express their ideas about the structure of the set of rational numbers, to externalize their visual representations of the structure of rational numbers, to see their fellow students' ideas, to realize that these can be quite different than their own and to argue with other students about the structure of the set of natural numbers.

Hatano & Iganaki (2003) discussed a similar set of strategies to enhance learners' conceptual change. The first strategy is to make learners recognize the inadequacy of their understanding. The second strategy is encouraging learners to participate in dialogical interaction, such as discussion or reciprocal teaching. The third strategy is freeing learners from the pressure to follow the externally set standards or to get permission from the external authorities. Finally, the fourth strategy is to ensure that learners value their understandings.

Another strategy for conceptual change is the constructivist model named as the Conceptual Change Model, which is discussed by Stepans (1994). This model includes six steps: (1) aiding the students to be aware of their thinking, to make predictions and to commit to problem-solving, (2) encouraging the students to express about their beliefs and share ideas with others, (3) helping students to test their ideas in small groups, (4) organizing class discussions to interrelate the new concepts with the prior knowledge, (5) helping the students to make connections with concepts from other disciplines, (6) helping students not to be limited by the existing concepts but find connection between the new ideas and other ideas.

Engaging students with cognitive conflict is another method to enhance students' conceptual change. Cognitive conflict or cognitive dissonance takes place when there is an explicit discrepancy between the learners' prior knowledge and the new knowledge they acquire. Cognitive conflict is the process where the students are directly confronted with their misconceptions and the conceptions that need to be changed. Limon (2001) discusses three steps to enhance cognitive conflict in learners. The first step consists of identifying the learners' current state of knowledge. The second step is to confront students with contradictory information and making the contradiction explicit. Finally, the third step is to evaluate the degree of change between students' prior ideas or beliefs and new conceptions. Confronting the learners with conflict is beneficial to students' conceptual change because it helps them to highlight the differences between their ideas and the new ones. Conceptual conflict does not always produce conceptual change. Sometimes the conflict remains unnoticed because students are overconfident or have the illusion of understanding (Merenluoto & Lehtinen, 2004).

Need for Conceptual Change in Mathematics Education

Many educators believe that Mathematics is a constructively built discipline where the students' new knowledge is highly related to their prior knowledge. Tirosh and Tsamir (2004) questioned whether the conceptual change theory is applicable in Mathematics education. They predicted cases in mathematics where the prior knowledge of students is not sufficient for the development of the understanding of a mathematical notion and that radical reorganization of existing knowledge structures is likely to be required. Such cases are expected in extensions of sets of numbers (from naturals to real numbers to complex numbers), relations (from equations to inequalities), and operations (multiplication and Cartesian product). Other candidates are probability and calculus (Tirosh & Tsamir, 2004). This section discusses the different instances of mathematics education, where a conceptual change is needed.

Numbers

Numbers are the core of mathematics curricula around the world. Natural numbers, rational numbers, real numbers, and complex numbers are introduced from kindergarten to grade 12 gradually. Each set of numbers have properties different from other sets. The transition from natural numbers to rational numbers to real numbers than to complex numbers can be challenging for students and teachers. Students from their early academic years are taught the properties of natural numbers. In later years, the transfer from natural numbers to rational numbers requires a conceptual change (Merenluoto & Lehtinen, 2004). At a later stage, after studying calculus, students may have difficulties when they transmit their conceptions from rational numbers to real numbers (Lehtinen, Merenluoto and Kasanen, 1997). Similar difficulties may arise in the process of transferring conceptions from real numbers to complex numbers.

Besides the properties of different sets of numbers, operations on numbers can be considered a field where conceptual change is needed. Multiplying natural numbers always produce bigger numbers. Many students generalize this fact erroneously to multiplication with rational numbers. (Prediger, 2008). Thus, conceptual change is needed in the transfer from the multiplication of natural numbers to the multiplication of rational numbers.

Algebra

A conceptual change is needed when passing from arithmetic to algebra. When students are introduced to algebra, they find it difficult to assign meaning to new symbols and to assign new meanings to old symbols, which were used in the context of arithmetic

(Christou, Vosniadou, & Vamvakoussi, 2007). Christou et al. (2007) found that the students' experience with numbers strongly influences their interpretation of the use of literal symbols in algebra in the context of arithmetic. Students interpreted the sign of the algebraic expressions as the actual sign of the numbers that they represent (Christou et al., 2007).

Geometry

Geometry is a field of mathematics where the prior knowledge of students is considered to be fundamental. The concepts in geometry are interrelated and highly associated with the students' prior knowledge. But even in geometry, the conceptual change is needed. Van Dooren, De Bock, Hessels, Janssens, and Verschaffel (2004) applied the conceptual change methods to overpass the illusion of linearity in calculating areas and volumes.

Problem Solving

Problem-solving is a domain of mathematics where students may generalize the concepts that they have been taught. However, Abu Sarar and Al-Migdady (2014) showed that the conceptual change model improves the ability of students to solve mathematical problems.

Calculus

In calculus, misconceptions in the concept of limits are so common among students. The misconceptions develop particularly about limits because students do not apply the definition of limits; instead, they interpret it based on their previous experiences and their prior knowledge. The conceptual change model is useful in the teaching of limits (Liang, 2016). The second instance of conceptual change in calculus is needed for the concept of tangent lines. Students apply the previously acquired properties of the tangent of circles to tangents of functions. The definition of the tangent line for circles is not sufficient to define the tangent line for curves; there is a need for a broader definition. This implies that a conceptual change is required when shifting from geometry to calculus (Biza, Souyoul & Zachariades, 2006).

Probability

Another domain where the conceptual change theory can be applied is probability. Castro (1998) found that the instructional method, which focused on conceptual change, enhanced students' skills of calculations, and intuitive probability reasoning. In probability, the conceptual change method is needed because a discrepancy occurs between the experiences of the students about probability and the formal definitions of probability.

Even though many research studies are conducted to analyze the different principles, methods, and strategies to enhance conceptual change in different mathematical subjects, the literature lacks a thorough analysis of the application of conceptual change methods in trigonometry.

Need for Conceptual Change in Trigonometry

In many countries' Mathematics curricula, trigonometry is considered to be an important category of mathematics instruction (Delice & Roper, 2006). It is used in mathematics, as well as in other scientific disciplines. A solid understanding of trigonometric functions is required in calculus and analysis (Demir, 2012). However, trigonometry has a complex nature. It combines different algebraic, geometric, and graphical concepts and procedures. Trigonometric functions are operations that cannot be expressed as algebraic formulae involving arithmetical procedures. For this reason, students have trouble in reasoning about such operations and viewing these operations as functions (Breidenbach, Dubinsky, Hawk, & Nichols, 1992). This complexity makes it challenging for students to understand trigonometry conceptually (Demir, 2012). Many researchers show that students have fragmented and incomplete understanding of trigonometric functions (Weber, 2005; Brown, 2005; Challenger, 2009).

There are three contexts of teaching trigonometry: The right triangle context, the unit circle context, and the functions and graphs context (Demir, 2012, Brown, 2015). In the first context, trigonometric concepts are defined as ratios in right triangles. In this context, trigonometry is limited to angles in degrees that are smaller than 90°. In the unit circle context, trigonometric concepts are defined as Cartesian coordinates in a plane. This expands trigonometry to include any angle, positive or negative, in degrees or radians. In the third context, trigonometric concepts are defined as graph functions, and they illustrate trigonometric functions in the domain of real numbers (Demir, 2012). Kendall and Stacey (1997) compared the unit circle method and the ratio method. They found that the ratio method was more effective, resulting in better performance and retention in trigonometry. Thompson (2008) emphasized that the problem of teaching trigonometry is that the different contexts of teaching trigonometry are unrelated and suggested to enhance coherence among different trigonometric contexts. Demir (2012) developed a model based on developing coherent connections among the three different contexts of trigonometry. In his study, Demir (2012) found that the new model was more effective in terms of enhancing a connected understanding of trigonometric functions (Demir, 2012).

Trigonometry is an abstract and non-intuitive subject which can result in many difficulties for the students; they develop many misconceptions (Gur, 2009). Students, irrespective of their cognitive ability, are subject to error in solving problems in trigonometry (Usman & Husaini, 2017). Delice (2002) thinks that errors in trigonometry are not random but result from misconceptions and think that those misconceptions should be identified and studied. Gur (2009) categorized the misconceptions in trigonometry into the following five categories: misused data, misinterpreted language, logically invalid inference, distorted definition, and technical mechanical errors. In addition, some students have difficulties in the measure of the angle in radians (Orhun, 2015). Others have difficulties in determining the domain of trigonometric functions. Many students have not learned to perceive a real number as an angle in trigonometric functions. They did not understand the concept of numerical trigonometry (Orhun, 2015). The most frequent errors made by students in solving problems in trigonometry include comprehension error, transformation error, and process skill error (Usman & Husaini, 2017). Comprehension errors occur when students do not understand how to approach a given trigonometric problem. The transformation errors occur during the computation process, especially during multiplication. Process skill errors occur during the manipulation of trigonometric ratios (Usman & Husaini, 2017). Brown (2005) identified different cognitive obstacles to understand trigonometry. They include a limited conception of the rotation of an angle and failure to relate the rotation on the unit circle to a point on the graph of the sine or the cosine function.

Computer-Assisted Instruction and GeoGebra

Computer-assisted instruction (CAI) or computer-assisted teaching (CAT) can be defined as an interactive teaching strategy that includes the use of computers and helps to present the teaching materials to students in a user-friendly format. These strategies promote learning, as well as to detect and remedy misconceptions (Gurbuz & Birgin, 2012). CAI is used in teaching mathematics and science, as well as many other disciplines. In the last decades, many softwares that support teaching and learning mathematics have been developed. Those softwares are of two forms: Computer Algebra Systems (CAS) and Dynamic Geometry Softwares (DGS). The CAS focus on the manipulation of symbolic expressions, the coordinates, and the equations. The DGS focus on the relationships between points, lines, and circles (Hohenwarter & Jones, 2007). Derive and Maple are examples of CAS, while Cabri and Sketchpad are examples of DGS. CAS and DGS greatly influence mathematics education. However, until the last decades, they were not used in an integrated manner (Kutluca, 2013). In the last decades, there has been a great evolution in mathematical softwares (Kutluca, 2013). Many forms of CAS have included graphing capacities to help to visualize Mathematics. Many forms of DGS have included elements of algebraic symbolization to be useful for more mathematical problems

(Hohenwarter & Jones, 2007). GeoGebra is a software that can be considered as both CAS and DGS. GeoGebra can be defined as CAS because it includes the symbolic and visualization features such as direct coding of equations, coordinates, and defining functions algebraically. It can also be defined as DGS since it includes concepts such as points, segments, lines, conic segments, and provides dynamic relationships between the concepts. (Kutluca, 2013)

GeoGebra contributes to developing the mathematical competencies of students, and their achievement in different fields of mathematics (Albaladejo, Garcia & Codina, 2015; Reis, 2010; Zengin, Furkan & Kutluca, 2012). Also, it contributes to the development of the students' conceptual understanding of different mathematical concepts (Kutluca, 2013; Zulnaidi and Zakaria, 2012). In particular, GeoGebra impacts students' achievement positively in trigonometry (Zengin, Furkan & Kutluca, 2012). It also motivates the students while learning trigonometry (Abdul Rahman & Puteh, 2016). Kepceoğlu and Yavuz (2016) showed that GeoGebra contributes to the development of the concept of periodicity of trigonometric functions.

In general, the CAI has positive effects on the overall performance of students in trigonometry (Naidoo & Govender, 2014; Blackett & Tall, 1991). The use of the Geometer's Sketchpad, the graphing calculators, and visualizations software impacts positively the students' understanding of trigonometry (Johari, Chan, Ramli & Ahmat, 2010; Choi-Koh, 2003 & Steckroth, 2007). Lofti and Mafi (2012) introduced the COTACSI software for trigonometry teaching and found that it has positive effects on the learning of trigonometry.

Computer-Assisted Instruction and Conceptual Change in Mathematics and Sciences

CAI may promote conceptual change in mathematics. In general, CAI may promote conceptual change in two ways: students realize discrepancies between their original ideas and alternative ideas, and they develop metacognitive skills such as planning, self-regulating, and monitoring (Liu & Hmelo-Silver, 2007). The CAI enhances the remediation of students' misconceptions in mathematics (Gurbuz and Bergin, 2012). Some softwares are designed to engage the students in cognitive conflicts (Cumming, Zangari & Thomason, 1995). Liu (2010) developed a CAI based on the cognitive conflict theory to correct students' statistical misconceptions.

Computers have many advantages that make them suitable to use during conceptual change instruction in science. One of those advantages is that they can simulate scientific phenomena effectively. Many unobservable scientific events can be visually represented and made accessible to students through the use of computers (Hameed, Hackling & Garnett, 1993). Many researchers showed that CAI is an effective conceptual change strategy in teaching Physics and Chemistry (Hameed, Hackling & Garnett, 1993; Talib, Matthew & Secombe, 2005; Trundle & Bell, 2009; Wiser & Amin, 2001). In some cases, CAI, based on the conceptual change model, promotes remedial of students' misconceptions and produces significant and lasting conceptual changes (Hameed, Hackling & Garnett, 1993). Using microcomputer simulations and conceptual change strategies together provides a significant conceptual change in physics (Zietsman & Hewson, 1986). Also, conceptual change instruction accompanied by demonstration and computer-assisted concept mapping causes a better acquisition of scientific conceptions (Yavuz, 2005).

Ozmen, Demircioglu, and Demircioglu (2009) think that the use of conceptual change texts and the use of computer animations have insufficiencies if used separately. So they suggested combining the two methods. Combining the use of the conceptual change texts and animations resulted in the remedial of many concepts and enhanced a better understanding of different chemical concepts in grade 11. Similar results were found when those methods are combined in grade 6 (Ozmen, 2011). Tas, Gulen, Oner, and Ozyurek (2015) compared the effects of the use of conceptual change texts and the use of web designed conceptual change texts. They showed that both methods decreased the students' conceptual errors in water chemistry. But this decrease was much more significant for the web-assisted conceptual change texts.

Sander and Heib (2014) compared the students' learning performance in trigonometry in three different sections. The first section included non-interactive learning, the second section included a conflict inducing interactive computer-supported learning, and the third section included interactive computer-supported learning that reduced the occurrence of cognitive conflict. The students of the section that included a conflict inducing interactive learning achieved significantly better results than the other two groups.

Sanger & Greenbowe (2000) compared the effect of both computer animations of microscopic chemical processes occurring in a galvanic cell and conceptual change instruction based on chemical demonstrations on college students' conceptions of current flow in electrolyte solutions. They compared the results of the verbal, conceptual questions, and visual questions for the control, animation, conceptual change, and animation/conceptual change groups. They found out that conceptual change instruction was effective at remedying student misconceptions for verbal, conceptual questions. On the other hand, animations did not affect students' responses to visual or verbal conceptual questions. An animation/conceptual change interaction for verbal, conceptual questions suggests that animations may cause distraction when the questions do not require students to visualize. They concluded that the effectiveness of the combination of computer animations and conceptual change instruction in changing students' conceptions might be attributed more to the use of conceptual change instruction and less to the use of computer animations, but they realized that their study had some limitations which are necessary to address in future research to make more definitive conclusions.

Yumusak, Maras & Sahin (2015) studied the effects of the use of computer-assisted instruction (CAI), conceptual change texts (CCT), computer-assisted instruction with conceptual change texts (CAI+CCT), and the use of traditional teaching method (TTM) on removing the misconceptions of science teacher candidates on the subject of radioactivity. They found out that CAI, CCT, and CAI + CCT groups were more successful than the traditional group in terms of removing misconceptions on radioactivity. Also, they found that combining the CAI and CCT had better results than the use of CAI and CCT alone. Finally, they did not found significant differences between the results of CAI and CCT groups.

Conclusion

The literature review shows that the development of instructional designs based on the conceptual change theory helps the students to remedy many misconceptions in science and mathematics. On the other hand, CAI helps students to achieve higher in science and mathematics. Many researchers studied the effect of the combination of the conceptual change methods and CAI in mathematics and science (Yumusak et al., 2015; Sanger et al., 2000; Tas et al., 2015; Ozmen et al. 2009; Ozmen, 2011; Hameed et al., 1993; Zietsman et al., 1986; Yavuz, 2005; Sander et al., 2014; Gurbuz et al., 2012; Cumming et al., 1995; Liu, 2010).

Trigonometry is a school discipline where students commonly face many difficulties and develop many misconceptions. However, research studies combining conceptual change methods and CAI in trigonometry are scarce. The combination of the conceptual change methods and GeoGebra software may help to remedy the common misconceptions, to overcome those difficulties and to promote students' conceptual change in trigonometry.

CHAPTER 3

METHODOLOGY

This chapter presents the research methods that were used in this research study to examine the impact of the three instructional designs (GeoGebra based explicit conceptual change (Geo + CC), GeoGebra based (Geo) and explicit conceptual change (CC)) on the conceptual enrichment, the overall conceptual change and the conceptual change in each of the concepts of periodicity, boundedness, non-monotonicity and non-linearity of trigonometric functions of students of different achievement levels. Also, it presents the research methods that were used to describe and analyze qualitatively the conceptual change associated with the three instructional designs.

This chapter presents the participants and the sampling procedures first. Then it describes the variables, the research designs, and the procedures of the interventions. Finally, it includes a description of the instruments, the data collection methods, and the data analysis procedures.

Participants

The participants of this research study are 51 grade 10 students from two K-12 Armenian private schools in Lebanon. The schools are selected based on the convenience of the researcher, who was the mathematics teacher in those schools. In this chapter, the schools are named as school A and school B. Both schools follow the Lebanese curriculum with extra classes of Armenian literature and Armenian history. The students of the two schools are Lebanese citizens of Armenian origin. The majority of the students are of middle socio-economic status.

In school A, grade 10 students are divided into two sections. By random selection, the first section was assigned to the Geo + CC instructional design, and the second section was assigned to Geo instructional design. The language of instruction of mathematics in school A is in French. On the other hand, school B has one section of grade 10 that was assigned to CC instructional design. In this school, the language of instruction of mathematics is English. All the students of the three sections have participated in this study without any exclusion.

The three sections include students of different achievements levels. The students had basic background knowledge in trigonometry covered in grade 9. Before the study, the researcher trained the students of the sections assigned to Geo + CC and Geo instructional designs to be able to use GeoGebra throughout the research study. This training was limited to providing the skills of using the basic tools of GeoGebra and was not related to trigonometry, particularly.

Variables

Independent Variable

Instructional design is the independent variable and varies in how the instruction of trigonometry is designed in the three sections. The three values of the instructional design are GeoGebra based explicit conceptual change (Geo + CC), GeoGebra based (Geo), and explicit conceptual change (CC).

Table 3.1

Distribution of the Students of the Three Instructional Designs based on Gender, School, and Language of Instruction

Instructional Design	Geo + CC		Geo		CC	
Gender	Male	Female	Male	Female	Male	Female
Frequency	12	6	14	5	6	8
School	A		A		B	
Language of instruction	French		French		English	

Dependent Variables

Two dependent variables are used: conceptual change and conceptual enrichment. The *conceptual enrichment* is the acquisition of the concepts that are not discrepant from the students' previous concepts; those concepts can be deduced from their prior knowledge through assimilation. Tardiff, Bascandziev, Carey and Zaitchik (2020) defined knowledge enrichment to involve learning information that can be represented with one's current conceptual repertoire.

The *conceptual enrichment* is measured by the structured conceptual enrichment items of the conceptual enrichment and conceptual change posttest (See Appendix 1)

The *conceptual change* is the acquisition of the concepts that are discrepant from the students' previous concepts; those concepts cannot be deduced directly from the students' prior knowledge. It requires accommodation of knowledge. Tariff et al. (2020) defined conceptual construction to involves acquiring knowledge that can only be represented in terms of concepts one does not yet possess.

This research study examines the conceptual change in the concepts of boundedness, periodicity, non-monotonicity, and non-linearity of trigonometric functions. *The conceptual change in the concept of boundedness* is the acquisition of the concept that the trigonometric functions have a limited image, unlike the other mathematical functions that the students have previously studied. The students have to acquire the concept that the sine and cosine functions are in [-1;1]. *The conceptual change in the concept of periodicity* is the acquisition of the concept that trigonometric functions repeat identically after each period, unlike the other mathematical functions that the students have studied previously. For example, they have to acquire that the numbers x and x + 2kπ are represented on the same point on the trigonometric circle; hence their sine and cosine are the same. *The conceptual change in the concept of non-monotonicity* is the acquisition of the concept that the trigonometric functions are increasing in some quadrants but decreasing in others, unlike other monotonic mathematical functions that the students have studied previously. *The conceptual change in the concept of non-linearity of trigonometric functions* is the acquisition of the concept that the trigonometric functions are not linear with the sum and multiplication by a real number. For example, the students have to acquire that $\sin(x + y) \neq sinx + siny$ and $\sin(ax) \neq asinx$.

The conceptual change of each of the following concepts is measured by the semi-structured items of the conceptual enrichment and conceptual change posttest. The semi-structured items include a comparison of the acquired knowledge with the students' prior knowledge. Thus, they measure the conceptual change explicitly in each of the concepts independently.

The overall conceptual change is the acquisition of trigonometric properties that include the concepts of boundedness, periodicity, non-monotonicity, and non-linearity of trigonometric functions. The acquisition of those concepts requires conceptual change. It is measured by the structured conceptual change items of the conceptual enrichment and conceptual change posttest. Those items do not compare students' acquired and previous knowledge explicitly; they do not measure the students' conceptual change explicitly; instead, they measure the properties of trigonometric functions that require a conceptual change in each of the above-mentioned concepts. In some cases, one item measures properties that require a conceptual change in more than one concept.

Moderator Variable

The *student achievement level* that represents the students' performance level is used as a moderator variable. It categorizes the students into low achievers, average students, and high achievers. The score on the pretest is used to categorize the students according to their achievement level. Students in the first quartile were classified as low achievers, students in the second and third quartiles as average achievers, and students in the fourth quartile as the high achievers.

Research Design

The mixed method of quantitative and qualitative approaches is applied in this study. A quasi-experimental 3x3 factorial design is used to measure the impact of the instructional designs on the conceptual enrichment, the overall conceptual change, and the conceptual change in each of the concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions of students of different achievement levels. The design is quasi-experimental research because it was not practical to reconstitute the class sections by random assignment of students to different instructional designs. Each section was assigned to the instructional design randomly. Statistical methods were used to mitigate the lack of random assignment.

The quantitative analysis aims to examine the differences in the impact of the three instructional designs on conceptual enrichment, the overall conceptual change, and the conceptual change in each of the concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions of students of different achievement levels.

The qualitative analysis aims to study in details the students' conceptual change in each of the concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions. It aims to identify the themes associated with the students' cognitive processes as defined by Revised Bloom Taxonomy (Krathwohl, 2002) on each of the concepts and to compare their frequencies in the three instructional designs. The Bloom's revised taxonomy is used because it includes metacognitive skills which may have an impact on the conceptual change of the students.

Procedures of the Intervention

The Geo + CC instructional design included the use of the GeoGebra software and the explicit conceptual change method. In this instructional design, the conceptual change was taught explicitly to students; learners were involved in activities where they recognized the need for the conceptual change and compared the new situations to their prior knowledge. The students used the GeoGebra software on their phones or on their personal computers during most of the activities of the study (See Appendix 2). Table 3.2 shows the main objectives of the activities conducted during each session of the Geo + CC instructional design.

In the CC instructional design, the instruction included the use of the conceptual change methods without the use of GeoGebra. All the conceptual change activities of the Geo + CC instructional design were included in the CC instructional design. Appendix 3 shows the conceptual change activities in the Geo + CC and CC instructional designs. The difference between those instructional designs is that in CC instructional design, the students constructed the trigonometric circle on a paper, and they drew the angles on that circle rather than using the GeoGebra software (See Appendix 4).

The Geo instructional design included the use of GeoGebra without the use of conceptual change strategies. The main difference between this instructional design and the other two instructional designs is that the students did not compare the new aspects that need conceptual change to their prior knowledge; thus, the conceptual change was taught implicitly. Extra examples and exercises replaced those comparisons (See Appendix 5)

Table 3.2

The Main Objectives of the Six Sessions of the Geo + CC Instructional Design

Session	Objectives
Session 1	• Familiarize to trigonometry and its uses • Define the trigonometric functions on a right triangle • Apply the definitions of the trigonometric functions to different mathematical and other situations • Show and use trigonometric identities
Session 2	• Define the trigonometric circle • Define the periodicity of the trigonometric functions. • Compare the periodicity of trigonometric functions to the non-periodic number line. • Recognize that the image of trigonometric functions is a bounded interval. • Compare this situation to other functions that have unbounded images. • Recognize the non-monotonicity of trigonometric functions • Compare this to other monotonic and non-monotonic functions
Session 3	• Define the quadrants of the trigonometric circle and the sign of the trigonometric functions in each quadrant. • Find out the quadrants where trigonometric functions are increasing and the quadrants where they are decreasing. • Compare this situation to other monotonic functions
Session 4	• Find out the trigonometric functions of some remarkable angles • Calculate trigonometric expressions. • Find out the non-linearity of trigonometric functions with addition and compare it to other linear operators • Find out the non-linearity of trigonometric functions with the multiplication with a real number and compare it to other linear operators.
Session 5	• Find out the relations of the trigonometric functions of associated angles • Find out the periodicity of trigonometric functions • Find out the non-linearity of the trigonometric functions with addition and compare it to other linear operators.
Session 6	• Calculate trigonometric lines of remarkable angles and their associated angles. • Simplify trigonometric expressions • Apply the non-linearity of trigonometric functions with the addition and the multiplication by a real number

The teacher of the three instructional designs was the researcher himself. In school A, the mathematics teacher of grade 9 observed two lessons of both of the sections and checked the conformity of the instruction to the instructional design. In school B, the science teacher of grade 8 did the observations during two lessons and checked the conformity of the instruction to the instructional design. A discsussion was held after those sessions with the teachers who confirmed the conformity of the instruction to the suggested lesson plans.

Data Collection Tools

Prior Knowledge Pretest

The prior knowledge pretest measures the prior knowledge of students in trigonometry. This test helps to overcome the limitations of the non-random assignment of students. In addition, it helps to categorize the students into low achievers, average students, and high achievers. It includes the basic definitions and properties of trigonometry that the students of grade 9 are introduced. In addition, it includes some concepts of non-linearity and non-monotonicity of functions that students already know (See Appendix 6).

Conceptual Enrichment and Conceptual Change Posttest

This test is designed to measure the conceptual enrichment and the conceptual change of students in trigonometry. It includes three sections. The first two sections include structured items, while the third section includes semi-structured items (See Appendix 1).

Structured items.

The first section of the posttest includes structured short answer questions where the students had to show their work. Some of those items measure the students' conceptual enrichment and are called conceptual enrichment items. Other items measure the students' understanding of trigonometric properties that require conceptual change and are called conceptual change items. The second section includes true or false structured questions where the students had to show their work. All the items of the second section are conceptual change items.

The conceptual enrichment items of the first section measure the ability of the students to use trigonometric functions, to find unknown sides of triangles and to show trigonometric identities. Also, they measure if the students are able to define trigonometric expressions of angles using the unit circle; to find out the principal measure and the quadrant of an angle and to find out the sign of trigonometric functions. Finally, they assess the ability of the students to calculate trigonometric expressions involving remarkable and associated angles.

The conceptual change items of the first and the second sections measure the students' understanding of trigonometric properties that include the concepts of boundedness, periodicity, non-monotonicity, and non-linearity of trigonometric functions. Those items do not compare students' acquired and previous knowledge explicitly; they do not measure the students' conceptual change explicitly; rather, they measure the students' understanding that requires conceptual change.

Semi-structured items.

The third section of the conceptual enrichment and conceptual change posttest includes four semi-structured questions, where the students had to show all the details of their reflection. These items measure the students' conceptual change in each of the above-mentioned concepts independently. They include a comparison of the acquired knowledge with the students' prior knowledge.

Validity and Reliability

The content validity of the instruments was established through the examination of grade 10 teachers. The researcher explained the research study, its objective, procedures, and instruments to three grade 10 mathematics teachers independently. The teachers stated whether each item measures its corresponding stated objective or not. The teachers had a positive degree of agreement of 90% for the items of the prior knowledge pretest and a positive degree of agreement of 88% for the items of the conceptual enrichment and conceptual change posttest.

The Cronbach alpha was used to check the internal reliability of the instruments. For the prior knowledge pretest, the Cronbach alpha was acceptable ($\alpha = .70$), and for the conceptual enrichment and conceptual change posttest, the Cronbach alpha was high ($\alpha = .86$).

Data Collection and Analysis Procedures

Data Collection Procedures

Before the intervention, the students of all three sections took the prior knowledge test, which lasted 30 minutes. The intervention included six sessions of 45 minutes over six days. The conceptual enrichment and conceptual change posttest, which lasted 90 minutes, was conducted after the intervention.

Data Analysis Procedures

Quantitative data analysis procedures.

Two-way ANCOVA was used to test for (1) differences in the impact of the three instructional designs on each of the conceptual enrichment, the overall conceptual change and the conceptual change in each of the concepts of boundedness, periodicity, non-monotonicity and non-linearity of trigonometric functions (2) interaction between the instructional design and student achievement level, controlling for the pretest score. The posttest score on the conceptual enrichment structured items was used as a measure of conceptual enrichment. The posttest score on the conceptual change structured items was used as a measure of the overall conceptual change. The four semi-structured items are graded using the conceptual change rubric (See Appendix 7). The conceptual change rubric is developed by the researcher. Its validity is established by the comparison of the grading of the researcher to the grading of another teacher. The four posttest scores on the four semi-structured questionnaires were used as measures of conceptual change in the four concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions. In this research study, p-values less than 0.1 are considered to be significant because of the small number of participants. Sample size strongly influences the p-value of a test. An effect that fails to be significant at a specified level alpha in a small sample can be significant in a larger sample. (Moore, McCabe, & Craig, 2016)

For a significant difference in the two-way ANCOVA, post hoc comparisons using the Scheffe test were used to identify pairwise differences in the impact of the three instructional designs on each of the dependent variables (conceptual enrichment, overall conceptual change and conceptual change in each of the concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions).

Qualitative data analysis procedures.

The qualitative data analysis followed the following steps:

1. Reading and coding the written answers to the semi-structured questionnaires. The coding scheme was based on the top-down deductive approach, and the conceptual change rubric was used as a framework for coding (See Appendix 7).
2. Each answer was coded according to the rubric.
3. Codes were grouped into different categories based on the different levels of the rubric.
4. The categories were then used to generate the themes by mapping each category to the cognitive process of the revised Bloom's taxonomy (Krathwohl, 2002).
5. Color coding was used to regroup the students' answers based on the emerging themes.

CHAPTER 4

RESULTS

The research study examines the differences of the impact of the three instructional designs (GeoGebra based explicit conceptual change (Geo + CC), GeoGebra based (Geo) and explicit conceptual change (CC)) on the conceptual enrichment, the overall conceptual change and the conceptual change in each of the concepts of periodicity, boundedness, non-monotonicity and non-linearity of trigonometric functions of students of different achievement levels. It also aims to describe and analyze qualitatively the conceptual change associated with the three instructional designs.

In this chapter, the quantitative and qualitative results of the study are presented. In the quantitative part, the differences of the impacts of the three instructional designs on each of the dependent variables (conceptual enrichment, overall conceptual change, and conceptual change in each of the concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions) are identified. In the qualitative part, the results of the semi-structured questionnaires of each of the concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions are described, categorized, and analyzed. The frequencies of the identified emerging themes are compared across instructional designs.

Quantitative Results

Two-way ANCOVA was used to test for (1) differences in the impact of the three instructional designs on each of the conceptual enrichment and the overall conceptual change in trigonometric functions of Lebanese grade 10 students (2) interaction between the instructional design and student achievement level, controlling for the pretest score. The posttest score on the conceptual enrichment test in trigonometric functions (Appendix 1) was used as a measure of conceptual enrichment. The posttest score on the conceptual change test in trigonometric functions (Appendix 1) was used as a measure of the overall conceptual change.

Two-way ANCOVA was used to test for (1) differences in the impact of the three instructional designs on each of the four targeted concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions of Lebanese grade 10 students (2) interaction between the instructional design and student achievement level, controlling for the pretest score. The four posttest scores on the four semi-structured questionnaires (Appendix 1) were used as measures of conceptual change in the four concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions.

For a significant difference in the two-way ANCOVA, post hoc comparisons using the Scheffe test were used to identify pairwise differences in the impact of the three instructional designs on each of the dependent variables (conceptual enrichment, overall conceptual change and conceptual change in each of the concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions).

Impact on Conceptual Enrichment and Overall Conceptual Change

The ANCOVA (Table 4.1) indicates that there is a significant difference ($p < .10$) in the impact of the three instructional designs on the students' overall conceptual change after controlling for the pretest scores. However, there is no significant difference in the overall conceptual change associated with student achievement level. The ANCOVA also shows that there is no significant interaction between the instructional design and achievement level on the students' overall conceptual change.

Table 4.1
ANCOVA of Instructional design by Achievement Level on Students' Overall Conceptual Change

Source	*df*	*MS*	*F*	*p*
Instructional design	2	57.37	2.48*	.09
Achievement level	2	4.66	.20	.82
Instructional design x Achievement level	4	35.85	1.55	.21

* $p < .10$

Post hoc comparisons using the Scheffe test (Table 4.2) indicates that *Geo + CC* instructional design has significantly (p < .10) more impact on the students' overall conceptual change than each of the Geo instructional design alone or CC instructional design alone. However, there is no significant difference between the Geo and CC instructional designs on the students' overall conceptual change.

The ANCOVA (Table 4.3) indicates that there is no significant difference ($p < .10$) in the impact of the three instructional designs on the students' conceptual enrichment after controlling for the pretest scores. But there is a significant difference ($p < .10$) in the conceptual enrichment associated with student achievement level. The ANCOVA also shows that there is no significant interaction between the instructional design and achievement level on the students' conceptual enrichment.

Table 4.2

Scheffe Pairwise Comparisons of Instructional Designs on Students' Overall Conceptual Change

Instructional design	Instructional design	Mean difference	*p*
Geo + CC	Geo	3.90*	0.06
	CC	3.90*	0.09
Geo	CC	.00	1.00

*p <.10

Table 4.3

ANCOVA of Instructional Design by Achievement Level on Students' Conceptual Enrichment

Source	*df*	*MS*	*F*	*p*
Instructional design	2	6.91	.75	.48
Achievement level	2	28.70	3.13*	.05
Instructional design x Achievement level	4	10.43	1.14	.35

*p <.10

* p <.05

In conclusion, there is evidence that the impact of combining the Geo and CC instructional designs on students' overall conceptual change is significantly more than each of Geo and CC instructional designs alone. However, there is no evidence to support a similar conclusion for conceptual enrichment. Moreover, the instructional design does not impact the overall conceptual change or conceptual enrichment of students with different achievement levels differently.

Impact of Instructional Designs on the Conceptual Change in the Four Targeted Concepts

In this section, the results of the two-way ANCOVA tests for each of the four targeted concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions of students of different achievement levels are presented. For a significant difference in the two-way ANCOVA, the results of the Scheffe pairwise comparison tests are presented.

Periodicity of trigonometric functions.

The ANCOVA (Table 4.4) indicates that there is a significant difference ($p < .05$) in the impact of the three instructional designs on the students' conceptual change in the concept of periodicity of trigonometric functions after controlling for the pretest scores. However, there is no significant difference in the conceptual change in the concept of periodicity of trigonometric functions associated with student achievement level. The ANCOVA also shows that there is no significant interaction between the instructional design and achievement level on the students' conceptual change in the concept of periodicity of trigonometric functions.

Table 4.4

ANCOVA of Instructional Design by Achievement Level on the Students' Conceptual Change in the Concept of Periodicity of Trigonometric Functions

Source	*df*	*MS*	*F*	*p*
Instructional design	2	13.97	7.17*	.00
Achievement level	2	1.46	.75	.48
Instructional design x Achievement level	4	2.20	1.13	.36

* $p < .05$

Post hoc comparisons using the Scheffe test (Table 4.5) indicate that each of Geo + CC and CC instructional designs has significantly ($p < .05$) more impact on the students' conceptual change in the concept of periodicity of trigonometric functions than the Geo instructional design. However, there is no significant difference between the Geo + CC and CC instructional designs on the students' conceptual change in the concept of periodicity of trigonometric functions.

Table 4.5

Scheffe Pairwise Comparisons of Instructional Designs on the Students' Conceptual Change in the Concept of Periodicity of Trigonometric Functions

Instructional design	Instructional design	Mean difference	*P*
Geo + CC	Geo	1.45*	.01
	CC	-.46	.65
Geo	CC	-1.91*	.00

*$p < .05$

Thus, there is support to the assertion that the CC instructional design, combined with Geo instructional design or stand-alone, has significantly more impact on the students' conceptual change in the concept of periodicity of trigonometric functions than the Geo instructional design. However, the instructional design does not impact the conceptual change in the concept of periodicity of trigonometric functions of students with different achievement levels differently.

Boundedness of trigonometric functions.

The ANCOVA (Table 4.6) indicates that there is a significant difference ($p < .05$) in the impact of the three instructional designs on the students' conceptual change in the concept of boundedness of trigonometric functions after controlling for the pretest scores. However, there is no significant difference in the conceptual change in the concept of boundedness of trigonometric functions associated with student achievement level. The ANCOVA also shows that there is no significant interaction between the instructional design and achievement level on the students' conceptual change in the concept of boundedness of trigonometric functions.

Post hoc comparisons using the Scheffe test (Table 4.7) indicate that the *Geo + CC* instructional design has significantly ($p < .05$) more impact on the students' conceptual change in the concept of boundedness of trigonometric functions than the CC instructional design. However, there is no significant difference between each of the Geo + CC and Geo instructional designs, as well as the Geo and CC instructional designs on the students' conceptual change in the concept of boundedness of trigonometric functions.

Table 4.6

ANCOVA of Instructional Design by Achievement Level on the Students' Conceptual Change in the Concept of Boundedness of Trigonometric Functions

Source	*df*	*MS*	*F*	*p*
Instructional design	2	10.81	4.43*	.02
Achievement level	2	1.16	.48	.62
Instructional design x Achievement level	4	2.36	.97	.44

**p* <.05

Table 4.7

Scheffe Pairwise Comparisons of Instructional Designs on the Students' Conceptual Change in the Concept of Boundedness of Trigonometric Functions

Instructional design	Instructional design	Mean difference	*p*
Geo + CC	Geo	.91	.22
	CC	1.85*	.01
Geo	CC	.94	.24

**p* <.05

Thus the impact of combining the Geo and CC instructional designs on students' conceptual change in the concept of boundedness is significantly more than the CC instructional design alone. Even though the impacts of the Geo and CC instructional designs on students' conceptual change in the concept of boundedness are not significantly different, but when combined the impact is mainly due to the Geo instructional design. On the other hand, the instructional design does not impact the conceptual change in the concept of boundedness of trigonometric functions of students with different achievement levels differently.

Non-monotonicity of trigonometric functions.

The ANCOVA (Table 4.8) indicates that there is no significant difference ($p < .10$) in the impact of the three instructional designs on the students' conceptual change in the concept of non-monotonicity of trigonometric functions after controlling for the pretest scores. Also, the ANCOVA shows that there is no significant difference in the conceptual change in the concept of non-monotonicity of trigonometric functions associated with student achievement level. Besides, it shows that there is no significant interaction between the instructional design and achievement level on the students' conceptual change in the concept of non-monotonicity of trigonometric functions.

Table 4.8

ANCOVA of Instructional Design by Achievement Level on the Students' Conceptual Change in the Concept of Non-Monotonicity of Trigonometric Functions

Source	*df*	*MS*	*F*	*P*
Instructional design	2	.47	.18	.84
Achievement level	2	.18	.07	.94
Instructional design x Achievement level	4	2.51	.94	.45

Thus, the three instructional designs do not impact differently on the students' conceptual change in the concept of the non-monotonicity of trigonometric functions. Also, they do not affect the conceptual change in the concept of non-monotonicity of trigonometric functions of students with different achievement levels differently.

Non-linearity of trigonometric functions.

The ANCOVA (Table 4.9) indicates that there is a significant difference ($p < .05$) in the impact of the three instructional designs on the students' conceptual change in the concept of non-linearity of trigonometric functions after controlling for the pretest scores. However, there is no significant difference in the conceptual change in the concept of non-linearity of trigonometric functions associated with student achievement level. On the other hand, the ANCOVA also shows that there is a significant interaction ($p < .05$) between the instructional design and achievement level on the students' conceptual change in the concept of non-linearity of trigonometric functions.

Table 4.9

ANCOVA of Instructional Design by Achievement Level on the Students' Conceptual Change in the Concept of Non-Linearity of Trigonometric Functions

Source	*df*	*MS*	*F*	*P*
Instructional design	2	6.77	8.99*	.00
Achievement level	2	.82	1.09	.35
Instructional design x Achievement level	4	3.17	4.21*	.01

* $p < .05$

Post hoc comparisons using the Scheffe test (Table 4.10) indicates that each of Geo + CC and CC instructional designs has significantly ($p<.05$ and $p< .10$ respectively) more impact on the students' conceptual change in the concept of non-linearity of trigonometric functions than the Geo instructional design. However, there is no significant difference between the Geo + CC and CC instructional designs on the students' conceptual change in the concept of non-linearity of trigonometric functions.

Table 4.10

Scheffe Pairwise Comparisons of Instructional Designs on the Students' Conceptual Change in the Concept of Non-Linearity of Trigonometric Functions

Instructional design	Instructional design	Mean difference	*p*
Geo + CC	Geo	1.15*	.01
	CC	.42	.52
Geo	CC	-.73**	.07

* p <.05

** p <.10

Post hoc comparisons using the Scheffe test (Table 4.11) indicates that there is no significant difference between the three instructional designs on the low achievers' conceptual change in the concept of non-linearity.

Post hoc comparisons using the Scheffe test (Table 4.12) indicates that each of Geo + CC and CC instructional designs has significantly ($p<.05$) more impact on the average students' conceptual change in the concept of non-linearity of trigonometric functions than the Geo instructional design. However, there is no significant difference between the Geo + CC and CC instructional designs on the students' conceptual change in the concept of non-linearity of trigonometric functions.

Table 4.11

Scheffe Pairwise Comparisons of Instructional Designs on Low Achievers' Conceptual Change in the Concept of Non-linearity of Trigonometric Functions

Instructional design	Instructional design	Mean difference	*p*
Geo + CC	Geo	-.80	.30
	CC	-.92	.13
Geo	CC	-0.13	.96

Table 4.12

Scheffe Pairwise Comparisons of Instructional Designs on Average Students' Conceptual Change in the Concept of Non-linearity of Trigonometric Functions

Instructional design	Instructional design	Mean difference	*p*
Geo + CC	Geo	1.88*	.00
	CC	-.33	0.83
Geo	CC	-1.55*	.02

Post hoc comparisons using the Scheffe test (Table 4.13) indicate that Geo + *CC* instructional design has significantly ($p<.05$) more impact on the high achievers' conceptual change in the concept of non-linearity of trigonometric functions than the Geo instructional design. However, there is no significant difference between each of the Geo + CC and CC instructional designs, as well as the Geo and CC instructional designs on the high achievers' conceptual change in the concept of non-linearity of trigonometric functions.

Table 4.13

Scheffe Pairwise Comparisons of Instructional Designs on High Achievers' Conceptual Change in the Concept of Non-linearity of Trigonometric Functions

Instructional design	Instructional design	Mean difference	*p*
Geo + CC	Geo	1.68*	.09
	CC	1.10	.38
Geo	CC	-0.58	.79

* $p < .10$

Thus, there is evidence that the CC instructional design, combined with the Geo instructional design or stand-alone, has more impact on the students' conceptual change in the concept of non-linearity of trigonometric functions than the Geo instructional design. A more in-depth analysis shows that a similar conclusion is correct for average students, but it is not true for low and high achievers. For the low achievers, the three instructional designs do not impact the students' conceptual change in the concept of the non-linearity of trigonometric functions differently. For high achievers, the impact of combining the Geo and CC instructional designs on students' conceptual change in the concept of non-linearity is significantly more than each of Geo and CC instructional designs alone. The Graph 1 represents the Scheffe interaction of the instructional design by achievement level on the conceptual change in the concept of non-linearity of trigonometric functions.

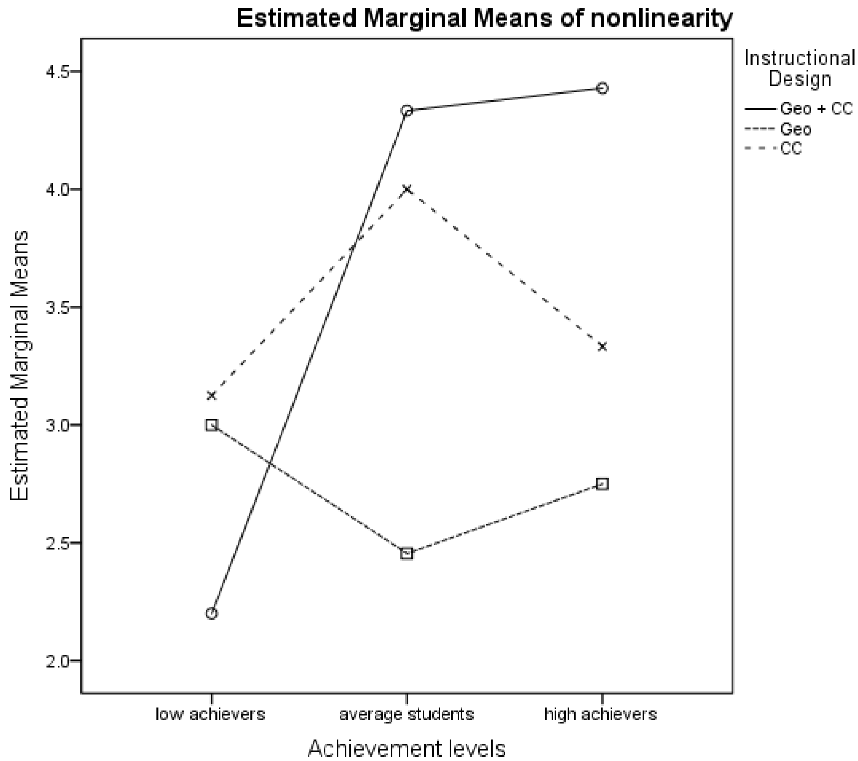

Figure 4.1: Scheffe interaction of instructional design by achievement level on the conceptual change in the concept of non-linearity of trigonometric functions

Summary of Quantitative Results

The three instructional designs impact the students' overall conceptual change in trigonometry differently. The impact of combining the Geo and CC instructional designs on students' overall conceptual change is significantly more than each of Geo and CC instructional designs alone. However, the three instructional designs do not impact the students' conceptual enrichment differently. Moreover, they do not have differential impacts on the overall conceptual change or conceptual enrichment of students with different achievement levels.

The impact of the instructional design differs for the targeted concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions. The CC instructional design, combined with Geo instructional design or stand-alone, has significantly more impact than the Geo instructional design on the students' conceptual change in the concepts of periodicity and non-linearity of trigonometric functions. The impact of combining the Geo and CC instructional designs on the students' conceptual change in the concept of boundedness is significantly more than the CC instructional design alone. When the two instructional designs are combined, the impact is mainly due to the Geo instructional design. The three instructional designs do not impact the students' conceptual change in the concept of non-monotonicity of trigonometric functions differently.

Finally, the instructional design does not impact the conceptual change in each of the concepts of periodicity, boundedness, and non-monotonicity of trigonometric functions of students with different achievement levels differently. But it does impact the conceptual change in the concept of non-linearity of trigonometric functions of students with different achievement levels differently.

Table 4.14 summarizes the impact of the instructional designs on the overall conceptual change and the conceptual change in each of the concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions.

Table 4.14

Summary of Scheffe for the Instructional Designs on the Overall Conceptual Change and the Conceptual Change in Each of the Concepts of Periodicity, Boundedness, non-Monotonicity, and non-Linearity of Trigonometric Functions

Instructional design I	Instructional Design J	Overall Conceptual Change (I-J)	Periodicity (I-J)	Boundedness (I-J)	Non-monotonicity (I-J)	Non-linearity (I-J)
Geo + CC	Geo	*	**	nsd	Nsd	**
	CC	*	nsd	**	Nsd	nsd
CC	Geo	nsd	**	nsd	Nsd	*

** positive significant difference *(p<.05)*
* positive significant difference *(p<.10)*
nsd no significant differences

Qualitative Results

This section presents the qualitative analysis of the students' written answers for the semi-structured questionnaire on each of the concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions. In this section, the emerging themes, the categories, and the cognitive processes are defined. Also, the frequencies of the emerging themes in the three instructional designs are compared for each of the four concepts.

Emerging Themes

The themes emerged from the students' answers to the semi-structured questionnaires using a top-down deductive approach. The conceptual change rubric

(Appendix 7) was used as a framework for coding. The codes were grouped into different categories (refer to the second column of Table 4.15). The categories were then used to generate the themes by mapping each category to the cognitive process of the revised Bloom's taxonomy. (refer to table 4.15 and chapter 3) .

The first theme is the non-recognition of conceptual change in trigonometric functions. This theme is defined by blank answers, incorrect answers, or application of the students' prior knowledge for each of the concepts of boundedness, periodicity, non-monotonicity, and non-linearity of trigonometric functions. The second theme is the recognition of the conceptual change in trigonometric functions. This theme is on the lowest level of the revised Bloom's taxonomy; it requires only remembering. The third theme is explaining or exemplifying the conceptual change in trigonometric functions. It includes the answers of the students who explain the property that requires conceptual change correctly and the students who provide examples and counterexamples explaining the property that requires conceptual change. This theme is at the level of understanding in revised Bloom's taxonomy. Finally, the fourth theme is the differentiation of the conceptual change in trigonometric functions from the students' prior knowledge. This theme is at the level of analyzing in Bloom's revised taxonomy. Table 4.15 summarizes those definitions.

In the next section, for each of the concepts of boundedness, periodicity, non-monotonicity, and non-linearity of trigonometric functions, examples of the coding of the students' answers of different categories are presented. Also, the frequencies of each of the themes of non-recognition, recognition, explanation/exemplification, and differentiation of conceptual change are compared in the three instructional designs.

Table 4.15

Emerging Themes, Categories, and their Cognitive Process in Bloom's Revised Taxonomy

Theme	Category	Cognitive Process
Non-recognition of conceptual change in trigonometric functions	- No answer or irrelevant answer - - Application of the students' prior knowledge to the trigonometric functions	
Recognition of conceptual change in trigonometric functions	- Identification of conceptual change in trigonometric functions	Remember
Explaining/Exemplifying conceptual change in trigonometric functions	- Explanation of conceptual change - Use of examples and counterexamples that explain the conceptual change in trigonometric functions	Understand
Differentiating between examples that require conceptual change and examples of the students' prior knowledge	- Differentiation of the property that requires a conceptual change from the students' prior knowledge	Analyze

Boundedness.

Below are examples of students' answers to the question about the boundedness of trigonometric functions: What are the possible values of the sine and cosine functions? Why? Compare this to other mathematical functions that you have dealt with (such as the double, the radical, the square functions)

Examples of student responses.

- **Category 1**: **Identification of conceptual change in trigonometric functions**:

  Example 1: sin et cos entre -1 et 1 [the student can recognize that the sine function is bounded in [-1;1]

  Example 2: Les valeurs possibles des fonctions sinus et cosinus est [-180;180] [the student can recognize that the sine function is bounded, but he/she considers the boundaries to be [-180 and 180].

- **Category 2: Explanation of conceptual change in trigonometric functions**

  Example 1: car le rayon du cercle trigonometrique est 1. [the student explains the conceptual change in trigonometric functions by stating that the trigonometric circle has a radius 1]

  Example 2: car $sin^2x + cos^2x = 1$ donc il peut que sin^2x ou cos^2x est 1 et l'autre est 0 mais sin^2x et cos^2x ne peuvent pas etre inférieur a 1. [the student explains the conceptual change by using the Pythagoras theorem in trigonometry]

- **Category 3: Differentiation of the property that requires a conceptual change from the prior knowledge**

  Example: Un double, une racine carrée et un carré peuvent etre plus grand que 1

  soit x=2 2x =4>1

  Soit x =4 $\sqrt{4} = 2 > 1$

  Soit x=2 $x^2 = 4 > 1$

  [The student states that the square root, the double and the square can be bigger than one and they provide examples]

Comparison of the three instructional designs on the emerging themes on the concept of boundedness of trigonometric functions. Table 4.16 shows that the distributions of the themes for the three instructional designs are different with the cell of non-recognition for the Geo + CC instructional design having significantly less percentage than what is expected by chance (adjusted residual <-2), whereas the CC instructional design has higher percentage of non-recognition and lower percentage of recognition than what is expected by chance (adjusted residual >2). Also, the mode of the Geo + CC is the cell of recognition, whereas the mode for the CC instructional design is the cell of non-recognition.

The Geo + CC and Geo instructional designs have many similarities. Both instructional designs have their mode in the theme of recognition of conceptual change. The Geo + CC and Geo instructional designs are not different in the cells of recognition, explaining/exemplifying, and differentiation of conceptual change. The distributions of the themes for the Geo + CC and Geo instructional designs are different for the cell of non-recognition only. The Geo + CC instructional design, has significantly less percentage than what is expected by chance (adjusted residual <-2).

Moreover, the themes of non-recognition and recognition have 70% of the total distribution, whereas the percentages of the students' answers that were categorized in the themes of explaining/exemplifying and differentiating have much lower percentages. In those two themes, the highest frequencies are for the Geo + CC instructional design followed by the Geo instructional design and, finally, the CC instructional design.

Table 4.16:

Frequencies and Percentages of Student Responses According to the Themes of Conceptual change of the Concept of Boundedness

	Non-recognition		Recognition		Explaining /exemplifying		Differentiating		Total	
Geo + CC	4**	15.4%	13	50%	4	15.4%	5	19.2%	26	100%
Geo	9	37.5%	10	41.7%	2	8.3%	3	12.5%	24	100%
CC	12*	80%	2**	13.3%	1	6.7%	0	0.0%	15	100%
Total	25	38.5%	25	38.5%	7	10.8%	8	12.3%	65	100%

*Adjusted residual >2
**Adjusted residual < -2

Those results show that the Geo + CC and Geo instructional designs have higher distributions on the cognitive processes that are higher on the revised Bloom's taxonomy than the CC instructional design for the concept of boundedness of trigonometric functions

Periodicity

Below are examples of students' answers to the question about the periodicity of trigonometric functions: On a given number line, is it possible to find two

different numbers that are represented by the same point? Why? On a given trigonometric circle, is it possible to find two different angles that have the same extremity? Why?

Examples of student responses.

- **Category 1**: **Identification of conceptual change in trigonometric functions**:

  Example "Sur un cercle trigonométrique donné, il est certainement possible de trouver deux réels différents qui sont représentés par le même point "

[the student states that it is certainly possible to find two different real numbers that are represented by the same point on a trigonometric circle]

- **Category 2: Explanation of conceptual change in trigonometric functions**

  Example: « car le cercle trigonometrique est périodique » [the student explains by stating that the trigonometric circle is periodic]

- **Category 3: Identification of examples and counterexamples that explains the conceptual change in trigonometric functions**

  Example: Comme par exemple 30 et 390 sur le cercle c'est le meme point [the student provides the example of 30 and 390 which are presented by the same point on the trigonometric circle]

- **Category 4: Differentiation of the property that requires a conceptual change from the prior knowledge**

  Example: Non sur un axe numerique un point est representé par un seul réel; par exemple si le réel est 1, ce réel est sur l'abscisse, il n'a pas une autre place sur l'axe [the student compares the trigonometric circle to the number line and states that on a number line each real number is presented by only one point. The student also provides an example, he/she states that the real number one has a unique abscissa and can not be presented in another place on the number line]

Comparison of the three instructional designs on the emerging themes on the concept of periodicity of trigonometric functions. Table 4.17 shows that the distributions of the themes for the three instructional designs are different for the cell of non-recognition for the CC instructional design having significantly less percentage than what is expected by chance (adjusted residual <-2) whereas the Geo instructional design has higher percentage of non-recognition and lower percentage of explaining/exemplifying than what is expected by chance (adjusted residual >2). Also, the modes of the Geo + CC and CC instructional designs are in the themes of recognition and differentiation, followed by the theme of explaining/exemplifying. Very few answers of the Geo + CC and CC instructional designs are categorized in the theme of non-recognition, whereas the mode for the Geo instructional design is in the theme of non-recognition, which has a frequency of around 50%.

Table 4.17:

Frequencies and Percentages of Student Responses According to the Themes of Conceptual change of the Concept of Periodicity

Instructional design	Non-recognition		Recognition		Explaining /exemplifying		Differentiating		Total	
Geo + CC	4	9.3%	14	32.6%	11	25.6%	14	32.6%	43	100%
Geo	13*	46.4%	6	21.4%	2**	7.1%	7	-0.8%	28	100%
CC	2**	5.6%	12	33.3%	10	27.8%	12	33.3%	36	100%
Total	19	17.8%	32	29.9%	23	21.5%	33	30.8%	107	100%

*Adjusted residual >2
**Adjusted residual < -2

Those results show that the Geo + CC and CC instructional designs have higher distributions on the cognitive processes that are higher on the revised Bloom's taxonomy than the Geo instructional design for the concept of periodicity of trigonometric functions.

Non-monotonicity.

Below are examples of students' answers to the question about the non-monotonicity of trigonometric functions: If x<y is sinx<siny? Explain your answer.

Compare this situation to the double function, to the cosine function, and the square function.

Examples of student responses

- **Category 1**: **Application of the students' prior knowledge:**

  Example: If $x<y$ then $sinx <siny$ [the student thinks that the order is conserved with trigonometric functions, he/she applies his/her prior knowledge of the conservation of order]

- **Category 2**: **Identification of conceptual change in trigonometric functions**:

  Example: Si $x<y$ donc sinx n'est pas necessairement plus grand que siny [the student states that the order is not necessarily conserved]

- **Category 3: Explanation of conceptual change in trigonometric functions**

  Example: Si x et y se trouvent dans le 1ere quadrant $sinx<siny$

  2eme quadrant $sinx>siny$

  3eme quadrant $sinx>siny$

  4eme quadrant $siny>sinx$

[the student explains the non-montonicity of trigonometric functions, by stating the quadrants where the order is conserved and the quadrant where the order is not conserved.]

- **Category 4: Identification of examples and counterexamples that explains the conceptual change in trigonometric functions**

  Example : x=90 y =270 x<y mais sinx =1 siny= -1 sinx >siny

  [the student provides a counterexample as an explanation. He/she provides an example where x<y but sinx>siny]

- **Category 5: Differentiation of the property that requires a conceptual change from the prior knowledge**

  Example: $Si\ x < y\ alors\ 2x < 2y$ Si $0 < x < y$ alors $x^2 < y^2$

  $Si\ x < y < 0\ alors\ x^2 > y^2$

  Si $x < y$

  1er quadrant cosy<cosx

  2eme quadrant cosy<cosx

  3eme quadrant cosx<cosy

  4eme quadrant cosx>cosy

[the student compares the non-monotonicity of the sine function to the monotonicity of the double function, and the non-monotonicity of the square and cosine functions]

Comparison of the three instructional designs on the emerging themes on the concept of non-monotonicity of trigonometric functions. The Table 4.18 shows that the distributions of the themes for the three instructional designs are different with the cell of differentiation for the Geo instructional design having significantly less percentage than what is expected by chance (adjusted residual<-2), whereas the CC instructional design has higher percentage of non-recognition (adjusted residual > 2) and lower percentage of explaining/exemplifying than what is expected by chance (adjusted residual <-2). Also, the modes of the Geo + CC and Geo instructional designs are in the theme of recognition, whereas the mode for the CC instructional design is in the theme of non-recognition.

Moreover, the themes of non-recognition and recognition have 70% of the total distribution, whereas the percentages of the students' answers that were categorized in the themes of explaining/exemplifying and differentiating have much lower percentages.

Table 4.18

Frequencies and Percentages of Student Responses According to the Themes of Conceptual Change of the Concept of Non-Monotonicity

Instructional design	Non-recognition		Recognition		Explaining /exemplifying		Differentiating		Total	
Geo + CC	6	22.2%	12	44.4%	6	22.2%	3	11.1 %	27	100%
Geo	7	25%	12	42.9%	9	32.1%	0**	0.0%	28	100%
CC	9*	52.9%	5	29.4%	0**	0.0 %	3	17.6%	17	100%
Total	22	30.6%	29	40.3%	15	20.8%	6	8.3%	72	100%

*Adjusted residual >2

**Adjusted residual < -2

Those results show that for the concept of non-monotonicity of trigonometric functions, the Geo + CC instructional design has higher distributions on the cognitive processes that are higher on the revised Bloom's taxonomy than the CC instructional design. Similarly, the Geo instructional design has higher distributions on the cognitive

processes that are higher on the revised Bloom's taxonomy than the CC instructional design except for the theme of differentiation. For the theme of differentiation which is the highest on Bloom's revised taxonomy, the Geo instructional design has less percentage than what is expected by chance

Non-linearity.

Below are examples of students' answers to the question about the non-linearity of trigonometric functions: Is sin3x =3sinx? Explain your answer. Compare this situation to other mathematical functions that you have dealt with.

Examples of student responses.

- **Category 1: Application of students' prior knowledge**

  Example: sin3x=3sinx [the student thinks that the sine function is linear]

- **Category 2: Identification of conceptual change in trigonometric functions**:

  Example: sin3x≠ 3sinx [the student stated that sin3x≠ 3sinx]

- **Category 3: Explanation of conceptual change in trigonometric functions**

  Example 1: Dans sin3x c'est l'angle x qu'on a multiplié par 3 Dans 3sinx l'angle reste x c'est le sinus qu'on a multiplié par 3 [the student states that in sin3x the angle is multiplied by 3, but in 3 sinx the angle is the same, but the sine is multiplied by 3]

  Example 2: The sine is not linear [the student states that the sine function is not linear]

- **Category 4: Identification of examples and counterexamples that explains the conceptual change in trigonometric functions**

  Example: Car par exemple sin(3*30)=sin90 est différent de 3sin30 [the student provides a counterexample and states that sin(3*30)$\neq$ 3sin30]

- **Category 5: Differentiation of the property that requires a conceptual change from the prior knowledge**

  Example: Since it is not like algebra: (2*3)x=3(2x) [the student compares the trigonometric functions to algebra and provides an example in Algebra where the linearity holds]

Comparison of the three nstructional designs on the emerging themes on the concept of non-linearity of trigonometric functions. Table 4.19 shows that the distributions of the themes for the three instructional designs are different for the cell of differentiation for the Geo instructional design having significantly less percentage than what is expected by chance (adjusted residual<-2). On the other hand, the modes of the three instructional designs are in the theme of recognition, which has around 41% of the overall distribution. No significant differences exist in the distributions of the themes of non-recognition, recognition, and explaining/exemplifying the conceptual change.

The Geo + CC and CC instructional designs do not have differences in the distributions of the cognitive processes. Geo instructional design has similar distributions for the themes of non-recognition, recognition, and explaining/exemplifying. On the other hand, the Geo instructional design has less percentage than what is expected by chance for the theme of differentiation, which is the highest on Bloom's revised taxonomy.

Table 4.19

Frequencies and Percentages of Student Responses According to the Themes of Conceptual Change of the Concept of Non Linearity

Instructional design	Non-recognition		Recognition		Explaining /exemplifying		Differentiating		Total	
Geo + CC	5	14.7%	13	38.2%	6	17.6%	10	29.4%	34	100%
Geo	7	28.0%	12	48.0%	6	24.0%	0**	0.0%	25	100%
CC	3	10.3%	11	37.9%	8	27.6%	7	24.1%	29	100%
Total	15	17.0%	36	40.9%	20	22.7%	17	19.3%	88	100%

*Adjusted residual >2

**Adjusted residual < -2

Conclusion of the Qualitative Analysis

The qualitative analysis, which aimed at the identification of the cognitive processes among the three instructional designs, shows that the saliency of the emerging themes differs for each of the four concepts in the three instructional designs.

For the concept of boundedness, the Geo + CC and Geo instructional designs have higher distributions on the cognitive processes that are higher on the revised Bloom's taxonomy than the CC instructional design. On the other hand, for the concept of periodicity, the Geo + CC and CC instructional designs have higher distributions on the cognitive processes that are higher on the revised Bloom's taxonomy than the Geo instructional design.

For the concept of non-monotonicity of trigonometric functions, the Geo + CC instructional design has higher distributions on the cognitive processes that are higher on the revised Bloom's taxonomy than the CC instructional design. Similarly, the Geo instructional design has higher distributions on the cognitive processes that are higher on the revised Bloom's taxonomy than the CC instructional design except for the theme of differentiation. For the theme of differentiation which is the highest on Bloom's revised taxonomy, the Geo instructional design has less percentage than what is expected by chance

For the concept of non-linearity, the Geo + CC and CC instructional designs do not have differences in the distributions of the cognitive processes. Geo instructional design has similar distributions for the themes of non-recognition, recognition and explaining/exemplifying. On the other hand, the Geo instructional design has less percentage than what is expected by chance for the theme of differentiation which is the highest on Bloom's revised taxonomy.

CHAPTER 5

DISCUSSION

In this research study, the quantitative analysis aimed to examine the differences of the impact of the three instructional designs (GeoGebra based explicit conceptual change (Geo + CC), GeoGebra based (Geo) and explicit conceptual change (CC)) on the conceptual enrichment, the overall conceptual change and the conceptual change in each of the concepts of periodicity, boundedness, non-monotonicity and non-linearity of trigonometric functions of students of different achievement levels. On the other hand, the qualitative analysis aimed to identify the students' cognitive processes associated with their conceptual change among the three instructional designs.

This chapter discusses the results of this research study. It discusses the impact of the instructional design on the overall conceptual change and the conceptual change in each of the concepts of periodicity, boundedness, non-monotonicity, and non-linearity of trigonometric functions. It also discusses the influence of the instructional design on the conceptual change of students of different achievement levels, as well as their conceptual enrichment, and their conceptual enrichment according to their level of achievement level. Then the emerging themes and their saliency for each of the four concepts in the three instructional designs are discussed. Finally, this chapter identifies its limitations and implications for practice and further research.

Impact of Instructional Design on Overall Conceptual Change

The three instructional designs (GeoGebra based (Geo), explicit conceptual change (CC), and GeoGebra based explicit conceptual change (Geo + CC)) impact students' overall conceptual change in trigonometry differently. The impact of combining the Geo and CC instructional designs on students' overall conceptual change is more than the impact of each of Geo and CC instructional designs alone.

The research study of Sander and Heib (2014) is relevant to this research study in the sense that it is the only research in the field of trigonometry that deals with conceptual change and cognitive conflict. Sander and Heib (2014) compared students' learning performance in trigonometry in three different sections. Those sections were similar to the three instructional designs of this research study. The first section included non-interactive learning, the second section contained a conflict inducing interactive computer-supported learning, and the third section included interactive computer-supported learning that reduced the occurrence of cognitive conflict. The students of the section that included a conflict inducing interactive learning achieved significantly better results than the other two groups during the test after six weeks. This result is similar to that found in this research study which shows that combining Geo + CC had better results than each method used alone even though the conflict inducing and reducing methods in Sander's and Heib's research study (2014) were very different from the conceptual change methods of this research study.

The results of this research study are discussed in terms of the conceptual change theory and the constructivist theory. Tirosh and Tsamir (2004) predicted cases in mathematics where the prior knowledge of students is not sufficient for the development of the understanding of a mathematical notion, and that radical reorganization of existing knowledge structures is likely to be required. The conceptual change theory suggests that learners must be dissatisfied with an initial conception to abandon it and accept a scientific conception for successful conceptual change (Posner et al., 1982). A method suggested by the conceptual change theorists is the explicit comparison of learners' new knowledge and their prior knowledge (Van Dooren et al., 2004; Stepans, 1994; Limon, 2001). In this research study, the Geo + CC instructional design has more impact than the Geo instructional design on the learners' overall conceptual change. This result is compatible with the conceptual change theory. On the other hand, the Geo + CC instructional design has better results than the CC instructional design on the students' overall conceptual change; this shows a discrepancy with the conceptual change theory. The conceptual change methods are not enough to enhance learners' conceptual change; other factors are needed.

The results of this research study may also be interpreted in terms of the constructivist theory in two ways. First, constructivism's main idea is that learners actively construct new understandings using what they already know (Bada, 2015). In this research study, learners have constructed their new knowledge comparing it to their prior knowledge. The conceptual change is based on the contrast with the prior knowledge rather than its smooth generalization. Second, the use of technology in the classroom engages the students to construct their knowledge actively. This research study includes the use of GeoGebra, which can be considered to be a tool for the constructivist approach. Learners use GeoGebra individually or by groups; they draw curves, check results, and construct knowledge. The use of explicit conceptual change methods and the use of GeoGebra are considered to be two different strategies of the constructivist theory. In this context, their combination has promoted a better impact on learners' overall conceptual change than each method used alone.

Impact of Technology on Conceptual change

Trundle and Bell (2009) showed that CAI is an effective conceptual change strategy in teaching physics. This research study shows that the Geo instructional design alone has less impact on the students' overall conceptual change than the Geo + CC instructional design. In this research study, the GeoGebra does not seem to be a tool to enhance conceptual change when it is used alone. The results of this research study do not seem to be compatible with the results of the researches that show that CAI is an effective conceptual change strategy in science education.

Other researchers combined conceptual change methods and CAI and found a better understanding of different concepts in chemistry. (Ozmen, Demircioglu & Demircioglu, 2009; Ozmen, 2011; Tas, Gulen, Oner & Ozyurek, 2015) Others found similar results in physics (Zietsman & Hewson, 1986; Yavuz, 2005; Yumusak, Maras & Sahin (2015) Hameed, Hackling & Garnett, 1993; Talib, Matthew & Secombe, 2005; Trundle & Bell, 2009; Wiser & Amin, 2001). This research study found similar results in mathematics. The Geo + CC instructional design has a better impact than the Geo and CC instructional designs on the overall conceptual change in trigonometry.

Yumusak, Maras & Sahin (2015) found that combining the CAI and the conceptual change methods had better results than the use of CAI and conceptual change methods alone. Also, they found that the results of combining the two methods or using each method alone were better than the results of the control group. Finally, they have not found differences between the results of groups where the two methods were applied separately. Those results are very similar to the results of this research study, where the Geo + CC had a better impact on the students' overall conceptual change than Geo and CC instructional designs. Also, no difference was found between the Geo and CC instructional designs. The main difference with this research study is that Yumusak, Maras, and Sahin (2015) used a control group. Also, the participants were teacher candidates rather than being school students.

Impact of Instructional Design on Periodicity, Non-linearity, Boundedness, and Non-Monotonicity

The effect of the instructional designs on the targeted concepts of periodicity, non-linearity, boundedness, and non-monotonicity of trigonometric functions is not the same. The CC instructional design, combined with Geo instructional design or stand-alone, has more impact than the Geo instructional design on students' conceptual change in the concepts of periodicity and non-linearity of trigonometric functions. Those concepts are discrepant from the learners' prior knowledge. Learners were able to understand the definitions of those concepts through explicit comparison with their prior knowledge. They defined the notion of non-linearity by comparing it to linear operators and identified the concept of periodicity by comparing it to the non-periodicity of the number line. On the other hand, the non-linearity and the periodicity are not visual concepts in the unit circle context of teaching trigonometry that is used in this research study. Hence GeoGebra has little role in enhancing the conceptual change in those concepts. Those results show discrepancies with the research by Kepceoğlu and Yavuz (2016), who showed that GeoGebra contributes to the learners' development of the concept of periodicity of trigonometric functions. In their research study, Kepceoğlu and Yavuz (2016) used the functions and graphs context of teaching trigonometry rather than the unit circle context. They represented the sine and cosine on GeoGebra as functions of their angles. In this context, GeoGebra promoted better conceptual learning of the concept of periodicity of trigonometric functions. The context of graphing and functions is more visual than the context of the unit circle. The use of the unit circle context may be one of the reasons why GeoGebra did not affect the students' conceptual change in the concept of periodicity of trigonometric functions.

The impact of combining the Geo and CC instructional designs on students' conceptual change in the concept of boundedness is more than the effect of CC instructional design alone. When the two instructional designs are combined, the impact is mainly due to the GeoGebra. Dragging the angles on GeoGebra and visualizing that the sine and cosine functions are bounded helped the students to understand the concept of boundedness of trigonometric functions. On the other hand, the explicit comparison to other unbounded functions was not beneficial. Even though, until grade 10, learners have dealt with unbounded functions, they did not realize that they are unbounded. They were not familiar with the concept of boundedness or unboundedness.

Finally, the three instructional designs do not impact students' conceptual change in the concept of non-monotonicity of trigonometric functions differently. Until this lesson, students have not learned the notion of variations of functions. It is challenging to introduce the non-monotonicity of trigonometric functions conceptually to students who do not know the concept of variations of functions. The comparison to the students' prior knowledge was not beneficial. On the other hand, better results were expected for the Geo and Geo + CC instructional designs because the non-conservation of order can be visualized clearly on GeoGebra. It consists of comparing the ordinates based on the comparison of the abscissa on the unit circle. This comparison was included in an activity of the Geo + CC and Geo instructional designs. These instructional designs did not have a better impact than the CC instructional design. An explanation of this can be the fact that the paper and pencil comparisons of the ordinates based on the comparison of the abscissa of the CC instructional design have replaced the comparisons through GeoGebra. Another explanation may be the fact that the non-monotonicity of trigonometric functions and the comparison of trigonometric functions were taught only in one activity in the three instructional designs. It did not have any application in the other parts of the unit. Also, learners have not understood the importance of non-monotonicity and order conservation in some quadrants and the non-conservation in others.

Differential Impact of Instructional Design on Students' Conceptual Change According to Level of Achievement

The three instructional designs do not impact the overall conceptual change and the conceptual change in each of the concepts of periodicity, boundedness, and non-monotonicity of trigonometric functions of students with different achievement levels differently. On the other hand, they do impact the conceptual change in the concept of non-linearity of trigonometric functions of students with different achievement levels differently.

For the low achievers, the three instructional designs do not impact the students' conceptual change in the concept of non-linearity of trigonometric functions differently. The low achievers do not master their prior knowledge. Also, they were not able to master the tools of GeoGebra and conceptual change methods. Both of the methods were new to them, and they were not familiar with working with them.

For average students, the CC instructional design, combined with the Geo instructional design or stand-alone, has more impact on the students' conceptual change in the concept of non-linearity of trigonometric functions than the Geo instructional design. For average students, the use of explicit conceptual change methods helped the students to define the concept of non-linearity. The concept of linearity is established in the prior knowledge of the students. Comparing the non-linearity of trigonometric functions to other linear and non-linear operators helped the students to acquire the concept of non-linearity of trigonometric functions.

For high achievers, the impact of combining the Geo and CC instructional designs on students' conceptual change in the concept of non-linearity of trigonometric functions is significantly more than each of Geo and CC instructional designs alone. The high achievers mastered their prior knowledge and adapted to the conceptual change methods and the use of GeoGebra software quickly.

This result is similar to the results of the research study by Sander and Heib (2014), who found that for the interactive computer-supported conflict inducing group, students with high prior knowledge showed a significantly better performance in trigonometry than those with low prior knowledge. The high achievers and average students benefit more than the low achievers when the CAI and explicit conceptual change methods are combined.

Those results are discrepant from the results of this research study for the overall conceptual change and the conceptual change in each of the concepts of boundedness, periodicity, and non-monotonicity of students of different achievement levels. For those concepts, the instructional designs have a similar impact, regardless of the students' achievement level. The concept of boundedness is visualized clearly on GeoGebra. It is not directly related to the students' prior knowledge. Hence, no difference was found for students of different achievement levels. For the concept of periodicity, students were able to compare the periodic unit circle to the non-periodic number line. This comparison is easy for all the students because they are used to use the number line regardless of their achievement level. The non-monotonicity of trigonometric functions is a challenging concept for all students. Hence there was no difference between students of different achievement levels. The case of the concept of non-linearity of trigonometric functions was different. The contrast with their prior knowledge helped the students to define the non-linearity of trigonometric functions. The low achievers who did not master their prior knowledge may be confused by this comparison. The non-linearity is not a visual concept. The GeoGebra did not have an important role for the acquisition of the concept of non-linearity. But it created some confusion for average students. The high achievers, who mastered their prior knowledge, were not influenced by this confusion.

The impact of the students' achievement level on their conceptual change in the concept of non-linearity of trigonometric functions is in line with the principles of the constructivist theory. The students' prior knowledge impacts their understanding and learning. Students who have low prior knowledge have some difficulties in constructing new knowledge. On the other hand, the difference from the previous knowledge does not affect the students' overall conceptual change and their conceptual change in each of the concepts of boundedness, periodicity, and non-monotonicity. From a first glance, this shows a discrepancy with the constructivist theory; the students' prior knowledge has not influenced their conceptual change and their construction of the new knowledge. On the other hand, this can also be interpreted from another perspective. The review of the prior knowledge and its contrast with the acquired knowledge during the conceptual change method has minimized its impact. Students focus on their prior knowledge that is needed in the construction of new knowledge. In this context, the results can be considered in line with the constructivist theory.

These results can be interpreted in terms of the conceptual change theory too. They show that the use of GeoGebra software and the explicit conceptual change methods promote conceptual change in trigonometry regardless of the achievement level of the students. The low achievers, average students and high achievers benefit from those methods.

Impact on Conceptual Enrichment

The results of this research study show that there is no difference in the effect of the three instructional designs on the students' conceptual enrichment in trigonometry. The conceptual enrichment items do not include properties of trigonometric functions that are discrepant from the students' prior knowledge. They are based on simple generalizations of the concepts that students already know; or on definitions that are new and unrelated to the students' prior knowledge. The conceptual enrichment items require procedural understanding rather than conceptual understanding.

The literature lacks research studies that explore the impact of CAI and conceptual change methods on the students' conceptual enrichment and conceptual change separately. The majority of the research studies that combined the CAI and explicit conceptual change methods investigated the impact of this combination on the students' conceptual change (Tas, Gulen, Oner & Ozyurek, 2015; Zietsman & Hewson, 1986; Yavuz, 2005, Ozmen, Demircioglu & Demircioglu, 2009; Ozmen, 2011; Tas, Gulen, Oner & Ozyurek, 2015, Yumusak, Maras & Sahin, 2015). Sander and Heib (2014) studied the influence of the combination of the CAI and explicit conceptual change methods on students' learning in trigonometry. The students' learning in trigonometry did not require conceptual change. It is similar to the conceptual enrichment described in this research study. The results of the posttest were similar to the results of this research study. They showed that no significant differences existed between the conflict inducing interactive computer-supported group, the non-interactive group, and the interactive computer-supported group that reduced the occurrence of cognitive conflict.

Impact on Conceptual Enrichment of Students According to Level of Achievement

The three instructional designs do not impact the conceptual enrichment of students of different achievement levels differently. The students of all achievement levels have acquired the conceptual enrichment similarly. In this research study, the conceptual enrichment items require a procedural understanding and include definitions and simple generalizations of the students' prior knowledge. Therefore the differences in the students' prior knowledge did not impact their conceptual enrichment.

The results of this research study are not similar to the results of the research study by Sander and Heib (2014), who found that for the interactive computer-supported conflict inducing group students with high prior knowledge showed a significantly better performance in trigonometry learning than those with low prior knowledge. Students with high prior knowledge have an advantage over students with low prior knowledge.

This research study represents a discrepancy with the constructivist theory too. In the constructivist theory, the students' prior knowledge influences on their construction of the new knowledge (Philips, 1995). Thus differences in the students' prior knowledge enhance differences in their learning. This discrepancy with the constructivist theory and the research study by Sander and Heib (2014) exists because the majority of the items that measure the students' conceptual enrichment are not related to their prior knowledge. They are new concepts that are introduced during the three interventions. Many of them included procedural knowledge. The intervention was sufficient to enhance the conceptual enrichment of students of different achievement levels.

Emerging Themes and their Saliency for each of the Four Concepts in the Three Instructional Designs

Four themes emerge from qualitative analysis of students' self-reported conceptions of their conceptual change in each of the concepts of boundedness, periodicity, non-linearity, and non-monotonicity of trigonometric functions. These four themes are non-recognition of conceptual change, recognition of conceptual change, explaining/exemplifying the conceptual change and differentiation of conceptual change.

For the concept of periodicity, the results of the qualitative analysis show that the Geo + CC and CC instructional designs have higher distributions on the cognitive processes that are higher on the revised Bloom's taxonomy than the Geo instructional design. Those results are in line with those of the quantitative analysis where the CC instructional design, combined with Geo instructional design or stand-alone, has significantly more impact on the students' conceptual change in the concept of periodicity of trigonometric functions than the Geo instructional design. The conceptual change methods with or without the use of GeoGebra help the students to define the concept of periodicity. This method of determining the periodicity of trigonometric functions helped the students to have a better conceptual change and to acquire higher skills in revised Bloom's taxonomy.

For the concept of boundedness, the results of the qualitative analysis show that the Geo + CC and Geo instructional designs have higher distributions on the cognitive processes that are higher on the revised Bloom's taxonomy than the CC instructional design. Those results are in line with those of the quantitative part. In the quantitative analysis, combining the Geo and CC instructional designs had more impact on the students' conceptual change in the concept of boundedness than the CC instructional design alone. Also, no significant differences existed between the Geo + CC and Geo instructional designs.

The results of the qualitative part show that GeoGebra has an essential role in the visualization of the concept of boundedness. It enhances the students to achieve higher objectives on Bloom's revised taxonomy. This result is similar to the research study by Kausar, Choudhry, and Gujjar (2008), who found that CAI enhanced the improvement of the students' skills that are higher on Bloom's taxonomy.

For the concept of non-monotonicity of trigonometric functions, the Geo + CC instructional design has higher distributions on the cognitive processes that are higher on the revised Bloom's taxonomy than the CC instructional design. Similarly, the Geo instructional design has higher distributions on the cognitive processes that are higher on the revised Bloom's taxonomy than the CC instructional design except for the theme of differentiation.

Those results are not in line with the results of the quantitative part where no significant differences existed between the three instructional designs. The concept of non-monotonicity and the non-conservation of order can be visualized clearly on GeoGebra. It consists of comparing the ordinates based on the comparison of the abscissa on the unit circle. Also, the concept of non-monotonicity is not established in the students' prior knowledge. Hence the qualitative part shows that GeoGebra used with the conceptual change methods or used alone has better results than the conceptual change methods except for the theme of differentiation. For the theme of differentiation, which is the highest on Bloom's revised taxonomy, the Geo instructional design has less percentage than what is expected by chance. The students of Geo + CC and CC instructional designs had an advantage for the theme of differentiation. Their interventions included similar comparisons.

For the concept of non-linearity, the results of the qualitative analysis show that the cognitive process of differentiation is more frequent in the Geo + CC and CC instructional designs than in the Geo instructional design. The differentiation is the highest on Bloom's revised taxonomy. This result is in line with those of the quantitative part where the CC instructional design, combined with Geo instructional design or stand-alone, has significantly more impact on the students' conceptual change in the concept of non-linearity of trigonometric functions than the Geo instructional design. The non-linearity is a concept where apparent discrepancies exist between the students' acquired knowledge and their prior knowledge. Hence the conceptual change methods with or without the use of GeoGebra helped the students to understand the concept of non-linearity.

On the other hand, the three instructional designs do not have differences in the distributions of the themes of non-recognition, recognition, and explaining/exemplifying conceptual change. This result is not in line with those of the quantitative part. It indicates that the highest frequency of the students recognizes the need for conceptual change for the three instructional designs.

This result can be explained by the fact that the concept of non-linearity is assessed in the semi-structured questionnaire directly. The students assisted in the posttest after the interventions immediately, so they probably remembered the non-linearity of trigonometric functions. The items of the structured questionnaire, which were analyzed quantitatively, did not ask for the non-linearity directly; but they asked to apply the concept of non-linearity. This may be the reason for the partial alignment of the quantitative and qualitative results.

The results of the qualitative analysis and the quantitative analysis are aligned for the concepts of boundedness and periodicity of trigonometric functions. However, the quantitative and the qualitative results are partially aligned in the concept of non-linearity of trigonometric functions, and not aligned in the concept of non-monotonicity. This alignment or partial alignment shows that the students of the instructional designs that had better conceptual change had cognitive processes that are higher on Bloom's revised taxonomy. The cognitive processes that are higher on Bloom's revised taxonomy have an influence on the students' conceptual change. This can be interpreted in the context of the revised Bloom's taxonomy. One important extension to Bloom's taxonomy was the inclusion of metacognitive knowledge in the revised Bloom's Taxonomy (Radmehr &

Drake, 2019). The conceptual change methods and the explicit comparison of the new knowledge to their prior knowledge can be considered as metacognitive skills. Students know the difference between the new knowledge and their prior knowledge; they are aware of the construction of knowledge. In the revised Bloom's taxonomy, the metacognitive knowledge is the highest on the knowledge dimensions. Hence the alignment between the results of the quantitative part about the conceptual change and the results of the qualitative part about the cognitive processes of the revised Bloom's taxonomy is expected. The students who achieved higher on the knowledge dimension also performed higher on the cognitive process dimension.

Limitations

This research study has a few limitations. One limitation of this research study is that the teacher that guided the interventions of the three instructional designs is the researcher himself for practical reasons. To avoid biases, mathematics and science teachers observed two lessons and checked the conformity of the instruction to its unit plan.

Few limitations are related to the number of participants and their assignment to different instructional designs. The small number of the participants of this research study may be an obstacle to the generalization of the results of this research study. Another limitation of this research study is the non-random assignment of the students to the three instructional designs. It was not practical to reconstitute the schools' class sections by random assignment of students to different instructional designs. The prior knowledge pretest and the use of ANCOVA helped to overcome the limitations of the non-random assignment of students. A last limitation related to the selection of the participants was the difference of the language of instruction in the three instructional designs. In one of the schools, the language of instruction of mathematics is English and in the other it is French. This was done for practical reasons. To avoid the problems that would be created because of the use of different languages, the students of the three instructional designs and the teacher used their native language during the sessions frequently.

Few limitations were related to the research method and the length of the intervention. The absence of the control group is a limitation related to the research method. It was done for practical reasons. The absence of the control group did not allow us to determine the definitive impact of Geo, CC, and Geo + CC instructional designs on the dependent variables of this research study. But, it allowed comparing the effects of the Geo + CC, the Geo, and CC instructional designs. Another limitation is the length of the interventions. The interventions consisted of six sessions only. This may not be enough to enhance the conceptual change for students. The students' conceptual change may require more time. But the results of the study showed that even with six sessions, the instructional design had a significantly different impact on the students' conceptual change.

Another limitation that rose after the analysis of the results was the absence of the follow-up test. No significant differences existed between the conceptual enrichment of the students of the three instructional designs because the students remembered the concepts that require conceptual enrichment. A follow-up test would guarantee that students do not merely remember the procedural methods. It would have different results in conceptual enrichment. Also, it would show whether the students' conceptual change is lasting for more extended periods.

Another limitation is the use of the right triangle and the unit circle approach of teaching trigonometry. The graphs and functions context is much more visual. If it were used in the intervention, the more impact of GeoGebra would have more impact.

Recommendations for Further Research and Implications for Practitioners

This research study had many essential findings and limitations that should be considered in future studies. Research studies that combine the CAI and the explicit conceptual change methods are scarce in mathematics education. This research study found promising results of the combination of the CAI and the conceptual change methods in trigonometry. Further research is suggested about this combination in trigonometry and other domains of mathematics. To be able to generalize the results of this research study and to have more definitive conclusions about the impact of this combination, a bigger sample of students, a control group, and more extended studies can be used.

Another recommendation is to study the effect of GeoGebra while using the graph and functions context of teaching trigonometry. In this context, GeoGebra is expected to have more results than in the unit circle and right triangle contexts that are used in this research study. Researchers can also study the impact of the combination of GeoGebra and the explicit conceptual change methods on the concept of non-monotonicity of trigonometric functions. In this research study, there was not an agreement between the quantitative and qualitative parts concerning the concept of non-monotonicity of trigonometric functions.

Another recommendation would be to include a follow-up test in a similar study and to study the link between conceptual change and conceptual enrichment. An immediate conceptual change might enhance conceptual enrichment in the long run. A final recommendation is to investigate the impact of the students' level of cognitive process on the revised Bloom's taxonomy on their the conceptual change. More particularly it is recommended to study the impact of students' metacognitive skills on their conceptual change in trigonometry.

This research study has implications for practitioners too. It suggests the use of GeoGebra software in mathematics classrooms. Also, it proposes to combine the GeoGebra with conceptual change methods, especially when teaching concepts that are discrepant from the students' prior knowledge.

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APPENDICES

APPENDIX 1

CONCEPTUAL ENRICHMENT AND CONCEPTUAL CHANGE POSTTEST

Duration: 90 minutes

Learning Outcomes to be assessed:

1) Conceptual enrichment outcomes
   a) Students are able to use trigonometric lines to find missing sides in a triangle
   b) Students are able to define trigonometric lines on the unit circle
   c) Students are able to calculate trigonometric expressions involving remarkable angles and associated angles.
   d) Students are able to show trigonometric identities.
   e) Students are able to find out the principal measure of angles.
   f) Students are able to find out the signs of trigonometric functions.
   g) Given a trigonometric function of an angle, students are able to find the trigonometric functions of its associated arcs.
2) Conceptual change outcomes
   a) Students are able to find out the boundedness of the image of trigonometric functions.
   b) Students are able to apply the periodicity of trigonometric functions.
   c) Students are able to apply the non linearity of trigonometric functions with addition and multiplication with a real number.

d) Students are able to apply the non monotonicity of trigonometric functions.

e) The students are able to compare bounded and unbounded functions and realize the conceptual change from unbounded functions to bounded trigonometric functions.

f) Students are able to compare periodic and non-periodic functions and realize the conceptual change from non-periodic functions to periodic trigonometric functions.

g) The students are able to compare monotonic and non-monotonic functions and realize the conceptual change from monotonic functions to non-monotonic functions.

h) Students are able to compare functions that are linear with the multiplication by a real number to functions that are not linear and realize the conceptual change from functions that are not linear with multiplication by a real number to functions that are linear multiplication by a real number.

The posttest:

The usage of calculators is not allowed.

Section 1

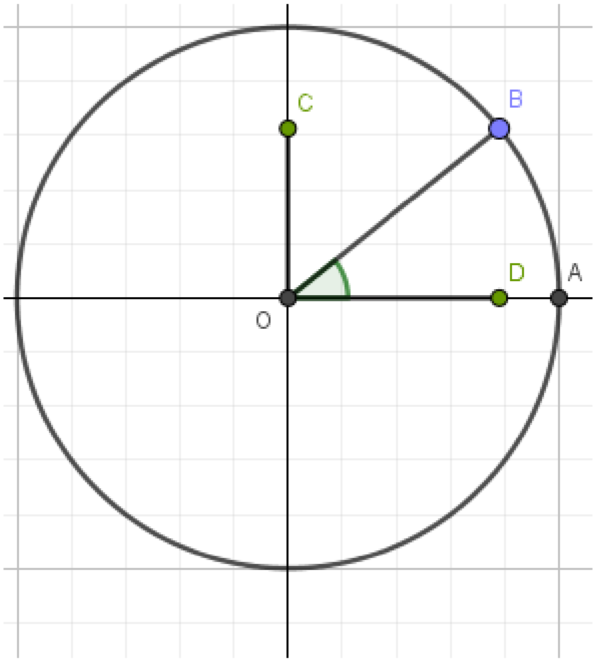
For the items of this section, read carefully the given and answer in the blank area. Show all the details of your work.

Item 1: Conceptual enrichment item

Let ABC be a triangle right at A, such that BC=4cm and B=30°. Find AC:

Objective: Students are able to use trigonometric functions to find missing sides in a triangle

Item 2: Conceptual enrichment item



Define sinA$\widehat{O}\,B$ and cos$A\widehat{O}\,B$ on this trigonometric circle.

Objective: Students are able to define trigonometric lines of angles using unit circle

Item 3: Conceptual enrichment item

Calculate the following trigonometric expressions

3.1 Sin $\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{6}\right)$

3.2 sin($\frac{2\pi}{3}$)+ cos$\left(\frac{4\pi}{3}\right)$

3.3 cos($\frac{25\pi}{4}$)+ sin $\left(\frac{61\pi}{6}\right)$

Objective: Students are able to calculate trigonometric expressions involving remarkable angles and associated angles.

In some cases, they are able to find out the principal measure of an angle

Item 4: Conceptual enrichment item

Show the following trigonometric identities

4.1 $(sinx + cosx)^2 + (sinx - cosx)^2 = 2$

4.2 $\frac{1}{sin^2x} - \frac{1}{tan^2x} = 1$

Objective: Students are able to show trigonometric identities

<u>Item 5:</u> Conceptual enrichment item

Find the principal measure of the following angles:

$\frac{13\pi}{6}$, $\frac{19\pi}{6}$, 1405°, 1812°.

Objective: Students are able to find out the principal measure of angles

<u>Item 6:</u> Conceptual enrichment item

Find the sign of the following trigonometric lines. Show the details of your work

cos(-30)

sin($\frac{5\pi}{6}$)

cos(210)

cos(170)

sin(700)

sin($\frac{43\pi}{6}$)

Objectives: Students are able to find out the signs of trigonometric functions

Students are able to find out the principal measure and the quadrants of angles

<u>Item 7:</u> Conceptual change item

Write 3 angles whose sine is equal to sin50

Objective: Students are able to apply the periodicity of trigonometric functions

Item 8: Conceptual change item

If $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$ find $\cos\frac{6\pi}{5}$ and $\cos\left(\frac{-\pi}{5}\right)$

Objectives:

Given a trigonometric line of an angle students are able to find the trigonometric lines of its associated arcs.

Students are able to apply the non linearity of trigonometric functions

Students are able to find out the quadrants of angles and the sign of their trigonometric lines

Item 9: Arrange the following trigonometric lines in an increasing order.

9.1 $\text{Sin}(60)$ $\sin(390)$ $\sin(-40)$ $\sin(170)$ $\sin(200)$
9.2 $\cos(60)$ $\cos(390)$ $\cos(-40)$ $\cos(170)$ $\cos(200)$

Objectives: Students are able to apply the non monotonicity of trigonometric lines

They are able to apply the periodicity of trigonometric lines and find the principal measures

Section 2

For the items of this section, state whether each of the expressions are true or false. Show the details of your work.

Item 10: Conceptual change item

10.1 $\cos(30) = \frac{\cos(60)}{2}$
10.2 $\cos(120) = -\frac{1}{2}$
10.3 $\cos(2x) = 2cosx$
10.4 $\sin(\frac{7\pi}{3}) = \frac{\sqrt{3}}{2}$

10.5 $cos(180-45) = -1 - \frac{\sqrt{2}}{2}$

10.6 $\cos\left(\frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) = \cos(2\pi)$

Objective: Students are able to calculate trigonometric expressions involving remarkable angles and associated angles

Students are able to apply the non linearity of trigonometric functions with addition and multiplication by a real number

Item 11: Conceptual change item

11.1 Sin(30)=sin(390)
11.2 Sin(220)=sin(40)
11.3 Cos(40)=cos(400)

Objective: Students are able to apply the periodicity of trigonometric functions

Item 12: Conceptual change item

If $sinx > \frac{1}{2}$ then $sin2x > 1$

Objective: Students are able to find out the boundedness of the image of trigonometric functions.

They are able to apply the non linearity of trigonometric functions with multiplication by a real number

Item 13: Conceptual change item

13.1 Sin(42)<sin(82)

13.2 Cos(42)<cos(82)

13.3 Sin(-31)<sin(-62)

13.4 Sin(122)<sin (155)

13.5 Cos(201)<cos(215)

13.6 Sin(382) > sin(23

13.7 Cos(374) > cos(13)

13.8 If x<y then sinx<siny

13.9 If x<y then cosx<cosy

Objective: Students are able to compare trigonometric lines

They are able to apply the non monotonicity of trigonometric functions

In some cases, they are able to apply the periodicity of trigonometric functions and find the principal measure.

Section 3

For the items in this section, answer the following questions in the blank area. Write all the details of your reflection.

Item 14:

What are the possible values of the sine and cosine functions? Why? Compare this to other mathematical functions that you have dealt with (such as the double, the radical, the square functions)

Objectives:

The students are able to find out the boundedness of the sine and the cosine functions.

The students realize the conceptual change from unbounded functions to bounded trigonometric functions.

Item15:

On a given number line is it possible to find two different numbers that are represented by the same point? Why? On a given trigonometric circle is it possible to find two different angles that have the same extremity? Why?

Objectives:

Students are able to distinguish the periodic trigonometric circle and the non-periodic number line.

Students realize the conceptual change from non-periodic functions to periodic trigonometric functions.

Item16

If $x<y$ is $\sin x<\sin y$? Explain your answer. Compare this situation to the double function, to the cosine function and to the square function.

Objectives:

The students are able to apply the non-monotonicity of the trigonometric functions.

The students realize the conceptual change from monotonic functions to non-monotonic functions.

Item17

Is $\sin 3x = 3\sin x$? Explain your answer. Compare this situation to other mathematical functions that you have dealt with.

Objectives:

Students are able to apply the non-linearity of trigonometric functions with multiplication by a real number.

Students realize the conceptual change from functions that are not linear with multiplication by a real number to functions that are linear with addition and multiplication by a real number

APPENDIX 2

UNIT 1: TEACHING TRIGONOMETRY WITH CONCEPTUAL CHANGE METHODS USING GEOGEBRA

Objectives of the unit:

The students will define the trigonometric lines in a right triangle

They will define the trigonometric lines on the trigonometric circle

They should find out the formulae of the trigonometry

They should use those formulae to find other trigonometric identities

They should find out the signs of the trigonometric lines in each quadrant

They should find out the trigonometric lines of some remarkable angles

They should find out the properties of trigonometric lines

They should find out the trigonometric lines of associated angles

Specific learning outcomes

The students should understand the non-linearity of trigonometric lines with addition and compare it to other linear operators.

They should understand the non-linearity of trigonometric lines with multiplication with a real number and compare it to other linear operators.

They should be able to compare trigonometric lines by comparing the angles and understand the non monoticity of trigonometric functions in a period and its difference from other monotonic functions.

They should be able to understand the periodicity of trigonometric functions and compare it to other non periodic functions.

Students should be able to present the trigonometric circle and the trigonometric lines on GeoGebra

Lessons of the unit:

Lesson 1: Trigonometric lines in a right triangle

Lesson 2: Trigonometric lines on the unit circle

Lesson 3: Quadrants of the trigonometric circle and signs of trigonometric lines

Lesson 4: Trigonometric lines of particular angles, non linearity with the addition and with the multiplication

Lesson 5: Associated Angles

Lesson 6: Application to associated angles

<u>Lesson 1</u>: Trigonometric Lines in a right triangle and on the trigonometric circle

<u>Learning Outcomes:</u>

The students should be able to:

- Define the trigonometric ratios in a right triangle
- Calculate missing sides of triangles using trigonometric ratios
- Use trigonometric ratios to solve real-life problems
- Find out the trigonometry formulae
- Show and use trigonometric identities

<u>Instructional Procedures:</u>

<u>Activity 1:</u> Introduction about trigonometry and its use

The teacher holds a discussion about trigonometry and its use. The students discuss several examples of the use of trigonometry in real-life situations.

<u>Activity 2:</u> Trigonometric Ratios in a right triangle

The students receive a file in GeoGebra and a worksheet. (Appendix 1) In the GeoGebra file, a right triangle is constructed with fixed angles, the students have to keep the angle fixed and drag a point to have different triangles with different dimensions, and they have to complete the worksheet accordingly. After completing the worksheet, the students are introduced to the definitions of the trigonometric lines in a right triangle.

<u>Activity 3:</u> Use of trigonometric lines to find missing measures

The students will be given worksheets that they have to fill. The worksheets include triangles with missing sides, and they have to use the corresponding trigonometric lines to find the missing sides. (Appendix 2) They have to work in pairs; then they have to discuss their answers.

The students will be given real-life situations with missing sides, and they have to use trigonometric lines to find the missing sides. (Appendix 3) They have to work in pairs; then, they share their ideas.

<u>Activity 4:</u> Trigonometric identities

The students should work in pairs to find out the trigonometric formulae. They should be given a right triangle and will be asked to apply the Pythagorean Theorem. They should be

asked to derive other trigonometric formulae from the Pythagorean Theorem. ($sin^2\alpha = \frac{tan^2\alpha}{tan^2\alpha+1}$ and $cos^2\alpha = \frac{1}{1+tan^2\alpha}$)

Students should solve an exercise about trigonometric identities. (Appendix 4)

Lesson 2: Trigonometric lines on the unit circle

Learning Outcomes:

The students should be able to:

- Define the trigonometric circle
- Construct a trigonometric circle on GeoGebra
- Define trigonometric lines on the trigonometric circle using GeoGebra
- Find out the periodicity of trigonometric functions and compare them to other non-periodic functions.
- Find out that the image of the sine and cosine are bounded intervals and compare them to other functions that have non-bounded images.
- Compare trigonometric lines of increasing and decreasing angles. Find out the non-monotonicity of trigonometric functions in a period and compare it to other monotonic functions.

Instructional Procedures:

Activity 1 The trigonometric lines on the trigonometric circle

- The students are introduced to the trigonometric circle, and they construct a trigonometric circle on GeoGebra
- A whole-class discussion is held to define the trigonometric lines on the trigonometric circle.
- The students have to construct an angle on the trigonometric circle and to find its trigonometric lines.
- They have to drag the angle to find the trigonometric lines of other angles.

Activity 2: Periodicity and principal measure

In this activity, the students are introduced to the idea of principal measure. They should notice that many angles have the same extremity on the trigonometric circle. The conceptual change from the non-periodic number line to the periodic trigonometric circle will be taught explicitly. The students have to compare the case of the number line to the case of the trigonometric circle. (Appendix 5) The students will define the principal measure of an angle, and they notice that angles that have the same principal measure have the same trigonometric lines. Hence they notice the importance of studying the principal measure of an angle. They will be given angles. They will construct them on GeoGebra and find their principal measure (Appendix 6). Also, they should notice that the trigonometric lines of an angle are in $[-1; 1]$. In general, all the functions that the students have dealt with have unbounded images. The conceptual change from functions with unbounded images to functions with bounded images is taught explicitly. The students should work on a worksheet about this conceptual change. (Appendix 7)

<u>Activity 3</u> Non-monotonicity of trigonometric functions

In this activity, the students have to construct the trigonometric circle and an angle on GeoGebra. They should drag the angle and find out the trigonometric lines of the corresponding angles. In addition, they should compare the trigonometric lines of different angles. They should find out whether the sine and cosine increase when the angle increases. The conceptual change is taught explicitly in this activity. A comparison with other monotonic functions should be held explicitly.

The students should work on an exercise about the trigonometric lines; the exercise aims to compare trigonometric lines of different angles. (Appendix 8)

Lesson 3: Quadrants of the trigonometric circle and signs of trigonometric lines

Objectives:

The students should be able to:

- Define the quadrants of the trigonometric circle
- Find the quadrant of an angle given its quadrant
- Find out the sign of trigonometric lines in each quadrant
- Compare trigonometric lines of angles in different quadrants

Instructional Procedures:

Activity 1:

The students should construct the trigonometric circle on GeoGebra. They should construct an angle and drag it. They should find out the intervals for each quadrant. The angles should be in degrees and radians.

The students should solve an exercise. The exercise aims to find out the quadrants of the given angles. (Appendix 9)

Activity 2:

On the same GeoGebra file, the students should drag the angles and find out the signs of the trigonometric lines in each quadrant.

The students should be given angles, and they should find out the sign of each trigonometric line. (Appendix 10)

Activity 3:

The students should compare the trigonometric lines of different angles on GeoGebra. They should find out in which quadrant the trigonometric lines increase as the angle increases and in which quadrant they decrease. They should compare this situation with other monotonic functions that have constant monotonicity in a given interval. They should do an exercise about comparing angles in the same and different quadrants. (Appendix 11)

Lesson 4: Trigonometric lines of particular angles, non-linearity with the addition and with the multiplication

Objectives:

The students should be able to:

- Find out the trigonometric lines of some remarkable angles using GeoGebra
- Calculate some trigonometric expressions
- Find out the non-linearity of trigonometric functions with addition and compare it to other linear operators
- Find out the non-linearity of trigonometric functions with the multiplication with a real number and compare it to other linear operators.

Instructional Procedures:

Activity 1:

The students should construct the trigonometric circle on GeoGebra, and they should find the trigonometric lines of the remarkable angles (30-45-60-0-90-180-270). They should construct a table that summarizes the results

The students should calculate trigonometric expressions, including the remarkable angles. (Appendix 12)

Activity 2:

The students should be asked to compare trigonometric lines of some angles and the trigonometric lines of the sum of the angles. (ie they should compare sin(60)+sin(30) and sin(90)) In addition, they should compare the product of the trigonometric line by a real and the trigonometric line of the angle multiplied by the same real (ie, sin(2x30) and 2sin30). They should compare this situation to the case of the associativity of multiplication. The conceptual change is taught explicitly in this activity. They should solve an exercise at the end of this activity. (Appendix 13)

Lesson 5: Associated angles

Objectives:

The students should be able to:

- Find out the relations of the trigonometric lines of associated angles
- Find out the periodicity of trigonometric lines
- Find out the non-linearity of the trigonometric lines with addition and compare it to other linear operators.

Instructional Procedures:

Activity 1:

The students should construct on GeoGebra an angle x and 180-x. They should find on the trigonometric circle the corresponding trigonometric lines, and they should compare them.

Similarly, they should construct 180+x, -x, 90-x, 90+x, and they should find their trigonometric lines. They should be given the angles in degrees and in radians.

They should apply the non-linearity with addition in each case and the non-linearity with multiplication in each case. (ie compare sin(180+x) and sin(180)+sinx and compare cos(-x) and -1cos(x)) The conceptual change method will be applied to this activity. A discussion will be held about other linear operators.

Activity 2

The students should find out the periodicity of trigonometric lines. They should apply the non-linearity with addition. (Compare sin(360+x) and sin360 + sinx) The conceptual change method will be applied to this activity. A discussion will be held about other linear operators.

Activity 3:

The students will be given an exercise where they should compare the trigonometric lines of associated angles, and they should find out the above-mentioned non-linearities. In each case, a discussion will be held about other linear operators (Appendix 14)

Lesson 6: Application of associated angles

Objectives

The students should be able to:

- Calculate trigonometric lines of remarkable angles and their associated angles
- Simplify trigonometric expressions
- Given a trigonometric line of an angle, find the trigonometric lines of its associated arcs.

Instructional Procedures

Activity 1:

The students will solve an exercise involving the calculation of the associated angles of remarkable angles. They will check their results on the trigonometric circle on GeoGebra. A discussion will be held about other linear operators. The conceptual change will be taught explicitly. (Appendix 15)

Activity 2:

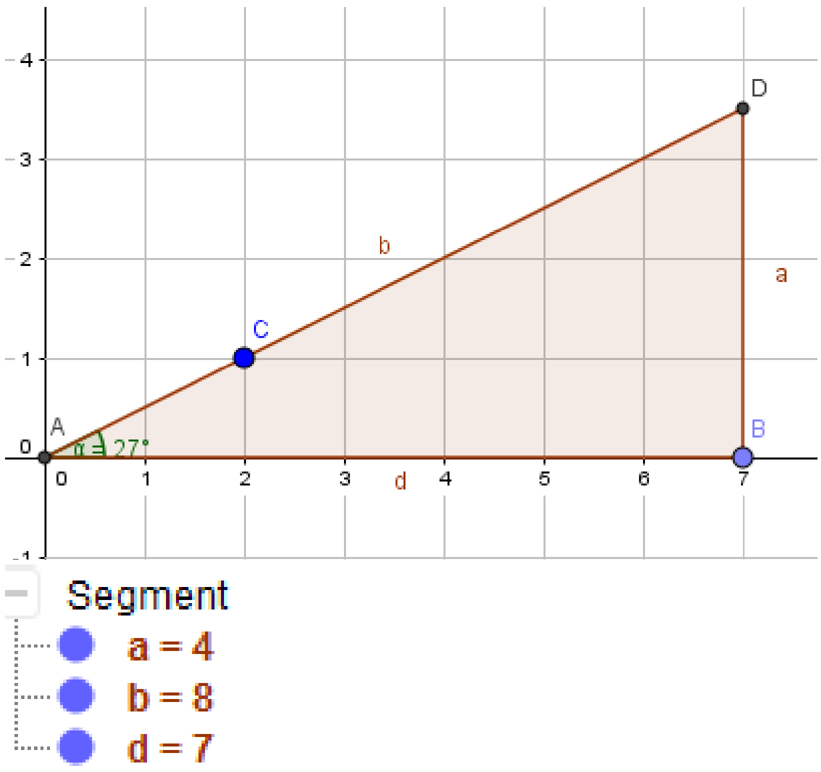
The students will be given a trigonometric line of an angle, and they should find the trigonometric lines of its associated arcs. They should check the non-linear properties of the trigonometric functions. They will check the results on the trigonometric circle on GeoGebra. A discussion will be held about other linear operators. The conceptual change will be taught explicitly. (Appendix 16)

Activity 3:

A review of the main concepts of the unit will be held.

Appendices of Unit 1

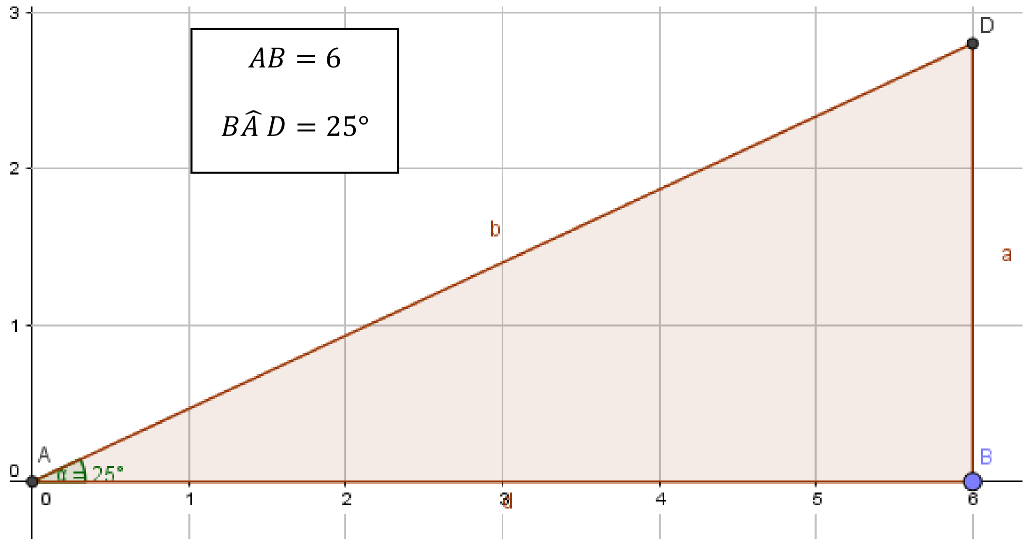
Appendix 1:



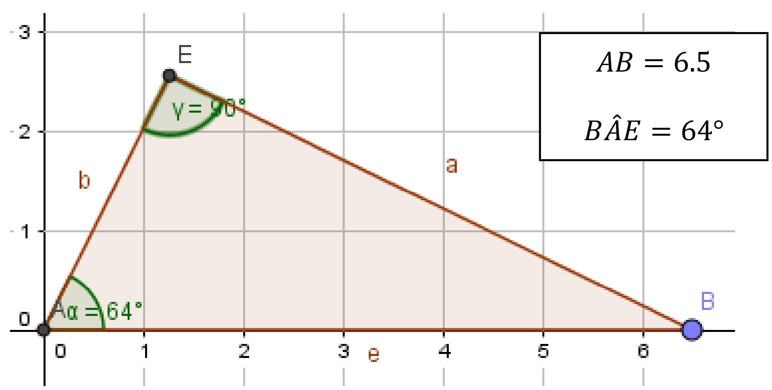
Drag B on the x-axis and complete this table

d	4	5	6`	7	8
a					
b					
d/b					
a/b					
a/d					

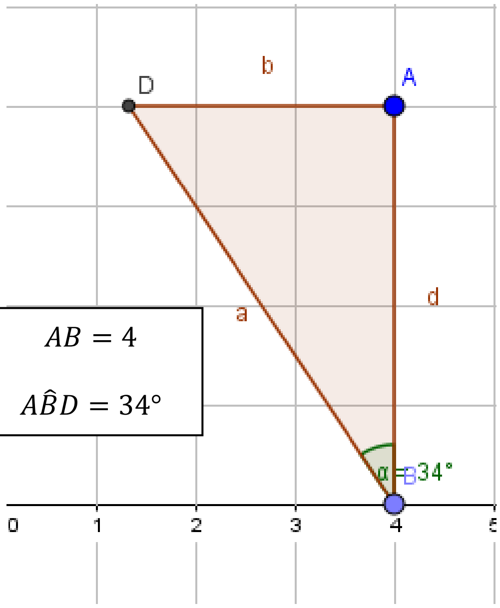
Appendix 2:



Calculate AD



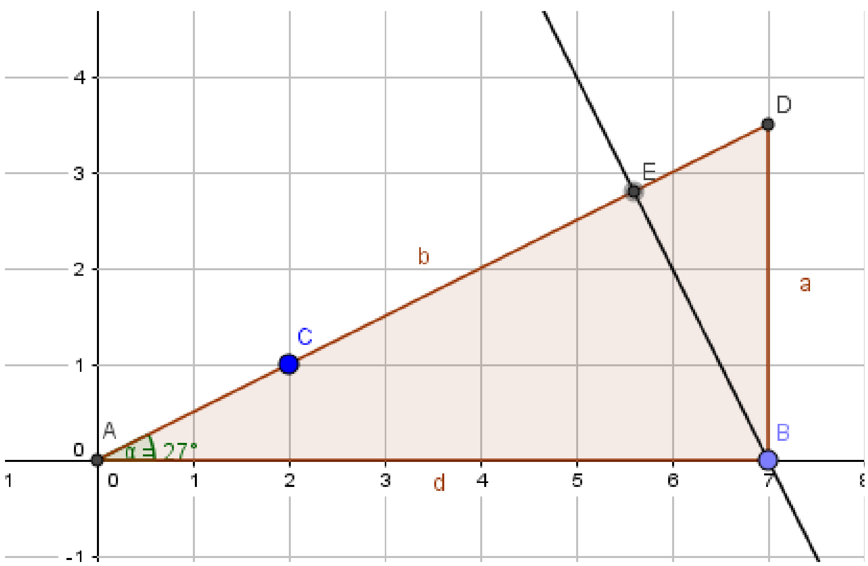
Calculate BE.



$AB = 4$

$A\hat{B}D = 34°$

Calculate AD



$AB = 7$

$B\hat{A}D = 27°$

(BE) *is the height of* ABD

Calculate BD, BE, and AE.

Appendix 3:

In the context of the steepness of a road, the grade or percentage grade to be the tangent of the angle α. The grade of a given road is 10%.

1- What is the rise BC if the distance AB is 2km
2- For which distance the rise is 200m.

Appendix 4: Show the following trigonometric identities (extract from the grade 10 textbook)

1) $(sin\alpha + cos\alpha)^2 - 2sin\alpha cos\alpha = 1$
2) $(sin\alpha + cos\alpha)^2 + (sin\alpha - cos\alpha)^2 = 2$
3) $tan\alpha + cot\alpha = \frac{1}{sin\alpha cos\alpha}$
4) $tan^2\alpha(1 + cot^2\alpha) = \frac{1}{cos^2\alpha}$
5) $1 - sin\alpha cos\alpha tan\alpha = cos^2\alpha$
6) $cos^4\alpha - sin^4\alpha + \frac{tan^2\alpha}{1+tan^2\alpha} - \frac{1}{1+tan^2\alpha} = 0$
7) $cot^2\alpha - cos^2\alpha = cot^2\alpha cos^2\alpha$
8) $tan^2\alpha + cot^2\alpha + 2 = \frac{1}{sin^2\alpha cos^2\alpha}$

Appendix 5

Construct the trigonometric circle. In this circle, construct the angles 30, 60, 390, 420. Mark the extremities of the angles by B, C, D, and E. What do you notice? Is it possible to have two points that have the same extremity?

Find their trigonometric lines. What do you notice?

Plot the points F(30), G(60) H(390), and I(420) on a one-dimensional axis. What do you notice? Is it possible to have two numbers that have the same location on the axis?

Compare the case of the number line and the case of a trigonometric circle.

Appendix 6

Find the principal measure of the following angles then find the trigonometric lines of those angles

300

730

1120

-320

Appendix 7:

Is it possible to find an angle which sine and cosine are bigger than 1?

Is it possible to find an angle in which sine and cosine are less than -1?

What do you deduce?

Is it possible to find a number which double is bigger than 1, bigger than 10, 100, 1000?

Is it possible to find a number which double is less than -1, -10,-100,-1000?

Compare this situation with the case of the sine and cosine.

Is it possible to find a number of which square root is bigger than 1, bigger than 10, 100, 1000?

Is it possible to find a number of which square root is less than 0?

Compare this situation with the case of the sine and cosine.

Appendix 8:

1) Answer by true or false and explain your answer using the GeoGebra.

a) $\sin(40) < \sin(60)$

b) $\cos(40) < \cos(60)$

c) Sin(40)<sin(150)

d) Cos(40)<cos(150)

e) Sin(190)<sin(220)

f) Cos(190)<cos(220)

g) Sin(30)<sin(300)

h) Cos(30)<cos(300)

2) Compare
   $2 \times 40 \ \ and\ 2 \times 60$
   $40 + 30$ and 60+30
   $2 \times 40 \quad and \quad 2 \times 150$
   40+30 and $150 + 30$

What is difference in the case of the comparison of the trigonometric lines of the corresponding angles found in part1 (a-b-c-d)

Appendix 9: In which quadrant is situated the extremity of the following angles:

$260°$

$132°$

$-300°$

$182°$

$470°$

$1300°$

$\frac{7\pi}{3}$

$\frac{-\pi}{4}$

$\frac{16\pi}{3}$

Appendix 10:

Find the sign of each of the trigonometric lines of the following angles:

$30°, 120°, -65°, \frac{\pi}{4}, \frac{10\pi}{3}, \frac{9\pi}{8}, 200°$

Appendix 11

Construct a trigonometric circle and angle. Construct the sine and the cosine of a given angle. Drag the angle and find the quadrants in which the sine increases when the angle increases and the angles where the angle decreases as the angle decreases.

Let a=3 and b =4. Compare $\sqrt{a}$ and $\sqrt{b}$. What can you deduce? What is the difference between the case of radicals and the case of trigonometric functions?

Compare $\frac{1}{a}$ and $\frac{1}{b}$. What can you deduce? What is the difference between the case of inverses and the case of trigonometric functions?

Compare a^2 and b^2. What can you deduce? What about negative numbers?

What is the difference between the case of squares and the case of trigonometric functions?

True or false. Explain your thinking.

Sin120<sin150

Sin60<sin80

Cos120<cos150

Cos60<cos80

Sin30<sin120

Sin150<sin320

Appendix 12: Calculate the exact value of each of the following expressions

A = $\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)$

$B = \sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right)$

$C = \cos(\pi) + \sin\left(\frac{\pi}{4}\right) + tan0$

$$D = \frac{tan0 + tan60}{1 - tan0tan60}$$

Appendix 13

1) Calculate sin45, sin90 and sin180.

Compare 2sin45 and sin90.

Compare 2sin90 and sin180.

What do you deduce?

Calculate

Sin30, sin60, and sin90. Compare sin60+sin30 and sin90

What do you deduce?

Calculate 2(30)+2(60) and 2(30+60).

In general we have 2x +2y =2(x+y).

How is this different from the case of trigonometric functions?

True or False

sin(60)=2sin(30)

$\frac{cos180}{3} = \frac{cos120}{2}$

$5sinx = sin5x$

$\frac{sin2x}{2} \times \frac{1}{cosx} = tanx$

Appendix 14

Calculate the following expressions and compare

sin $\left(\frac{7\pi}{6}\right)$ and sin (π)+sin $(\frac{\pi}{6})$

sin $(\frac{3\pi}{4})$ and $\sin(\pi) - \sin$ $(\frac{\pi}{4})$

sin $\left(\frac{13\pi}{3}\right)$ and sin (4π)+sin $(\frac{\pi}{3})$

What do you notice?

Appendix 15

Calculate the following expressions:

$$A = \sin\left(\frac{23\pi}{6}\right) + \cos\left(\frac{13\pi}{3}\right)$$

$$B = \frac{\cos\left(\frac{-\pi}{3}\right) + \sin\left(\frac{11\pi}{3}\right)}{\cos\left(\frac{25\pi}{6}\right) + \sin\left(\frac{47\pi}{6}\right)}$$

Appendix 16: Exercise associated angles with variable

If $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$, calculate $\sin\frac{\pi}{5}$, $\cos\frac{6\pi}{5}$, $\sin\left(\frac{11\pi}{5}\right)$

Compare $\cos\frac{6\pi}{5}$ *and* $\cos(\pi) + \cos\left(\frac{\pi}{5}\right)$

$\sin\left(\frac{11\pi}{5}\right)$ and $\sin(2\pi) + \sin\left(\frac{\pi}{5}\right)$

APPENDIX 3

TABLE SHOWING THE CONCEPTUAL CHANGE ACTIVITIES IN THE LESSONS OF UNITS 1 AND 2.

Lesson2	
Activity 2	-Conceptual change from non-periodic number line to periodic trigonometric circle - Conceptual change from functions with unbounded images to trigonometric functions with bounded images
Activity 3	Conceptual change from monotonic functions to functions that are not monotonic in the period.
Lesson 3	
Activity 3	Conceptual change from monotonic functions to functions that are not monotonic
Lesson 4	
Activity 2	Conceptual change from linear operators to non-linear operators
Lesson 5	
Activity 1	Conceptual change from linear operators to non-linear operators
Activity 2	Conceptual change from linear operators to non-linear operators
Activity 3	Conceptual change from linear operators to non-linear operators
Lesson 6	
Activity 1	Conceptual change from linear operators to non-linear operators
Activity 2	Conceptual change from linear operators to non-linear operators

APPENDIX 4

UNIT 2: TEACHING TRIGONOMETRY WITH CONCEPTUAL CHANGE METHODS WITHOUT THE USE OF GEOGEBRA

Objectives of the unit:

The students will define the trigonometric lines in a right triangle

They will define the trigonometric lines on the trigonometric circle

They should find out the formulae of the trigonometry

They should use those formulae to find other trigonometric identities

They should find out the signs of the trigonometric lines in each quadrant

They should find out the trigonometric lines of some remarkable angles

They should find out the properties of trigonometric lines

They should find out the trigonometric lines of associated angles

Specific Learning Outcomes

The students should understand the non-linearity of trigonometric lines with addition and compare it to other linear operators.

They should understand the non-linearity of trigonometric lines with multiplication with a real number and compare it to other linear operators.

They should be able to compare trigonometric lines by comparing the angles and understand the non-monotonicity of trigonometric functions in a period and compare it to other monotonic functions.

They should be able to understand the periodicity of trigonometric functions and compare it to other non periodic functions.

Lessons of the unit:

Lesson 1: Trigonometric lines in a right triangle

Lesson 2: Trigonometric lines on the trigonometric circle

Lesson 3: Quadrants of the trigonometric circle and signs of trigonometric lines

Lesson 4: Trigonometric lines of particular angles, non linearity with the addition and with the multiplication

Lesson 5: Associated Angles

Lesson 6: Application to associated angles

Lesson 1: **Trigonometric Lines in a right triangle and on the trigonometric circle**

Learning Outcomes:

The students should be able to:

- Define the trigonometric ratios in a right triangle
- Calculate missing sides of triangles using trigonometric ratios
- Use trigonometric ratios to solve real-life problems
- Find out the trigonometry formulae
- Show and use trigonometric identities

Instructional Procedures:

Activity 1: Introduction about trigonometry and its use

The teacher holds a discussion about trigonometry and its use. The students discuss several examples of the use of trigonometry in real-life situations.

Activity 2: Trigonometric Ratios in a right triangle

The students have to construct right triangles with equal angles but different sides. They have to calculate the quotients of the sides in the different triangles. (Appendix 1) After completing the worksheet, the students are introduced to the definitions of the trigonometric lines in a right triangle.

Activity 3: Use of trigonometric lines to find missing measures

The students will be given worksheets that they have to fill. The worksheets include triangles with missing sides, and they have to use the corresponding trigonometric lines to find the missing sides. (Appendix 2) They have to work in pairs; then they have to discuss their answers.

The students will be given real-life situations with missing sides, and they have to use trigonometric lines to find the missing sides. (Appendix 3) They have to work in pairs; then, they share their ideas.

Activity 4:

The students should work in pairs to find out the trigonometric formulae. They should be given a right triangle and will be asked to apply the Pythagorean Theorem. They should be

asked to derive other trigonometric formulae from the Pythagorean Theorem. ($sin^2\alpha = \frac{tan^2\alpha}{tan^2\alpha+1}$ and $cos^2\alpha = \frac{1}{1+tan^2\alpha}$)

Students should solve an exercise about trigonometric identities. (Appendix 4)

Lesson 2: Trigonometric lines on the unit circle

Learning Outcomes:

The students should be able to:

- Define the trigonometric circle
- Construct a trigonometric circle on GeoGebra
- Define trigonometric lines on the trigonometric circle using GeoGebra
- Find out the periodicity of trigonometric functions and compare them to other non-periodic functions.
- Find out that the image of the sine and cosine are bounded intervals and compare them to other functions that have unbounded images.
- Compare trigonometric lines of increasing and decreasing angles. Find out the non-monotonicity of trigonometric functions in a period and compare it to other monotonic functions.

Instructional Procedures:

<u>Activity 1:</u> The trigonometric lines on the trigonometric circle

- The students are introduced to the trigonometric circle, and they draw a trigonometric circle
- A whole-class discussion is held to define the trigonometric lines on the trigonometric circle.
- The students have to construct an angle on the trigonometric circle and to find its trigonometric lines.
- They have to construct other angles on the trigonometric circle to find their trigonometric lines.

<u>Activity 2:</u> Periodicity and principal measure

In this activity, the students are introduced to the idea of principal measure. They should notice that many angles have the same extremity on the trigonometric circle. The conceptual change from the non-periodic number line to the periodic trigonometric circle will be taught explicitly. The students have to compare the case of the number line to the case of the trigonometric circle. (Appendix 5) The students will define the principal measure of an angle, and they notice that angles that have the same principal measure have the same trigonometric lines. Hence they notice the importance of studying the principal measure of an angle. They will be given angles, and they will be asked to find their principal measure. They will construct those angles on GeoGebra and to find their trigonometric lines. (Appendix 6). Also, they should notice that the trigonometric lines of an angle are in $[-1; 1]$. In general, all the functions that the students have dealt with have unbounded images. The conceptual change from functions with unbounded images to functions with bounded images is taught explicitly. The students should work on a worksheet. (Appendix 7)

Activity 3 Trigonometric properties

In this activity, the students have to construct the trigonometric circle and an angle. They draw other angles and find out their trigonometric lines. They should compare the trigonometric lines of different angles. They should find out whether the sine and cosine increase when the angle increases. The conceptual change is taught explicitly in this activity. A comparison with other monotonic functions should be held explicitly.

(Appendix 8)

Lesson 3: Quadrants of the trigonometric circle and signs of trigonometric lines

Objectives:

The students should be able to:

- Define the quadrants of the trigonometric circle
- Find the quadrant of an angle given its quadrant
- Find out the sign of trigonometric lines in each quadrant
- Compare trigonometric lines of angles in different quadrants

Instructional Procedures:

Activity 1:

The students should construct the trigonometric circle. They should construct different angles They should find out the intervals for each quadrant. The angles should be in degrees and radians.

The students should solve an exercise. The aim of the exercise is to find out the quadrants of the given angles. (Appendix 9)

Activity 2:

On the trigonometric circle that they drew, the students should construct angles in each quadrant and find out the signs of the trigonometric lines in each quadrant.

The students should be given angles, and they should find out the sign of each trigonometric line. (Appendix 10)

Activity 3:

The students should compare the trigonometric lines of different angles on the trigonometric circle. They should find out in which quadrant the trigonometric lines increase as the angle increases and in which quadrant they decrease. They should compare this situation with other monotonic functions that have constant monotonicity in a given interval. They should do an exercise about comparing angles in the same and different quadrants. (Appendix 11)

Lesson 4: Trigonometric lines of particular angles, non-linearity with the addition and with the multiplication

Objectives:

The students should be able to:

- Find out the trigonometric lines of some remarkable angles
- Calculate some trigonometric expressions
- Find out the non-linearity of trigonometric functions with addition and compare it to other linear operators
- Find out the non-linearity of trigonometric functions with the multiplication with a real number and compare it to other linear operators.

Instructional Procedures:

Activity 1:

The students should construct the trigonometric circle, and they should find the trigonometric lines of the remarkable angles (30-45-60-0-90-180-270) They should construct a table that summarizes the results

The students should calculate trigonometric expressions, including the remarkable angles. (Appendix 12)

Activity 2:

The students should be asked to compare trigonometric lines of some angles and the trigonometric line of the sum of the angles. (ie they should compare sin(60)+sin(30) and sin(90)) Also, they should compare the product of the trigonometric line by a real and the trigonometric line of the angle multiplied by the same real (ie, sin(2x30) and 2sin30). They should compare this situation to the case of the associativity of multiplication. The conceptual change is taught explicitly in this activity. (Appendix 13)

Lesson 5: Associated angles

Objectives:

The students should be able to:

- Find out the relations of the trigonometric lines of associated angles
- Find out the periodicity of trigonometric lines
- Find out the non-linearity of the trigonometric lines with the addition

Instructional Procedures:

Activity 1:

The students should construct on the trigonometric circle an angle x and 180-x. They should find on the trigonometric lines of those angles, and they should compare them.

Similarly, they should construct 180+x, -x, 90-x, 90+x, and they should find their trigonometric lines.

They should apply the non-linearity with addition in each case and the non-linearity with multiplication in each case. (ie compare sin(180+x) and sin(180)+sinx and compare cos(-x) and -1cos(x)) The conceptual change method will be applied to this activity. A discussion will be held about other linear operators.

Activity 2

The students should find out the periodicity of trigonometric lines. They should apply the non-linearity with addition. (Compare sin(360+x) and sin360 + sinx) The conceptual change method will be applied to this activity. A discussion will be held about other linear operators.

Activity 3:

The students will be given an exercise should find out the non-linearity, as mentioned earlier, by using the associated angles. A discussion will be held about other linear operators. (Appendix 14)

Lesson 6: Application of associated angles

Objectives

The students should be able to:

- Calculate trigonometric lines of remarkable angles and their associated angles
- Simplify trigonometric expressions
- Given a trigonometric line of an angle, find the trigonometric lines of its associated arcs.

Instructional Procedures

Activity 1:

The students will solve an exercise involving the calculation of the associated angles of remarkable angles. (Appendix 15)

Activity 2:

The students will be given a trigonometric line of an angle, and they should find the trigonometric lines of its associated arcs. They should check the non-linear properties of the trigonometric functions. They will check the results on the trigonometric circle on GeoGebra. (Appendix 16)

Activity 3:

A review of the main concepts of the unit will be held. The idea of conceptual change should be stressed and discussed the needs and the reason for this change.

Appendices of Unit 2

Appendix 1:

Construct a triangle ABC right at B such that AB = 8cm and the angle BCA = 30°.

Plot the points D, E, F and G on [AB] such that AD=7cm, AE=6cm, AF=5cm, and AG=4cm

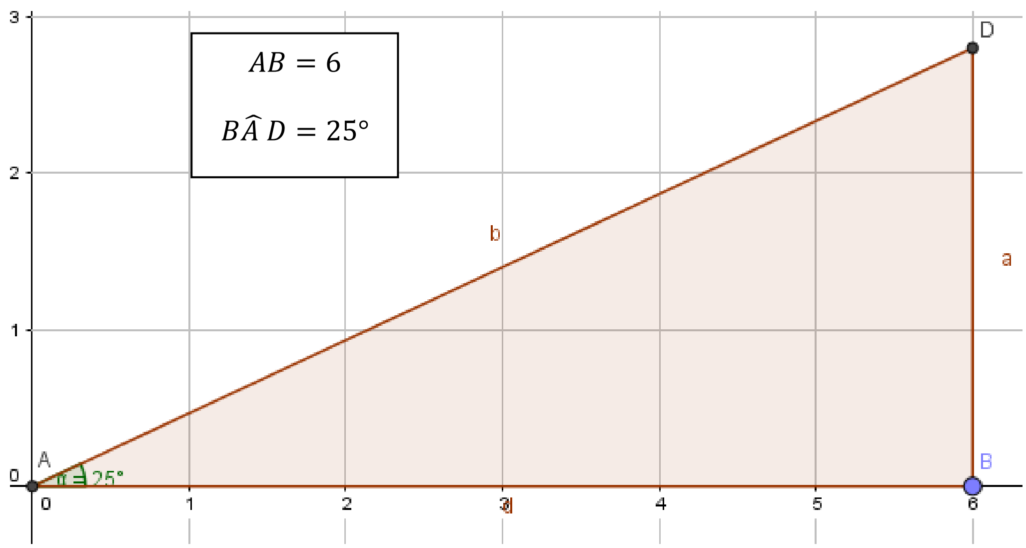
Draw the lines perpendicular to (AB) from D, E, F, and G. The following lines cut [BC] at H, I, J, and K.

Calculate and compare $\frac{AB}{AC}, \frac{AD}{AH}, \frac{AE}{AI}, \frac{AF}{AJ}, \frac{AG}{AK}$.

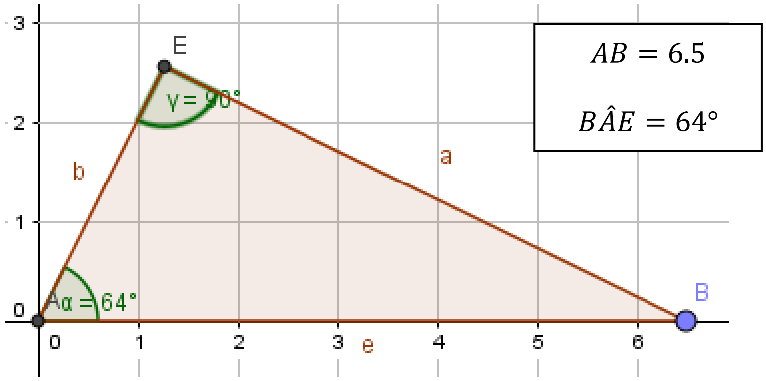
Calculate and compare $\frac{GK}{AK}, \frac{FJ}{AJ}, \frac{EI}{AI}, \frac{DH}{AH}, \frac{BC}{AC}$.

Calculate and compare $\frac{GK}{AG}, \frac{FJ}{AF}, \frac{EI}{AE}, \frac{DH}{AD}, \frac{BC}{AB}$

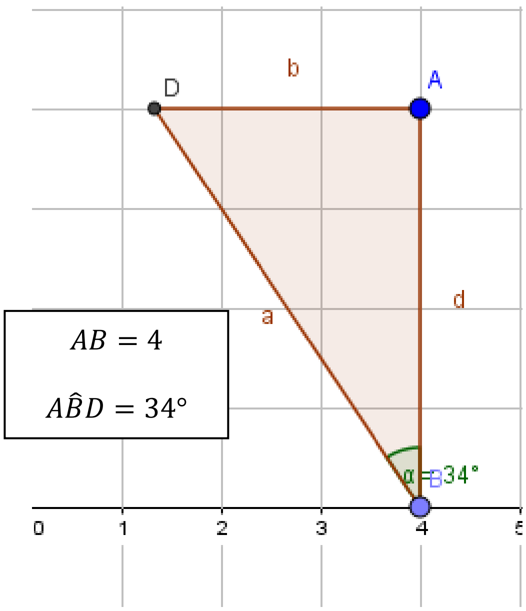
Appendix 2:



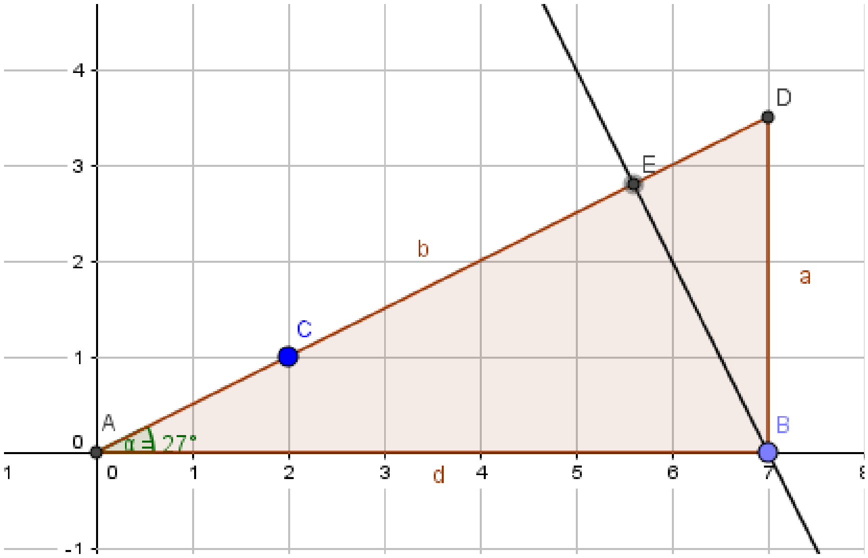
Calculate AD



Calculate BE.



Calculate AD



$AB = 7$

$B\widehat{A}D = 27°$

(BE) *is the height of ABD*

Calculate BD, BE, and AE.

Appendix 3:

In the context of the steepness of a road, the grade or percentage grade to be the tangent of the angle α. The grade of a given road is 10%.

1- What is the rise BC if the distance AB is 2km
2- For which distance the rise is 200m.

Appendix 4: Show the following trigonometric identities (extract from the grade 10 textbook)

9) $(sin\alpha + cos\alpha)^2 - 2sin\alpha cos\alpha = 1$
10) $(sin\alpha + cos\alpha)^2 + (sin\alpha - cos\alpha)^2 = 2$
11) $tan\alpha + cot\alpha = \frac{1}{sin\alpha cos\alpha}$
12) $tan^2\alpha(1 + cot^2\alpha) = \frac{1}{cos^2\alpha}$
13) $1 - sin\alpha cos\alpha tan\alpha = cos^2\alpha$
14) $cos^4\alpha - sin^4\alpha + \frac{tan^2\alpha}{1+tan^2\alpha} - \frac{1}{1+tan^2\alpha} = 0$
15) $cot^2\alpha - cos^2\alpha = cot^2\alpha cos^2\alpha$
16) $tan^2\alpha + cot^2\alpha + 2 = \frac{1}{sin^2\alpha cos^2\alpha}$

Appendix 5

Construct the trigonometric circle. On this circle, construct the angles 30, 60, 390, 420. Mark the extremities of the angles by B, C, D, and E. What do you notice? Is it possible to have two points that have the same extremity?

Find their trigonometric lines. What do you notice?

Plot the points F(30), G(60) H(390), and I(420) on a one-dimensional axis. What do you notice? Is it possible to have two numbers that have the same location on the axis?

Compare the case of the number line and the case of a trigonometric circle.

Appendix 6

Find the principal measure of the following angles then find the trigonometric lines of those angles.

300

730

1120

-320

Appendix 7

Is it possible to find an angle which sine and cosine are bigger than 1?

Is it possible to find an angle which sine and cosine are less than -1?

What do you deduce?

Is it possible to find a number which double is bigger than 1, bigger than 10, 100, 1000?

Is it possible to find a number which double is less than -1, -10,-100,-1000?

Compare this situation with the case of the sine and cosine.

Is it possible to find a number which square root is bigger than 1, bigger than 10, 100, 1000?

Is it possible to find a number which square root is less than 0?

Compare this situation with the case of the sine and cosine.

Appendix 8

Answer by true or false and explain your answer by drawing the angle on the trigonometric circle.

a) $\sin(40) < \sin(60)$

b) $\cos(40) < \cos(60)$

c) Sin(40)<sin(150)

d) Cos(40)<cos(150)

e) Sin(190)<sin(220)

f) Cos(190)<cos(220)

g) Sin(30)<sin(300)

h) Cos(30)<cos(300)

3) Compare
   $2 \times 40 \ \ and \ 2 \times 60$
   40^2 and 60^2
   $2 \times 40 \ \ and \ \ 2 \times 150$
   40^2 and 150^2

What is different in the case of the comparison of the trigonometric lines of the corresponding angles found in part1 (a-b-c-d)

Appendix 9: In which quadrant is situated the extremity of the following angles:

$260°$

$132°$

$-300°$

$182°$

$470°$

$1300°$

$\frac{7\pi}{3}$

$\frac{-\pi}{4}$

$\frac{16\pi}{3}$

Appendix 10

Find the sign of each of the trigonometric lines of the following angles:

$30°, 120°, -65°, \frac{\pi}{4}, \frac{10\pi}{3}, \frac{9\pi}{8}, 200°$

Appendix 11

Construct a trigonometric circle and angle. Construct the sine and the cosine of a given angle. Construct two angles in each quadrant and find the quadrants in which the sine increases when the angle increases and the angles where the angle decreases as the angle decreases.

Let a=3 and b =4. Compare $\sqrt{a}$ and $\sqrt{b}$. What can you deduce? What is the difference between the case of radicals and the case of trigonometric functions?

Compare $\frac{1}{a}$ and $\frac{1}{b}$. What can you deduce? What is the difference between the case of inverses and the case of trigonometric functions?

Compare a^2 *and* b^2. What can you deduce? What about negative numbers?

What is the difference between the case of squares and the case of trigonometric functions?

True or false. Explain your thinking.

Sin120<sin150

Sin60<sin80

Cos120<cos150

Cos60<cos80

Sin30<sin120

Sin150<sin320

Appendix 12: Calculate the exact value of each of the following expressions

A = $\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)$

$B = \sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right)$

$C = \cos(\pi) + \sin\left(\frac{\pi}{4}\right) + tan0$

$$D = \frac{tan0 + tan60}{1 - tan0tan60}$$

Appendix 13

2) Calculate sin45, sin90 and sin180.

Compare 2sin45 and sin90.

Compare 2sin90 and sin180.

What do you deduce?

Calculate

Sin30, sin60 and sin90. Compare sin60+sin30 and sin90

What do you deduce?

Calculate 2(30)+2(60) and 2(30+60).

In general we have 2x +2y =2(x+y).

How is this different from the case of trigonometric functions?

True or False

sin(60)=2sin(30)

$\frac{cos180}{3} = \frac{cos120}{2}$

$5sinx = sin5x$

$\frac{sin2x}{2} \times \frac{1}{cosx} = tanx$

Appendix 14: Calculate the following expressions and compare

$\sin\left(\frac{7\pi}{6}\right)$ and $\sin(\pi)+\sin(\frac{\pi}{6})$

$\sin(\frac{3\pi}{4})$ and $\sin(\pi) - \sin(\frac{\pi}{4})$

$\sin\left(\frac{13\pi}{3}\right)$ and $\sin(4\pi)+\sin(\frac{\pi}{3})$

What do you notice?

Appendix 15:

Calculate the following expressions:

$$A = \sin\left(\frac{23\pi}{6}\right) + \cos(\frac{13\pi}{3})$$

$$B = \frac{\cos\left(\frac{-\pi}{3}\right) + \sin\left(\frac{11\pi}{3}\right)}{\cos\left(\frac{25\pi}{6}\right) + \sin\left(\frac{47\pi}{6}\right)}$$

Appendix 16:

If $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$, calculate $\sin\frac{\pi}{5}, \cos\frac{6\pi}{5}, \sin(\frac{11\pi}{5})$

Compare $\cos\frac{6\pi}{5}$ and $\cos(\pi) + \cos\left(\frac{\pi}{5}\right)$

$\sin(\frac{11\pi}{5})$ and $\sin(2\pi) + \sin(\frac{\pi}{5})$

APPENDIX 5

UNIT 3: TEACHING TRIGONOMETRY WITH GEOGEBRA WITHOUT CONCEPTUAL CHANGE METHODS

Objectives of the unit:

The students will define the trigonometric lines in a right triangle

They will define the trigonometric lines on the trigonometric circle

They should find out the formulae of the trigonometry

They should use those formulae to find other trigonometric identities

They should find out the signs of the trigonometric lines in each quadrant

They should find out the trigonometric lines of some remarkable angles

They should find out the properties of trigonometric lines

They should find out the trigonometric lines of associated angles

Specific Learning Outcomes:

Students should be able to present the trigonometric circle and the trigonometric lines in GeoGebra

The students should understand the non-linearity of trigonometric lines with the addition

They should understand the non-linearity of trigonometric lines with multiplication with a real number

They should be able to compare trigonometric lines by comparing the angles and understand the non-monotinicity of trigonometric functions in a period

They should be able to understand the periodicity of trigonometric functions

Lessons of the unit:

Lesson 1: Trigonometric lines in a right triangle and on the trigonometric circle

Lesson 2: Trigonometric properties and identities

Lesson 3: Quadrants of the trigonometric circle and signs of trigonometric lines

Lesson 4: Trigonometric lines of particular angles, non-linearity with the addition and with the multiplication

Lesson 5: Associated Angles

Lesson 6: Application to associated angles

Lesson 1: **Trigonometric Lines in a right triangle**

Learning Outcomes:

The students should be able to:

- Define the trigonometric ratios in a right triangle
- Calculate missing sides of triangles using trigonometric ratios
- Use trigonometric ratios to solve real-life problems
- Find out the trigonometry formulae
- Show and use trigonometric identities

Instructional Procedures:

Activity 1: Introduction about trigonometry and its use

The teacher holds a discussion about trigonometry, and its use. The students discuss several examples about the use of trigonometry in real life situations.

Activity 2: Trigonometric Ratios in a right triangle

The students receive a file in GeoGebra and a worksheet. (Appendix 1) In the GeoGebra file, a right triangle is constructed with fixed angles, the students have to keep the angle fixed and drag a point to have different triangles with different dimensions, and they have to complete the worksheet accordingly. After completing the worksheet, the students are introduced to the definitions of the trigonometric lines in a right triangle.

Activity 3: Use of trigonometric lines to find missing measures

The students will be given worksheets that they have to fill. The worksheets include triangles with missing sides, and they have to use the corresponding trigonometric lines to find the missing sides. (Appendix 2) They have to work in pairs. Then they have to discuss their answers.

The students will be given real-life situations with missing sides, and they have to use trigonometric lines to find the missing sides. (Appendix 3) They have to work in pairs and share their ideas.

Activity 4: Trigonometric identities

The students should work in pairs to find out the trigonometric formulae. They should be given a right triangle and will be asked to apply the Pythagorean Theorem. They should be

asked to derive other trigonometric formulae from the Pythagorean Theorem. ($sin^2\alpha = \frac{tan^2\alpha}{tan^2\alpha+1}$ and $cos^2\alpha = \frac{1}{1+tan^2\alpha}$)

Students should solve an exercise about trigonometric identities. (Appendix 4)

Lesson 2: Trigonometric lines on the trigonometric circle

Learning Outcomes:

The students should be able to:

- Define the trigonometric circle
- Construct a trigonometric circle on GeoGebra
- Define trigonometric lines on the trigonometric circle using GeoGebra
- Find out the periodicity of trigonometric functions
- Find out that the image of the sine and cosine are bounded intervals.
- Compare trigonometric lines of increasing and decreasing angles. Find out the non-monotonicity of trigonometric functions in a period.

Instructional Procedures:

Activity 1 The trigonometric lines on the trigonometric circle

- The students are introduced to the trigonometric circle, and they construct a trigonometric circle on GeoGebra
- A whole-class discussion is held to define the trigonometric lines on the trigonometric circle.
- The students have to construct an angle on the trigonometric circle and to find its trigonometric lines.
- They have to drag the angle to find the trigonometric lines of other angles.

Activity 2: Periodicity and principal measure

In this activity, the students are introduced to the idea of principal measure. They should notice that many angles have the same extremity on the trigonometric circle. (Appendix 5)The students will define the principal measure of an angle, and they notice that angles that have the same principal measure have the same trigonometric lines. Hence they notice the importance of studying the principal measure of an angle. They will be given angles, and they will be asked to find their principal measure. They will construct those angles on GeoGebra and find their trigonometric lines. (Appendix 6). Also, they should notice that the trigonometric lines of an angle are in $[-1; 1]$.

Activity 3: Trigonometric properties

In this activity, the students have to construct the trigonometric circle and an angle on GeoGebra. They should drag the angle and find out the trigonometric lines of the corresponding angles. They should compare the trigonometric lines of different angles. They should find out whether the sine and cosine increase when the angle increases.

The students should work on an exercise about the trigonometric lines. The exercise aims to compare trigonometric lines of different angles. (Appendix 7)

Lesson 3: Quadrants of the trigonometric circle and signs of trigonometric lines

Objectives:

The students should be able to:

- Define the quadrants of the trigonometric circle
- Find the quadrant of an angle given its quadrant
- Find out the sign of trigonometric lines in each quadrant

Instructional Procedures:

Activity 1:

The students should construct the trigonometric circle on GeoGebra. They should construct an angle and drag it. They should find out the intervals for each quadrant. The angles should be in degrees and radians.

The students should solve an exercise. The aim of the exercise is to find out the quadrants of the given angles. (Appendix 8)

Activity 2:

On the same GeoGebra file, the students should drag the angles and find out the signs of the trigonometric lines in each quadrant.

The students should be given angles, and they should find out the sign of each trigonometric line. (Appendix 9)

Activity 3:

The students should compare the trigonometric lines of different angles on GeoGebra. They should find out in which quadrant the trigonometric lines increase as the angle increases and in which quadrant they decrease. They should do an exercise about comparing angles in the same and different quadrants (Appendix 10)

Lesson 4: Trigonometric lines of particular angles

Objectives:

The students should be able to:

- Find out the trigonometric lines of some remarkable angles using GeoGebra
- Find out the non-linearity of trigonometric functions with addition.
- Find out the non-linearity of trigonometric functions with the multiplication with a real number.
- Calculate some trigonometric expressions

Instructional Procedures:

Activity 1:

The students should construct the trigonometric circle on GeoGebra, and they should find the trigonometric lines of the remarkable angles (30-45-60-0-90-180-270) They should construct a table that summarizes the results

The students should calculate trigonometric expressions, including the remarkable angles. (Appendix 11)

Activity 2:

The students would calculate many additional trigonometric expressions. The conceptual change would be clear from those examples, but it will not be taught explicitly. (Appendix 12)

Lesson 5: Associated angles

Objectives:

The students should be able to:

- Find out the relations of the trigonometric lines of associated angles
- Find out the periodicity of trigonometric lines

Instructional Procedures:

Activity 1:

The students should construct on GeoGebra an angle x and 180-x. They should find on the trigonometric circle the corresponding trigonometric lines, and they should compare them.

Similarly they should construct 180+x, -x, 90-x, 90+x, and they should find their trigonometric lines.

Activity 2

The students should find out the periodicity of trigonometric lines. They should find out the trigonometric lines of 360+x.

Activity 3:

The students would be given an exercise where they have to calculate the trigonometric lines of associated angles. The non linearity and the conceptual change are not taught explicitly.

(Appendix 13)

Lesson 6: Application of associated angles

Objectives

The students should be able to:

- Calculate trigonometric lines of remarkable angles and their associated angles
- Simplify trigonometric expressions
- Given a trigonometric line of an angle find the trigonometric lines of its associated arcs.

Instructional Procedures

Activity 1:

The students will solve an exercise involving the calculation of the associated angles of remarkable angles. (Appendix 14)

Activity 2:

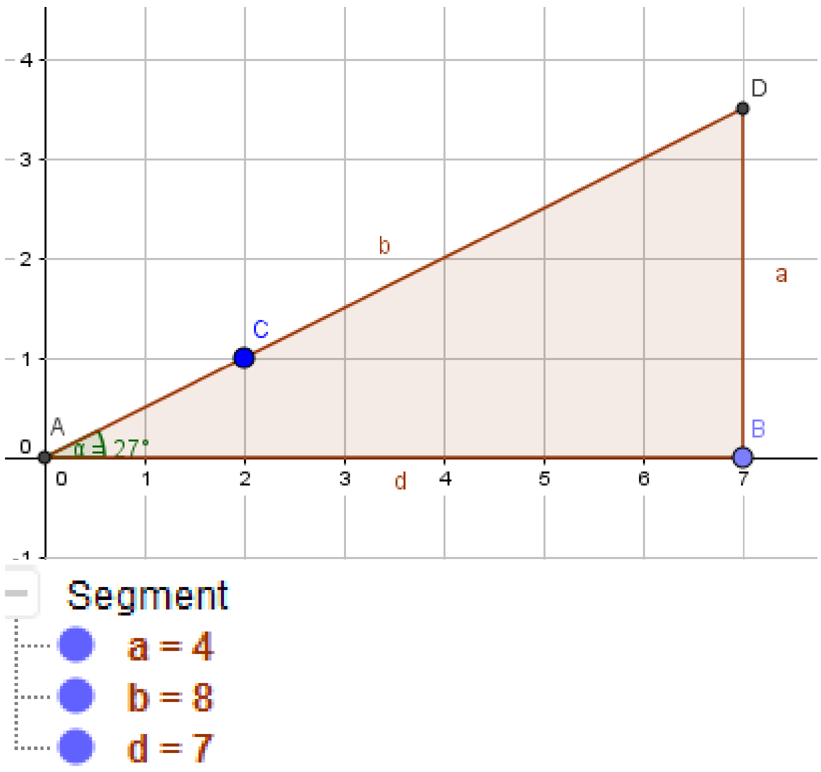
The students will be given a trigonometric line of an angle, and they should find the trigonometric lines of its associated arcs. (Appendix 15)

Activity 3:

A review of the main concepts of the unit will be held.

Appendices of Unit 3

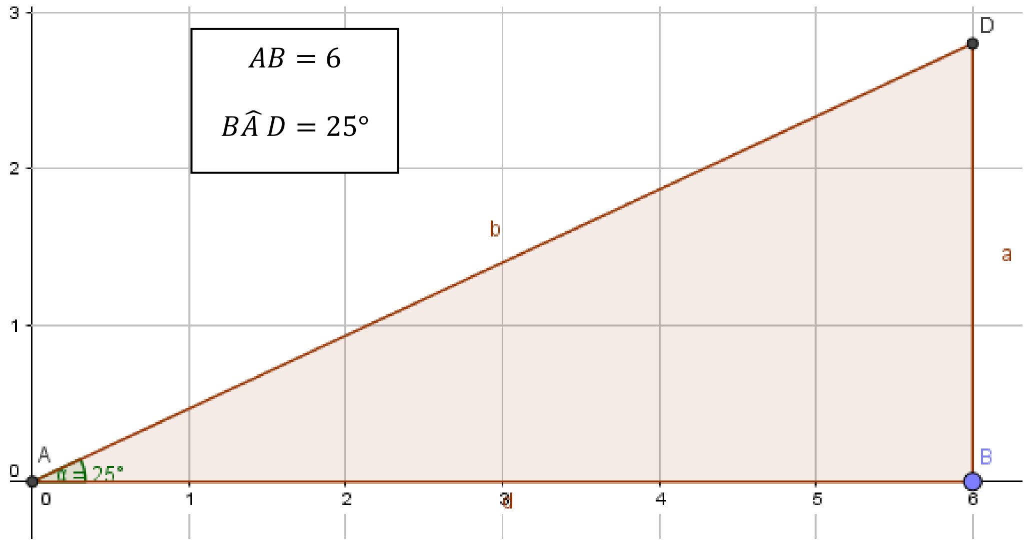
Appendix 1:



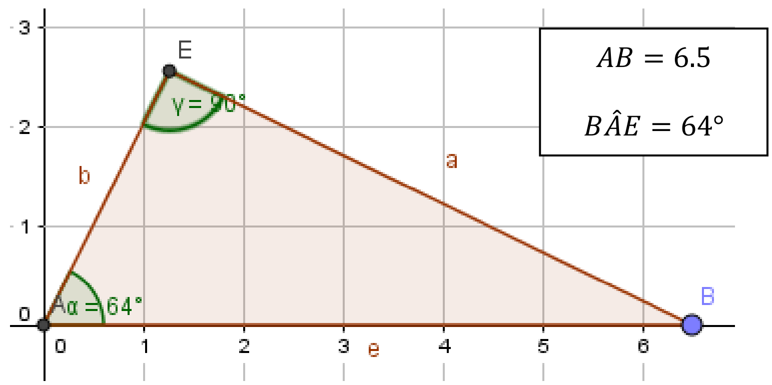
Drag B on the x-axis and complete this table

d	4	5	6`	7	8
a					
b					
d/b					
a/b					
a/d					

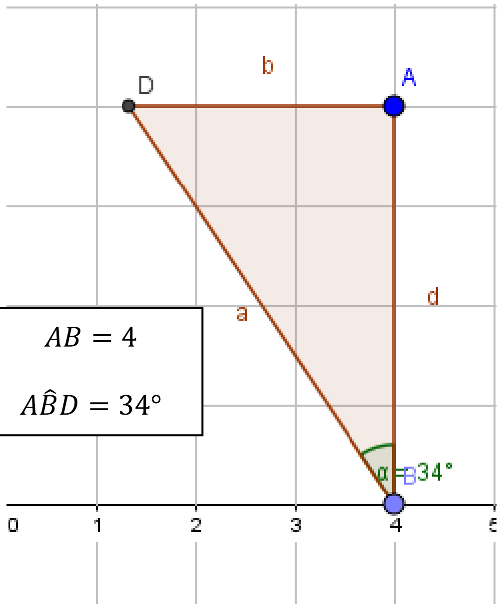
Appendix 2:



Calculate AD



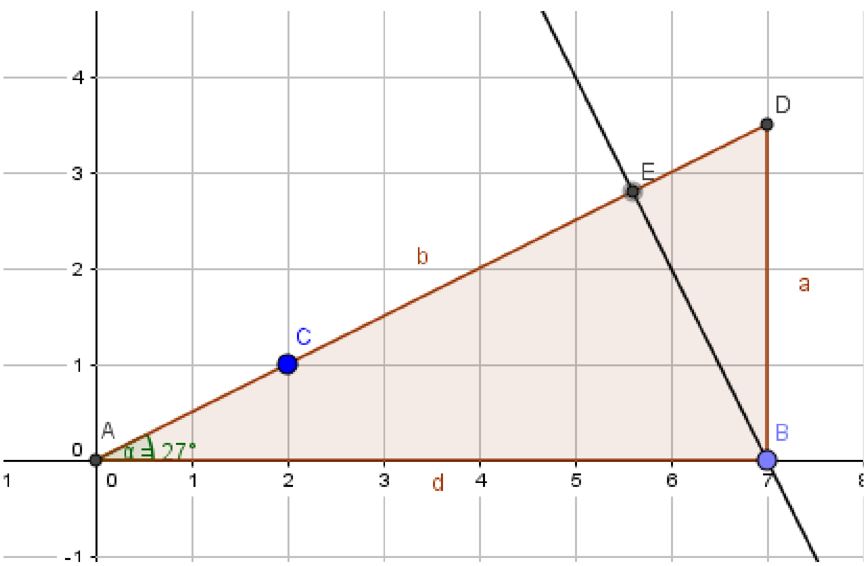
Calculate BE.



$AB = 4$

$A\hat{B}D = 34°$

Calculate AD



$AB = 7$

$B\widehat{A}\,D = 27°$

(BE) *is the height of* ABD

Calculate BD, BE, and AE.

Appendix 3:

Exercise 1

In the context of the steepness of a road, the grade or percentage grade to be the tangent of the angle α. The grade of a given road is 10%.

What is the rise BC if the distance AB is 2km

3- For which distance the rise is 200m.

B

Appendix 4: Show the following trigonometric identities (extract from the grade 10 textbook)

A

C

17) $(sin\alpha + cos\alpha)^2 - 2sin\alpha cos\alpha = 1$
18) $(sin\alpha + cos\alpha)^2 + (sin\alpha - cos\alpha)^2 = 2$
19) $tan\alpha + cot\alpha = \frac{1}{sin\alpha cos\alpha}$
20) $tan^2\alpha(1 + cot^2\alpha) = \frac{1}{cos^2\alpha}$
21) $1 - sin\alpha cos\alpha tan\alpha = cos^2\alpha$
22) $cos^4\alpha - sin^4\alpha + \frac{tan^2\alpha}{1+tan^2\alpha} - \frac{1}{1+tan^2\alpha} = 0$
23) $cot^2\alpha - cos^2\alpha = cot^2\alpha cos^2\alpha$
24) $tan^2\alpha + cot^2\alpha + 2 = \frac{1}{sin^2\alpha cos^2\alpha}$

Appendix 5

Construct the trigonometric circle. On this circle construct the angles 30, 60, 390, 420. Mark the extremities of the angles by B, C, D and E. What do you notice? Is it possible to have two points that have the same extremity?

Find their trigonometric lines. What do you notice?

Appendix 6

Find the principal measure of the following angles.

300

730

1120

-320

Appendix 7:

Answer by true or false and explain your answer using the GeoGebra.

a) $\sin(40) < \sin(60)$

b) $\cos(40) < \cos(60)$

c) Sin(40)<sin(150)

d) Cos(40)<cos(150)

e) Sin(190)<sin(220)

f) Cos(190)<cos(220)

g) Sin(30)<sin(300)

h) Cos(30)<cos(300)

Appendix 8: In which quadrant is situated the extremity of the following angles:

260°

132°

$-300°$

182°

$470°$

$1300°$

$\frac{7\pi}{3}$

$\frac{-\pi}{4}$

$\frac{16\pi}{3}$

Appendix 9:

Find the sign of each of the trigonometric lines of the following angles:

$30°, 120°, -65°, \frac{\pi}{4}, \frac{10\pi}{3}, \frac{9\pi}{8}, 200°$

Appendix 10

Construct a trigonometric circle and angle. Construct the sine and the cosine of a given angle. Drag the angle and find the quadrants in which the sine increases when the angle increases and the quadrants where the angle decreases as the angle decreases.

True or false. Explain your thinking.

Sin120<sin150

Sin60<sin80

Cos120<cos150

Cos60<cos80

Sin30<sin120

Sin150<sin320

Cos320<cos350

Cos210<cos240

Cos3210<cos3254

Appendix 11: Calculate the exact value of each of the following expressions

A = $\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)$

$B = \sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right)$

$C = \cos(\pi) + \sin\left(\frac{\pi}{4}\right) + tan0$

$$D = \frac{tan0 + tan60}{1 - tan0tan60}$$

Appendix 12

3) Calculate sin45, sin90 and sin180.

Compare 2sin45 and sin90.

Compare 2sin90 and sin180.

What do you deduce?

Calculate

Sin30, sin60, and sin90. Compare sin60+sin30 and sin90

What do you deduce?

True or False

sin(60)=2sin(30)

$\frac{cos180}{3}=\frac{cos120}{2}$

$5sinx = sin5x$

$$\frac{sin2x}{2} \times \frac{1}{cosx} = tanx$$

Appendix 13: Calculate the following expressions and compare

$\sin\left(\frac{7\pi}{6}\right)$ and $\sin(\pi)+\sin(\frac{\pi}{6})$

$\sin(\frac{3\pi}{4})$ and $\sin(\pi) - \sin(\frac{\pi}{4})$

$\sin\left(\frac{13\pi}{3}\right)$ and $\sin(4\pi)+\sin(\frac{\pi}{3})$

What do you notice?

Appendix 14:

Calculate the following expressions:

$$A = \sin\left(\frac{23\pi}{6}\right) + \cos\left(\frac{13\pi}{3}\right)$$

$$B = \frac{\cos\left(\frac{-\pi}{3}\right) + \sin\left(\frac{11\pi}{3}\right)}{\cos\left(\frac{25\pi}{6}\right) + \sin\left(\frac{47\pi}{6}\right)}$$

Appendix 15: Exercise associated angles with variable

If $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$, calculate $\sin\frac{\pi}{5}, \cos\frac{6\pi}{5}, \sin\left(\frac{11\pi}{5}\right)$

Compare $\cos\frac{6\pi}{5}$ $\quad and \ \cos(\pi) + \cos\left(\frac{\pi}{5}\right)$

$\sin\left(\frac{11\pi}{5}\right)$ and $\sin(2\pi) + \sin\left(\frac{\pi}{5}\right)$

APPENDIX 6

PRIOR KNOWLEDGE TEST

Duration: 30 minutes

Learning Outcomes to be assessed:

Conceptual Enrichment Outcomes:

a) Students define the trigonometric functions in a right triangle.
b) Students apply the definition of trigonometric functions to find out a missing side of a triangle.
c) They recognize the trigonometric functions of particular angles.
d) Students are able to find out that the sine of an angle is equal to the cosine of its complementary angle.
e) Students are able to find out that $sin^2x + cos^2x = 1$ for any angle x.
f) Students recognize that the trigonometric lines can be negative.
g) They recognize that the trigonometric lines can be defined for angles bigger than 90°.

Conceptual Change Outcomes:

a) Students recognize that the sine and the cosine of an acute angle are positive.
b) Students recognize that the sine and the cosine of an angle are less than 1.
c) Students are able to apply the non-linearity of trigonometric functions.
d) Students are able to apply the non-linearity of inverse, square, and radical functions with addition.
e) Students are able to apply the non-monotonicity of square functions.

The Prior Knowledge Test

The usage of calculators is not allowed.

Section 1

For items 1 to 4, ABC is a triangle right at A.

For each of the items 1 to 4, put a circle around the correct answer. Choose only one of the answers for each question.

Item1: Conceptual enrichment item

1.1 sinB=:

a) $\frac{AB}{AC}$

b) $\frac{AC}{AB}$

c) $\frac{AC}{BC}$

d) $\frac{BC}{AC}$

Objective: Students define the sine and the tangent of an angle

Item 2: Conceptual enrichment item

2.1 If BC=4cm and B=30°. AB=

a) 2cm

b) 8cm

c) $4\sqrt{3}$ cm

d) $2\sqrt{3}$ cm

2.2 If AC=4cm and B=30°. BC=:

a) 2cm

b) 8cm

c) $4\sqrt{3}cm$

d) $2\sqrt{3}$cm

Objective: Students are able to apply the definition of trigonometric lines to find a missing side

They recognize the trigonometric functions of particular angles.

Item 3: Conceptual enrichment item

Which of the following expressions is correct:

a. sinB=cosA
b. sinA=cosB
c. sinC=cosB
d. sinC=cosA

Objective: Students recognize the property that the sine of an angle is equal to the cosine of its complementary angle.

Item 4: Conceptual enrichment item

If sinB = $\frac{1}{3}$ then cosB=:

a) $\frac{2}{3}$
b) $\frac{2\sqrt{2}}{3}$
c) $\pm\frac{2\sqrt{2}}{3}$
d) $\pm\frac{2}{3}$

Objective: Students are able to apply the Pythagoras' identity in trigonometry.

They recognize that the trigonometric lines are positive for acute angles

Section 2

For the items of this section, state whether each of the following statements is true or false.

Show the details of your work.

Item 5: Conceptual change item

5.1 $sin^2\alpha + cos^2\alpha = 1$

5.2 $sin^2 45 + cos^2 45 = 1$

5.3 $sin^2 2\alpha + cos^2 2\alpha = 2$

Objective: Students are able to apply the Pythagoras' identity in Trigonometry

They are able to apply the non- linearity of trigonometric functions with addition.

Item 6: Conceptual change item

6.1 $\frac{1}{2} + \frac{1}{3} = \frac{1}{5}$

6.2 $(2 + 3)^2 = 25$

6.3 $\sqrt{9 + 16} = 5$

6.4 $\frac{1}{x} + \frac{1}{y} = \frac{1}{x+y}$ For any real numbers x and y.

6.5 $(2 + x)^2 = 4 + x^2$ for any real number x.

6.6 $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$ for any real numbers x and y.

Objective: Students are able to apply the non-linearity of the inverse, square, square root functions with addition.

Item 7: Conceptual change item

1) If sin40=0.64   then sin80=1.28
2) If sin30=0.5   then sin60=1

Objectives: Students are able to apply the non-linearity of trigonometric functions with multiplication by a real number

They recognize that trigonometric functions are bounded

They recognize the trigonometric functions of particular angles

Item 8: Conceptual change item

If $x < y \; then \; x^2 < y^2$ for any real numbers x and y.

Objective: Students are able to apply the non-monotonicity of the square function.

Item 9: Conceptual change item

1) For any x sinx<1
2) For any x sinx>0

Objective: Students recognize that trigonometric functions are bounded

They recognize that trigonometric functions can be negative

Item 10: Conceptual enrichment item

For x>90 sinx is undefined

Objective: Students recognize that the trigonometric lines can be defined for angles bigger than 90°.

APPENDIX 7

RUBRIC FOR ANALYZING THE SEMI-STRUCTURED QUESTIONS OF THE CONCEPTUAL ENRICHMENT AND CONCEPTUAL CHANGE TEST

Item 14: Boundedness of trigonometric functions

What are the possible values of the sine and cosine functions? Why? Compare this to other mathematical functions that you have dealt with (such as the double, the radical, the square functions)

Objectives:

The students are able to find out the boundedness of the sine and the cosine functions.

The students realize the conceptual change from unbounded functions to bounded trigonometric functions.

Answer key:

1) Possible Values of sine and cosine function: $-1 \leq sinx \leq 1$ $-1 \leq cosx \leq 1$

2) Why? On the trigonometric, the sine and cosine can not exceed -1 or 1 because the trigonometric circle has a radius of 1.

3) Comparison to other functions:

The sine and cosine are bounded other functions are not bounded

4) Comparison to the double, radical, and square functions:

- any number can be a double]-∞; +∞[
- Only positive numbers can be radicals [0;+∞[
- Only positive numbers can be squares [0;+∞[

Rubric for evaluating the item 14

	1	2	3
Boundedness of trigonometric functions	The students do not state that trigonometric functions are bounded	The students state that trigonometric functions are bounded, but they do not know the values by which they are bounded	The students state that trigonometric functions are bounded in [-1;1]
Explanation	The students have no explanation or have an incorrect explanation	The students explain that the boundedness is due to the fact that trigonometric lines are defined on the trigonometric circle, which has a radius one, but they make mistakes in their explanations *(*such as neglecting the negative values of the trigonometric functions, assigning for the radius negative values…)	The students explain that the boundedness is due to the fact that trigonometric lines are defined on the trigonometric circle which has a radius 1
Comparison to other functions	The students are not able to differentiate the bounded trigonometric functions from other unbounded functions	The students are able to differentiate the bounded trigonometric functions from one unbounded function correctly. (*they make mistakes in the other functions or they compare only to one function)*	The students are able to differentiate the bounded trigonometric functions from many other unbounded functions correctly.

Item15: Periodicity of trigonometric functions

On a given number line, is it possible to find two different numbers that are represented by the same point? Why? On a given trigonometric circle, is it possible to find two different angles that have the same extremity? Why?

Objectives:

Students are able to distinguish the periodic trigonometric circle and the non-periodic number line.

Students realize the conceptual change from non-periodic functions to periodic trigonometric functions.

Answer key:

1) Two numbers can not be represented by the same point on the number line since the number line is not periodic.
2) Two angles can have the same extremity on the trigonometric circle since the trigonometric circle is periodic.

Rubric for item 15

	1	2	3
Periodicity of the trigonometric circle	The students do not state that the trigonometric circle is periodic *(they have no answer or incorrect answer)*	The students state that two numbers can be represented by the same point on the trigonometric circle but do not explain the reason of this.	The students state that two numbers can be represented by the same point on the trigonometric circle and they can explain that this is due to the periodicity of trigonometric functions (*they may use the word periodic, explain the idea of periodicity without using the word or they may provide examples or figures)*
Non periodicity of the number line	The students do not state that on a number line two numbers can not rebe presented by the same point *(they have no answer or incorrect answer)*	The students state that on a number line two numbers can not be represented by the same point but do not explain the reason of this.	The students state that on a number line two numbers can not be represented by the same point and they can explain that this is due to the non periodicity of the number line *(they may use the word non-periodic, explain the idea of non periodicity without using the word or they may provide examples or figures)*

Item16: Non monotonicity of trigonometric functions

If $x<y$ is $\sin x<\sin y$? Explain your answer. Compare this situation to the double function, to the cosine function, and to the square function.

Objectives:

The students are able to apply the non-monotonicity of the trigonometric functions.

The students realize the conceptual change from monotonic functions to non-monotonic functions.

Answer key:

1) It depends on the quadrant

If $x<y$ then $\sin x<\sin y$ in quadrants I and IV

If $x<y$ then $\sin x>\sin y$ in quadrants II and III

Additional explanations may be given by figures of the trigonometric circle.

2) Double: If $x<y$, then $2x<2y$ for any real numbers x and y.

3) Square: It depends on the sign of x and y.

If $x>y>0$ then $x^2>y^2$ but if $x<y<0$ then $x^2>y^2$

4) Cosine: It depends on the quadrant (like the sine, but different quadrants)

If $x<y$ then $\cos x<\cos y$ in quadrants III and IV

If $x<y$ then $\cos x>\cos y$ in quadrants I and II

Rubric for item 16:

	1	2	3
Non-monotonicity of the sine function	The students do not state that if $x<y$, then $sinx<siny$ is correct in some quadrants but incorrect in others.	The students state that if $x<y$, then it is not necessarily true that $sinx<siny$, but they do not explain it and relate it to quadrants.	The students state that if $x<y$, then $sinx<siny$ is correct in some quadrants but incorrect in others.
Monotonicity of the double function	The students do not state that the sine function's behavior with the order sign is different from the double function's behavior, they do not state that if $x<y$ then $2x<2y$ for any real numbers x and y.	The students state that the sine function's behavior with the order sign is different from the double function's behavior but do not explain the difference.	The students state that the double function's behavior with the order sign is different from the sine function's behavior. They state that if $x<y$ then $2x<2y$ for any real numbers x and y.
Monotonocity of the square function for	The students do not state that the sine	The students state that the sine	The students state that the square

negative values and positive values.	function's behavior with the order sign is different from the square function's behavior. They do not state if 0<x<y then $x^2 < y^2$ but if $x < y < 0$ then $x^2 > y^2$	function's behavior with the order sign is different from the square function's behavior but do no explain the difference.	function's behavior with the order sign is different from the sine function's behavior. They state that if 0<x<y then $x^2 < y^2$ but if $x < y < 0$ then $x^2 > y^2$
Non-monotonicity of the cosine function	The students do not state that the cosine function is non-monotonic like the sine function and they do not state that the quadrants where the cosine is increasing and the quadrants where it is decreasing are different from that of the sine.	The students state that the sine function's behavior with the order sign is different (if they talk about the quadrants) or similar (if they talk about the non-monotonicity) to the sine function without giving clear explanations.	The students state that the cosine function is non-monotonic like the sine function, and they state that the quadrants where the cosine is increasing and the quadrants where it is decreasing are different from that of the sine.

<u>Item17</u>: Non linearity of trigonometric functions

Is sin3x =3sinx? Explain your answer. Compare this situation to other mathematical functions that you have dealt with.

Objectives:

Students are able to apply the non-linearity of trigonometric functions with multiplication by a real number.

Students realize the conceptual change from functions that are not linear with multiplication by a real number to functions that are linear with addition and multiplication by a real number

Answer key

1) Sin3x≠ 3sinx
2) Explanation by a counterexample: $\sin(\frac{\pi}{3}) \neq \frac{1}{3}\sin(\pi)$
3) Comparison to algebra (the double function) 2(3x)=3*2*x .

Rubric for item 17:

	1	2	3
Non-linearity of the sine function	The students do not state that sin3x$\neq 3sinx$	The students state that sin3x$\neq 3sinx$ but they can not explain why.(or they just say that trigonometry is not linear)	The students state that $sin3x \neq 3sinx$ and provide a counterexample.
Comparison to other linear functions	The students do not provide a linear function as an example and do not compare trigonometry from other domains of mathematics.	The students state that algebra is linear, but trigonometry is not linear without providing a clear example.	The students provide a linear function as an example and compare it to trigonometry