AMERICAN UNIVERSITY OF BEIRUT

INTEGRABLE GENERATORS OF LIE ALGEBRAS OF VECTOR FIELDS ON THE KORAS–RUSSELL CUBIC THREEFOLD

by
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My journey in AUB started during the fall semester of the academic year 2020–2021. It wasn't a conventional semester, given that a pandemic was roaming on Earth, so we were forced to proceed the academic year virtually. In my first semester I enrolled in two courses, Complex Analysis (Math 304) and Rings and Modules (Math 341). The analysis course was taught by professor Faruk Abi-Khuzam. Dr. Abi-Khuzam taught me a great deal. He used to tell us in his recordings that one must get their hands "dirty" while studying analysis, and by "dirty" he meant that we must solve many computational exercises in order to get a feeling out of the theory. I would like to thank Dr. Abi-Khuzam for his efforts for teaching me a wonderful theory and for giving me advice for tackling tough problems. Dr. Khuri-Makdisi was my Algebra instructor. I can talk about him all day if I like to. In our very first lecture he told us something that I will never forget. He told us that in graduate school you have to start building "muscles", and by "muscles" he meant "mathematical muscles" which means we have to solve every single problem that we can get our hands on, it also means that we have to read the textbook daily. I would like to thank Dr. Khuri-Makdisi for his advice, office hours and for his support in my academic journey. Moreover, Dr. Khuri-Makdisi was one of the committee members in my master's thesis. During my defense, when I struggled to answer his question, he told me something that I will keep under my ears till the rest of my days, he said something along this line "You should reach a point in your career that even if a tiger is chasing you, you should answer questions without hesitation".

We move forward to the spring semester of 2021. I enrolled in two courses, Measure Theory (Math 303) and Calculus on Manifolds (Math 306), these courses were taught by professor Bassam Shayya and professor Giuseppe Della Sala. Dr. Shayya in my opinion deserves the title "the great explainer".
I don't recall an instant during his lectures that I didn't understand a concept, he talked like it was, as he would say, "piece of cake". I would like to thank Dr. Shayya for all his advice and efforts, my love for analysis is due to him. Dr. Della Sala started his lecture with the following sentence, "Why do you think there are people who believe the earth is flat?". He answered this question in a few lectures by explaining the concept of a manifold, which are mathematical objects that look like $\mathbb{R}^n$ locally. Dr. Della Sala is a kind soul, who helped me a lot

these past couple of years, especially during his office hours and as a committee member to my master's thesis, he has my gratitude.

Fall 2021 was the "back to normal semester", we came back to campus, so the courses were no longer online. I took two courses, Stochastic Processes (Math 338) and Discrete Models for PDEs (Math 350). These courses were taught by professor Abbas Alhakim and professor Nabil Nassif. Dr. Alhakim is one the most interesting human being I have ever met. He's a probabilist and also a great poet. I was so fascinated by this course, which made me have at least ten new questions in each lecture. Thank you Dr. Alhakim for making me appreciate the theory of probability. Dr. Nassif, one of the most optimistic person I know, taught my first graduate applied math course. Honestly, I wasn't a big fan of applied mathematics. However, Dr. Nassif explained in a way that made me appreciate the field of numerical PDEs. His programming assignments were so helpful in getting a feel out of the theoretical part of the course. I would like to thank Dr. Nassif for his optimistic advice, and for all the pink scratch papers he gave me while explaining a concept on them with a pencil.

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I would like to thank my friends and family for helping me out all these years during my academic journey. From undergrad to grad school, they had my back and gave me their support. I want to thank my mom, Hasmik Daniel, without her, this journey would had been a lot more difficult. Thanks mom!

Dr. Bertrand was my academic advisor. His academic advice and leadership were so helpful in my stay at the mathematics department, so I would like to thank him for all his efforts.

Finally, the most important acknowledgment goes to my thesis advisor, Dr. Rafael Andrist. I would like to thank Dr. Andrist for his mentor-ship, patience and for all those meetings we had from the beginning of summer 2021 till the end of spring 2022. We shared a few laughs, and he taught me what ethical research is all about. Because of him, I learned several complex variables, and started reading published papers about the topic which were quite fascinating. I couldn't have done this without him, he has all my gratitude.

ABSTRACT
OF THE THESIS OF

Estepan Kirakos Ashkarian for Master of Science
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Title: Integrable Generators of Lie Algebras of Vector Fields on the Koras–Russell Cubic Threefold

The Koras–Russell cubic threefold is a complex-affine manifold that is diffeomorphic to the three-dimensional complex-Euclidean space, but not algebraically isomorphic to the three-dimensional complex-affine space as an affine variety. We study the Lie algebra of polynomial vector fields on the Koras–Russell cubic threefold; We prove that the compositions of the flows of a list of complete vector fields approximate every holomorphic automorphism that is in the path-connected component of the identity.

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Chapter 1

Introduction

In this Masters thesis, we are mainly concerned about complete vector fields on the complex submanifold X of $\mathbb{C}^4$, where $X = \{(x, y, z, w) \in \mathbb{C}^4 : x^2y + x + z^2 + w^3 = 0\}$ is the Koras–Russell cubic threefold. This three dimensional complex submanifold was discovered in the process of proving the so called Linearization Conjecture in dimension 3.

In 2001, Dror Varolin introduced the notion of the density property in his paper [16]. A complex manifold has the density property if the Lie algebra generated by complete holomorphic vector fields is dense in the Lie algebra of all holomorphic vector fields. In other words, every holomorphic vector field on the complex manifold can be approximated uniformly on compacts by Lie combinations of complete holomorphic vector fields. Mathematicians in this field started to study complex manifolds in order to determine if they have the density property. Some examples would be $\mathbb{C}^n$ for $n \geq 2$, $\mathrm{SL}_2(\mathbb{C})$, the Calogero–Moser spaces and a family of spaces given by $\{x^2y = a(z)+xb(z)\}$ where $z = (z_0, \ldots, z_n)$, $\deg_{z_0} a \leq 2$ and $\deg_{z_0} b \leq 1$. Notice that the Koras–Russell cubic is part of the family in the last example.

In Chapter 1, we give some background material from complex analysis of one variable, several complex variables, the similarities and differences between the two subjects. Then we move on to category theory in order to define sheaves and coherent sheaves. The language of sheaves and coherent sheaves will be used for the proof that the Koras–Russell cubic has the density property in Chapter 5.

Chapter 2 is also preliminary topics, which is more focused on vector fields and their flow maps. We give some known approximation theorems concerning flow maps. Finally, we introduce the main definition of this chapter, which is algorithms of vector fields. Later on we prove that under some conditions a flow map can be approximated uniformly on compacts by these algorithms. Finally we state and prove the most important theorem of this chapter which says that the flow map of any vector field in the Lie subalgebra of finitely many complete vector fields can be approximated uniformly on compacts by holomorphic automorphisms, these holomorphic automorphisms being the flow maps of the complete vector fields in the Lie subalgebra generated by the complete holomorphic vector fields.

The third chapter is rather humble in nature. We give five vector fields U, V, W, E and H on the Koras–Russell cubic, and we show that they are complete, which means that their flow maps are defined for all time. The first four vector fields will turn out to play a crucial role in Chapter 5.

In Chapter 4, our main goal is to give the proof of the density property of the Koras–Russell cubic. We start by defining shears and overshears, then we give a proof of the Kaliman–Kutzschebauch formula. The Andersén–Lempert theorem is stated without proof, and finally a proof is given for the density property.

Chapter 2

Definitions and Results from Several Complex Variables

The main objective of this chapter is to give a humble overview of complex analysis of one variable, several complex variables, and their differences and similarities. Moreover, we introduce the language of category theory in order to define presheaves, sheaves and coherent sheaves. We make use of the following references. See [13, Chapter 1, 2], [14, Chapter 14], [9, Chapter 1] and [15, Chapter 7, 9].

2.1 Complex Euclidean Space

For $n \in \mathbb{N}^+$ the $n-$dimensional complex Euclidean space is denoted by $\mathbb{C}^n$ where:

$$\mathbb{C}^n = \{z = (z_1, \dots, z_n) : \ z_j \in \mathbb{C} \ \text{ for all } 1 \leq j \leq n\}$$

is the Cartesian product of n copies of $\mathbb{C}$. $\mathbb{C}^n$ can be viewed as an $n-$dimensional complex vector space, equipped with the Hermitian inner product defined by:

$$(z, w) = \sum_{i=1}^{n} z_i \bar{w}_i \ \ z, w \in \mathbb{C}^n$$

The induced norm by the inner product $(\cdot, \cdot)$ is $|z| = \sqrt{(z, z)}$, which in hand also defines a distance d on $\mathbb{C}^n$ given by $\mathrm{d}(z, w) = |z - w|$.
The open ball of center $a \in \mathbb{C}^n$ and radius $r > 0$ is given by:

$$B(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}$$

Often it is convenient to use another system of neighborhoods: the open polydisc $P(a, r)$ of multiradius $r = (r_1, \dots, r_n)$, $r_j > 0$ and center $a \in \mathbb{C}^n$ is the product of n open discs in C, that is $P(a, r) = D(a_1, r_1) \times \cdots \times D(a_n, r_n)$.

$$P(a, r) = \{z \in \mathbb{C}^n : |z_j - a_j| < r_j, 1 \leq j \leq n\}$$

Notice that $P(a,(r_1,\dots,r_n)) \subset B(a,R)$ whenever $\sum_{j=1}^n r_j^2 < R^2$, and that $B(a,R) \subset P(a,(r_1,\dots,r_n))$ for $R \le \min\{r_j : 1 \le j \le n\}$.

2.1.1 Cauchy-Riemann equations

Recall that in the theory of complex analysis on $\mathbb{C}$ a complex valued function $f\colon D \to \mathbb{C}$ defined by $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ where $u, v\colon \mathbb{R}^2 \to \mathbb{R}$, is said to satisfy the Cauchy-Riemann equations if

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.\end{aligned}$$

Moreover, if we introduce the partial differential operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

Then,

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}}(z) = 0 &\iff \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) f(z) = 0\\ &\iff \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)(u(x,y) + iv(x,y)) = 0\\ &\iff \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0\\ &\iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\end{aligned}$$

Notation 2.1.1. *Turning to $\mathbb{C}^n = \mathbb{R}^{2n}$ with coordinates $z_j = x_j + iy_j$ we introduce the following notation, which are partial differential operators:*

$$\begin{aligned}\frac{\partial}{\partial z_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \frac{1}{i}\frac{\partial}{\partial y_j}\right)\\ \frac{\partial}{\partial \bar{z}_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \frac{1}{i}\frac{\partial}{\partial y_j}\right)\end{aligned}$$

Definition 2.1.2. *Let $D \subset \mathbb{C}^n$ be open. A function $f\colon D \to \mathbb{C}$ is called holomorphic on D if $f \in C^1(D)$ and f satisfies the system of partial differential equations known to be the Cauchy-Riemann equations:*

$$\frac{\partial f}{\partial \bar{z}_j}(z) = 0 \quad \textit{for } 1 \le j \le n \quad \textit{and } z \in D$$

Note that $C^1(D)$ is the space of continuously differentiable complex valued functions on D.

Notation 2.1.3. *The space of holomorphic functions on D is denoted by $\mathcal{O}(D)$.*

Remark 2.1.4. *If a function $f\colon D \to \mathbb{C}$ satisfies the Cauchy–Riemann equations from Definition 2.1.2 then it also satisfies the Cauchy–Riemann equations in the z_j-coordinate for any j, that is the map $f_{z_j}(\lambda) = f(z_1, \dots, z_{j-1}, \lambda, z_{j+1}, \dots, z_n)$ is holomorphic.*

One might ask if the converse is true. Hartogs gave an answer in 1906, proving that any $f\colon D \to \mathbb{C}$ which is holomorphic in each variable separately is also holomorphic as in the sense of Definition 2.1.2. One should also note the strength of Hartogs' theorem by observing the following example.

Example 2.1.5. *The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(0) = 0$ and $f(x, y) = \frac{xy}{x^4+y^4}$ for $(x, y) \neq (0, 0)$ is C^∞(even stronger than this actually, it is real analytic) in each variable separately, but is not bounded at 0.*

Now, we shall introduce the standard multi-index notation.
For $D \subset \mathbb{R}^n$, open and $k \in \mathbb{N}$, let $C^k(D)$ denote the space of k times continuously differentiable complex valued functions on D. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, one sets

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \dots \alpha_n!$$

$$x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$$

We say that $\alpha \geq 0 (> 0)$ if $\alpha_j \geq 0 (> 0)$ for $1 \leq j \leq n$.

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

For $f \in C^k(D)$, $k < \infty$, we define the C^k-norm of f over D by

$$|f|_{k,D} = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq k}} \sup_{x \in D} |D^\alpha f(x)|$$

The space $B^k(D) = \{f \in C^k(D) : |f|_k < \infty\}$ is complete in the C^k norm $|\cdot|_k$ and hence is a Banach space.
The multi-index notation extends to the partial differential operators as follows: for α, $\beta \in \mathbb{N}^n$,

$$D^{\alpha\bar{\beta}} = \frac{\partial^{|\alpha|+|\beta|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n} \partial \bar{z_1}^{\beta_1} \dots \partial \bar{z_n}^{\beta_n}}.$$

2.2 Results from Complex Analysis which are Generalized in Several Complex Variables

2.2.1 Cauchy Integral Formula on the Polydisc

In the theory of one complex variable, the Cauchy integral formula states that for any holomorphic function $f\colon D(a,r) \to \mathbb{C}$ we have:

$$f(z) = (2\pi i)^{-1} \int_{\partial D(a,r_0)} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for all $z \in D(a, r_0)$ where $r_0 < r$.

Now we generalize this formula in the context of several complex variables.

Theorem 2.2.1. *Let $P = P(a,r)$ be a polydisc in $\mathbb{C}^n$ with multiradius $r = (r_1, \dots, r_n)$. Suppose that $f \in C(\bar{P})$ and f is holomorphic in each variable separately, i.e. for each $z \in \bar{P}$ and $1 \le j \le n$, the function*

$$f_{z_j}(\lambda) = f(z_1, \dots, z_{j-1}, \lambda, z_{j+1}, \dots, z_n)$$

is holomorphic on $\{\lambda \in \mathbb{C} : |\lambda - a_j| < r_j\}$. Then

$$f(z) = (2\pi i)^{-n} \int_{b_o P} \frac{f(\zeta)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n \quad \text{for } z \in P$$

where $b_o P = \{\zeta \in \mathbb{C}^n : |\zeta_j - a_j| = r_j, 1 \le j \le n\}$.

Proof. The method of proof is done by induction on the dimension of the complex Euclidean space. For $n = 1$, it follows from the classical Cauchy integral formula. Suppose $n > 1$, and that the statement of the theorem is true for $n-1$ variables. Let $z \in P$ be fixed, apply the inductive hypothesis with respect to $(z_2, \cdots, z_n)$, obtaining

$$f(z_1, z_2, \cdots, z_n) = (2\pi i)^{-n+1} \int_{b_o P'(a',r')} \frac{f(z_1, \zeta_2, \dots, \zeta_n)}{(\zeta_2 - z_2) \cdots (\zeta_n - z_n)} d\zeta_2 \cdots d\zeta_n \tag{2.1}$$

where $a' = (a_2, \cdots, a_n)$, $r' = (r_2, \cdots, r_n)$. For $\zeta_2, \cdots, \zeta_n$ fixed, the case $n = 1$ gives us

$$f(z_1, \zeta_2, \dots, \zeta_n) = (2\pi i)^{-1} \int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta_1, \dots, \zeta_n)}{\zeta_1 - a_1} d\zeta_1 \tag{2.2}$$

In general, in terms of the standard parametrization we have

$$\zeta_j = a_j + r_j e^{i\theta_j}, \ 0 \le \theta_j \le 2\pi, 1 \le j \le n$$

of $b_o(P)$, one has

$$\int_{b_oP} g(\zeta)d\zeta_1\cdots d\zeta_n = i^n r_1\cdots r_n \int_{[0,2\pi]^n} g(\zeta(\theta))\mathrm{e}^{i\theta_1}\cdots\mathrm{e}^{i\theta_n}d\theta_1\cdots d\theta_n$$

for any $g \in C(b_oP)$. Now, we substitute 2.2 into 2.1 and transform the iterated integral over $\{|\zeta_1 - a_1| = r_1\} \times b_oP'(a', r')$ into an integral over b_oP using the aforementioned parametrization. □

2.2.2 Analyticity of Holomorphic Functions

In the theory of one complex variable, if we have $f\colon \Omega \subset \mathbb{C} \to \mathbb{C}$ holomorphic, then for any open disc $D(z_0, R) \subset \Omega$ we can write a power series expansion of f

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad \text{for all } z \in D(z_0, R)$$

where,

$$a_n = (2\pi i)^{-1}\int_{\partial D(z_0,R)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta$$

Moreover, $a_n = \frac{f^{(n)}(z_0)}{n!}$. Due to these results, one can show the following estimate which is called Cauchy's estimate

$$|a_n| < \frac{1}{R^n}\cdot \sup_{|\zeta - z_0| = R} |f(\zeta)|$$

A power series in n complex variables $z_1, \ldots z_n$ centered at the point $a \in \mathbb{C}^n$ is a multiple series $\sum_{v\in\mathbb{N}^n} b_v$ with terms

$$b_v = c_v(z-a)^v = c_{v_1,\ldots,v_n}(z_1 - a_1)^{v_1}\cdots(z_n - a_n)^{v_n}$$

where $c_v \in \mathbb{C}$ for $v \in \mathbb{N}^n$.

Definition 2.2.2. *The multiple series $\sum_{v\in\mathbb{N}^n} b_v$ is called convergent if*

$$\sum_{v\in\mathbb{N}^n} |b_v| = \sup\left\{\sum_{v\in\Lambda} |b_v| : \Lambda \text{ finite}\right\} < \infty.$$

Definition 2.2.3. *The domain of convergence $\Omega = \Omega(\{c_v\})$ of the power series $\sum_{v\in\mathbb{N}^n} c_v(z-a)^v$ is the interior of the set of points $z \in \mathbb{C}^n$ for which the power series converges.*

Theorem 2.2.4. *Let $f \in \mathcal{O}(P(a,r))$. Then the Taylor series of f at a converges to f on $P(a,r)$, that is*

$$f(z) = \sum_{v\in\mathbb{N}^n} \frac{D^v f(a)}{v!}(z-a)^v \quad \text{for } z \in P(a,r)$$

Proof. See Theorem 1.18 in Range [13]. □

The more general analogue of the Cauchy estimate in the context of several complex variables is the following.

Theorem 2.2.5. *Let $f \in \mathcal{O}(P(a,r))$. Then, for $\alpha \in \mathbb{N}^n$,*

$$|D^\alpha f(a)| \leq \frac{\alpha!}{r^\alpha} |f|_{P(a,r)}.$$

Proof. See Theorem 1.6 in Range [13]. □

2.3 Holomorphic Maps

Let $D \subset \mathbb{C}^n$ be open and consider a map $F \colon D \to \mathbb{C}^m$. By writing $F = (f_1, \ldots, f_m)$ and $f_k = u_k + iv_k$, where u_k, v_k are real valued functions on D, we can view $F = (u_1, v_1, \ldots, u_m, v_m)$ as a map from $D \subset \mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$. If F is differentiable at $a \in D$, its differential $dF(a) \colon \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ is a linear transformation with matrix representation given by the (real) Jacobian matrix

$$J_{\mathbb{R}}(F) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \cdots & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \cdots & \frac{\partial v_1}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial v_m}{\partial x_1} & \frac{\partial v_m}{\partial y_1} & \cdots & \frac{\partial v_m}{\partial y_n} \end{pmatrix}$$

evaluated at a.

The map $F \colon D \to \mathbb{C}^m$ is called holomorphic if its (complex) components $f_1, \ldots, f_m$ are holomorphic functions on D. If F is holomorphic, its differential $F'(a)$ at $a \in D$ is a complex linear map $\mathbb{C}^n \to \mathbb{C}^m$, with complex matrix representation

$$F'(a) = \begin{pmatrix} \frac{\partial f_1}{\partial z_1}(a) & \cdots & \frac{\partial f_1}{\partial z_n}(a) \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial z_1}(a) & \cdots & \frac{\partial f_m}{\partial z_1}(a) \end{pmatrix}$$

We call $F'(a)$ the derivative (or complex Jacobian matrix) of the holomorphic map F at a.

Definition 2.3.1. *We say F is nonsingular at $a \in D$ if $F'(a)$ has maximal rank; F is nonsingular on D, if F is nonsingular at every $a \in D$.*

2.4 The Riemann Mapping Theorem

In complex analysis of one variable, there is a deep result which is called the Riemann mapping theorem.

Definition 2.4.1. *Let $\Omega \subset \mathbb{C}$ be an open set. Let γ_0 and γ_1 be two curves in Ω such that $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$. We say that γ_0 is $\Omega-$homotopic to γ_1 if there exists a continuous mapping $H\colon [0,1] \times [0,1] \to \Omega$ such that*

1. *For all $t \in [0,1]$, $H(t,0) = \gamma_0(t)$ and $H(t,1) = \gamma_1(t)$.*
2. *For all $s \in [0,1]$, $H(0,s) = H(1,s)$*

Definition 2.4.2. *A region Ω in the complex plane is simply connected if any pair of curves in Ω with the same initial and end points are homotopic.*

Definition 2.4.3. *Let Ω_1 and Ω_2 be two regions in $\mathbb{C}$. The two regions are said to be conformally equivalent if there exists a $\varphi \in \mathcal{O}(\Omega_1)$ such that φ is one-to-one and $\varphi(\Omega_1) = \Omega_2$.*

The definition above actually implies that the inverse of φ is holomorphic on Ω_2, and hence φ is a biholomorphism of Ω_1 and Ω_2.

Theorem 2.4.4 (The Riemann Mapping Theorem)**.** *Every simply connected region $\Omega \subset \mathbb{C}$ such that $\Omega \neq \emptyset$ and $\Omega \neq \mathbb{C}$ is conformally equivalent to $D(0,1)$.*

Proof. See Theorem 14.8 [14]. □

The Riemann Mapping Theorem implies that a simply connected region in the complex plane is either $\mathbb{C}$ or biholomorphic to the open unit disc $D(0,1)$. One might ask if the analogue of this results holds in the context of several complex variables. The answer is negative. In 1907, Henri Poincaré gave a proof. In his proof, he computed the groups of holomorphic automorphisms of the ball and the bidisc (Note that the bidisc is a polydisc in $\mathbb{C}^2$) and compared them.

Theorem 2.4.5. *There exists no biholomorphic map*

$$F\colon P(0,1) \to B(0,1)$$

between the polydisc and the ball in $\mathbb{C}^n$ if $n > 1$.

2.5 Hartogs' Extension Phenomenon

In the theory of complex analysis of one variable the function $f\colon \mathbb{C}^* \to \mathbb{C}^*$ defined by $f(z) = \frac{1}{z}$ can't be extended to a holomorphic function which is also holomorphic at 0. In several complex variables, that is not the case. In 1906 Hartogs discovered the first example exhibiting the remarkable extension properties of holomorphic functions in more than one variable. It is this phenomenon, more than anything else, which distinguishes function theory of several variables from the classical one-variable theory.

Theorem 2.5.1. *Let $n \geq 2$ and suppose that $0 < r_j < 1$ for $1 \leq j \leq n$. Then every function f holomorphic on the domain*

$$\begin{aligned} H(r) = &\{z \in \mathbb{C}^n : |z_j| < 1 \quad \text{for } j < n, \;\; r_n < |z_n| < 1\} \\ &\cup \{z \in \mathbb{C}^n : |z_j| < r_j \quad \text{for } j < n, \;\; |z_n| < 1\} \end{aligned}$$

has a unique holomorphic extension $\hat{f}$ to the polydisc $P(0,1)$.

Theorem 2.5.2. *Let $n \geq 2$ and suppose U is a neighborhood of the boundary bP of a polydisc $P \subset \mathbb{C}^n$, such that $U \cap P$ is connected. Then every $f \in \mathcal{O}(U)$ has a holomorphic extension to P.*

Corollary 2.5.3. *Let U be open in $\mathbb{C}^n$ and $a \in U$. If $n \geq 2$, then every $f \in \mathcal{O}(U - \{a\})$ extends holomorphically across a.*

Basically, this corollary tells us that in $\mathbb{C}^n$ for $n \geq 2$ there is no such thing as isolated singularities, in contrast to the situation in one variable, for instance consider the function $f(z) = \frac{1}{z}$.

2.6 Complex Submanifolds and Analytic Sets

In this subsection we follow the textbook of Range [13, Chap I, Section 2.6. and Section 3.2.].

Definition 2.6.1. *A set $M \subset \mathbb{C}^n$ is called a* ***complex submanifold*** *of $\mathbb{C}^n$, if for every point $P \in M$ there is a holomorphic coordinate system $(w_1, \dots, w_n)$ on a neighborhood U of P, and an integer k, $0 \leq k \leq n$, such that*

$$M \cap U = \{z \in U : w_j(z) = 0 \quad \text{for } j > k\}.$$

The integer k is called the complex dimension of M at P, and it is denoted by $k = \dim_{\mathbb{C}} M_p$.

Notice that $\dim_{\mathbb{C}} M_p$ is locally a constant on M, and hence is constant on each connected component of M. The dimension of M is defined by

$$\dim M = \sup \dim_{\mathbb{C}} M_p.$$

Theorem 2.6.2. *A subset M of $\mathbb{C}^n$ is a complex submanifold if and only if for every $P \in M$ there is a neighborhood U of P, an open ball $B^k(a, \varepsilon) \subset \mathbb{C}^k$, and a nonsingular holomorphic map $H \colon B^k(a, \varepsilon) \to \mathbb{C}^n$, such that*

$$H(B^k(a, \varepsilon)) = M \cap U$$

A map H which satisfies all the conditions stated above is called a local parametrization of M at P.

Proof. See Theorem 2.8 in Range [13]. □

Theorem 2.6.3. *Let $D \subset \mathbb{C}^n$ and suppose that $F\colon D \to \mathbb{C}^m$ is nonsingular. Then for every $a \in D$ the level set*

$$L_a(F) = \{z \in D : F(z) = F(a)\}$$

is a complex submanifold of dimension $\max(0, n-m)$ *at every point.*

Proof. See Theorem 2.9 in Range [13]. □

Example 2.6.4. *The Koras–Russell Cubic is a complex submanifold of dimension 3. Indeed, consider the function $F\colon \mathbb{C}^4 \to \mathbb{C}$ defined by $F(x,y,z,w) = x^2y + x + z^2 + w^3$. Let (x_0, y_0, z_0, w_0) be an arbitrary point in $\mathbb{C}^4$, notice that*

$$F'(x_0, y_0, z_0, w_0) = (2x_0y_0 + 1, x_0^2, 2z_0, 3w_0^3)$$

so F is nonsingular on $\mathbb{C}^4$ since the rank of $F'(x_0, y_0, z_0, w_0)$ is one which is maximal. Note that the rank is at least one, if $y_0 = 0$ then it's obvious. If $y_0 \neq 0$ and we we require $2x_0y_0 + 1 = 0$ and $x_0^2 = 0$ we will get a contradiction. Therefore, $L_0(F) = \{(x,y,z,w) \in \mathbb{C}^4 : F(x,y,z,w) = F(0) = 0\}$, which is the Koras Russell Cubic. Hence, it is a complex submanifold of dimension 3.

Definition 2.6.5 ([13, Chapter II page 68])**.**
Let $K \subset D \subset \mathbb{C}^n$. Its holomorphically convex hull $\hat{K}_{\mathcal{O}(D)}$ in D is defined by

$$\hat{K}_{\mathcal{O}(D)} = \{z \in D : |f(z)| \leq |f|_K \quad \textit{for all } f \in \mathcal{O}(D)\}.$$

Moreover, $K \subset D$ is called $\mathcal{O}(D)$-convex if $\hat{K}_{\mathcal{O}(D)} = K$.

Definition 2.6.6 ([8, Definition 5.1.3])**.**
Let M be a complex submanifold of $\mathbb{C}^n$. M is said to be a Stein manifold if M has a countable basis for open sets and the following three properties hold.

1. *M is $\mathcal{O}(M)$-convex.*
2. *Given two distinct points P, $Q \in M$, there is $f \in \mathcal{O}(M)$ with $f(P) \neq f(Q)$.*
3. *For every $P \in M$ there is a holomorphic coordinate system in a neighborhood of P which is given by global holomorphic functions in $\mathcal{O}(M)$.*

Theorem 2.6.7. *A complex manifold is a Stein manifold if and only if it is biholomorphic to a closed complex submanifold of $\mathbb{C}^n$.*

Proof. See Forstnerič [5, Page 49]. □

Example 2.6.8. *The Koras–Russell cubic is a closed complex submanifold of $\mathbb{C}^4$, as it is a zero set of the continuous defining function $F(x,y,z,w) = x^2y + x + z^2 + w^3$. Hence, the Koras–Russell cubic is Stein.*

Definition 2.6.9. *A subset A of a region $\Omega \subset \mathbb{C}^n$ is called* ***analytic*** *in Ω if A is closed in Ω and if for every $p \in A$ there are open neighborhoods U_p of p in Ω and a holomorphic map $H_p \colon U_p \to \mathbb{C}^{l_p}$, such that*

$$U_p \cap A = \{z \in U_p : H_p(z) = 0\}$$

Stated differently, $U_p \cap A$ is the common zero set of the components $h_1^{(p)}, \ldots, h_{l_p}^{(p)}$ of H_p.

Let A_1 and A_2 be two analytic sets in $\Omega \subset \mathbb{C}^n$. Then it follows from the definition that $A_1 \cup A_2$ and $A_1 \cap A_2$ are analytic sets in Ω.

Definition 2.6.10. *An analytic sets is said to be* ***reducible*** *if A can be written as $A = A_1 \cup A_2$ where A_1 and A_2 are analytic, non-empty and not equal to A. A is said to be* ***irreducible*** *if A is not reducible.*

Definition 2.6.11. *A point $p \in A$ of an analytic set is called a* ***regular point*** *of A if there is a neighborhood U of p, such that $A \cap U$ is a complex submanifold of U, and it is called a* ***singular point*** *otherwise.*

Definition 2.6.12. *A subset E of $D \subset \mathbb{C}^n$ is thin, if for every point $p \in D$ there is a ball $B(p, \varepsilon)$ and a function $f \in \mathcal{O}(B(p, \varepsilon))$ such that f is not constant and $f(z) = 0$ on $B(p, \varepsilon) \cap E$.*

Notice that if $E \subset D$ is thin, its closure in D is also thin, and by the Identity theorem, E is nowhere dense.

Theorem 2.6.13. *Let A be an analytic set in the connected region D in $\mathbb{C}^n$. If $A \neq D$ then A is thin.*

Proof. Suppose that A is not thin. For each $p \in A$ we choose a connected neighborhood U_p and a holomorphic map $H_p \colon U_p \to \mathbb{C}^{l_p}$ such that $U_p \cap A = \{z \in U_p : H_p(z) = 0\}$. Given that A is not thin, there exists $p \in A$, such that $H_p \equiv 0$ on U_p. Hence, $U_p \cap A = U_p$, and the interior $\mathring{A}$ of A is not empty. Let $q \in b\mathring{A} \cap D$. Then $\mathring{A} \cap U_q$ is open and nonempty, and the components of H_q are zero on $\mathring{A} \cap U_q$. By the Identity Theorem, the components of H_q are zero on U_q. This implies that $U_q \subset A$, so $q \in \mathring{A}$ and $\mathring{A}$ is closed in D. Since D is connected, and $\mathring{A}$ is clopen and nonempty, then $\mathring{A} = D$. □

Notation 2.6.14.

- *The set of regular points is denoted by $\mathfrak{R}(A)$. Moreover, $\mathfrak{R}(A)$ it is the maximal complex submanifold contained in A.*
- *The set of singular points is denoted by $\mathfrak{S}(A) = A - \mathfrak{R}(A)$.*

2.7 Sheaves and Coherent Sheaves

Definition 2.7.1. *A category* $\mathbf{C}$ *consists of*

1. *A class* $\operatorname{ob}\mathbf{C}$ *of objects (usually denoted as* $X, Y, Z, etc.$*)*
2. *For each ordered pair of objects* (X, Y)*, a set* $\hom_{\mathbf{C}}$ *whose elements are called morphisms with domain* X *and codomain* Y*.*
3. *A composition map, whenever* $X, Y, Z \in \operatorname{ob}\mathbf{C}$*, we have a map*

$$\hom_{\mathbf{C}}(X, Y) \times \hom_{\mathbf{C}}(Y, Z) \to \hom_{\mathbf{C}}(X, Z) \text{ where } (f, g) \mapsto gf.$$

which satisfy the following conditions:

- *If* $(X, Y) \neq (X', Y')$*, then* $\hom_{\mathbf{C}}(X, Y)$ *and* $\hom_{\mathbf{C}}(X', Y')$ *are disjoint.*
- *If* $f \in \hom_{\mathbf{C}}(X, Y)$*,* $g \in \hom_{\mathbf{C}}(Y, Z)$ *and* $h \in \hom_{\mathbf{C}}(Z, W)$*, then* $h(gf) = (hg)f$*.*
- *For every* $X \in \operatorname{ob}\mathbf{C}$*, there exists* $1_X \in \hom_{\mathbf{C}}(X, X)$ *with the property that, for every* $f \in \hom_{\mathbf{C}}(X, Y)$*,* $f1_X = f$*, and for every* $g \in \hom_{\mathbf{C}}(Z, X)$*,* $1_X g = g$*.*

Example 2.7.2. *The following are some examples of categories.*

1. **Set** *where* $\operatorname{ob}\mathbf{Set}$ *is all sets, and the morphisms in* $\hom_{\mathbf{Set}}(X, Y)$ *are all functions* $f\colon X \to Y$*.*
2. **Grp** *where* $\operatorname{ob}\mathbf{Grp}$ *is all groups, and the morphisms in* $\hom_{\mathbf{Grp}}(X, Y)$ *are the group homomorphisms* $f\colon X \to Y$*.*

Definition 2.7.3. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be two categories. A covariant functor* $F\colon \mathbf{C} \to \mathbf{D}$ *is:*

1. *For each* $X \in \operatorname{ob}\mathbf{C}$ *we have* $FX \in \operatorname{ob}\mathbf{D}$*.*
2. *For each morphism* $f \in \hom_{\mathbf{C}}(X, Y)$ *we have* $Ff \in \hom_{\mathbf{D}}(FX, FY)$*, such that*

$$F1_X = 1_{FX} \quad \text{and} \quad F(gf) = FgFf$$

Example 2.7.4 (The forgetful functor and the abelianization of groups)**.**

1. *The forgetful functor. For example* $F\colon \mathbf{Grp} \to \mathbf{Set}$*. For any group* $G \in \mathbf{Grp}$*,* $FG = G$ *where* G *is viewed as a set. For any group homomorphism* $\phi\colon G \to H$ *we have* $F\phi\colon G \to H$ *viewed as a function and no longer a group homomorphism.*

2. *Let* $G \in \mathbf{Grp}$*. Let* $[G, G] = \{[x, y] = xyx^{-1}y^{-1} : x, y \in G\}$ *be the commutator subgroup. The functor* $F\colon \mathbf{Grp} \to \mathbf{Ab}$ *is defined by* $FG = G/[G, G]$*.*

Definition 2.7.5. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be two categories. A contravariant functor* $F\colon \mathbf{C} \to \mathbf{D}$ *is characterized by the following:*

1. *For each* $X \in \operatorname{ob}\mathbf{C}$ *we have* $FX \in \operatorname{ob}\mathbf{D}$*.*

2. *For each morphism* $f \in \hom_{\mathbf{C}}(X, Y)$ *we have* $Ff \in \hom_{\mathbf{D}}(FY, FX)$*, such that*
$$F1_X = 1_{FX} \quad \text{and} \quad F(gf) = FfFg$$

Example 2.7.6 (Continuous functions and the dualization functor)**.**

1. $F\colon \mathrm{Top} \to \mathrm{Ring}$*, then for any topological space* X*, we define* $FX = C(X)$ *the ring of continuous functions. Let* $f\colon X \to Y$ *be continuous function (which are the morphisms in* Top*), then* $Ff = f^*\colon C(Y) \to C(X)$*. If* $\phi\colon Y \to \mathbb{R} \in C(Y)$*, then* $f^*\phi = \phi \circ f$*.*

2. *Let* k *be a fixed field. We define the dualization functor,* $F\colon \mathrm{Vect/k} \to \mathrm{Vect/k}$*, such that* $FV = V^*$*, where* V^* *is the dual space of the vector space* V*. Now let* $T \in \hom_{\mathrm{Vect/k}}(V, W)$*, then* $FT = T^*\colon W^* \to V^*$ *such that foe every* $f \in W^*$*,* $T^*f = f \circ T$*.*

Definition 2.7.7. *Let* $(X, \mathcal{T})$ *be a topological space. Consider the collection of all open subsets of* X *to be a category, i.e.* $\mathbf{C} = \{U \subset X : U \in \mathcal{T}\}$*, where the objects of this category are the open sets* $U \subset X$*, the morphisms are the inclusion, that is* $f \in \hom_{\mathbf{C}}(U, V)$ *means that* $U \subset V$*, and the compositions are defined by the transitivity of the inclusion. Then a presheaf on* X *is a contravariant functor from the category* $\mathbf{C}$ *to the category of abelian groups* $\mathbf{Ab}$*.*

Let S be a presheaf on X. S assigns to each open set $U \subset X$ an abelian group $S(U)$, and to each inclusion map $f \in \hom_{\mathcal{C}}(U, V)$ a group homomorphism $\rho_{U,V}\colon S(V) \to S(U) \in \hom_{\mathrm{Ab}}(S(V), S(U))$ which is called a restriction map, where $\rho_{U,U} = \mathrm{Id}$ for each open set U and $\rho_{U,W} = \rho_{U,V} \circ \rho_{V,W}$ for every $U \subset V \subset W$.

In category theory, we also have the concept of morphisms between functors, so consider the morphism $\Phi\colon S \to T$ between the the presheaves S and T on X. Φ assigns a morphism $\Phi_U\colon S(U) \to T(U)$ to each open set $U \subset X$ in a way which commutes with restrictions.

Example 2.7.8. *If X is any topological space and G is a fixed abelian group, then we may define a presheaf called the constant presheaf by assigning G to each non-empty open set $U \subset X$ and 0 to the empty set.*

Example 2.7.9. *The presheaf $\mathcal{C}$ of continuous functions is the contravariant functor which assigns to each open set $U \subset X$ the algebra of continuous complex valued functions $\mathcal{C}(U)$ and to each inclusion map $U \subset V$ the usual restriction map $\rho_{U,V} \colon \mathcal{C}(V) \to \mathcal{C}(U)$ where $f \mapsto f|_U$ for every $f \in \mathcal{C}(V)$.*

Example 2.7.10. *The presheaf $\mathcal{T}$ of holomorphic sections of the tangent bundle is also a contavariant functor which assigns to each open set $U \subset X$ the module of holomorphic vector fields $\mathcal{T}(U)$ and to each inclusion map $U \subset V$ the usual restriction map $\rho_{U,V} \colon \mathcal{T}(V) \to \mathcal{T}(U)$ where $\Theta \mapsto \Theta|_U$ for every $\Theta \colon V \to TV \in \mathcal{T}(V)$.*

If $U \subset V$ then the image of $s \in S(V)$ under $\rho_{U,V} \colon S(V) \to S(U)$ will be denoted by $s|_U$ and will be called the restriction of S to U.

Definition 2.7.11. *If S is a presheaf on X, then S is called a sheaf if the following conditions are satisfied for each open subset $U \subset X$ and each open cover $\mathcal{V}$ of U:*

1. *If $s \in S(U)$ and $s|_V = 0$ for all $V \in \mathcal{V}$, then $s = 0$.*
2. *If $\{s_V \in S(V)\}_{V \in \mathcal{V}}$ is a collection of elements with the property that $s_V|_{V \cap W} = s_W|_{V \cap W}$ for each pair $V, W \in \mathcal{V}$ then there is an $s \in S(U)$ such that $s|_V = s_V$ for every $V \in \mathcal{V}$.*

Definition 2.7.12. *Let $(X, \mathcal{T})$ be a topological space. Given a sheaf S, it is said to be a coherent sheaf if for every $p \in X$, for every neighborhood U of p in X, there exists a neighborhood $U' \subset U$ of p and a map $\varphi_{U'}$ such that*

$$\mathcal{O}^m(U') \xrightarrow{\varphi_{U'}} S(U') \longrightarrow 0$$

is exact.

Moreover, for any such $\varphi_{U'}$, there exists $U'' \subset U'$ a neighborhood of p, and a map $\psi_{U''}$ such that

$$\mathcal{O}^n(U'') \xrightarrow{\psi_{U''}} \mathcal{O}^m(U'') \xrightarrow{\varphi_{U'}|_{U''}} S(U'') \longrightarrow 0$$

is exact.

Definition 2.7.13. *We say that the sheaf S is locally free if $\varphi_{U'}$ is an isomorphism.*

Now we can show that, locally free implies coherence. Indeed if $\varphi_{U'}$ is an isomorphism, then the kernel of $\varphi_{U'}$ is trivial. Then we choose $\psi_{U''}$ to be the zero map and hence we have

$$\mathcal{O}^n(U'') \xrightarrow{\psi_{U''}} \mathcal{O}^m(U'') \xrightarrow{\varphi_{U'}|_{U''}} S(U'') \longrightarrow 0$$

is exact.

Chapter 3

Vector Fields and Flows

3.1 Generalities

In this section we will define vector fields on manifolds and the flows they determine. Moreover, we will prove a few results concerning approximation of flows. We use the following references. See [10, Chapter 4], [10, Chapter 17] and [5, Chapter 1].

Definition 3.1.1. *Let X be a manifold. A vector field on X is a section of the tangent bundle TX of X. That is, a vector field is a mapping $V: X \to TX$ where $V_p \in T_pX$ for all $p \in X$.*

Definition 3.1.2. *Let X be a smooth manifold. Given a differentiable path $\gamma: I \to X$ where I is an interval of $\mathbb{R}$. An integral curve of a vector field V on X is a path $\gamma: I \to X$ such that:*

$$\frac{d\gamma}{dt}(t) = V_{\gamma(t)} \quad \forall t \in I$$

The initial value problem (the flow equation)

$$\dot{x} = V_x \quad x(0) = x^0$$

asks for an integral curve which passes through the point x^0 at time $t = 0$. In local coordinates $x = (x_1, ..., x_n)$ on X with

$$V_x = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j} \quad x^0 = (x_1^0, ..., x_n^0)$$

the flow equation is equivalent to the system of autonomous ordinary differential equations:

$$\dot{x_j} = a_j(x_1, ..., x_n) \quad x_j(0) = x_j^0 \quad j = 1, ...n$$

Example 3.1.3. *Let* $W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ *on* $\mathbb{R}^2$*. Let* $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ *be a smooth curve, written in standard coordinates as* $\gamma(t) = (x(t), y(t))$*, then for* γ *to be an integral curve, it must satisfy* $\dot{\gamma}(t) = W_{\gamma(t)}$*, which is equivalent to solving the following system of ODEs:*

$$\dot{x}(t) = -y(t) \quad \dot{y}(t) = x(t)$$

The solution of the system above yields a family of integral curves of W*:*

$$\gamma(t) = (a\cos(t) - b\sin(t), a\sin(t) + b\cos(t))$$

However, if we specify an initial condition such as $x(0) = 0$ *and* $y(0) = 1$ *then the integral curve* $\gamma(t) = (-\sin(t), \cos(t))$ *is the unique solution of the flow equation satisfying the initial conditions.*

If V is Lipschitz continuous, then for every $p \in X$ there exists a neighborhood $U \subset X$ of p and a number $t_0 > 0$ such that the flow equation has a unique solution

$$x(t, x^0) = \phi_t(x^0) \quad \text{for every } x^0 \in U \quad \text{and for every } |t| < t_0.$$

This solution, and its $t-$derivative are continuous in (t, x^0). The map $t \mapsto \phi_t(x)$ is called the **local flow** of V. For a fixed $t \in \mathbb{R}$ the map ϕ_t is a diffeomorphism of its domain $\Omega_t \subset X$ onto $\phi_t(\Omega_t)$, called the **time-t map**. These maps satisfy the group law

$$\phi_t \circ \phi_s = \phi_{t+s}, \quad t, s \in \mathbb{R}$$

on the set X where both sides are defined.

Theorem 3.1.4 (Grönwall's Inequality)**.** *Let* $f, g\colon [a, b) \to [0, \infty)$ *be non-negative continuous functions which satisfy the following:*

$$f(t) \leqslant A + \exp\left(\int_a^t f(\tau)g(\tau)\, d\tau\right) \text{ for some } A \geq 0$$

Then,

$$f(t) \leqslant A \cdot \exp(\int_a^t g(\tau)\, d\tau) \quad \text{for all } t \in [a, b)$$

Proof. If $A > 0$:
Let $h(t) = A + \int_a^b f(\tau)g(\tau)\, d\tau$, since f and g are non-negative and $A > 0$, then $h(t) > 0$.
By the fundamental theorem of calculus, we have $h'(t) = f(t)g(t) \leq h(t)g(t)$. Therefore $h'(t)/h(t) \leq g(t)$. Integrating both sides we get:

$$f(t) \leq h(t) \leq A\exp(\int_a^t g(\tau)\, d\tau)$$

If $A = 0$:
Let $\epsilon > 0$. We set $h(t) = \epsilon + \int_a^b f(\tau)g(\tau)\, d\tau$, by the same steps as above, we will have:

$$f(t) \leq h(t) \leq \epsilon \cdot \exp(\int_a^t g(\tau)\, d\tau)$$

which implies that $f \equiv 0$. □

Theorem 3.1.5. *Let V be a time-dependent continuous vector field on a domain $\Omega \subset \mathbb{R}^{1+n}$ satisfying a uniform Lipschitz estimate with Lipschitz constant $B > 0$:*

$$|V_t(x) - V_t(y)| \leq B|x - y|$$

Then for any $s \in \mathbb{R}$ and any pair of points $x, y \in \Omega_s$ we have:

$$|\Phi_{t,s}(x) - \Phi_{t,s}(y)| \leq \mathrm{e}^{B \cdot |t-s|}|x - y|$$

for all t such that the trajectories exist and remain in the domain Ω_t.

Proof. Let $f(t) = |\Phi_{t,s}(x) - \Phi_{t,s}(y)|$. Without loss of generality, we assume that $t \geq s$.

Recall that

$$\frac{\partial}{\partial t}\Phi_{t,s}(x) = V_t(\Phi_{t,s}(s))$$

We integrate from t to s and we get:

$$\Phi_{t,s}(x) - \Phi_{s,s}(x) = \int_s^t V_\tau(\Phi_{\tau,s}(x))d\tau$$

$$\Phi_{t,s}(y) - \Phi_{s,s}(y) = \int_s^t V_\tau(\Phi_{\tau,s}(y))d\tau$$

Where $\Phi_{s,s}(x) = x$ and $\Phi_{s,s}(y) = y$. We get:

$$f(t) = \left| x + \int_s^t V_\tau(\Phi_{\tau,s}(x))d\tau - y - \int_s^t V_\tau(\Phi_{\tau,s}(y))d\tau \right|$$

By the triangular inequality, the uniform Lipschitz condition and the definition of $f(t)$ we have:

$$\begin{aligned} f(t) &\leq |x - y| + \int_s^t |V_\tau(\Phi_{\tau,s}(x)) - V_\tau(\Phi_{\tau,s}(y))|\, d\tau \\ &\leq |x - y| + B \cdot \int_s^t |\Phi_{\tau,s}(x) - \Phi_{\tau,s}(y)| d\tau \\ &= |x - y| + B \cdot \int_s^t f(\tau) d\tau \end{aligned}$$

Note that $f(s) = |x-y|$, so $f(t) \leq f(s) + B \cdot \int s^t f(\tau)d\tau$. By Grönwall's Inequality, we have:

$$f(t) \leq |x - y|\mathrm{e}^{B \cdot |t-s|}$$

Note that if we assume $s > t$, the proof remains the same, but we work with $-V$ instead of V. □

Lemma 3.1.6 (Escape Lemma)**.** *Let V be a smooth vector field on a smooth manifold X. If γ is an integral curve of V whose maximal domain is not all of $\mathbb{R}$, then the image of γ cannot lie in any compact subset of X.*

Proof. Let (a, b) be the maximal domain of γ where $-\infty \leq a < 0 < b \leq \infty$. Let $p = \gamma(0)$ and let Φ denote the flow of V, so $\gamma = \Phi(\cdot, p) = \Phi.(p)$.

Assume that $b < \infty$ and $\gamma((a, b)) \subset K$ where K is a compact subset of X. Let $\{t_i\}$ be a sequence of times approaching b from below. By our assumption $\{\gamma(t_i)\}$ lies in K, and since K is compact, then there exists a subsequence of $\{\gamma(t_i)\}$ converging to a point $q \in X$. Let U be a relatively compact neighborhood of q and let $\epsilon > 0$ such that Φ is defined on $(-\epsilon, \epsilon) \times U$. We choose i large enough so that $\gamma(t_i) \in U$ and $t_i > b - \epsilon$. Now we define $\sigma\colon (a, t_i + \epsilon) \to X$ by:

$$\sigma(t) = \begin{cases} \gamma(t) & \text{if}\, a < t < b \\ \Phi_{t-t_i} \circ \Phi_{t_i}(p) & \text{if}\, t_i - \epsilon < t < t_i + \epsilon \end{cases}$$

Note that these two definitions agree where they overlap, because $\Phi_{t-t_i} \circ \Phi_{t_i}(p) = \Phi_t(p) = \gamma(t)$ by the group law of Φ. Therefore, σ is an integral curve extending γ, which contradicts the maximality! We had assumed that the maximal domain is (a, b). ☐

Remark 3.1.7. *Let V be a vector field with flow map Φ_t. Let $t_1 = \inf\{t \in \mathbb{R} : \Phi_t \text{ exists}\} < \infty$. So, the flow map exists for $t \in [0, t_1)$. Since Φ_t is an integral curve of V whose maximal domain is not all of $\mathbb{R}$, then by the escape lemma, it has to leave any compact.*

Theorem 3.1.8. *Let X be a compact manifold. Then every smooth vector field on X is complete, which means that the flow map of the vector field is defined for all time.*

Proof. Let X be a compact manifold, and V a smooth vector field on X. By the converse of the preceding lemma, every smooth vector field is complete. ☐

Assume that $\Omega_0 = \{x \in X : (0, x) \in \Omega\} \neq \emptyset$. We fix a compact set $K \subset \Omega_0$, and let $t_0 > 0$ such that the flow $\Phi_t(x) = \Phi_{t,0}(x)$ exists and remain in Ω_t when $x \in K$ and $t \in [0, t_0]$. Set $K_t = \Phi_t(K) \subset \Omega_t$. For any $\epsilon > 0$ we let

$$K(\epsilon) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) = \inf_{y \in K} |x - y| < \epsilon\}$$
$$S(\epsilon) = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : 0 \leq t \leq t_0, \operatorname{dist}(x, K_t) < \epsilon\}$$

Set $\eta_0 = (1 + t_0)\mathrm{e}^{Bt_0} > 1$ where B is the Lipschitz constant from Theorem 3.1.5. Choose $\epsilon_0 > 0$ sufficiently small so that $S(\epsilon\eta_0) \Subset \Omega$.

Theorem 3.1.9. *Assume that for some $\epsilon \in (0, \epsilon_0)$ we have a continuous map $V^\epsilon\colon \Omega \to \mathbb{R}^n$(a time dependent vector field) satisfying:*

$$\|V - V^\epsilon\|_{L^\infty(S(\epsilon\eta_0))} \leq \epsilon$$

Then, the flow $\Phi_t^\epsilon(x)$ *of* V_t^ϵ *with* $\Phi_0^\epsilon(x) = x$ *exists for all* $x \in K(\epsilon)$ *and for all* $t \in [0, t_0]$ *and it satisfies the estimate:*

$$\|\Phi_t - \Phi_t^\epsilon\|_{L^\infty(K(\epsilon))} \leq t_0 \mathrm{e}^{Bt} \|V - V^\epsilon\|_{L^\infty(S(\epsilon\eta_0))}, t \in [0, t_0]$$

Proof. Let $A(\epsilon) = \|V - V^\epsilon\|_{L^\infty(S(\epsilon\eta_0))}$.
Let $x \in K(\epsilon)$. We set $f(t) = |\Phi_t(x) - \Phi_t^\epsilon(x)|$. Since $f(0) = |\Phi_0(x) - \Phi_0^\epsilon(x)| = |x - x| = 0$, $\frac{d}{dt}\Phi_t(x) = V_{\Phi_t(x)}$ and by the triangular inequality we have:

$$\begin{aligned}
f(t) &= \left| \int_0^t [V_s(\Phi_s(x) - V_s^\epsilon(\Phi_s^\epsilon(x)] ds \right| \\
&\leq \int_0^t |V_s(\Phi_s(x)) - V_s^\epsilon(\Phi_s^\epsilon(x))| ds \\
&= \int_0^t |V_s(\Phi_s(x)) - V_s(\Phi_s^\epsilon(x)) + V_s(\Phi_s^\epsilon(x)) - V_s^\epsilon(\Phi_s^\epsilon(x))| ds \\
&\leq \int_0^t |V_s(\Phi_s(x)) - V_s(\Phi_s^\epsilon(x))| ds + \int_0^t |V_s(\Phi_s^\epsilon(x)) - V_s^\epsilon(\Phi_s^\epsilon(x))| ds
\end{aligned}$$

By the hypothesis that V satisfies the uniform Lipschitz estimate with Lipschitz constant B, we have:

$$\begin{aligned}
f(t) &\leq B \cdot \int_0^t |\Phi_s(x) - \Phi_s^\epsilon(x)| ds + \int_0^t |V_s(\Phi_s^\epsilon(x)) - V_s^\epsilon(\Phi_s^\epsilon(x))| ds \\
&= B \cdot \int_0^t f(s) ds + \int_0^t |V_s(\Phi_s^\epsilon(x)) - V_s^\epsilon(\Phi_s^\epsilon(x))| ds
\end{aligned}$$

Now, suppose that $\Phi_t^\epsilon(x) \in K_t(\epsilon\eta_0)$:

$$\Phi_t^\epsilon(x) \in K_t(\epsilon\eta_0) \Leftrightarrow \mathrm{dist}(\Phi_t^\epsilon(x), K_t) < \epsilon\eta_0 \Leftrightarrow (t, \Phi_t^\epsilon(x)) \in S(\epsilon\eta_0)$$

So, we have:

$$f(t) \leq B \cdot \int_0^t f(s) ds + A(\epsilon) \cdot t$$

therefore, by Grönwall's inequality we have:

$$f(t) \leq A(\epsilon) t_0 \mathrm{e}^{Bt}$$

Let's show that $\Phi_t^\epsilon \in K_t(\epsilon\eta_0)$.

The preceding theorem gives us $\Phi_t(x) \in K_t(\epsilon e^{Bt})$. Assume that Φ_t^ϵ exists for $t \in [0, t_0]$, we have that:

$$\begin{aligned}
\operatorname{dist}(\Phi_t^\epsilon(x), K_t) &= \operatorname{dist}(\Phi_t^\epsilon(x) - \Phi_t(x) + \Phi_t(x), K_t) \\
&\leq |\Phi_t^\epsilon(x) - \Phi_t(x)| + \operatorname{dist}(\Phi_t(x), K_t) \\
&\leq A(\epsilon) t_0 e^{Bt} + \epsilon e^{Bt} \\
&\leq \epsilon t_0 e^{Bt} + \epsilon e^{Bt} \\
&\leq \epsilon(t_0 + 1) e^{Bt_0} = \epsilon \eta_0
\end{aligned}$$

Note that we used the assumption that $A(\epsilon) \leq \epsilon$ and the definition of η_0.

Now, to get a contradiction, assume that Φ_t^ϵ does not exist for some $t \in [0, t_0]$. By the local existence theorem, Φ_t^ϵ exists for small enough $t > 0$. Let $t_1 = \inf\{t > 0 : \Phi_t^\epsilon \text{ does not exist}\}$, therefore Φ_t^ϵ is well defined for $t \in [0, t_1)$. By Remark 3.1.7 following the Escape lemma, we have that:

$$\lim_{t \to t_1} \|\Phi_t^\epsilon(x)\| = \infty$$

We showed earlier that if Φ_t^ϵ exists, then:

$$\|\Phi_t - \Phi_t^\epsilon\|_{L^\infty(K(\epsilon))} \leq t_0 e^{Bt} \|V - V^\epsilon\|_{L^\infty(S(\epsilon\eta_0))} = C < \infty \text{ for } 0 \leq t < t_1$$

However, $\|\Phi_t^\epsilon\| = \|\Phi_t^\epsilon - \Phi_t + \Phi_t\| \leq \|\Phi_t^\epsilon - \Phi_t\| + \|\Phi_t\| \leq C + C_0$. Note that $\|\Phi_t\| \leq C_1$ since Φ_t is continuous on the compact $[0, t_0]$ hence an upper bound exists. We reached a contradiction, therefore our assumption is false! Hence, Φ_t^ϵ exists for all $t \in [0, t_0]$, which implies that $\Phi_t^\epsilon \in K_t(\epsilon\eta_0)$. □

Definition 3.1.10. *Let Φ_t be the flow of a vector field V. The Lie derivative $L_V W$ of a vector field W with respect to V is defined by:*

$$L_V W = \left.\frac{d}{dt}\right|_{t=0} \Phi_t^* W$$

Proposition 3.1.11. *Let V and W be vector fields on X where Φ_t is the flow of V. Then:*

$$\frac{d}{dt} \Phi_t^* W = \Phi_t^*(L_V W) \tag{3.1}$$

Moreover, if $L_V W = 0$ then, $\Phi_t^* W = W$ for all t

Proof. Let $s = t + u$, then by the group law of flows we have:

$$\Phi_s = \Phi_{t+u} = \Phi_u \circ \Phi_t$$

So that,

$$\Phi_s^* W = \Phi_t^*(\Phi_u^* W)$$

Now we differentiate w.r.t. s at $s = t$ and we get:

$$\frac{d}{dt}\Phi_t^* W = \left.\frac{d}{du}\right|_{u=0} \Phi_t^*(\Phi_u^* W) = \Phi_t^*\left(\left.\frac{d}{du}\right|_{u=0} \Phi_u^* W\right) = \Phi_t^*(L_V W)$$

So, if $L_V W = 0$ we have:

$$\frac{d}{dt}\Phi_t^* W = 0$$

we integrate from 0 to t:

$$0 = \int_0^t \frac{d}{ds}\Phi_s^* W ds = \Phi_t^* W - \Phi_0^* W$$

Finally we have $\Phi_t^* W = W$. □

Proposition 3.1.12. *Let $f\colon X \to Y$ be a smooth map and $\widetilde{V}$ and $\widetilde{W}$ be vector fields on Y such that $V = f^*\widetilde{V}$ and $W = f^*\widetilde{W}$, then:*

$$f^*(L_{\widetilde{V}}\widetilde{W}) = L_V W$$

Proof. Let $\widetilde{\Phi}_t$ be the flow of $\widetilde{V}$. Then, $f \circ \Phi_t = \widetilde{\Phi}_t \circ f$, since

So, we have:

$$\Phi_t^* W = \Phi_t^* f^* W = (f \circ \Phi_t)^* \widetilde{W} = (\widetilde{\Phi}_t \circ f)^* \widetilde{W} = f^*(\widetilde{\Phi}_t^* \widetilde{W})$$

Now we differentiate with respect to t at $t = 0$, and we get:

$$L_V W = \left.\frac{d}{dt}\right|_{t=0} \Phi_t^* W = \left.\frac{d}{dt}\right|_{t=0} f^*(\widetilde{\Phi}_t^* \widetilde{W}) = f^*\left(\left.\frac{d}{dt}\right|_{t=0} \widetilde{\Phi}_t^* \widetilde{W}\right) = f^*(L_{\widetilde{V}}\widetilde{W})$$

Moreover, if $f\colon X \to Y$ is a diffeomorphism, then:

$$L_{f_* V} f_* W = f_*(L_V W)$$

Indeed, since $\widetilde{V} = (f^*)^{-1} V = f_* V$ and $\widetilde{W} = (f^*)^{-1} V = f_* W$ we have:

$$L_{f_* V} f_* W = (f^*)^{-1}(f^*(L_{\widetilde{V}}\widetilde{W})) = (f^*)^{-1}(L_V W) = f_*(L_V W)$$ □

Theorem 3.1.13 (Canonical form theorem)**.** *Let V be a C^1 vector field on an n-dimensional manifold X. If $V_p \neq 0$ for some $p \in X$ then there exist local coordinates $u = (u_1, \ldots, u_n)$ in a neighborhood of p such that $V = \frac{\partial}{\partial u_1}$.*

Proof. By the way coordinate vector fields are defined on a manifold, a smooth chart (U, φ) will satisfy the conclusion of the theorem if $(\varphi^{-1})_*(\frac{\partial}{\partial u_1}) = V$.

Let's choose a smooth local coordinate $x = (x_1, \ldots, x_n)$ around p, we may think of X as an open set $U \subset \mathbb{R}^n$, and V as a vector field on U. Since $V_p \neq 0$ we may assume that V has a nonzero x_1-component at p.

Let $\Phi\colon \Omega \to U$ be the flow of V. There exist $\epsilon > 0$ and an open neighborhood $U_0 \subset U$ of p, such that $(-\epsilon, \epsilon) \times U_0 \subset \Omega$. Let $S_0 \subset U_0$ such that $S_0 = U_0 \cap \{x_1 = 0\}$, and define $S \subset \mathbb{R}^{n-1}$ by

$$S = \{(u_2, \ldots, u_n) : (0, u_2, \ldots, u_n) \in S_0\}.$$

Define a smooth map $\psi\colon (-\epsilon, \epsilon) \times S \to U$ by

$$\psi(t, u_2, \ldots, u_n) = \Phi_t(0, u_2, \ldots, u_n)$$

First we will show that ψ pushes $\frac{\partial}{\partial t}$ forward to V. For any $(t_0, u_0) \in (-\epsilon, \epsilon) \times S$, we have

$$\begin{aligned} \left(\psi_* \left.\frac{\partial}{\partial t}\right|_{(t_0,u_0)}\right) f &= \left.\frac{\partial}{\partial t}\right|_{(t_0,u_0)} (f \circ \psi) \\ &= \left.\frac{\partial}{\partial t}\right|_{t=t_0} (f(\Phi_t(0, u_0))) \\ &= V_{\psi(t_0,u_0)} f \end{aligned}$$

On the other hand, when ψ is restricted to $\{0\} \times S$,
$\psi(0, u_2, \ldots, u_n) = \Phi_0(0, u_2, \ldots, u_n) = (0, u_2, \ldots, u_n)$, so

$$\psi_* \left.\frac{\partial}{\partial u_i}\right|_{(0,0)} = \left.\frac{\partial}{\partial x_i}\right|_p, \quad i = 2, \ldots, n.$$

Thus at $(0, 0)$, ψ_* takes the basis
$\left(\left.\frac{\partial}{\partial t}\right|_{(0,0)}, \left.\frac{\partial}{\partial u_2}\right|_{(0,0)}, \ldots, \left.\frac{\partial}{\partial u_n}\right|_{(0,0)}\right)$ to $\left(V_p, \left.\frac{\partial}{\partial x_2}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p\right)$. Since V_p has a nonzero x_1-component, this is also a basis, so ψ_* is an isomorphism. Therefore, by the inverse function theorem, there are neighborhoods W of $(0, 0)$ and Y of p such that $\psi\colon W \to Y$ is a diffeomorphism.

Let $\varphi = \psi^{-1}\colon Y \to W$, which is a smooth coordinate map.The equation $\left(\psi_* \left.\frac{\partial}{\partial t}\right|_{(t_0,u_0)}\right) f = V_{\psi(t_0,u_0)} f$ says precisely that V is the coordinate vector field $\frac{\partial}{\partial t}$ in these coordinates. With t renamed to u_1, this is what we wanted to prove. $\square$

Let V and W be smooth vector fields on a smooth manifold M. We define $[V, W]\colon C^\infty(M) \to C^\infty(M)$, which is called the Lie bracket of V and W, defined by

$$[V, W]f = VWf - WVf$$

Lemma 3.1.14. *Let V, W be smooth vector fields on a smooth manifold M, and let $V = \sum V^i \frac{\partial}{\partial x_i}$ and $W = \sum W^j \frac{\partial}{\partial x_j}$ be the coordinate expressions for V and W in terms of some smooth local coordinates (x_i) for M. Then $[V, W]$ has the following coordinate expression:*

$$[V, W] = \sum_i \sum_j \left(V^i \frac{\partial W^j}{\partial x_i} - W^i \frac{\partial V^j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

Proof. See [10, Lemma 4.13]. □

Lemma 3.1.15. *For every $f, g \in C^\infty(M)$, for every smooth vector field V, W we have:*

$$[fV, gW] = fg[V, W] + (fVg)W - (gWf)V$$

Proof. See [10, Lemma 4.15]. □

Proposition 3.1.16. *Let V and W be C^1 vector fields on a manifold X, then:*

$$L_V W = [V, W]$$

Proof. Let $R(V) = \{p \in X : V_p \neq 0\}$. By the continuity of V, $R(V)$ is an open set in X and $\operatorname{supp} V = \overline{R(V)}$.
First we will show that $L_V W = [V, W]$ on $R(V)$. Let $p \in R(V)$. We can choose local coordinates $x = (x_1, \dots, x_n)$ such that $V = \frac{\partial}{\partial x_1}$ with flow $\Phi_t(x_1, \dots, x_n) = (x_1 + t, x_2, \dots, x_n)$ by Theorem 3.1.13 page 26. Let $W = \sum_{j=1}^n b_j(x) \frac{\partial}{\partial x_j}$, It's easy to see that the matrix representation of Φ_t^* is the identity matrix for a fixed t at any given point in these coordinates. Let $u \in U$ where $(U, (x_1, \dots, x_n))$ is the local coordinate chart. Then:

$$\begin{aligned}
(\Phi_t^*) W_{\Phi_t(u)} &= \Phi_t^* \left(\sum_{j=1}^n b_j(x) \left. \frac{\partial}{\partial x_j} \right|_{\Phi_t(u)} \right) \\
&= \sum_{j=1}^n b_j(x) \Phi_t^* \left(\left. \frac{\partial}{\partial x_j} \right|_{\Phi_t(u)} \right) \\
&= \sum_{j=1}^n b_j(x) (\Phi_{-t})_* \left(\left. \frac{\partial}{\partial x_j} \right|_{\Phi_t(u)} \right)
\end{aligned}$$

Such that,

$$(\Phi_t^*) W_{\Phi_t(u)} = \sum_{j=1}^n b_j(x) \left(\left. \frac{\partial}{\partial x_j} \right|_u \right)$$

Using the definition of Lie derivative we get:

$$\begin{aligned}
(L_V W)_u &= \frac{d}{dt}\bigg|_{t=0} (\Phi_t)^* W_{\Phi_t(u)} \\
&= \frac{d}{dt}\bigg|_{t=0} \left(\sum_{j=1}^{n} b_j(x_1 + t, x_2, ..., x_n) \left.\frac{\partial}{\partial x_j}\right|_u \right) \\
&= \sum_{j=1}^{n} \frac{\partial b_j}{\partial x_1}(x_1, ..., x_n) \left.\frac{\partial}{\partial x_j}\right|_u
\end{aligned}$$

Therefore, by comparing this to the expression of the Lie bracket in coordinates, we see that $L_V W = [V, W]$.

Second, we will show that $L_V W = [V, W]$ on $\operatorname{supp} V$. This follows from the continuity and he fact that $\operatorname{supp} V$ is the closure of $R(V)$.

Finally we will show that $L_V W = [V, W]$ on $X - \operatorname{supp} V$. Let $p \in X - \operatorname{supp} V$. Then $V \equiv 0$ on a neighborhood U of p, that is $[V, W] = L_V W = 0$ on U, since $[V, W] = VW - WV$ and the flow map of the zero vector field is a constant. $\square$

Theorem 3.1.17. *If V and W are vector fields with flows Φ_t, Ψ_t, then $[V, W] = 0$ if and only if $\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$ holds on the domain of the composition.*

Proof. First let us assume that $\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$. Let x be in the domain of the composition, we have:

$$\frac{d}{ds}(\Phi_t \circ \Psi_s)(x) = d_{\Psi(x)}\Phi_t \frac{d}{ds}\Psi_s(x) = (d_{\Psi_s(x)}\Phi_t)(W_{\Psi_s(x)}) = ((\Phi_t)_* W)_{\Phi_t(\Psi_s(x))}$$

$$\frac{d}{ds}\bigg|_{s=0}(\Phi_t \circ \Psi_s)(x) = ((\Phi_t)_* W)_{\Phi_t(x)}$$

On the other hand we have:

$$\frac{d}{ds}\bigg|_{s=0} \Psi_s \circ \Phi_t(x) = W_{\Phi_t(x)}$$

Therefore, $((\Phi_t)_* W)_{\Phi_t(x)} = W_{\Phi_t(x)}$ for all t. Let $y = (\Phi_t^{-1})(x) = \Phi_{-t}(x)$. Since the above identity is true for all t, by replacing $\Phi_t(x)$ by $\Phi_{-t}(x)$ we can see that:

$$(\Phi_t^* W)_y = W_y \iff \Phi_t^* W = W$$

Differentiating at $t = 0$ we get:

$$\frac{d}{dt}\bigg|_{t=0} \Phi_t^* W = L_V W$$

$$\frac{d}{dt}\bigg|_{t=0} W = 0$$

$\Longrightarrow L_V W = 0$
$\Longrightarrow [V, W] = 0$ by Proposition 3.1.16.
Conversely, we assume that $[V, W] = 0$ (i.e. $L_V W = 0$).
By Equation (3.1) page 25 we have:

$$\frac{d}{dt}\Phi_t^* W = \Phi_t^*(L_V W)$$

and since $L_V W = 0$ we have:

$$\Phi_t^* W = W \Longleftrightarrow (\Phi_t)_* W = W_{\Phi_t}$$

Consider the path $\mathbb{R} \ni s \mapsto \gamma(s) = \Phi_t(\Psi_s(x))$. We have:

$$\frac{d\gamma}{ds}(s) = (\Phi_t)_* \frac{d}{ds}\Psi_s(x) = (\Phi_t)_* W_{\Psi_s(x)} = W_{\Phi_t(\Psi_s(x))} = W_{\gamma(s)}$$

So, γ is an integral curve of W.
Now we consider the path $\mathbb{R} \ni s \mapsto \sigma(s) = \Psi_s(\Phi_t(x))$. We have:

$$\frac{d\sigma}{ds}(s) = W_{\sigma(s)}$$

So, σ is an integral curve of W. Moreover, we have:

$$\gamma(0) = \Phi_t(\Psi_0(x)) = \Phi_t(x)$$

$$\sigma(0) = \Psi_0(\Phi_t(x)) = \Phi_t(x)$$

Therefore, by the uniqueness of integral curves, we have $\gamma(s) = \sigma(s) \Longrightarrow \Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$. □

Theorem 3.1.18. *If V and W are vector fields with flows Φ_t, Ψ_t respectively and $t > 0$, then:*

$$[V, W]_x = \left.\frac{d}{dt}\right|_{t=0} \Psi_{-\sqrt{t}} \circ \Phi_{-\sqrt{t}} \circ \Psi_{\sqrt{t}} \circ \Phi_{\sqrt{t}}(x)$$

Proof. Recall the Taylor expansion formula of $\Phi_t(x)$ and $\Psi_t(x)$ at $t = 0$:

$$\Phi_t(x) = \Phi_0(x) + t\dot{\Phi}_0(x) + \frac{t^2}{2}\ddot{\Phi}_0(x) + O(t^3)$$
$$\Psi_t(x) = \Psi_0(x) + t\dot{\Psi}_0(x) + \frac{t^2}{2}\ddot{\Psi}_0(x) + O(t^3)$$

First, notice that $\frac{d}{dt}\Phi_t(x) = V_{\Phi_t(x)}$ and $\frac{d}{dt}\Psi_t(x) = W_{\Psi_t(x)}$.
Second, we see that $\frac{d^2}{dt^2}\Phi_t(x) = J_{\Phi_t(x)}V \cdot V_{\Phi_t(x)}$ and $\frac{d^2}{dt^2}\Psi_t(x) = J_{\Psi_t(x)}W \cdot W_{\Psi_t(x)}$.
So,

$$\Phi_t(x) = x + tV_x + \frac{t^2}{2} J_x V \cdot V_x + O(t^3)$$

$$\Psi_t(x) = x + tW_x + \frac{t^2}{2} J_x W \cdot W_x + O(t^3)$$

First we calculate $\Psi_{\sqrt{t}} \circ \Phi_{\sqrt{t}}(x)$:

$$\begin{aligned}
\Psi_{\sqrt{t}}(\Phi_{\sqrt{t}}(x)) &= \Phi_{\sqrt{t}}(x) + \sqrt{t} W_{\Phi_{\sqrt{t}}(x)} + \frac{t}{2} J_{\Phi_{\sqrt{t}}(x)} \cdot W_{\Phi_{\sqrt{t}}(x)} + O(t\sqrt{t}) \\
&= x + \sqrt{t} V_x + \sqrt{t} W_x \\
&\quad + t\left(\frac{J_x V}{2} \cdot V_x + J_x W \cdot V_x + \frac{J_x W}{2} \cdot W_x\right) + O(t\sqrt{t})
\end{aligned}$$

Second, we calculate $\Psi_{-\sqrt{t}} \circ \Phi_{-\sqrt{t}}(x)$:

$$\begin{aligned}
\Psi_{-\sqrt{t}}(\Phi_{-\sqrt{t}}(x)) &= \Phi_{-\sqrt{t}}(x) - \sqrt{t} W_{\Phi_{-\sqrt{t}}(x)} + \frac{t}{2} J_{\Phi_{-\sqrt{t}}(x)} \cdot W_{\Phi_{-\sqrt{t}}(x)} + O(t\sqrt{t}) \\
&= x - \sqrt{t} V_x + \frac{t}{2} J_x V \cdot V_x - \sqrt{t} W_x \\
&\quad + t J_x W . V_x + \frac{t}{2} J_x W \cdot W_x + O(t\sqrt{t})
\end{aligned}$$

Finally, we will calculate $\Psi_{-\sqrt{t}} \circ \Phi_{-\sqrt{t}} \circ \Psi_{\sqrt{t}} \circ \Phi_{\sqrt{t}}(x)$:

$$\begin{aligned}
\Psi_{-\sqrt{t}}(\Phi_{-\sqrt{t}}(\Psi_{\sqrt{t}}(\Phi_{\sqrt{t}}(x)))) &= x + \sqrt{t} V_x + \frac{t}{2} J_x V \cdot V_x + \sqrt{t} W_x + t J_x W \cdot V_x \\
&\quad + \frac{t}{2} J_x W \cdot W_x - \sqrt{t}(V_x + \sqrt{t} J_x V \cdot V_x \\
&\quad + \sqrt{t} J_x V \cdot W_x) + \frac{t}{2} J_x V \cdot V_x \\
&\quad - \sqrt{t}\left(W_x + \sqrt{t} J_x W \cdot V_x + \sqrt{t} J_x W \cdot W_x\right) \\
&\quad + t J_x W \cdot V_x + \frac{t}{2} J_x W \cdot W_x + O(t\sqrt{t}) \\
&= x + t J_x W \cdot V_x - t J_x V \cdot W_x + O(t\sqrt{t}) \\
&= x + t \cdot [V, W]_x + O(t\sqrt{t})
\end{aligned}$$

And so, we have:

$$[V, W]_x = \left.\frac{d}{dt}\right|_{t=0} \Psi_{-\sqrt{t}} \circ \Phi_{-\sqrt{t}} \circ \Psi_{\sqrt{t}} \circ \Phi_{\sqrt{t}}(x) \qquad \square$$

Remark 3.1.19. *Theorem 3.1.18 is stated only for $t > 0$. What if $t < 0$?. We know that $\Phi_{-t} = \widetilde{\Phi_t}$ is the flow map of the vector field $-V$ and $\Psi_{-t} = \widetilde{\Psi_t}$ is the* *flow map of $-W$. Let $t<0$, then $-t>0$. This means that $\widetilde{\Psi}_{-\sqrt{-t}}=\Psi_{\sqrt{-t}}$ and $\widetilde{\Phi}_{-\sqrt{-t}}=\Phi_{\sqrt{-t}}$. So, by Theorem 3.1.18 we have:*

$$\begin{aligned}[V,W]_x=[-V,-W]_x&=\left.\frac{d}{dt}\right|_{t=0}\widetilde{\Psi}_{-\sqrt{-t}}\circ\widetilde{\Phi}_{-\sqrt{-t}}\circ\widetilde{\Psi}_{\sqrt{-t}}\circ\widetilde{\Phi}_{\sqrt{-t}}(x)\\&=\left.\frac{d}{dt}\right|_{t=0}\Psi_{\sqrt{-t}}\circ\Phi_{\sqrt{-t}}\circ\Psi_{-\sqrt{-t}}\circ\Phi_{-\sqrt{-t}}(x)\end{aligned}$$

Therefore in general, for any $t\in\mathbb{R}$ we have:

$$[V,W]_x=\left.\frac{d}{dt}\right|_{t=0}\Psi_{-\operatorname{sgn}(t)\sqrt{|t|}}\circ\Phi_{-\operatorname{sgn}(t)\sqrt{|t|}}\circ\Psi_{\operatorname{sgn}(t)\sqrt{|t|}}\circ\Phi_{\operatorname{sgn}(t)\sqrt{|t|}}(x)$$

3.2 Algorithms and Computing Flows

The main purpose of this subsection is to show that a flow of a vector field V which is determined by finitely many vector fields $V_1,\ldots,V_m$ can be approximated by flow maps of these finitely many vector fields vector fields. See [5, Chapter 4].

Definition 3.2.1. *Let V be a continuous vector field on a manifold X, let $A(t,x)=A_t(x)$ be a continuous map $A\colon U\to X$ where $\{0\}\times X\subset U\subset[0,\infty)\times X$ is open. A is said to be an algorithm for V if:*

1. *$A_0(x)=x\ \forall x\in X$.*
2. *$V_x=\left.\frac{\partial}{\partial t}\right|_{t=0}A_t(x)\ \forall x\in X$*
3. *$A_t(x)$ is C^1 in t with derivatives continuous in (t,x)*

Theorem 3.2.2. *Let V be a Lipschitz continuous vector field with flow Φ_t on a manifold X. Let Ω be the fundamental domain of V and $\Omega_+=\Omega\cap([0,\infty)\times X)$. If A is an algorithm for V, then for all $(t,x)\in\Omega_+$ the n-th iterate $A^n_{\frac{t}{n}}(x)$ of the map $A_{\frac{t}{n}}$ is defined for sufficiently large $n\in\mathbb{N}$ ($n=n(x,t)$), and we have that*

$$\lim_{n\to\infty}A^n_{\frac{t}{n}}(x)=\Phi_t(x)$$

The convergence is uniform on compacts in Ω_+. Conversely, if $A^n_{\frac{t}{n}}(x)$ is defined and converges for $0\le t\le t_0$ then $(t_0,x)\in\Omega_+$ and $\lim_{n\to\infty}A^n_{\frac{t}{n}}(x)=\Phi_t(x)$.

Proof. For $X=\mathbb{R}^m$.
Let $p\in\mathbb{R}^m$ be fixed, and suppose that $\Phi_t(p)$ exists for $t\in[0,t_0]$.
Let $C=\{\Phi_t(p):t\in[0,t_0]\}$. Notice that $C=\Phi(\{p\}\times[0,t_0])$, so C is a compact subset of $\mathbb{R}^m$.

Choose compacts $L_1 \subset L_2 \subset \mathbb{R}^m$ such that $C \subset \mathring{L}_1 \subset L_1 \subset \mathring{L}_2 \subset L_2$.
We claim that there is a compact neighborhood $K \subset \mathring{L}_1$ of p such that $\forall x \in K$ the flow $\Phi_t(x)$ exists for $t \in [0, t_0]$ and remains in L_1. Indeed, Φ is uniformly continuous on L_1 since Φ is continuous on the compact set L_1, that is for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in L_1$ and for all $t \in [0, t_0]$ we have $\|\Phi_t(x) - \Phi_t(y)\| < \epsilon$ whenever $\|x - y\| < \delta$. Let $\epsilon = \frac{1}{2}\operatorname{dist}(C, X - L_1)$ and let $K = \overline{B(p, \delta)}$. So K is indeed compact and is a subset of $\mathring{L}_1$. Now, let $x \in K$, we will show that $\Phi_t(x) \in L_1$. Since $x \in K$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset B(p, \delta)$ such that $\|x_n - x\| \to 0$ as $n \to \infty$ (Note that by the continuity of $\Phi_t(\cdot)$ we have $\|\Phi_t(x_n) - \Phi_t(x)\| \to 0$ as $n \to \infty$). Now,

$$\begin{aligned} \|x_n - p\| < \delta &\Rightarrow \|\Phi_t(x_n) - \Phi_t(p)\| < \epsilon \\ &\Rightarrow \Phi_t(x_n) \in B(\Phi_t, p) \text{ for all } n \in \mathbb{N} \\ &\Rightarrow \Phi_t(x) \in \overline{B(\Phi_t(p), \epsilon)} \subset L_1 \end{aligned}$$

The definition of an algorithm tells us that:

$$A_0(x) = x \qquad \left.\frac{\partial}{\partial t}\right|_{t=0} A_t(x) = V_x$$

We also know that:

$$\Phi_0(x) = x \qquad \left.\frac{\partial}{\partial t}\right|_{t=0} \Phi_t(x) = V_x$$

$$\begin{aligned} &\Rightarrow \left.\frac{\partial}{\partial t}\right|_{t=0} (A(t, x) - \Phi_t(x)) = 0 \\ &\Rightarrow \lim_{t\to 0} \frac{(A(t, x) - \Phi_t(x)) - (A_0(x) - \Phi_0(x))}{t - 0} = 0 \\ &\Rightarrow \lim_{t\to 0} \frac{A(t, x) - \Phi_t(x)}{t} = 0 \\ &\Rightarrow |\Phi_t(x) - A_t(x)| = o(t) \text{ uniformly on } L_2 \end{aligned}$$

Let's show that why do we have $|\Phi_t(x) - A_t(x)| = o(t)$ uniformly on L_2. Let $F(t, x) = \frac{A(t,x) - \Phi_t(x)}{t}$. Let $x_0 \in L_2$ and let $\varepsilon > 0$. By the continuity of $F(t, x)$, there exists $\gamma > 0$ such that $|F(t, y) - F(t, x_0)| < \frac{\varepsilon}{2}$ whenever $y \in B(x_0, \gamma)$. Moreover, there exists a $\delta > 0$ such that $|F(t, x_0)| < \frac{\varepsilon}{2}$ whenever $|t| < \delta$.
So, $|F(t, y)| < |F(t, y) - F(t, x_0)| + |F(t, x_0)| < \varepsilon$ whenever $y \in B(x_0, \gamma)$ and $|t| < \delta$. We do this for every x_0, and we get an open cover for the set L_2. By the compactness of L_2 there exists a finite sub-cover where $L_2 \subset \bigcup_{i=1}^{N} B(x_i, \gamma_i)$. Since any $y \in L_2$ is contained in one of the balls $B(x_i, \gamma_i)$ we have $|F(t, y)| < \varepsilon$ for $|t| < \delta$.

Now, we fix $n \in \mathbb{N}$ and let $x \in K$. Assume that for the moment that the orbits,

$$y_0 = x, \quad y_1 = A_{\frac{t}{n}}(y_0), \quad y_2 = A_{\frac{t}{n}}(y_1), \ldots, \quad y_n = A_{\frac{t}{n}}(y_{n-1})$$

exist and they lie in L_2.

Lemma 3.1.5 gives us the estimate:

$$|\Phi_t(x) - \Phi_t(y)| \leq \mathrm{e}^{\beta t}|x - y|$$

Also, notice that $\Phi^k_{\frac{t}{n}}(x) = \Phi_{\frac{kt}{n}}(x)$ by the group law for $k = 1, \dots, n$.

$$\underline{\text{Claim}} : \Phi_{\frac{kt}{n}}(x) - A^k_{\frac{t}{n}} = \sum_{j=1}^{k} (\Phi^{k-j}_{\frac{t}{n}}(\Phi_{\frac{t}{n}}(y_{j-1})) - \Phi^{k-j}_{\frac{t}{n}}(A_{\frac{t}{n}}(y_{j-1})))$$

Indeed,

$$\begin{aligned}
\sum_{j=1}^{k} (\Phi^{k-j}_{\frac{t}{n}}(\Phi_{\frac{t}{n}}(y_{j-1})) - \Phi^{k-j}_{\frac{t}{n}}(A_{\frac{t}{n}}(y_{j-1}))) =&\Phi^{k-1}_{\frac{t}{n}}(\Phi_{\frac{t}{n}}(y_0)) - \Phi^{k-1}_{\frac{t}{n}}(A_{\frac{t}{n}}(y_0)) \\
&+\Phi^{k-2}_{\frac{t}{n}}(\Phi_{\frac{t}{n}}(y_1)) - \Phi^{k-2}_{\frac{t}{n}}(A_{\frac{t}{n}}(y_1)) \\
&\vdots \\
&+\Phi_{\frac{t}{n}}(\Phi_{\frac{t}{n}}(y_{k-2})) - \Phi_{\frac{t}{n}}(A_{\frac{t}{n}}(y_{k-2})) \\
&+\Phi_{\frac{t}{n}}(y_{k-1}) - A_{\frac{t}{n}}(y_{y_{k-1}}) \\
=&\Phi_{\frac{kt}{n}}(x) - A^k_{\frac{t}{n}}(x)
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow |\Phi_{\frac{kt}{n}}(x) - A^k_{\frac{t}{n}}(x)| &\leq \sum_{j=1}^{k} |\Phi^{k-j}_{\frac{t}{n}}(\Phi_{\frac{t}{n}}(y_{j-1})) - \Phi^{k-j}_{\frac{t}{n}}(A_{\frac{t}{n}}(y_{j-1}))| \\
&\leq \sum_{j=1}^{k} \mathrm{e}^{\beta \frac{t(k-j)}{n}} |\Phi_{\frac{t}{n}}(y_{j-1}) - A_{\frac{t}{n}}(y_{j-1})| \\
&\leq \mathrm{e}^{\beta \frac{tk}{n}} \cdot k \cdot o\left(\frac{t}{n}\right), k = 1, \dots, n \\
\Longrightarrow |\Phi_{\frac{kt}{n}}(x) - A^k_{\frac{t}{n}}(x)| &\leq \mathrm{e}^{\beta \frac{tk}{n}} \cdot k \cdot o\left(\frac{t}{n}\right), k = 1, \dots, n
\end{aligned}$$

Now we will show that the orbits do exist and they lie in L_2 by induction. First of all, note that A is continuous and L_2 is compact. So there exists some time t_2 such that $A_t(x)$ exists for $x \in L_2$ and $t \in [0, t_2]$. Choose n large enough (that is $\frac{t}{n} < t_2$) so that $A_{\frac{t}{n}}$ exists.

For $k = 0$, we have $y_0 = x$ which exists trivially.

Now assume that $y_k = A_{\frac{t}{n}}(y_{k-1})$ is well defined and lies in L_2. We will show that $y_{k+1} = A_{\frac{t}{n}}(y_k)$ is well defined and lies in L_2.

Indeed, y_{k+1} is well defined since y_k lies in L_2 by the induction hypothesis. We just need to show that y_{k+1} lies in L_2. Notice that $\lim_{n\to\infty} \mathrm{e}^{\beta \frac{tk}{n}} \cdot k \cdot o(\frac{t}{n}) = 0$, that is, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n > N$ we have $\left|e^{\beta \frac{tk}{n}} \cdot k \cdot o(\frac{t}{n})\right| < \epsilon$. For every $x \in K$ set $C_x = \{\Phi_t(x) : t \in [0, t_2]\}$ let $\epsilon = \frac{1}{2} \inf_{x \in K}(\text{dist}(C_x, X - L_2))$. So we will have that:

$$|\Phi_{\frac{kt}{n}}(x) - A^k_{\frac{t}{n}}(x)| < \epsilon$$

which shows that $A_{\frac{t}{n}}(y_k) \in L_2$.

Now, for the converse statement we have assumed that the iterates $A^n_{\frac{t}{n}}(x)$ are well-defined and converges for $0 \leq t \leq t_0$. We just need to show that $\lim_{n\to\infty} A^n_{\frac{t}{n}}(x) = \Phi_t(x)$. The flow map Φ_t exists for sufficiently small t, that is, set $t_1 = \inf\{t > 0 : \Phi_t \text{ does not exist}\}$, so Φ_t exists for $t \in [0, t_1)$. Moreover, by using the inequality shown above for $k = n$;

$$|\Phi_t(x) - A^n_{\frac{t}{n}}(x)| \leq e^{\beta t} \cdot n \cdot o\left(\frac{t}{n}\right)$$

we have $\lim_{n\to\infty} A^n_{\frac{t}{n}}(x) = \Phi_t(x)$ for $t \in [0, t_1)$. But, we wish to show that this is true for $t \in [0, t_0]$. For the sake of reaching a contradiction, suppose that Φ_t does not exist for some $t \in [0, t_0]$. By Remark 3.1.7 of the Escape lemma, we have that:

$$\lim_{t\to t_1} \|\Phi_t(x)\| = \infty$$

We showed earlier that if Φ_t exists, then:

$$|\Phi_t(x) - A^n_{\frac{t}{n}}(x)| \leq e^{\beta t} \cdot n \cdot o\left(\frac{t}{n}\right) \quad \text{for } t \in [0, t_1)$$

For fixed $n \in \mathbb{N}$ we have,

$$\begin{aligned} \|\Phi_t(x)\| &= \|\Phi_t(x) - A_{\frac{t}{n}}(x) + A_{\frac{t}{n}}(x)\| \\ &\leq \|\Phi_t(x) - A_{\frac{t}{n}}(x)\| + \|A_{\frac{t}{n}}(x)\| \\ &\leq e^{\beta t} \cdot n \cdot \mathrm{o}\left(\frac{t}{n}\right) + \|A_{\frac{t}{n}}(x)\| \end{aligned}$$

Allowing $t \to t_1$, we get infinity on the left side and a constant on the right side, contradiction! Therefore our assumption is false, hence Φ_t exists for $t \in [0, t_0]$, which implies that $\lim_{n\to\infty} A_{\frac{t}{n}}(x) = \Phi_t(x)$. □

Proposition 3.2.3. *Let V and W be vector fields with flows Φ_t and Ψ_t respectively. Then,*

1. *$\Phi_t \circ \Psi_t$ is an algorithm for $V + W$.*

2. *$\Psi_{-\sqrt{t}} \circ \Phi_{-\sqrt{t}} \circ \Psi_{\sqrt{t}} \circ \Phi_{\sqrt{t}}$ is an algorithm for $[V, W]$.*

Proof. 1. By the Taylor expansion of flows:

$$\Phi_t(x) = x + tV_x + O(t^2)$$

$$\Psi_t(x) = x + tW_x + O(t^2)$$

so that, $\Phi_t(\Psi_t(x)) = x + tW_x + tV_x + O(t^2)$.

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (\Phi_t \circ \Psi_t)(x) = V_x + W_x = (V + W)_x$$

Also, $(\Phi_0 \circ \Psi_0)(x) = \Phi_0(x) = x$. Therefore: $A(t, x) = \Phi_t \circ \Psi_t(x)$ is an algorithm for $V + W$.

Let ζ_t be the flow of $V + W$, then by Theorem 3.2.2 we have:

$$\lim_{n\to+\infty} (\Phi_{\frac{t}{n}} \circ \Psi_{\frac{t}{n}})^n(x) = \zeta_t(x)$$

2. Let $A(t, x) = (\Psi_{-\sqrt{t}} \circ \Phi_{-\sqrt{t}} \circ \Psi_{\sqrt{t}} \circ \Phi_{\sqrt{t}})(x)$. By Theorem 3.1.18 we have:

$$\left.\frac{\partial}{\partial t}\right|_{t=0} A(t, x) = [V, W]_x, \quad A(0, x) = x$$

so, $A(t, x)$ is an algorithm for $[V, W]$.

Let η_t be the flow map of $[V, W]$. So by Theorem 3.2.2 we have:

$$\lim_{n\to+\infty} (\Psi_{-\sqrt{\frac{t}{n}}} \circ \Phi_{-\sqrt{\frac{t}{n}}} \circ \Psi_{\sqrt{\frac{t}{n}}} \circ \Phi_{\sqrt{\frac{t}{n}}})^n(x) = \eta_t(x)$$

□

Remark 3.2.4. *In Proposition 3.2.3 part 1, we can assume Φ_t and Ψ_t to be algorithms of V and W respectively, the same result will follow.*

By a repeated application of this proposition and the preceding theorem we have the following important result. But first, we need a couple of definitions.

Definition 3.2.5. *A Lie algebra $\mathfrak{g}$ over a field F is a F-vector space endowed with a map called the Lie bracket from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$, usually denoted by $(V, W) \mapsto [V, W]$, that satisfies the following properties for all V_1,V_2,$V_3 \in \mathfrak{g}$:*

1. *Bilinearity: For $a, b \in F$*

$$[aV_1 + bV_2, V_3] = a[V_1, V_3] + b[V_2, V_3]$$

$$[V_3, aV_1 + bV_2] = a[V_3, V_1] + b[V_3, V_2]$$

2. *Antisymetry:*

$$[V_1, V_1] = 0$$

3. Jacobi Identity:

$$[V_1, [V_2, V_3]] + [V_2, [V_3, V_1]] + [V_3, [V_1, V_2]] = 0$$

Definition 3.2.6. *Let $\mathfrak{g}$ be a Lie algebra over F. Then a linear subspace $U \subset \mathfrak{g}$ is a Lie subalgebra if U is closed under the Lie bracket of $\mathfrak{g}$:*

$$[V_1, V_2] \in U \quad \forall V_1, V_2 \in U$$

Definition 3.2.7. *Let $\mathfrak{g}$ be a Lie algebra over F, and let $U \subset \mathfrak{g}$. We call $\langle U \rangle$ the Lie subalgebra generated by U, where:*

$$\langle U \rangle = \bigcap \{ I \subset \mathfrak{g} : I \text{ is a Lie subalgebra of } \mathfrak{g} \text{ containing } U \}$$

Corollary 3.2.8. *Let $V_1, \ldots, V_m$ be $\mathbb{R}$-complete holomorphic vector fields on a complex manifold X. Let $V \in \langle V_1, \ldots, V_m \rangle$. Assume that K is a compact set in X and $t_0 > 0$ is such that the flow $\Phi_t(x)$ of V exists for every $x \in K$ and for all $t \in [0, t_0]$. Then Φ_{t_0} is a uniform limit on K of a sequence of compositions of time forward maps of the vector fields $V_1, \ldots, V_m$. In particular Φ_{t_0} can be approximated uniformly on K by holomorphic automorphisms of X.*

Proof. We will prove the following result using induction.
First, let us define the sets that we need in order to proceed.

$$\begin{aligned} U_0 &= U = \{V_1, \ldots, V_m\} \\ U_{k+1} &= \operatorname{span}\{[W_1, W_2], W_1 : W_1, W_2 \in U_k\} \end{aligned}$$

We claim that $\langle U \rangle = \bigcup_{k \in \mathbb{N}} U_k$. To prove this, we have to show that $\bigcup_{k \in \mathbb{N}} U_k$ is the smallest Lie algebra containing U. It is trivial to see that $U \subset \bigcup_{k \in \mathbb{N}} U_k$. Now we have to show that $\bigcup_{k \in \mathbb{N}} U_k$ is a Lie algebra. Notice that $U_0 \subset U_1 \subset \cdots \subset U_k \subset U_{k+1} \subset \ldots$. So, $\bigcup_{k \in \mathbb{N}} U_k$ is a Lie algebra.

We just have to show that $\bigcup_{k \in \mathbb{N}} U_k$ is the smallest Lie algebra containing U. Indeed, let L be a Lie algebra containing U, and let $W \in \bigcup_{k \in \mathbb{N}} U_k \Rightarrow \exists k \in \mathbb{N} : W \in U_k \Rightarrow W$ is an element formed by taking a certain number of successive $[\cdot, \cdot]$-operations on the vector fields in U, but this implies that $W \in L$, since L is closed under taking $[\cdot, \cdot]$'s. Therefore $\langle U \rangle = \bigcup_{k \in \mathbb{N}} U_k$.

Step 1: The base case, the flow of every element in U_0 can be approximated uniformly on compacts by a finite number of composition of complete flow maps of the vector fields $V_1, \ldots, V_m$. Well, this is quite trivial, there is no need for an approximation, since the flow is equal to itself everywhere, particularly on compacts.

Step 2: The inductive step, suppose that the flow map of every element in U_k can be approximated uniformly on compacts by a finite number of composition of complete flow maps of the vector fields $V_1, \ldots, V_m$, we need to show that the same is true for every flow map in U_{k+1}.

For $V = [W_1, W_2]$ such that $W_1, W_2 \in U_k$. Let $\Phi_t^{W_1}$ and $\Phi_t^{W_2}$ be the flow maps for W_1 and W_2 respectively. By the induction hypothesis, there exists flow maps $\varphi_t^1, \ldots, \varphi_t^r, \psi_t^1, \ldots, \psi_t^s$ of the complete vector fields $V_1, \ldots, V_m$, such that for any given $\epsilon > 0$ we have:

$$\sup_{x \in K, t \in [0,t_0]} (\mathrm{dist}(\Phi_t^{W_1}(x), (\varphi_t^r \circ \cdots \circ \varphi_t^1)(x)) < \epsilon$$
$$\sup_{x \in K, t \in [0,t_0]} (\mathrm{dist}(\Phi_t^{W_2}(x), (\psi_t^s \circ \cdots \circ \psi_t^1)(x)) < \epsilon$$

By part 2 of the preceding proposition we have that, $\Phi^{W_2}_{-\sqrt{t}} \circ \Phi^{W_1}_{-\sqrt{t}} \circ \Phi^{W_2}_{\sqrt{t}} \circ \Phi^{W_1}_{\sqrt{t}}$ is an algorithm for $V = [W_1, W_2]$. Moreover the preceding theorem gives us the following:

$$\lim_{n \to \infty} (\Phi^{W_2}_{-\sqrt{\frac{t}{n}}} \circ \Phi^{W_1}_{-\sqrt{\frac{t}{n}}} \circ \Phi^{W_2}_{\sqrt{\frac{t}{n}}} \circ \Phi^{W_1}_{\sqrt{\frac{t}{n}}})^n(x) = \zeta_t(x)$$

uniformly on K, where ζ_t is the flow map of V.

Since the convergence is uniform on K we have, $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$; $\forall n > N$ and $\forall x \in K$ $\mathrm{dist}((\Phi^{W_2}_{-\sqrt{\frac{t}{n}}} \circ \Phi^{W_1}_{-\sqrt{\frac{t}{n}}} \circ \Phi^{W_2}_{\sqrt{\frac{t}{n}}} \circ \Phi^{W_1}_{\sqrt{\frac{t}{n}}})^n(x), \zeta_t(x)) < \frac{\epsilon}{2}$. Let

$$\begin{aligned} f_n(x) = ((&\psi^s_{-\sqrt{\frac{t}{n}}} \circ \cdots \circ \psi^1_{-\sqrt{\frac{t}{n}}}) \circ (\varphi^r_{-\sqrt{\frac{t}{n}}} \circ \cdots \circ \varphi^1_{-\sqrt{\frac{t}{n}}}) \circ (\psi^s_{\sqrt{\frac{t}{n}}} \circ \cdots \circ \psi^1_{\sqrt{\frac{t}{n}}}) \circ \\ &(\varphi^r_{\sqrt{\frac{t}{n}}} \circ \cdots \circ \varphi^1_{\sqrt{\frac{t}{n}}}))^n(x) \\ g_n(x) = (&\Phi^{W_2}_{-\sqrt{\frac{t}{n}}} \circ \Phi^{W_1}_{-\sqrt{\frac{t}{n}}} \circ \Phi^{W_2}_{\sqrt{\frac{t}{n}}} \circ \Phi^{W_1}_{\sqrt{\frac{t}{n}}})^n(x) \end{aligned}$$

We wish to show that $\lim_{n \to \infty} f_n(x) = \zeta_t(x)$ uniformly on K.

Let $x \in K$, by the triangular inequality we have:

$$\begin{aligned} \|\zeta_t(x) - f_n(x)\|_K &= \|\zeta_t(x) - f_n(x) - g_n(x) + g_n(x)\|_K \\ &\leq \|\zeta_t(x) - g_n(x)\|_K + \|f_n(x) - g_n(x)\|_K \\ &\leq \epsilon + \|f_n(x) - g_n(x)\|_K \end{aligned}$$

Now, concerning $\|f_n(x) - g_n(x)\|_K$

Consider the map defined by $(\mathrm{End}(X))^2 \times \mathbb{R} \to \mathrm{End}(X)$ defined by $(\Phi, \Psi, t) \mapsto (\Psi_{-\sqrt{\frac{t}{n}}} \circ \Phi_{-\sqrt{\frac{t}{n}}} \circ \Psi_{\sqrt{\frac{t}{n}}} \circ \Phi_{\sqrt{\frac{t}{n}}})$ which is continuous in the compact open topology. So, $\|\zeta_t(x) - f_n(x)\|_K < \epsilon$.

Finally let us deal with linear combinations. Suppose $V = W_1 + W_2$, such that $W_1, W_2 \in U_k$ and let $\Phi_t^{W_1}, \Phi_t^{W_2}$ be the flow maps of W_1 and W_2 respectively. We follow the same method of proof that we did concerning the Lie bracket $[\cdot, \cdot]$. The only difference is that the algorithm for V is $\Phi_t^{W_1} \circ \Phi_t^{W_2}$. $\square$

Chapter 4

Vector Fields On The Koras–Russell Cubic

In this chapter we study some particular vector fields on the Koras–Rusell cubic. Moreover we show that they are complete.

In [11], Leuenberger defines vector fields on a space X as follows. Let $n \geq k \geq 0$ and let $a, b \in \mathbb{C}[z_0, \dots, z_n]$ such that $\deg_{z_i}(a) \leq 2$ and $\deg_{z_i}(b) \leq 1$ for all $i \leq k$. Let $\bar{z} = (z_0, \dots, z_n)$ and $X = \{x^2 y = a(\bar{z}) + x b(\bar{z})\}$. Now, we define the following vector fields on X:

$$v_x^i = (\frac{\partial a}{\partial z_i} + x\frac{\partial b}{\partial z_i})\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z_i} \text{ and } v_y^j = (\frac{\partial a}{\partial z_j} + x\frac{\partial b}{\partial z_j})\frac{\partial}{\partial x} + (2xy - b(\bar{z}))\frac{\partial}{\partial z_j}$$

for $0 \leq i \leq n$ and $0 \leq j \leq k$ and moreover, let

$$v_z = a(\bar{z})x\frac{\partial}{\partial x} - (2a(\bar{z})y - xyb(\bar{z}) + b^2(\bar{z}))\frac{\partial}{\partial y}$$

Notice that, the Koras–Russell Cubic is this space X for $a(\bar{z}) = -z_0^2 - z_1^3$ and $b(\bar{z}) = -1$ where $\bar{z} = (z_0, z_1)$. We will rename these vector fields (associated to the Koras–Russell cubic) from v_x^0, v_x^1, v_y^0, v_z to U, V, W, E . They turn out to be complete vector fields, that is, their flow maps are defined for all time.

Theorem 4.0.1. *The following four vector fields*

$$\begin{aligned}
U &= -2z\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}\\
V &= -3w^2\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial w}\\
W &= -2z\frac{\partial}{\partial x} + (2xy+1)\frac{\partial}{\partial z})\\
E &= -x(z^2+w^3)\frac{\partial}{\partial x} - (-2(z^2+w^3)y + xy + 1)\frac{\partial}{\partial y}\\
H &= 6x\frac{\partial}{\partial x} - 6y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z} + 2w\frac{\partial}{\partial w}
\end{aligned}$$

on $X = \{(x,y,z,w) \in \mathbb{C}^4 : x^2y + x + z^2 + w^3 = 0\}$ *are complete.*

Proof.

1. Let $\gamma(t) = (x(t), y(t), z(t), w(t))$ be an integral curve for the vector field $U = -2z\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}$. This implies that $\dot\gamma(t) = U_{\gamma(t)}$, which is equivalent to the following system of Ordinary differential equations,

$$\begin{aligned}
\dot x(t) &= 0\\
\dot y(t) &= -2z(t)\\
\dot z(t) &= x^2(t)\\
\dot w(t) &= 0
\end{aligned}$$

$\implies x(t) = c_1$, $\dot y(t) = -2(c_1^2t + c_3)$, $z(t) = c_1^2t + c_3$ and $w(t) = c_4$. So, we have:

$$\begin{aligned}
x(t) &= c_1\\
y(t) &= -c_1^2t^2 + c_3t + c_2\\
z(t) &= c_1^2t + c_3\\
w(t) &= c_4
\end{aligned}$$

$\implies \gamma(t) = (c_1, -c_1^2t^2 + c_3t + c_2, c_1^2t + c_3, c_4)$. Therefore the flow map of U is given by,

$$\Phi_t(x,y,z,w) = (x, -x^2t^2 - 2zt + y, x^2t + z, w)$$

which is defined for all time.

2. Let $\delta(t) = (x(t), y(t), z(t), w(t))$ be an integral curve for the vector field V. So, $\dot\delta(t) = V_{\gamma(t)}$, which is equivalent to the following system of ordinary differential equations,

$$\begin{aligned}
\dot x(t) &= 0\\
\dot y(t) &= -3w^2(t)\\
\dot z(t) &= 0\\
\dot w(t) &= x^2(t)
\end{aligned}$$

$\Longrightarrow x(t) = c_1$, $y(t) = -c_1^4 t^3 - 3c_1^2 c_4 t^2 - 3c_4^2 t + c_2$, $z(t) = c_3$, $w(t) = c_1^2 t + c_4$. $\Longrightarrow \delta(t) = (c_1, -c_1^4 t^3 - 3c_1^2 c_4 t^2 - 3c_4^2 t + c_2, c_3, c_1^2 t + c_4)$. Therefore the flow map of V is given by,

$$\Psi_t(x, y, z, w) = (x, -x^4 t^3 - 3w^2 t - 3x^2 w t^2 + y, z, x^2 t + w)$$

which is defined for all time.

3. Consider the vector field W. Let $\epsilon(t) = (x(t), y(t), z(t), w(t))$ be an integral curve for W. So, $\dot{\epsilon}(t) = W_{\epsilon(t)}$, that is:

$$\begin{aligned}
\dot{x}(t) &= -2z(t) \\
\dot{y}(t) &= 0 \\
\dot{z}(t) &= 2x(t)y(t) + 1 \\
\dot{w}(t) &= 0
\end{aligned}$$

This implies that $y(t) = c_2$ and $w(t) = c_4$. Now, $\dot{z}(t) = 2c_2 x(t) + 1 \Rightarrow \ddot{z}(t) = 2c_2 \dot{x}(t) \Rightarrow \ddot{z}(t) = -4c_2 z(t)$. We have to solve the following ODE:

$$\ddot{z}(t) + 4c_2 z(t) = 0$$

The corresponding characteristic equation is $m^2 + 4c_2 = 0$ which implies that $m = \pm i \cdot 2\sqrt{c_2}$. So, this yields

$$z(t) = A_1 \cos(2\sqrt{c_2} t) + A_2 \sin(2\sqrt{c_2} t)$$

after taking the derivative of $z(t)$ we get:

$$\dot{z}(t) = -2A_1 \sqrt{c_2} \sin(2\sqrt{c_2} t) + 2A_2 \sqrt{c_2} \cos(2\sqrt{c_2} t)$$

but, $\dot{z}(t) = 2c_2 x(t) + 1$, which means:

$$\begin{aligned}
x(t) &= -A_1 \frac{\sqrt{c_2}}{c_2} \sin(2\sqrt{c_2} t) + A_2 \frac{\sqrt{c_2}}{c_2} \cos(2\sqrt{c_2} t) - \frac{1}{2c_2} \\
x(t) &= -A_1 \frac{1}{\sqrt{c_2}} \sin(2\sqrt{c_2} t) + A_2 \frac{1}{\sqrt{c_2}} \cos(2\sqrt{c_2} t) - \frac{1}{2c_2}
\end{aligned}$$

$\Longrightarrow \epsilon(t) = (-A_1 \frac{1}{\sqrt{c_2}} \sin(2\sqrt{c_2} t) + A_2 \frac{1}{\sqrt{c_2}} \cos(2\sqrt{c_2} t) - \frac{1}{2c_2}, c_2, A_1 \cos(2\sqrt{c_2} t) + A_2 \sin(2\sqrt{c_2} t), c_4)$.

Let $\varphi_t(x, y, z, w)$ be the flow map of W. We have, $\varphi_0(x, y, z, w) = (x, y, z, w)$ and $\dot{\varphi}_t(x, y, z, w) = W_{\varphi_t(x,y,z,w)}$. So,

$$\begin{aligned}
x &= \frac{A_2}{\sqrt{c_2}} - \frac{1}{2c_2} = \frac{A_2}{\sqrt{y}} - \frac{1}{2y} \\
y &= c_2 \\
z &= A_1 \\
w &= c_4
\end{aligned}$$

Therefore, $A_2 = x\sqrt{y} + \frac{1}{2\sqrt{y}}$, which gives us our flow map:

$$\varphi_t(x,y,z,w) = (\frac{z}{\sqrt{y}}\sin(2\sqrt{y}t) + x\cos(2\sqrt{y}t) + \frac{1}{2y}(\cos(2\sqrt{y}t)-1), y,$$
$$z\cos(2\sqrt{y}t) + (x\sqrt{y} + \frac{1}{2\sqrt{y}})\sin(2\sqrt{y}t), w)$$

Notice the, the flow map is well defined for all time. However, we have a singularity at $y = 0$. As it turns out, it is a removable singularity since,

$$\begin{aligned}
\frac{\sin(2\sqrt{y}t)}{\sqrt{y}} &= \frac{1}{\sqrt{y}}\,[2\sqrt{y}t - \frac{(2\sqrt{y}t)^3}{3!} + \frac{(2\sqrt{y}t)^5}{5!} - \dots] \\
&= 2t - \frac{4t^3y}{3} + \frac{4t^5y^2}{15} - \dots \\
\frac{1}{y}[\cos(2\sqrt{y}t) - 1] &= \frac{1}{y}[-\frac{(2\sqrt{y}t)^2}{2!} + \frac{(2\sqrt{y}t)^4}{4!} - \frac{(2\sqrt{y}t)^6}{6!} + \dots] \\
&= -2t^2 + \frac{2t^4}{3}y - \frac{4t^6y^2}{45} + \dots
\end{aligned}$$

Finally, the complex square root is also well defined, since $\frac{1}{-\sqrt{y}}\sin(-\sqrt{y}) = \frac{1}{\sqrt{y}}\sin(\sqrt{y})$ and $\cos(2(-\sqrt{y})t) = \cos(2\sqrt{y}t)$. That is, the result is the same regardless if we chose $-\sqrt{\cdot}$ or $+\sqrt{\cdot}$.

4. Consider the vector field

$$E = -x(z^2+w^3)\frac{\partial}{\partial x} - (-2(z^2+w^3)y + xy + 1)\frac{\partial}{\partial y}$$

We want to show that it's complete. Let $\zeta_t = (x(t), y(t), z(t), w(t))$ be a integral curve for E. Then,

$$\begin{aligned}
\dot{x}(t) &= -x(t)(z^2(t) + w^3(t)) \\
\dot{y}(t) &= 2(z^2(t) + w^3(t))y(t) - x(t)y(t) - 1 \\
\dot{z}(t) &= 0 \\
\dot{w}(t) &= 0
\end{aligned}$$

So, we have that $z(t) = c_3$, $w(t) = c_4$ and $x(t) = c_1\mathrm{e}^{-(c_3^2+c_4^3)\cdot t}$. To find $y(t)$, we used some rudimentary ODE techniques. Let $\lambda = c_3^2 + c_4^3$.

$$\begin{aligned}
\dot{y}(t) - 2\lambda y(t) + c_1\exp(-\lambda\cdot t)y(t) + 1 &= 0 \\
\dot{y}(t) - (2\lambda - c_1\mathrm{e}^{-\lambda t})y(t) + 1 &= 0
\end{aligned}$$

The corresponding homogeneous equation is:

$$\dot{y}(t) - (2\lambda - c_1\mathrm{e}^{-\lambda t})y(t) = 0$$

By separation of variables we get:

$$y_0(t) = c_2 \mathrm{e}^{2\lambda t + \frac{c_1}{\lambda}\mathrm{e}^{-\lambda t}}$$

Now we wish to find a particular solution $y_p(t)$, so that $y(t) = y_0(t) + y_p(t)$. Let $y_p(t) = u(t)y_1(t)$ where $y_1(t) = \mathrm{e}^{2\lambda t + \frac{c_1}{\lambda}\mathrm{e}^{-\lambda t}}$. So,

$$\begin{aligned}
y_p(t) &= u(t)\mathrm{e}^{2\lambda t + \frac{c_1}{\lambda}\mathrm{e}^{-\lambda t}} \\
\dot{y}_p(t) &= \dot{u}(t)y_1(t) + u(t)\dot{y}_1(t) \\
&= \dot{u}(t)\mathrm{e}^{2\lambda t + \frac{c_1}{\lambda}\mathrm{e}^{-\lambda t}} + u(t)(2\lambda - c_1\mathrm{e}^{-\lambda t})\mathrm{e}^{2\lambda t + \frac{c_1}{\lambda}\mathrm{e}^{-\lambda t}}
\end{aligned}$$

By substituting y_p and $\dot{y}_p$ in the ODE, we get:

$$\begin{aligned}
\dot{y}_p(t) - (2\lambda + c_1\mathrm{e}^{-\lambda t})y_p(t) - 1 = 0 \\
\dot{u}(t)\mathrm{e}^{2\lambda t + \frac{c_1}{\lambda}\mathrm{e}^{-\lambda t}} + u(t)(2\lambda - c_1\mathrm{e}^{-\lambda t})\mathrm{e}^{2\lambda t + \frac{c_1}{\lambda}\mathrm{e}^{-\lambda t}} \\
-(2\lambda - c_1\mathrm{e}^{-\lambda t})u(t)\mathrm{e}^{2\lambda t + \frac{c_1}{\lambda}\mathrm{e}^{-\lambda t}} &= -1 \\
\dot{u}(t)\mathrm{e}^{2\lambda t + \frac{c_1}{\lambda}\mathrm{e}^{-\lambda t}} &= -1
\end{aligned}$$

Finally, all is left to solve is the following:

$$\dot{u}(t) = -\mathrm{e}^{-2\lambda t + \frac{c_1}{\lambda}\mathrm{e}^{-\lambda t}}$$

After integration we have:

$$u(t) = -\frac{(\mathrm{e}^{-\lambda t} + \frac{\lambda}{c_1})}{c_1}\mathrm{e}^{-\frac{c_1}{\lambda}\mathrm{e}^{-\lambda t}}$$

Therefore,

$$y(t) = c_2\mathrm{e}^{2\lambda t + \frac{c_1}{\lambda}\mathrm{e}^{-\lambda t}} - \frac{1}{c_1}(\mathrm{e}^{\lambda t} + \frac{\lambda}{c_1}\mathrm{e}^{2\lambda t})$$

So finally we have

$$\zeta(t) = (c_1\mathrm{e}^{-(c_3^2 + c_4^3)\cdot t}, c_2\mathrm{e}^{2\lambda t + \frac{c_1}{\lambda}\mathrm{e}^{-\lambda t}} - \frac{1}{c_1}(\mathrm{e}^{\lambda t} + \frac{\lambda}{c_1}\mathrm{e}^{2\lambda t}), c_3, c_4)$$

Let $\psi_t(x, y, z, w)$ be the flow map of V. Then, by the definition of flow map, we have that, $\psi_0(x, y, z, w) = (x, y, z, w)$ and $\dot{\psi}_t(x, y, z, w) = V_{\psi_t(x,y,z,w)}$. Therefore,

$$\begin{aligned}
&\psi_t(x, y, z, w) = \\
&\quad (x\mathrm{e}^{-(z^2+w^3)t}, (y + \frac{1}{x}(1 + \frac{z^2 + w^3}{x}))\mathrm{e}^{\frac{x}{z^2+w^3}(\mathrm{e}^{-(z^2+w^3)t} - 1) + 2(z^2+w^3)t} \\
&\qquad - \frac{1}{x}(\mathrm{e}^{(z^2+w^3)t} + \frac{z^2 + w^3}{x}\mathrm{e}^{2(z^2+w^3)t}), z, w)
\end{aligned}$$

First of all, notice that the flow is defined for all time. Second, it appears that we have a few singularities. Since we're in the Koras–Russell Cubic, let's use the fact that $z^2 + w^3 = -(x^2y + x)$ for the second component of the flow map. We get

$$\begin{aligned}(y + \frac{1}{x}(1 + \frac{-(x^2y+x)}{x}))\mathrm{e}^{\frac{x}{z^2+w^3}(\mathrm{e}^{-(z^2+w^3)t}-1)+2(z^2+w^3)t} \\ -\frac{1}{x}\mathrm{e}^{(z^2+w^3)t}(1+\frac{-(x^2y+x)}{x}\mathrm{e}^{(z^2+w^3)t}) \\ = -\frac{1}{x}\mathrm{e}^{(z^2+w^3)t}(1-\mathrm{e}^{(z^2+w^3)t} - xy\mathrm{e}^{(z^2+w^3)t}) \\ = -\frac{1}{x}\mathrm{e}^{(z^2+w^3)t}(1-\mathrm{e}^{-(x^2y+x)t}) + y\mathrm{e}^{2(z^2+w^3)t}\end{aligned}$$

By Taylor expanding $\frac{1}{x}(1 - \mathrm{e}^{-(x^2y+x)t})$ we see that the singularity at $x = 0$ vanishes. Finally,the flow map of E is defined for all time, has no singularities, and is give as follows:

$$\psi_t(x, y, z, w) = (x\mathrm{e}^{-(z^2+w^3)t}, -\frac{1}{x}(\mathrm{e}^{(z^2+w^3)t} + \frac{z^2+w^3}{x}\mathrm{e}^{2(z^2+w^3)t}), z, w)$$

5. Consider the vector field

$$H = 6x\frac{\partial}{\partial x} - 6y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z} + 2w\frac{\partial}{\partial w}$$

H is a vector field on the Koras–Russell cubic since:

$$\begin{aligned}H(x^2y + x + z^2 + w^3) &= 6x(2xy + 1) - 6y(x^2) + 3z(2z) + 2w(3w^2) \\ &= 6(x^2y + x + z^2 + w^3) \\ &= 0\end{aligned}$$

Now, let's show that H is a complete vector field.
Let $\sigma(t) = (x(t), y(t), z(t), w(t))$ be an integral curve for H. That is, $\dot{\sigma}(t) = H_{\sigma(t)}$, which is equivalent to the following system of ordinary differential equations,

$$\begin{aligned}\dot{x}(t) &= 6x(t) \\ \dot{y}(t) &= -6y(t) \\ \dot{z}(t) &= 3z(t) \\ \dot{w}(t) &= 2w(t)\end{aligned}$$

So, we can easily see that $\sigma(t) = (c_1\mathrm{e}^{6t}, c_2\mathrm{e}^{-6t}, c_3\mathrm{e}^{3t}, c_4\mathrm{e}^{2t})$. And finally, the flow map of H is given by,

$$\rho_t(x, y, z, w) = (x\mathrm{e}^{6t}, y\mathrm{e}^{-6t}, z\mathrm{e}^{3t}, w\mathrm{e}^{2t})$$

which is complete, as it is defined for all time. $\square$

Remark 4.0.2. *Notice that $[U, V] = 0$, indeed:*

$$\begin{aligned}
[U, V] &= \left[-2z\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}, -3w^2\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial w}\right] \\
&= \left[-2z\frac{\partial}{\partial y}, -3w^2\frac{\partial}{\partial y}\right] + \left[-2z\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial w}\right] \\
&\quad + \left[x^2\frac{\partial}{\partial z}, -3w^2\frac{\partial}{\partial y}\right] + \left[x^2\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial w}\right] \\
&= 0
\end{aligned}$$

Chapter 5

The Density Property of the Koras–Russell Cubic

The main purpose of this chapter is to show that the Koras–Russell cubic X has the density property. We will need a few ingredients. Namely, we will show that $\mathrm{Aut}_{\mathrm{hol}}(X)$ acts transitively on X, (V, W) are compatible pairs, and W_p is a generating set of T_pX for a generic point $p \in X$. In this chapter, we make use of the following references: Leuenberger [11], Andrist and Kutzschebauch [4] and Munkres [12].

5.1 Shears and Overshears

Definition 5.1.1 ([4, Definition 3.1]).
Let X be a complex manifold and let Θ be a $\mathbb{C}$-complete vector field on X,i.e. the flow map of Θ exists for all complex times. A vector field $f \cdot \Theta$, $f \in \mathcal{O}(X)$ is called a Θ-shear vector field if $\Theta(f) = 0$. It is called a Θ-overshear vector field if $\Theta^2(f) = 0$.

Proposition 5.1.2. *Let X be a complex manifold and let Λ be a $\mathbb{C}-$complete vector field on X, then all $\Lambda-$shear fields are $\mathbb{C}-$complete. In fact, if ψ_t denotes the flow map of Λ, the the flow map ζ_t of $f \cdot \Lambda$ is given by*

$$\zeta_t(z) = \psi_{t \cdot f(z)}(z)$$

Proof. We need to show that $\zeta_t(z)$ is the flow map of the vector field $\Theta = f \cdot \Lambda$. First, it's trivial to see that $\zeta_0(z) = \psi_{0 \cdot f(z)}(z) = \psi_0(z) = z$. Now, we need to show that $\dot{\zeta}_t(z) = \Theta_{\zeta_t(z)}$. Indeed, $\dot{\zeta}_t(z) = \frac{d}{dt}(\zeta_t(z)) = \frac{d}{dt}(\psi_{t \cdot f(z)}(z)) = f(z)\dot{\psi}_{t \cdot f(z)}(z) = f(z)\Lambda_{\psi_{t \cdot f(z)}} = \Theta_{\zeta(z)}$. □

Lemma 5.1.3 ([4, Lemma 3.3]).
Let X be a complex manifold and let V be a $\mathbb{C}$-complete vector field on X, then

all V-overshear vector fields are $\mathbb{C}$-complete as well. In fact, if ϕ_t denotes the flow map of V, then the flow map ψ_t of $f \cdot V$ is given by

$$\psi_t(z) = \phi_{\varepsilon(tV_z(f))\cdot tf(z)}(z)$$

where $\varepsilon\colon \mathbb{C} \to \mathbb{C}$ is given by

$$\varepsilon(\zeta) = \sum_{k=1}^{\infty} \frac{\zeta^{k-1}}{k!} = \frac{\mathrm{e}^{\zeta} - 1}{\zeta}$$

Proof. First, let us compute $\frac{d}{dt}\psi_t(z)$. Note that, $\varepsilon(t\Theta_z(f)) \cdot tf(z) = \frac{\mathrm{e}^{t\Theta_z f}-1}{\Theta_z f} f(z)$.

$$\begin{aligned}
\frac{d}{dt}\psi_t(z) &= \frac{d}{dt}\phi_{\frac{\mathrm{e}^{t\Theta_z f}-1}{\Theta_z f} f(z)}(z) \\
&= \dot{\phi}_{\frac{\mathrm{e}^{t\Theta_z f}-1}{\Theta_z f} f(z)}(z) \cdot \frac{\Theta_z f \mathrm{e}^{t\Theta_z f}}{\Theta_z f} f(z) \\
&= \Theta_{\psi_t(z)} \cdot f(z)\mathrm{e}^{t\Theta_z f}
\end{aligned}$$

Second, we will calculate $\frac{d}{dt} f(\psi_t(z))$.

$$\begin{aligned}
\frac{d}{dt} f(\psi_t(z)) &= d_{\psi_t(z)} f \circ \dot{\phi}_{\frac{\mathrm{e}^{t\Theta_z f}-1}{\Theta_z f} f(z)}(z) \cdot f(z)\mathrm{e}^{t\Theta_z f} \\
&= f(z)\mathrm{e}^{t\Theta_z f} \cdot \Theta_{\psi_t(z)} f
\end{aligned}$$

Note that $\frac{d}{dt}\Theta_{\psi_t(z)} f = 0$ because $\Theta^2(f) = 0$. We can compare higher order derivatives:

$$\begin{aligned}
\frac{d^m}{dt^m} f(\psi_t(z)) &= f(z)\mathrm{e}^{t\Theta_z f} \cdot (\Theta_z f)^{m-1} \cdot \Theta_{\psi_t(z)} f \\
\frac{d^m}{dt^m} f(z) \cdot \mathrm{e}^{t\Theta_z f} &= f(z)\mathrm{e}^{t\Theta_z f} \cdot (\Theta_z f)^m
\end{aligned}$$

Notice that for $t = 0$ and for all $m \in \mathbb{N}$ the values above agree. Therefore, $f(\psi_t(z)) = f(z)\mathrm{e}^{t\Theta_z f}$ and $\frac{d}{dt}\psi_t(z) = f(z) \cdot \Theta_{\psi_t(z)}$. $\square$

Remark 5.1.4. *Using Lemma 5.1.3, we can say the following about the vector fields from Theorem 4.0.1:*

1. *If $f(x, y, z, w) = x^l w^n$ for some $l, m \in \mathbb{N}$, then $f \in \ker U$, that is: fU is a $U-$shear vector field. Moreover, if $f(x, y, z, w) = zx^l w^n$ for some $l, m \in \mathbb{N}$, then $f \in \ker U^2$ that is, fU is a $U-$overshear.*

2. *If $f(x, y, z, w) = x^l z^n$ for some $l, n \in \mathbb{N}$, then $f \in \ker V$, that is: fV is a $V-$shear vector field. Moreover, if $f(x, y, z, w) = wx^l z^n$ for some $l, m \in \mathbb{N}$, then $f \in \ker V^2$ that is, fV is a $V-$overshear vector field.*

5.2 The Kaliman–Kutzschebauch Formula

Proposition 5.2.1 (Kaliman–Kutzschebauch formula)**.** *Let Θ and Λ be two vector fields on a manifold X. Then,*

$$\begin{aligned}[hf\Theta, g\Lambda] - [f\Theta, hg\Lambda] &= -fg\Theta(h)\Lambda - fg\Lambda(h)\Theta \\ &= -fg\,(\Theta(h)\Lambda - \Lambda(h)\Theta)\end{aligned}$$

Proof.

$$\begin{aligned}[hf\Theta, g\Lambda] &= hfg[\Theta, \Lambda] + hf\Theta(g)\Lambda - g\Lambda(hf)\Theta \\ &= hfg[\Theta, \Lambda] + hf\Theta(g)\Lambda - g(f\Lambda(h) + h\Lambda(f))\Theta \\ &= hfg[\Theta, \Lambda] + hf\Theta(g)\Lambda - gf\Lambda(h)\Theta - gh\Lambda(f)\Theta \\ [f\Theta, hg\Lambda] &= fhg[\Theta, \Lambda] + f\Theta(hg)\Lambda - hg\Lambda(f)\Theta \\ &= fhg[\Theta, \Lambda] + f(g\Theta(h) + h\Theta(g))\Lambda - hg\Lambda(f)\Theta \\ &= fhg[\Theta, \Lambda] + fg\Theta(h)\Lambda + fh\Theta(g)\Lambda - hg\Lambda(f)\Theta\end{aligned}$$

Therefore,

$$[hf\Theta, g\Lambda] - [f\Theta, hg\Lambda] = -fg\Theta(h)\Lambda - fg\Lambda(h)\Theta$$ □

Remark 5.2.2. *Suppose that we add the assumption that $\Theta(f) = \Lambda(g) = 0$, $\Theta^2(h) = 0$ but $\Theta(h) \neq 0$, and $\Lambda(h) = 0$. Then,*

$$[hf\Theta, g\Lambda] - [f\Theta, hg\Lambda] = -fg\Theta(h)\Lambda$$

we can observe that, $hf\Theta$, $g\Lambda$, $f\Theta$, and $hg\Lambda$ are all complete vector fields, if Θ and Λ were complete. Using this variant of the Kaliman–Kutzschebauch formula is a strategic way to go, since we can find new vector fields using complete ones.

Remark 5.2.3. *The reason we stated the Kaliman–Kutzschebauch formula is because of the missing commutator on the right hand side. In Theorem 4.0.1, $[U, V] = 0$. However, we don't have the same for the rest, i.e. $[U, W]$, $[U, E]$, $[V, W]$, $[V, E]$, $[W, E] \neq 0$.*

5.3 The Density Property

Definition 5.3.1. *Let X be a complex manifold. Let $f\colon X \to X$ be a holomorphic map. We say that f is a holomorphic automorphism if f is bijective and f^{-1} is holomorphic.*

Notation 5.3.2.

1. *We will denote the space of holomorphic automorphisms on X by $\mathrm{Aut}_{\mathrm{hol}}(X)$.*

2. *We will denote the space of holomorphic vector fields on X by $\mathrm{VF}_{\mathrm{hol}}(X)$.*

3. *We will denote the set of complete holomorphic vector fields on X by $\mathrm{CVF}_{\mathrm{hol}}(X)$.*

Definition 5.3.3 ([12, Page 82])**.**
A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

Definition 5.3.4 ([12, Page 285])**.**
Let X and Y be topological spaces. If K is a compact subspace of X and U is an open subset of Y, define

$$S(K,U) = \{f : f \in \mathcal{C}(X,Y) \text{ and } f(K) \subset U\}.$$

The sets $S(K,U)$ form a subbasis for a topology on $\mathcal{C}(X,Y)$ that is called the compact open topology.

Definition 5.3.5 ([16, Section 1])**.** *Let X be a Stein manifold. If the Lie algebra* $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$ *generated by the complete holomorphic vector fields* $\mathrm{CVF}_{\mathrm{hol}}(X)$ *on X is dense (in the compact open topology) in the Lie algebra of all holomorphic vector fields* $\mathrm{VF}_{\mathrm{hol}}(X)$*, then X has the density property.*

Note that $\mathbb{C}^n$ for $n \geq 2$ also has the algebraic density property. Note that a complex algebraic manifold is said to have the algebraic density property if the Lie algebra generated by the complete algebraic vector fields on the manifold coincides with the Lie algebra of all algebraic vector fields. In fact, in his paper, Andrist [3] shows that the Lie algera of polynomial vector fields on $\mathbb{C}^n$ can be generated by three complete polynomial vector fields.

The main implication of the density property is the so called Andersén–Lempert theorem. The proof can be found in Andersén–Lempert [2], Forstnerič–Rosay [6], [7] and Varolin [16].

Theorem 5.3.6 (Andersén–Lempert Theorem)**.** *Let X be a Stein manifold with the density property. Let $\Omega \subset X$ be a Stein open subset and let $\varphi\colon [0,1] \times \Omega \to X$ be a $\mathcal{C}^1$-smooth map such that*

1. $\varphi_0\colon \Omega \to X$ is the natural embedding.

2. $\varphi_t\colon \Omega \to X$ is holomorphic and injective for every $t \in [0,1]$.

3. $\varphi_t(\Omega)$ is a Runge subset of X for every $t \in [0,1]$.

Then for every $\varepsilon > 0$ and for every compact $K \subset \Omega$, there exists a continuous family $\Phi\colon [0,1] \to \mathrm{Aut}(X)$ such that $\Phi_0 = \mathrm{Id}_X$ and $||\varphi_t - \Phi_t||_K < \varepsilon$.
Moreover, these automorphisms can be chosen to be compositions of flows of completely integrable generators of any dense Lie subalgebra $\mathfrak{g}$ of $\mathrm{Lie}(\mathrm{VF}_{\mathrm{hol}}(X))$.

Remark 5.3.7 (Andersén–Lempert Theorem and Corollary 3.2.8)**.**
Notice that in Corollary 3.2.8, given a vector field $V \in \mathrm{Lie}(\{V_1, \ldots, V_m\})$, where the V_i's are complete, we can approximate uniformly the flow map of V by compositions of the flow maps of the V_i's. In the Andersén–Lempert Theorem, we can approximate any automorphism in the path–connected component of the identity, by compositions of the flow maps of the vector fields in the dense Lie subalgebra of $\mathrm{Lie}(\mathrm{VF}_{\mathrm{hol}}(X))$.

Lemma 5.3.8. *$f(z,w)E \in \mathrm{CVF}_{\mathrm{hol}}(X)$ for every $f \in \mathbb{C}[z,w]$.*

Proof. We have shown in Theorem 4.0.1 that E is complete.
Moreover, $E(f(z,w)) = 0$ for all $f \in \mathbb{C}[z,w]$, then $f(z,w)E$ is an $E-$shear vector field, whose flow map is given by $\eta_t = \psi_{t \cdot f}$ hence it's complete. □

Definition 5.3.9. *Let G be a group with identity e and let X be a nonempty set. A left group action of G on X is a map from $G \times X \to X$, defined by $(g,x) \mapsto g \cdot x$ that satisfies the following two conditions:*

1. For every $g_1, g_2 \in G$ and for every $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

2. For every $x \in X$, $e \cdot x = x$.

Definition 5.3.10. *We say that a group G acts transitively on a set X if there exists $x \in X$ such that $G \cdot x = X$.*

Note: Here, our set is $X = \{(x,y,z,w) \in \mathbb{C}^4 : x^2y + x + z^2 + w^3 = 0\}$ and our group $G = \mathrm{Aut}_{\mathrm{hol}}(X)$. The action of $\mathrm{Aut}_{\mathrm{hol}}(X)$ on X is given by, $(f,(x,y,z,w)) \mapsto f(x,y,z,w)$, where $f \in \mathrm{Aut}_{\mathrm{hol}}(X)$ and $(x,y,z,w) \in X$.

Lemma 5.3.11. *The group $\mathrm{Aut}_{\mathrm{hol}}(X)$ acts transitively on $X - \{x = 0\}$.*

Proof. Let $F_c = \{(x,y,z,w) \in X : x = c\}$ where $c \neq 0$. Note that $\bigcup_{c \in \mathbb{C}^*} F_c = X - \{x = 0\}$. So, to show transitivity, we need to show that for any points p, $q \in X - \{x = 0\}$ there exists an automorphism $f \in \mathrm{Aut}_{\mathrm{hol}}(X)$ such that $f(p) = q$.
Claim: For any two points p,$q \in F_c$ for some $c \neq 0$ there exists an automorphism $g \in \mathrm{Aut}_{\mathrm{hol}}(X)$ such that $g(p) = q$.

Indeed, let $p = (c, y_1, z_1, w_1) \in F_c$ and $q = (c, y_2, z_2, w_2) \in F_c$. Recall that the flow maps of the vector fields U and V from Theorem 4.0.1 are given by:

$$\Phi_t(x, y, z, w) = (x, -x^2t^2 - 2zt + y, x^2t + z, w)$$
$$\Psi_t(x, y, z, w) = (x, -x^4t^3 - 3w^2t - 3x^2wt^2 + y, z, x^2t + w)$$

We will find $t, s, \tau \in \mathbb{C}$ such that $\Phi_\tau \circ \Psi_s \circ \Phi_t(p) = q$.

$$\begin{aligned}\Phi_\tau \circ \Psi_s \circ \Phi_t(p) = (c, -c^2\tau^2 - 2(c^2t + z_1)\tau - c^4s^3 - 3w_1^2s - 3c^2w_1s^2 - c^2t^2 \\ - 2z_1t + y_1, c^2(t + \tau) + z_1, c^2s + w_1)\end{aligned}$$

Let $s = \frac{w_2 - w_1}{c^2}$ and $\tau + t = \frac{z_2 - z_1}{c^2}$. Now, if we substitute s and $t = \frac{z_2 - z_1}{c^2} - \tau$ in the equation:

$$-c^2\tau^2 - 2(c^2t + z_1)\tau - c^4s^3 - 3w_1^2s - 3c^2w_1s^2 - c^2t^2 - 2z_1t + y_1 = y_2$$

we get a polynomial in τ. By the fundamental theorem of algebra, there exists a τ_0 which satisfies the polynomial equation. Hence, we have a found $\tau_0, t_0, s \in \mathbb{C}$ such that $\Phi_{\tau_0} \circ \Psi_s \circ \Phi_{t_0}(p) = q$.
Claim: Let c_1, $c_2 \in \mathbb{C}^*$ such that $c_1 \neq c_2$. Then, for every $p \in F_{c_1}$ and for every $q \in F_{c_2}$ there exists an automorphism $h \in \mathrm{Aut}_{\mathrm{hol}}(X)$ such that $h(p) = q$.
Consider the vector field E from Theorem 4.0.1 and its flow map, which was given by:

$$\psi_t(x, y, z, w) = (x\mathrm{e}^{-(z^2+w^3)t}, -\frac{1}{x}(\mathrm{e}^{(z^2+w^3)t} + \frac{z^2 + w^3}{x}\mathrm{e}^{2(z^2+w^3)t}), z, w)$$

Let $p = (c_1, y_1, z_1, w_1) \in X - \{x = 0\}$ and let $q = (c_2, y_2, z_2, w_2) \in X - \{x = 0\}$. If $z_1^2 + w_1^3 = -c_1^2y_1 - c_1 = -c_1(c_1y_1 - 1) \neq 0$. Notice that the first component of $\psi_t(p)$ is $c_1\mathrm{e}^{-(z_1^2+w_1^3)t}$. By the surjectivity of the exponential map we can find a $t_0 \in \mathbb{C}$ such that $c_1\mathrm{e}^{-(z_1^2+w_1^3)t} = c_2$, and then using the first claim we can flow $\psi_{t_0}(p)$ to q using the flow maps of U and V. If on the other hand, we have $c_1y_1 = 1$, first we flow out p from the curve $c_1y_1 = 1$ in F_{c_1} using the flow maps of U and V, and then we proceed like in the first case. Therefore for every points p,$q \in X - \{x = 0\}$ we found an automorphism $f \in \mathrm{Aut}_{\mathrm{hol}}(X)$ such that $f(p) = q$. ☐

Theorem 5.3.12. $\mathrm{Aut}_{\mathrm{hol}}(X)$ *acts transitively on* X.

Proof. In the preceding lemma, we showed that for any points p, $q \in X - \{x = 0\}$ there exists an automorphism $f \in \mathrm{Aut}_{\mathrm{hol}}(X)$ such that $f(p) = q$. Now, we will consider a point $p = (0, y_1, z_1, w_1) \in \{x = 0\}$ and we need to show that for any point $q = (x_2, y_2, z_2, w_2) \in X$ there exists an automorphism $f \in \mathrm{Aut}_{\mathrm{hol}}(X)$ such that $f(p) = q$. First let us consider $q = (c, y_2, z_2, w_2) \in F_c$ for some $c \neq 0$. Let W be the vector field from Theorem 4.0.1, with flow map:

$$\varphi_t(x, y, z, w) = \Big(\frac{z}{\sqrt{y}} \sin(2\sqrt{y}t) + x \cos(2\sqrt{y}t) + \frac{1}{2y}(\cos(2\sqrt{y}t) - 1), y,$$
$$z \cos(2\sqrt{y}t) + (x\sqrt{y} + \frac{1}{2\sqrt{y}}) \sin(2\sqrt{y}t), w\Big)$$

Notice that, the first component of $\varphi_t(p)$ is:

$$\frac{z_1}{\sqrt{y_1}} \sin(2\sqrt{y_1}t) + \frac{1}{2y_1}(\cos(2\sqrt{y_1}t) - 1)$$

If $z_1 = 0$, then there exists $t_0 \in \mathbb{C}^*$, $\varphi_{t_0}(p) \in F_{c'}$ for some $c' \in \mathbb{C}^*$. Also, if $z_1 \neq 0$, the Taylor expansion of the first component of $\varphi_t(p)$ is:

$$\left(2z_1 t - t^2\right) + \left(\frac{t^4}{3} - \frac{4t^3}{3} z_1\right) y_1 + \left(\frac{4t^5}{15} z_1 - \frac{2t^6}{45}\right) y_1^2 + \dots$$

Hence, there exists a $t_0 \in \mathbb{C}^*$ such that $\varphi_{t_0}(p) \in F_{c''}$ for some $c'' \in \mathbb{C}^*$. So, regardless what the value of z_1 is, there exists $c_1 \in \mathbb{C}^*$ such that $\varphi_{t_0}(p) \in F_{c_1}$ for some $t_0 \in \mathbb{C}^*$. Now, by the second claim of the preceding lemma, we know that there exits an automorphism $f \in \mathrm{Aut}_{\mathrm{hol}}(X)$ such that $f(\varphi_{t_0}(p)) = q$.

If $q = (0, y_2, z_2, w_2) \in F_0$, then there exists $t_0, s_0 \in \mathbb{C}^*$ such that $\varphi_{t_0}(p) \in F_{c_2}$ and $\varphi_{s_0}(p) \in F_{c_3}$ for some $c_2, c_3 \in \mathbb{C}^*$. Again, by the second claim of the preceding lemma there exists an automorphism $f \in \mathrm{Aut}_{\mathrm{hol}}(X)$ such that $f(\varphi_{t_0}(p)) = \varphi_{s_0}(q)$. And so, $\varphi_{-s_0}(f(\varphi_{t_0}(p))) = q$. □

Definition 5.3.13 ([11, Definition 2.9]).
A semi-compatible pair is a pair (ν, μ) of complete vector fields such that the closure of the linear span of the product of the kernels $\ker \nu \cdot \ker \mu$ *contains a non-trivial ideal $I \subset \mathcal{O}(X)$. We call I a compatible ideal of (ν, μ).*

Note: The ideal I is not unique. In most applications $\mathcal{O}(X)$ itself serves as the ideal.

Definition 5.3.14 ([11, Definition 2.11]). *A semi-compatible pair (ν, μ) is called a compatible pair if there is a holomorphic function $h \in \mathcal{O}(X)$ with $\nu(h) \in \ker \nu - 0$ and $h \in \ker \mu$. We call h a compatible function of the pair (ν, μ).*

Example 5.3.15. *On $\mathbb{C}^n$, for $n \geq 2$ with coordinates $z = (z_1, \dots, z_n)$ the pair of vector fields $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ are compatible with $h = z_1$ and $I = \mathcal{O}(\mathbb{C}^n)$, moreover $\frac{\partial}{\partial z_2}$ is a generating set for each tangent space.*

Lemma 5.3.16. *The pair (V, W) is a compatible pair.*

Proof. First, let us show that the pair (V, W) is a semi-compatible pair. Indeed the kernel of V contains the functions depending on x, z and the kernel of W contains the functions depending on y, w; thus the closure of $\operatorname{span}\{\ker V \cdot \ker W\}$ is equal to $\mathcal{O}(X)$ and in particular contains an ideal.

For (V, W) to be a compatible pair, we need a function $h \in \ker W$ such that $V(h) \in \ker V - 0$. Let $h = w \in \ker W$, and $V(w) = x^2 \neq 0$ and $V(w) \in \ker V$. $\square$

Definition 5.3.17. *Let G be a group acting on a set X. Let $x \in X$ be a fixed element of the group. The subgroup $G_x = \{g \in G : g \cdot x = x\}$ is called the stabilizer of x.*

Definition 5.3.18 ([11, Definition 2.4])**.**
Let $p \in X$. A set $\Omega \subset T_pX$ is called a generating set for T_pX if the orbit of Ω of the induced action of the stabilizer $\mathrm{Aut}_{\mathrm{hol}}(X)_p = \{f \in \mathrm{Aut}_{\mathrm{hol}}(X) : f(p) = p\}$ contains a basis of T_pX.

Lemma 5.3.19. *For a generic point $p \in X = \{(x, y, z, w) \in \mathbb{C}^4 : x^2y + x + z^2 + w^3 = 0\}$, the vector W_p is a generating set for T_pX.*

Proof. It is sufficient to find one point since spanning is an open condition and hence will fail only on a thin set. Let $p = (x_0, y_0, 1, 0) \in X$ where $x_0 \neq 0$. Let $\nu \in \mathrm{CVF}_{\mathrm{hol}}(X)$ and $f \in \ker \nu$ with $f(p) = 0$. Consider the flow map of ν given by $\zeta_t \in \mathrm{Aut}_{\mathrm{hol}}(X)$. Let $\eta_t = \zeta_{tf}$ be the flow map of $f\nu$. Notice that the time-1 map of $f\nu$ is given by ζ_f where $(\zeta_f)(q) = \zeta_{f(q)}(q)$. Consider the induced action by these time-1 maps on T_pX which is given by $v \mapsto v + v(f)\nu(p)$. To see this, let's Taylor expand $\zeta_t(q)$.

$$\zeta_t(q) = q + t\nu_q + O(t^2)$$

Also, we have that

$$f(q) = f(p) + J_pf \cdot (q - p) + O(\|q - p\|^2) = J_pf \cdot (q - p) + O(\|q - p\|^2)$$

Finally,

$$\zeta_{f(q)}(q) = q + f(q)\nu_q + O(f(q)^2)$$

Combining all the linear terms, and considering $\nu_q = \nu_p +$ non–linear terms we get:

$$\begin{aligned}
\zeta_{f(q)}(q) &= q + (J_pf \cdot (q - p))\,\nu_q + O(f(q)^2) + O(\|q - p\|^2) \\
&= q + (J_pf \cdot (q - p))\,\nu_q + O\left(\left\|J_pf \cdot (q - p) + O(\|q - p\|^2)\right\|^2\right) \\
&\quad + O(\|q - p\|^2) \\
&= p + (q - p) + (J_pf \cdot (q - p))\,\nu_p + O(\|q - p\|^2)
\end{aligned}$$

We conclude that

$$d_p\zeta_f(v) = v + d_pf(v)\cdot\nu_p$$

The vector fields U,$V \in \mathrm{CVF}_{hol}(X)$ with $f = x - x_0 \in \ker U, \ker V$, with time-1 maps given by Φ_{x-x_0},$\Psi_{x-x_0} \in \mathrm{Aut_{hol}}(X)_p$.

$$\Phi_{x-x_0}(x,y,z,w) = (x, y - x^2(x-x_0)^2x^2 - 2z(x-x_0), z + x^2(x-x_0), w)$$

$$\begin{aligned}\Psi_{x-x_0}(x,y,z,w) = (x, y - x^4(x-x_0)^3 - 3x^2w(x-x_0)^2 -& \\ 3w^2(x-x_0), z, w + x^2(x-x_0))&\end{aligned}$$

The orbit of W_p under the $\mathrm{Aut_{hol}}(X)_p-$action are given by: W_p, $W_p - 2U_p$ and $W_p - 2V_p$ due to Id, Φ_{x-x_0} and $\Psi_{x-x_0} \in \mathrm{Aut_{hol}}(X)_p$. In vector notation W_p, $W_p - 2U_p$ and $W_p - 2V_p$ correspond to

$$\begin{pmatrix} -2 \\ 0 \\ 2x_0y_0+1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -2x_0^2+2x_0y_0+1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 2x_0y_0+1 \\ -2x_0^2 \end{pmatrix}$$

Since these three vectors are independent in T_pX, they form a basis. □

Now that we have proved $\mathrm{Aut_{hol}}(X)$ acts transitively on X, (V,W) are compatible pairs, and $\{W_p\}$ for a generic $p \in X$ is a generating for T_pX, we can prove the density property for the Koras–Russell cubic.

Definition 5.3.20. *Let X be a complex manifold of dimension n. The subset $Y \subset X$ is said to be Runge if, for every $K \subset Y$ compact, for every $f \in \mathcal{O}(Y)$ and for every $\epsilon > 0$ there exists $F \in \mathcal{O}(X)$ such that $||f - F||_K < \epsilon$.*

Lemma 5.3.21. *Let $\{s_i\}_{i=1}^N \subset \mathcal{T}(X)$ where $\mathcal{T}(X)$ is the set of global sections of the tangent bundle. Let $p \in X$ and $\mathfrak{m}_p = \{f \in \mathcal{O}(X) : f(p) = 0\}$. If $\{s_i + \mathfrak{m}_p\mathcal{T}(X)\}_{i=1}^N$ span the vector space $\mathcal{T}(X)/\mathfrak{m}_p\mathcal{T}(X) \cong T_pX$, then the localizations $(s_i)_p$ generate $\mathcal{T}_p$.*

Proof. Since $s_1|_p, s_2|_p, \dots s_N|_p$ span the tangent space T_pX where the vector space T_pX has dimension $d \le N$, then without loss of generality, we can say that $s_1|_p, \dots s_d|_p$ is a basis of T_pX. Let $((z_1,\dots,z_d),U)$ be a coordinate neighborhood of p. That is to say, in local coordinates we can write $s_i = \sum_{j=1}^d f_i^j \frac{\partial}{\partial z_j}$ on a neighborhood W of p. At p we know that:

$$\begin{vmatrix} f_1^1 & f_1^2 & \cdots & f_1^d \\ f_2^1 & f_2^2 & \cdots & f_2^d \\ \vdots & \vdots & & \vdots \\ f_d^1 & f_d^2 & \cdots & f_d^d \end{vmatrix} \neq 0$$

So, the determinant is not zero in a neighborhood as well. Let $\Theta = \sum_{j=1}^{d} g_j \frac{\partial}{\partial z_j}$ on a neighborhood of p, then we can write $\Theta = \sum_{i=1}^{d} h_i s_i$ in a neighborhood of p, where the h_i are holomorphic in that neighborhood. $\square$

Lemma 5.3.22. *If the elements $(s_i)_p$ generate the stalks $\mathfrak{F}_p$ for all points $p \in X$. Then every global section $\nu \in \mathfrak{F}(X)$ is of the form $\sum f_i s_i$ for some global holomorphic functions $f_i \in \mathcal{O}(X)$.*

Proof. See [13, Theorem 6.25]. $\square$

Remark 5.3.23. *The proof of Lemma 5.3.22 requires Theorem B of Cartan which states that $H^p(X, \mathfrak{F}) = 0$ for all $p > 0$ where X is a Stein manifold and $\mathfrak{F}$ is a coherent sheaf.*

Corollary 5.3.24. *Given $\Theta \in \mathcal{T}(X)$ a holomorphic vector field on X we can write $\Theta = \sum_{i=1}^{N} g_i s_i$ where $g_i \in \mathcal{O}(X)$ and $s_1, \ldots, s_N$ are the sections from Lemma 5.3.21.*

Proof. By taking the coherent sheaf $\mathfrak{F}$ from Lemma 5.3.22 to be the tangent sheaf $\mathcal{T}$ concludes the proof. Note that the tangent sheaf is coherent since it is locally free. $\square$

Lemma 5.3.25. *Let $Y \subset X$ be a domain of X which is Runge and Stein. If the elements $(s_i)_p$ generate the stalks $\mathcal{F}_p$ for all points $p \in Y$, then every global section $\nu \in \mathcal{T}(X)$ can be uniformly approximated on compacts $K \subset Y$ by global sections of the form $\sum f_i s_i$ for some global holomorphic functions $f_i \in \mathcal{O}(X)$.*

Proof. Let $\nu \in \mathcal{T}(X)$ be a global section, consider its restriction $\nu|_Y \in \mathcal{T}(Y)$. By Corollary 5.3.24 we have $\nu|_Y = \sum_{i=1}^{N} g_i s_i$ for some holomorphic functions $g_i \in \mathcal{O}(Y)$. Since Y is given to be a Runge domain and $K \subset Y$ is compact, let $\epsilon > 0$, there exists $f_i \in \mathcal{O}(X)$ such that $||f_i - g_i||_K < \frac{\epsilon}{N} \cdot \sup ||s_i||_K$. Thus, $||\nu - \sum_{i=1}^{N} f_i s_i|| \leq \sup ||s_i||_K \sum_{i=1}^{N} ||f_i - g_i||_K < \epsilon$. Therefore we have approximated the global section ν by sections $\sum_{i=1}^{N} f_i s_i$ uniformly on compacts $K \subset Y$. $\square$

Lemma 5.3.26. *The submodule $\mathcal{O}(X) \cdot x^2 \cdot W$ of $\mathrm{VF}_{\mathrm{hol}}(X)$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$.*

Proof. Notice that $\mathcal{O}(X) = \mathrm{span}\{\ker V \cdot \ker W\}$. Let $f \in \ker V$ and $g \in \ker W$, then by Proposition 5.1.2 and Proposition 5.1.3 fV, fwV, gW, $gwW \in \mathrm{CVF}_{\mathrm{hol}}(X)$. By the Kaliman Kutzschebauch formula we have:

$$[fV, gwW] - [fwV, gW] = fgx^2W \in \mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X)).$$

Thus an arbitrary element $\sum(f_i g_i)x^2W \in \mathcal{O}(X) \cdot x^2 \cdot W$ with $f_i \in \ker V$ and $g_i \in \ker W$ is contained in $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$. $\square$

Theorem 5.3.27. *The Koras–Russell cubic X has the density property.*

Proof. We know that W_p for a generic $p \in X$ is a generating set. Moreover, we saw that W_p,$W_p - 2V_p$ and $W_p - 2U_p$ form a basis for the tangent space T_pX. Let $A = \{a \in X : T_aX \neq \text{span}\{W_a, W_a - 2V_a, W_a - 2U_a\}\}$ which is analytic. From here on let $\nu_1 = W$, $\nu_2 = W - 2V$, $\nu_3 = W - 2U$.

Let $\bigcup K_i = X$ be an exhaustion of X by $\mathcal{O}-$convex compacts. For any $K = K_i$, let Y be a neighborhood of K which is Stein and Runge, and moreover, the closure of Y is compact. (The existence of Y can be found in Theorem 5.1.6 and Theorem 5.2.8 from Lars Hörmander's textbook [8]).

Claim: After adding finitely many complete vector fields alongside W, we get that $Y \cap A = \emptyset$.

Sub-claim: $\bar{Y} \cap A$ is a finite union of irreducible analytic subsets.

First, we prove the sub-claim. Notice that $\bar{Y} \cap A$ is a closed subset of the compact set $\bar{Y}$. For every $p \in \bar{Y} \cap A$ there exists U_p a neighborhood of p and a holomorphic map $H_p \colon U_p \to \mathbb{C}^{l_p}$, such that $U_p \cap A = \{z \in U_p : H_p(z) = 0\}$. The open sets U_p form an open covering of the compact set $\bar{Y} \cap A$. By compactness, there exists a finite sub-cover $U_1, \dots, U_N$. Assume to get a contradiction that there are infinitely many irreducible analytic subsets. Then, by the pigeonhole principle there exists a U_{j_0} which contains infinitely many irreducible analytic subsets, call them $A_1, A_2, \dots$. For each irreducible analytic subset A_j choose a point $a_j \in A_j$ such that $a_j \notin A_k$ for all $k \neq j$. Let p be an accumulation point of the sequence $\{a_j\}_{j=1}^{\infty}$. By [13, Chapter I.E.3.9, page 40] there is a polydisc $P(p, \delta)$ such that $A \cap P(p, \delta)$ can be written as a finite union of irreducible analytic sets. This contradicts our assumption, and hence ends the proof of the sub-claim.

Let $A_0 \subset A$ be an irreducible component of maximal dimension. Let $a \in A_0$ and $\phi \in \text{Aut}_{\text{hol}}(X)$ such that $\phi(a) \in Y - A$, we can do this by transitivity. Since $\phi(a) \notin A$ then, $\text{span}\{\nu_1(\phi(a)), \nu_2(\phi(a)), \nu_3(\phi(a))\} = T_{\phi(a)}X$ by taking pullbacks we get $\text{span}\{(\phi^*\nu_1)(a), (\phi^*\nu_2)(a), (\phi^*\nu_3)(a)\} = T_aX$. Thus after adding these pullbacks among the vector field ν_1, ν_2, ν_3, the component $A_0 \cap Y$ is replace by finitely many components of lower dimension. Repeating the same procedure inductively we get after finitely many steps a list of complete vector fields $\nu_1, \dots, \nu_N$ such that $A \cap Y = \emptyset$.

Let $\mathcal{T}$ be the tangent sheaf. It is coherent because it is locally free. Since the vectors $\nu_i(a)$ span T_aX for all $a \in Y$, then by Lemma 5.3.21 the assumption of Corollary 5.3.24 holds. Therefore, by Lemma 5.3.25 every vector field on X can be approximated uniformly on K by elements of the form $\sum f_i\nu_i$ for some holomorphic functions $f_i \in \mathcal{O}(X)$. Also, by Lemma 5.3.26 the submodule generated by the finite list of ν_i's is contained in the closure of $\text{Lie}(\text{CVF}_{hol}(X))$ (Note that this property still holds after enlarging the list of vector fields since we're pulling back W using diffeomorphisms; uniform approximation on compacts is preserved by pullbacks of diffeomorphisms and automorphisms). Therefore every holomorphic vector field is in the closure of $\text{Lie}(\text{CVF}_{hol}(X))$, which means X has the density property. □

Remark 5.3.28. *The proofs of Lemma 5.3.8, Lemma 5.3.11, Theorem 5.3.12, Lemma 5.3.16, Lemma 5.3.19, Lemma 5.3.21, Lemma 5.3.25, Lemma 5.3.26 and Theorem 5.3.27 can be found in the paper of Leuenberger* [11]. *However, we have modified the proofs for the Koras–Russell cubic, since Leuenberger gave a more general proof for a family of submanifolds, where the Koras–Russell cubic is just one member of that family.*

Chapter 6

List of Generators

Lemma 6.0.1. *Let* $U = -2z\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}$ *and* $V = -3w^2\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial w}$*. Using the complete vector fields* U*,* V*,* zU*,* wz^2V*,* wz^2xV*,*wV*,* w^2zU *and* w^2zxU*, we can generate the following vector fields:*

$$\begin{aligned}
&wz^2x^iV, zw^2x^iU && \text{for } i = 0, 1, 2, \dots \\
&z^2x^iV, w^2x^iU && \text{for } i = 2, 3, 4, \dots \\
&zx^iV, wx^iU && \text{for } i = 4, 5, 6, \dots \\
&x^iV, x^iU && \text{for } i = 6, 7, 8, \dots
\end{aligned}$$

Proof. 1. First, let us generate wz^2x^iV using zU, wz^2V, wz^2xV.

$$\begin{aligned}
[zU, wz^2V] &= zU(wz^2)V = 2z^2wx^2V \\
[zU, 2z^2wx^2V] &= zU(2z^2wx^2)V = 4z^2wx^4V \\
[zU, 4z^2wx^4V] &= zU(4z^2wx^4)V = 8z^2wx^6V \\
&\vdots \\
[zU, 2^{i-1}z^2wx^{2i-2}V] &= zU(2^{i-1}z^2wx^{2i-2})V = 2^iz^2wx^{2i}V
\end{aligned}$$

So, using zU and wz^2V we were able to generate $z^2wx^{2i}V$. Now we will generate $z^2wx^{2i+1}V$ using zU and wz^2xV.

$$\begin{aligned}
[zU, wz^2xV] &= zU(wz^2x)V = 2z^2wx^3V \\
[zU, 2z^2wx^3V] &= zU(2z^2wx^3)V = 4z^2wx^5V \\
[zU, 4z^2wx^5V] &= zU(4z^2wx^5)V = 8z^2wx^7V \\
&\vdots \\
[zU, 2^{i-1}z^2wx^{2i-1}V] &= zU(2^{i-1}z^2wx^{2i-1})V = 2^iz^2wx^{2i+1}V
\end{aligned}$$

Therefore, using zU, wz^2V and wz^2xV we have generated z^2wx^iV for $i = 0, 1, 2, \dots$

Similarly we will generate the vector fields of the form zw^2x^iU using wV, zw^2U and zw^2xU.

Now, we will get the even powers in x, $zw^2x^{2i}V$ using wV and zw^2U.

$$
\begin{aligned}
[wV, zw^2U] &= wV(zw^2)U = 2zw^2x^2U \\
[wV, 2zw^2x^2U] &= wV(2zw^2x^2)U = 4zw^2x^4U \\
[wV, 4zw^2x^4U] &= wV(4zw^2x^4)U = 8zw^2x^6U \\
&\vdots \\
[wV, 2^{i-1}zw^2x^{2i-2}U] &= wV(2^{i-1}zw^2x^{2i-2})U = 2^izw^2x^{2i}U
\end{aligned}
$$

So,using wV and zw^2U we have generated $zw^2x^{2i}U$. Now, we will generate $zw^2x^{2i+1}U$ using wV and zw^2xU.

$$
\begin{aligned}
[wV, zw^2xU] &= wV(zw^2)U = 2zw^2x^3U \\
[wV, 2zw^2x^3U] &= wV(2zw^2x^3)U = 4zw^2x^5U \\
[wV, 4zw^2x^5U] &= wV(4zw^2x^5)U = 8zw^2x^7U \\
&\vdots \\
[wV, 2^{i-1}zw^2x^{2i-1}U] &= wV(2^{i-1}zw^2x^{2i-1})U = 2^izw^2x^{2i+1}U
\end{aligned}
$$

Therefore, using wV, zw^2U and zw^2xU we have generated zw^2x^iU for $i = 0, 1, 2 \ldots$

2. Now, we will add U and V to our list of vector fields to get rid of the z and w terms from zw^2x^iU and wz^2x^iV respectively.
First, using V and z^2wx^iV for $i = 0, 1, 2, \ldots$ we get:

$$
\begin{aligned}
[V, z^2wV] &= V(wz^2)V = z^2x^2V \\
[V, z^2wxV] &= V(wz^2x)V = z^2x^3V \\
[V, z^2wx^2V] &= V(wz^2x^2)V = z^2x^4V \\
&\vdots \\
[V, z^2wx^iV] &= V(z^2wx^i) = z^2wx^{i+2}V
\end{aligned}
$$

$$
\begin{aligned}
[U, zw^2U] &= U(zw^2)U = w^2x^2U \\
[U, zw^2xU] &= U(zw^2x)U = w^2x^3U \\
[U, zw^2x^2U] &= U(zw^2x^2)U = w^2x^4U \\
&\vdots \\
[U, zw^2x^iU] &= U(zw^2x^i)U = w^2x^{i+2}U
\end{aligned}
$$

Therefore, using V, U, wz^2x^iV and zw^2x^iU for $i = 0, 1, 2, \ldots$ we have generated z^2x^iV and w^2x^iU for $i = 2, 3, \ldots$

3. Now, using the vector fields that we have, particularly U,V, z^2x^iV and w^2x^iU for $i = 2, 3, \dots$ we will generate zx^iV and wx^iU for $i = 4, 5, \dots$
   Apply $[U, \cdot]$ to z^2x^iV for $i = 2, 3, \dots$ and we get the vector fields of the form zx^iV for $i = 4, 5, \dots$
   Apply $[V, \cdot]$ to w^2x^2U and we get the vector fields of the form wx^iU for $i = 4, 5, \dots$
   Finally, we apply $[U, \cdot]$ and $[V, \cdot]$ to zx^iV and wx^iU for $i = 4, 5, \dots$ respectively, and we get the vector fields of the form x^iV and x^iU for $i = 6, 7, \dots$ □

Lemma 6.0.2. *Given the vector fields U,V,W and E from Theorem 4.0.1, the vector fields used and found from Lemma 6.0.1 and the complete vector fields z^2V,wE,z^2xV,E,w^2U,zE, w^2xU, W and wW we can generate the following vector fields:*

$$\begin{aligned} &z^2x^iE \quad \textit{for } i = 2, 3, 4, \dots \\ &w^2x^iE \quad \textit{for } i = 2, 3, 4, \dots \\ &z^2x^iW \quad \textit{for } i = 2, 3, 4, \dots \end{aligned}$$

Proof. 1. By the Kaliman–Kutzschebauch formula, we have:
   $[hfV, gE] - [fV, hgE] = -fgV(h)E$ where $V(h) \neq 0$, $V^2(h) = 0$ and E(h)=0. Let $h = w$ and $g = 1$,

$$\begin{aligned} f = z^2 &\Longrightarrow [wz^2V, E] - [z^2V, wE] = -z^2x^2E \\ f = z^2x &\Longrightarrow [wz^2xV, E] - [z^2xV, wE] = -z^2x^3E \\ f = z^2x^2 &\Longrightarrow [wz^2x^2V, E] - [z^2x^2V, wE] = -z^2x^4E \\ &\vdots \\ f = z^2x^i &\Longrightarrow [wz^2x^iV, E] - [z^2x^iV, wE] = -z^2x^{i+2}E \end{aligned}$$

   So, using z^2V,E,wE and z^2xV we were able to generate z^2x^iE for $i = 2, 3, \dots$

2. By the Kaliman–Kutzschebauch formula, we have:
   $[hfU, gE] - [fU, hgE] = -fgU(h)E$ where $U(h) \neq 0$, $U^2(h) = 0$ and $E(h) = 0$. Let $h = z$ and $g = 1$,

$$\begin{aligned} f = w^2 &\Longrightarrow [zw2^2U, E] - [w^2V, zE] = -w^2x^2E \\ f = w^2x &\Longrightarrow [zw^2xU, E] - [w^2xV, zE] = -w^2x^3E \\ f = w^2x^2 &\Longrightarrow [zw^2x^2U, E] - [w^2x^2U, zE] = -w^2x^4E \\ &\vdots \\ f = w^2x^i &\Longrightarrow [zw^2x^iU, E] - [w^2x^iU, zE] = -w^2x^{i+2}E \end{aligned}$$

So, using E,w^2U,zE and w^2xU we were able to generate w^2x^iE for $i = 2, 3, \ldots$

3. By the Kaliman–Kutzschebauch formula, we have:
$[hfV, gW] - [fV, hgW] = -fgV(h)W$ where $V(h) \neq 0$, $V^2(h) = 0$ and $W(h) = 0$. Let $h = w$ and $g = 1$,

$$
\begin{aligned}
f = z^2 &\Longrightarrow [wz2^2V, W] - [z^2V, wW] = -z^2x^2W \\
f = z^2x &\Longrightarrow [wz^2xV, W] - [z^2xV, wW] = -z^2x^3W \\
f = z^2x^2 &\Longrightarrow [wz^2x^2V, W] - [z^2x^2V, wW] = -z^2x^4W \\
&\vdots \\
f = z^2x^i &\Longrightarrow [wz^2x^iV, W] - [z^2x^iV, wW] = -z^2x^{i+2}W
\end{aligned}
$$

So, using W and wW we were able to generate z^2x^iW for $i = 2, 3, \ldots$ $\square$

Remark 6.0.3. *Note that we can't make a symmetric argument using the Kaliman Kutzschebauch formula for U and W, since if we take $h = z$, which is an overshear of U, we will have $W(z) = 2xy + 1 \neq 0$.*

Lemma 6.0.4. *Given the vector fields U, V from Theorem 4.0.1, the vector fields used and found from Lemma 6.0.1, and the vector fields used in Lemma 6.0.2, we can generate the following vector fields:*

$$
\begin{aligned}
&z^2x^iU \quad \text{for } i = 8, 9, 10, \ldots \\
&zx^iU \quad \text{for } i = 10, 11, 12 \ldots \\
&w^2x^iV \quad \text{for } i = 8, 9, 10, \ldots \\
&wx^iV \quad \text{for } i = 10, 11, 12, \ldots \\
&z^3wx^iV \quad \text{for } i = 10, 11, 12, \ldots \\
&z^3x^iV \quad \text{for } i = 12, 13, 14, \ldots \\
&w^3zx^iU \quad \text{for } i = 10, 11, 12, \ldots \\
&w^3x^iU \quad \text{for } i = 12, 13, 14, \ldots
\end{aligned}
$$

Proof. 1. By the Kaliman–Kutzschebauch formula, we have:
$[hfV, gU] - [fV, hgU] = -fgV(h)U$ where $V(h) \neq 0$, $V^2(h) = 0$ and $U(h) = 0$ Let $h = w$ and $g = x^6$,

$$
\begin{aligned}
f = z^2 &\Longrightarrow [wz2^2V, x^6U] - [z^2V, wx^6U] = -z^2x^8U \\
f = z^2x &\Longrightarrow [wz^2xV, x^6U] - [z^2xV, wx^6U] = -z^2x^9U \\
f = z^2x^2 &\Longrightarrow [wz^2x^2V, x^6U] - [z^2x^2V, wx^6U] = -z^2x^10U \\
&\vdots \\
f = z^2x^i &\Longrightarrow [wz^2x^iV, x^6U] - [z^2x^iV, wx^6U] = -z^2x^{i+2}U
\end{aligned}
$$

So, using previously known vector field we were able to generate z^2x^iU for $i = 8, 9, 10, \ldots$ Moreover, if we apply $[U, \cdot]$ on z^2x^iU for $i = 8, 9, 10, \ldots,$ we will generate zx^iU for $i = 10, 11, 12, \ldots$

2. By the Kaliman–Kutzschebauch formula, we have:
$[hfU, gV] - [fU, hgV] = -fgU(h)V$ where $U(h) \neq 0$, $U^2(h) = 0$ and $V(h) = 0$. Let $h = z$ and $g = x^6$,

$$
\begin{aligned}
f = w^2 &\Longrightarrow [zw^2U, x^6V] - [w^2U, zx^6V] = -w^2x^8V \\
f = w^2x &\Longrightarrow [zw^2xU, x^6V] - [w^2xU, zx^6V] = -w^2x^9V \\
f = w^2x^2 &\Longrightarrow [zw^2x^2U, x^6V] - [w^2x^2U, zx^6V] = -w^2x^10V \\
&\vdots \\
f = w^2x^i &\Longrightarrow [zw^2x^iU, x^6U] - [w^2x^iU, zx^6V] = -w^2x^{i+2}V
\end{aligned}
$$

So, using previously known vector field we were able to generate w^2x^iV for $i = 8, 9, 10, \ldots$ Moreover, if we apply $[V, \cdot]$ on w^2x^iV for $i = 8, 9, 10, \ldots,$ we will generate wx^iV for $i = 10, 11, 12, \ldots$

3. Consider the vector fields wz^2V and z^2x^iU for $i = 8, 9, 10, \ldots$ We apply $[\cdot, wz^2V]$ to the vector fields z^2x^iU for $i = 8, 9, 10, \ldots$ and we get z^3wx^iV for $i = 10, 11, 12, \ldots$ Moreover, we apply $[V, \cdot]$ on the vector fields z^3wx^iV for $i = 10, 11, 12, \ldots$ and we get z^3x^iV for $i = 12, 13, 14, \ldots$

4. Consider the vector fields zw^2U and w^2x^iV for $i = 8, 9, 10, \ldots$ We apply $[\cdot, zw^2U]$ on the vector fields w^2x^iV for $i = 8, 9, 10, \ldots$ and we get w^3zx^iU for $i = 10, 11, 12, \ldots$ Moreover, we apply $[U, \cdot]$ on the vector fields w^3zx^iU and we get the w^3x^iU for $i = 12, 13, 14, \ldots$

□

Lemma 6.0.5. *Given the vector fields U, V and W. Using the complete vector fields xU, yW, ywW and all the needed vector fields from the previous lemmas, we* *can generate the following vector fields:*

$$\begin{array}{l}
z^3x^iU \quad \textit{for } i = 2,3,4,\ldots \\
z^3w^2x^iU \quad \textit{for } i = 2,3,4,\ldots \\
z^4w^2x^iU \quad \textit{for } i = 2,3,4,\ldots \\
yz^2x^iW \quad \textit{for } i = 4,5,6,\ldots \\
yx^2W \\
zxyW \\
y^2xzW \\
y^2xzwW \\
yz^3x^iU \quad \textit{for } i = 4,5,6,\ldots \\
z^3w^2x^iU \quad \textit{for } i = 4,5,6,\ldots \\
z^4x^iW \quad \textit{for } i = 4,5,6,\ldots \\
z^6x^iW \quad \textit{for } i = \\
z^3x^iV \quad \textit{for } i = 2,3,4,\ldots \\
z^5wx^iV \quad \textit{for } i = 2,3,4,\ldots \\
z^5x^iV \quad \textit{for } i = 2,3,4\ldots \\
yz^3x^iV \quad \textit{for } i = 4,5,6,\ldots \\
z^3w^2x^iV \quad \textit{for } i = 4,5,6,\ldots \\
z^3w^2x^iW \quad \textit{for } i = 4,5,6,\ldots \\
yz^4w^2x^iW \quad \textit{for } i = 5,6,7,\ldots
\end{array}$$

Proof. 1. By the Kaliman–Kutzschebauch formula we have:

$$[hfW, gU] - [fW, hgU] = -fgW(h)U - fgU(h)W$$

For $g = 1$, $h = x$ and $f = z^2x^i$ for $i = 2,3,4,\ldots$ we get, z^3x^iU for $i = 2,3,4,\ldots$

For $g = w^2$, $h = x$ and $f = z^2x^i$ for $i = 2,3,4,\ldots$ we get, $z^3w^2x^iU$ for $i = 2,3,4,\ldots$

For $g = zw^2x^i$ for $i = 0,1,2,\ldots$, $h = x$ and $f = z^2x^2$ we get, $z^4w^2x^iU$ for $i = 2,3,4,\ldots$

2. By the Kaliman–Kutzschebauch formula we have:

$$[fhV, gW] - [fV, ghW] = -fgV(h)W - fgW(h)V$$

For $g = y$, $h = w$ and $f = z^2x^i$ for $i = 2,3,4,\ldots$ we get yz^2x^iW for $i = 4,5,6,\ldots$

For $f = 1$, $h = 2$ and $g = y$ we have yx^2W. Moreover, we can get the following vector fields: $[W, yx^2W] = zxyW$, $[yW, yx^2W] = y^2xzW$ and $[ywW, yx^2W] = y^2xwzW$.

3. Again, by the Kaliman–Kutzchebauch formula for U and W, for $f = 1$, $h = x$ and $g = yz^2x^i$ for $i = 4, 5, 6, \ldots$ we get yz^3x^iU for $i = 4, 5, 6, \ldots$. Moreover, $[V, yz^3x^iU] = z^3w^2x^iU$ for $i = 4, 5, 6, \ldots$
For $h = y$, $g = 1$ and $f = z^3x^i$ we get z^4x^iW for $i = 4, 5, 6, \ldots$.
For $f = z^3x^4$, $h = y$ and $g = z^2x^i$ we get z^6x^iW for $i = 8, 9, 10, \ldots$

4. Using the Kaliman–Kutzschebauch formula for V and W, for $f = 1$, $h = x$ and $f = z^2x^i$ for $i = 2, 3, 4, \ldots$ we get z^3x^iV for $i = 2, 3, 4, \ldots$
For $g = z^2x^2$, $h = x$ and $f = z^2wx^i$ for $i = 0, 1, 2, \ldots$ we get z^5wx^iV for $i = 2, 3, 4, \ldots$
For $h = x$, $f = z^2$ and $g = z^2x^i$ for $i = 2, 3, 4, \ldots$ we get z^5x^iV for $i = 2, 3, 4, \ldots$
For $f = 1$, $h = x$ and $g = yz^2x^i$ for $i = 4, 5, 6, \ldots$ we get yz^3x^iV for $i = 4, 5, 6, \ldots$ Moreover, $[V, yz^3x^iV] = z^3w^2x^iV$ for $i = 4, 5, 6, \ldots$
For $h = y$, $g = 1$ and $f = z^3x^i$ for $i = 4, 5, 6, \ldots$ we get $z^3w^2x^iW$ for $i = 4, 5, 6, \ldots$
For $h = y$, $g = yxz$ and $f = z^3x^i$ for $i = 4, 5, 6, \ldots$ we get $yz^4w^2x^iW$ for $i = 5, 6, 7, \ldots$ □

In all these computations we have used the complete vector fields U, V, zU, wz^2V, wz^2xV,wV, w^2zU, w^2zxU, z^2V,wE, z^2xV, E, w^2U, zE, w^2xU, W, wW, xU, yW and ywW. We let $\mathcal{V} = \{\nu_i\}_{i=1}^{20}$ to be these complete vector fields, and $\{\eta_t^i\}_{i=1}^{20}$ be the flow maps of the vector fields in $\mathcal{V}$. Let $\mathcal{W}$ be all the vector fields we have found using $\mathcal{V}$, where $\mathcal{W} \subset \mathrm{Lie}(\mathcal{V}) \subset \mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$.

Note that, to achieve the density property for the Koras–Russell cubic we need three crucial steps. The first being that $\mathrm{Aut}_{\mathrm{hol}}(X)$ acts transitively on X, where we used the vector fields U, V, W and E. Second, we showed that W_p for a generic point $p \in X$ is a generating set for T_pX. Here we used xU and xV. Finally, we showed that the submodule $\mathcal{O}(X) \cdot x^2 \cdot W$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$, where we used the vector fields x^kz^lV, wx^kz^lV and y^mw^nW for any $k, l, m, n \in \mathbb{N}$. Denote the collection of all the vector fields need to prove the density property by $\mathcal{D}$, and let $\{\zeta_t^i\}$ be the flow maps of the vector fields in $\mathcal{D}$.

Notice that using the twenty vector fields in $\mathcal{V}$, we were able to find the vector fields $\mathcal{M} = \{\mu^i_j\}_{j=1}^7$ given by

$$
\begin{aligned}
&wx^iz^2V \quad i = 0, 1, 2, \ldots \\
&z^2x^iV \quad i = 2, 3, 4, \ldots \\
&zx^iV \quad i = 4, 5, 6, \ldots \\
&x^iV \quad i = 6, 7, 8, \ldots \\
&wx^iV \quad i = 10, 11, 12, \ldots \\
&x^iz^3V \quad i = 2, 3, 4, \ldots \\
&wx^iz^5V \quad i = 2, 3, 4, \ldots
\end{aligned}
$$

which are all required to prove the density property of the threefold. Let Δ be the flow maps of the vector fields in $(\mathcal{D} - \mathcal{M}) \cup \mathcal{V}$.

Let F be an automorphism on X, which is in the path–connected component of the identity. That is, there exists $\varphi_t \colon X \to X$, such that $\varphi_0 = \mathrm{Id}$ and $\varphi_1 = F$. Note that φ_t satisfies the three conditions of the Andersén–Lempert theorem, namely $\Omega = X$ and $\varphi_t(X) = X$ is a Runge subset of X for every $t \in [0, 1]$ trivially. So φ_t can be approximated by a composition of flows from Δ, in particular, we can approximate F by compositions of the flows in Δ.

Theorem 6.0.6. *Any automorphism F in the path-connencted component of the identity map can be approximated uniformly by compositions of flow maps from Δ.*

Bibliography

[1] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, tensor analysis, and applications*, 2nd ed., Applied Mathematical Sciences, vol. 75, Springer-Verlag, New York, 1988. MR960687

[2] Erik Andersén and László Lempert, *On the group of holomorphic automorphisms of* $\mathbf{C}^n$, Invent. Math. **110** (1992), no. 2, 371–388, DOI 10.1007/BF01231337. MR1185588

[3] Rafael B. Andrist, *Integrable generators of Lie algebras of vector fields on* $\mathbb{C}^n$, Forum Math. **31** (2019), no. 4, 943–949, DOI 10.1515/forum-2018-0204. MR3975669

[4] Rafael B. Andrist and Frank Kutzschebauch, *The fibred density property and the automorphism group of the spectral ball*, Math. Ann. **370** (2018), no. 1-2, 917–936, DOI 10.1007/s00208-017-1520-8. MR3747506

[5] Franc Forstnerič, *Stein manifolds and holomorphic mappings*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 56, Springer, Cham, 2017. The homotopy principle in complex analysis. MR3700709

[6] Franc Forstnerič and Jean-Pierre Rosay, *Approximation of biholomorphic mappings by automorphisms of* $\mathbf{C}^n$, Invent. Math. **112** (1993), no. 2, 323–349, DOI 10.1007/BF01232438. MR1213106

[7] ———, *Erratum: "Approximation of biholomorphic mappings by automorphisms of* $\mathbf{C}^{\mathbf{n}}$*" [Invent. Math.* **112** *(1993), no. 2, 323–349; MR1213106 (94f:32032)]*, Invent. Math. **118** (1994), no. 3, 573–574, DOI 10.1007/BF01231544. MR1296357

[8] Lars Hörmander, *An introduction to complex analysis in several variables*, 3rd ed., North-Holland Mathematical Library, vol. 7, North-Holland Publishing Co., Amsterdam, 1990. MR1045639

[9] Nathan Jacobson, *Basic algebra. II*, 2nd ed., W. H. Freeman and Company, New York, 1989. MR1009787

[10] John M. Lee, *Introduction to smooth manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013. MR2954043

[11] Matthias Leuenberger, *(Volume) density property of a family of complex manifolds including the Koras-Russell cubic threefold*, Proc. Amer. Math. Soc. **144** (2016), no. 9, 3887–3902, DOI 10.1090/proc/13030. MR3513546

[12] James R. Munkres, *Topology*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition of [MR0464128]. MR3728284

[13] R. Michael Range, *Holomorphic functions and integral representations in several complex variables*, Graduate Texts in Mathematics, vol. 108, Springer-Verlag, New York, 1986. MR847923

[14] Walter Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987. MR924157

[15] Joseph L. Taylor, *Several complex variables with connections to algebraic geometry and Lie groups*, Graduate Studies in Mathematics, vol. 46, American Mathematical Society, Providence, RI, 2002. MR1900941

[16] Dror Varolin, *The density property for complex manifolds and geometric structures*, J. Geom. Anal. **11** (2001), no. 1, 135–160, DOI 10.1007/BF02921959. MR1829353