AMERICAN UNIVERSITY OF BEIRUT

ANALYSIS AND IMPLEMENTATION FOR AN IN TIME EULER IMPLICIT - SPACE FINITE ELEMENT APPROXIMATION TO A HASEGAWA-MIMA PLASMA MODEL

by

ADEL MOUNZER SALEH

A thesis
submitted in partial fulfillment of the requirements
for the degree of Master of Engineering
to the Department of Electrical and Computer Engineering
of Maroun Semaan Faculty of Engineering and Architecture
at the American University of Beirut

Beirut, Lebanon
January 2022

AMERICAN UNIVERSITY OF BEIRUT

ANALYSIS AND IMPLEMENTATION FOR AN IN TIME EULER IMPLICIT - SPACE FINITE ELEMENT APPROXIMATION TO A HASEGAWA-MIMA PLASMA MODEL

by

ADEL MOUNZER SALEH

Approved by:

Dr. Nabil Nassif, Professor — Advisor

Mathematics

Dr. Sophie Moufawwad, Assistant Professor — Member of Committee

Mathematics

Dr. Bassam Shayya, Professor — Member of Committee

Mathematics

Bassam Shayya

Dr. Ghassan Antar, Professor — Member of Committee

Physics

GAntar

Date of thesis defense: Jan 27, 2022

AMERICAN UNIVERSITY OF BEIRUT

THESIS RELEASE FORM

Student Name: Saleh Adel Mounzer
Last First Middle

I authorize the American University of Beirut, to: (a) reproduce hard or electronic copies of my thesis; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes

___ **As of the date of submission of my thesis**

✓ **After 1 year from the date of submission of my thesis .**

___ **After 2 years from the date of submission of my thesis .**

___ **After 3 years from the date of submission of my thesis .**

Signature   02/07/2022
Date

This form is dated and signed when asked to submit the final document to ScholarWorks. DELETE THIS NOTE WHEN SIGNED

Acknowledgements

First and foremost, I thank my family and friends for being very patient and supportive during these rough times in the last two years. I wouldn't have been able to finish without them and I owe my health and well-being to them the most.

Second, I would like to thank the research team on the Hasegawa-Mima equation, namely my advisor Dr. Nabil Nassif, my co-advisor Dr. Sophie Moufawwad, and Mr. Hagop Karakazian, for introducing me to this very interesting and challenging topic, and for their wonderful and inspiring contributions to the problem. In particular, I thank my advisor Dr. Nassif and co-advisor Dr. Moufawwad for being very patient, helpful and available during the writing process. I am very grateful to their coaching, which set me on the right track, and very much our lengthy discussions, from which I learned quite a lot. They have taught me to enjoy all aspects of PDEs, all the way from the theoretical background in Functional Analysis to the numerical implementation of the Finite Element Method. I also thank Dr. Antar for introducing me the physics behind the problem, constantly providing feedback to the simulations, and mostly for inviting to the 2020 summer school on Plasma Physics in Tunis.

Third, I would like to thank some professors who essentially shaped my mathematical path: Dr. Bassam Shayya, Dr. Giuseppe Della Sala and Dr. Richard Aoun for teaching me Analysis, Measure Theory and Topology. I enjoyed very much their enthusiastic lecturing and teaching style, and learned even more from our (quite lengthy!) mathematical discussion.

An Abstract of the Thesis of

Adel Mounzer Saleh for Master of Science
Major: Mathematics

Title: Analysis and Implementation for an Time Euler Implicit - Space Finite Element Approximation To a Hasegawa-Mima Plasma Model

In Magnetohydrodynamics, the Hasegawa-Mima equation appears as a model for pseudo three-dimensional turbulence of a confined plasma inside a tokamak reactor. The Hasegawa-Mima equation, bearing close resemblence with the 2d Navier-Stokes equation for an incompressible fluid, is given

$$\frac{\partial}{\partial t}(\Delta u - u) - ((\nabla u \times \hat{z}) \cdot \nabla) \left(\Delta u - \ln \frac{n_0}{\omega_{\mathrm{ic}}} \right) = 0.$$

In this thesis, we apply an in time Euler implicit - space Finite Element Galerkin method to obtain solutions to this equation that satisfy *periodic boundary values* over the square $\Omega = [0, L] \times [0, L]$. Furthermore, we use the method to build a numerical scheme for simulation. We prove the convergence of this scheme with minimal constraints on the time step τ as a function of the mesh size h.

Furthermore, we search for initial data u_0 for which the solution u is a traveling wave in y-direction. Those initial data turn out to be solutions for a semi-linear elliptic equation, which can be simulated using Newton-Galerkin method, which is a discretization of the Newton method in Banach spaces. We then use these initial data as input in the numerical scheme to showcase that they indeed correspond to traveling waves.

Contents

List of Figures

Notation

a.e	Almost everywhere
PBCs	Periodic boundary conditions over a square (defined in (1.6))
$X^\star$	Topological dual of a Banach space X
$L(X,Y)$	All linear maps from X to Y.
$\mathcal{B}(X)$	All bilinear forms on $X \times X$.
$W^{m,p}(\Omega)$	Sobolev space of m times weakly differentiable L^p functions
$H^m(\Omega)$	Shorthand for the Sobolev space $W^{m,2}(\Omega)$
$\lvert \cdot \rvert_{H^m}$	The H^m semi-norm
$C_c^\infty(\Omega)$	Smooth functions that are compactly supported in Ω
$H_0^1(\Omega)$	The completion of $C_c^\infty(\Omega)$ in $H^1(\Omega)$
$H^{-1}(\Omega)$	The dual space of $H_0^1(\Omega)$
Tr	The trace operator $\mathrm{Tr} : H^1(\Omega) \to H^{1/2}(\Omega)$
$C_P^\infty(\Omega)$	Smooth functions over a square that satisfy PBCs
$H_P^1(\Omega)$	The closure of $C_P^\infty(\Omega)$ in $H^1(\Omega)$
$H_P^m(\Omega)$	Functions in $H^m(\Omega)$ whose first derivatives are in $H_P^{m-1}(\Omega)$
$L^p(0,T;X)$	Bochner space of p-integrable X-valued maps on $[0,T]$
$L^\infty(0,T;X)$	Bochner space of essentialy bounded X-valued maps on $[0,T]$
$\lVert \cdot \rVert_{L^p,X}$	The norm of the space $L^p(0,T,X)$
$C(I,X)$	X-valued continuous maps on an interval I
$\{f,g\}$	Poisson bracket of two differentiable functions f and g
$\vec{V}(f)$	Perpendicular gradient of f, usually written as $\nabla_\perp f$
$X \hookrightarrow Y$	X injects continuously in Y
$\rightharpoonup, \overset{\star}{\rightharpoonup}$	Weak and weak* convergence respectively.

Chapter 1

Introduction

1.1 Origins of the problem in plasma physics

The Charney-Hasegawa-Mima equation (1.3) arises in the context of plasma physics, and in particular during the process of *magnetic plasma confinement* used in a tokamak reactor. In short, this relatively novel method uses very powerful magnetic fields to heat a plasma to temperatures of over 100 million °C while keeping it confined in space [1]. This allows for fusion to take place and thus generating immense amount of thermonuclear power with relatively low hydrogen fuel consumption.

However, this confinement process is difficult to maintain as plasma can become easily turbulent and unstable at such high energy levels. Furthermore, any impurity in the plasma can destroy the entire process. The record for longest running plasma confinement is held by the *Tore Supra tokamak* at 6 minutes and 30 seconds [2]. With this important challenge in mind, numerous efforts have been devised for the development of mathematical models describing plasma during it's confinement in order to better understand the evolution of turbulence and instabilities.

One of these models is the Hasegawa-Mima equation. It was derived in 1977 by Hasegawa and Mima [3] from the continuity and ion momentum balance equations, under the assumption that the plasma behaves like a cold fluid. But instead of describing the velocity field of the ions, the term u in (1.1) describes *the electrostatic potential* of the plasma, which can be shown related to the total velocity through perturbation theory, as noted in [4]. In it's original normalized from, the Hasegawa-Mima equation in the plane is written as

$$\frac{\partial}{\partial t}(-\Delta u + u) + \big((\nabla u \times \hat{\mathbf{z}}) \cdot \nabla\big)\,(\Delta u - p) = 0, \tag{1.1}$$

where $\hat{\mathbf{z}}$ is the unit normal to the ambient magnetic field and we are given $p := \ln(n_0/\omega_{\mathrm{ic}})$ with n_0 being the background particle density and ω_{ic} the ion-cyclotron frequency. It is important to note that the solution u describes *pseudo-three-dimensional turbulence*, and therefore it is a function of two spatial variables only. The Hasegawa-Mima equation equation is closely related to the two dimensional Navier-Stokes equation for an incompressible fluid which is given by

$$\frac{\partial}{\partial t}(\Delta u) - [(\nabla u \times \hat{\mathbf{z}}) \cdot \nabla] \nabla^2 u = 0.$$

The extra terms in the Hasegawa-Mima equation (1.1) are due to the presence of compressibilty induced by the parallel electron motion [3]. Furthermore, the Hasegawa-Mima and Navier-Stoke's description of the plasma are identical under certain conditions on the perpendicular wave number. Another common form of (1.1) is obtained by noticing that $\nabla u \times \hat{\mathbf{z}} = -\vec{V}(u) = -\nabla_{\perp} u$ so that (1.1) becomes

$$\frac{\partial}{\partial t}(-\Delta u + u) - \{u, \Delta u\} - \{u, p\} = 0, \tag{1.2}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. Throughout this thesis, we will study equation (1.3) under the assumption background particle density is given by $n_0(x, y) = e^{\hat{k}x+\gamma}$, a function of x only. In this case, (1.2) becomes

$$\boxed{\frac{\partial}{\partial t}(-\Delta u + u) = \{u, \Delta u\} + \hat{k}\frac{\partial u}{\partial y}.} \tag{1.3}$$

Except for in literature review where each source studies some variation of the Hasegawa-Mima equation, (1.3) will be the equation under consideration throughout the thesis.

1.2 Literature review

Since it's derivation in 1977, the Hasegawa-Mima equation and it's variants have been subject to numerous mathematical, numerical, physical and experimental investigations. In the following list, unless otherwise stated, the domain Ω of the solutions of the Hasegawa-Mima equation is assumed to be the entire plane $\mathbb{R}^2$.

- In 1976, Larichev and Reznik [5] were amongst the first to prove the existence of exact traveling wave solutions called *dipole vortices* or *Modons*.
- In 1987, Nycander [6] proved the existence of a larger class of dipole and *monopole* vortices as solutions to a two boundary value problem involving a semi-linear elliptic equation on a bounded domain and a linear equation on the complement of that domain.
- In 1989, Shivamoggi [7] proved the existence of exact solutions in $\mathbb{R}^2$ assuming that the *vorticity* is proportional to u. This assumption on u linearizes equation (1.1) and the obtained solutions exhibit traveling wave behaviour.

- In 1998, Grauer [8] gave an energy estimate of a perturbed Hasegawa-Mima equation with periodic boundary values on a rectangular domain Ω. His estimate predicted energy saturation occurs after some finite time T.

- In 2003, Paumond [9] used the technique of *parabolic regularization* and analytic semi-groups to prove that if $u_0 \in H^m(\mathbb{R}^2)$ with $m \geq 4$ then for any $T > 0$ there is a unique $u \in L^\infty(0, T; H^m(\mathbb{R}^2)) \cap C([0, T]; H^1(\mathbb{R}^2))$ that solves (1.1).

- In 2004, Guo and Han [10] proved that given $u_0 \in H^m(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$ with $m \geq 3$, then there exists a *unique* classical solution $u \in L^\infty(0, T; H^m(\mathbb{R}^2))$ with $u_t \in L^\infty(0, T; H^{m-1}(\mathbb{R}^2))$ for any $T > 0$. However, if $u_0 \in H^2(\mathbb{R}^2)$, then only a weak solution of (1.3) is obtained with the same regularity as the case $m \geq 3$ (but with $m = 2$ of course), but with the additional information that $u \in L^\infty(0, T; W^{2,p}(\mathbb{R}^2))$ for all $2 \leq p < \infty$. This weak solution is unique if u_0 also belongs to $W^{2,\infty}(\mathbb{R}^2)$, and if in addition $k = 0$ in (1.3) then $u \in L^\infty(0, T; H^2(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2))$.

- In 2008, Hounkonnou and Kabir [11] implemented the *Lie Symmetry reduction* method to obtain existence of many families of exact global solutions that can be expressed in terms of Bessel functions and trigonometric functions.

- In 2009, Boling and Daiwen [12] proved the existence and stability of steady wave solutions to a variant of the Hasegawa-Mima equation given by $\partial_t(\Delta u - u) + \{u, \Delta u - u\} = 0$, with u going to zero as $|x|, |y| \to \infty$. A steady wave solution u for such an equation satisfies $u = f(\Delta u - u)$, for some function f. These steady waves correspond to critical points of the *Casimir* energy functional $I(u) = \int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2) + \int_{\mathbb{R}^2} F(\Delta u - u)$, where F is the anti-derivative of f.

- In 2016, Karakazian [13] in their MS thesis proved using methods similar to [9], [10] that a unique solution $u \in L^2(0, T; H^m_P(\Omega)) \cap C([0, T]; L^2(\Omega))$ to (1.3) exists given that the initial data is in $H^m_P(\Omega)$ with $m \geq 4$, where $\Omega = [0, L] \times [0, L]$. See section 1.3 for the definition of the spaces $H^m_P(\Omega)$.

- In 2018, Karakazian and Nassif [14] used a Petrov-Galerkin approach based on the Fourier basis of $H^1_P(\Omega)$ with $\Omega = [0, L] \times [0, L]$ to obtain local existence of a local solution $u \in L^2(0, T; H^2_P(\Omega))$ to a weak formulation of (1.3) knowing that $u_0 \in H^3_P(\Omega)$ and $(I - \Delta)u_0 \in L^\infty(\Omega)$. See section 1.3 for the formulation. Even though regularity results using this approach are not stronger than the ones in [15], this approach paved the way for a robust numerical scheme to be implemented later.

- In 2018, Guo, Li, and Han [16] studied the stochastic Hasegawa-Mima equation with additive noise over a square, with the solution satisfying periodic boundary values. This equation generates a continuous *random dynamical system* for which they have shown the existence of a global attractor in $H^3_P(\Omega) \cap W^{2,\infty}_P(\Omega)$.

- In 2021, Karakazian, Moufawad, and Nassif [17] used a Petrov-Galerkin approach based on the Finite Element Method to obtain both an existence result and a numerical scheme for simulation. Since this thesis is essentially a continuation of their work, a rather lengthy discussion about their results is found in Section 1.3.

This literature review is most likely not exhaustive, but covers moderately well the research done on this equation.

1.3 Mathematical framework

In this section we set up the framework of the thesis by defining all the needed function spaces for the variational approach and the Petrov-Galerkin scheme based on the finite element method. Let us write (1.3),which will be the equation under consideration for the rest of the thesis, in the form used in [14], [17]. In particular, we are interested in solutions on the square $\Omega = [0, L] \times [0, L]$ which are periodic on the boundary of the square at each time t.

Definition 1.1. Let $\Omega = [0, L] \times [0, L]$ and $\Gamma = \partial\Omega = \bigcup_{j=1}^{4} \Gamma_j$, where the Γ_j's are the sides of the square ordered by tracing the Γ counterclockwise starting from the bottom side. A function $f : \Omega \to \mathbb{R}$ is said to said to satisfy periodic boundary conditions, or PBCs for short, if $f_{|\Gamma}$ satisfies the following

$$(f_{|\Gamma})_{|\Gamma_1} = (f_{|\Gamma})_{|\Gamma_3} \quad \text{and} \quad (f_{|\Gamma})_{|\Gamma_2} = (f_{|\Gamma})_{|\Gamma_4}. \tag{1.4}$$

So we want a $T > 0$ and a function $u = u(x, y, t) : \Omega \times [0, T] \to \mathbb{R}$ that solves (1.3) such that $u(t)$, $\partial_x u(t)$ and $\partial_y u(t)$ satisfy PBCs for all $t \in [0, T]$. Second, we can write equation (1.3) as a coupled elliptic-hyperbolic system as follows. Let $w := -\Delta u + u$ and replace in (1.3) to obtain

$$\begin{cases} \dfrac{\partial w}{\partial t} = \vec{V}(u) \cdot \nabla w + \hat{k}\dfrac{\partial u}{\partial y}, & \text{on } \Omega \times [0, T], \\ -\Delta u + u = w, & \text{on } \Omega \times [0, T], \\ u(t),\ \partial_x u(t),\ \text{and } \partial_y(t) \text{ satisfy PBCs}, & \text{for all } t \in [0, T]. \end{cases} \tag{1.5}$$

Notice that the first equation is hyperbolic in w and the second is elliptic in u. Now to apply the Galerkin method, we need to put (1.5) in variational form, and therefore we introduce the needed function spaces.

Definition 1.2 (Periodic Sobolev spaces)**.** Let $C_P^\infty(\Omega)$ be the set of all function in $C^\infty(\Omega)$ that satisfy PBCs. We define

$$H_P^1(\Omega) := \overline{C_P^\infty(\Omega)} = \{u \in H^1(\Omega) : \mathrm{Tr}(u) \text{ satisfies PBCs a.e on } \Gamma\}. \tag{1.6}$$

where $\mathrm{Tr} : H^1(\Omega) \to H^{1/2}(\Omega)$ is the trace operator For $m \geq 2$ we have

$$H_P^m(\Omega) := \left\{ f \in H^m(\Omega) : \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \in H_P^{m-1}(\Omega), \text{ for all multiindex } \alpha \text{ with } |\alpha| = m \right\}.$$

These spaces are Hilbert spaces with usual inner products and mimic all the desired properties of their parent spaces, cf. [13]. We also need the well known Bochner spaces.

Definition 1.3 (Bochner spaces)**.** Let X be a Banach space. For $1 \leq p \leq \infty$, we define $L^p(0, T; X)$ as the space of almost everywhere defined functions from $[0, T]$ to X, equipped with the norm

$$\|u\|_{L^p,X} := \begin{cases} \left(\int_0^T \|u(t)\|_X^p \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \mathrm{essup}_{t \in [0,T]} \|u(t)\|_X, & \text{if } p = \infty. \end{cases} \tag{1.7}$$

When using a Galerkin approach to seek a solution for the Hasegawa-Mima equation, one of the desired consequences of periodicity is that the elliptic equation can be put variational form without boundary terms. Indeed, one can readily show that the $H_P^1(\Omega)$ variational formulation of (1.5), obtained by multiplying (1.5) by $v \in H_P^1(\Omega)$ and integrating over Ω for each t, is stated as follows.

Strong-time weak-space formulation. Given initial data $u_0 \in H_P^2(\Omega)$, seek $u : [0, T] \to H_P^1(\Omega)$ and $w : [0, T] \to L^2(\Omega)$ such that for all $(v, t) \in H_P^1(\Omega) \times [0, T]$ one has

$$\begin{cases} \left\langle \frac{dw}{dt}(t), v \right\rangle_{L^2} = - \left\langle \vec{V}(u(t)) \cdot \nabla w(t), v \right\rangle_{L^2} + \hat{k} \left\langle \frac{\partial u(t)}{\partial y}, v \right\rangle_{L^2}, \\ \langle u(t), v \rangle_{H^1} = \langle w(t), v \rangle_{L^2}. \end{cases} \tag{1.8}$$

In this form, one would look for $w \in C^1([0, T]; L^2(\Omega)) \cap L^2(0, T; H_P^1(\Omega))$. These regularities are difficult to obtain directly through the Galerkin method, and therefore we weaken them by integrating the first equation on $[0, t]$ for $t \in [0, T]$ and using Fubini's theorem to get

$$\begin{cases} \langle w(t) - w(0), v \rangle_{L^2} = - \int_0^t \left\langle \vec{V}(u(s)) \cdot \nabla w(s), v \right\rangle_{L^2} + \hat{k} \left\langle \frac{\partial u(s)}{\partial y}, v \right\rangle_{L^2} ds, \\ \langle u(t), v \rangle_{H^1} = \langle w(t), v \rangle_{L^2}, \quad \forall (v, t) \in H_P^1(\Omega) \times [0, T]. \end{cases} \tag{1.9}$$

Therefore, a solution w that satisfies the hyperbolic equation in (1.9) has to be in $L^2(0, T; H_P^1(\Omega))$. However, as a consequence of periodic boundary values, the regularity of w can be weakened even further. This is due to the following fact.

Proposition 1.1 (Skew-symmetry)**.** For all $(u, v, w) \in H^2_P(\Omega) \times H^1_P(\Omega) \times H^1_P(\Omega)$, we have that

$$\left\langle \vec{V}(u) \cdot \nabla w, v \right\rangle_{L^2} = -\left\langle \vec{V}(u) \cdot \nabla v, w \right\rangle_{L^2}. \tag{1.10}$$

Proof. Sobolev embeddings with Holder's inequality guarantee that the above quantities are well defined. Indeed, recall that $H^1(\Omega) \subset L^r(\Omega)$ for all $1 \le r < \infty$ in dimension two and Ω bounded. Since $\vec{V}(u) \in H^1(\Omega)^2$, $v, w \in H^1(\Omega)$ and $\nabla u, \nabla v \in L^2(\Omega)^2$ then Holder's inequality implies that for any p, q such that $1/p + 1/q + 1/2 = 1$ we have that $\left|\left\langle \vec{V}(u) \cdot \nabla w, v \right\rangle\right| \le \|\vec{V}(u)\|_p \|\nabla w\|_2 \|v\|_q$. Also, by equality of the second mixed weak derivatives of u, it is clear that $\vec{V}$ is divergence free, ie $\nabla \cdot \vec{V}(u) = 0$. Without loss of generality, one can assume that $u \in C^\infty_P(\Omega)$ and the general case follows by density. Applying Green's formula to the vector field $\mathbf{\Gamma} := w\vec{V}(u)$ gives (1.10). □

With equation (1.10) in mind, we could replace the non-linear term in (1.9) with $\langle \vec{V}(u(s)) \cdot \nabla v, w(s) \rangle_{L^2}$, and ask for the weaker regularity $w \in L^2(0, T; L^2(\Omega))$. However, this term is no longer guaranteed to be finite. To resolve this, we choose a more regular test space, namely we assume that $v \in W^{1,\infty}_P(\Omega)$ instead of $H^1_P(\Omega)$, so that the non-linear term makes sense for all $v \in W^{1,\infty}_P(\Omega)$. Finally, we arrive at the main variational formulation of the thesis, which we write as an evolution problem as follows.

Weak-time weak-space with skew-symmetry. Given $u_0 \in H^2_P(\Omega)$ and $w_0 = (I - \Delta)u_0 \in L^2(\Omega)$, seek $u, w : [0, T] \to H^1_P(\Omega)$ with initial conditions $u(0) = u_0$ and $w(0) = w_0$ such that for all $v \in W^{1,\infty}_P(\Omega)$ and $t \in [0, T]$ one has

$$\begin{cases} \langle w(t) - w(0), v \rangle_{L^2} = \displaystyle\int_0^t \left\langle \vec{V}(u(s)) \cdot \nabla v, w(s) \right\rangle_{L^2} + \hat{k} \left\langle \frac{\partial u(s)}{\partial y}, v \right\rangle_{L^2} ds, \\ \langle u(t), v \rangle_{H^1} = \langle w(t), v \rangle_{L^2}. \end{cases} \tag{1.11}$$

With this formulation, one would can readily discrtize (1.11) using a FEM based Galerkin method for which *a priori* estimates are now much easier to obtain.[(1)] Indeed, starting with spacial discretization, we pick a small mesh size $h > 0$ and restrict the test space in (1.11) to the finite dimensional space $V^h_P(\Omega) \subset W^{1,\infty}_P(\Omega)$ of all periodic finite element corresponding to a uniform triangulation $\mathcal{T}_h$ of Ω, as introduced in Definition 2.1. As for temporal discretization, we pick a small time step $\tau > 0$ and consider the following

[(1)] As opposed (1.8) where one would need to assume that $w_0 \in H^1_P(\Omega)$ and find bounds on the H^1-norm of $w(t)$, which is not obvious without additional assumptions on w_0, c.f [13] where the authors further assume that $w_0 \in L^\infty(\Omega)$.

integral approximations

$$\begin{aligned}
&\int_t^{t+\tau} \left\langle \vec{V}(u(s)) \cdot \nabla v, w(s) \right\rangle ds = \tau \left\langle \vec{V}(u(t+\tau)) \cdot \nabla v, w(t+\tau) \right\rangle + \epsilon(v,\tau), \\
&\int_t^{t+\tau} \hat{k} \left\langle \frac{\partial u(s)}{\partial y}, v \right\rangle ds = \tau \hat{k} \left\langle \frac{\partial u(t+\tau)}{\partial y}, v \right\rangle + \delta(v,\tau).
\end{aligned} \tag{1.12}$$

Under these approximations, we obtain a fully implicit finite element discretization of (1.11) as follows.

Fully implicit discrete-time FE formulation. Given the following:

(i) initial data $u_0 \in H_P^2(\Omega)$ and $w_0 = (I - \Delta)u_0 \in L^2(\Omega)$,

(ii) $h > 0$ denoting the mesh size of the FEM triangulation,

(iii) $\tau > 0$ a denoting the time step,

seek $u_h, w_h : [0,T] \to V_P^h(\Omega)$ such that $u_h(0) = \pi_h(u_0)$ and $w_h(t) = \pi_h(w_0)$, for all $v \in V_P^h(\Omega)$, and for all $v \in V_P^h(\Omega)$ one has

$$\begin{cases}
\langle w_h(t+\tau) - w_h(t), v \rangle_{L^2} = \tau \left\langle \vec{V}(u_h(t+\tau)) \cdot \nabla v, w_h(t+\tau) \right\rangle + \tau \hat{k} \left\langle \dfrac{\partial u_h(t+\tau)}{\partial y}, v \right\rangle, \\
\langle u_h(t), v \rangle_{H^1} = \langle w_h(t), v \rangle, \quad \text{for all } t \in [0, T-\tau), \\
u_h(t) = u_h(\tau) \text{ and } w_h(t) = w_h(\tau) \text{ for all } t \in (0, \tau].
\end{cases} \tag{1.13}$$

Before we conclude the introduction with the thesis outline, let us mention the following regularity result for elliptic problems over domains with corners which will be used frequently throughout the thesis.

Theorem 1.1 (Grisvard [18], Theorem 3.2.1.2)**.** Let D be any convex bounded open subset of $\mathbb{R}^n$. Given $f \in L^2(D)$, there is a unique $u \in H^2(D) \cap H_0^1(D)$ such that

$$-\Delta u + u = f.$$

Furthermore, there is a constant C_E independent of f and u such that $\|u\|_{H^2} \leq C_E \|f\|_{L^2}$.

With a slight modification to the proof of the above result, we also have the following periodic equivalent of the above theorem.

Theorem 1.2 (Periodic elliptic regularity)**.** Given $f \in L^2(\Omega)$, where $\Omega = [0, L] \times [0, L]$, there is a unique $u \in H_P^1(\Omega) \cap H^2(\Omega)$ such that

$$-\Delta u + u = f.$$

One can also find a constant C_E such that $\|u\|_{H^2} \leq C_E \|f\|_{L^2}$, independently of the choice of f and u.

1.4 Thesis outline

In Chapter 2, we showcase and continue the work done by Karakazian, Moufawad, and Nassif [17] on the fully implicit FE scheme.

- Section 2.1 We showcase the existence of a solution to (1.13).
- Section 2.2 We show the existence of a sequence of fully implicit solutions that converge to a solution of (1.11).

In Chapter 3, we study traveling waves of (1.1).

- Section 3.1. We derive from (1.5) an equation that produces traveling waves satisfying PBCs.
- Section 3.2. We prove the existence of traveling waves satisfying PBCs.
- Section 3.3. We suggest a Newton-Galerkin numerical scheme for simulating the initial data that produces traveling waves.

In Chapter 4, we use the numerical schemes developed in previous chapters for simulation.

- Section 4.1 We suggest a semi-linearized approach for solving the fully implicit numerical scheme obtained in Chapter 2.
- Section 4.2 We test the suggested method in Section 3.3 for various non-linearities.
- Section 4.3 We check that the initial data provided by the Newton-Galerkin method does indeed give traveling wave.

Chapter 2

On the convergence of the fully implicit numerical Scheme

2.1 Deriving the numerical scheme

The numerical scheme in this thesis is based on the finite element Galerkin method or simply finite element method. With a chosen mesh size h, and a meshing $\mathcal{T}_h$ of the domain Ω, this method entails discretizing (1.11), so that we look for solutions (u_h, w_h) contained in a finite dimensional subspace V^h of $H^1_P(\Omega)$, whose dimension is inversely proportional to h. As the mesh size h goes to zero, and the corresponding finite dimensional subspace V^h becomes larger, one expects that the finite dimensional solutions (u_h, v_h) to converge in some appropriate sense to the real solution (u, w).

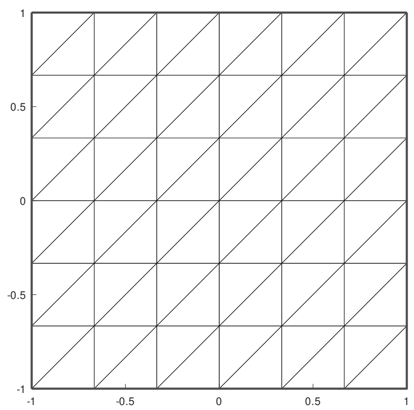
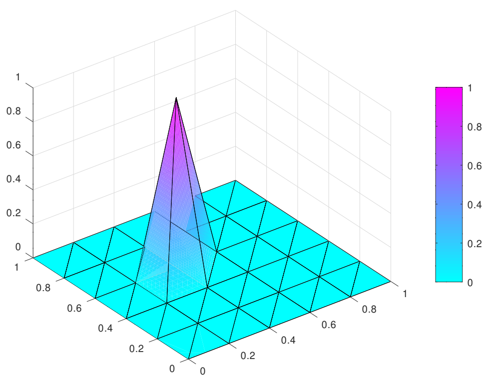


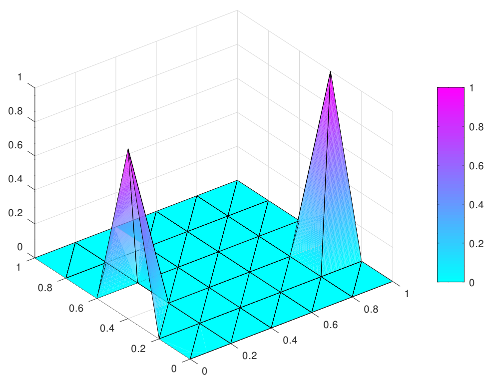
Figure 2.1: A uniform triangular mesh of Ω.

Definition 2.1 (Periodic finite elements)**.** Choose $n \in \mathbb{N}$ corresponding to the number of nodes (i.e vertices) on each segment of the square and let $h = \sqrt{2}L/n$. Consider a uniform triangulation $\mathcal{T}_h$ of Ω, as seen in Figure 2.1, with h the longest side of any triangle in the mesh. Defined the space of periodic finite elements $V_P^h(\Omega)$ corresponding to $\mathcal{T}_h$ as

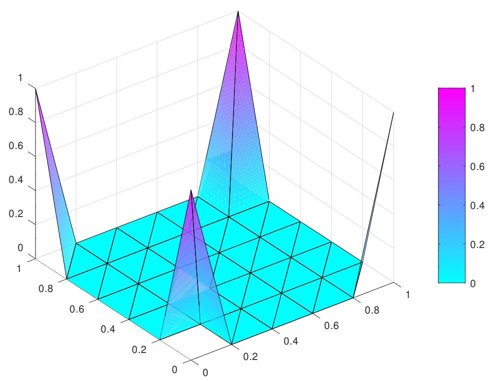
$$V_P^h(\Omega) := \left\{ v \in H_P^1(\Omega) : v_{|K} \text{ is linear for every triangle } K \in \mathcal{T}_h \right\}.$$



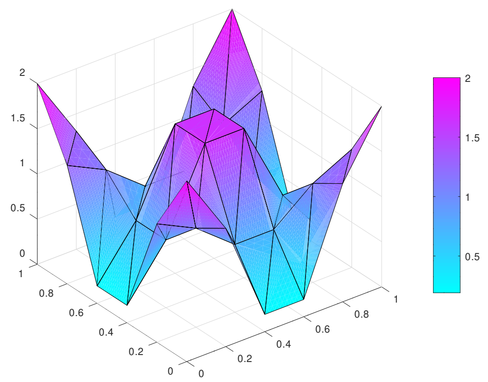
(a) Typcial interior basis element.



(b) Typical boundary basis element.



(c) Extremal boundary basis element.



(d) A typical periodic element.

Figure 2.2: The space $V_P^h(\Omega)$ visualized with $\Omega = [0, 1]^2$.

In particular, $V_P^h(\Omega)$ is the subspace of $H_P^1(\Omega)$ of continuous functions that are linear on each of the triangle in the triangulation. A typical element in $V_P^h(\Omega)$ is plotted in Figure 2.2d. It is easy to deduce from figure (4.1) and the conditions imposed by (1.6) that there are

- $(n-2)^2$ elements in $V_P^h(\Omega)$ such that each one equals one on some interior node and 0 on all other nodes, as in Figure 2.2a.

- $2(n-2)$ elements in $V_P^h(\Omega)$ that are equal to one on two opposing boundary nodes and zero otherwise, as in Figure 2.2b.

- one element that is equal to one on the four corners of the domain and zero otherwise, as in Figure 2.2c.

These elements clearly form a basis for $V_P^h(\Omega)$ and hence

$$d := \dim V_P^h = (n-2)^2 + 2(n-2) + 1 = (n-1)^2.$$

As usual for the Galerkin method, we look for a finite dimensional equivalent of (1.11) by replacing the requirement (1.11) hold for all $v \in W_P^{1,\infty}(\Omega)$ to holding for all $v \in V_P^h(\Omega)$. This insinuates looking for solutions u and v as such that $u(t), w(t) \in V_P^h(\Omega)$ for all $t \in [0,T]$. Then, the semi-discrete finite dimensional equivalent of (1.11) reads as follows.

Semi-Discrete FE Formulation. Given $u_0 \in H_P^1(\Omega)$ with $w_0 = (I - \Delta)u_0 \in L^2(\Omega)$, seek functions $u_h, w_h : [0,T] \to V_P^h(\Omega)$ such that $u_h(0) = \pi_h(u_0)$ and $w_h(0) = \pi_h(w_0)$, and for all $v \in V_P^h(\Omega)$ one has

$$\begin{cases} \left\langle \dfrac{dw_h}{dt}(t), v \right\rangle_{L^2} = \left\langle \vec{V}(u_h(t)) \cdot \nabla v, w_h(t) \right\rangle_{L^2} + \hat{k} \left\langle \partial_y u_h(t), v \right\rangle_{L^2}, \\ \left\langle u_h(t), v \right\rangle_{H^1} = \left\langle w_h(t), v \right\rangle_{L^2}, \end{cases} \tag{2.1}$$

which implies that

$$\begin{cases} \left\langle w_h(t) - w_h(0), v \right\rangle = \displaystyle\int_0^t \left\langle \vec{V}(u_h(s)) \cdot \nabla v, w_h(s) \right\rangle + \hat{k} \left\langle \partial_y u_h(s), v \right\rangle \; ds, \\ \left\langle u_h(t), v \right\rangle_{H^1} = \left\langle w_h(t), v \right\rangle_{L^2}, \end{cases} \tag{2.2}$$

which is also means that for any $\tau > 0$ and $t \in [0, T - \tau)$,

$$\begin{cases} \left\langle w_h(t+\tau) - w_h(t), v \right\rangle = \displaystyle\int_t^{t+\tau} \left\langle \vec{V}(u_h(s)) \cdot \nabla v, w(s) \right\rangle + \hat{k} \left\langle \partial_y u_h(s), v \right\rangle ds, \\ \left\langle u_h(t), v \right\rangle_{H^1} = \left\langle w_h(t), v \right\rangle. \end{cases} \tag{2.3}$$

Naturally, the previous two equations require at least that $u \in L^2(0,T; H^1(\Omega))$ and $w \in L^2(0,T; L^2(\Omega))$, so that these are the solution spaces on which we apply the Galerkin analysis. Now using the fact that (2.1) holds for all $v \in V_P^h(\Omega)$ if and only if it holds for all basis elements of $V_P^h(\Omega)$, letting $B_P^h = \{\varphi_1, \ldots, \varphi_d\}$ be a basis of $V_P^h(\Omega)$ so that we can write

$$u(t) := \sum_{i=1}^d u_i(t)\varphi_i \quad w(t) = \sum_{i=1}^d w_i(t)\varphi_i, \quad u_h(0) = \pi_h(u_0),$$

then one obtains the matrix form of (2.1) as follows.

Semi-discrete matrix form. Given $u_0 \in H^2_P(\Omega)$ with $w_0 = (I - \Delta)u_0 \in L^2(\Omega)$, and $h > 0$, seek functions $U_h, W_h : [0,T] \to \mathbb{R}^{d_h}$ such that

$$U_h(0) = \left(\pi_h^{(1)} u_0, \dots, \pi_h^{(d)} u_0\right) \quad \text{and} \quad W_h(0) = \left(\pi_h^{(1)} w_0, \dots, \pi_h^{(d)} w_0\right)$$

and such that for all $t \in [0,T]$ we have

$$\begin{cases} M\dfrac{dW_h}{dt}(t) = S(U_h(t))W_h(t) + kRU_h(t), \\ KU_h(t) = MW_h(t). \end{cases} \tag{2.4}$$

where $M = [\langle \varphi_i, \varphi_j \rangle]_{i,j=1}^d$ is the mass matrix, $K = \left[\langle \varphi_i, \varphi_j \rangle_{H^1}\right]_{i,j=1}^d$ is the sum of the mass and stiffness matrices, $R = [\langle \partial_y \varphi_i, \varphi_j \rangle]_{i,j=1}^d$, and $S : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is the matrix valued *linear* map given by

$$S(x) = \left[\sum_{k=1}^d x_k \left\langle \vec{V}(\varphi_k) \cdot \nabla \varphi_i, \varphi_j \right\rangle \right]_{i,j=1}^d, \quad x = (x_1, \dots, x_d).$$

It has been already shown in [17] using the Picard-Lindelöf theorem that such a pair (U_h, W_h) satisfying (2.4) exists for each h and that there is a sequence of solution pairs $\{(U_{h_n}, W_{h_n})\}_{n \geq 1}$ satisfying (2.4) for each n, such that the corresponding sequence of solutions pairs $\{(u_{h_n}, w_{h_n})\}_{n \geq 1}$ converges to a solution pair (u, w) of equation (1.11). However, to obtain a numerical scheme, one would still need to discretize (2.1) and (2.4) with respect to time. Therefore, we go to (2.3) and use the approximations given by (1.12) to obtain (1.13), which we rewrite here for the sake of convenience.

Fully implicit discrete-time FE formulation. Given the following:

(i) initial data $u_0 \in H^2_P(\Omega)$ and $w_0 = (I - \Delta)u_0 \in L^2(\Omega)$,

(ii) h denoting the mesh size of the FEM triangulation,

(iii) τ denoting some time step,

seek $u_h, w_h : [0,T] \to V^h_P(\Omega)$ such that $u_h(t) = \pi_h(w_0)$ and $w_h(t) = \pi_h(w_0)$, and for all $v \in V^h_P(\Omega)$ one has

$$\begin{cases} \langle w_h(t+\tau) - w_h(t), v \rangle_{L^2} = \tau \left\langle \vec{V}(u_h(t+\tau)) \cdot \nabla v, w_h(t+\tau) \right\rangle_{L^2} & \\ \qquad\qquad + \tau \hat{k} \left\langle \partial_y u_h(t+\tau), v \right\rangle_{L^2}, & \text{for all } t \in (\tau, T-\tau], \\ \langle u_h(t), v \rangle_{H^1} = \langle w_h(t), v \rangle_{L^2}, & \text{for all } t \in (\tau, T-\tau], \\ u_h(t) = u_h(\tau) \text{ and } w_h(t) = w_h(\tau), & \text{for all } t \in (0, \tau]. \end{cases} \tag{2.5}$$

This last condition defines u_h and w_h as left-continuous-right-limit step functions, ie

$$u_h(t) = \begin{cases} \sum_j u_h(j\tau) \cdot \mathbf{1}_{E_j}(t), & t \neq 0, \\ \pi_h(u_0), & t = 0, \end{cases} \qquad w_h(t) = \begin{cases} \sum_j w_h(j\tau) \cdot \mathbf{1}_{E_j}(t), & t \neq 0, \\ \pi_h(w_0), & t = 0. \end{cases} \tag{2.6}$$

where $\mathbf{1}_{E_j}$ is the indicator function of the set $E_j := ((j-1)\tau, j\tau]$. In matrix form, this is equivalent to the following.

Fully-implicit-time FE-space, matrix formulation. Given initial data $u_0 \in H^2_P(\Omega)$, $h > 0$ denoting the mesh size of the FE triangulation, and $\tau > 0$ denoting a time step, seek functions $U_h, W_h : [0,T] \to \mathbb{R}^d$ such that $U_h(0) = U_h^{(0)}$ and the following hold

(a) For all $t \in [0,T]$ one has that $KU_h(t) = MW_h(t)$.

(b) For all $t \in (\tau, T-\tau]$ one has

$$MW_h(t+\tau) = MW_h(t) + \tau S(U_h(t+\tau))W_h(t+\tau) + \tau\hat{k}RU_h(t+\tau), \tag{2.7}$$

(c) For all $t \in (0,\tau]$ one has $U_h(t) = U_h(\tau)$.

The existence and uniqueness of (U_h, W_h) satisfying (2.7), and therefore the existence of (u_h, w_h) satisfying, is shown by writing (2.7) in fixed point format through

$$\begin{cases} W_h(t+\tau) = G(W_h(t+\tau)), \quad \text{where} \\ G(\mathbf{x}) := M^{-1}A^{-1}\left(\mathbf{z} + \tau M^{-1}S(K^{-1}M\mathbf{x})\right)\mathbf{x}, \text{ with} \\ A := I - \tau\hat{k}RK^{-1}, \quad \mathbf{z} := W(t). \end{cases} \tag{2.8}$$

provided that A is invertible, which is indeed the case when a constraint is added to τ. Therefore, existence becomes a matter of applying the Leray-Shauder fixed point theorem, as showcased in the next proposition.

Proposition 2.1 (Karakazian, Moufawad, and Nassif [17])**.** For $\tau \leq (2\hat{k})^{-1}$, the fixed point problem (2.8) has a solution.

Proof. A straightforward argument shows that the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by

$$a(u,v) = \langle u, v \rangle_{H^1} - \tau\hat{k}\,\langle \partial_y u, v \rangle_{L^2},$$

is coercive when $\tau \leq (2\hat{k})^{-1}$ and in particular it is coercive when restricted finite dimensional subspace $V_P^h(\Omega)$. Now let $w \in V_P^h(\Omega)$ with coordinate vector $\mathbf{w} \in \mathbb{R}^d$. The

Lax-Milgram theorem guarantees the existence of a unique $u \in V_P^h(\Omega)$ with coordinate vector $\mathbf{u} \in \mathbb{R}^d$ such that

$$a(u, v) = \langle w, v \rangle\,, \quad \text{for all } v \in V_P^h(\Omega), \quad \text{i.e} \quad (K - \tau \hat{k} R)\mathbf{u} = M\mathbf{w}.$$

This implies that $K - \tau\hat{k}R$ is invertible and therefore $A = (K - \tau\hat{k}R)K^{-1}$ is also invertible. Hence, (2.7) can be put in fixed point format through (2.8).
Now we prove the existence of a fixed point of G. Indeed, let $\mathbf{x} \in \mathbb{R}^d$ be such that $\mathbf{x} = \lambda G(\mathbf{x})$ for some $0 \le \lambda \le 1$. This means that

$$(I - \tau\hat{k}RK^{-1})M\mathbf{x} = \lambda(M\mathbf{z} - \tau S(K^{-1}M\mathbf{x}))\mathbf{x}.$$

Multiplying by $\mathbf{x}^T$ on both sides, using the fact that $S(\mathbf{v})$ is skew-symmetric for all $\mathbf{v} \in \mathbb{R}^d$, and the triangle inequality, we obtain

$$\|\mathbf{x}\|_M^2 - \tau\hat{k}\,\left|\mathbf{x}^T RK^{-1}M\mathbf{x}\right| = \lambda\|\mathbf{x}\|_M\|\mathbf{z}\|_M. \tag{2.9}$$

Now let $\mathbf{y} \in \mathbb{R}^d$ be such that $\mathbf{y} = K^{-1}M\mathbf{x}$ so that $\|\mathbf{y}\|_K \le \|\mathbf{x}\|_M$ by elliptic regularity. Then $\mathbf{x}$ and $\mathbf{y}$ are the coordinate vectors of two elements $v_\mathbf{x}$ and $u_\mathbf{y}$ in $V_P^h(\Omega)$ which satisfy the following

$$|\mathbf{x}^T R\mathbf{y}| = |\,\langle \partial_y u_\mathbf{y}, v_\mathbf{x}\rangle\,| \le \|u_\mathbf{y}\|_{H^1}\|v_\mathbf{x}\|_{L^2} = \|\mathbf{y}\|_K\|\mathbf{x}\|_M \le \|\mathbf{x}\|_M^2,$$

so that for $\tau \le (2\hat{k})^{-1}$ equation (2.9) implies

$$\|\mathbf{x}\|_M \le \frac{\lambda}{1 - \tau\hat{k}}\|\mathbf{z}\|_M \le \frac{1}{1 - \tau\hat{k}}\|\mathbf{z}\|_M \le 2\|\mathbf{z}\|_M.$$

Therefore, the set of all such $\mathbf{x}$ is bounded in the $\|\cdot\|_M$ norm by $2\|z\|_M$, and hence the Leray-Schauder fixed point theorem applies to give a fixed point $\mathbf{x}$ of G. □

Remark. When $\tau \le Ch^{5/2}$ with C depending on $\hat{k}^{-1}$ and $\|\mathbf{z}\|_M^{-1} = \|W(t)\|_M^{-1}$, the function G defined in (2.8) becomes a contraction, and hence the obtained fixed point is unique. For a proof of this see [17].

2.2 A priori estimates and convergence of fully-implicit system

In this section, we exploit periodicity to derive a priori estimates and use them to prove the existence of a pair (u, w) satisfying (2.2), which will be the limit of a sequence of pairs (u_n, w_n) satisfying (2.5) for some (h_n, τ_n). More precisely, we prove the following theorem.

Theorem 2.1. There are real sequences $\{h_n\}$ and $\{\tau_n\}$ with $\tau_n \in \Theta(h_n)$, a sequence $\{(u_n, w_n)\}$ of solution pairs that solve (2.5) for the given (h_n, τ_n), and a pair

$$(u, w) \in L^\infty(0, T; H_P^1(\Omega)) \times L^\infty(0, T; L^2(\Omega))$$

that solves (1.11) such that $u_n \overset{\star}{\rightharpoonup} u$ and $w_n \overset{\star}{\rightharpoonup} w$.

We note that the method to be used in the sequel is almost identical to the one used in [17], and is more or less standard in the Galerkin method, which dictates finding a priori estimates from the finite dimensional equation (2.5) and extracting a subsequence that converges to a solution of the original equation (2.2). First of all, let us mention the "skew-symmetry" property of the Poisson term for elements in $V_P^h(\Omega)$.

Theorem 2.2 (Karakazian, Moufawad, and Nassif [17])**.** For all $u, v, w \in V_P^h(\Omega)$ we have that

$$\left\langle \vec{V}(u) \cdot \nabla v, w \right\rangle = -\left\langle \vec{V}(u) \cdot \nabla w, v \right\rangle,$$

and in particular this implies that $\left\langle \vec{V}(u) \cdot \nabla w, w \right\rangle = 0$.

Proposition 2.2. There is a constant $C_a = C_a(T, \|w_0\|_{L^2}, \hat{k})$ and an $N \in \mathbb{N}$ such that for all $n \geq N$, for any $h \in \mathbb{R}^+$, and given the solution pair (u, w) satisfying (2.5) for h and $\tau_n = T/n$, one has

$$\|u(t)\|_{H^1} \leq \|w(t)\|_{L^2} \leq C_a. \tag{2.10}$$

Proof. The fact that $\|u(t)\|_{H^1} \leq \|w(t)\|_{L^2}$ follows from simply plugging $v = u(t)$ in (2.5) and using Cauchy-Shwarz. On the other hand, plugging $v = w(t+\tau) \in V_P^h(\Omega)$ in the first equation of (2.5) and using Theorem 2.2 to cancel the Poisson bracket term, one has that

$$\|w(t+\tau)\|_{L^2}^2 = \langle w(t), w(t+\tau)\rangle_{L^2} + \tau\hat{k} \left\langle \frac{\partial u(t+\tau)}{\partial y}, w(t+\tau) \right\rangle_{L^2},$$

then use triangle inequality and Cauchy-Shwarz to obtain

$$\|w(t+\tau)\|_{L^2} \leq \|w(t)\|_{L^2} + \tau\hat{k} \left\| \frac{\partial u(t+\tau)}{\partial y} \right\|_{L^2} \leq \|w(t)\|_{L^2} + \tau\hat{k}\|w(t+\tau)\|_{L^2},$$

where we have used

$$\left\| \frac{\partial u(t+\tau)}{\partial y} \right\|_{L^2} \leq \|u(t+\tau)\|_{H^1} \leq \|w(t+\tau)\|_{L^2}.$$

Now since $t \in [(j-1)\tau, j\tau)$ for some $1 \leq j \leq n$, we have that

$$\|w(t)\|_{L^2} \leq \frac{\|w(0)\|_{L^2}}{(1-\tau\hat{k})^j} \leq \frac{\|w(0)\|_{L^2}}{(1-\tau\hat{k})^n} = \frac{\|w(0)\|_{L^2}}{(1-\frac{T}{n}\hat{k})^n} \leq \underbrace{2^{\hat{k}T}\|w(0)\|_{L^2}}_{:=C_a},$$

where the last inequality in the above equation holds whenever $\tau \leq (2\hat{k})^{-1}$, as desired. $\square$

Notation. Before we start with the proof of convergence of the numerical scheme, we introduce some notation to be used as shorthands in the following.

(i) For $n \in \mathbb{N}$ and $v \in H^1_P(\Omega)$ set

$$h_n = \sqrt{2}L/n \quad \text{and} \quad \pi_n v := \pi_{h_n} v \in V_P^{h_n}(\Omega), \tag{2.11}$$

where $\pi_{h_n}(v)$ is the interpolant of v_n in $V_P^h(\Omega)$.

(ii) For $m \in \mathbb{N}$, set $\tau_m = T/m$. For fixed $t \in [0,T]$, we let $k_m = k_m(t)$ and $\delta_m = \delta_m(t)$ be the numbers such that

$$t \in ((k_m - 1)\tau_m, k_m\tau_m] \quad \text{and} \quad \delta_m := k_m\tau_m - t. \tag{2.12}$$

(iii) Given (u_n, w_n) satisfying (2.5) for some (h_n, τ_m), we define set each $0 \le j \le k_m$, we define

$$u_n^{(j)} := \begin{cases} u_n(j\tau_m), & \text{if } j < k_m, \\ u_n(t), & \text{if } j = k_m. \end{cases} \qquad w_n^{(j)} := \begin{cases} w_n(j\tau_m), & \text{if } j < k_m, \\ w_n(t), & \text{if } j = k_m. \end{cases} \tag{2.13}$$

(iv) For $(u, v, w) \in H^1(\Omega) \times W^{1,\infty}(\Omega) \times L^2(\Omega)$, define the following

$$P_v(u, w) := \left\langle \vec{V}(u) \cdot \nabla v, w \right\rangle, \quad L_v(u) := \left\langle \frac{\partial u}{\partial y}, v \right\rangle,$$

and we have

$$|P(u, v, w)| \le \|u\|_{H^1} \|v\|_{W^{1,\infty}} \|w\|_{L^2} \quad \text{and} \quad |L(u, v)| \le \|u\|_{H^1} \|v\|_{L^2}.$$

Lemma 2.1 (Error estimation)**.** Let $t \in [0,T]$ and $v \in W_P^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega)$. With the N given in Proposition 2.2, define a sequence $\{(h_n, \tau_n)\}_{n \ge N}$ by

$$h_n := \frac{\sqrt{2}L}{n} \quad \text{and} \quad \tau_n \in \Theta(h_n).$$

Let $\pi_n v$ be the interpolant of v in $V_P^{h_n}(\Omega)$ and let (u_n, w_n) be the solution pair of (2.5) corresponding to (h_n, τ_n). Then we have that

$$\langle w_n(t) - w_n(0), v \rangle_{L^2} = \sum_{j=1}^{k_n} \left\langle w_n^{(j)} - w_n^{(j-1)}, \pi_n v \right\rangle_{L^2} + \alpha_n, \tag{2.14}$$

where $\alpha_n, \beta_n \in \mathcal{O}\big(\|v - \pi_n v\|_{L^2}\big)$ and $\gamma_n \in \mathcal{O}\big(\|v - \pi_n v\|_{W^{1,\infty}}\big)$.

Proof. For equation (2.14), one has that

$$\begin{aligned}\langle w_n(t) - w_n(0), v\rangle_{L^2} &= \left\langle w_n^{(k_n)} - w_n^{(0)}, v\right\rangle_{L^2} \\ &= \left\langle w_n^{(k_n)} - w_n^{(0)}, \pi_n v\right\rangle_{L^2} + \left\langle w_n^{(k_n)} - w_n^{(0)}, v - \pi_n v\right\rangle_{L^2} \\ &= \sum_{j=1}^{k_n} \left\langle w_n^{(j)} - w_n^{(j-1)}, \pi_n v\right\rangle_{L^2} + \alpha_n,\end{aligned}$$

where $\alpha_n = \langle w_n^{(k_n)} - w_n^{(0)}, v - \pi_n v\rangle_{L^2}$. Furthermore, by a priori estimate (2.10) and Cauchy-Shwarz one has

$$|\alpha_n| \leq (\|w_n^{(k_n)}\|_{L^2} + \|w_n(0)\|_{L^2})\|v - \pi_n v\|_{L^2} \leq 2C_a\|v - \pi_n v\|_{L^2} \in \mathcal{O}\big(\|v - \pi_n v\|_{L^2}\big). \quad \square$$

Lemma 2.2. Let $t \in [0, T]$ and $v \in W_P^{1,\infty}(\Omega)$. With the N given in Proposition 2.2 and $\{(h_n, \tau_n)\}_{n \geq N}$ given as in Lemma 2.1, and (u_n, w_n) a solution pair of (2.5) corresponding to (h_n, τ_n), we have that

$$\int_0^t L(u_n(s), v)ds = \tau_n \sum_{j=1}^{k_n} L(u_n^{(j)}, \pi_n v) + \beta_n, \tag{2.15}$$

where $\beta_n \in \mathcal{O}\big(\|v - \pi_n v\|_{L^2}\big)$.

Proof. From the definition of u_n in (2.5) as a step function on $[0, T]$, and using the notation (2.11)-(2.13), we infer that equation (2.15) holds because

$$\begin{aligned}\int_0^t L(u_n(s), v)ds &= \int_0^{k_n\tau_n} L(u_n(s), v)ds - \int_t^{k_n\tau_n} L(u_n(s), v)ds \\ &= \int_0^{k_n\tau} L\left(\sum_{j=1}^{n} u_n^{(j)} \cdot \mathbf{1}_{E_j}(s), v\right) ds - \epsilon_n L(u_n^{(k_n)}, v)ds \\ &= \sum_{j=1}^{n} \int_0^{k_n\tau_n} \mathbf{1}_{E_j}(s)\, L\left(u_n^{(j)}, v\right) ds - \epsilon_n L(u_n^{(k_n)}, v)ds \\ &= \sum_{j=1}^{k_n} \tau_n L(u_n^{(j)}, v) - \epsilon_n L(u_n^{(k_n)}, v) \\ &= \sum_{j=1}^{k_n} \tau_n L(u_n^{(j)}, \pi_n v) + \beta_n\end{aligned}$$

where β_n is given by

$$\beta_n = \sum_{j=1}^{k_n} \tau_n L(u_n^{(j)}, v - \pi_n v) - \epsilon_n L(u_n^{(k_n)}, v).$$

Now observe that by the definition of k_n one has $k_n\tau_n \in \Theta(h_n^{-1})\Theta(h_n) = \Theta(1)$. Therefore, we have that

$$\begin{aligned}
|\beta_n| &\leq \tau_n \sum_{j=1}^{k_n} \left|L\left(u_n^{(j)}, v - \pi_n v\right)\right| + \epsilon_n |L(u_n^{(k_n)}, v)| \\
&\leq \tau_n \|v - \pi_n v\|_{L^2} \sum_{j=1}^{k_n} \left\|\partial_y u_n^{(j)}\right\|_{L^2} + \epsilon_n \left\|\partial_y u_n^{(k_n)}\right\|_{L^2} \|v\|_{L^2} \\
&\leq \tau_n \|v - \pi_n v\|_{L^2}\, k_n C_a + \epsilon_n C_a \|v\|_{L^2} \\
&\in \mathcal{O}\big(\|v - \pi_n v\|_{L^2}\big) + \mathcal{O}(\epsilon_n) = \mathcal{O}\big(\|v - \pi_n v\|_{L^2}\big). \qquad \square
\end{aligned}$$

Lemma 2.3. Let $t \in [0, T]$ and $v \in W_P^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega)$. With the N given in Proposition 2.2 and $\{(h_n, \tau_n)\}_{n \geq N}$ given as in Lemma 2.1, and (u_n, w_n) a solution pair of (2.5) corresponding to (h_n, τ_n), we have that

$$\int_0^t P(u_n(s), v, w_n(s))ds = \tau_n \sum_{j=1}^{k_n} P\big(u_n^{(j)}, \pi_n v, w_n^{(j)}\big) + \gamma_n, \tag{2.16}$$

where $\gamma_n \in \mathcal{O}\big(\|v - \pi_n v\|_{W^{1,\infty}}\big)$.

Proof. From the definitions of u_n and w_n in (2.5) as a step functions on $[0, T]$, and using the the notation (2.12)-(2.13), we infer that equation (2.16) holds because

$$\begin{aligned}
\int_0^{k_n\tau_n} P(u_n(s), v, w_n(s))ds &= \int_0^{k_n\tau_n} \sum_{i,j=1}^{n} \mathbf{1}_{E_i}(s) \cdot \mathbf{1}_{E_j}(s) \cdot P(u_n^{(i)}, v, w_n^{(j)})ds \\
&= \int_0^{k_n\tau_n} \sum_{i,j=1}^{k_n} \delta_{ij}(s) \cdot P(u_n^{(i)}, v, w_n^{(j)})ds \\
&= \int_0^{k_n\tau_n} \sum_{j=1}^{k_n} P(u_n^{(j)}, v, w_n^{(j)})ds \\
&= \sum_{j=1}^{k_n} \tau_n P(u_n^{(j)}, v, w_n^{(j)})ds,
\end{aligned}$$

and therefore

$$\begin{aligned}\int_0^t P(u_n(t), v, w_n(t))ds &= \int_0^{k_n\tau_n} P(u_n(s), v, w_n(s))ds - \int_t^{k_n\tau_n} P(u_n(s), v, w_n(s))ds \\ &= \sum_{j=1}^{k_n} \tau_n P(u_n^{(j)}, v, w_n^{(j)}) - \epsilon_n P(u_n^{(k_n)}, v, w_n^{(k_n)}) \\ &= \sum_{j=1}^{k_n} \tau_n P(u_n^{(j)}, \pi_n v, w_n^{(j)}) + \gamma_n,\end{aligned}$$

where

$$\gamma_n = \sum_{j=1}^{k_n} \tau_n P(u_n^{(j)}, v - \pi_n v, w_n^{(j)}) - \epsilon_n P(u_n^{(k_n)}, v, w_n^{(k_n)}).$$

With a priori estimate (2.10) and Cauchy-Shwarz, one has that

$$\begin{aligned}|\gamma_n| &\leq \tau_n \sum_{j=1}^{k_n} \left|P\left(u_n^{(j)}, v - \pi_n v, w_n^{(j)}\right)\right| + \epsilon_n |P(u_n^{(k_n)}, v, w_n^{(k_n)})| \\ &\leq \tau_n \|v - \pi_n v\|_{W^{1,\infty}} \sum_{j=1}^{k_n} \left\|u_n^{(j)}\right\|_{H^1} \|w_n^{(j)}\|_{L^2} + \epsilon_n \|v\|_{W^{1,\infty}} \left\|u_n^{(k_n)}\right\|_{H^1} \|w_n^{(k_n)}\|_{L^2} \\ &\leq \tau_n k_n C_a^2 \|v - \pi_n v\|_{W^{1,\infty}} + C_a^2 \epsilon_n \|v\|_{W^{1,\infty}} \\ &\in \mathcal{O}\big(\|v - \pi_n v\|_{W^{1,\infty}}\big) + \mathcal{O}(\epsilon_n) = \mathcal{O}\big(\|v - \pi_n v\|_{W^{1,\infty}}\big).\end{aligned}$$

□

Proposition 2.3. Let $\{(h_n, \tau_n)\}_{n\geq N}$ be defined as in Lemma 2.1 and let $\{(u_n, w_n)\}_{n\geq N}$ be the corresponding sequence of solution pairs of (2.5). There is a subsequence $\{(u_{n_m}, w_{n_m})\}$, an element $u \in L^2(0, T; H^1_P(\Omega))$, and an element $w \in L^\infty(0, T; L^2(\Omega))$ such that

(i) $w_{n_m} \overset{\star}{\rightharpoonup} w$ in the weak* topology on $L^\infty(0, T; L^2(\Omega))$.

(ii) $w_{n_m} \rightharpoonup w$ in the weak topology on $L^2(0, T; L^2(\Omega))$.

(iii) $u_{n_m} \overset{\star}{\rightharpoonup} u$ in the weak* topology on $L^\infty(0, T; H^1_P(\Omega))$.

(iv) $u_{n_m} \rightharpoonup u$ in the weak topology on $L^2(0, T; H^1_P(\Omega))$.

Proof. In this proof we will need the Banach-Alaoglu and Rellich-Kondrachov theorems (Theorems 3.16 and 9.16 resp. in [19]).

(i) Proposition 2.2 tells us that the sequence $\{w_n\}$ is bounded in $L^\infty(0, T; L^2(\Omega))$, which is the dual space of $L^1(0, T; L^2(\Omega))$. Therefore, the Banach-Alaoglu theorem tells us that there is a subsequence $\{w_m\} = \{w_{m_n}\}_{n\geq N}$ and an element $w \in L^\infty(0, T; L^2(\Omega))$ such that $w_{m_n} \overset{\star}{\rightharpoonup} w$.

(ii) It is clear that the w obtained in (i) is in $L^2(0,T;L^2(\Omega))$. The fact that $w_{n_m} \overset{\star}{\rightharpoonup} w$ means that for every $v \in L^1(0,T;L^2(\Omega))$, and thus in particular for all $v \in L^2(0,T;L^2(\Omega))$, one has that

$$\int_0^T \langle w_{n_m}(s), v\rangle_{L^2}\, ds \to \int_0^T \langle w(s), v\rangle\, ds.$$

Then weak convergence $u_n \rightharpoonup u$ follows since every continuous linear functional on $L^2(0,T;L^2(\Omega))$ is of the form $\int_0^T \langle \cdot, g\rangle_{L^2}\, ds$ for some $g \in L^2(0,T;L^2(\Omega))$,

(iii) Let $\{u_{n_m}\}$ be the subsequence of $\{u_n\}$ corresponding to $\{w_{n_m}\}$. Then for each m, consider the unique function $z_{n_m} : [0,T] \to H^1_P(\Omega) \cap H^2(\Omega)$ such that

$$\langle z_{n_m}(t), v\rangle_{H^1} = \langle w_{n_m}(t), v\rangle_{L^2}, \quad \text{for all } v \in H^1_P(\Omega) \text{ and } t \in [0,T].$$

This sequence in bounded in $L^2(0,T;H^2_P(\Omega))$ because by Theorem 1.2 and Proposition 2.2 one has that

$$\|z_{n_m}\|_{L^2,H^2} = \left(\int_0^T \|z_{n_m}(t)\|^2_{H^2}\right)^{\frac{1}{2}} \le \left(\int_0^T C_E^2 \|w_{n_m}(t)\|^2_{L^2}\right)^{\frac{1}{2}} \le \sqrt{T} C_E C_a.$$

There is a subsequence $\{z_{n_{m_k}}\}$, which we relabel as $\{z_{n_m}\}$, and an element $u \in H^1_P(\Omega)$ such that $z_{n_m} \to u$ pointwise on almost everywhere on $L^2(0,T;H^1_P(\Omega))$. Furthermore, by part (iii) of Proposition 2.4, there is a constant K such that

$$\begin{aligned} \|z_{n_m}(t) - u_{n_m}(t)\|_{H^1} &= \|z_{n_m}(t) - \pi_{h_{n_m}}(z_{n_m}(t))\|_{H^1} \le K h_{n_m} |z_{n_m}(t)|_{H^2} \\ &\le K h_{n_m} C_E \|w_{n_m}(t)\|_{L^2} \\ &\le K h_{n_m} C_E C_a, \end{aligned}$$

Hence, we also have that $\|u_{m_n} - z_{m_n}\|_{L^2,H^1} \to 0$ and thus $u_{n_m} \rightharpoonup u$.

(iv) Follow the exact same line of reasoning as in (ii). □

Corollary 2.2.1. Fix any $v \in W^{1,\infty}_P(\Omega)$ and consider the sequence $\{(u_{n_m}, w_{n_m})\}$ obtained in the previous proposition. Then for all $t \in [0,T]$ we have that

$$\int_0^t L_v(u_m(s)) \to \int_0^t L_v(u(s)), \quad \int_0^t P_v(u_m(s), w_m(s)) \to \int_0^t P_v(u(s), w(s)). \tag{2.17}$$

Furthermore, there is a subsequence $\{w_{n_{m_k}}\}$, which we relabel as $\{w_{m_n}\}$, which satisfies

$$\langle w_{m_k}(t) - w_{m_k}(0), v\rangle \to \langle w(t) - w(0), v\rangle, \quad \text{for each } t \in [0,T]. \tag{2.18}$$

Proof. The functional $u \mapsto \int_0^t L(u(s), v)$ is continuous on $L^2(0, T; H_P^1(\Omega))$ since

$$\left| \int_0^t L(u(s), v) \right| \leq \|v\|_{L^2} \int_0^t \|u(s)\|_{H^1} ds \leq \|v\|_{L^2} \|u\|_{L^2, H^1}.$$

Also, the bilinear map $(u, w) \mapsto \int_0^t P_v(u(s), w(s))$ is bounded and hence continuous on the product space $L^2(0, T; L^2(\Omega)) \times L^2(0, T; H_P^1(\Omega))$ since

$$\begin{aligned} \left| \int_0^t P(u(s), v, w(s)) ds \right| &\leq \|v\|_{W^{1,\infty}} \int_0^t \|u(s)\|_{H^1} \|w(s)\|_{L^2} ds \\ &\leq \|v\|_{W^{1,\infty}} \cdot \|u(s)\|_{L^2, H^1} \cdot \|w(s)\|_{L^2, L^2}. \end{aligned}$$

Therefore, (2.17) follows immediately from the previous proposition since $w_n \rightharpoonup w$ and $u_n \rightharpoonup u$ in their respective spaces. Now define the following real valued functions on $[0, T]$ as

$$\begin{aligned} f_{n_m}(t) &= \int_0^t \langle w_{n_m}(s), v \rangle \, ds, & f(t) &= \int_0^t \langle w_{n_m}(s), v \rangle \, ds, \\ f'_{n_m}(t) &= \langle w_{n_m}(t), v \rangle \,, & f'(t) &= \langle w_{n_m}(t), v \rangle \,. \end{aligned}$$

From either parts (i) or (ii) in Proposition 2.3, one deduces that $f_{n_m} \to f$. Furthermore, the conditions of the Arzela-Ascoli theorem for the sequence $\{f'_{n_m}\}$ are met since

- A priori estimate (2.10) tells us that the sequence $\{f'_{n_m}\}$ is uniformly bounded on $[0, T]$ with $|f'_{n_m}| \leq C_a \|v\|_{L^2}$.
- The sequence $\{f'_{n_m}\}$ is uniformly equicontinuous. Indeed, pick $1 \leq s < t \leq T$, and let $j, k \in \mathbb{N}$ be the integers such $s \in ((j-1)\tau_n, j\tau_n]$ and $t \in ((k-1)\tau_n, k\tau_n]$. Then we have that

$$\begin{aligned} |f'_{n_m}(t) - f'_{n_m}(s)| &= |\langle w_{n_m}(t) - w_{n_m}(s), v \rangle| = \left| \left\langle w_{n_m}^{(k)} - w_{n_m}^{(j)}, v \right\rangle \right| \\ &= \left| \tau_{n_m} \sum_{i=j+1}^{k} P(u_{n_m}^{(i)}, v, w_{n_m}^{(i)}) + L(u_{n_m}^{(i)}, v) \right| \\ &\leq \tau_{n_m} \sum_{i=j+1}^{k} \left(\|u_{n_m}^{(i)}\|_{H^1} \|w_{n_m}^{(i)}\|_{L^2} \|v\|_{W^{1,\infty}} + \|u_{n_m}^{(i)}\|_{H^1} \|v\|_{L^2} \right) \\ &\leq (k-j)\tau_{n_m} \left(C_a^2 \|v\|_{W^{1,\infty}} + C_a \|v\|_{L^2} \right) \\ &\leq C(t-s) \left(C_a^2 \|v\|_{W^{1,\infty}} + C_a \|v\|_{L^2} \right). \end{aligned}$$

Therefore, we can apply the Arzela-Ascoli theorem and obtain a subsequence $\{f'_{n_{m_k}}\}$ that converges uniformly on $[0, T]$ to f'. By relabeling $\{f'_{n_{m_k}}\}$ to $\{f'_{n_m}\}$, equation (2.18) readily follows. $\square$

We also need some well-established approximation properties of finite elements, whose proofs can be found for instance in Ciarlet [20].

Proposition 2.4 (FE approximation properties)**.** Suppose that $v \in H^2(\Omega)$ and let v_n be the interpolant of v defined (2.11). Then we have the following

(i) The sequence $\{v_n\}$ converges to v in $H^1(\Omega)$.

(ii) If in addition $v \in W^{2,\infty}(\Omega)$, then $\{v_n\}$ converges to v in $W^{1,\infty}(\Omega)$.

(iii) If $z \in H^1(\Omega)$ solves

$$\langle z, \phi \rangle_{H^1} = \langle v, \phi \rangle_{L^2}, \quad \text{for all } \phi \in H^1_P(\Omega)$$

and z_h solves

$$\langle z_h, \phi \rangle_{H^1} = \langle v, \phi \rangle_{L^2}, \quad \text{for all } \phi_h \in V^h(\Omega)$$

then $z \in H^2(\Omega)$, $\pi_h(z) = z_h$, and there is constant K independent of h such that

$$\|z - z_h\|_{H^1} \leq Kh|z|_{H^2}.$$

Proof of Theorem 2.1. Let $\{(u_{n_m}, w_{n_m})\}$ be the sequence obtained in Corollary 2.2.1 and (u, w) be the limit pair. For the sake of convenience, we relabel this sequence as $\{(u_n, w_n)\}$ Assume that $v \in W^{1,\infty}_P(\Omega) \cap W^{2,\infty}(\Omega)$. Using Lemmas 2.1-2.3, for each $t \in [0, T]$ we have that

$$\langle w_n(t) - w_n(0), v \rangle = \sum_{j=1}^{k_n} \left\langle w_n^{(j)} - w_n^{(j-1)}, v_n \right\rangle + \alpha_n, \tag{2.19}$$

$$\int_0^t P(u_n(s), v, w_n(s))ds = \tau_n \sum_{j=1}^{k_n - 1} P(u_n^{(j)}, \pi_n v, w_n^{(j)}) + \beta_n, \tag{2.20}$$

$$\int_0^t L(u_n(s), v)ds = \tau_n \sum_{j=1}^{k_n - 1} L(u_n^{(j)}, \pi_n v) + \gamma_n. \tag{2.21}$$

Subtracting (2.19) and (2.20) from (2.21) and then using the definition of (u_n, w_n) in (2.5) to cancel the sums, one has that

$$\langle w_n(t) - w_n(0), v \rangle - \int_0^t P(u_n(s), v, w_n(s))ds - \int_0^t L(u_n(s), v)ds$$
$$= \alpha_n - \beta_n - \gamma_n.$$

where α_n, β_n and γ_n go to zero as $n \to \infty$ by Proposition 2.4. Therefore, by taking the limit as $n \to \infty$, using Corollary 2.2.1, and Lemmas 2.1-2.3, one obtains

$$\langle w(t) - w(0), v \rangle - \int_0^t P_v(u(s), w(s))ds - \int_0^t L_v(u(s))ds = 0,$$

ie the pair (u, v) solves (1.11) for all $v \in W_P^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega)$. Since $W_P^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega)$ is dense in $W_P^{1,\infty}(\Omega)$, we also obtain that (u, v) solves (1.11) for all $v \in W_P^{1,\infty}(\Omega)$. $\square$

Chapter 3

Periodic traveling waves of the Hasegawa-Mima equation

In Magentohydrodynamics, stationary traveling waves seem to be of particular interest. For instance, several authors have studied traveling waves of the Hasegawa-Mima equation, called *Modons* or *Dipole Vortices* [4], [6], [7], [12], [21]. Therefore, our goal in this chapter is to try and reproduce, albeit approximately, some of the results found in the literature on traveling waves. Thus, in view of our setting, we ask the following question: for which initial data $u_0 \in H^2_P(\Omega)$ will the solution u of (1.11) be a traveling wave?

In Section 3.1, we introduce the problem and show that traveling waves of (1.1) are in fact solutions to a semi-linear elliptic equation. In Section 3.2, we prove the existence of periodic traveling waves by solving the semi-linear elliptic equation. In section 3.3, we suggest the usage of numerical scheme based on the Newton-Galerkin method and apply it to some examples. In section 3.4, we compare and contrast some of our results to the ones found in the literature.

3.1 Traveling waves as solutions to a semi-linear elliptic equation

Let us start by deriving from (1.5) the equation that governs traveling waves. If one looks for traveling wave solution of (1.5) equation traveling in the y direction with constant velocity c, one would seek a solution of the form

$$u(x, y, t) = \Psi(\xi, \zeta), \quad (\xi, \zeta) = (x, y - ct), \quad \forall (x, y, t) \in \Omega \times [0, \infty).$$

The definition of w in (1.5) also implies that there is a function ϕ such that $w(x, y, t) = \phi(\xi, \zeta)$. Therefore, with $\nabla = (\partial_\xi, \partial_\zeta)$ and $\vec{V} = (-\partial_\zeta, \partial_\xi)$, and replacing in (1.5), one obtains the following system

$$\begin{cases} -c\partial_\zeta \phi + \vec{V}(\Psi) \cdot \nabla \psi = \hat{k}\partial_\zeta \Psi, \\ -\Delta\Psi + \Psi = \phi. \end{cases}$$

If we define $\Psi_c = \Psi - c\xi$, then we can rewrite the above system as

$$\begin{cases} \vec{V}(\Psi_c) \cdot \nabla \phi = \hat{k}\partial_\zeta \Psi_c, \\ -\Delta\Psi_c + \Psi_c = \phi - c\xi. \end{cases}$$

Now if $\phi_{\hat{k}} = \phi + \hat{k}\xi$, then by plugging in the previous equation we obtain

$$\begin{cases} \vec{V}(\Psi_c) \cdot \nabla \phi_{\hat{k}} = 0, \\ -\Delta\Psi_c + \Psi_c = \phi_{\hat{k}} - (\hat{k} + c)\xi. \end{cases}$$

Now notice that for any $f \in C^1(\mathbb{R})$, one has that

$$\vec{V}(f(\Psi_c)) \cdot \nabla \Psi_c = f'(\Phi_c)\vec{V}(\Psi_c) \cdot \nabla \Psi_c = 0.$$

Therefore, if $\phi_{\hat{k}} = f(\Psi_c)$ for some $f \in C^1(\mathbb{R})$, then the pair $(\Psi_c, \phi_{\hat{k}})$ satisfies the previous system of equations provided that the function $\Psi = \Psi_c + c\xi$ satisfies the following semi-linear elliptic boundary values problem.

Hasegawa-Mima traveling wave form. Given an arbitrary function $f \in C^1(\mathbb{R})$ and the strip $\Lambda = (0, L) \times (-\infty, L)$, seek $\Psi : \overline{\Lambda} \to \mathbb{R}$ such that

$$\begin{cases} -\Delta\Psi + \Psi = f(\Psi - c\xi) - \hat{k}\xi, & \text{on } \Lambda, \\ \text{B.C's inherited from } u, & \text{on } \partial\Lambda, \\ \text{A.C's inherited from } u, & \text{as } \zeta \to -\infty. \end{cases} \tag{HM-Travel}$$

The abbreviations B.C's and A.C's mean boundary conditions and asymptotic conditions respectively. We remark that equation (HM-Travel) is already found in the literature [4], [6], [21], but without showcasing how it's derived.

Now suppose that we are looking for a traveling wave solutions that solve HM-Travel, but we require that $u(t)$ and $\nabla u(t)$ satisfy PBCs on $\overline{\Omega}$ for all $t \in [0, \infty)$. Then, from the equality $u(x, 0, t) = u(x, L, t)$ for all $(x, t) \in [0, L] \times [0, \infty)$ one deduces that

$$\Psi(x, -ct) = \Psi(x, L - ct), \quad \text{for all } (x, t) \in [0, L] \times [0, \infty),$$

and therefore

$$\Psi(\xi, \zeta) = \Psi(\xi, L + \zeta), \quad \text{for all } \zeta > 0,$$

In other words, Ψ is L-periodic in the ζ variable. This reduces the problem of solving (HM-Travel) on the semi-infinite strip Λ to solving on the square $\Omega = [0, L] \times [0, L]$, and then simply extend Ψ to the whole of Λ by L-periodicity. Therefore, equation (HM-Travel) becomes the following

$$\begin{cases} -\Delta\Psi + \Psi = f\big(\Psi - c\xi\big) - \hat{k}\xi, \quad \text{on } \Omega = [0, L] \times [0, L], \\ \Psi \text{ and } \nabla\Psi \text{ satisfy periodic boundary values.} \end{cases} \tag{3.1}$$

3.2 Existence of traveling waves in $H_P^2(\Omega)$.

In this section we prove the existence of a solution to equation (3.1) by putting it in fixed point format and using the Schauder Fixed Point Theorem. More specifically, we prove the following theorem.

Proposition 3.1. Suppose here are constants $\alpha, \beta, \gamma \in \mathbb{R}^+$ such that the non-linearity $f \in C^1(\mathbb{R})$ satisfies

$$|f(x)| \leq \alpha + \beta|x|^\gamma, \quad \text{for all } x \in \mathbb{R}. \tag{C1}$$

Then (3.1) has a weak, not necessarily unique solution.

The proof requires some basic results first.

Lemma 3.1. Suppose that $u \in H_P^1(\Omega) \cap H^2(\Omega)$ such that

$$\int_\Gamma (\nabla u \cdot \vec{n})\, v = 0, \qquad \text{for all } v \in H_P^1(\Omega). \tag{3.2}$$

Then $u \in H_P^2(\Omega)$, i.e $\nabla u \in H_P^1(\Omega)^2$.

Proof. Let $\varphi \in C_c^\infty([0, L])$ so that φ can be considered as a function on the curves

$$\Gamma_1 = [0, L] \times \{0\}, \;\; \Gamma_2 = \{L\} \times [0, L], \;\; \Gamma_3 = [0, L] \times \{L\}, \Gamma_4 = \{0\} \times [0, L],$$

through the parametrizations

$$\alpha_1(t) = (t, 0), \;\; \alpha_2(t) = (L, t), \;\; \alpha_3(t) = (t, L), \;\; \alpha_4(t) = (0, t), \;\; t \in [0, L].$$

Now let $v \in H^1(\Omega)$ be such that $v_{|\Gamma_1} = v_{|\Gamma_3} = \varphi$ and $v_{|\Gamma_2} = v_{|\Gamma_4} = 0$, so that clearly we have $v \in H_P^1(\Omega)$. Then, equation (3.2) applied with this particular v reduces to

$$\int_0^L \left(\frac{\partial u}{\partial y}\Big|_{\Gamma_1}(t) - \frac{\partial u}{\partial y}\Big|_{\Gamma_3}(t) \right) \varphi(t) dt = 0.$$

Since $\varphi \in C_c^\infty([0,L])$ is arbitrary, we get that $\partial_y u_{|\Gamma_1} = \partial_y u_{|\Gamma_3}$. A similar argument shows that $\partial_x u_{|\Gamma_2} = \partial_x u_{|\Gamma_4}$ almost everywhere. On the other hand, we have that

$$\int_0^L \left(\frac{\partial u}{\partial x}\Big|_{\Gamma_1}(t) - \frac{\partial u}{\partial x}\Big|_{\Gamma_3}(t) \right) \varphi(t)dt = -\int_0^L \left(u_{|\Gamma_2}(t) - u_{|\Gamma_4}(t) \right) \varphi'(t)dt = 0,$$

where we have used that $u \in H_P^1(\Omega)$. Therefore, we obtain $\partial_x u_{|\Gamma_1} = \partial_y u_{|\Gamma_3}$ and an identical argument shows that $\partial_y u_{|\Gamma_2} = \partial_y u_{|\Gamma_4}$. This concludes the proof. □

We want to study the existence of solutions to this problem. We start with the following proposition.

Proposition 3.2. For every $g \in L^2(\Omega)$, there is a unique (weak) solution to the elliptic problem

$$\begin{cases} -\Delta u + u = g, \\ \quad u \in H_P^2(\Omega). \end{cases} \tag{3.3}$$

Proof. Multiplying (3.3) by $v \in H_P^1(\Omega)$, using the periodicity of the derivatives of u and Green's formula, we have that the variational formulation of (3.3) is

$$\langle u, v \rangle_{H^1} = \langle g, v \rangle_{L^2}, \quad \text{for all } v \in H_P^1(\Omega).$$

Therefore, the Lax-Milgram theorem applies and gives the unique solution $u \in H_P^1(\Omega)$ to (3.3). Elliptic regularity on convex polygons (ie Theorem 1.2) gives $u \in H^2(\Omega)$. Finally, Green's formula yields $\int_\Gamma (\nabla u \cdot \vec{n})v = 0$ for all $v \in H_P^1(\Omega)$, so that by the previous lemma one gets $u \in H_P^2(\Omega)$ as desired. □

Proof of Proposition 3.1. Let $\mathcal{E} : L^2(\Omega) \to H_P^2(\Omega)$ be the solution operator of (3.3) (ie $\mathcal{E} : g \mapsto u$). Then, consider the maps $\phi_{\hat{k}}, \phi_{\hat{c}} \in L^\infty(\Omega)$ and $\Psi_{\hat{k}} \in H_P^2(\Omega)$ by

$$\phi_{\hat{k}} : (\xi, \zeta) \mapsto -\hat{k}\xi, \quad \phi_c : (\xi, \zeta) \mapsto -c\xi, \quad \Psi_{\hat{k}} = \mathcal{E}(\phi_{\hat{k}}).$$

The maps ϕ_c and $\Psi_{\hat{k}}$ are bounded, and hence they are in $L^{2\gamma}(\Omega)$, where γ is given in (C1). Therefore, it makes sense to define the map $f_{\hat{k},c} : L^{2\gamma}(\Omega) \to L^2(\Omega)$

$$f_{\hat{k},c} : u \mapsto f\big(u(\cdot) + \phi_c(\cdot) + \Psi_{\hat{k}}(\cdot)\big).$$

Indeed, the range of $f_{\hat{k},c}$ is $L^2(\Omega)$ since

$$|f_{\hat{k},c}(u)|^2 = |f(u + \Psi_{\hat{k}} + \phi_c)|^2 \leq 4\alpha + 4\beta^2(u + \Psi_{\hat{k}} + \phi_c)^{2\gamma},$$

and therefore

$$\|f_{\hat{k},c}(u)\|_{L^2} \leq \sqrt{4\alpha|\Omega| + 4\beta^2\|u + \Psi_{\hat{k}} + \phi_c\|_{L^{2\gamma}}} < \infty, \quad \text{for all } u \in L^{2\gamma}(\Omega).$$

Furthermore, using Dominated Convergence and the continuity of f, one can show that $f_{\hat{k},c}$ is continuous.[(1)] With all of the maps defined above, we can now put equation HM-Travelin fixed point format on $H^2_P(\Omega)$. By defining $\Phi = \Psi - \Psi_{\hat{k}}$ and plugging in (HM-Travel), one has that

$$-\Delta\Phi + \Phi = f_{\hat{k},c}(\Phi), \quad \text{i.e} \quad \langle \Phi, v\rangle_{H^1} = \left\langle f_{\hat{k},c}(\Phi), v\right\rangle_{L^2}, \quad \forall v \in H^1_P(\Omega). \tag{3.4}$$

Now, consider the operator $\mathcal{K} : H^2_P(\Omega) \to H^2_P(\Omega)$ as a composition of the following maps

$$\mathcal{K} : H^2_P(\Omega) \xhookrightarrow{i} L^{2\gamma}(\Omega) \xrightarrow{f_{\hat{k},c}} L^2(\Omega) \xrightarrow{\mathcal{E}} H^2_P(\Omega),$$

where i is the continuous Sobolev embedding (Theorem , which is also compact by the Rellich-Kondrachov theorem. Therefore, we can deduce that $\mathcal{K}$ is continuous and compact since $f_{\hat{k},c}$ is continuous and $\mathcal{E}$ is continuous by elliptic regularity. Also, it is clear that Φ is a fixed point of $\mathcal{K}$, ie that $\Phi = \mathcal{K}(\Phi)$. Therefore, Shauder's Fixed Point Theorem guarantees the existence of Φ, and hence also of Ψ. □

Remark. The semi-linear elliptic equation

$$\begin{cases} -\Delta u(x) = f(x, u(x)), & x \in \Omega \subset \mathbb{R}^d, \\ \Omega \text{ bounded or unbounded}, & u \in H^1_0(\Omega), \end{cases} \tag{3.5}$$

has been thoroughly studied under a wide variety of assumptions on the non-linearity f, especially in dimension $d \geq 3$ where some of the above arguments fail due to the critical Sobolev exponent. For theoretical background about existence, regularity, and what assumptions on f are needed to guarantee them, one can refer to the books by Badiale and Serra [22] and Ambrosetti and Malchiodi [23]. Galerkin methods have also been applied in the study of (3.5) in [24], [25], and they have been shown to converge, for instance, when f' is strictly bounded between two successive eigenvalues of $-\Delta$.

3.3 Suggesting Newton-Galerkin numerical scheme

When dealing with semi-linear problems such as (3.4) and (3.5), one possible way to approximate the solution is through *Newton-Galerkin method* [26]–[28]. This method is roughly a discretization of the Newton method in Banach spaces, where the latter stipulates finding a root of some function $F : X \to Y$, for some Banach spaces X and Y, by using the usual Newton iterative procedure.

$$\text{Given } u_n \in X, \text{ seek } u_{n+1} \in X \text{ s.t } F'(u_n)(u_{n+1} - u_n) = -F(u_n), \tag{3.6}$$

[(1)] The operator $f_{\hat{k},c}$ is called the Nemistki operator associated with the function $f_{\hat{k},c}(x, u(x)) = f(u(x) + \Psi_{\hat{k}}(x) + \phi(x))$ for $x \in \mathbb{R}^2$.

where the function $F' : X \to L(X,Y)$ is the Fréchet derivative of F. The appeal of this method in the context of semi-linear equations is that it generates *linear* elliptic equations at each iterative step, as will be apparent in the following. But before we write down the procedure, let us point out important differences between the method suggested here and the one found in the literature.

1. The Newton-Galerkin method is usually applied to Dirichlet problems of the form (3.5), and so X is usually $H_0^1(\Omega)$, while in our setting we need $X = H_P^1(\Omega)$.

2. Most of the proofs of convergence are based on the form $-\Delta u = f(u)$ and not $(I - \Delta)u = f(u)$, as in our case.

3. A lot of the interesting results in the literature assume that $f(x,0) = 0$ for all $x \in \mathbb{R}^d$ in (3.5). This is because in that case, the (3.5) always has the trivial solution $u = 0$. However, this is not the case here since $f_{\hat{k},c}(x,0) = f(\Phi_{\hat{k}}(x) + \phi_c(x))$ which need not at all be zero for all $x \in \mathbb{R}^2$.

Ideally, we should be able to work with $H_P^1(\Omega)$ similarly to $H_0^1(\Omega)$ because both spaces can be identified with their duals through a boundary value operator. Indeed, the weak operator $I-\Delta : H_P^1(\Omega) \to H_P^1(\Omega)^\star$ defined by $(I-\Delta)(u)(v) = \langle u, v\rangle_{H^1}$ for all $u, v \in H_P^1(\Omega)$ is an isomorphism due to Riesz-Fréchet, so Point 1 needn't be a problem. However, Points 2 and 3 entail the need to modify the usual arguments used in the literature. In any case, in view of the Proposition 3.1, one is still encouraged to try to use the Newton-Galerkin method to the current setting due to how simple it is to implement in our context.

We now introduce the method, with it's goal being to approximate a solution of (3.4). To do so, we need that Φ is a root of some function between Banach spaces. Indeed, let $J : H_P^1(\Omega) \to \mathbb{R}$ be the functional given by

$$J(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^2}^2 - \int_\Omega F_{\hat{k},c}(u),$$

where $F_{\hat{k},c}(u) = F(u + \Psi_{\hat{k}} + \phi_c)$ and F is the anti-derivarive of the non-linearity $f_{\hat{k},c} \in C^1(\mathbb{R})$. Then one can show that J is Fréchet differentiable (cf. Section 1.3.2 of [22]) and that the Fréchet derivative $J' : H_P^1(\Omega) \to H_P^1(\Omega)^\star$ is given by

$$J'(u)v = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv - \int_\Omega f_{\hat{k},c}(u)v, \quad \text{for all } u, v \in H_P^1(\Omega). \tag{3.7}$$

Therefore, a function Φ satisfies (3.4) if and only if it satisfies $J'(\Phi) = 0$, and so we apply the Newton method in Banach spaces to approximate a root of $J' : H_P^1(\Omega) \to H_P^1(\Omega)^\star$. Hence, we need to compute the second derivative of J''. A similar argument to the one found in [22], [23] says that J' is Fréchet differentiable and that $J'' : H_P^1(\Omega) \to$

$L(H^1_P(\Omega), H^1_P(\Omega)^\star) \simeq \mathcal{B}(H^1_P(\Omega))$[2] is given by

$$J''(u)(v,w) = \int_\Omega \nabla v \cdot \nabla w + \int_\Omega vw - \int_\Omega f'_{\hat{k},c}(u)vw, \quad \text{for all } u,v,w \in H^1_P(\Omega). \tag{3.8}$$

Therefore, the Newton iteration is given as follows.

Newton procedure for finding roots of $J' : H^1_P(\Omega) \to H^1_P(\Omega)^\star$. Given an approximation $\Phi_n \in H^1_P(\Omega)$ of Φ, find an improved guess $\Phi_{n+1} \in H^1_P(\Omega)$ by solving the equation

$$J''(\Phi_n)(\Phi_{n+1} - \Phi_n) = -J'(\Phi_n), \;\text{ ie } J''(\Phi_n)(\Phi_{n+1} - \Phi_n)v = -J'(\Phi_n)v, \; \forall v \in H^1_P(\Omega),$$

and with J'' identified with a bilinear form as in (3.8) this means

$$J''(\Phi_n)(\Phi_{n+1}, v) = J''(\Phi_n)(\Phi_n, v) - J'(\Phi_n)(v), \quad \text{for all } v \in H^1_P(\Omega).$$

Then by using the definitions of J' and J'' from (3.7) and (3.8) respectively, one has that

$$\begin{cases} \text{Given } \Phi_n \in H^1_P(\Omega), \text{ seek } \Phi_{n+1} \in H^1_P(\Omega) \text{ such that for all } v \in H^1_P(\Omega), \\ \displaystyle\int_\Omega \nabla\Phi_{n+1} \cdot \nabla v + \int_\Omega \big(1 - f'_{\hat{k},c}(\Phi_n)\big)\Phi_{n+1} v = \int_\Omega \big(f_{\hat{k},c}(\Phi_n) - f'_{\hat{k},c}(\Phi_n)\Phi_n\big)v. \end{cases} \tag{3.9}$$

Combined with the Galerkin discretization, we obtain the following

$$\begin{cases} \text{Given } \Phi^{(h)}_n \in V^h_P(\Omega), \text{ seek } \Phi^{(h)}_{n+1} \in V^h_P(\Omega) \text{ such that for all } v \in V^h_P(\Omega), \\ \displaystyle\int_\Omega \nabla\Phi^{(h)}_{n+1} \cdot \nabla v + \int_\Omega \big(1 - f'_{\hat{k},c}(\Phi^{(h)}_n)\big)\Phi^{(h)}_{n+1} v = \int_\Omega \big(f_{\hat{k},c}(\Phi^{(h)}_n) - f'_{\hat{k},c}(\Phi^{(h)}_n)\Phi^{(h)}_n\big)v. \end{cases} \tag{3.10}$$

The last equation yields the so called *Newton-Galerkin* method. Before we showcase the application of this method to some specific periodic boundary value problems, let us point out some difficulties and limitations of this method. The first issue that comes to mind is in equation (3.9), and in particular the existence and uniqueness of Φ_{n+1} given Φ_n. This can be guaranteed for instance when

$$f' \leq 1, \quad \text{and therefore } f'_{\hat{k},c} \leq 1. \tag{C2}$$

so that Lax-Milgram applies in (3.9). Incidentally, this condition also implies that $J''(u) : H^1_P(\Omega) \to H^1_P(\Omega)^\star$ is in fact invertible, an essential condition for using convergence theorems of the Newton method.

[2] $\mathcal{B}(H^1_P(\Omega))$ is the collection of all continuous bilinear forms on $H^1_P(\Omega)$, which is in bijection with $L(H^1_P(\Omega), H^1_P(\Omega)^\star)$.

This brings us to the second problem to resolve and that is convergence of equations (3.6), (3.9), and (3.10). Let us remark first that for general non-linearities f, it is well known that the solutions of (3.4) and (3.5) are not unique, cf. [22], [23], [26], [27] and the references therein. This poses an important challenge to the Newton method, namely that the iterates might jump from the basin of attraction of one root to the basin of attraction of another. Besides the obvious need to choose an initial close enough to the desired root, a solution to this challenge is suggested in [27], [29], where damping is applied to each Newton iterate by multiplying the term $-F(u_n)$ in (3.6) with a suitable damping parameter δ_n to guarantee that one remains in the attractor of the sough after root.

Nevertheless, there are simple enough situations when (3.4) has a unique solution, for example when f (and thus $\widetilde{f}$) satisfies the following assumptions

$$f(t)t \leq 0, \quad \text{and} \quad (f(t) - f(s))(t-s) \leq 0, \quad \text{for all } s, t \in \mathbb{R}. \tag{C3}$$

In this case, the functional J becomes strictly convex and coercive (cf. Theorem 1.6.6 of [22]), and therefore it's critical point Φ is unique. For our purpose, we will use (3.10) to simulate Φ with f satisfying (C3).

3.4 Localised traveling waves on $\mathbb{R}^2$

In the physics literature, traveling wave solutions for the Hasegawa-Mima equation (1.3) on $\mathbb{R}^2$ traveling in the y-direction, called *Modons* or *Monopole/Dipole Vortices*, were obtained through a similar reasoning to that of Section 2.1. In fact, we have the following two equations appear in the literature

$$\begin{aligned} [4], [6] &: \quad -\Delta\Psi = f(\Psi - c\xi) - (v_0 + c)\xi, \qquad \text{on } \mathbb{R}^2, \qquad (3.11)\\ [21] &: \quad -\Delta\Psi + \Psi = f(\Psi - c\xi) - \xi. \qquad \text{on } \mathbb{R}^2. \qquad (3.12) \end{aligned}$$

where c is the wave velocity and v_0 is a constant depending on the particle density. We will compare these two equations with (HM-Travel), which was derived in Section 3.1, but we take the domain $\mathbb{R}^2$ instead of the strip, and we restate the problem as

$$-\Delta\Psi + \Psi = f(\Psi - c\xi) - \hat{k}\xi, \quad \text{on } \mathbb{R}^2. \tag{3.13}$$

For both equations (3.11) and (3.12), the solution Ψ was assumed to be *localized*, which according to Crotinger [30] means that $\Psi \to 0$ as $\zeta \to \pm\infty$. By fixing ξ and letting $\zeta \to \infty$ in either (3.11) or (3.12), all authors observed that f has to be linear. Implicitly assumed in their arguments is that $\Delta\Psi$ is continuous and $\lim_{\xi,\zeta\to\infty} \Delta\Psi = 0$, or else the conclusion is obviously not true. Let us start off this section by demonstrating that the same holds true under relaxed assumptions on Ψ.

Proposition 3.3. Choose $f \in C(\mathbb{R})$ satisfying (C1) and let $h \in L^1_{\text{loc}}(\mathbb{R}^2)$ be a function of the ξ-variable only. Suppose that a function $\Psi \in H^1(\mathbb{R}^2)$ satisfies the following

$$\int_{\mathbb{R}^2} \nabla\Psi \cdot \nabla v + \int_{\mathbb{R}^2} \Psi v = \int_{\mathbb{R}^2} \big(f(\Psi - c\xi) - h(\xi)\big)v, \quad \text{for all } v \in C_c^\infty(\mathbb{R}^2). \tag{3.14}$$

Then we have that

$$f(\xi) = h(-\xi/c), \quad \text{for all } \xi \in \mathbb{R}.$$

The reason why we chose $H^1(\mathbb{R}^2)$ is because for any $u \in H^1(\mathbb{R}^2)$, one has that '$\lim_{|\xi|,|\zeta|\to\infty} u(\xi,\zeta) = 0$'. But first we need to make precise the notion of 'limit at infinity' for functions in $H^1(\mathbb{R}^2)$.

Definition 3.1. For fixed $\xi \in \mathbb{R}$, let $\mathrm{Tr}_\xi : H^1(\mathbb{R}^2) \to L^1(\mathbb{R})$ be the *trace along lines* operator such that for all $\varphi \in C_c^\infty(\mathbb{R}^2)$ and all $\zeta \in \mathbb{R}$ one has that $\mathrm{Tr}(\varphi)(\zeta) = \varphi(\xi,\zeta)$.

Therefore, when we say "$\lim_{\zeta\to\infty} u(\xi,\zeta)$", it is to be understood in the sense of traces along lines. Let us now state an important property of the trace along lines for functions in $H^1(\mathbb{R}^2)$.

Theorem 3.1 (ACL characterisation)**.** Let $u \in H^1(\mathbb{R}^2)$. Then for almost all $\xi \in \mathbb{R}$, one has that $\mathrm{Tr}_\xi(u) \in H^1(\mathbb{R}) \subset C(\mathbb{R})$. This implies that $\lim_{\zeta\to\pm\infty} \mathrm{Tr}_\xi(u)(\zeta) = 0$ for almost all $\xi \in \mathbb{R}$.

The 'ACL' alias stands for *Abolutely Continuous on almost all Lines*. See Section 1.1.3 of the book by Maz'ya and Shaposhnikova [31] for a proof of this fact.

Lemma 3.2. Let $u \in H^1(\mathbb{R}^2)$. Then for any $E \subset \mathbb{R}^2$ of finite measure, there is a sequence $\{\zeta_n\}$ in $\mathbb{R}$ with $\zeta_n \to \infty$, and a function $\bar{u} \in H^1(\mathbb{R}^2)$ such that for all $n \in \mathbb{N}$ we have $|u(\cdot,\cdot+\zeta_n)| \le \bar{u}$ a.e on E.

Proof of Theorem 3.3. Choose an arbitrary function $\varphi \in C_c^\infty(\mathbb{R})$. Fix $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi > 0$ on the interior of it's support. Let $\psi \in C_c^\infty(\mathbb{R}^2)$ be given by $\psi(\xi,\zeta) = \varphi(\xi)\phi(\zeta)$ and let K be the support of ψ. With $u = \Psi$ and $E = K$ in the above lemma, we get a sequence $\{\zeta_n\}$ such and an function $\bar{\Psi} \in H^1(\mathbb{R}^2)$ such that $\zeta_n \to \infty$ and

$$|\Psi(\cdot,\cdot+\zeta_n)| \le \bar{\Psi}, \;\; \text{a.e on K}.$$

Now define $\phi_n \in C_c^\infty(\mathbb{R})$ and $\psi_n \in C_c^\infty(\mathbb{R}^2)$ by

$$\phi_n(\zeta) := \phi(\zeta - \zeta_n) \;\text{ and }\; \psi_n(\xi,\zeta) = \varphi(\xi)\phi_n(\zeta), \quad \text{for } (\xi,\zeta) \in \mathbb{R}^2.$$

Replace v by ψ_n in (3.14) to obtain

$$\int_{\mathbb{R}^2} \left(\frac{\partial\Psi}{\partial\xi}\phi_n\varphi' + \frac{\partial\Psi}{\partial\zeta}\varphi\phi_n' + \Psi\varphi\phi_n \right) = \int_{\mathbb{R}^2} \big(f(\Psi - c\xi) - h\big)\varphi\phi_n. \tag{3.15}$$

Using Fubini's theorem and integration by parts, we have that

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{\partial \Psi}{\partial \xi}\varphi' \phi_n &= -\int_{\mathbb{R}} \phi_n(\zeta)\left(\int_{\mathbb{R}} \frac{\partial \Psi}{\partial \xi}(\xi,\zeta)\varphi(\xi)d\xi\right)d\zeta \\ &= -\int_{\mathbb{R}} \phi_n(\zeta)\left(\int_{\mathbb{R}} \Psi(\xi,\zeta)\varphi''(\xi)d\xi\right)d\zeta \\ &= -\int_{\mathbb{R}^2} \varphi''\phi_n\Psi,\end{aligned}$$

and similarly one has that

$$\int_{\mathbb{R}^2} \frac{\partial \Psi}{\partial \zeta}\varphi\phi_n' = -\int_{\mathbb{R}^2} \Psi\varphi\phi_n'.$$

Note that in the above we have made constant use of the fact that the integrands have compact support and thus allowing us to cancel boundary terms. Putting everything together one obtains

$$-\int_{\mathbb{R}^2}\Big(\varphi''\phi_n + \varphi\phi_n'' - \varphi\phi_n\Big)\Psi = \int_{\mathbb{R}^2}\Big(f\big(\Psi - c\xi\big) - h(\xi)\Big)\varphi\phi_n.$$

Using the change of variables $(\xi, \zeta - \zeta_n) \to (\xi,\zeta)$ in the above equation and letting $\Psi_n(\xi,\zeta) = \Psi(\xi, \zeta+\zeta_n)$ yields

$$\begin{aligned}&-\iint_{\mathbb{R}^2}\big(\underbrace{\varphi''(\xi)\phi(\zeta) + \varphi(\xi)\phi''(\xi) - \varphi(\xi)\phi(\zeta)}_{:=F(\xi,\zeta)}\big)\Psi_n(\xi,\zeta)\,d\xi d\zeta \\ &\qquad = \iint_{\mathbb{R}^2}\underbrace{\phi(\zeta)\varphi(\xi)}_{:=G(\xi,\zeta)}\Big(f\big(\Psi_n(\xi,\zeta) - c\xi\big)\Big)\,d\xi d\zeta - \iint_{\mathbb{R}^2}\varphi\phi h(\xi)d\xi d\zeta\end{aligned} \tag{3.16}$$

We want to apply Dominated Convergence to be able to take limits in (3.16). This is is possible because

$$|F\Psi_n| = \mathbf{1}_K|F\Psi_n| \le \mathbf{1}_K \cdot \|F\|_{L^\infty}|\bar{\Psi}| \in L^1(\mathbb{R}^2),$$

and with assumption (C1) on f, we have that

$$|Gf(\Psi_n - c\xi)| = \mathbf{1}_K|Gf(\Psi_n - c\xi)| \le \|G\|_{L^\infty}\left(\alpha\mathbf{1}_K + \mathbf{1}_K\,\beta|\bar{\Psi}|^\gamma + \mathbf{1}_K\,\beta|c\xi|^\gamma\right) \in L^1(\mathbb{R}^2).^{(3)}$$

Hence, by taking the limits as $n \to \infty$ in (3.16) and applying Dominated Convergence

(3) Recall that $\bar{\Psi} \in H^1(\mathbb{R}^2)$, and therefore $\bar{\Psi} \in L^\gamma(\mathbb{R}^2)$ for any $\gamma \ge 2$ by Sobolev embeddings.

we get

$$
\begin{aligned}
0 &= \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(\zeta)\Big(f\big(\underbrace{\lim_{n\to\infty}\Psi(\xi,\zeta+n)}_{=0}-c\xi)\big)-h(\xi)\Big)\varphi(\xi)d\xi d\zeta \\
&= \underbrace{\int_{\mathbb{R}}\phi(\zeta)d\zeta}_{\neq 0}\cdot\int_{\mathbb{R}}\big(f\big(-c\xi\big)-h(\xi)\big)\varphi(\xi)d\xi.
\end{aligned}
\tag{3.17}
$$

Since φ is chosen arbitrarily, it follows that $f(-c\xi)-h(\xi)=0$ as desired. □

Since equations (HM-Travel), (3.11), and (3.12) can be put in variational formulation in similar form to (3.14), Theorem 3.3 tells us that f is in fact linear. In the case of equation (3.13) we have that $f(\xi)=-\hat{k}\xi/c$ for all $\xi\in\mathbb{R}$, and therefore this equation becomes

$$
\Delta\Psi=(1+\hat{k}/c)\Psi, \qquad \text{on } \mathbb{R}^2. \tag{3.18}
$$

In other words, we have that Ψ is an eigenvalue of Δ with eigenvalue $\lambda=(1+\hat{k}/c)$. Eigenvalues of the unbounded Laplace operator $\Delta: W^{2,p}(\mathbb{R}^2)\subset L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)$ for $p>2n/(n-1)$ are characterized in the following theorem due to Talenti [32].

Theorem 3.2. Consider the operator $\Delta: W^{2,p}(\mathbb{R}^d)\subset L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)$ with $p>2n/(n-1)$. If $\lambda<0$ and Y_k is any spherical harmonic[(4)] of degree k, the the problem $\Delta u=\lambda u$ has a solution of the form

$$
u_\lambda(x)=\left(|\lambda|^{\frac{1}{2}}|x|\right)^{-\frac{d}{2}+1} J_{\frac{d}{2}+k-1}\left(|\lambda|^{\frac{1}{2}}|x|\right)Y_k\left(\frac{x}{|x|}\right),\quad x\in\mathbb{R}^d.^{(5)} \tag{3.19}
$$

Applying this to our case, one sees that the solution Ψ of equation (3.18) with the condition $1+\hat{k}/c<0$ is simply given by

$$
\Psi_k(\eta)=J_k\left(|1+\hat{k}/c|^{\frac{1}{2}}|\eta|\right)Y_k\left(\frac{\eta}{|\eta|}\right),\quad \eta=(\xi,\zeta)\in\mathbb{R}^2. \tag{3.20}
$$

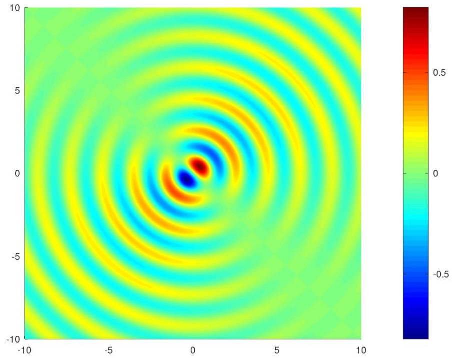
In Figure (3.1), we plot two different eigenfunctions Ψ's for the same eigenvalue $(1+\hat{k}/c)$ with $\hat{k}=12$ and $c=-1$, which are values we constantly use in the simulations that are presented in Chapter 4. In Figures 3.1a and 3.1b, we plot the eigenfunctions Ψ_1 and Ψ_2 corresponding to the eigenvalue $\lambda=(1-k/c)=-11$ and to the spherical harmonics

$$
Y_1=\frac{x+y}{\sqrt{x^2+y^2}},\quad Y_2=\frac{10x^2+2xy-10y^2}{x^2+y^2},
$$

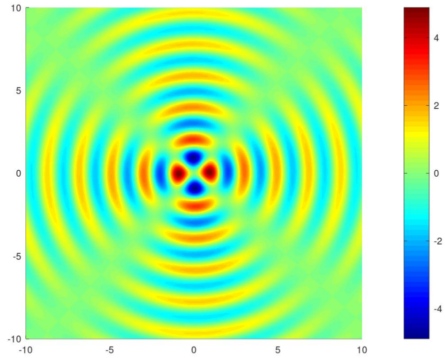
respectively.

[(4)] A spherical harmonic Y_k of degree k is such that $|x|^k Y_k(x/|x|)$ is a homogeneous and harmonic polynomial of degree k.

[(5)] J_k is the Bessel function of the first kind.



(a) Eigenfunction Ψ_1 corresponding to Y_1.



(b) Eigenfunction Ψ_2 corresponding to Y_2.

Figure 3.1: Examples of solutions of (3.18) corresponding to eigenvalue $\lambda = -11$

Remark. The eigenfunctions u_λ obtained in Theorem 3.19 are not in $L^2(\mathbb{R}^2)$ for any $\lambda < 0$ and therefore do not have finite energy norm.

Chapter 4

Algorithm and Numerical Simulations

4.1 Semi-linearized approach for fully-implicit scheme

In this section we describe the algorithm used for all of the simulations in the the thesis. In order to solve the fully implicit equation (2.5), we use the following semi-linear approach to obtain a simple computational formulation.

Computational Procedure. Given a mesh size h, time step τ, and initial data $u_0 \in H^2_P(\Omega)$, seek a sequence of pairs $\{(u_h^{(j)}, w_h^{(j)}) \in V_P^h(\Omega) \times V_P^h(\Omega) : j = 0, 1, 2, \ldots\}$ such that $u_h^{(0)} = \pi_h u_0$, for all $j = 0, 1, 2, \ldots$, and for all $v \in V_P^h(\Omega)$ one has

$$\begin{cases} \left\langle w_h^{(j+1)}, v \right\rangle_{L^2} - \tau \left\langle \vec{V}(u_h^{(j)}) \cdot \nabla v, w_h^{(j+1)} \right\rangle_{L^2} = \left\langle w_h^{(j)}, v \right\rangle_{L^2} + \tau \hat{k} \left\langle \partial_y u_h^{(j)}, v \right\rangle_{L^2}, \\ \left\langle u_h^{(j+1)}, v \right\rangle_{H^1} = \left\langle w_h^{(j+1)}, v \right\rangle_{L^2}, \quad \text{for all } v \in V_P^h(\Omega). \end{cases} \tag{4.1}$$

In matrix form, this is equivalent to the following. Given U_0 to be the component of the vector of $\pi_h(u_0)$ in $V_P^h(\Omega)$, seek $\{(U_h^{(j)}, W_h^{(j)}) \in \mathbb{R}^{d_h} \times \mathbb{R}^{d_h} : j = 0, 1, 2, \ldots\}$ such that $U_h^0 = U_0$ and for all $j = 0, 1, 2, \ldots$ one has that

$$\begin{cases} \left(M - S(U_h^{(j)})\right) W_h^{(j+1)} = M W_h^{(j)} + \tau \hat{k} R U_h^{(j)}, \\ K U_h^{(j)} = M W_h^{(j)}. \end{cases} \tag{4.2}$$

Remark. We also have a priori estimates for (4.1) as long as $j\tau \in [0, T]$. In fact, by plugging in $v = w_h^{(j+1)}$ in (4.1), using Theorem 2.2 to cancel the non-linear term, and then using Cauchy-Shwarz, one obtains

$$\|w_h^{(j+1)}\|_{L^2} \leq \|w_h^{(j)}\|_{L^2} + \tau \hat{k} \|u_h^{(j)}\|_{H^1} \leq (1 + \tau \hat{k}) \|w_h^{(j)}\|_{L^2} \leq C e^{\hat{k}T} \|w_0\|_{L^2}. \tag{4.3}$$

This computational procedure (4.1) is implemented in the software FreeFem++ [33] and the plots are generated in Octave through the ffmatlib.

4.2 Testing Newton-Galerkin method

In this section we test the proposed Newton-Galerkin method through (3.10), which we restate here for the sake of convenience. First of all, let $v = \sum_{j=1}^{d} v_j \varphi_j \in V_P^h(\Omega)$ where $\varphi_j \in \mathcal{B}_h$ and $\mathcal{B}_h$ is the basis for $V_P^h(\Omega)$ defined in Section 2.1, and $v_j \in \mathbb{R}$. For a function $f : \mathbb{R} \to \mathbb{R}$, we define a function $f_h : V_P^h(\Omega) \to V_P^h(\Omega)$ by

$$f_h(v) = \sum_{j=1}^{d} f(v_j)\varphi_j.$$

Computational Procedure. Given an initial guess $\Phi_0 \in H_P^2(\Omega)$ of a function Φ that solves (3.4) and $\Phi_{h,0} = \pi_h \Phi_0$, seek $\Phi_{h,n+1} \in V_P^h(\Omega)$ such that for all $v \in V_P^h(\Omega)$ one has that

$$a_{h,n}(\Phi_{h,n+1}, v) = F_{h,n}(v). \tag{4.4}$$

where

$$a_{h,n}(\Phi_{h,n+1}, v) = \int_\Omega \nabla\Phi_{h,n+1} \cdot \nabla v + \int_\Omega (1 - f_h'(\Phi_{h,n}))\Phi_{h,n+1} v,$$

and

$$F_{h,n}(v) = \int_\Omega \big(f_h(\Phi_{h,n}) - f_h'(\Phi_{h,n})\Phi_{h,n}\big)v,$$

and

$$f_h(\cdot) := f_h\big(\,\cdot + \pi_h(\Psi_{\hat{k}}) + \pi_h(cx)\big), \quad f_h'(\cdot) := f_h'\big(\,\cdot + \pi_h(\Psi_{\hat{k}}) + \pi_h(cx)\big).$$

Afterwards, we plot $\Psi_{n,h} = \Phi_{n,h} + \pi_h \Psi_{\hat{k}}$,

Example 1. We start with the function $f(x) = -\arctan(x)$ since f satisfies conditions (C1), (C2) and (C3). Let us point out first that in this case, the solution Ψ is unique by the discussion in Section 3.3. With mesh size $h = 1/64$ and the initial guess $\Phi_0^h = 0$, after about 4 iterations the isovalues of stop changing.

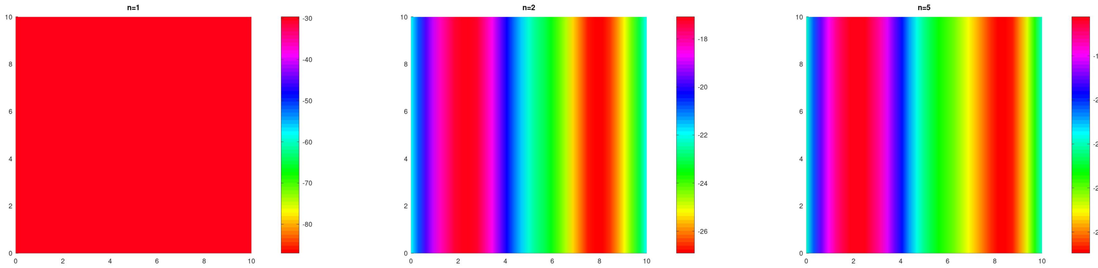


Figure 4.1: Newton-Galerkin method applied to $-\Delta\Psi + \Psi = -\arctan(\Psi - c\xi) - \hat{k}\xi$.

Example 2. With this example, we purposely pick a function $f(x)$ that does not satisfy (C3), but we still obtain convergence of the Newton-Galerkin method in some cases. Namely, we choose $f(x) = -x^2$. With $\Omega = [0,4] \times [0,4]$, mesh size $h = 1/64$ and the initial guess $\Phi_0 = 0$, after about 15 iterations the isovalues of stop changing. The plots are in Figure 4.2.

Example 3. With this example we intend to demonstrate the chaotic behaviour of the Newton if condition (C2) is omitted. Pick $f(x) = -x^3$ and $\Omega = [0,1] \times [0,1]$ with $h = 1/64$. This function satisfies (C1) and (C3) but not (C2), and therefore the Newton iterates are not guaranteed to have a solution. Starting with the initial guess $\Phi_0 = 0$, it becomes apparent that even after 12 iteration no convergence appears. The plots are seen in Figure 4.3.

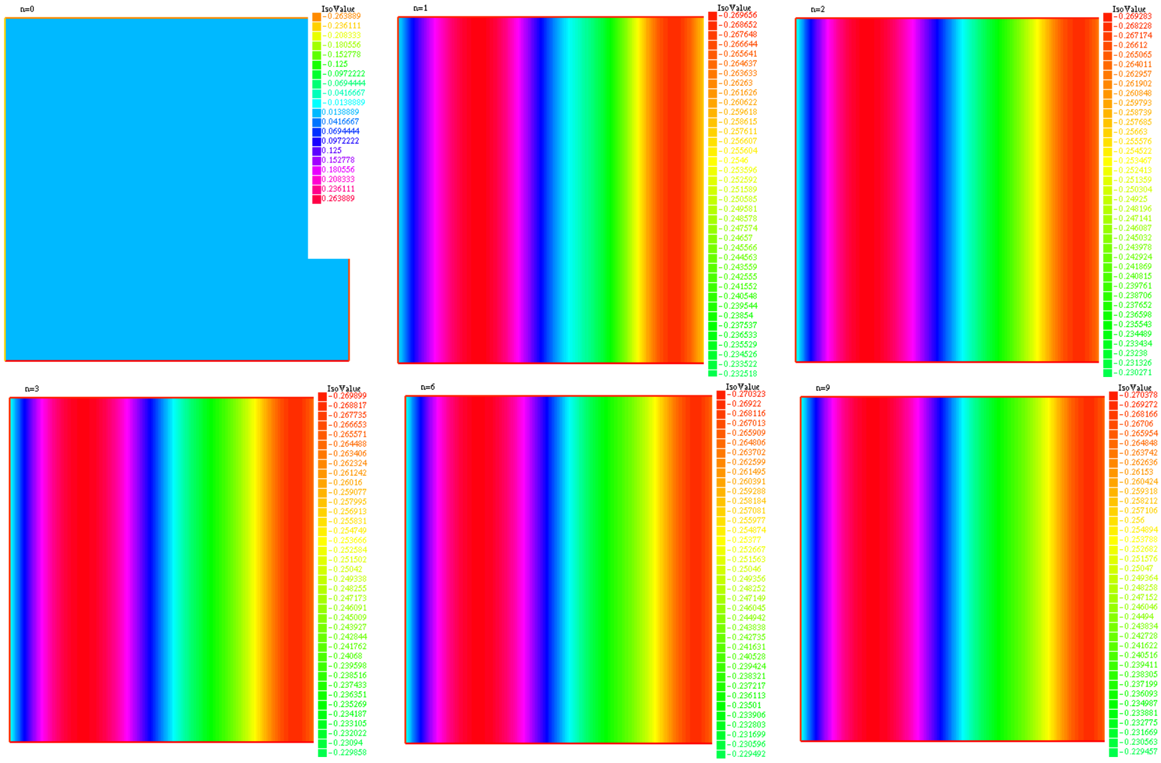


Figure 4.2: Newton-Galerkin method applied to $-\Delta\Psi + \Psi = -(\Psi - c\xi)^2 - \hat{k}\xi$.

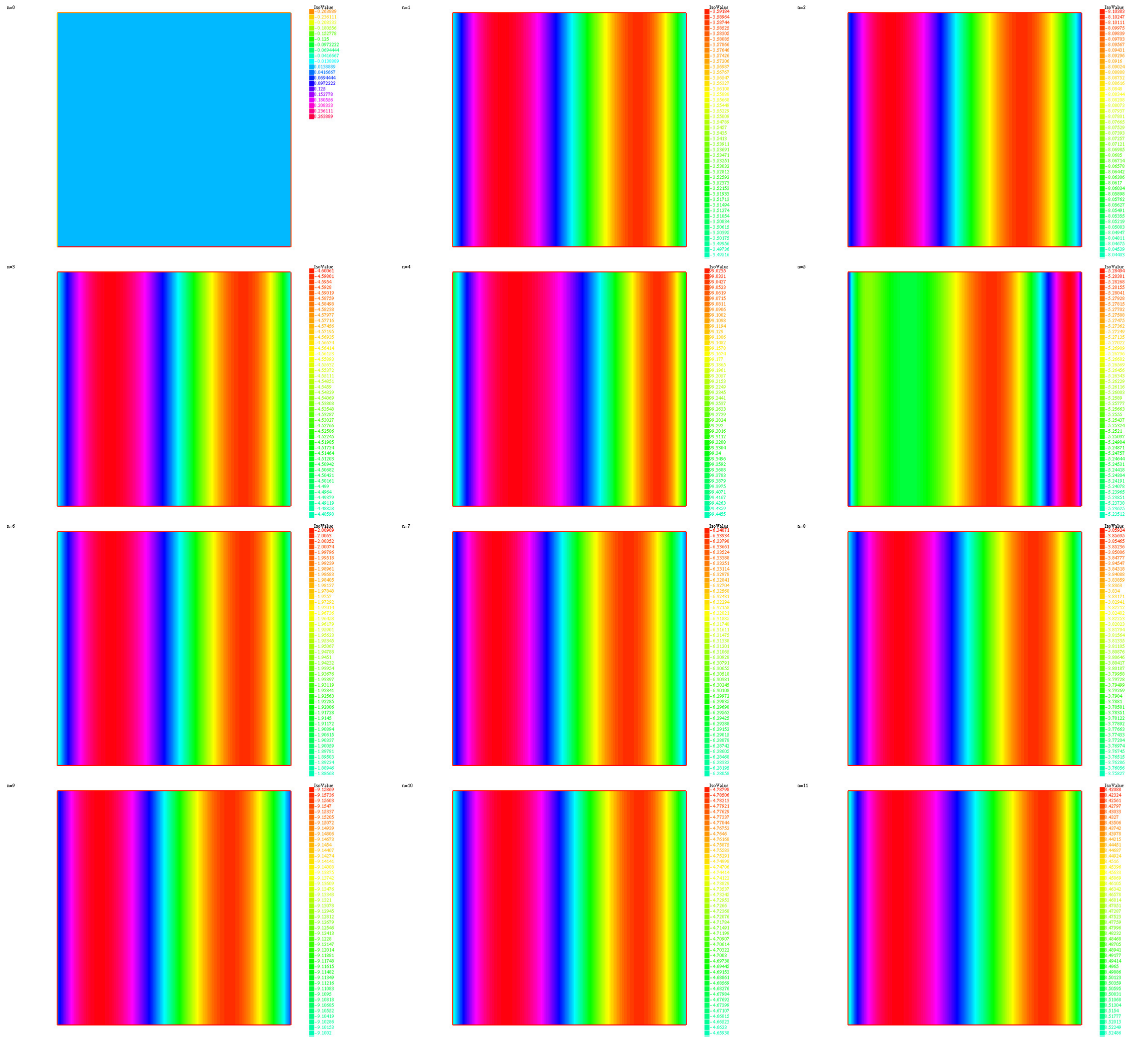


Figure 4.3: Newton-Galerkin method applied to $-\Delta\Psi + \Psi = -(\Psi - c\xi)^3 - \hat{k}\xi$.

4.3 Simulation of traveling waves

Using the Newton-Galerkin method defined in the previous section, we approximate Ψ and check that if Ψ given as initial data to (4.1) then one indeed obtains a traveling wave.

Example 1. We choose $\Omega = [0, 10] \times [0, 10]$, mesh size $h = 64$, time step $\tau = 0.1$, and $f(x) = \arctan(x)$ in (4.1). The obtained solution Ψ is then given as initial data. For $t \in [0, 10]$, the simulation does not change, so it either stationary or traveling in the y-direction. After adding noise at $t = 11$, the traveling wave behaviour becomes apparent to the naked eye, implying that u was in fact traveling in the y-direction. See Figure 4.4 for the simulation.

Example 2. We choose $\Omega = [0, 4] \times [0, 4]$, mesh size $h = 64$, time step $\tau = 0.2$, and $f(x) = \arctan(x)$ in (4.1). The obtained solution Ψ is then given as initial data. The same exact phenomena as Example 1 appears after noise is added at around $t = 10$. See Figure 4.5 for the simulation.

Example 3. We choose $\Omega = [0, 1] \times [0, 1]$, mesh size $h = 64$, time step $\tau = 0.2$, and $f(x) = -x^3$ in (4.1). Notice that this function does not satisfy (C2). The obtained solution Ψ is then given as initial data. Even though the initial data was not obtained from converging Newton iterates, the simulated function still exhibits traveling waves behavior. See Figure 4.6 for the simulation.

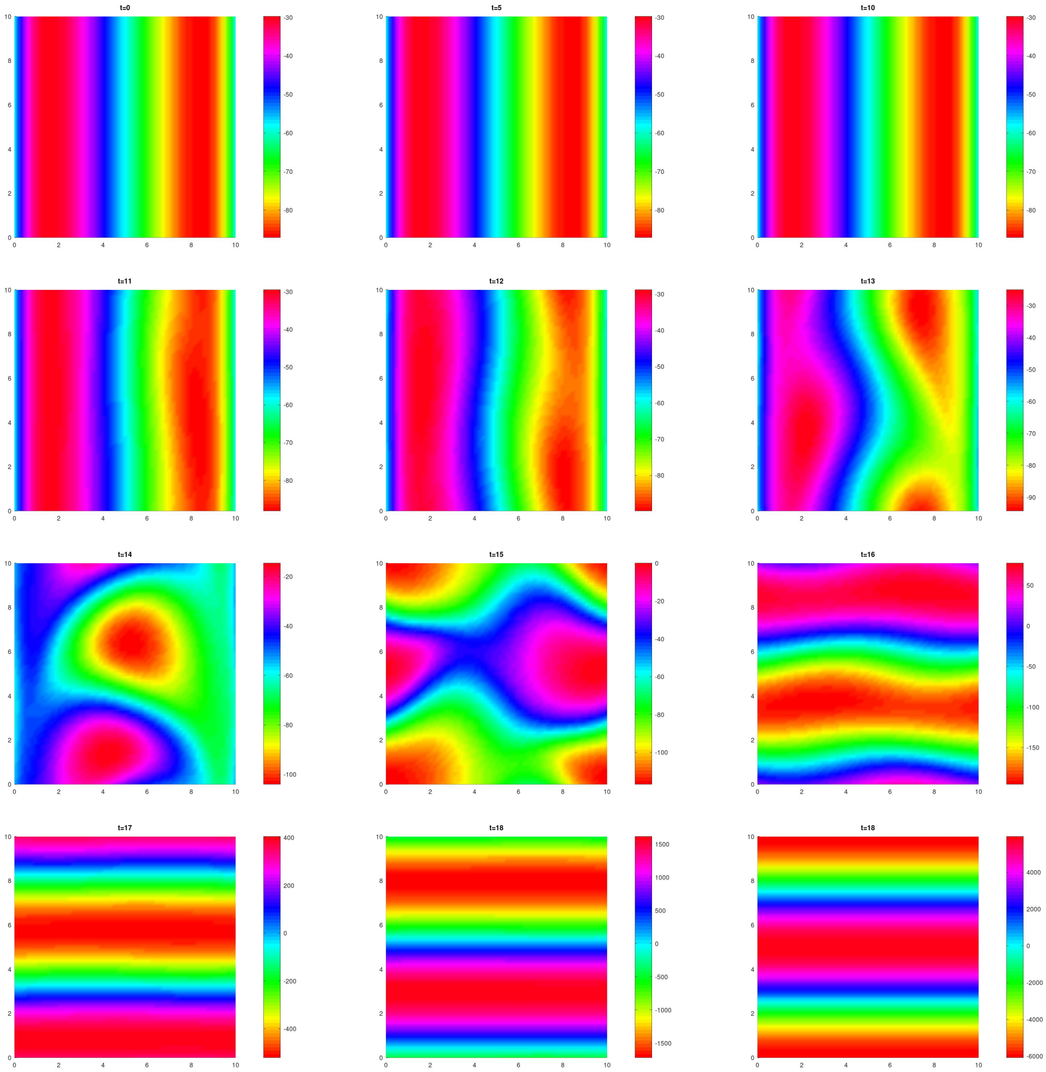


Figure 4.4: Simulation of traveling wave on $\Omega = [0, 10] \times [0, 10]$ with $h = 1/64$ and with with initial data $u_0 = \Psi_{h,4}$ obtained by applying 4 Newton-Galerkin iterations to the equation $-\Delta\Psi + \Psi = \arctan(\Psi - c\xi) - \hat{k}\xi$.

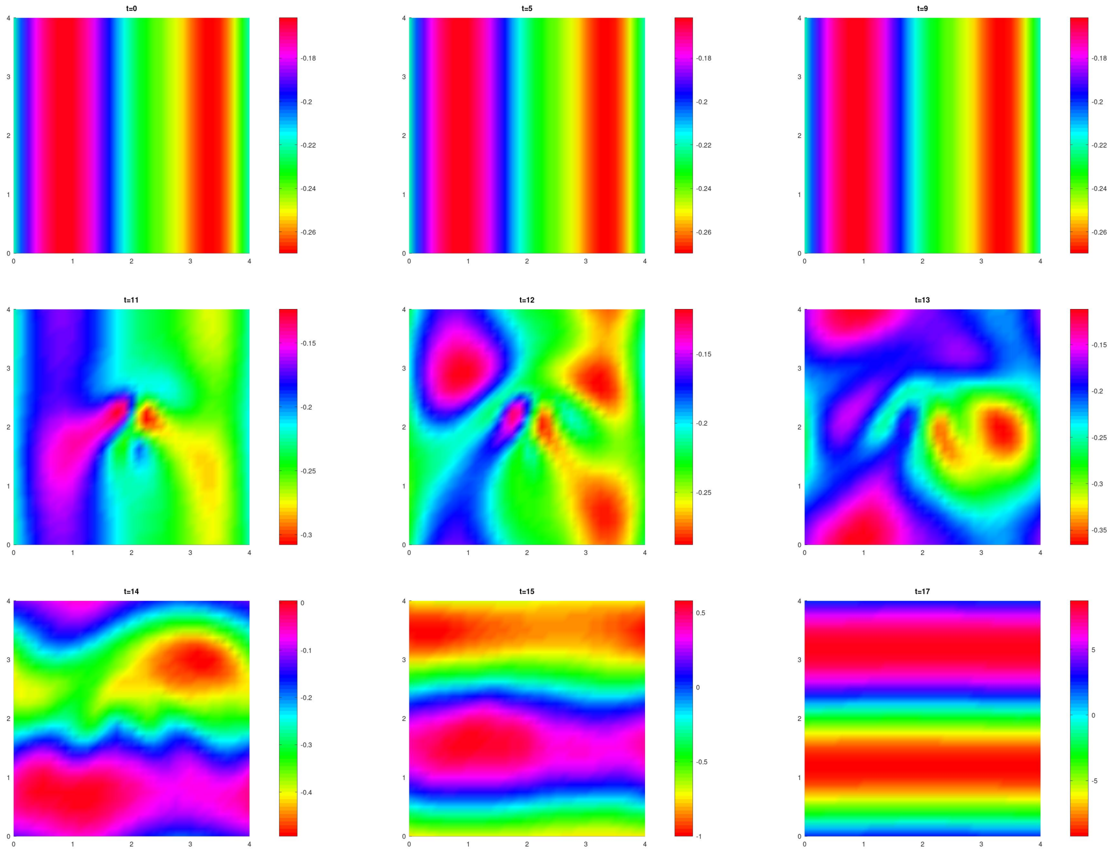


Figure 4.5: Simulation of traveling wave on $\Omega = [0,4] \times [0,4]$ with $h = 1/64$ and with with inital data $\Psi_{h,15}$ obtained by applying 15 Newton-Galerkin interations to the equation $-\Delta\Psi + \Psi = -(\Psi - c\xi)^2 - \hat{k}\xi$.

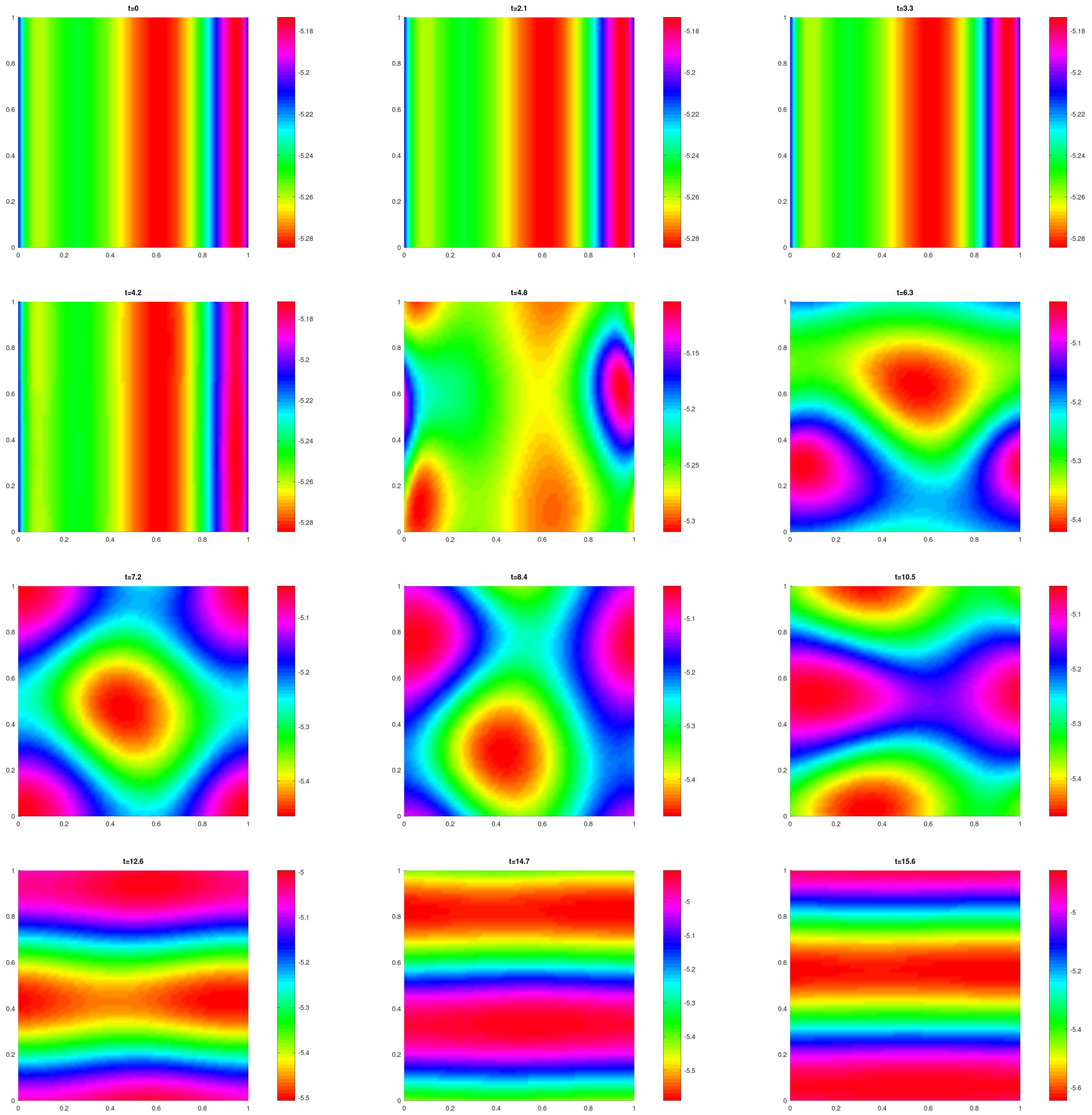


Figure 4.6: Simulation of traveling wave on $\Omega = [0, 1] \times [0, 1]$ with $h = 1/64$ and with with inital data $\Psi_{h,5}$ obtained by applying 5 Newton-Galerkin interations to the equation $-\Delta\Psi + \Psi = -(\Psi - c\xi)^2 - \hat{k}\xi$.

Chapter 5

Concluding Remarks

We summarize Here the work done this thesis.

In Chapter 2, using similar methods to [17], we have proven the existence of a sequence of pairs $\{(u_n, w_n)\}$ solving (2.5) with $h = h_n = \sqrt{2}L/n$ and $\tau = \tau_n \in \Theta(h_n)$ that converge weakly to a solution pair (u, v) of the time integral FE formulation of (1.11). However, in the simulations we have used a semi-linearized version of (2.5) which is given by (4.1). Since we also have a priori estimates for (4.1) given in (4.3), one could work out a similar argument to that in Section 2.2 to show that an appropriate sequence of solutions $\{(\tilde{u}_n, \tilde{w}_n)\}$ to (4.1) converges to a solution of (1.11).

In Chapter 3, we have obtained periodic traveling waves of the Hasegawa-Mima equation by analyzing the semi-linear elliptic equation given in (3.1), for which we have shown existence in Proposition 3.1, following similar arguments to the ones in the literature [22], [23]. We have used periodicity to show that we only need to solve for the initial data on the square $[0, L] \times [0, L]$, and then we suggested the use Newton-Galerkin method to numerically approximate the initial data. However, no proofs for convergence were given to that method. Furthermore, as already mentioned, one needs to apply appropriate damping to the Newton iterates and choose appropriate initial guess to guarantee convergence. This could be done by employing some of the ideas presented in [26], [27], [29].

Also, we have not been able to obtain similar traveling wave behavior to the one suggested in [4], [6], [21]. Therefore, more investigation should be done in this matter by choosing f, $\hat{k}$ and c appropriately in (3.1) to obtain dipole vortex structures or Modons or more generally any trapped structure.

Bibliography

[1] I. P. E. G. on Confin Transport, I. P. E. G. on Confin Database, and I. P. B. Editors, "Chapter 2: Plasma confinement and transport," *Nuclear Fusion*, vol. 39, no. 12, pp. 2175–2249, Dec. 1999. DOI: 10.1088/0029-5515/39/12/302. [Online]. Available: https://doi.org/10.1088%2F0029-5515%2F39%2F12%2F302.

[2] (). "Beyond iter, milestones around the world," [Online]. Available: https://www.iter.org/sci/BeyondITER.

[3] A. Hasegawa and K. Mima, "Pseudo-three-dimensional turbulence in magnetized nonuniform plasma," *The Physics of Fluids*, vol. 21, no. 1, pp. 87–92, 1978. DOI: 10.1063/1.862083.

[4] F. A. Hariri, "Simulating bi-dimensional turbulent plasmas using the hasegawa-mima model," M.S. thesis, American University of Beirut, 2010.

[5] V. Larichev and G. Reznik, "Two-dimensional rossby soliton: An exact solution," *Rep. USSR Acad. Sci.*, vol. 231, pp. 1077–1079, Jan. 1976.

[6] J. Nycander, "New stationary vortex solutions of the hasegawa–mima equation," *Journal of Plasma Physics*, vol. 39, no. 3, pp. 413–430, 1988. DOI: 10.1017/S0022377800026738.

[7] B. Shivamoggi, "Charney-hasegawa-mima equation: A general class of exact solutions," *Physics Letters A*, vol. 138, pp. 37–42, 1989.

[8] R. Grauer, "An energy estimate for a perturbed hasegawa–mima equation," *Nonlinearity*, vol. 11, May 1998. DOI: 10.1088/0951-7715/11/3/014.

[9] L. Paumond, "Some remarks on a hasegawa-mima-charney-obukhov equation," *Physica D Nonlinear Phenomena*, vol. 195, pp. 379–390, Aug. 2004. DOI: 10.1016/j.physd.2004.04.005.

[10] B. Guo and Y. Han, "Existence and uniqueness of global solution of the hasegawa–mima equation," *Journal of Mathematical Physics*, vol. 45, no. 4, pp. 1639–1647, 2004. DOI: 10.1063/1.1667607. eprint: https://doi.org/10.1063/1.1667607. [Online]. Available: https://doi.org/10.1063/1.1667607.

[11] M. Hounkonnou and M. Kabir, “Hasegawa-mima-charney-obukhov equation: Symmetry reductions and solutions,” *Int. J. Contemp. Math. Sciences*, vol. 3, pp. 145–157, Jan. 2008.

[12] G. Boling and H. Daiwen, “Existence and stability of steady waves for the hasegawa-mima equation,” *Boundary Value Problems*, vol. 2009, Jan. 2009. DOI: `10.1155/2009/509801`.

[13] H. Karakazian, “Local existence and uniqueness of the solution to the 2d hasegawa-mima equation with periodic boundary conditions,” M.S. thesis, American University of Beirut, Feb. 2016.

[14] H. Karakazian and N. Nassif, “Local existence of an h_p^3 solution to the hasegawa-mima plasma equation,” Apr. 2019.

[15] H. Gao and A. Zhu, “The global strong solutions of hasegawa-mima-charney-obukhov equation,” *Journal of Mathematical Physics*, vol. 46, no. 8, p. 083 517, 2005. DOI: `10.1063/1.2008208`. eprint: `https://doi.org/10.1063/1.2008208`. [Online]. Available: `https://doi.org/10.1063/1.2008208`.

[16] C. X. Guo, C. Y. Li, and Y. Q. Han, “Dynamical behaviors of stochastic hasegawa-mima equation in torus,” *Journal of Mathematical Physics*, vol. 59, no. 2, p. 021 508, 2018. DOI: `10.1063/1.5006446`. eprint: `https://doi.org/10.1063/1.5006446`. [Online]. Available: `https://doi.org/10.1063/1.5006446`.

[17] H. Karakazian, S. Moufawad, and N. Nassif, “A robust computational model for the hasegawa-mima wave equation,” 2020.

[18] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, ser. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 2011.

[19] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, ser. Universitext. Springer New York, 2010.

[20] P. Ciarlet, *The Finite Element Method for Elliptic Problems*, ser. ISSN. Elsevier Science, 1978, ISBN: 9780080875255. [Online]. Available: `https://books.google.com.lb/books?id=TpHfoXnpKvAC`.

[21] J. Crotinger and T. Dupree, “Trapped structures in drift wave turbulence,” Ph.D. dissertation, Massachusetts Institute of Technology, Sep. 1992. DOI: `10.1063/1.860160`.

[22] M. Badiale and E. Serra, *Semilinear Elliptic Equations for Beginners*. Springer, 2011, ISBN: 9780857292285. [Online]. Available: `https://books.google.com.lb/books?id=LXkBswEACAAJ`.

[23] A. Ambrosetti and A. Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2007, ISBN: 9780521863209. [Online]. Available: https://books.google.com.lb/books?id=4O6tfmZtxQAC.

[24] E. M. Harrell and W. J. Layton, "L2 estimates for galerkin methods for semilinear elliptic equations," *SIAM Journal on Numerical Analysis*, vol. 24, no. 1, pp. 52–58, 1987, ISSN: 00361429. [Online]. Available: http://www.jstor.org/stable/2157384.

[25] M. H. Schultz, "L2 error bounds for the rayleigh-ritz-galerkin method," *SIAM Journal on Numerical Analysis*, vol. 8, no. 4, pp. 737–748, 1971, ISSN: 00361429. [Online]. Available: http://www.jstor.org/stable/2949603.

[26] J. Neuberger and J. Swift, "Newton's method and morse index for semilinear elliptic pdes," *International Journal of Bifurcation and Chaos*, vol. 11, pp. 801–820, Mar. 2001. DOI: 10.1142/S0218127401002444.

[27] M. Amrein and T. Wihler, "Fully adaptive newton-galerkin methods for semilinear elliptic partial differential equations," *SIAM Journal on Scientific Computing*, vol. 37, Jul. 2015. DOI: 10.1137/140983537.

[28] G. Arbaciauskas, "Application of finite volume and finite element methods to distributed optimal control of semi-linear elliptic equations," M.S. thesis, TU Delft, Jan. 2016.

[29] M. Amrein and T. Wihler, "An adaptive newton-method based on a dynamical systems approach," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, pp. 2958–2973, Sep. 2014. DOI: 10.1016/j.cnsns.2014.02.010.

[30] J. Crotinger, "Simulation of drift wave turbulence: Trapped structures and a new nonadiabatic electron model," 1989.

[31] V. Maz'ya and T. Shaposhnikova, *Sobolev Spaces: with Applications to Elliptic Partial Differential Equations*, ser. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2011, ISBN: 9783642155642. [Online]. Available: https://books.google.com.lb/books?id=vNx2I5xu1AAC.

[32] G. Talenti, "Spectrum of the laplace operator acting in l p (r n)·," Feb. 1971.

[33] F. Hecht, "New development in freefem++," *J. Numer. Math.*, vol. 20, no. 3-4, pp. 251–265, 2012, ISSN: 1570-2820. [Online]. Available: https://freefem.org/.