AMERICAN UNIVERSITY OF BEIRUT

PROFIT OPTIMIZATION IN THE PRICE SETTING NEWS-VENDOR MODEL WITH POISSON DEMAND

by

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ABSTRACT
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The Price Setting Newsvendor (PSN) model is one of the fundamental models for Operations Research and Management Science. In its most basic version, the Price Setting News-Vendor model allows retailers, who sell their products over a single selling period, to determine the optimal ordering quantity and selling price that maximize their expected profits. The set of products covered by the News-Vendor model can range from perishable food and bakery items to short life cycle items such as seasonal fashion goods or even electronics. In our model, we represent the demand on these products by a Poisson distribution, as some of them can face a low customer appeal due to their availability in a wide assortment or their occasional use. We also consider substitutable retail products, that are horizontally differentiated variants, under an additive-multiplicative demand setting and a logit consumer choice model. Under these settings, we propose a coordinate ascent algorithm that finds a local maximum of the profit function. Moreover, we devise optimality conditions that allow for checking whether the computed solution is a local or global optima. These conditions are used to develop a method that allows escaping local solutions that are not global. We validate our results by conducting various numerical experiments with random inputs of the model parameters. Finally, we evaluate the effectiveness of our approach compared to existing optimization heuristics applied under similar problem settings, and we suggest areas for improvement in future research.

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Chapter 1

Background and Motivation

The newsvendor model has been used since the eighteenth century in analysing inventory systems with perishable products (Morse & Kimball, 1951). In supply chain management, perishable products are products that lose their value or cannot be sold after a period of time such as newspapers, airplane tickets, fashion goods, food, and many more. The newsvendor model allows the retailers of perishable products to make informed inventory management decisions in order to maximize their profits. In its basic structure, the newsvendor model can run given little information such as the demand and the costs of over or understocking. Decision makers who implement this model can then maximize their profits; they are able to determine the optimal product quantity to order prior to the selling season. The significance of the newsvendor model and its insightful solutions have inspired researchers and students in the inventory management field throughout the years. (Arrow, Harris, & Marschak, 1951) amended the original model to run in the case where the demand on the perishable products is unknown. Since then, the demand on the products, that is uncertain in real life, was represented by a random variable in the newsvendor model. Thereafter, (Whitin, 1955), motivated by the significance of the product's selling price on the consumer's behavior, studied the newsvendor model with price effects, known as the Price Setting newsvendor (PSN) model. The (PSN) model allows decision makers to simultaneously determine the optimal inventory level and the selling price of their products to maximize their profits.

The (PSN) model has gained significant attention since then and was further extended in various directions. In this paper, we study the (PSN) problem with Poisson demand. First of all, we use the Poisson distribution to model the demand in cases where the inventory levels are discrete quantities. In this case, a continuous approximation of the discrete demand might yield to non integer values of the optimal inventory levels , which may be not feasible, whereas the discrete version of the (PSN) always yields to integer values, as shown in Figure 1.1.

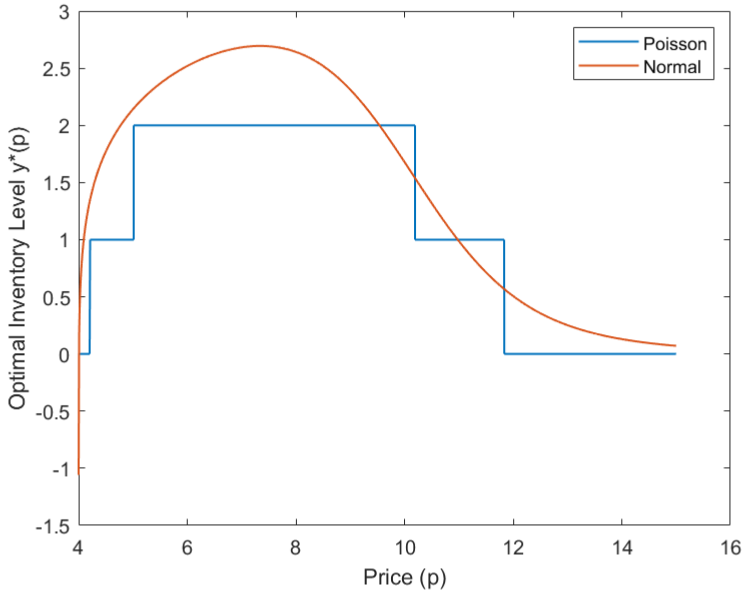


Figure 1.1: Optimal inventory level in the cases of Poisson and normal demand

Moreover, we find the Poisson distribution very useful in modelling the demand on some products under the newsvendor type inventory setting. We mainly note products in wide assortments, such as bakery or grocery items, and we also consider slow moving items such as seasonal fashion goods or apparel. Second, as discrete inventory quantity limits supply flexibility and requires additional demand shaping, the role of the selling price in matching supply and demand under the Poisson demand setting is greater than it is in the continuous case (Schulte & Sachs, 2019). For instance, under the Poisson demand setting, the optimal price exhibits discontinuous jumps when the problem parameters are varied as shown in Figure 1.2.

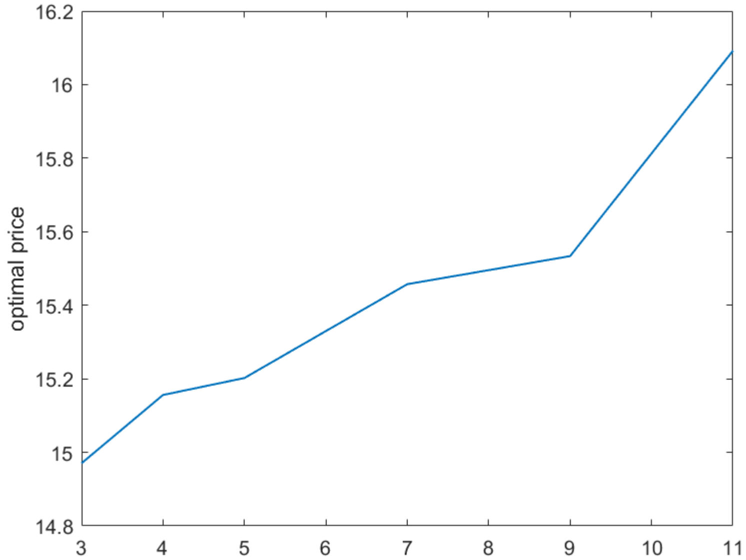
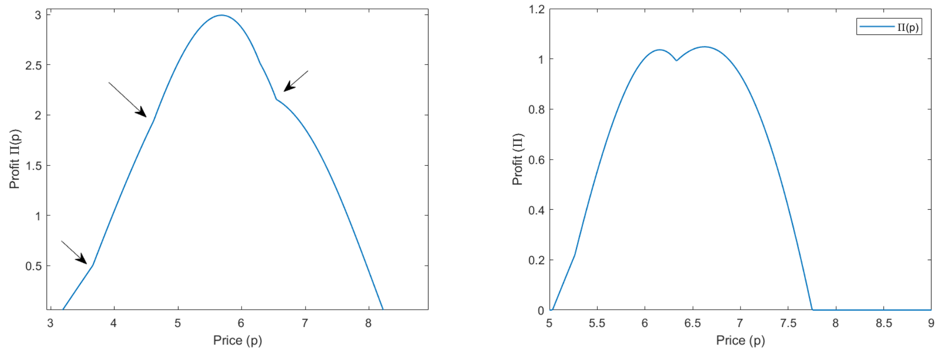


Figure 1.2: Discontinuous jumps in the optimal price subject to change in the problem's parameters

Note that these effects are hardly observable in the case of high demand rates and inventory levels, but they can be significant when the demand is sparse and consists of a few items (Schulte & Sachs, 2019). In addition to that, using the Poisson distribution allows us to combine the characteristics of additive and multiplicative demand models at the same time. Most of the literature on the (PSN) considers either an additive demand model, where the standard deviation is independent of the price, or a multiplicative model, where the demand coefficient of variation is not depending on the price. In that sense, our model under the Poisson demand setting can be seen as "additive-multiplicative" where both the demand variance and coefficient of variation are functions of the price. Finally, using the Poisson distribution in the (PSN) problem can sometimes lead to a non typical behavior of the profit function. For instance, the expected profit function in the (PSN) problem with Poisson demand is not continuous nor unimodular in the price as shown in Figure 1.3.



(a) (b)

Figure 1.3: (a) Discontinuity of the profit function (b) Non unimodularity of the profit function

This behavior has led some researchers on the topic to avoid working with cumbersome functions, hence they approximate the discrete demand by continuous distributions. However, (Axsäter, 2013) shows that continuous (normal) distributions can fail in approximating discrete demand models in case of low demand mean, specifically when the mean is smaller than ten units per period. Consequently, much of the literature on the (PSN) model focuses on continuous rather than discrete demand models.

Our contribution is to study the (PSN) problem under the Poisson demand setting and logit choice model in a multiple product assortment. We first apply the Coordinate Ascent Algorithm on the expected profit function to reach a local maximum in the price, and we then develop a heuristic to escape the local maximum into a higher maximum value. We finally demonstrate the efficiency of our proposed methods by running multiple numerical experiments and analysing our results.

Chapter 2

Literature Review

In this section, we discuss some previous work on the (PSN) model from three different research streams based on the demand models. We first start by the research on the (PSN) problem under the continuous demand setting and the additive or multiplicative demand models. We then discuss the literature done on more general demand models, such as the mixture of both additive and multiplicative demand settings, in the single and multi-product cases of the (PSN) model. At the end of this section, we discuss the recent research done on the (PSN) model under discrete (Poisson) demand setting.

(Arrow et al., 1951) introduce a model to derive the optimal inventory level and the reordering point for finished goods under exogenous pricing settings, the Classic News-Vendor model. In their model, they represent the demand by a random variable following a specified distribution, and they determine the optimal ordering quantity that maximizes the profit function. (Whitin, 1955) then introduces the Price Setting News-Vendor (PSN) model, that addresses the joint decision of setting the selling price and the inventory level in a single product assortment. For the single product case, (Whitin, 1955) adopts the continuous and linear-demand model $D(p)$=ap+b where p represents the selling price, and a and b are given constants. As an extension to (Whitin, 1955)'s work, (Mills, 1959) studies the effect of uncertainty for a monopolist producing a single commodity, and making short-run price decisions under static conditions. He considers an additive demand model $D(p, u)$, obtained by incorporating an additional random variable u to the original demand function $D(p, u)$ as follows: $D(p, u)$=$D(p) + u$, where u follows a known distribution independent of p. He then concludes that under the continuous and additive demand setting, the optimal selling price of a single product in a (PSN) model is lower than the deterministic price, corresponding to the risk free case where the demand is deterministic. Afterwards, (Ernst, 1971) & (Thowsen, 1975) prove the uniqueness of the optimal policy under the continuous additive demand setting. They base their results on reasonable assumptions for the single and multiple period inventory problems, where the demand function follows a second order Polya frequency distribution. Then, (Lau & Lau, 1988) & (Polatoğlu, 1991) show similar profit unimodularity results for demand functions that follow a uniform continuous distribution. On another hand, (Karlin & Carr, 1962) study the single product (PSN) model under the continuous and multiplicative demand setting where $D(p,u)$=$D(p).u$. They prove that the optimal selling price in their case is greater than the deterministic price. In that same context, (Nevins, 1966) shows that for each positive inventory level, there exists a unique price that maximizes the expected profit for normally distributed demand functions. Moreover, (Zabel, 1970) shows similar profit unimodularity results for exponential and uniform demand distributions.

As a generalization of these works, (Young, 1978) amends the (PSN) model for the continuous additive-multiplicative demand setting. The author represents the demand by a random variable of the form $D(p) = \alpha(p)\varepsilon+\beta(p)$ where $\alpha(p)$ and $\beta(p)$ are deterministic functions of price p, and ε is a random variable with a fixed density function $\Phi(\varepsilon)$. (Young, 1978) proves the existence and uniqueness of the optimal policy when $\Phi(\varepsilon)$ is log-concave or log-normal. The author explains that approximating the optimal policy of stochastic demand inventory problems with that of deterministic demand problems can lead to contrasting results between the additive and multiplicative cases, due to the way in which randomness is incorporated in the demand function. Thereafter, (Petruzzi & Dada, 1999) expand (Young, 1978)'s work by developing a unique framework of the (PSN) model for both the additive and multiplicative demand settings. The authors introduce a benchmark price variable that, when added to a premium price, will be equal to the optimal price value maximizing the expected profit at a fixed inventory level. Besides, they use a stocking factor variable, first introduced by (Silver, Pyke, Peterson, et al., 1998), that represents the number of standard deviations by which the stocking quantity deviates from the expected demand. They also address the multi-period models, thus revive the research on the (PSN) model to cover more general demand models. Subsequently, (Roels, 2013), (Kocabıyıkoğlu & Popescu, 2011), and (Lu & Simchi-Levi, 2013) derive concavity and monotonicity results for the single product (PSN) model under the additive-multiplicative demand setting. Nevertheless, (Yao, Chen, & Yan, 2006) also establish a general demand model that solves the joint pricing and assortment decision without using specific demand functions. They consider the demand D(p, ξ) to consist of two parts: The mean *y(p)* being continuous, non-negative, and twice differentiable with increasing price elasticity, and the stochastic factor ξ being a random variable independent of the price and having a generalized strict increasing failure rate. In a more general framework, (Raz & Porteus, 2006) develop a model, that is not necessarily a mixture of additive and multiplicative demand models, to solve the joint pricing and ordering problems. They approximate the demand distribution function by a number of representative fractiles, being linear piecewise functions of the price. Going forward, they determine the optimal price and inventory level values for each eligible fractile problem. Finally, they enumerate along the fractiles subproblems to determine the optimal selling price and inventory level.

In a slightly different context, (Maddah, Bish, & Tarhini, 2014) attempt to tackle the (PSN) model for discrete demand models. They approximate the Poisson distribution with a normal distribution with mean $\mu = \lambda$ and a variance $\sigma^2 = \lambda$ under an additive-multiplicative demand setting. They use a Taylor series approximation for the inventory cost, and show that the expected profit function is unimodular in the selling price. However, (Axsäter, 2013)'s numerical study proves that discrete demand distributions can not be sufficiently well approximated by continuous distributions for relatively low mean values. Motivated by (Axsäter, 2013)'s results, (Schulte & Sachs, 2019) develop an analytical approach to solve the (PSN) problem under the Poisson demand setting. They cover a broad class of demand models, mainly linear and logit models. (Schulte & Sachs, 2019) prove the unimodality of the expected profit function at a fixed inventory level in a single product assortment subject to a Poisson demand. Following from that, they obtain the optimal price value for each inventory level using standard optimization techniques. However, they use enumeration to determine the optimal solution, amongst the different solutions at fixed inventory levels, which is computationally exhaustive for large values of demand rates and multi-product scenarios.

The vast majority of the literature on the Price Setting News-Vendor (PSN) model covers the cases where the demand is represented by a continuous random variable. Nonetheless, the contributors to the existing literature body cover a wide range of distribution models to represent the continuous demand. We mainly note the normal, uniform, and exponential distribution models. Moreover, the existing literature body is very rich in terms of the different ways used to incorporate the price, as a variable, into the continuous demand's expression. We mainly note the additive, multiplicative, and even the mixture of both as a price-demand relationship model. However, little work is done here on the (PSN) model under discrete demand settings. (Maddah et al., 2014) determine an optimal solution for the discrete demand (PSN) problem, under the multi-product setting, by approximating the Poisson demand to a Normal distribution. On another hand, (Schulte & Sachs, 2019) also derive a solution for this problem, while preserving the discrete properties of the Poisson distribution, but they limit their results to the single product case and an acceptable range of demand rates. Consequently, no results had yet been derived to solve the discrete demand (PSN) model for scenarios of multiple products and various demand rates at the same time.

Chapter 3

Model Overview

In this section, we develop a generic model for the (PSN) problem with Poisson demand in a multi-product assortment. By using the Poisson distribution, we combine the characteristics of both additive and multiplicative demand settings. Moreover, we adopt (Maddah et al., 2014)'s logit choice model in determining the purchase probabilities.

We denote by Ω={1,2,3,...n} the set of horizontally differentiated variants from which retailers can compose their product line. Let $K \in \Omega$ be the assortment stocked by the store. We also denote by y_i the fixed inventory level of product $i \in K$, p and c the selling price and the cost of buying one unit of product $i \in K$ respectively. We denote by λ the given demand rate, it is usually obtained from historical demand data. We also adopt (Maddah et al., 2014)'s model of "popular sets", which are shown to be optimal under both exogenous and endogenous pricing by (Mahajan & van Ryzin, 1999) and (Maddah & Bish, 2007). We assume, in a product line of n products, the customer's reservation prices to satisfy $\alpha_1 \geq \alpha_2 \geq ...\alpha_n$, which implies that we only consider assortments $S_k = \{1, 2, ..., k\}$, $k = 1, ..., n$. Moreover, we work with logit demand models where a customer decides on buying product $i \in K$ to maximize the utility U_i of buying the product over the no purchase utility U_o. We define $U_i = \alpha - p + \epsilon$, and $U_o = \epsilon_o$; with α_i being the customer's reservation price and ϵ_i, ϵ_o being independent and identically distributed Gumbel random variables with mean zero and shape factor one (Guadagni & Little, 1983). Thus, the probability that one customer will buy product $i \in K$ is given by $q_i(p) = Pr\{U_i = \max_{j \in K\cup\{0\}} U_j\}$, and the no purchase probability is given by $q_o(p) = 1 - \sum_{i\in K} q_i(p)$ as follows:

$$q_i(p) = \frac{e^{\alpha_i - p}}{1 + \sum_{i\in K} e^{\alpha_i - p}} \quad \forall \ i \in K$$

and

$$q_0(p) = \frac{1}{1 + \sum_{i\in K} e^{\alpha_i - p}} \quad \forall \ i \in K$$

We represent the demand on product $i \in K$ by the price dependent random variable $D_i(p)$ that follows a Poisson distribution at a rate $\lambda q_i(p)$. Without loss of generality, we assume the products to have no salvage value and no holding costs. We then write the sales' expression of product $i \in K$ as the minimum between the ordered quantity y_i and the demand $D_i(p)$ as follows:

$$S_i(p, y_i) = \min(D_i(p), y_i). \tag{3.1}$$

The expected sales' expression of product $i \in K$ follows as such:

$$E(S_i(p, y_i)) = \lambda q_i(p)\Theta(y_i, \lambda q_i(p) + y_i(1 - \Theta(y_i, \lambda q_i(p)), \tag{3.2}$$

where Θ represents the Poisson cumulative distribution function. By expanding the above expression, we obtain

$$E(S_i(p, y_i)) = \sum_{j=0}^{j=y_i} j e^{-\lambda q_i(p)} \frac{[\lambda q_i(p)]^j}{j!} + y_i \sum_{j=y_i+1}^{j=\infty} e^{-\lambda q_i(p)} \frac{[\lambda q_i(p)]^j}{j!}. \tag{3.3}$$

We then simplify the equation above as follows:

$$E(S_i(p, y_i)) = \lambda q_i(p) \sum_{j=0}^{j=y_i-1} e^{-\lambda q_i(p)} \frac{[\lambda q_i(p)]^j}{j!} + y_i \sum_{j=y_i+1}^{j=\infty} e^{-\lambda q_i(p)} \frac{[\lambda q_i(p)]^j}{j!}. \tag{3.4}$$

Finally, we express the Poisson Cumulative Distribution function by Θ and write the expression of $E(S_i(p, y_i, K))$ as follows:

$$E(S_i(p, y_i)) = \lambda q_i(p)\Theta(y_i - 1, \lambda q_i(p)) + y_i(1 - \Theta(y_i, \lambda q_i(p))). \tag{3.5}$$

In the remaining of this report, we assume, without loss of generality, negligible holding costs and salvage values, and we write the expression of the expected profit function for product $i \in K$, $\Pi_i(p, y_i)$ as the difference between the revenue from selling $S_i(p, y_i)$ units at a unit price p and the cost of buying y_i units at a unit cost c as follows:

$$\Pi_i(p, y_i) = p\lambda q_i(p)\Theta(y_i - 1, \lambda q_i(p)) + p y_i(1 - \Theta(y_i, \lambda q_i(p))) - c y_i. \tag{3.6}$$

However, since our research aims to determine the optimal selling price and inventory level of product $i \in K$, we need to consider the inventory level as a variable rather than a fixed quantity. Based on well known results of the news-vendor model under discrete demand settings (Hadley & Whitin, 1963), we calculate the optimal inventory level, under the Poisson demand setting, as the smallest integer y that satisfies the following inequation:

$$\sum_{j=1}^{j=y} \Phi(j) \geq \frac{p - c}{p}, \tag{3.7}$$

where p and c are the product's price and purchasing cost, and Φ is the probability density function of the Poisson distribution. Following on from that, we write the expression of the optimal inventory level $y_i^*(p)$ for product i in assortment K as follows:

$$y_i^*(p) = \Theta^{-1}(1 - \frac{c}{p}, \lambda q_i(p)), \tag{3.8}$$

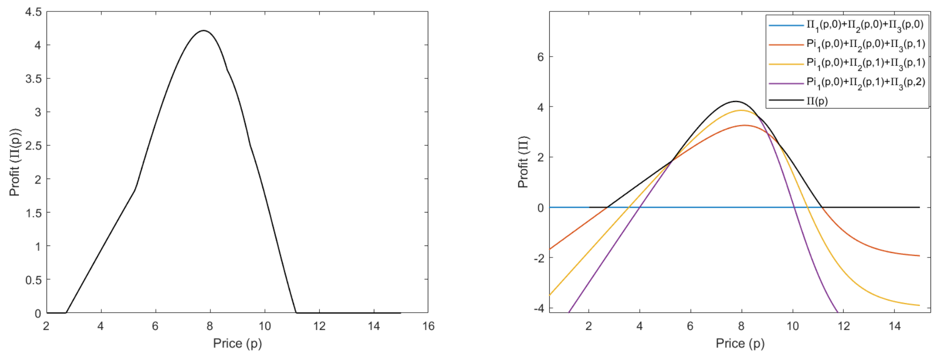
where Θ^{-1} is the inverse of the Poisson Cumulative Distribution function and $1-\frac{c}{p}$ is the critical fractile's expression. We now define the total profit function in an assortment K, $\Pi(p)$, as the the sum of the individual products' profit functions as follows:

$$\Pi(p) = \Pi(p, Y^*(p) = \sum_{i \in K} \Pi_i(p, y_i^*(p)), \tag{3.9}$$

where $Y^*(p) = \begin{bmatrix} y_1^*(p) & y_2^*(p) & \dots & y_K^*(p) \end{bmatrix}$ is the optimal inventory level vector in an assortment of K products. Finally, based on the property of the optimal inventory level, being the inventory level that maximizes the profit function, we rewrite the expression of the expected profit function, $\Pi(p)$, of equation (3.9) as follows:

$$\Pi(p) = \sum_{i \in K} \Pi_i(p, arg \max_{y_i} \Pi_i(p, y_i))) = \sum_{i \in K} \max_{y_i} \Pi_i(p, y_i)), \tag{3.10}$$

where $\Pi_i(p, y_i)$ is the profit function of product $i \in K$, at an inventory level $y_i \in \mathbf{N}$. In Figure 3.1, we show the breakdown of the expected profit function, $\Pi(p)$, in a 3 product assortment as a function of the individual products' profit functions.



(a) (b)

Figure 3.1: Break down of the expected profit function in a 3 product assortment

We now reduce the (PSN) problem with Poisson demand to solving equation 3.11.

$$\max_{p \geq 0} \Pi(p) = \max_{p \geq 0} \sum_{i \in K} \max_{y_i} \Pi_i(p, y_i). \ \forall y_i \in \mathbf{N} \tag{3.11}$$

Chapter 4

Solution Approach

In this section, we develop a solution approach to solve the problem in equation (3.11) in 3 steps. First, we find a local maximum of $\Pi(p)$ using the coordinate ascent algorithm. We then devise optimality conditions that allow for checking whether the computed solution is a local or global optima. Finally, we develop a method that allows escaping local solutions that are not global.

Local Optimization of $\Pi(p)$

In this part, we develop an algorithm that finds a local maximum of the expected profit function $\Pi(p)$. Before proceeding, we state the following Lemma on the existence of a local maximum of $\Pi(p)$.

Lemma 1 *If there exists a price value p such that $\Pi(p)_{|p>c} > 0$, then $\Pi(p)$ has at least one local maximum at a price value $p > c$.*

Proof 1 *We know from equation (3.6) that $\Pi_i(p, y_i)_{|y_i=0} = 0$. Moreover, since $y_i^*(p)_{|p=c} = 0$ and $lim_{p\mapsto\infty} y_i^*(p) = 0$, $\forall i \in K$, we can easily show that $\Pi(p)_{|p=c} = 0$ and $\lim_{p\to\infty} \Pi(p) = 0$. Therefore, if $\Pi(p)$ is positive at a price value $p > c$, then the function has at least one local maximum to the right of the unit cost c.*

Following on from that, we now develop an algorithm that converges to a local maximum of $\Pi(p)$ in the price p. In figure 3.1, we see that the functions, $\Sigma_{i\in K}\Pi_i(p, y_i)$, are well behaved and continuous in the price p, for different non-negative integer values of y_i. Moreover, numerical experiments show that in an assortment K, $\Sigma_{i\in K}\Pi_i(p, y_i)$ is not only continuous, but also unimodular in the price p for values of $p > c$. Therefore, we develop the following assumption to proceed with our solution approach.

Assumption 1 *In a given assortment K, the sum of the products' individual profit functions, $\Sigma_{i\in K}\Pi_i(p, y_i)$, is continuous and unimodular in the price p, if there exists at least one product i, with a non-negative integer value y_i.*

Remark 1 *We theoretically validate Assumption 2 in a single product assortment, as an application of (Schulte & Sachs, 2019)'s Theorem 3.7 for logit demand models. The unimodularity proof of* $\Sigma_{i\in K}\Pi(p, y_i)$ *in a single product assortment, where* $K = 1$*, is available in Appendix A.*

Following from that, we can now use an optimization technique from the group of Majorization-Minimization (Minorization-Maximization), also known as MM, algorithms. The MM class of algorithms consists of iterative optimization techniques that operate by creating a surrogate function that majorizes (minorizes) the objective function. At each iteration, the surrogate function is minimized (maximized), and the objective function is driven downhill (uphill). A special class of the MM algorithms, known as the Expected Minimization (Maximization) or EM algorithms, can be used to optimize the expected profit function $\Pi(p)$ in the problem of equation equation (3.10). At each iteration, the EM algorithm maximizes the lower bound and updates the initial guess on the objective function in an upward direction, unless the gradient is zero. In this case, the algorithm stops at a stationary point of the objective function.

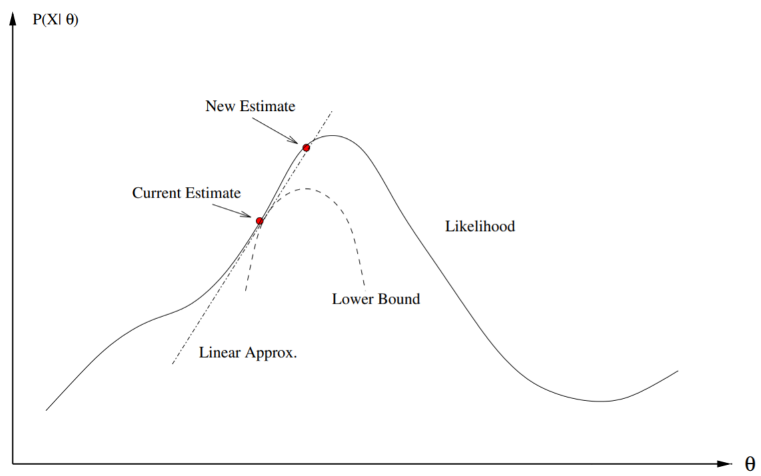
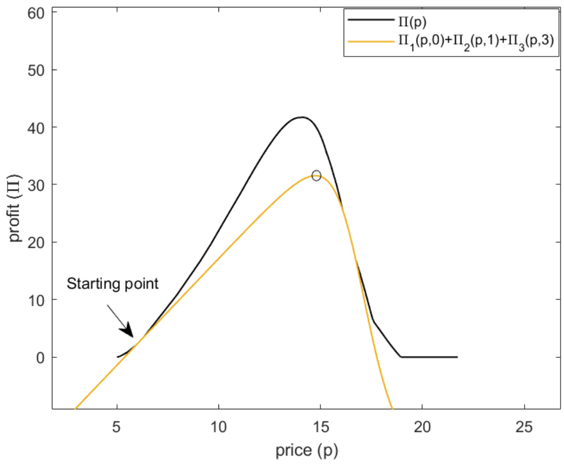


Figure 4.1: Lower bound optimization vs gradient ascent (Harpaz & Haralick, 2006)

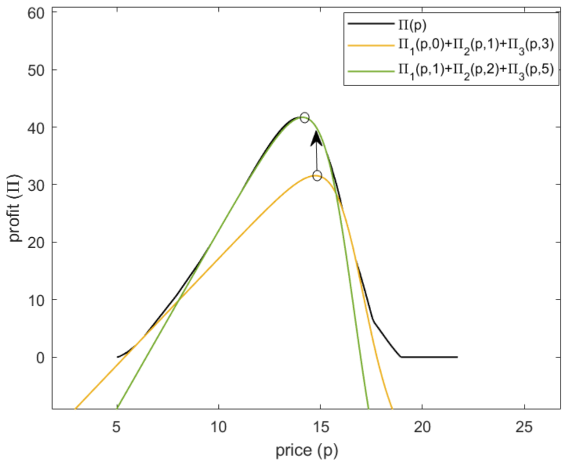
Under our problem settings, we can determine the lower bounds on the objective function $\Pi(p)$, at any price p, as the sum of the individual profit functions using equation (4.1).

$$\sum_{i\in K} \Pi_i(p, y_i^*(p)). \tag{4.1}$$

At a starting point p_0, we first compute the lower bound on the objective using equation (4.1). Second, since the functions in equation (4.1) are continuous and unimodular in the price at a fixed inventory level (Assumption 1), we can use standard optimization techniques, such as the gradient ascent, to reach their maximum values. We then update the initial price value on the objective function, and finally stop when the gradient on the objective is equal to zero as illustrated in 4.2.



(a)



(b)

Figure 4.2: EM as a lower bound maximization: (a) First Iteration (b) Second iteration

On the assumption that the lower bounds on the objective are unimodular functions in the price (Assumption 1), the EM algorithm, detailed above, converges to a local maximum of the expected profit function $\Pi(p)$ (Wu, 1983). In the next step, we divise optimality conditions to check whether the local maximum is an optimal solution, we then develop a heuristic to escape the local solutions that are not global. Note that in order to develop the optimality conditions and escape the local solution, we need the local maximization algorithm to operate on one side of the curve, i.e., we need to update the price in a consistent direction over the objective at each iteration. Given that the EM algorithm does not guarantee consistency in the price variation over the objective, as shown in Figure 4.2, we apply one of its extensions, the coordinate descent (ascent) algorithm, to reach a local maximum of $\Pi(p)$. The coordinate descent (ascent) allows us to move on one side of the objective function determined by the starting point. The algorithm uses an iterative methodology in which each iterate is obtained by fixing most components of the variable vector at their values from the current iteration, and approximately minimizing (maximizing) the objective over the other components (Wright, 2015). Since our objective function is essentially a function of the price and the inventory levels (equation (3.9)), the coordinate ascent operates by fixing one variable and maximizing over the other at each iteration. Starting at price value $p_0 \gtrsim c$, we maximize over the inventory level by computing $y_i^*(p_0)$ for each product $i \in K$. We then fix the inventory levels at $y_i^*(p_0)$ $\forall i \in K$, and apply one gradient step on the function $\Sigma_{i \in K} \Pi(p, y_i^*(p_0))$. At each iteration, we update the price in an increasing direction, when the gradient is not zero, and repeat the same procedure. Finally, we stop at a price value where the gradient of objective is equal to 0. In the following, we want to converge to the first local maximum of $\Pi(p)$ to the right of the unit cost c. Therefore, we start at a price value, p_0, slightly greater than the unit cost c. We define by $s > 0$ and $\xi \approx 0$, the step size and the precision parameters respectively to terminate the algorithm.

Algorithm 1 Coordinate Ascent CA (p_0, s, ξ, K)

Require: $p_0 \gtrsim c$, $s \geq 0$, $\xi \approx 0$

for $j = 0, \ldots,$ **do**

Compute $y_i^*(p_j) = \Theta^{-1}(1 - \frac{c}{p_j}, \lambda q_i(p_j))$

Compute $\nabla\Pi(p)|_{p=p_j} = \sum_{i \in K} \nabla\Pi_i(p, y_i^*(p_j))$

Update $p_{j+1} = p_j + s\ [\nabla\ \Pi(p)|_{p_j}]$

if $\nabla\Pi(p)|_{p=p_j} \leq \xi$ **then**

Break

end if

end for

Return p_j

Theorem 1 ***Local maximization***
In a multi-product assortment of the (PSN) problem covering a logit choice model under the Poisson demand setting, Algorithm 1 converges to a local maximum of $\Pi(p)$.

Proof 1 *The EM algorithm converges to a stationary point of the objective function under the following conditions (Wu, 1983):*

1. *Unconstrainted optimization problem*
2. *Unimodularity of the lower bound functions*

Based on Assumption 1, the lower bounds on the objective functions computed by Algorithm 1 are unimodular in the price p. Therefore, the EM algorithm converges to the global maximum of one of the unimodular lower bounds, hence a local maximum of the objective $\Pi(p)$.

The solution computed in Algorithm 1 can then be used to determine a price value that maximizes the expected profit function in a certain interval to the right of the unit cost c. Consequently, we can determine the inventory level at the local maximum using equation (3.8). However, we know that the price value obtained in Algorithm 1, is not necessarily the optimal solution that a decision maker is looking for. Therefore, we develop in the next section an optimality condition to check whether the local maximum obtained using Algorithm 1 is an optimal solution for equation (3.11).

Global Optimization of $\Pi(p)$

In this section, we generate a heuristic to determine $\Pi(p)$'s global maximum. We first start by analyzing the bahavior of $\Pi(p)$'s local maxima in a single product assortment and develop an optimality condition to check whether a local maximum is an optimal solution. We then generalize the optimality condition to the multi-product case under a set of reasonable assumptions. Finally, we develop a technique to escape the local maxima that are not global to higher values.

Optimality condition on the local maximum

Numerical experiments show that in some cases, the expected profit function $\Pi(p)$ can have more than one local maximum. We show an example in Figure 4.3 for a 3 product assortment, where the expected profit function $\Pi(p)$ has 2 local maxima.

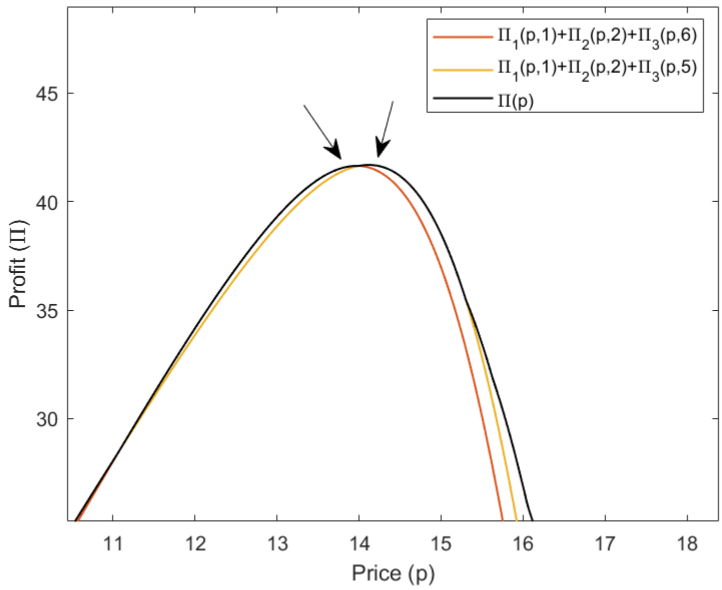


Figure 4.3: Case where $\Pi(p)$ has two maxima

Next, we analyze the behavior of the local maxima in a single product assortment and we develop a technique to check the optimality of a local solution based on (Schulte & Sachs, 2019)'s analytical results. We then prove the validity of our proposed technique in a multi-product assortment via numerical results and generalize it to the multi-product case.

Case of a single product assortment

In a single product assortment, the optimal prices, p_j^*, of the expected profit functions, $\Pi_1(p, j)$, decrease in the inventory level j, for all $j \in \mathbf{N}^*$ (Schulte & Sachs, 2019). Therefore, if $p_1 = CA(p_0 \gtrsim c, s, \xi, 1)$ is the first local maximum of $\Pi(p)$, to the right of the unit cost c, then a second local maximum might only occur at a price value p_2, such that $p_2 > p_1$ and $y_1^*(p_2) < y_1^*(p_1)$.

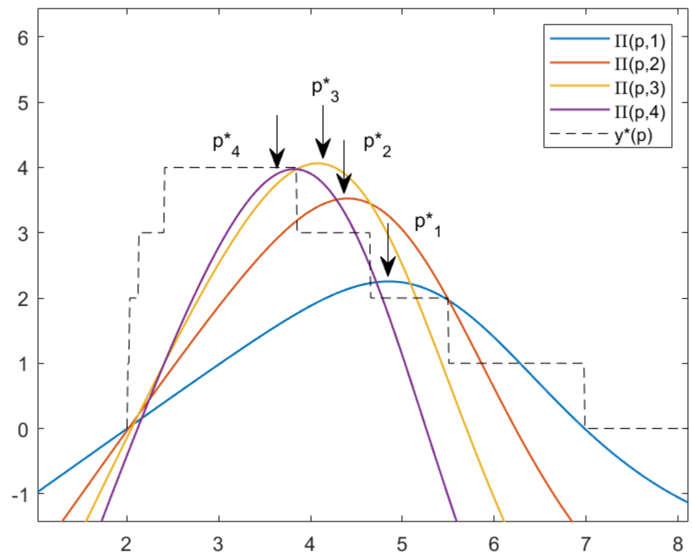


Figure 4.4: Optimal prices decreasing in the inventory level: case of a single product assortment

Given that $\Pi_1(p, j)$ is unimodular in the price for all $j \in \mathbf{N}$ (Schulte & Sachs, 2019), we can check whether $p_1 = CA(p_0 \gtrsim c, s, \xi, 1)$ is an optimal solution by comparing $\Pi(p)_{|p=p_1}$ to the maxima of $\Pi_1(p, j)$ for all nonnegative integer values of $j < y_1^*(p_1)$. Besides, since we are moving in a consistent direction to the right of the unit cost c, we can avoid enumerating over all the values of $j \in \mathbf{N}^*$ and escape to the first value that is higher than the local maximum. Therefore, we can start by checking the maximum of the function $\Pi(p, y_1^*(p_1) - 1)$, obtained by decreasing the optimal inventory level at p_1 by 1 unit. If the maximum of this function, at a price $p_2 > p_1$, is greater than the local maximum at p_1, then we escape the local maximum and move to p_2. We then apply the same procedure starting from p_2, and we only consider a local maximum, at a price p_2, to be a global solution when $\Pi(p)_{|p=p_2} > \max_p \Pi_1(p, j)$ for all $j < y_1^*(p_2)$.

Case of a multi-product assortment

In a multi-product assortment, we write the expected profit function, $\Pi(p)$, as the sum of the products' individual profit functions: equation (3.10). Therefore, we can extend the optimality condition from the single product case, by checking the maxima of the individual profits' summation functions following the methodology detailed in this section. First of all, since the property of the optimal price decreasing in the inventory level does not apply in the multi-product scenario, we need to check the maxima of the individual profits' summation functions obtained by increasing and decreasing the optimal inventory level of each product at the local maximum by 1 unit. For a local maximum at a price p_1, we determine these functions using equation (4.2).

$$\Pi(p, Y^*_{r\pm}(p_1)) = \sum_{i \in K, i \neq r} \Pi_i(p, y^*_i(p_1)) + \Pi_r(p, y^*_r(p_1) \pm 1) \ \ \forall r \in K \tag{4.2}$$

where $Y^*_{r\pm}(p_1)$ is the inventory level vector obtained by increasing/decreasing the optimal inventory level of product r at p_1, $y^*_r(p_1)$, by 1 unit.

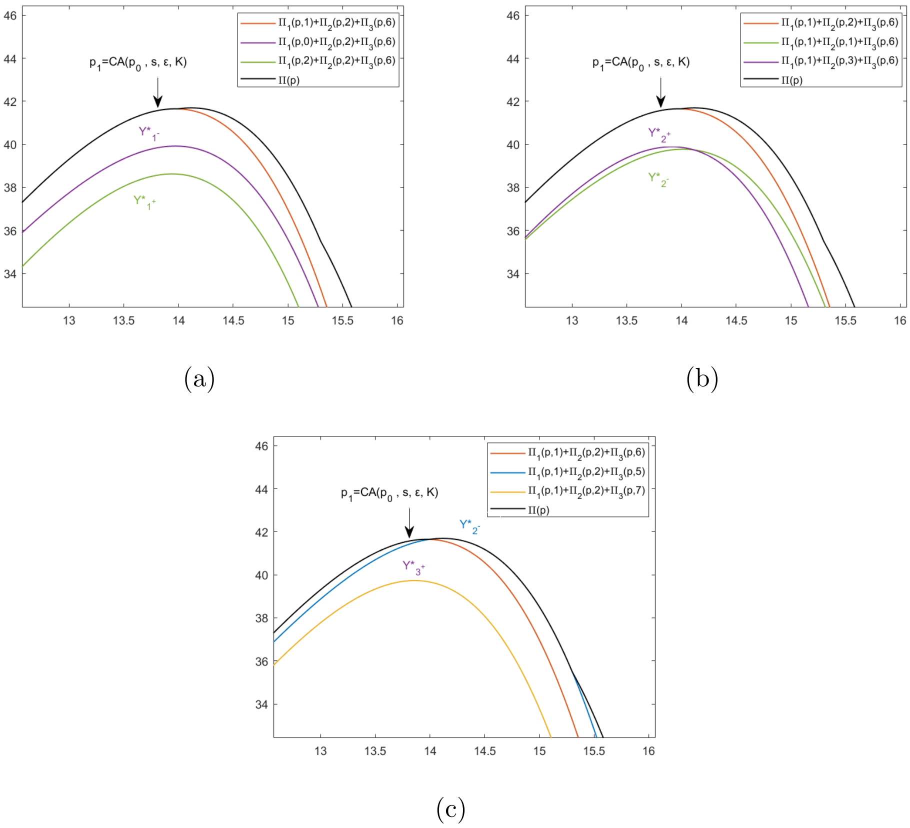


Figure 4.5: Numerical example in a 3 product assortment: (a) $\Pi(p, Y^*_{1\pm}(p_1)$ (b) $\Pi(p, Y^*_{2\pm}(p_1)$ (c) $\Pi(p, Y^*_{3\pm}(p_1)$

In the next step, we develop a function to compute the maxima of the functions in equation (4.2). Given that these functions are continuous and unimodular in the price p (Assumption 1), we can apply a well known optimization technique, the gradient descent (ascent), to find their maxima. We build our function to take as inputs a starting price value, which is a local maximizer of $\Pi(p)$, say p_j, and a product index $r \in K$. We also add the parameters of the gradient ascent algorithm, s and ξ for the step size and precision, respectively. For an input price p_j, a product index r, we denote by $GA^{\pm}(p_j, s, \xi, K, r)$ our maximization function that operates on the following objective:

$$\sum_{i \in K, i \neq r} \Pi_i(p, y^*_i(p_j)) + \Pi_r(p, y^*_r(p_j) \pm 1).$$

Similar to the single product case, our numerical results imply that, a local maximum at a price $p_1 = CA(p_0 \gtrsim c, s, \xi, 1)$ is a global maximum of $\Pi(p)$ when $\Pi(p)_{|p=p_1}$ is greater than the maxima of the functions $\Pi(p, Y^*_{r\pm}(p_1))$ in equation (4.2) for all the products $r \in K$. Following on from that, we next develop an

Algorithm 2 GA$^{\pm}$(p_j, s,ξ, K, r)

Require: $p_j > c$, $s \geq 0$, $\xi \gtrsim 0$, $r \in K$

Compute $y_r^*(p_j) = \Theta^{-1}(1 - \frac{c}{p_j}, \lambda q_r(p_j))$

Compute $\nabla Temp(p) = \nabla[\Sigma_{i \in K, i \neq r} \Pi_i(p, y_i^*(p_j)) + \Pi_r(p, y_r^*(p_j) \pm 1)]$

Set $p_{j+1} = p_j$

for $j = 0, \ldots$ **do**

  Set $p_j = p_{j+1}$

  Update $p_{j+1} = p_j + \nabla Temp(p)_{|p_j}$

  if $\nabla Temp(p)_{|p=p_j} \leq \xi$ **then**

    Break

  end if

end for

Return p_j

assumption to determine whether a local maximum of $\Pi(p)$ is a global maximum.

Assumption 2 *A local maximum of $\Pi(p)$ at p_1 is a global maximum iff*

$$\Pi(p)_{|p=p_1} > \Pi(p)_{|p=GA^{\pm}(p_1,s,\xi,K,i)} \;\; \forall i \in K$$

Heuristic to determine the global maximum of $\Pi(p)$

Based on the above, we now provide a heuristic approach to find $\Pi(p)$'s global maximum in 3 steps.

1. Local maximum: converge to the first local maximum of $\Pi(p)$ to the right of the unit cost c, $p_1 = CA(p_0 \gtrsim c, s, \xi, K)$
2. Optimality check: If $\Pi(p)_{|p=p_1} > \Pi(p)_{|p=GA^{\pm}(p_1,s,\xi,K,i)}$ $\forall i \in K$, then p_1 is a global maximizer of $\Pi(p)$
3. Escaping the local maxima that are not global: If the condition in step 2 is not satisfied, then we escape the local maximum to the first higher value that we reach by computing $\Pi(p)_{|p=GA^{\pm}(p_1,s,\xi,K,i)}$ among values of $i \in K$, as detailed in Algorithm 3.

Lastly, we determine the optimal inventory levels of the products at the global maximum, p^*, using equation (3.9) for each product $i \in K$.

Remark 2 *Complexity of Algorithm 3*
In an multi-product assortment of K products, the number of operations required to reach $\Pi(p)$'s global maximum, using Algorithm 3, does not exceed $2K$ operations. The number of operations increases linearly with the number of products in the assortment and is not affected by the problem's parameters. Meanwhile, (Schulte & Sachs, 2019)'s approach to determine $\Pi(p)$'s global maximum in a single product assortment requires enumeration over fixed values of the inventory levels below an upper limit $\hat{y}$. Note that $\hat{y}$ is determined as a function of problem's nonzero parameters such as the average demand rate, the product's purchasing and holding cost, and the salvage value. Consequently, the number of operations required in (Schulte & Sachs, 2019)'s can be very high in case of high demand rates or various combinations of the problem's parameters.

Algorithm 3 PSNP (p_0, s, ξ, K)

Require: $p_0 \gtrsim c$, $s \geq 0$, $\xi \gtrsim 0$

Compute $p^* = CA(p_0, s, \xi, K)$;

for $i = 1 : K$ **do**

Compute $p^*_{i-} = GA^-(p^*, s, \xi, K, i)$

if $\Pi(p)_{|p=p^*_{i-}} > \Pi(p)_{|p=p^*}$ **then**

Update $p^* = p^*_{i-}$

end if

Compute $p^*_{i+} = GA^+(p^*, s, \xi, K, i)$

if $\Pi(p)_{|p=p^*_{i+}} > \Pi(p)_{|p=p^*}$ **then**

Update $p^* = p^*_{i+}$

end if

end for

Return p^*

Numerical Experiments

We next validate the correctness of Algorithm 3 by conducting various numerical experiments with randomly generated input parameters. We consider a multiple product assortment where the product number K is set by the user. We then randomly generate the demand rate λ, the unit cost c, and the customers' reservation prices α_i, $\forall i \in K$, as follows:

- λ is a random variable greater than or equal to 1.
- c is random variable greater than or equal to 1.
- α_i's are random variables greater than c and increasing in i.

We then apply Algorithm 3 on the expected profit function $\Pi(p)$, of equation (3.10) and converge to a price value p^* that maximizes $\Pi(p)$ in the interval $\begin{bmatrix} c & \infty \end{bmatrix}$. We next highlight 3 different cases in which Algorithm 3 converges to $\Pi(p)$'s global maximum.

Case where $\Pi(p)$ has one local maximum

We now summarize our numerical results, in the case where $\Pi(p)$ has one maximum, in a 5 product assortment with the following parameters:

- unit cost $c = 3$
- average demand $\lambda = 4$
- customer reservation prices $\alpha_1 = 10$, $\alpha_2 = 11$, $\alpha_3 = 12$, $\alpha_4 = 13$, $\alpha_5 = 14$

In this specific example, Algorithm 1 converges to a price value $p^* = 12.4028$ at the combination of inventory levels $\mathbf{Y}(\mathbf{p}^*) = \begin{bmatrix} 0 & 0 & 1 & 1 & 3 \end{bmatrix}$. We next compute the value of $\Pi(p)$ at the local maximum, $\Pi(p)_{|p^*} = 19.3879$. In the next step of our heuristic, we compute the maxima of the individual profits' using the gradient ascent technique in Algorithm 2, and update p^* in table 4.1.

y_1	y_2	y_3	y_4	y_5	p*	$\max_p \Sigma_{i=1,2,3,4,5}(\Pi_i(p, y_i))$
1	0	1	1	3	12.4028	16.8897 < 19.3879
0	**1**	1	1	3	12.4028	17.7053 < 19.3879
0	0	**0**	1	3	12.4028	19.1264 < 19.3879
0	0	**2**	1	3	12.4028	16.8609 < 19.3879
0	0	1	**0**	3	12.4028	15.396 < 19.3879
0	0	1	**2**	3	12.4028	18.8995 < 19.3879
0	0	1	1	**2**	12.4028	17.5745 < 19.3879
0	0	1	1	**4**	12.4028	18.7944 < 19.3879

Table 4.1: Computing the global maximum of $\Pi(p)$: Case of one local maximum

We then validate our results by manual computation of $\Pi(p)$'s maximum value as shown in figure 4.6.

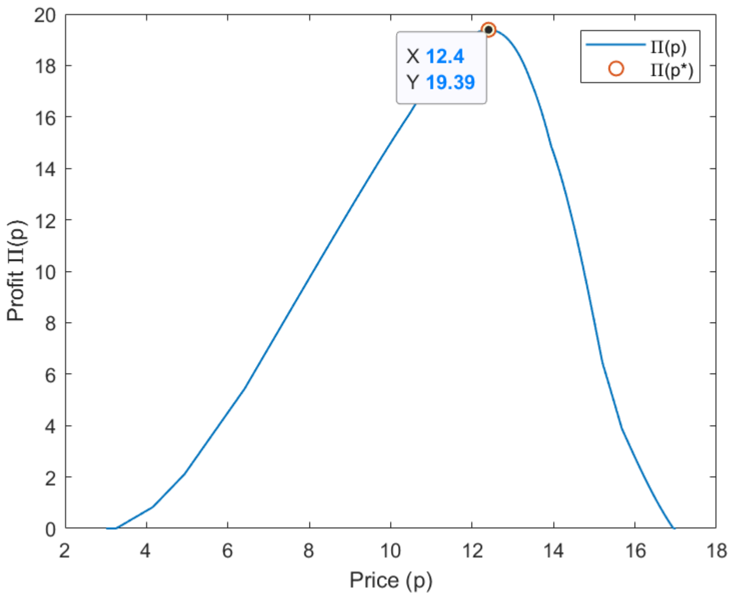


Figure 4.6: Maximizing $\Pi(p)$: Case of one local maximum

Case where $\Pi(p)$ has 2 local maxima

We now show the validity of our algorithm in the case where $\Pi(p)$ has more than one local maximum. For this example, we consider an assortment of 3 products with the following parameters:

- unit cost $c = 10$
- average demand $\lambda = 9$
- customer reservation prices $\alpha_1 = 16.2362$, $\alpha_2 = 18.5162$, $\alpha_3 = 19.7369$

Under these settings, Algorithm 1 converges to a price value $p^* = 17.938$ and a combination of the inventory levels $\mathbf{Y}(\mathbf{p}^*) = \begin{bmatrix} 0 & 1 & 6 \end{bmatrix}$. We next compute the value of $\Pi(p)$ at the local maximum, $\Pi(p)_{|p^*} = 35.555$. In the next step of our heuristic, we compute the maxima of the individual profits' summation functions using the gradient ascent technique in Algorithm 2, and update p^* in table 4.2. In this case, we escape the local maximum and update p^* whenever we reach a higher value.

y_1	y_2	y_3	p*	$\max_p(\Pi_1(p, y_1) + \Pi_2(p, y_2) + \Pi_3(p, y_3))$
1	1	6	17.938	28.544 < 35.555
0	**0**	6	17.938	30.6429 < 35.555
0	**2**	6	17.938	35.1086 < 35.555
0	1	**5**	18.173	35.6816 > 35.555

Table 4.2: Computing the global maximum of $\Pi(p)$: Case of 2 local maxima

We validate our results by computing the values of the optimal inventory levels of the 3 products at $p^* = 18.173$ using equation (3.9). We obtain $\mathbf{Y}(\mathbf{p}^*) = \begin{bmatrix} 0 & 1 & 5 \end{bmatrix}$, which confirms our solution. We next repeat the same procedure starting from $p^* = 18.173$ as shown in table 4.3.

y_1	y_2	y_3	p*	$\max_p(\Pi_1(p, y_1) + \Pi_2(p, y_2) + \Pi_3(p, y_3))$
1	1	5	18.173	28.6295 < 35.6816
0	**0**	5	18.173	30.7285 < 35.6816
0	**2**	5	18.173	35.0949 < 35.6816
0	1	**4**	18.173	33.2527 < 35.6816
0	1	**6**	18.173	35.555 < 35.6816

Table 4.3: Computing the global maximum of $\Pi(p)$: Case of 2 local maxima

We also verify our results by manual computation of $\Pi(p)$'s maximum value as shown in figure 4.7.

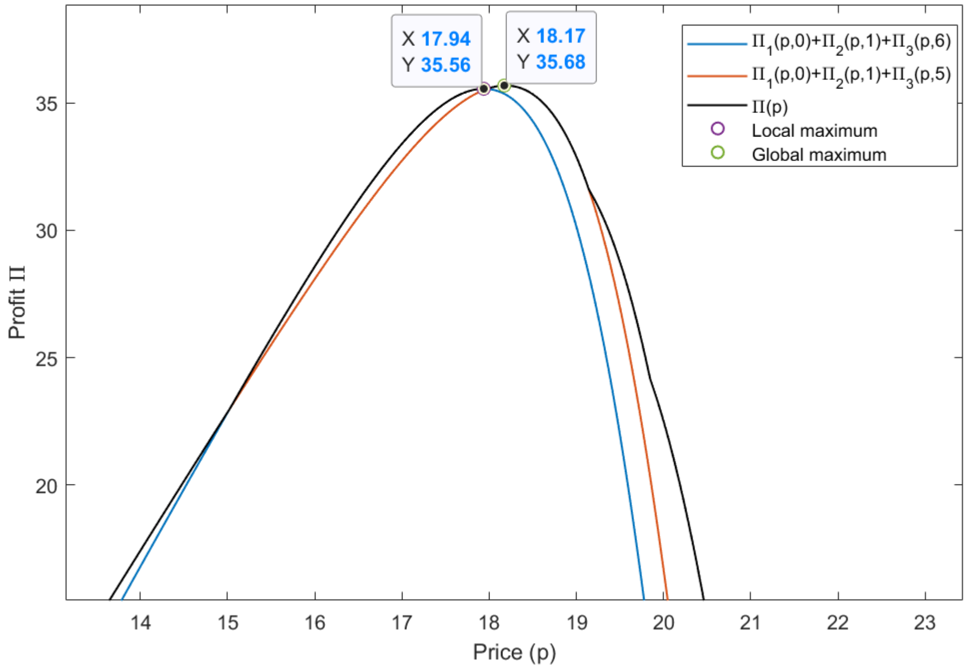


Figure 4.7: Maximizing $\Pi(p)$: Case of two local maxima

Following the same procedure, decision makers can optimize their expected profit functions using our algorithm by simply specifying the problem's parameters (K, $\alpha_i \forall i \in K$, c, λ), and determining the desired accuracy (s, n, ξ).

Chapter 5

Analytical and Numerical Insights

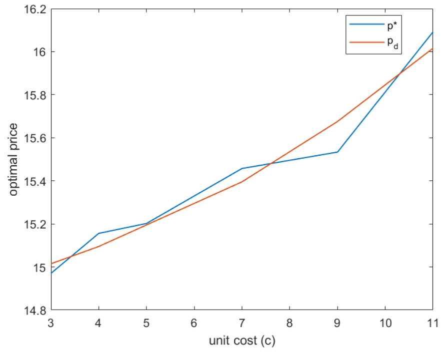
In this chapter, we provide some numerical and analytical insights by comparing our results to well known results of the (PSN) under the deterministic and normal demand settings.

Optimal price vs risk free price

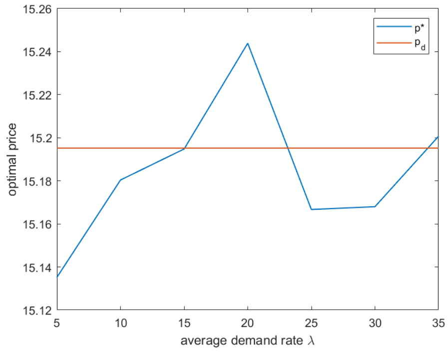
From a theoretical aspect, we see that there are some key differences between our model and the models that have been studied previously in the (PSN) problem. First, we notice, from our literature review, that most of the existing research considers demand stochasticity to be additive, multiplicative, or a mixture of both. This implies the use of 2 distinct price functions: one that describes the position (the mean), and the one that describes the shape of the demand distribution. In contrast, the stochasticity of the Poisson (PSN) cannot be classified either as additive or multiplicative nor can it be explained by a combination of the two. Instead, there is only one demand rate, $\lambda q_i(p)$, that defines both the position and the shape of the demand distribution (Schulte & Sachs, 2019). Following on from that, we now attempt to study the relationship between the optimal price p^*, and the risk free price p_r, also known as the deterministic price in problems with deterministic demand models. Recall from the literature review section that p^* is lower than p_r for additive demand models, but its higher in the case of multiplicative demand models. Note that in the additive case, the demand's variance is independent of the price and its coefficient of variation is decreasing in the price, whereas in the multiplicative case, the demand's coefficient of variation is independent of the price and its variance is decreasing in the price. For continuous demand models, (Petruzzi & Dada, 1999) explain the deviation of p^* from p_r as a way to reduce variability and mitigate risk. In other words, increasing the price, $p^* > p_r$, lowers the variance without increasing the coefficient of variation, whereas decreasing the price, $p^* < p_r$, lowers the coefficient of variation without increasing variance. However, in the case of Poisson demand, at a rate $\lambda q_i(p)$ for product $i \in K$, the change in the optimal price p^* leads the variance ($\lambda q_i(p)$), and coefficient variation ($\frac{1}{\sqrt{\lambda q_i(p)}}$), to move in opposite directions. Consequently, the prediction in the manner explained above for the continuous demand case is not possible under discrete demand settings. Furthermore, in discrete demand settings, the price has the additional function of compensating for the limited supply flexibility and, thus, improving the match between demand and supply (Schulte & Sachs, 2019). This function of the price is highly important when we change the problem's parameters. Therefore, the optimal price p^*, in our problem, can be above or below p_r depending on the problem's parameters. We next show, in figure 4.7,the optimal price p^* and the risk free price p_r as functions of the unit cost c, average demand rate λ, and customer reservation price α_i of product $i \in K$. We use the expression of the risk free profit function, $\Pi_r(p)$, defined by (Maddah et al., 2014) as follows:

$$\Pi_r(p) = \lambda(p - c)\frac{\sum\limits_{i \in K} e^{\alpha_i - p}}{1 + \sum\limits_{i \in K} e^{\alpha_i - p}} \tag{5.1}$$

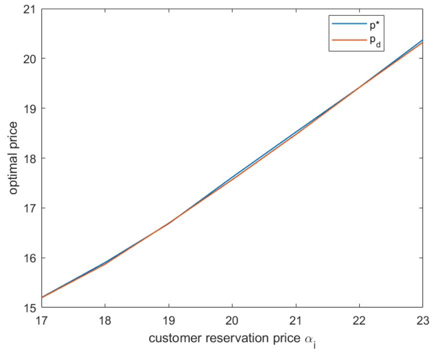
(Maddah et al., 2014) show that the function $\Pi_r(p)$ attains a unique maximum, p_r, and they provide a closed form expression to compute p_r as a function of the problem's parameters.



(a)



(b)



(c)

Figure 5.1: (a) p^* and p_r vs α (b) p^* and p_r vs c (c) p^* and p_r vs λ

Based on the analysis above, we summarize the behavior of the optimal price, p^*, and the risk free , p_r, subject to changes in the problem's parameters as shown in table 5.1.

Parameter	p^*	p_r
c	Fluctuating	Increasing
λ	Fluctuating	Insensitive
α_i	Increase	Increasing

Table 5.1: Characteristics of the optimal price p^* and the risk free price p_r

Accordingly, we conclude that there doesn't exist a constant pattern of the relationship between p^* and p_r in the (PSN) problem under Poisson demand settings. We also believe that the deviation of p^* from p_r is not a predictable behavior and varies with the problem's parameters.

Comparison to the normal demand setting

In order to highlight the usefulness of the (PSN) problem with Poisson demand, we next compare our results to a common approximation in the literature: The normal distribution. Much of the existing research considers the normal distribution a good approximation to the Poisson demand in many practical cases. However, we know that in the case of low demand rates, this approximation may not be accurate. Therefore, we compare our results to the case with a normal distribution under similar problem settings. Recall that under the normal demand settings, (Maddah et al., 2014) prove the unimodularity of the expected profit function, using a Taylor series approximation and under a set of reasonable assumptions. For the purpose of this analysis, we define the expected profit function under the normal demand setting as follows:

$$\Pi_n(p) = p\lambda q_i(p)(p-c) - p\phi\,(\Phi^{-1}(1-\frac{c}{p}))\sqrt{\lambda q_i(p)} \tag{5.2}$$

where ϕ and Φ respectively denote the probability density function and the cumulative distribution function of the standard normal distribution.

In order to understand how the resulting profit function under normal demand settings differ from the expected profit function under Poisson demand, we conduct a numerical study where we compare the optimal solution under both settings. In our study, we relatively consider low, mid, and high values for the parameters λ, c, and α=average(α_i) $\forall i \in K$, to study their effect on the profit functions. We summarize the data and the results from our numerical study in table 5.2.

λ	α	c	$\Pi^*(p)/\Pi_n^*(p)$
Low	Low	Low	0.9963
		Mid	0.8799
		High	-1.5348
	Mid	Low	0.9224
		Mid	0.8052
		High	0.4632
	High	Low	0.9597
		Mid	0.8775
		High	0.8711
Mid	Low	Low	0.9902
		Mid	0.9455
		High	1.2574
	Mid	Low	1.0000
		Mid	0.9505
		High	2.8793
	High	Low	1.0100
		Mid	0.9947
		High	0.8715
High	Low	Low	0.9982
		Mid	0.9853
		High	1.3487
	Mid	Low	1.0000
		Mid	0.9754
		High	1.4962
	High	Low	1.0000
		Mid	0.9746
		High	0.8932

Table 5.2: Ratio between heuristic profit: $\Pi^*(p)/\Pi_n^*(p)$

Based on the above ratio figures, we conclude that in most of the cases, the normal approximation performs well. Nonetheless, for certain parameter combinations, namely when λ is low, α is low, and c is high, the normal approximation fails. In these cases, the optimization heuristics under normal demand settings are either inapplicable or perform poorly. As a consequence, decision makers can misinterpret a profitable business opportunity as an unprofitable one. Therefore, we believe that its crucial for companies who face such parameter combinations, due to sparse demand, low margins and high purchasing costs, to consider the Poisson (PSN) instead of a deterministic or continuous approximation.

Chapter 6

Conclusion and Recommendations

In this paper, we develop a heuristic to maximize the profit of retailers selling from a product line under the News-Vendor type inventory setting and consumer choice. We consider the price and the inventory levels as decision variables, and the assortment size to be predetermined by the retailers. We prove that the expected profit function has at least one local maximum at a price value greater than the unit cost c, and under reasonably accepted assumptions, we apply classic optimization techniques to reach the local maximum value. We then show that in some cases, the expected profit function can have more than one local maximum, and we devise optimality conditions to check whether a local solution is global. As a result, we develop a heuristic approach to reach the expected profit function's global maximum, and we validate our results by numerical examples. We also show that the optimal price follows a discontinuous pattern, and we conclude that, there doesn't exist a predictable relationship between the optimal price and the risk free price. In addition, we accentuate the usefulness of the (PSN) problem with Poisson demand, by comparing our optimal profit with the profit that results from heuristically assuming demand to be deterministic or continuous (normal) when determining the selling price. Accordingly, we believe that several extensions of our model deserves further analysis. First, we can consider more general demand functions, especially discrete functions to represent the sparse demand on perishable products (e.g., the compound Poisson distribution) (Schulte & Sachs, 2019). Moreover, we cal also generalize the horizontally differentiated product line to a situation where the items in a product line may be classified into groups of horizontally differentiated items (Maddah et al., 2014). Finally, we can relax the assumption of horizontal differentiation by considering items with distinct unit costs and prices (Maddah et al., 2014).

Appendix A

Unimodularity of $\Pi_i(p, y_i)$

As per (Schulte & Sachs, 2019), a demand function $D(p)$ at a rate $\lambda(p)$ is said to be admissible if it satisfies the following conditions:

1. $\lambda(p)$ is monotonically decreasing in the price;
2. $\lim_{p\to\infty} p\lambda(p) = 0$;
3. $\lambda(p)$ has an inverse $\lambda(p)^{-1}$ that is log-concave on $[0\ \ \infty]$.

In this section, we represent $\lambda q_i(p)$ by the letter z. For a fixed non-negative inventory level y, we write the sales function, of any product $i \in K$, in terms of z as follows:

$$S_y(z) = min(D(z), y) \tag{A.1}$$

where $D(z)$ represents a random variable following a Poisson distribution at a rate z. Thus, we can rewrite the sales function's expression as follows:

$$S_y(z) = \sum_{x=0}^{x=y} \frac{x.e^{-z}.z^x}{x!} + \sum_{x=y+1}^{x=\infty} \frac{y.e^{-z}.z^x}{x!} \tag{A.2}$$

By replacing the summation expressions with the Poisson cumulative distribution function Φ, we can get the following expression of $S_y(z)$:

$$S_y(z) = z\Phi(y-1, z) + y(1 - \Phi(y, z)) \tag{A.3}$$

Next, we calculate $S_y(z)$'s first derivative with respect to z being equal to:

$$\frac{d(S_y(z))}{dz} = \Phi(y-1, z) \tag{A.4}$$

We then calculate its second derivative with respect to z:

$$\frac{d^2(S_y(z))}{dz^2} = -\varphi(y-1, z), \tag{A.5}$$

where φ represents the Poisson distribution density function.
As its second derivative with respect to z will be always negative, we can prove $S_y(z)$ to be concave in z.
In what follows, we want to prove the price function $p(z)$ to be concave in z by calculating its second derivative as well. As z is an invertible function in p, we write the function $p(z)$ as follows:

$$p(z) = \alpha - ln(\frac{z}{\lambda - z}) \tag{A.6}$$

We now calculate its first derivative with respect to z and obtain the following:

$$\frac{dp(z)}{dz} = -\frac{1}{z} - \frac{1}{\lambda - z} \tag{A.7}$$

We then compute its second derivative in z as follows:

$$\frac{d^2p(z)}{dz^2} = \frac{1}{z^2} - \frac{1}{(\lambda - z)^2} \tag{A.8}$$

In this case, the concavity result of $p(z)$ with respect to z holds for values of p lower than the consumer reservation price α; an assumption that we adopted throughout our study.
Therefore, as concavity implies log-concavity, we now have $S_y(z)$ and $p(z)$ to be log-concave in z, $\forall\ p \leq \alpha$. Next, we develop an expression for the profit function at a fixed inventory level in terms of z as follows:

$$\Pi_y(z) = S_y(z)p(z) - cy \tag{A.9}$$

As the second term of the above equation consists of constant values, independent of z, we can prove $\Pi_y(z)$ to be log-concave in z as it is the product of two functions proven log-concave in z. Not only does the log-concavity property of $\Pi_y(z)$ prove existence and uniqueness of the optimal solution z^*, it also allows us to obtain z^* using standard convex optimization techniques as explained by (Boyd & Vandenberghe, 2004). Consequently, as z is invertible in the price p, we determine the optimal price p^* as p^*=$z^{-1}(z^*)$.

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