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A FIXED-POINT ARGUMENT FOR EXISTENCE OF DICHROMATIC LENS

by
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Abstract
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Title: A Fixed-Point Argument For The Existence of Dichromatic Lens

In this paper, we are going to solve a system of functional differential equations and use Banach Fixed Point Theorem to prove existence and uniqueness of solution. This will help us in a problem in geometrical optics. Our goal is to be able to design a simple lens which refracts the superposition of dichromatic light back in one direction.

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Chapter 1

Introduction

The main concept in this thesis is related to optical physics, namely, chromatic aberration in lenses. In order to talk about chromatic aberration, we first introduce refraction of light. In his book "Opticks" [1], Newton defined the term "refrangibilty" of the rays of light as their ability to deviate from their original direction when passing from one transparent medium into another. Geometric optics usually consider the rays of light to be straight lines emitted from luminous body (or medium) to an illuminated body (or medium), and refraction of light the bending of these lines. In refraction, the angle of refraction depends on the angle of incidence and a quantity called the refractive index of the corresponding media. By definition, the refractive index of a medium is the ratio of the velocity of light in vacuum to its velocity in the medium. The refractive index varies with wavelength: it is higher for blue light than for red light [2, chapter 2]. This means that if the ray hitting the surface is not monochromatic, that is it is made of two or more wavelengths, the refractive index differs and consequently the rays have different refractive directions. This phenomena is called light dispersion, and from this behavior of optical materials, it follows that every property of a lens depending on its refractive index will also change with the wavelength creating an unfocused target. This undesirable phenomenon is called chromatic aberration [2, chapter 5]. In 1729, it was noted that chromatic aberration in lenses can be reduced notably by cementing lenses together made of different glasses. This lead to a renewed research into understanding the behavior of optical materials and discovering that the dispersion of light in glass varies from type to type. In 1760, the mathematician Klingenstierna developed a mathematical theory of achromatic lenses (a lens where chromatic aberration is notably reduced) [2, chapter 5]. Then more concepts and methods were developed. Recently, imatest handled chromatic aberration numerically using demosaicing algorithms [3].

In this thesis, we study the existence of a lens (2 faces) that eliminates chromatic aberration of a dichromatic source. The problem is divided into two cases.

The monochromatic case: we have three media 1, 2, and 3, a point source in medium 1 emitting monochromatic rays, and a curve σ_1 separating media 1 and 2. We construct a curve σ_2 separating media 2 and 3 such that the lens sandwiched between σ_1 and σ_2 refracts the monochromatic rays into a fixed direction in medium 3. This is done by applying Snell's Law at σ_1 and σ_2.

The dichromatic problem: here, the rays emitted from O are dichromatic of two colors red and blue, denoted by r and b. If these rays hit a lens with faces σ_1 and σ_2, then red and blue rays will refract differently at σ_1 and at σ_2 due to dispersion. The goal is to construct the lens in such a way that the dichromatic rays leave the lens with the same direction. From the monochromatic case, we get a family of lenses (σ_1, σ_r) and (σ_1, σ_b) corresponding to colors r and b such that these lenses refract red rays and blue rays respectively into a desired fixed direction. The goal of the problem is to find one single lens that would do the job for both colors. If σ_r is parametrized by f_r and σ_b by f_b, we intend to prove that f_r can be written as a reparametrization of f_b. In other words, we study the existence of a C^1 function ϕ such that $f_r(t) = f_b(\phi(t))$. Mathematically, the problem can be formulated through a system of functional differential equations. To analyze the system, we need fixed point theorems namely the Banach and Brouwer fixed point theorems.

In chapter 2, we introduce and prove these fixed point theorems and show two well-known applications to IVPs in Sections 2.2, 2.4. In Chapter 3, we use the Banach fixed point theorem to prove an existence and uniqueness result for system of functions ODE, this is originally due to [4] and later to [5]. We, in fact, prove an extension of this theorem in Section 3.4 allowing to handle cases that are inconclusive by the theorem proved in Section 3.1. In Chapter 4, we introduce the Snell's law and construct monochromatic system refracting monochromatic light uniformly into constant direction, we consider the case of 1 surface of separation in Section 4.3.1, and 2 surfaces of separation in Section 4.3.2. Finally, in Chapter 5, we introduce and formulate the dichromatic problem as a system of Functional Differential Equation using the construction in Section 5.2. We use Brouwer fixed point theorem to show necessary condition for existence of solutions, and then use the analysis in Chapter 3 to solve the corresponding system and find conditions for solving the model. We should mention that the monochromatic problem was handled by Friedman and Mcleod using fixed argument [6]. The problem was then solved by Rogers by solving a system of non-linear differential equations [4]. As for the dichromatic problem, the existence theorem is due to Rogers [4]. Later, in [5] a simpler proof of this theorem was provided. However, in this same paper it was shown that the existence theorem depends on the choice of norms on $\mathbb{R}^n$. In our thesis, we state and prove another existence theorem that doesn't have this dependence, and it then gives a more precise result.

Chapter 2

Fixed point theorems

In this chapter, we introduce two fixed point theorems and illustrate one of their most famous applications in solving initial value Problems.

2.1 Banach Fixed Point Theorem

Definition 2.1.1 (Contraction). *Let* $(V, \|\cdot\|)$ *be a normed vector space. We say that a map*

$$T : V \mapsto V$$

is a contraction if there exists a constant c*,* $0 < c < 1$*, such that*

$$\|T(x) - T(y)\| \leq c\|x - y\| \qquad \forall x, y \in V.$$

Theorem 2.1.2 (Banach Fixed Point Theorem). *Given a non-empty Banach vector space* $(V, \|\cdot\|)$*, if* $T : V \mapsto V$ *is a contraction, then there exists a unique* $x \in V$ *such that* $T(x) = x$*.*

Proof. Take $x_0 \in V$, and define the sequence $x_n = T(x_{n-1})$, $n \in \mathbb{N}$. We have

$$\|x_{n+1} - x_n\| = \|T(x_n) - T(x_{n-1})\| \leq c\|x_n - x_{n-1}\|,$$

where c, $0 < c < 1$, is the contraction constant of T. Recursively,

$$\|x_{n+1} - x_n\| \leq c\|x_n - x_{n-1}\| \leq c^2\|x_{n-1} - x_{n-2}\| \leq \cdots \leq c^n\|x_1 - x_0\|$$

We claim that (x_n) is Cauchy. In fact, for $m, n \in \mathbb{N}$, $m \geq n$

$$\begin{aligned}
\|x_m - x_n\| &= \|x_m - x_{m-1} + x_{m-1} - \cdots - x_{n+1} + x_{n+1} - x_n\| \\
&\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+1} - x_n\| \\
&\leq c^{m-1}\|x_1 - x_0\| + c^{m-2}\|x_1 - x_0\| + \cdots + c^n\|x_1 - x_0\| \\
&= \|x_1 - x_0\| c^n [1 + c + \cdots + c^{m-n-1}] \\
&\leq \|x_1 - x_0\| c^n \sum_{n=0}^{\infty} c^n \\
&= \frac{c^n}{1-c}\|x_1 - x_0\|
\end{aligned}$$

Since $0 < c < 1$ the right handside of the inequality goes to 0 as $n \to \infty$ concluding that (x_n) is Cauchy sequence.

V is Banach, so by completeness x_n converges to an element $x \in V$. We have $x_n = T(x_{n-1})$ and T is continuous then $T(x) = x$. □

2.2 Application: Picard's Theorem

We apply the Banach fixed theorem to show the following existence and uniqueness result of solutions to initial value problems.

Theorem 2.2.1 (Picard's Theorem). *Given an open subset $U \subseteq \mathbb{R}^2$, and $(x_0, y_0) \in U$, we consider the following initial value problem (IVP)*

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} .$$

If f is continuous on U and is Lipschitz continuous in the variable y, then $\exists \delta > 0$ such that the (IVP) has a unique solution $y := y(x)$ with $x \in [x_0 - \delta, x_0 + \delta]$.

Proof. Let δ, b be such that for $I = [x_0 - \delta, x_0 + \delta]$ and $J = [y_0 - b, y_0 + b]$, $I \times J \subseteq U$. Also, let α be the Lipschitz constant of the function f, that is,

$$|f(x, y_1) - f(x, y_2)| \leq \alpha |y_1 - y_2|, \qquad \forall (x, y_1), (x, y_2) \in U.$$

Since f is continuous on the compact set $I \times J$ then f is bounded, we denote $M = \max_{(x,y) \in I \times J} |f|$. Taking δ small enough, we may assume that $\alpha\delta < 1$, and $M\delta < b$.

Let $S := \{g \in C(I) : \|g - y_0\|_\infty \leq b\}$. S is non-empty as it contains the function $g(x) = y_0$. Since I is compact then $C(I)$ is complete with respect to the sup norm, and so S being closed we get that S is Banach.

Define the map $T : S \mapsto C(I)$ by $T(y) = T_y$, where

$$T_y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

We show that T is a contraction on S.

First, since f is continuous, then T_y is differentiable, so $Im(T) \subseteq C(I)$. Moreover

$$\|T_y - y_0\|_\infty = \sup\{|T_y(x) - y_0|, x \in I\} = \sup\left\{\left|\int_{x_0}^{x} f(t, y(t))dt\right|, x \in I\right\}.$$

Since $|f(t, y(t))| \leq M$ then

$$\left|\int_{x_0}^{x} f(t, y(t))dt\right| \leq M|x - x_0| \leq M\delta \leq b.$$

Therefore $\|T_y - y_0\| \leq b$, and consequently $Im(T) \subseteq S$.

To show that T is a contraction, notice that

$$\|T_{y_1} - T_{y_2}\|_\infty = \sup\{|T_{y_1}(x) - T_{y_2}(x)|, x \in I\} = \sup\left\{\left|\int_{x_0}^{x} f(t, y_1(t)) - f(t, y_2(t))dt\right|, x \in I\right\}$$

Since

$$|f(t, y_1(t)) - f(t, y_2(t))| \leq \alpha|y_1(t) - y_2(t)| \leq \alpha\|y_1 - y_2\|_\infty,$$

then

$$\left|\int_{x_0}^{x} f(t, y_1(t)) - f(t, y_2(t))dt\right| \leq \alpha\|y_1 - y_2\|_\infty \cdot |x - x_0| \leq \alpha\delta\|y_1 - y_2\|_\infty.$$

Therefore, $\|T_{y_1} - T_{y_2}\|_\infty \leq \alpha\delta\|y_1 - y_2\|_\infty$. Having $\alpha\delta < 1$, we conclude that T is a contraction.

Using Banach Fixed Point Theorem (Theorem 2.1.2), T has a unique fixed point, call it y^*. We have that

$$y^*(x) = T(y^*)(x) = T_{y^*}(x) = y_0 + \int_{x_0}^{x} f(t, y^*(t))\, dt.$$

Differentiating with respect to x and using the Fundamental Theorem of Calculus we get that $(y^*)'(x) = f(x, y^*(x))$. Also, $y^*(x_0) = y_0 + \int_{x_0}^{x_0} f(t, y^*(t))\, dt = y_0$. Hence, y^* solves the IVP.

To conclude uniqueness, notice that if $y := y(x)$ is another solution to the IVP then

$$T_y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt = y_0 + \int_{x_0}^{x} y'(t)\, dt = y_0 + y(x) - y_0 = y(x).$$

Then y is a fixed point of T, and by uniqueness of the fixed point in Theorem 2.1.2, we get $y = y^*$. □

2.3 Brouwer Fixed Point Theorem

In this section, we introduce another useful fixed point theorem due to Brouwer, then we prove Schauder fixed point theorem, the extension to infinite dimensional spaces.

Theorem 2.3.1. (The one-dimensional case) *Let f be a continuous function from $[a, b]$ to itself, then there exists a point $c \in [a, b]$ such that $f(c) = c$.*

Proof. Let $g(x) = f(x) - x$. Since f is continuous, then g is continuous. We have $g(a) = f(a) - a \geq 0$, and $g(b) = f(b) - b \leq 0$. Hence, by the Intermediate Value Theorem, there exists a point $c \in [a, b]$ such that $g(c) = 0$, that is, $f(c) = c$. □

In the rest of the section, we denote by $\mathbb{D}^n$ the closed unit ball in $\mathbb{R}^n$. We state the following result from algebraic topology whose proof can be found in Evan's book [7, chapter 8.1].

Theorem 2.3.2. (No-Retraction Theorem in n-dimensions) *There exists no continuous map $w : \mathbb{D}^n \mapsto \partial\mathbb{D}^n$ such that $w(x) = x$ for all $x \in \partial\mathbb{D}^n$.*

A consequence of Theorem 2.3.2 is the following $n-$dimensional fixed point result on unit balls.

Theorem 2.3.3. *Let f be a continuous function from $\mathbb{D}^n$ to itself, then there exists a point $x_0 \in \mathbb{D}^n$, such that $f(x_0) = x_0$.*

Proof. Assume that f has no fixed points. Define the map $w : \mathbb{D}^n \mapsto \partial\mathbb{D}^n$ by setting $w(x)$ to be the tip of the ray emanating from $f(x)$, passing through x and intersecting the boundary $\partial\mathbb{D}^n$. This map is well defined as $f(x) \neq x$ for all $x \in \mathbb{D}^n$. In addition, w is continuous and $w(x) = x$ for all $x \in \partial\mathbb{D}^n$ which, by Theorem 2.3.2, is a contradiction. We conclude that f has at least one fixed point in $\mathbb{D}^n$. □

Since every non-empty compact convex set is homeomorphic to a point or to the unit ball (Topology and Geometry by E.Bredon [8, chapter 1, section 16]), we then get the following version for general convex sets.

Theorem 2.3.4. (Brouwer's Theorem for general finite dimensional spaces] *Let $(X, \|\cdot\|)$ be a finite dimensional normed space, and K be a non-empty, compact, and convex subset of X, then any continuous map $T : K \mapsto K$ has at least one fixed point.*

The above result can be extended to Banach spaces obtaining then the following fixed point result named after Schauder.

Theorem 2.3.5. (Schauder Fixed Point Theorem- Version 1) *Let $(X, \|\cdot\|)$ be a Banach space, and K be a non-empty, compact, and convex subset of X, then any continuous map $T : K \mapsto K$ has at least one fixed point.*

Proof. Given $\epsilon > 0$. By compactness, there exist $x_1, x_2, \cdots, x_N \in K$ such that the balls $B_i := B(x_i, \varepsilon)$ form an open cover of K. We denote by K_ϵ the closed convex hull of the points $x_1, \cdots, x_N$, that is

$$K_\epsilon = \left\{ \sum_{i=1}^{N} \lambda_i x_i : 0 \leq \lambda_i \leq 1, \sum_{i=1}^{N} \lambda_i = 1 \right\}.$$

Since K is convex, then $K_\epsilon \subseteq K$. Moreover, K_ϵ is nonempty, finite dimensional, and compact.

We define the function $f_\epsilon : K \mapsto K_\epsilon$ by

$$f_\epsilon(x) = \frac{\sum_{i=1}^{N} dist(x, K \setminus B_i) x_i}{\sum_{i=1}^{N} dist(x, K \setminus B_i)}$$

where *dist* denotes the distance between two points. Note that the denominator is not zero since $K \subseteq \cup_{i=1}^{N} B_i$, and so f_ϵ is continuous. Also, we have that for every $x \in K$

$$\|f_\epsilon(x) - x\| = \left\| \frac{\sum_{i=1}^{N} dist(x, K \setminus B_i)(x_i - x)}{\sum_{i=1}^{N} dist(x, K \setminus B_i)} \right\| \leq \frac{\sum_{i=1}^{N} dist(x, K \setminus B_i) \|x_i - x\|}{\sum\limits_{i=1}^{N} dist(x, K \setminus B_i)} \leq \epsilon \tag{2.1}$$

Define then the map

$$T_\epsilon = f_\epsilon \circ T.$$

We have $T_\epsilon(K_\epsilon) \subseteq K_\epsilon$, T_ϵ is continuous, then by Theorem 2.3.4 applied on K_ϵ, there is a point $x_\epsilon \in K_\epsilon$ such that $T_\epsilon(x_\epsilon) = x_\epsilon$.

Consider $\epsilon_n = \frac{1}{n}$, and let $x_{\epsilon_n} \in K_{\epsilon_n}$ be a sequence of fixed points of T_{ϵ_n}. By compactness of K, x_{ϵ_n} has a converging subsequence $x_{\epsilon_{n_k}}$. Denote $x_0 = \lim_{k \to \infty} x_{\epsilon_{n_k}}$. Using (2.1), we have

$$\left\|x_{\epsilon_{n_k}} - T(x_{\epsilon_{n_k}})\right\| = \left\|T_{\epsilon_{n_k}}(x_{\epsilon_{n_k}}) - T(x_{\epsilon_{n_k}})\right\| = \left\|f_{\epsilon_{n_k}}(T(x_{\epsilon_{n_k}})) - T(x_{\epsilon_{n_k}})\right\| \leq \epsilon_{n_k} = \frac{1}{n_k}$$

Letting $k \to \infty$, we get $Tx_0 = x_0$. □

In fact a weaker version of Schauder fixed point theorem can be also proven and will be of use in the next section

Theorem 2.3.6. (Schauder Fixed Point Theorem- Version 2) *Let $(X, \|\cdot\|)$ be a Banach space and K a non-empty, closed, and convex subset of X. If $T : K \mapsto K$ is a continuous map such that $T(K)$ is precompact*[1]*, then T has a fixed point in K.*

[1] T is called a compact operator

Proof. We have $\overline{T(K)}$ is compact then from Rudin Functional analysis book [9, theorem 3.24], the closed convex hull C of $\overline{T(K)}$ is compact. We have by definition C is non-empty and convex. It is also clear that $C \subseteq K$, in fact since K is closed and $T(K) \subseteq K$, then $\overline{T(K)} \subseteq K$, and hence having that K is convex we conclude the inclusion. Notice finally that $T(C) \subseteq T(K) \subseteq C$, applying then Theorem 2.3.5 on $T|_C$ we get that T has a fixed point in C. □

2.4 Application: Peano's Existence Theorem

One of the applications of Schauder Fixed Point Theorem in ODEs is Peano's existence Theorem.

Theorem 2.4.1 (Peano's Existence Theorem). *Given an open subset $U \subseteq \mathbb{R}^2$, and $(x_0, y_0) \in U$, we consider the following initial value problem (IVP)*

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} .$$

If f is continuous on U, then $\exists \delta > 0$ such that the (IVP) has a solution $y := y(x)$ with $x \in [x_0 - \delta, x_0 + \delta]$.

Proof. We use the same setting of the proof of Picard's Theorem 2.2.1. $I = [x_0 - \delta, x_0 + \delta]$, and $J = [y_0 - b, y_0 + b]$ with $b, \delta > 0$ chosen so that $I \times J \subseteq U$. Let $M = \max_{(x,y) \in I \times J} |f(x,y)|$, and take δ small enough so that $\delta M < b$. Finally, let

$$S := \{g \in C(I, \mathbb{R}) : \|g - y_0\|_\infty \leq b\}.$$

S is non-empty and closed. Moreover, notice that if h and g are two functions in S and $\lambda \in [0,1]$, then $(1-\lambda)h + \lambda g \in C(I)$ and for all $x \in I$,

$$|(1-\lambda)h(x)+\lambda g(x)-y_0| = |(1-\lambda)(h(x)-y_0)+\lambda(g(x)-y_0)| \leq (1-\lambda)|h(x)-y_0|+\lambda|g(x)-y_0| \leq b.$$

Hence, $(1-\lambda)h(x) + \lambda g(x) \in S$, concluding that S is convex. Finally notice that S is bounded since for every $g \in S$, we have $\|g\|_\infty \leq b + |y_0|$.

Next, as in the proof of Theorem 2.2.1, consider the map $T : S \mapsto S$ given by $T(y) = T_y$ with

$$T_y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

We prove T is continuous[2]. Let y_n be a sequence in S that converges uniformly to a function y. We have for every $x \in I$

$$|T_{y_n}(x) - T_y(x)| = \left| \int_{x_0}^{x} f(t, y_n(t)) - f(t, y(t))dt \right| \leq \int_I |f(t, y_n(t)) - f(t, y(t))|dt,$$

[2]The proof of continuity here is different as we do not have necessarily a contraction

and so

$$\|T(y_n) - T(y)\|_\infty \leq \int_I |f(t, y_n(t)) - f(t, y(t))| dt.$$

We have $|f(t, y_n(t)) - f(t, y(t))| \leq 2M$, and $f(t, y_n(t)) \to f(t, y(t))$, hence, by the Dominated Convergence Theorem, we get

$$\lim_{n \to \infty} \|T(y_n) - T(y)\|_\infty = 0.$$

Therefore, $T : S \mapsto S$ is continuous.

To apply Theorem 2.3.6 it remains to show that $T(S)$ is precompact. We already have that $T(S) \subseteq S$ is bounded. Also, for every $y \in S$,

$$|T_y(x_2) - T_y(x_1)| = \left| \int_{x_1}^{x_2} f(t, y(t)) dt \right| \leq M|x_2 - x_1|.$$

Hence $T(S)$ is equicontinuous. Therefore, by Arzelà-Ascoli $T(S)$ is precompact.

We conclude from Theorem 2.3.6, that T has a fixed point $y^* \in S$ which following the proof of Theorem 2.2.1 is a solution to the IVP. □

Remark 2.4.2. In Peano's theorem, we do not have necessarily uniqueness. In fact, consider the the initial value problem

$$\begin{cases} y' = xy^{\frac{1}{3}} \\ y(0) = 0 \end{cases}$$

where $(x, y) \in U \subset \mathbb{R}^2$, and U is an open set containing $(0, 0)$. $f(x, y) = xy^{\frac{1}{3}}$ is continuous, so by Peano's theorem, there exist a neighborhood of $x = 0$ where a solution $y(x)$ exists. Notice that $y = 0$ is a solution, but it is not the only one as $y = \frac{x^3}{3\sqrt{3}}$ is another solution to the IVP.

Chapter 3

Solving system of Functional differential equations

In this chapter, we introduce the system of functional differential equations that will be used later to solve the dichromatic problem. In sections 3.1 and 3.2, we prove an existence and uniqueness theorem for solutions to the system. In section 3.3, some analysis on the theorem shows that the existence theorem depends on the choice of norms that could give inconclusive results. In section 3.4, we avoid these cases using another existence theorem which is stated and proved.

3.1 Existence of solutions

Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$, and H be a continuous map defined in an open domain in $\mathbb{R}^{4n+1}$ with values in $\mathbb{R}^n$ given by

$$H(X) = (h_1(X), h_2(X), \cdots, h_n(X))$$

with $X = (t, v^0, v^1, w^0, w^1); t \in \mathbb{R}$, and $v^0, v^1, w^0, w^1 \in \mathbb{R}^n$. We are interested in solving the system:

$$\begin{cases} Z'(t) = H(t, Z(t), Z(z_1(t)), Z'(t), Z'(z_1(t))) \\ Z(0) = 0 \end{cases}$$

with $Z(t) = (z_1(t), z_2(t), \cdots, z_n(t))$ a C^1 map from $\mathbb{R}$ to $\mathbb{R}^n$. We denoted by $V_Z(t)$ the vector

$$V_Z(t) = (t, Z(t), Z(z_1(t)), Z'(t), Z'(z_1(t))). \tag{3.1}$$

Note that if a solution exists in a neighborhood of $t = 0$, then

$$Z'(0) = H(0, \mathbf{0}, \mathbf{0}, Z'(0), Z'(0))$$

has a solution. In other words, there must exist a point $P = (p_1, p_2, \cdots, p_n) \in \mathbb{R}^n$ that solves the system

$$P = H(0, \mathbf{0}, \mathbf{0}, P, P)$$

For such a point P, we define $\mathcal{P} = (0, \mathbf{0}, \mathbf{0}, P, P)$ and the closed neighborhood in $\mathbb{R}^{4n+1}$

$$N_\epsilon(\mathcal{P}) = \{(t, v^0, v^1, w^0, w^1); \quad |t| + \|v^0\| + \|v^1\| + \|w^0 - P\| + \|w^1 - P\| \leq \epsilon\}$$

Assuming H is continuous in $N_\epsilon(\mathcal{P})$, we define

$$\alpha = \max\{\|H(X)\| : X \in N_\epsilon(\mathcal{P})\}. \tag{3.2}$$

Definition 3.1.1. *Given $\delta > 0$, $C^1([-\delta, \delta])$ is the set of C^1 functions $Z(t) : [-\delta, \delta] \mapsto \mathbb{R}^n$,with the norm*

$$\| \cdot \|_{C^1([-\delta,\delta])} = \max_{[-\delta,\delta]} \|Z(t)\| + \max_{[-\delta,\delta]} \|Z'(t)\|.$$

$(C^1([-\delta, \delta]), \| \cdot \|_{C^1([-\delta,\delta])})$ is a Banach space.

Definition 3.1.2. *Given $\epsilon > 0$ and $P = (p_1, p_2, \cdots, p_n)$ a solution to the system $P = H(0, \mathbf{0}, \mathbf{0}, P, P)$. We assume H is continuous in $N_\epsilon(\mathcal{P})$ and α is the constant defined in (3.2).*

For $\mu, \delta > 0$, we define $\mathcal{C}_{P,\mu}(\delta)$ to be the set of functions $Z(t)$ that satisfy the following properties for $t, \bar{t} \in [-\delta, \delta]$

1. *$Z \in C^1([-\delta, \delta])$.*
2. *$Z(0) = \mathbf{0}$, and $Z'(0) = P$.*
3. *$|z_1(t)| \leq |t|$.*
4. *$\|Z(t) - Z(\bar{t})\| \leq \alpha|t - \bar{t}|$.*
5. *$|z_1(t) - z_1(\bar{t})| \leq |t - \bar{t}|$.*
6. *$\|Z'(t) - Z'(\bar{t})\| \leq \mu|t - \bar{t}|$.*
7. *$V_Z(t) \in N_\epsilon(\mathcal{P})$.*

Lemma 3.1.3. *Let ϵ, H, α and P be as in Definition 3.1.2, assume further that $|p_1| \leq 1$. Then there exists $\delta_0 > 0$ such that $\mathcal{C}_{P,\mu}(\delta)$ is not empty for all $\delta \leq \delta_0$ and for all $\mu > 0$.*

Proof. Let $Z_0(t) = tP$. We have,

- Z_0 is C^1, $Z_0(0) = \mathbf{0}$, and $Z_0'(0) = P$.

- $|z_1(t)| = |tp_1| = |t| \cdot |p_1| \leq |t|$.
- Using (3.2), we have

$$\|Z_0(t) - Z_0(\bar{t})\| = \|P(t-\bar{t})\| = \|P\| \cdot |t-\bar{t}| = \|H(\mathcal{P})\| \cdot |t-\bar{t}| \leq \alpha|t-\bar{t}|.$$

- $|z_1(t) - z_1(\bar{t})| = |p_1(t-\bar{t})| = |p_1| \cdot |t-\bar{t}| \leq |t-\bar{t}|$.
- $\|Z_0'(t) - Z_0'(\bar{t})\| = \|P - P\| = 0 \leq \mu|t-\bar{t}|$.
- $V_{Z_0}(t) = (t, tP, (tp_1)P, P, P)$.

(1)-(6) in Definition 3.1.2 are satisfied by $Z_0(t)$. To obtain (7), we need $|t| + \|tP\| + \|tp_1P\| \leq \epsilon$. This is achieved by taking $|t| \leq \frac{\epsilon}{1+\|P\|+|p_1|\cdot\|P\|} := \delta_0$. □

Lemma 3.1.4. *Using the same setting of Definition 3.1.2, $\mathcal{C}_{P,\mu}(\delta)$ is complete for every $\mu, \delta > 0$ with respect to the $\|\cdot\|_{C^1([-\delta,\delta])}$ norm.*

Proof. Let Z^k be a Cauchy sequence in $\mathcal{C}_{P,\mu}(\delta)$. Since, $(C^1[-\delta,\delta], \|\cdot\|_{C^1[-\delta,\delta]})$ is complete, Z^k converges to a function Z in $C^1[-\delta,\delta]$, this means that Z^k and $(Z^k)'$ converge uniformly to Z and Z' respectively with respect to the chosen norm $\|\cdot\|$ on $\mathbb{R}^n$. We will show that $Z \in \mathcal{C}_{P,\mu}(\delta)$ by checking properties $(1)-(7)$ in Definition 3.1.2.

- By pointwise convergence, $Z(0) = \mathbf{0}$, and $Z'(0) = P$.
- For every $\epsilon_1 > 0$, there exists k such that $|z_1(t) - z_1^k(t)| < \epsilon_1$ for all $t \in [-\delta,\delta]$ so we get

  $$|z_1(t)| \leq |z_1(t) - z_1^k(t)| + |z_1^k(t)| < \epsilon_1 + |t|.$$

  Letting ϵ_1 go to zero, we get $|z_1(t)| \leq |t|$.
- For every $\epsilon_1 > 0$, there exists k such that $\left\|Z(t) - Z^k(t)\right\| < \epsilon_1$ for all $t \in [-\delta,\delta]$, so we get

  $$\|Z(t) - Z(\bar{t})\| \leq \left\|Z(t) - Z^k(t)\right\| + \left\|Z^k(t) - Z^k(\bar{t})\right\| + \left\|Z^k(\bar{t}) - Z(\bar{t})\right\| < 2\epsilon_1 + \alpha|t-\bar{t}|$$

  Letting ϵ_1 go to zero, we get $\|Z(t) - Z(\bar{t})\| \leq \alpha|t-\bar{t}|$.
- Similarly as above, we obtain (5) and (6).
- We have that for every $\epsilon_1 > 0$, there exists k such that for $t \in [-\delta,\delta]$,

  $$\left\|Z(t) - Z^k(t)\right\| < \epsilon_1, \left\|Z'(t) - (Z^k)'(t)\right\| < \epsilon_1, |z_1(t) - z_1^k(t)| < \epsilon_1.$$

using the fact that $|z_1(t)|, |z_1^k(t)| \leq |t| \leq \delta$,

$$
\begin{aligned}
&|t| + \|Z(t)\| + \|Z(z_1(t))\| + \|Z'(t) - P\| + \|Z'(z_1(t)) - P\| \\
&\leq |t| + \big(\big\|Z(t) - Z^k(t)\big\| + \big\|Z^k(t)\big\|\big) + \big(\big\|Z(z_1(t)) - Z^k(z_1^k(t))\big\| + \big\|Z^k(z_1^k(t))\big\|\big) \\
&\quad + \big(\big\|Z'(t) - (Z^k)'(t)\big\| + \big\|(Z^k)'(t) - P\big\|\big) + \big(\big\|Z'(z_1(t)) - (Z^k)'(z_1^k(t))\big\| \\
&\qquad + \big\|(Z^k)'(z_1^k(t)) - P\big\|\big) \\
&\leq \epsilon + 2\epsilon_1 + \big\|Z(z_1(t)) - Z^k(z_1^k(t))\big\| + \big\|Z'(z_1(t)) - (Z^k)'(z_1^k(t))\big\|
\end{aligned}
$$

We have

$$
\begin{aligned}
\big\|Z(z_1(t)) - Z^k(z_1^k(t))\big\| &\leq \big\|Z(z_1(t)) - Z(z_1^k(t))\big\| + \big\|Z(z_1^k(t)) - Z^k(z_1^k(t))\big\| \\
&< \alpha|z_1(t) - z_1^k(t)| + \epsilon_1 < (1+\alpha)\epsilon_1.
\end{aligned}
$$

Similarly, $\big\|Z'(z_1(t)) - (Z^k)'(z_1^k(t))\big\| < (1+\alpha)\epsilon_1$. We conclude that for every $t \in [-\delta, \delta]$

$$
|t| + \|Z(t)\| + \|Z(z_1(t))\| + \|Z'(t) - P\| + \|Z'(z_1(t)) - P\| \leq \epsilon + (4+2\alpha)\epsilon_1.
$$

Letting ϵ_1 go to zero, we get that Z verifies (7), concluding the proof of the lemma.

□

Theorem 3.1.5. *Given a norm $\|\cdot\|$ on $\mathbb{R}^n$ and a function H defined in R^{4n+1} as in the beginning of section 3.1, and assume that the system*

$$
P = H(0, \mathbf{0}, \mathbf{0}, P, P) \tag{3.3}
$$

has a solution $P = (p_1, p_2, \cdots, p_n)$, with

$$
|p_1| \leq 1. \tag{3.4}
$$

Define $\mathcal{P} = (0, \mathbf{0}, \mathbf{0}, P, P)$. Furthermore, assume there exists $\epsilon > 0$ such that

1. *H is uniformly Lipschitz in the variable t, i.e., there exists $\Lambda > 0$ such that*

$$
\big\|H(t, v^0, v^1, w^0, w^1) - H(\bar{t}, v^0, v^1, w^0, w^1)\big\| \leq \Lambda|t - \bar{t}| \tag{3.5}
$$

   for all $(t, v^0, v^1, w^0, w^1), (\bar{t}, v^0, v^1, w^0, w^1) \in N_\epsilon(\mathcal{P})$.

2. *H is uniformly Lipschitz in the variables v^0 and v^1, i.e., there exist positive constants L_0 and L_1 such that*

$$
\big\|H(t, v^0, v^1, w^0, w^1) - H(t, \bar{v^0}, \bar{v^1}, w^0, w^1)\big\| \leq L_0\big\|v^0 - \bar{v^0}\big\| + L_1\big\|v^1 - \bar{v^1}\big\| \tag{3.6}
$$

   for all $(t, v^0, v^1, w^0, w^1), (t, \bar{v^0}, \bar{v^1}, w^0, w^1) \in N_\epsilon(\mathcal{P})$.

3. *H is a uniform contraction in the variables w^0 and w^1, i.e., there exist constants C_0 and C_1 with $C_0 + C_1 < 1$ such that*

$$\left\|H(t, v^0, v^1, w^0, w^1) - H(t, v^0, v^1, \bar{w^0}, \bar{w^1})\right\| \leq C_0\left\|w^0 - \bar{w^0}\right\| + C_1\left\|w^1 - \bar{w^1}\right\| \tag{3.7}$$

for all $(t, v^0, v^1, w^0, w^1), (t, v^0, v^1, \bar{w^0}, \bar{w^1}) \in N_\epsilon(\mathcal{P})$.

4. *For all $X \in N_\epsilon(\mathcal{P})$,*

$$|h_1(X)| \leq 1. \tag{3.8}$$

Then there exists $\delta > 0$ and $Z \in C^1([-\delta, \delta])$ with $V_Z(t) \in N_\epsilon(\mathcal{P})$ that solves the system

$$\begin{cases} Z'(t) = H(t, Z(t), Z(z_1(t)), Z'(t), Z'(z_1(t))) \\ Z(0) = 0 \end{cases} \tag{3.9}$$

for $|t| \leq \delta$, with $Z'(0) = P$.

Proof. Let μ be a real number such that

$$\mu \geq \frac{\Lambda + (L_0 + L_1)\alpha}{1 - C_0 - C_1} \tag{3.10}$$

where α is the constant defined in (3.2). Using Lemma 3.1.3, there exists δ_0 such that $\mathcal{C}_{P,\mu}(\delta)$ is not empty for every $\delta \leq \delta_0$. Fixing μ satisfying (3.10), we denote in this proof for simplicity the set $\mathcal{C}_{P,\mu}(\delta)$ by $\mathcal{C}(\delta)$.

Let $\delta \leq \delta_0$. We define the map T on $\mathcal{C}(\delta)$ as follows

$$T(Z)(t) = \int_0^t H(s, Z(s), Z(z_1(s)), Z'(s), Z'(z_1(s)))\, ds.$$

As $V_Z(t) \in N_\epsilon(\mathcal{P})$ then the integrand is well defined. We will prove that T is a contraction on $\mathcal{C}(\delta)$. From Lemma 3.1.4, $\mathcal{C}(\delta)$ is complete then this will imply by Banach Fixed Point Theorem, Theorem 2.1.2, that T has a unique fixed point which will turn out to be the solution of the system.

We start by showing that $T(\mathcal{C}(\delta)) \subseteq \mathcal{C}(\delta)$. Take $Z(t) \in \mathcal{C}(\delta)$, and

$$W(t) = (w_1(t), w_2(t), \cdots, w_n(t)) = T(Z)(t).$$

We need to check properties $(1) - (7)$ in Definition 3.1.2.

- Since H is continuous in $N_\epsilon(\mathcal{P})$, then from the Fundamental Theorem of Calculus, W is continuously differentiable.

- Obviously, $W(0) = 0$, and

$$W'(0) = H(0, Z(0), Z(z_1(0)), Z'(0), Z'(z_1(0))) = H(0, \mathbf{0}, \mathbf{0}, P, P) = P.$$

- Using (3.8), we have

$$|w_1(t)| = \left| \int_0^t h_1(V_Z(s))dt \right| \leq |t - 0| = |t|,$$

and

$$|w_1(t) - w_1(\bar{t})| = \left| \int_0^t h_1(V_Z(s))dt - \int_0^{\bar{t}} h_1(V_Z(s))dt \right| = \left| \int_{\bar{t}}^t h_1(V_Z(s))dt \right| \leq |t - \bar{t}|$$

- Moreover, by (3.2)

$$\begin{aligned} \|W(t) - W(\bar{t})\| &= \left\| \int_0^t H(V_Z(s))dt - \int_0^{\bar{t}} H(V_Z(s))dt \right\| = \left\| \int_{\bar{t}}^t H(V_Z(s))dt \right\| \\ &\leq \left| \int_{\bar{t}}^t \|H(V_Z(s))\|dt \right| \leq \alpha|t - \bar{t}|. \end{aligned}$$

- By the Lipchitz properties (3.5), (3.6), and (3.7) of H ,

$$\begin{aligned} \|W'(t) - W'(\bar{t})\| &= \|H(V_Z(t)) - H(V_Z(\bar{t}))\| \\ &\leq \Lambda|t - \bar{t}| + L_0\|Z(t) - Z(\bar{t})\| + L_1\|Z(z_1(t)) - Z(z_1(\bar{t}))\| \\ &\quad + C_0\|Z'(t) - Z'(\bar{t})\| + C_1\|Z'(z_1(t)) - Z'(z_1(\bar{t}))\| \end{aligned}$$

Using properties $3, 4, 5$, and 6 in Definition 3.1.2, we get

$$\begin{aligned} \|W'(t) - W'(\bar{t})\| &\leq \Lambda|t - \bar{t}| + L_0\alpha|t - \bar{t}| + L_1\alpha|z_1(t) - z_1(\bar{t})| + C_0\mu|t - \bar{t}| + C_1\mu|z_1(t) - z_1(\bar{t})| \\ &\leq (\Lambda + (L_0 + L_1)\alpha + (C_0 + C_1)\mu)|t - \bar{t}|. \end{aligned}$$

By (3.10), $\Lambda + (L_0 + L_1)\alpha \leq (1 - C_0 - C_1)\mu$. Hence,

$$\|W'(t) - W'(\bar{t})\| \leq \mu|t - \bar{t}|.$$

- It remains to show that $V_W(t) \in N_\epsilon(\mathcal{P})$. Define

$$S_W(t) = |t| + \|W(t)\| + \|W(w_1(t))\| + \|W'(t) - P\| + \|W'(w_1(t)) - P\|.$$

We have to show that $S_W(t) \leq \epsilon$ by choosing δ properly. We have:

$$\|W(t)\| = \|W(t) - W(0)\| \leq \alpha|t| \leq \alpha\delta$$

Since $|w_1(t)| \leq |t| \leq \delta$, we get

$$\|W(w_1(t))\| \leq \alpha\delta.$$

We next calculate $\|W'(t)-P\|$. From (3.3) and the Lipschitz properties (3.5), (3.6) and (3.7) of H, and we get

$$\begin{aligned}\|W'(t)-P\| &= \|H(V_Z(t))-H(\mathcal{P})\| \\ &\leq \Lambda|t| + L_0\|Z(t)\| + L_1\|Z(z_1(t))\| + C_0\|Z'(t)-P\| + C_1\|Z'(z_1(t))-P\|.\end{aligned}$$

We have

$$\|Z(t)\| = \|Z(t)-Z(0)\| \leq \alpha|t| \leq \alpha\delta, \quad \|Z'(t)-P\| = \|Z'(t)-Z'(0)\| \leq \mu|t| \leq \mu\delta.$$

Since $|z_1(t)| \leq |t| \leq \delta$, then we also get

$$\|Z(z_1(t))\| \leq \alpha\delta, \quad \|Z'(z_1(t))-P\| \leq \mu\delta.$$

Hence,

$$\|W'(t)-P\| \leq [\Lambda + (L_0+L_1)\alpha + (C_0+C_1)\mu]\delta.$$

Since $|w_1(t)| \leq |t| \leq \delta$, then

$$\|W'(w_1(t))-P\| \leq [\Lambda + (L_0+L_1)\alpha + (C_0+C_1)\mu]\delta.$$

We conclude that

$$S_W(t) \leq \delta[1 + 2\Lambda + 2\alpha(1+L_0+L_1) + 2\mu(C_0+C_1)].$$

Choosing δ to be less than $\frac{\epsilon}{1+2\Lambda+2\alpha(1+L_0+L_1)+2\mu(C_0+C_1)}$, we get $V_W(t) \in N_\epsilon(\mathcal{P})$.

It remains to show that for such δ, T is a contraction. Let Z^1, Z^2 be in $\mathcal{C}(\delta)$, we prove that there exists a constant $0<c<1$ such that

$$\left\|T(Z^1)-T(Z^2)\right\|_{C^1([-\delta,\delta])} \leq c\left\|Z^1-Z^2\right\|_{C^1([-\delta,\delta])}$$

Let $W^1 = T(Z^1)$, $W^2 = T(Z^2)$. By the Fundamental theorem of calculus, we have

$$\left\|W^1(t)-W^2(t)\right\| \leq \left|\int_0^t \left\|(W^1)'(s)-(W^2)'(s)\right\|ds\right| \leq \delta \sup_{|t|\leq\delta}\left\|(W^1)'(t)-(W^2)'(t)\right\| \tag{3.11}$$

And similarly,

$$\left\|Z^1(t)-Z^2(t)\right\| \leq \delta \sup_{|t|\leq\delta}\|(Z^1)'(t)-(Z^2)'(t)\| \leq \delta\left\|Z^1-Z^2\right\|_{C^1([-\delta,\delta])}.$$

Next we will estimate $\|(W^1)'(t)-(W^2)'(t)\|$. From the Lipchitz properties (3.5), (3.6), (3.7) we get

$$
\begin{aligned}
\left\|(W^1)'(t)-(W^2)'(t)\right\| &= \|H(V_{Z^1}(t))-H(V_{Z^2}(t))\| \\
&\le L_0\left\|Z^1(t)-Z^2(t)\right\|+L_1\left\|Z^1(z_1^1(t))-Z^2(z_1^2(t))\right\| \\
&\quad +C_0\left\|(Z^1)'(t)-(Z^2)'(t)\right\|+C_1\left\|(Z^1)'(z_1^1(t))-(Z^2)'(z_1^2(t))\right\| \\
&\le (L_0\delta+C_0)\left\|Z^1-Z^2\right\|_{C^1([-\delta,\delta])}+L_1\left\|Z^1(z_1^1(t))-Z^2(z_1^2(t))\right\| \\
&\qquad +C_1\left\|(Z^1)'(z_1^1(t))-(Z^2)'(z_1^2(t))\right\|
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\left\|Z^1(z_1^1(t))-Z^2(z_1^2(t))\right\| &\le \left\|Z^1(z_1^1(t))-Z^2(z_1^1(t))\right\|+\left\|Z^2(z_1^1(t))-Z^2(z_1^2(t))\right\| \\
&\le \delta\left\|Z^1-Z^2\right\|_{C^1([-\delta,\delta])}+\alpha|z_1^1(t)-z_1^2(t)| \\
&\le \delta\left\|Z^1-Z^2\right\|_{C^1([-\delta,\delta])}+\alpha C_{\|.\|}\left\|Z^1(t)-Z^2(t)\right\| \qquad , \\
&\le \delta(\alpha C_{\|.\|}+1)\left\|Z^1-Z^2\right\|_{C^1([-\delta,\delta])}.
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|(Z^1)'(z_1^1(t))-(Z^2)'(z_1^2(t))\right\| &\le \left\|(Z^1)'(z_1^1(t))-(Z^2)'(z_1^1(t))\right\|+\left\|(Z^2)'(z_1^1(t))-(Z^2)'(z_1^2(t))\right\| \\
&\le \left\|Z^1-Z^2\right\|_{C^1([-\delta,\delta])}+\mu|z_1^1(t)-z_1^2(t)| \\
&\le \left\|Z^1-Z^2\right\|_{C^1([-\delta,\delta])}+\mu C_{\|.\|}\left\|Z^1(t)-Z^2(t)\right\| \\
&\le \delta(\mu C_{\|.\|}+1)\left\|Z^1-Z^2\right\|_{C^1([-\delta,\delta])},
\end{aligned}
$$

Here $C_{\|\cdot\|}$ is a constant that depends on the norm $\|\cdot\|$. Combining the above inequalities, we obtain

$$
\begin{aligned}
\left\|(W^1)'(t)-(W^2)'(t)\right\| &\le (L_0\delta+L_1\delta(\alpha C_{\|.\|}+1))+C_0+C_1(\mu C_{\|.\|}\delta+1))\left\|Z^1-Z^2\right\|_{C^1([-\delta,\delta])} \\
&= (M\delta+C_0+C_1)\left\|Z^1-Z^2\right\|_{C^1([-\delta,\delta])} \qquad (3.12)
\end{aligned}
$$

Combining (3.11), and (3.12) we get

$$
\left\|W^1(t)-W^2(t)\right\| \le \delta(M\delta+C_0+C_1)\left\|Z^1-Z^2\right\|_{C^1([-\delta,\delta])}.
$$

We conclude that:

$$
\left\|(TZ^1)(t)-(TZ^2)(t)\right\|_{C^1([-\delta,\delta])}=\left\|W^1-W^2\right\|_{C^1([-\delta,\delta])}\le(1+\delta)(M\delta+C_0+C_1)\left\|Z^1-Z^2\right\|_{C_1([-\delta,\delta])}.
$$

Since $C_0+C_1<1$, choosing δ to be small enough, we get that T is a contraction on $\mathcal{C}(\delta)$. Hence, by Banach Fixed Point Theorem (Theorem 2.1.2), T has a unique fixed point, that is, there is a unique function $Z^*(t)\in\mathcal{C}(\delta)$ such that

$$
Z^*(t)=(TZ^*)(t)=\int_0^t H(V_{Z^*}(s))ds.
$$

Differentiating both sides, and plugging $t=0$ we get that $Z^*(t)$ solves the system (3.9), for $|t|\le\delta$, and satisfies $(Z^*)'(0)=P$. □

3.2 Uniqueness of the Solution

In Theorem 3.1.5, we have proven that the fixed point of the contraction T is a solution to the system 3.9. If we can show that for any solution $Z(t)$ to the system, Z is a fixed point of T, then uniqueness follows from Theorem 2.1.2.

Theorem 3.2.1. *Under the assumptions of Theorem 3.1.5, the local solution $Z(t)$ to the system 3.9 with $|t| \leq \delta$, $Z'(0) = P$, and $V_Z(t) \in N_\epsilon(\mathcal{P})$ is unique.*

Proof. First, we prove that $Z \in \mathcal{C}(\delta)$ by checking the properties of Definition 3.1.2 where P is given and μ satisfies (3.10). Since H is continuous, $Z'(t)$ is continuous and hence $Z \in C^1([-\delta, \delta])$. We also have that Z solves the system then $z_1'(t) = h_1(V_Z(t))$, so from (3.8)

$$|z_1(t)| = \left| \int_0^t h_1(V_Z(s))ds \right| \leq |t|. \tag{3.13}$$

and,

$$|z_1(t) - z_1(\bar{t})| = \left| \int_{\bar{t}}^t h_1(V_Z(s))ds \right| \leq |t - \bar{t}|. \tag{3.14}$$

Also, for $|t|, |\bar{t}| \leq \delta$, we have from (3.2)

$$\|Z(t) - Z(\bar{t})\| = \left\| \int_{\bar{t}}^t H(V_Z(s))ds \right\| \leq \alpha|t - \bar{t}|, \tag{3.15}$$

and from the Lipchitz properties (3.5), (3.6), and (3.7) of H

$$\begin{aligned} \|Z'(t) - Z'(\bar{t})\| &= H(V_Z(t)) - H(V_Z(\bar{t})) \\ &\leq \Lambda|t - \bar{t}| + L_0\|Z(t) - Z(\bar{t})\| + L_1\|Z(z_1(t)) - Z(z_1(\bar{t}))\| \\ &\qquad + C_0\|Z'(t) - Z'(\bar{t})\| + C_1\|Z'(z_1(t)) - Z'(z_1(\bar{t}))\|. \end{aligned}$$

Using (3.13), (3.14) and (3.15), we get that for every $|t|, |\bar{t}| \leq \delta$

$$\|Z'(t) - Z'(\bar{t})\| \leq (\Lambda + (L_0 + L_1)\alpha)|t - \bar{t}| + C_0\|Z'(t) - Z'(\bar{t})\| + C_1\|Z'(z_1(t)) - Z'(z_1(\bar{t}))\|.$$

Now fix t and $\bar{t}$ and let $d = |t - \bar{t}|$. Let τ and $\bar{\tau}$ be such that $|\tau|, |\bar{\tau}| \leq \delta$, and $|\tau - \bar{\tau}| \leq d$. We have

$$|z_1(\tau) - z_1(\bar{\tau})| \leq |\tau - \bar{\tau}| \leq d.$$

Hence, substituting τ and $\bar{\tau}$ in the inequality gives

$$\begin{aligned} \|Z'(\tau) - Z'(\bar{\tau})\| &\leq (\Lambda + (L_0 + L_1)\alpha)|\tau - \bar{\tau}| + C_0\|Z'(\tau) - Z'(\bar{\tau})\| + C_1\|Z'(z_1(\tau)) - Z'(z_1(\bar{\tau}))\| \\ &\leq (\Lambda + (L_0 + L_1)\alpha)d + (C_0 + C_1) \sup_{|\tau|,|\bar{\tau}|\leq\delta, |\tau-\bar{\tau}|\leq d} \|Z'(\tau) - Z'(\bar{\tau})\|. \end{aligned}$$

Taking the supremum of the left hand side we get from (3.10)

$$\sup_{|\tau|,|\bar{\tau}|\leq\delta,|\tau-\bar{\tau}|\leq d} \|Z'(\tau) - Z'(\bar{\tau})\| \leq \frac{\Lambda + (L_0 + L_1)\alpha}{1 - C_0 - C_1} d \leq \mu d = \mu|t - \bar{t}|.$$

Hence, for every $|t|, |\bar{t}| \leq \delta$,

$$\|Z'(t) - Z'(\bar{t})\| \leq \sup_{|\tau|,|\bar{\tau}|\leq\delta,|\tau-\bar{\tau}|\leq d} \|Z'(\tau) - Z'(\bar{\tau})\| \leq \mu|t - \bar{t}|.$$

Therefore, $Z \in \mathcal{C}(\delta)$. Noticing that

$$T(Z)(t) = \int_0^t H(V_Z(t))\, dt = \int_0^t Z'(t)\, dt = Z(t) - Z(0) = Z(t),$$

then Z is a fixed point of T concluding uniqueness of the solution.

□

3.3 Estimate of the contraction constants C_0 and C_1

Recall the following preliminary definitions from linear algebra.

Definition 3.3.1. *Given a norm $\|\cdot\|$ on $\mathbb{R}^n$. We define the induced norm $|||\cdot|||$ on the space of $n \times n$ matrices by*

$$|||A||| = \max\{\|Av\| : v \in \mathbb{R}^n, \|v\| = 1\}.$$

Definition 3.3.2. *Given an $n \times n$ matrix A, the spectral radius R_A is the largest absolute value of the eigenvalues of A.*

Remark 3.3.3. Let λ be the eigenvalue of A with the largest absolute value, and x a corresponding eigenvector then for any norm $\|\cdot\|$ we have

$$|\lambda| \cdot \|x\| = \|\lambda x\| = \|Ax\| \leq |||A||| \cdot \|x\|$$

and hence, $R_A = |\lambda| \leq |||A|||$.

In fact, we have the following useful result due to Householder in "Matrices: Theory and Applications" [10, chapter 7].

Theorem 3.3.4. *Given a matrix A in $\mathbb{R}^n$, for every $\epsilon > 0$, there exists a norm $\|\cdot\|$ in $\mathbb{R}^n$ such that*

$$R_A \leq |||A||| < R_A + \epsilon.$$

Let H, and P be as defined at the beginning of Section 3.1 and $\mathcal{P} = (0, \mathbf{0}, \mathbf{0}, P, P)$, moreover we assume that H is C^1 in $N_\epsilon(\mathcal{P})$ for some $\epsilon > 0$. In this case, H is automatically Lipschitz in all the variables. We are interested in this section in the Lipschitz constants with respect to variables w^0 and w^1. We define the following $n \times n$ matrices

$$\nabla_{w^0} H = \left(\frac{\partial h_i}{\partial w_j^0}\right)_{1 \le i,j \le n}, \qquad \nabla_{w^1} H = \left(\frac{\partial h_i}{\partial w_j^1}\right)_{1 \le i,j \le n},$$

and we denote by DH the $n \times (4n+1)$ first order derivative matrix of H.

Proposition 3.3.5. *The Lipchitz property (3.7) is satisfied for $C_0 = \max_{N_\epsilon(\mathcal{P})} |||\nabla_{w^0} H|||$ and $C_1 = \max_{N_\epsilon(\mathcal{P})} |||\nabla_{w^1} H|||$.*

Proof. For (t, v^0, v^1, w^0, w^1) and $(t, v^0, v^1, \bar{w}^0, \bar{w}^1)$ in $N_\epsilon(\mathcal{P})$, and by the fundamental theorem of calculus

$$\begin{aligned}
&H(t, v^0, v^1, w^0, w^1) - H(t, v^0, v^1, \bar{w}^0, \bar{w}^1) \\
&\quad = \int_0^1 DH((1-s)(t, v^0, v^1, w^0, w^1) + s(t, v^0, v^1, \bar{w}^0, \bar{w}^1))(0, \mathbf{0}, \mathbf{0}, w^0 - \bar{w}^0, w^1 - \bar{w}^1)^t ds \\
&\quad = \int_0^1 \nabla_{w^0} H((1-s)(t, v^0, v^1, w^0, w^1) + s(t, v^0, v^1, \bar{w}^0, \bar{w}^1))(w^0 - \bar{w}^0)^t ds \\
&\qquad + \int_0^1 \nabla_{w^1} H((1-s)(t, v^0, v^1, w^0, w^1) + s(t, v^0, v^1, \bar{w}^0, \bar{w}^1))(w^1 - \bar{w}^1)^t ds
\end{aligned}$$

Hence we get

$$\begin{aligned}
&\left\|H(t, v^0, v^1, w^0, w^1) - H(t, v^0, v^1, \bar{w}^0, \bar{w}^1)\right\| \\
&\le \int_0^1 \left\|\nabla_{w^0} H((1-s)(t, v^0, v^1, w^0, w^1) + s(t, v^0, v^1, \bar{w}^0, \bar{w}^1))(w^0 - \bar{w}^0)^t\right\| ds \\
&\quad + \int_0^1 \left\|\nabla_{w^1} H((1-s)(t, v^0, v^1, w^0, w^1) + s(t, v^0, v^1, \bar{w}^0, \bar{w}^1))(w^1 - \bar{w}^1)^t\right\| ds \\
&\le \int_0^1 |||\nabla_{w^0} H((1-s)(t, v^0, v^1, w^0, w^1) + s(t, v^0, v^1, \bar{w}^0, \bar{w}^1))||| \cdot \left\|w^0 - \bar{w}^0\right\| ds \\
&\quad + \int_0^1 |||\nabla_{w^1} H((1-s)(t, v^0, v^1, w^0, w^1) + s(t, v^0, v^1, \bar{w}^0, \bar{w}^1))||| \cdot \left\|w^1 - \bar{w}^1\right\| ds \\
&\le \max_{N_\epsilon(\mathcal{P})} |||\nabla_{w^0} H||| \cdot \left\|w^0 - \bar{w}^0\right\| + \max_{N_\epsilon(\mathcal{P})} |||\nabla_{w^1} H||| \cdot \left\|w^1 - \bar{w}^1\right\|.
\end{aligned}$$

□

Proposition 3.3.6. *For any values C_0 and C_1 satisfying the Lipchitz property (3.7), we have $C_0 \ge |||\nabla_{w^0} H(\mathcal{P})|||$, $C_1 \ge |||\nabla_{w^1} H(\mathcal{P})|||$*

Proof. Let $v \in \mathbb{R}^n$ such that $\|v\| = 1$. For $s > 0$ small, we have $(0, \mathbf{0}, \mathbf{0}, P + sv, P) \in N_\epsilon(\mathcal{P})$, and hence

$$\|H(0, \mathbf{0}, \mathbf{0}, P, P) - H(0, \mathbf{0}, \mathbf{0}, P + sv, P)\| \leq C_0|s|. \tag{3.16}$$

Applying the mean value theorem for each component, we get

$$h_i(0, \mathbf{0}, \mathbf{0}, P + sv, P) - h_i(0, \mathbf{0}, \mathbf{0}, P, P) = \nabla h_i(\theta_i) \cdot (0, \mathbf{0}, \mathbf{0}, sv, \mathbf{0}) = s \begin{bmatrix} \frac{\partial h_i}{\partial w_1^0} \\ \frac{\partial h_i}{\partial w_2^0} \\ \vdots \\ \frac{\partial h_i}{\partial w_n^0} \end{bmatrix} (\theta_i) \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

where $\theta_i = (0, \mathbf{0}, \mathbf{0}, P + s_{0i}v, P)$, $0 < s_{0i} < s$ is a point in $N_\epsilon(\mathcal{P})$. Hence,

$$\frac{h_i(0, \mathbf{0}, \mathbf{0}, P + sv, P) - h_i(0, \mathbf{0}, \mathbf{0}, P, P)}{s} = \sum_{j=1}^{n} \frac{\partial h_i}{\partial w_j^0}(\theta_i) \cdot v_j.$$

Then, replacing in (3.16)

$$C_0 \geq \left\| \frac{H(0, \mathbf{0}, \mathbf{0}, P + sv, P) - H(0, \mathbf{0}, \mathbf{0}, P, P)}{s} \right\| = \left\| \begin{bmatrix} \sum_{i=1}^n \frac{\partial h_1}{\partial w_i^0}(\theta_1) \cdot v_i \\ \sum_{i=1}^n \frac{\partial h_2}{\partial w_i^0}(\theta_2) \cdot v_i \\ \vdots \\ \sum_{i=1}^n \frac{\partial h_n}{\partial w_i^0}(\theta_n) \cdot v_i \end{bmatrix} \right\|$$

As s tends to zero, θ_i tends to $\mathcal{P}$ and we get

$$C_0 \geq \left\| \begin{bmatrix} \sum_{i=1}^n \frac{\partial h_1}{\partial w_i^0}(\mathcal{P}).v_i \\ \sum_{i=1}^n \frac{\partial h_2}{\partial w_i^0}(\mathcal{P}).v_i \\ \vdots \\ \sum_{i=1}^n \frac{\partial h_n}{\partial w_i^0}(\mathcal{P}) \cdot v_i \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{\partial h_1}{\partial w_1^0} & \frac{\partial h_1}{\partial w_2^0} & \cdots & \frac{\partial h_1}{\partial w_n^0} \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \frac{\partial h_n}{\partial w_1^0} & \frac{\partial h_n}{\partial w_2^0} & \cdots & \frac{\partial h_n}{\partial w_n^0} \end{bmatrix} (\mathcal{P}) \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\| = \|\nabla_{w^0} H(\mathcal{P}) \cdot v\|$$

Taking the supremum over all $v \in \mathbb{R}^n$ with $\|v\| = 1$, we get $C_0 \geq \|\!|\nabla_{w^0} H(\mathcal{P})\|\!|$.

The inequality for C_1 follows similarly. □

Remark 3.3.7. By Proposition 3.3.6, and Remark 3.3.3 we have

$$C_0 + C_1 \geq \|\!|\nabla_{w^0} H(\mathcal{P})\|\!| + \|\!|\nabla_{w^1} H(\mathcal{P})\|\!| \geq R_{w^0} + R_{w^1}.$$

where R_{w^0} and R_{w^1} are spectral radii of the Jacobian matrices $\nabla_{w^0} H(\mathcal{P})$ and $\nabla_{w^1} H(\mathcal{P})$ respectively. Hence, if the sum $R_{w^0} + R_{w^1}$ is bigger than one, then it is not possible to find a norm $\|\cdot\|$ in $\mathbb{R}^n$ for which the contraction property (3.7) holds, and Theorem 3.1.5 cannot be applied. On the other hand, if there exists a norm on $\mathbb{R}^n$ such that

$$\|\!|\nabla_{w^0} H(\mathcal{P})\|\!| + \|\!|\nabla_{w^1} H(\mathcal{P})\|\!| < 1,$$

then by continuity of H, there exists $N_\epsilon(\mathcal{P})$ such that

$$\max_{N_\epsilon(\mathcal{P})} |||\nabla_{w^0} H||| + \max_{N_\epsilon(\mathcal{P})} |||\nabla_{w^1} H||| < 1$$

and hence giving values to C_0 and C_1 by Proposition 3.3.5 to which we can apply Theorem 3.1.5.

3.4 Another existence result

Motivated by Remark 3.3.7, we are going in this section to state and prove another existence and uniqueness theorem in the case when H is continuously differentiable which does not depend on the choice of norms but will require the inequality (3.4) to be strict.

Theorem 3.4.1. *Given $H : U \to \mathbb{R}^n$ continuously differentiable, where $U \subseteq \mathbb{R}^{4n+1}$. Assume $P = (p_1, \cdots, p_n)$ is a solution to the system $P = H(0, \mathbf{0}, \mathbf{0}, P, P)$ with*

$$|p_1| < 1, \tag{3.17}$$

Set $\mathcal{P} = (0, \mathbf{0}, \mathbf{0}, P, P)$. If the matrix $Id - \nabla_{w^0} H(\mathcal{P})$ is invertible and the spectral radius of the matrix $[Id - \nabla_{w^0} H(P)]^{-1} \nabla_{w^1} H(\mathcal{P})$ is strictly less than one, then there exists $\delta > 0$, and a map $Z : [-\delta, \delta] \mapsto \mathbb{R}^n$ that solves the system 3.9. Moreover, Z is the unique solution in $[-\delta, \delta]$ satisfying $Z'(0) = P$.

Proof. Since $R_{[Id - \nabla_{w^0} H(P)]^{-1} \nabla_{w^1} H(\mathcal{P})} < 1$, then from Theorem 3.3.4 there exists a norm $\|\cdot\|$ in $\mathbb{R}^n$ such that

$$\left|\left|\left| [Id - \nabla_{w^0} H(P)]^{-1} \nabla_{w^1} H(\mathcal{P}) \right|\right|\right| < 1. \tag{3.18}$$

Define the function $G : U \mapsto \mathbb{R}^n$ by

$$G(t, v^0, v^1, w^0, w^1) = w^0 - H(t, v^0, v^1, w^0, w^1).$$

We have

$$G(\mathcal{P}) = G(0, \mathbf{0}, \mathbf{0}, P, P) = P - H(0, \mathbf{0}, \mathbf{0}, P, P) = 0,$$

and

$$\nabla_{w^0} G(\mathcal{P}) = Id - \nabla_{w^0} H(\mathcal{P})$$

which is given to be invertible. Then, by the Implicit Function Theorem, there exist an open neighborhood V_{ϵ_1} of $(0, \mathbf{0}, \mathbf{0}, P)$

$$V_{\epsilon_1} = \{(t, v^0, v^1, w^1) \in \mathbb{R}^{3n+1} : |t| + \left\|v^0\right\| + \left\|v^1\right\| + \left\|w^1 - P\right\| < \epsilon_1\}$$

and an open neighborhood O_{ϵ_2} of P

$$O_{\epsilon_2} = \{w^0 \in \mathbb{R}^n : \left\|w^0 - P\right\| < \epsilon_2\}$$

and a unique continuously differentiable function $F : V_{\epsilon_1} \mapsto O_{\epsilon_2}$ satisfying

$$F(0,\mathbf{0},\mathbf{0},P) = P, \text{ and } G(t,v^0,v^1,F(t,v^0,v^1,w^1),w^1) = 0 \ \forall (t,v^0,v^1,w^1) \in V_{\epsilon_1}.$$

Let $\mathcal{W} = \{X = (t,v^0,v^1,w^0,w^1); \quad (t,v^0,v^1,w^1) \in V_{\epsilon_1}, w^0 \in O_{\epsilon_2}\}$. We define the function $\tilde{F} : \mathcal{W} \to \mathbb{R}^n$ by

$$\tilde{F}(t,v^0,v^1,w^0,w^1) = F(t,v^0,v^1,w^1).$$

We will show that $\tilde{F}$ satisfies the conditions of Theorem 3.1.5.

- We have $\tilde{F}(0,\mathbf{0},\mathbf{0},P,P) = F(0,\mathbf{0},\mathbf{0},P) = P$ with $|p_1| < 1$, and so (3.3) and (3.4) follow.
- Since $\tilde{F}$ is continuously differentiable, then $\tilde{F}$ satisfies the Lipchitz conditions (3.5), (3.6), and (3.7) with respect to any norm in $\mathbb{R}^n$.
- Since $|\tilde{F}_1(\mathcal{P})| = |F_1(0,\mathbf{0},\mathbf{0},P)| = |p_1| < 1$ and $\tilde{F} \in C^1$, then taking ϵ_1 small enough we get that $|\tilde{F}(X)| \le 1$ for $X \in V_{\epsilon_1}$, and (3.8) follows
- It remains to show that $\tilde{F}$ is actually a contraction in the variables w^0 and w^1. Since F is independent of w^0

  $$\nabla_{w^0}\tilde{F}(\mathcal{P}) = \nabla_{w^0}F(0,\mathbf{0},\mathbf{0},P) = [0].$$

  Also, by the Implicit Function Theorem

  $$\frac{\partial F}{\partial w_j^1}(0,\mathbf{0},\mathbf{0},P) = -[\nabla_{w_0}G(\mathcal{P})]^{-1} \times \frac{\partial G}{\partial w_j^1}(\mathcal{P}).$$

  This gives

  $$\nabla_{w^1}\tilde{F}(\mathcal{P}) = \nabla_{w^1}F(0,\mathbf{0},\mathbf{0},P) = -[\nabla_{w_0}G(\mathcal{P})]^{-1}\nabla_{w^1}G(\mathcal{P}) = [Id - \nabla_{w^0}H(\mathcal{P})]^{-1}\nabla_{w^1}H(\mathcal{P})$$

  We conclude from (3.18)

  $$\left|\left|\left|\nabla_{w^0}\tilde{F}(\mathcal{P})\right|\right|\right| + \left|\left|\left|\nabla_{w^1}\tilde{F}(\mathcal{P})\right|\right|\right| = 0 + |||[Id - \nabla_{w^0}H(P)]^{-1}\nabla_{w^1}H(\mathcal{P})||| < 1.$$

  Hence using remark 3.3.7, for ϵ small enough, there exist two constants C_0 and C_1 such that $\tilde{F}$ is a uniform contraction in w^0 and w^1 in a neighborhood $N_\epsilon(\mathcal{P})$ contained in $\mathcal{W}$.

Applying Theorem 3.1.5 for $\tilde{F}$, there exists $\delta > 0$ and a unique $Z \in C^1([-\delta,\delta])$ with $V_Z(t) \in N_\epsilon(\mathcal{P})$ that solves the system

$$\begin{cases} Z'(t) = \tilde{F}(V_Z(t)) \\ Z(0) = 0 \end{cases} \tag{3.19}$$

for $|t| \leq \delta$, with $Z'(0) = P$.

Observe

$$\begin{aligned}
&Z'(t) - H(t, Z(t), Z(z_1(t)), Z'(t), Z'(z_1(t))) \\
&\quad = G(t, Z(t), Z(z_1(t)), Z'(t), Z'(z_1(t))) \\
&\quad = G(t, Z(t), Z(z_1(t)), \tilde{F}(t, Z(t), Z(z_1(t)), Z'(t), Z'(z_1(t))), Z'(z_1(t))) \\
&\quad = G(t, Z(t), Z(z_1(t)), F(t, Z(t), Z(z_1(t)), Z'(z_1(t))), Z'(z_1(t))) \\
&\quad = 0
\end{aligned}$$

Hence, $Z(t)$ is a solution of the system (3.9) with $Z'(0) = P$.

Finally, to prove uniqueness we assume that $W(t)$ solves (3.9) for $t \in [-\delta, \delta]$ with $W'(0) = P$. We have for $|t| < \delta$,

$$G(t, W(t), W(w_1(t)), W'(t), W'(w_1(t))) = W'(t) - H(t, W(t), W(w_1(t)), W'(t), W'(w_1(t))) = 0.$$

By uniqueness of F in the Implicit Function Theorem,

$$W'(t) = F(t, W(t), W(w_1(t)), W'(w_1(t))) = \tilde{F}(t, W(t), W(w_1(t)), W'(t), W'(w_1(t))),$$

By the uniqueness of Theorem 3.1.5 applied to system (3.19), $W = Z$ in $[-\delta, \delta]$.

□

Chapter 4

Uniformly refracting surfaces and lenses

Snell's Law is the law that governs refraction of light. In sections 4.1 and 4.2 of this chapter, we introduce Snell's Law in two dimensions and in three dimensions. Then in section 4.3, we solve the monochromatic problem that was described in the introduction.

4.1 Snell's Law in Two Dimensions

In this section, we introduce the most familiar form of Snell's law.

Let Γ be a curve in $\mathbb{R}^2$ that separates two homogeneous and isotropic media 1 and 2. Let v_1, and v_2 be the velocities of light in medium 1 and medium 2 respectively. The index of refraction of medium 1 is defined as $n_1 = \frac{c}{v_1}$, where c is the speed of light in vacuum. Similarly, the refractive index of medium 2 is $n_2 = \frac{c}{v_2}$. If an incident light ray with unit direction x traveling in medium 1 hits Γ at a point of incidence P, and if ν is the normal to Γ at P going toward medium 2, then this ray is refracted in the unit direction m through medium 2 according to the equation

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where θ_1 is the angle formed by x and ν, and θ_2 is the angle formed by m and ν. Let $\kappa = \frac{n_2}{n_1}$, Snell's law becomes

$$\sin\theta_1 = \kappa \sin\theta_2. \tag{4.1}$$

If $\kappa > 1$, the refracted ray bends towards the normal, figure 4.1. In this case we have

$$\sin\theta_1 = \kappa \sin\theta_2 > \sin\theta_2$$

The maximum value of θ_1 is $\pi/2$, so the maximum value attained by $\sin\theta_1$ is 1. In this case we always have refraction, and the maximum refracted angle is $\theta_2 = \arcsin\left(\frac{1}{\kappa}\right)$ that corresponds to $\theta_1 = \pi/2$.

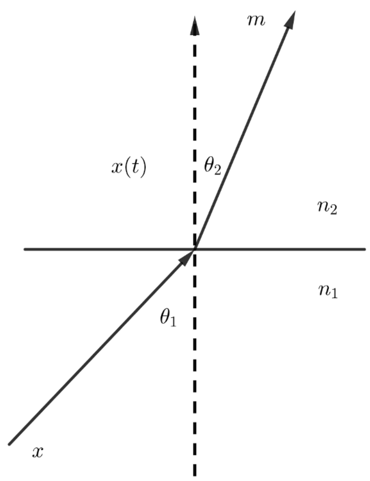

Figure 4.1: Refraction $\kappa > 1$

If $\kappa < 1$, the refracted ray bends away from the normal. In this case we have

$$\sin\theta_1 = \kappa \sin\theta_2 < \sin\theta_2.$$

The maximum value of θ_2 is $\pi/2$, so the maximum value of $\sin\theta_2$ is 1. This corresponds to the case where $\sin\theta_1 = \kappa$. Since the sine inverse function is increasing, we conclude that the maximum angle of incidence is $\theta_1 = \arcsin\kappa$. This angle is called the critical angle and is denoted by θ_c, figures 4.2 and 4.3. If the incident angle is larger than the critical angle, there is no refraction. In this case we have total reflection, figure 4.4.

4.2 Snell's Law in Three Dimensions (Vector Form)

Now we discuss the general form of Snell's Law that is the vector form.

Let Γ be a surface in $\mathbb{R}^3$ separating two homogeneous and isotropic media 1 and 2. Let n_1 and n_2 be the corresponding refractive indices. If an incident light ray with unit direction x traveling in medium 1 hits Γ at a point of incidence P, and if ν is the normal to Γ at P going towards medium 2, then this ray is refracted in the unit direction m through medium 2 according to the equation

$$n_1(x \times \nu) = n_2(m \times \nu).$$

Again $\kappa = \frac{n_2}{n_1}$, Snell's law becomes

$$(x \times \nu) = \kappa(m \times \nu). \tag{4.2}$$

We mention two consequences of (4.2):

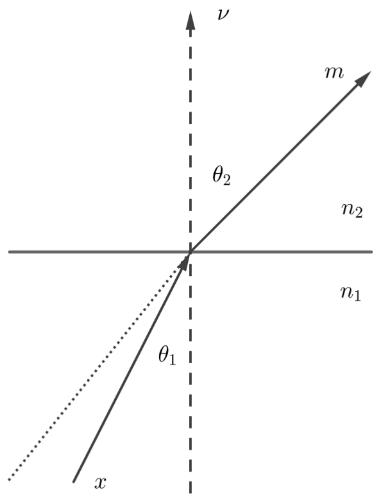

Figure 4.2: $\theta_1 < \theta_c$

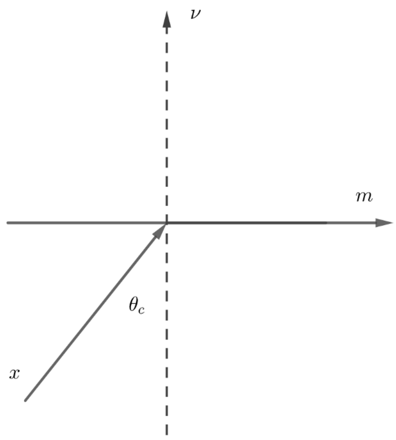

Figure 4.3: $\theta_1 = \theta_c$

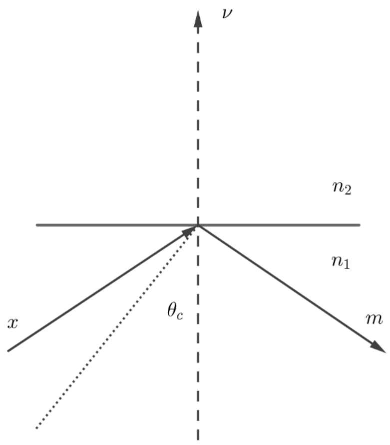

Figure 4.4: $\theta_1 > \theta_c$

- The vectors x, m, and ν belong to the same plane. In fact,

$$m \cdot (x \times \nu) = m \cdot (\kappa(m \times \nu)) = \kappa m \cdot (m \times \nu) = 0$$

  and so m belongs to the plane generated by x and ν and passing through P, this is called the plane of incidence.

- Snell's law in two dimensions (4.1) is recovered in the plane of incidence. In fact, taking modulus in (4.2) we get

$$\|x\| \cdot \|\nu\| \sin \theta_1 = \kappa \|m\| \cdot \|\nu\| \sin \theta_2.$$

Replacing the unit vectors norms with 1 we obtain (4.1).

Now, we are going to find a formula for the refracted unit direction m. From (4.2), $x - \kappa m$ is parallel to the normal ν, that is, there exists $\lambda \in \mathbb{R}$ such that

$$x - \kappa m = \lambda \nu \tag{4.3}$$

Dotting (4.3) with ν yields

$$\lambda = x \cdot \nu - \kappa m \cdot \nu = \cos\theta_1 - \kappa \cos\theta_2 \tag{4.4}$$

We have that

$$\cos\theta_2 = m \cdot \nu = \sqrt{1 - \sin^2\theta_2} = \sqrt{1 - \frac{1}{\kappa^2}\sin^2\theta_1} = \sqrt{1 - \frac{1}{\kappa^2}(1 - (x \cdot \nu)^2)}$$

Replacing in (4.4) we get

$$\lambda = x \cdot \nu - \kappa\sqrt{1 - \frac{1}{\kappa^2}(1 - (x \cdot \nu)^2)} = x \cdot \nu - \sqrt{\kappa^2 - 1 + (x \cdot \nu)^2} \tag{4.5}$$

Note that if $\kappa > 1$ we have $\kappa^2 - 1 + (x \cdot \nu)^2) > 0$ and refraction always occurs as expected. But if $\kappa < 1$, then to have refraction we need $(x \cdot \nu)^2 \geq 1 - \kappa^2$, and so

$$x \cdot \nu \geq \sqrt{1 - \kappa^2} \tag{4.6}$$

(4.6) is equivalent to the condition $\sin\theta_1 \leq \kappa$, that is, $\theta_1 \leq \arcsin(\kappa)$. This angle is called the critical angle. To have refraction, the angle of incidence cannot go beyond this value.

We conclude that the refracted ray (if it exists) is given by

$$m = \frac{1}{\kappa}(x - \lambda\nu) \tag{4.7}$$

with λ given in (4.5).

Remark 4.2.1. Dotting (4.7) with x we get

$$\kappa x \cdot m = 1 - \lambda x \cdot \nu = 1 - (x \cdot \nu)^2 + (x \cdot \nu)\sqrt{\kappa^2 - 1 + (x \cdot \nu)^2}$$

Observe that for $\kappa < 1$, the function $x \cdot m$ is increasing in $x \cdot \nu$ for $x \cdot \nu \geq \sqrt{1 - \kappa^2}$ which is then equivalent to

$$x \cdot m \geq \kappa \tag{4.8}$$

For the case $\kappa > 1$, we use the principle of reversibility of refraction of light. In this case, m becomes the incident ray and x the refracted ray. $\kappa' = \frac{n_1}{n_2} = \frac{1}{\kappa}$. Then we have:

$$x \cdot m \geq \kappa' = \frac{1}{\kappa}$$

In conclusion, a light ray in medium 1 with unit direction x could be refracted by some surface into the unit direction m in medium 2 if and only if $m \cdot x \geq \kappa$ when $\kappa < 1$; and if and only if $m \cdot x \geq \frac{1}{\kappa}$, when $\kappa > 1$.

4.3 Two Dimensional Monochromatic Problem

4.3.1 *The case of one face*

Consider two media 1 and 2. Given a point O in medium 1 and m a fixed unit vector in medium 2. Let D be the closed sub-interval $[-\pi/2, \pi/2]$. Rays are emitted from O with the unit direction $x(t) = (\sin t, \cos t)$, $\quad t \in D$. The goal is to construct a curve of separation Γ such that all the rays with direction $x(t)$, $t \in D$ are refracted by Γ into the constant direction m, figure 4.5. Remark 4.2.1

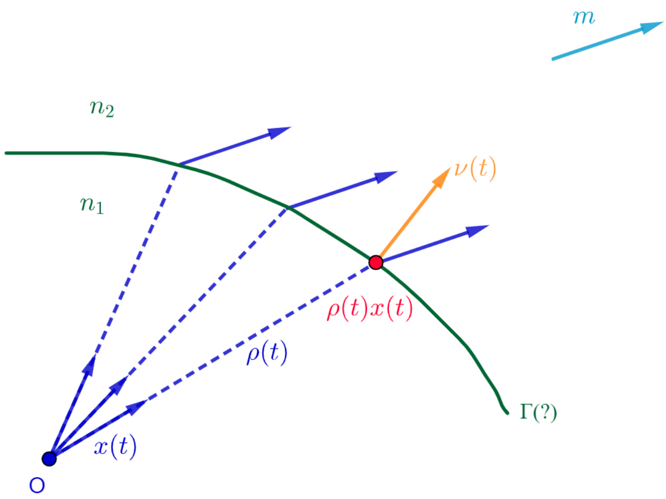

Figure 4.5: Refracting the rays by Γ into m

gives a restriction on t, i.e, on the interval D: x and m should satisfy $x(t)\cdot m \geq \kappa$ for the case $\kappa < 1$, and $x(t) \cdot m \geq \frac{1}{\kappa}$ for the case $\kappa > 1$.

Suppose that $r(t) = \rho(t)x(t)$, $t \in D$ is the parametrization of Γ where $\rho(t)$ is the distance between the origin and the incident point of $x(t)$ at Γ. Since $x(t) - \kappa m$ is parallel to the normal of Γ at the point of contact, we have

$$r'(t) \cdot (x(t) - \kappa m) = 0$$

that is,

$$[\rho'(t)x(t) + \rho(t)x'(t)] \cdot (x(t) - \kappa m) = 0$$

Since $x(t)$ is unit, $\|x\|^2 = x(t) \cdot x(t) = 1$. Differentiating we get, $2x'(t) \cdot x(t) = 0$. Hence the equation becomes

$$\rho'(t) - \kappa\rho'(t)m \cdot x(t) - \kappa\rho(t)m \cdot x'(t) = 0$$

that is,

$$[\rho(t)(1 - \kappa m \cdot x(t))]' = 0$$

Then we have

$$\rho(t) = \frac{b}{1 - \kappa m \cdot x(t)}$$

for some $b \in \mathbb{R}$, finding then the parametrization of the curves that uniformly refract point source rays into a constant direction m. In the case where $\kappa < 1$, the curve Γ is a part of an ellipse with axis of direction m, whereas in the case of $\kappa > 1$, Γ is a piece of a hyperbola sheet about the axis of direction m. A proof by calculation is given in [11].

4.3.2 *The case of two faces*

Consider three media 1,2, and 3 with n_1, n_2, and n_3 the respective indices of refraction. For simplicity, we assume that $n_1 = n_3 = 1$ and $n_2 > 1$ because this is the case we will study later. Given σ_1 the curve separating medium 1 and medium 2 and a point source O in medium 1. w is a fixed unit vector in medium 3. Let D be the closed sub-interval $[-\pi/2, \pi/2]$. Rays are omitted from O with the unit direction $x(t) = (\sin t, \cos t), \quad t \in D$. We ask the following question: Can we find a curve σ_2 separating medium 2 and medium 3 such that the lens with lower face σ_1 and upper face σ_2 refracts all the incident rays emitted from O into the direction w, see figure 4.6. As in the previous section, we use the polar

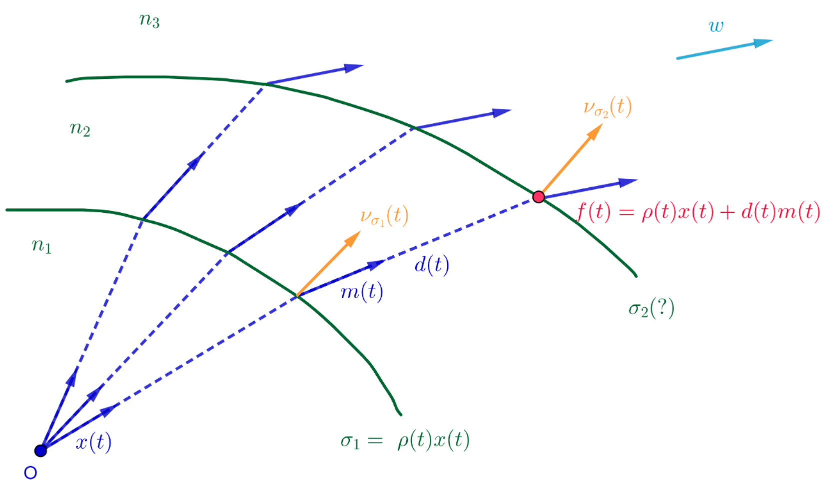

Figure 4.6: Lens refracting monochromatic rays into w

parametrization $\rho(t)x(t)$ for σ_1, $t \in D$ with $\rho(t)$ a given C^2 positive function. At σ_1, since $\kappa_1 = \frac{n_2}{n_1} = n_2 > 1$, refraction always occurs. Now let $m(t)$ be the refracted rays corresponding to $x(t)$ at σ_1. $m(t)$ can be found from (4.7) in terms of the incident direction $x(t)$ and the outer unit normal $\nu_{\sigma_1}(t)$ to σ_1. The formula of $\nu_{\sigma_1}(t)$ can be obtained in terms of ρ and ρ' from the following general result.

Proposition 4.3.1. *Given a curve σ parametrized by $\rho(t)x(t)$ where $x(t) = (\sin t, \cos t)$ and with $\rho \in C^1$, the outer unit normal to the curve is given by*

$$\nu(t) = \frac{1}{\sqrt{\rho(t)^2 + \rho'(t)^2}}(\rho(t)\sin t - \rho'(t)\cos t, \rho'(t)\sin t + \rho(t)\cos t).$$

Proof. A tangent vector to the curve at the point $\rho(t)x(t)$ is

$$(\rho(t)x(t))' = \rho'(t)x(t) + \rho(t)x'(t) = (\rho'(t)\sin t + \rho(t)\cos t, \rho'(t)\cos t - \rho(t)\sin t).$$

Noticing that

$$\|(\rho(t)x(t))'\| = \rho^2(t) + \rho'^2(t),$$

and using that ν is outer, i.e, $x \cdot \nu \geq 0$ we conclude the proof of the Proposition. □

At σ_2, since $\kappa_2 = \frac{n_3}{n_2} = \frac{1}{n_2} < 1$, we should have from (4.8)

$$m(t) \cdot w \geq \kappa_2 \quad \forall t \in D.$$

This means there are restrictions on $w, x(t)$, and $\rho(t)$.

We assume that these conditions are satisfied, and we parametrize σ_2 by

$$f(t) = \rho(t)x(t) + d(t)m(t) \tag{4.9}$$

$d(t)$ being the distance crossed by the incident ray $x(t)$ inside the lens. To find σ_2, it is enough to find a formula for $d(t)$.

Snell's law at σ_2 implies that $m(t) - \frac{1}{n_2}w$ is parallel to $\nu_{\sigma_2}(t)$ for every t. Here ν_{σ_2} is the outer unit normal to σ_2 at t. This means that

$$(m(t) - \frac{1}{n_2}w) \cdot f'(t) = 0 \quad \forall t \in D \tag{4.10}$$

We have

$$m(t) \cdot f'(t) = m(t) \cdot [(\rho(t)x(t))' + (d(t)m(t))'] = m(t) \cdot [\rho(t)x(t)]' + m(t) \cdot [d(t)m(t)]',$$

From (4.7), and using the fact that $\|x(t)\| = 1$ and that ν_{σ_1} is the normal to $\sigma_1 = \{\rho(t)x(t)\}$, we get

$$m(t) \cdot [\rho(t)x(t)]' = \frac{1}{n_2}(x(t) - \lambda_1\nu_{\sigma_1}(t)) \cdot (\rho(t)x(t))' = \frac{1}{n_2}x(t) \cdot (\rho'(t)x(t) + \rho(t)x'(t)) = \frac{1}{n_2}\rho'(t)$$

Since $\|m(t)\| = 1$

$$m(t) \cdot [d(t)m(t)]' = m(t) \cdot [d'(t)m(t) + d(t)m'(t)] = d'(t).$$

Hence (4.10) becomes

$$0 = d'(t) + \frac{1}{n_2}\rho'(t) - \frac{1}{n_2}w \cdot f'(t) = \left(d(t) + \frac{1}{n_2}\rho(t) - \frac{1}{n_2}w \cdot f(t)\right)'$$

Replacing $f(t)$ with its value in (4.9), we get that

$$d(t) = \frac{C - \rho(t)(1 - w \cdot x(t))}{n_2 - w \cdot m(t)}, \tag{4.11}$$

with C a constant. Given the lower face of a lens, we then obtain a family of curves so that the corresponding lenses uniformly refract incident rays emitted from a point source into a constant direction. See Aspherical Lens Design [12] for more similar examples and figures and Design of Pairs of reflectors [13] for the case of reflectors and hybrid systems.

Chapter 5

Refraction of Dichromatic Ray Into a Fixed Direction

In the previous chapter, we studied how we can adjust a lens so that the monochromatic incident rays refract uniformly into a fixed direction at every point. Given the lower face, we were able to derive a parametrization for the upper one.

In this chapter, we assume that the incident rays emitted from a point source are dichromatic, so the rays are of two colors (i.e. two different wavelengths), say r and b. Since different wavelengths give different refraction indices, at a point of incidence of dichromatic ray, the corresponding colors will refract differently. In this section, we study the existence of a single lens which refracts rays of the two different colors into a fixed direction.

The setting of the problem is as follows. We are given a fixed unit vector w in $\mathbb{R}^2$ and three media 1, 2, and 3, and we assume media 1 and 3 are vaccuum i.e. $n_1 = n_3 = 1$. O is a point source in medium 1, and w is a unit direction in medium 3. Consider a closed interval $D \subseteq [-\pi/2, \pi/2]$. Dichromatic rays with colors r and b are emitted from O with unit direction $x(t) = (\sin t, \cos t)$, $t \in D$. The goal is to construct σ_1 separating medium 1 and 2 and σ_2 separating medium 2 and 3 so that rays with both colors leave the lens enclosed by σ_1 and σ_2 with direction w. In this case both the lower and upper faces of the lens are unknown, and the incident rays $x(t)$ are refracted differently for the colors b and r. We assume that the refractive indices of medium 2 corresponding to the colors b and r are $n_b, n_r > 1$ with $n_b > n_r$, see figure 5.1.

Given a lower face σ_1 parametrized by $\rho(t)x(t)$, the incident dichromatic rays with direction $x(t)$ are dispersed by σ_1 into two rays with colors r and b and with corresponding unit directions $m_r(t)$ and $m_b(t)$. We know from Remark 4.2.1 that if $m_r(t) \cdot w \geq \dfrac{1}{n_r}$ and $m_b(t) \cdot w \geq \dfrac{1}{n_b}$, then there exist curves σ_r parametrized by f_r, and σ_b parametrized by f_b such that the lenses (σ_1, σ_r) refract the rays with color r into the direction w and the the len (σ_1, σ_b) refract the rays with color b into w. Note from (4.11), that f_r and f_b are not unique. We ask then the

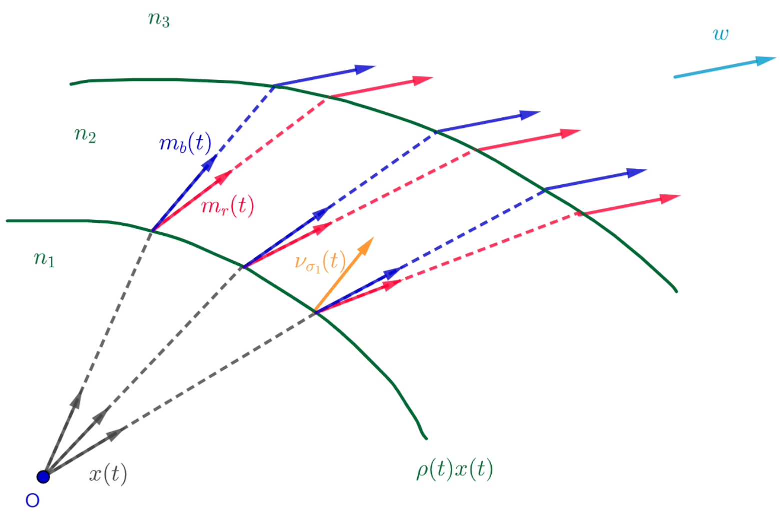

Figure 5.1: Lens refracting dichromatic rays into w

following question: is it possible to find ρ, f_r and f_b such that f_r can be obtained by a re-parametrization of f_b? That is, is there ρ, f_r, f_b and a continuous map $\phi : D \mapsto D$ such that

$$f_r(t) = f_b(\phi(t)) \quad \forall t \in D. \tag{5.1}$$

In case this function exists, we have that $f_r(D) \subseteq f_b(D)$. This means that the lens (σ_1, σ_b) refracts all the rays with both colors into direction w, but there might be some points on f_b that are not reached by the rays with color r.

5.1 Necessary Condition for the Existence of the Lens

We find a neccessary condition on w for the existence of a solution to the dichromatic problem.

Lemma 5.1.1. *If the problem described above is solvable, then $w = x(t)$ for some $t \in D$.*

Proof. If there exists a continuous function $\phi : D \mapsto D$ satisfying (5.1), then since D is a closed interval, by Brouwer's Fixed Point Theorem 2.3.1, there exists a point $t_0 \in D$ such that $\phi(t_0) = t_0$. By (5.1), this gives $f_r(t_0) = f_b(t_0)$. This means from (4.9) that the refracted rays $m_r(t_0)$ and $m_b(t_0)$ coincide. Since the two rays have different wavelengths, this can only happen if $m_r(t_0) = m_b(t_0) = x(t_0)$ which means that the incident ray is along the normal of σ_1 at t_0. Again since $m_r(t_0) = m_b(t_0)$ and are both refracted into w then they coincide with the normal of σ_2 at t_0 concluding that $x(t_0) = m_r(t_0) = m_b(t_0) = w$. □

Therefore, if w doesn't belong to the set $x(D)$, the problem is not solvable.

Without loss of generality, we can assume that $w = e = (0, 1)$. In fact, the system can be rotated in order to get this setting.

Corollary 5.1.2. *If the problem is solvable for $w = e$, then $0 \in D$.*

Proof. As in the argument of Lemma 5.1.1, there is a point $t_0 \in D$ such that $x(t_0) = (0, 1)$. Then $(\sin t_0, \cos t_0) = (0, 1)$ obtaining that $t_0 = 0$. □

We shall then in the rest of the chapter assume by means of rotation that $w = e$ and study the dichromatic problem when D is a closed subinterval of $[-\pi/2, \pi/2]$ containing 0 in its interior.[1]

5.2 Deriving the system of Functional Differential Equations

Assume a solution exists to the dichromatic problem as follows: having $D \subseteq [-\pi/2, \pi/2]$ a closed interval containing 0, there exists a $\rho \in C^2(D)$ positive, $\phi \in C^1(D)$, and constants C_r and C_b such that $f_r(t) = f_b(\varphi(t))$. Geometrically this implies that the lens with lower face $\sigma_1 = \{\rho(t)x(t)\}$ and upper face σ_2 parametrized by $f_b(t)$ refracts dichromatic rays with colors r and b emitted from the origin with unit direction $x(t) = (\sin t, \cos t), t \in D$ into the direction e. Recall our assumption that the medium around the lens is vacuum and the medium of the lens has refraction indices n_b and n_r corresponding to each color b and r with $n_b > n_r > 1$.

We denote by ν_{σ_1} and ν_{σ_2} the outer unit normal vectors of σ_1 and σ_2 respectively. Rays with direction $x(t)$ are dispersed by σ_1 at the point $\rho(t)x(t)$ into the unit directions $m_r(t)$ and $m_b(t)$ corresponding to each color r and b. $m_r(t)$ and $m_b(t)$ can be obtained using Snell's law (4.7). We also have $f_r(t) = \rho(t)x(t)+d_r(t)m_r(t)$ and $f_b(t) = \rho(t)x(t) + d_b(t)m_b(t)$ with d_r and d_b given by (4.11).

From Corollary 5.1.2, we have that $t = 0$ is a fixed point of ϕ, i.e.

$$\phi(0) = 0 \tag{5.2}$$

and

$$x(0) = \nu_{\sigma_1}(0) = \nu_{\sigma_2}(0) = m_r(0) = m_b(0) = (0, 1). \tag{5.3}$$

Since $f_r(t) = f_b(\phi(t))$ then $f_r(0) = f_b(\phi(0)) = f_b(0)$, and so

$$\rho(0)x(0) + d_r(0)m_r(0) = \rho(0)x(0) + d_b(0)m_b(0)$$

and hence

$$d_r(0) = d_b(0). \tag{5.4}$$

[1] We can still assume that 0 is in the boundary of D and solve the problem similarly to the left or right of zero, however we chose for simplicity of notations to discuss the problem in an interval $[-\delta, \delta]$ with $\delta > 0$.

Proposition 4.3.1 and the fact that $\nu_{\sigma_1}(0) = (0, 1)$ imply that

$$\rho'(0) = 0. \tag{5.5}$$

We set $\rho_0 = \rho(0)$, and $d_0 = d_b(0) = d_r(0)$. Consider the map $Z : D \mapsto \mathbb{R}^3$ defined by $Z(t) = (z_1(t), z_2(t), z_3(t))$, with

$$z_1(t) = \phi(t), \quad z_2(t) = \rho(t) - \rho_0, \quad z_3(t) = \rho'(t).$$

From (5.2) and (5.5), $Z(0) = \mathbf{0}$. In this section we derive a system of functional differential equations of the form

$$\begin{cases} Z'(t) &= H(t, Z(t), Z(z_1(t)), Z'(t), Z'(z_1(t))) \\ Z(0) &= \mathbf{0} \end{cases}, \tag{5.6}$$

with $H = H(t, v^0, v^1, w^0, w^1) := (h_1, h_2, h_3)$, a C^1 map in a neighborhood of

$$(0, Z(0), Z(z_1(0)), Z'(0), Z'(z_1(0))) = (0, \mathbf{0}, \mathbf{0}, Z'(0), Z'(0)),$$

where from (5.5), $Z'(0) = (\phi'(0), 0, \rho''(0))$. Our goal is to find H.

5.2.1 *Auxiliary funtions*

We introduce the following auxiliary functions, that will be used to derive the map H. We are given a color r with refractive index $n_r > 1$, ρ_0, d_0 positive real values, $t \in D$, $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in \mathbb{R}^3$

List 1

$$A_r(v) = \frac{1-n_r^2}{(v_2+\rho_0)+\sqrt{(n_r^2-1)|(v_2+\rho_0,,v_3)|^2+(v_2+\rho_0)^2}}$$

$$\mu_r(t,v) = \frac{1}{n_r}[\sin t - A_r(v)((v_2+\rho_0)\sin t - v_3\cos t)]$$

$$\tau_r(t,v) = \frac{1}{n_r}[\cos t - A_r(v)(v_3\sin t + (v_2+\rho_0)\cos t)]$$

$$D_r(t,v) = \frac{(n_r-1)d_0-(v_2+\rho_0)(1-\cos t)}{n_r-\tau_r(t,v)}$$

$$F_{r1}(t,v) = (v_2+\rho_0)\sin t + D_r(t,v)\mu_r(t,v)$$

$$F_{r2}(t,v) = (v_2+\rho_0)\cos t + D_r(t,v)\tau_r(t,v)$$

$$\Lambda_r(t,v) = \sqrt{1+\frac{1}{n_r^2}-\frac{2}{n_r}\tau_r(t,v)}$$

List 2

$$\tilde{A}_r(v,w) = \frac{(A_r(\mathbf{v}))^{\mathbf{2}}}{n_r^2-1}\left[v_2 + \frac{(n_r^2-1)((v_2+\rho_0)w_2+v_3w_3)+(v_2+\rho_0)w_2}{\sqrt{(n_r^2-1)((v_2+\rho_0)^2+v_3^2)+(v_2+\rho_0)^2}}\right]$$

$$\begin{aligned}\tilde{\mu}_r(t,v,w) = \frac{1}{n_r}[&\cos t - A_r(v)(w_2\sin t + (v_2+\rho_0)\cos t + v_3\sin t - w_3\cos t) \\ &-\tilde{A}_r(v,w)((v_2+\rho_0)\sin t - v_3\cos t)]\end{aligned}$$

$$\begin{aligned}\tilde{\tau}_r(t,v,w) = \frac{1}{n_r}[&-\sin t - A_r(v)(w_2\cos t - (v_2+\rho_0)\sin t + w_3\sin t + v_3\cos t) \\ &-\tilde{A}_r(v,w)((v_2+\rho_0)\cos t + v_3\sin t)]\end{aligned}$$

$$\begin{aligned}\tilde{D}_r(t,v,w) = &\frac{-n_r[w_2(1-\cos t)+(v_2+\rho_0)\sin t]+\tau_r(t,\mathbf{v})[w_2(1-\cos t)+(v_2+\rho_0)\sin t]}{(n_r-\tau_r(t,v))^2} \\ &+\frac{\tilde{\tau}_r(t,v,w)[(n_r-1)d_0-(v_2+\rho_0)(1-\cos t)]}{(n_r-\tau_r(t,v))^2}\end{aligned}$$

$$\tilde{F}_{r1}(t,v,w) = w_2\sin t + (v_2+\rho_0)\cos t + \tilde{D}_r(t,v,w)\mu_r(t,v) + D_r(t,v)\tilde{\mu}_r(t,v,w)$$

$$\tilde{F}_{r2}(t,v,w) = w_2\cos t - (v_2+\rho_0)\sin t + \tilde{D}_r(t,v,w)\tau_r(t,v) + D_r(t,v)\tilde{\tau}_r(t,v,w)$$

$$\tilde{\Lambda}_r(t,v,w) = -\frac{\tilde{\tau}_r(t,v,w)}{n_r\Lambda(t,v)}$$

Denoting $\Delta_r = \dfrac{n_r}{n_r-1}$, we have the following values at $t = 0$, $v = \mathbf{0}$ and $w \in \mathbb{R}^3$

List 3

$$A_r(\mathbf{0}) = \frac{1-n_r}{\rho_0}, \quad \mu_r(0,\mathbf{0}) = 0, \quad \tau_r(0,\mathbf{0}) = 1, \quad D_r(0,\mathbf{0}) = d_0, \quad F_{r1}(0,\mathbf{0}) = 0, \quad F_{r2}(0,\mathbf{0}) = \rho_0 + d_0,$$

$$\Lambda_r(0,\mathbf{0}) = \frac{1}{\Delta_r}, \quad \tilde{A}_r(\mathbf{0},v) = \frac{n_r-1}{\rho_0^2}v_2, \quad \tilde{\mu}_r(0,\mathbf{0},w) = 1 - \frac{1}{\Delta_r}\frac{w_3}{\rho_0}, \quad \tilde{\tau}_r(0,\mathbf{0},w) = 0, \quad \tilde{D}_r(0,\mathbf{0},v) = 0,$$

$$\tilde{F}_{r1}(0,\mathbf{0},v) = \rho + d_0\left(1 - \frac{1}{\Delta_r}\frac{w_3}{\rho_0}\right), \quad \tilde{F}_{r2}(0,\mathbf{0},w) = v_2, \quad \tilde{\Lambda}_r(0,\mathbf{0},w) = 0.$$

Remark 5.2.1. We then deduce from List 3 that all the functions in List 1 and 2 are analytic in a neighborhood of $t = 0$. Moreover, one can verify that for every C^1 field $V(t)$ in this neighborhood satisfying $V(0) = \mathbf{0}$

$$\frac{d}{dt}A_r(V(t)) = \tilde{A}_r(V(t), V'(t)) \qquad \frac{d}{dt}\mu_r(t,V(t)) = \tilde{\mu}_r(t,V(t),V'(t))$$

$$\frac{d}{dt}\tau_r(t,V(t)) = \tilde{\tau}_r(t,V(t),V'(t)) \qquad \frac{d}{dt}D_r(t,V(t)) = \tilde{D}_r(t,V(t),V'(t))$$

$$\frac{d}{dt}F_{r1}(t,V(t)) = \tilde{F}_{r1}(t,V(t),V'(t)) \qquad \frac{d}{dt}F_{r2}(t,V(t)) = \tilde{F}_{r2}(t,V(t),V'(t))$$

$$\frac{d}{dt}\Lambda_r(t,V(t)) = \tilde{\Lambda}_r(t,V(t),V'(t))$$

5.2.2 *Finding h_i's*

We write the variables in terms of t, Z, and Z' using the functions in Lists 1 and 2.

From Snell's law (4.7), and (4.5) at $\rho(t)x(t)$ we have that

$$m_r(t) = \frac{1}{n_r}(x(t) - \Phi_{n_r}(x\cdot\nu)\nu(t)) \tag{5.7}$$

with

$$\Phi_{n_r}(x\cdot\nu) = x\cdot\nu - \sqrt{n_r^2 - 1 + (x\cdot\nu)^2} = \frac{1-n_r^2}{x\cdot\nu + \sqrt{n_r^2-1+(x\cdot\nu)^2}},$$

and from Proposition 4.3.1,

$$\begin{aligned}
x\cdot\nu &= \frac{1}{\sqrt{\rho(t)^2+\rho'(t)^2}}(\sin t, \cos t)\cdot(\rho(t)\sin t - \rho'(t)\cos t, \rho'(t)\sin t + \rho(t)\cos t) \\
&= \frac{1}{\sqrt{\rho(t)^2+\rho'(t)^2}}\left(\rho(t)\sin^2 t - \rho'(t)\sin t\cos t + \rho'(t)\cos t\sin t + \rho(t)\cos^2 t\right) \\
&= \frac{\rho(t)}{\sqrt{\rho^2(t)+\rho'^2(t)}}.
\end{aligned}$$

Then we get using List 1

$$\Phi_{n_r}(x\cdot\nu) = \frac{1-n_r^2}{\frac{\rho(t)}{\sqrt{\rho^2(t)+\rho'^2(t)}} + \sqrt{n_r^2 - 1 + \frac{\rho^2(t)}{\rho^2(t)+\rho'^2(t)}}} = \frac{(1-n_r^2)(\sqrt{\rho^2(t)+\rho'^2(t)})}{\rho(t)+\sqrt{\rho^2(t)+(n_r^2-1)(\rho^2(t)+\rho'^2(t))}}$$
$$= \frac{1-n_r^2}{(z_2+\rho_0)+\sqrt{(n_r^2-1)((z_2+\rho_0)^2+z_3^2)+(z_2+\rho_0)^2}}\sqrt{(z_2+\rho_0)^2+z_3^2}$$
$$= A_r(Z(t))\,|(z_2+\rho_0,z_3)|$$

Writing $m_r(t) = (m_{r1}(t), m_{r2}(t))$, we deduce from (5.7), and Proposition 4.3.1

$$\begin{aligned} m_{r1}(t) &= \frac{1}{n_r}[\sin t - A_r(Z(t))(\rho(t)\sin t - \rho'(t)\cos t)] \\ &= \frac{1}{n_r}[\sin t - A_r(Z(t))((z_2+\rho_0)\sin t - z_3\cos t)] \\ &= \mu_r(t, Z(t)), \end{aligned} \tag{5.8}$$

$$\begin{aligned} m_{r2}(t) &= \frac{1}{n_r}[\cos t - A_r(Z(t))(\rho'(t)\sin t + \rho(t)\cos t)] \\ &= \frac{1}{n_r}[\cos t - A_r(Z(t))(z_3\sin t + (z_2+\rho_0)\cos t)] \\ &= \tau_r(t, Z(t)) \end{aligned} \tag{5.9}$$

The refracted ray with color r propagates inside the lens and hits σ_2 at the point $f_r(t) = \rho(t)x(t) + d_r(t)m_r(t)$, where from (4.11), and (5.9)

$$d_r(t) = \frac{C_r - \rho(t)(1-\cos t)}{n_r - m_{r2}(t)} = \frac{C_r - (z_2+\rho_0)(1-\cos t)}{n_r - \tau_r(t, Z(t))}$$

Plugging $t = 0$ in the formula of d_r and using List 3, we get $C_r = (n_r - 1)d_0$. Hence from List 1

$$d_r(t) = \frac{(n_r-1)d_0 - (z_2+\rho_0)(1-\cos t)}{n_r - \tau_r(t, Z(t))} = D_r(t, Z(t)), \tag{5.10}$$

Writing $f_r(t) = (f_{r1}(t), f_{r2}(t))$, we get from (5.8), (5.9), (5.10)

$$f_{r1}(t) = (z_2+\rho_0)\sin t + D_r(t, Z(t))\mu_r(t, Z(t)) = F_{r1}(t, Z(t)) \tag{5.11}$$
$$f_{r2}(t) = (z_2+\rho_0)\cos t + D_r(t, Z(t))\tau_r(t, Z(t)) = F_{r2}(t, Z(t) \tag{5.12}$$

At σ_2, from Snell's Law (4.7) with $m_r(t)$ as the incident direction and $e = (0,1)$ the refracted direction, we have

$$m_r(t) - \frac{1}{n_r}e = \lambda_{2,r}\nu_{\sigma_2}.$$

Since $\frac{1}{n_r} < 1$ then from (4.5) $\lambda_{2,r} > 0$ and so taking absolute values in above equation yields to

$$\lambda_{2,r} = \left| m_r(t) - \frac{1}{n_r} e \right| = \sqrt{1 + \frac{1}{n_r^2} - \frac{2}{n_r} \tau(t, Z(t))} = \Lambda_r(t, Z(t)) \tag{5.13}$$

Remark 5.2.2. From Remark 5.2.1, we have

$$m'_{1r}(t) = \tilde{\mu}_r(t, Z(t), Z'(t)), \quad m'_{r2}(t) = \tilde{\tau}_r(t, Z(t), Z'(t)),$$

$$d'_r(t) = \tilde{D}_r(t, Z(t), Z'(t)), \quad f'_{r1}(t) = \tilde{F}_{r1}(t, Z(t), Z'(t)),$$

$$f'_{r2}(t) = \tilde{F}_{r2}(t, Z(t), Z'(t)), \quad \lambda'_{2,r}(t) = \tilde{\Lambda}_r(t, Z(t), Z'(t)).$$

Notice also the formula for the variables related to the rays with color b can be obtained from above with replacing n_r in (5.8), (5.9), (5.10), (5.11), (5.12), (5.13), and in the formula above by n_b.

We are ready to calculate z'_1, z'_2, z'_3 as functions of $(t, Z(t), Z(z_1(t)), Z'(t), Z'(z_1(t)))$. We will need the following result.

Lemma 5.2.3.

$$f'_{b1}(0) \neq 0.$$

Proof. Assume by contradiction that $f'_{b1}(0) = 0$. Differentiating (5.1) and using (5.2) we obtain $f'_{r1}(0) = f'_{b1}(0) = 0$. From Remark 5.2.2, and List 3

$$f'_{r1}(0) = \tilde{F}_{r1}(0, \mathbf{0}, Z'(0)) = \rho_0 + d_0 \left(1 - \frac{1}{\Delta_r} \frac{z'_3}{\rho_0}\right) = \rho_0 + d_0 \left(1 - \frac{1}{\Delta_r} \frac{\rho''(0)}{\rho_0}\right) = 0,$$

so $\rho''(0) = \rho_0 \Delta_r \left(\frac{\rho_0}{d_0} + 1\right)$. Similarly, using that

$$f'_{b1}(0) = \rho_0 + d_0 \left(1 - \frac{1}{\Delta_b} \frac{\rho''(0)}{\rho_0}\right) = 0,$$

we get $\rho''(0) = \rho_0 \Delta_b \left(\frac{\rho_0}{d_0} + 1\right)$. Equating both obtained formulas for $\rho''(0)$ leads to a contradiction since $\rho_0, d_0 > 0$ and

$$\Delta_r = \frac{n_r}{n_r - 1} = 1 + \frac{1}{n_r - 1} > \Delta_b. \tag{5.14}$$

□

Calculating h_1. Recall $z_1'(t) = \phi'(t)$. We have $f_{r1}(t) = f_{b1}(\phi(t))$. Differentiate both sides

$$f_{r1}'(t) = f_{b1}'(\phi(t)) \cdot \phi'(t)$$

We have from Lemma 5.2.3 and (5.2) that at $t = 0$, $f_{b1}'(\phi(0)) = f_{b1}'(0) \neq 0$ then by continuity there exists a neighborhood of $t = 0$ such that $f_{b1}(\phi(t)) \neq 0$ concluding from Remark 5.2.2 that for t in this neighborhood

$$z_1'(t) = \phi'(t) = \frac{f_{r1}'(t)}{f_{b1}'(z_1(t))} = \frac{\tilde{F_{r1}}(t, Z(t), Z'(t))}{\tilde{F_{b1}}(z_1(t), Z(z_1(t)), Z'(z_1(t)))} := h_1(t, Z(t), Z(z_1(t)), Z'(t), Z'(z_1(t)))$$

where

$$h_1(t, v^0, v^1, w^0, w^1) = \frac{\tilde{F}_{r1}(t, v^0, w^0)}{\tilde{F}_{b1}(v_1^0, v^1, w^1)} \tag{5.15}$$

Calculating h_2. We have $z_2(t) = \rho(t) - \rho_0$, then

$$z_2'(t) = \rho'(t) = z_3(t) := h_2(t, Z(t), Z(z_1(t)), Z'(t), Z'(z_1(t)))$$

where

$$h_2(t, v^0, v^1, w^0, w^1) = v_3^0 \tag{5.16}$$

Calculating h_3. At σ_2, the rays $m_r(t)$ and $m_b(\phi(t))$ refract at the point $f_r(t)$ into the vector $e = (0, 1)$. By Snell's Law (4.3) at that point

$$m_r(t) - \frac{1}{n_r} e = \lambda_{2,r} \nu_{\sigma_2}(t), \text{ and } \qquad m_b(\phi(t)) - \frac{1}{n_b} e = \lambda_{2,b} \nu_{\sigma_2}(t)$$

We have already shown that since $n_r, n_b > 1$ then $\lambda_{2,r}, \lambda_{2,b} > 0$ and so

$$\frac{m_r(t) - \frac{1}{n_r}}{\lambda_{2,r}(t)} = \frac{m_b(\phi(t)) - \frac{1}{n_b}}{\lambda_{2,b}(\phi(t))}$$

Taking the first components we get

$$m_{r1}(t) \lambda_{2,b}(\phi(t)) = m_{b1}(\phi(t)) \lambda_{2,r}(t),$$

and so using (5.8) and (5.13)

$$\mu_r(t, Z) \Lambda_b(z_1, Z(z_1)) = \mu_b(z_1, Z(z_1)) \Lambda_r(t, Z)$$

Differentiating with respect to t, and using Remark 5.2.2 we get

$$\begin{aligned} &\tilde{\mu}_r(t, Z, Z') \Lambda_b(z_1, Z(z_1)) + \mu_r(t, Z) \tilde{\Lambda}_b(z_1, Z(z_1), Z'(z_1)) z_1' \\ &\qquad = \tilde{\mu}_b(z_1, Z(z_1), Z'(z_1)) \Lambda_r(t, Z) z_1' + \mu_b(z_1, Z(z_1)) \tilde{\Lambda}_r(t, Z, Z') \end{aligned} \tag{5.17}$$

so

$$\tilde{\mu}_r(t,Z,Z') = \frac{z_1'\left[\tilde{\mu}_b(z_1,Z(z_1),Z'(z_1))\Lambda_r(t,Z) - \mu_r(t,Z)\tilde{\Lambda}_b(z_1,Z(z_1),Z'(z_1))\right] + \mu_b(z_1,Z(z_1))\tilde{\Lambda}_r(t,Z,Z')}{\Lambda_b(z_1,Z(z_1))}$$

Plugging the formula for $\tilde{\mu}_r$ (see List 2)

$$\frac{1}{n_r}[\cos t - A_r(Z)(z_2'\sin t + (z_2+\rho_0)\cos t + z_3\sin t - z_3'\cos t) - \tilde{A}_r(Z,Z')((z_2+\rho_0)\sin t - z_3\cos t)]$$
$$= \frac{z_1'[\tilde{\mu}_b(z_1,Z(z_1),Z'(z_1))\Lambda_r(t,Z) - \mu_r(t,Z)\tilde{\Lambda}_b(z_z,Z(z_1),Z'(z_1))] + \mu_b(z_1,Z(z_1))\tilde{\Lambda}_r(t,Z,Z')}{\Lambda_b(z_1,Z(z_1))}$$

Solving for $z_3'(t)$, we get

$$\begin{aligned} z_3'(t) = \frac{1}{\cos t A_r(Z)}\Big[&-\cos t + \tilde{A}_r(Z,Z')((z_2+\rho_0)\sin t - z_3\cos t)) \\ &+ n_r \frac{z_1'[\tilde{\mu}_b(z_1,Z(z_1),Z'(z_1))\Lambda_r(t,Z) - \mu_r(t,Z)\tilde{\Lambda}_b(z_1,Z(z_1),Z'(z_1))] + \mu_b(z_1,Z(z_1))\tilde{\Lambda}_r(t,Z,Z')}{\Lambda_b(z_1,Z(z_1))}\Big] \\ &+ \frac{z_2'\sin t + (z_2+\rho_0)\cos t + z_3\sin t}{\cos t} := h_3(t,Z(t),Z(z_1(t)),Z'(t),Z'(z_1(t))) \end{aligned}$$

where

$$\begin{aligned} h_3(t,v^0,v^1,w^0,w^1) = \frac{1}{\cos t A_r(v^0)}\Big[&-\cos t + \tilde{A}_r(v^0,w^0)((v_2^0+\rho_0)\sin t - v_3^0\cos t) \\ &+ n_r \frac{w_1^0[\tilde{\mu}_b(v_1^0,v^1,w^1)\Lambda_r(t,v^0) - \mu_r(t,v^0)\tilde{\Lambda}_b(v_1^0,v^1,w^1)] + \mu_b(v_1^0,v^1)\tilde{\Lambda}_r(t,v^0)}{\Lambda_b(v_1^0,v^1)}\Big] \\ &+ \frac{w_2^0\sin t + (v_2^0+\rho_0)\cos t + v_3^0\sin t}{\cos t} \end{aligned} \tag{5.18}$$

5.3 Finding a Solution for the Dichromatic Problem From the Solution of the System

Before proving the existence of solution for the system computed in the previous section, we show in this section that a solution to the system (if it exists) solves the stated optic problem.

Theorem 5.3.1. *Let $\rho_0 > 0$ and $d_0 > 0$ be given and $H = (h_1, h_2, h_3)$ with h_i given in* (5.15), (5.16), *and* (5.18). *Assume the system*

$$P = H(0, \mathbf{0}, \mathbf{0}, P, P)$$

has a solution $P = (p_1, p_2, p_3)$ with $0 < |p_1| < 1$ and that H is smooth in a neighborhood of $\mathcal{P} = (0, \mathbf{0}, \mathbf{0}, P, P)$. Assume also that $Z(t) = (z_1(t), z_2(t), z_3(t))$ is a C^1 solution to the system of functional equations, with $Z'(0) = P$. Define

$$\rho(t) = z_2(t) + \rho_0, \qquad \phi(t) = z_1(t).$$

Then, there exists $\delta > 0$ such that $\phi : [-\delta, \delta] \mapsto [-\delta, \delta]$ and $f_r(t) = f_b(\phi(t))$ where $f_r(t) = \rho(t)x(t) + d_r(t)m_r(t)$, $f_b(t) = \rho(t)x(t) + d_b(t)m_b(t)$ and

$$d_r(t) = \frac{C_r - \rho(t)(1 - \cos t)}{n_r - e \cdot m_r(t)}, \qquad d_b(t) = \frac{C_b - \rho(t)(1 - \cos t)}{n_b - e \cdot m_b(t)} \tag{5.19}$$

with $C_r = (n_r - 1)d_0$ and $C_b = (n_b - 1)d_0$ and $x(t) = (\sin t, \cos t)$ are the emitted rays which refract at the curve $\rho(t)x(t)$ into the rays $m_r(t)$ and $m_b(t)$ corresponding to the colors r and b respectively. Moreover, for $t \in [-\delta, \delta]$, f_r and f_b have normal vectors, and

$$\rho(t), d_r(t), d_b(t) > 0, \qquad m_r(t) \cdot e \geq \frac{1}{n_r}, \text{ and } m_b(t) \cdot e \geq \frac{1}{n_b}. \tag{5.20}$$

Proof. The proof will go in steps.
Step 1: Calculating $z_3(t)$.
From (5.16) and since $Z(t)$ solves the system we have that

$$z_2'(t) = h_2(t, Z(t), Z(z_1(t)), Z'(t), Z'(z_1(t))) = z_3(t).$$

Then,

$$\rho'(t) = z_3(t) \tag{5.21}$$

Step 2: Showing that $\phi : [-\delta, \delta] \mapsto [-\delta, \delta]$ for some $\delta > 0$.
We have $\phi(t) = z_1(t)$, and

$$z_1'(0) = |p_1| = \lim_{t \to 0} \left| \frac{z_1(t) - z(0)}{t - 0} \right| = \lim_{t \to 0} \left| \frac{z_1(t)}{t} \right| < 1,$$

there exists $\delta > 0$ such that $|\phi(t)| \leq |t|$ for $|t| < \delta$.

Step 3: For δ sufficiently small, (5.20) holds.
By continuity, it is enough to look at the values at $t = 0$. We have

$$\rho(0) = z_2(0) + \rho_0 = \rho_0 > 0.$$

Also from (5.21), $\rho'(0) = z_3(0) = 0$. Using Proposition 4.3.1, the normal to $\sigma = \{\rho(t)x(t)\}$ at $t = 0$ is $\nu(0) = (0, 1)$. By Snell's law, we obtain that

$$m_r(0) = m_b(0) = (0, 1). \tag{5.22}$$

Hence, $e \cdot m_r(0) = 1 > \frac{1}{n_r}$ and $e \cdot m_b(0) = 1 > \frac{1}{n_b}$. Finally, from (5.19)

$$d_r(0) = \frac{C_r}{n_r - 1} = d_0 = \frac{C_b}{n_b - 1} = d_b(0) > 0 \tag{5.23}$$

Choosing $\delta > 0$ sufficiently small, (5.20) holds.

Step 4: For $t \in [-\delta, \delta]$, $f_r(t) = (F_{r1}(t, Z(t)), F_{r2}(t, Z(t)))$ and $f_b(t) = (F_{b1}(t, Z(t)), F_{b2}(t, Z(t)))$. First, we show that

$$m_r(t) = (\mu_r(t, Z(t)), \tau_r(t, Z(t))) \tag{5.24}$$

m_r is the refracted ray of $x(t)$ by $\rho(t)x(t)$, then from (5.7) and Proposition 4.3.1 we have

$$\begin{aligned} m_r(t) = \frac{1}{n_r}\Big[&(\sin t, \cos t) \\ &- \frac{(1-n_r^2)}{\rho(t) + \sqrt{\rho^2(t) + (n_r^2 - 1)(\rho^2(t) + \rho'^2(t))}}((\rho(t)\sin t - \rho'(t)\cos t, \rho'(t)\sin t + \rho(t)\cos t))\Big] \end{aligned}$$

Using that Step 1, List 1, and $\rho(t) = z_2(t) + \rho_0$ we get

$$m_{r1}(t) = \frac{1}{n_r}[\sin t - A_r(Z(t))((z_2(t) + \rho_0)\sin t - z_3(t)\cos t)] = \mu_r(t, Z(t)),$$

$$m_{r2}(t) = \frac{1}{n_r}[\cos t - A_r(Z(t))(z_3(t)\sin t + (z_2(t) + \rho_0)\cos t)] = \tau_r(t, Z(t)).$$

Therefore, replacing $\rho(t) = z_2(t) + \rho_0$ in (5.19), it follows that $d_r(t) = D_r(t, Z(t))$. Now,

$$\begin{aligned} f_r(t) &= \rho(t)x(t) + d_r(t, Z(t))m_r(t) \\ &= \rho(t)(\sin t, \cos t) + D_r(t, Z(t))(\mu_r(t, Z(t)), \tau_r(t, Z(t))) \\ &= (F_{r1}(t, Z(t)), F_{r2}(t, Z(t))). \end{aligned}$$

Similarly, we get $f_b(t) = (F_{b1}(t, Z(t)), F_{b2}(t, Z(t))$

Step 5: For $t \in [-\delta, \delta]$, $F_{r1}(t, Z(t)) = F_{b1}(z_1(t), Z(z_1(t)))$.
From (5.15), we have that

$$\tilde{F}_{r1}(t, Z(t), Z'(t)) = z_1'(t)\tilde{F}_{b1}(z_1(t), Z(z_1(t)), Z'(z_1(t)))$$

Integrating with respect to t and using Remark 5.2.1 we get

$$F_{r1}(t, Z(t)) = F_{b1}(z_1(t), Z(z_1(t))) + c$$

At $t = 0$, we have from List 3 that $F_{r1}(0, \mathbf{0}) = F_{b1}(0, \mathbf{0}) = 0$, so $c = 0$, and the result follows.

Step 6: For δ small enough, $f_{r1}'(t) \neq 0$ and $f_{b1}'(t) \neq 0$ for $t \in [-\delta, \delta]$.

This means that f_r and f_b have normal vectors in this interval. By continuity of these functions, it is enough to show that the result holds for $t = 0$. From (5.15), $\tilde{F}_{1b}(0, \mathbf{0}, P) \neq 0$. Using Remark 5.2.1 and Step 4, we get

$$f'_{b1}(0) = \tilde{F}_{b1}(0, \mathbf{0}, Z'(0)) = \tilde{F}_{b1}(0, \mathbf{0}, P) \neq 0.$$

Having $p_1 \neq 0$ and using step 5 at $t = 0$, we get

$$f'_{r1}(0) = z'_1(0) f'_{b1}(0) = p_1 f'_{b1}(0) \neq 0.$$

Step 7: The two vectors $m_r(t) - \frac{1}{n_r}e$ and $m_b(z_1(t)) - \frac{1}{n_b}e$ are colinear for $t \in [-\delta, \delta]$.
Since $z'_3(t) = h_3(t, Z(t), Z(z_1(t)), Z'(t), Z'(z_1(t)))$, and from the calculation of h_3, we have

$$\begin{aligned}\tilde{\mu}_r(t, Z, Z')\Lambda_b(z_1, Z(z_1)) &+ \mu_r(t, Z)\tilde{\Lambda}_b(z_1, Z(z_1), Z'(z_1))z'_1 \\ &= \tilde{\mu}_b(z_1, Z(z_1), Z'(z_1))\Lambda_r(t, Z)z'_1 + \mu_b(z_1, Z(z_1))\tilde{\Lambda}_r(t, Z, Z')\end{aligned}$$

Integrating, and using Remark 5.2.1,

$$\mu_r(t, Z)\Lambda_b(z_1, Z(z_1)) = \mu_b(z_1, Z(z_1))\Lambda_r(t, Z) + c \tag{5.25}$$

From List 3, $\mu_r(0, \mathbf{0}) = \mu_b(0, \mathbf{0}) = 0$, $c = 0$. Squaring the resulting equality

$$\mu_r(t, Z)^2\Lambda_b(z_1, Z(z_1))^2 = \mu_b(z_1, Z(z_1))^2\Lambda_r(t, Z)^2 \tag{5.26}$$

Since $\|m_r(t)\| = \|m_b(t)\| = 1$, we have from (5.24)

$$\mu_r(t, Z(t))^2 + \tau_r(t, Z(t))^2 = 1 = \mu_b(z_1, Z(z_1))^2 + \tau_b(z_1, Z(z_1))^2$$

and then using the formula of Λ_r in List 1

$$\Lambda_r(t, Z)^2 = 1 + \frac{1}{n_r^2} - \frac{2}{n_r}\tau_r(t, Z) = 1 - \tau_r(t, Z)^2 + \left(\frac{1}{n_r} - \tau_r(t, Z)\right)^2 = \mu_r(t, Z)^2 + \left(\frac{1}{n_r} - \tau_r(t, Z)\right)^2$$

Similarly, since from step 1 $|z_1(t)| \leq |t|$

$$\Lambda_b(z_1, Z(z_1))^2 = \mu_b(z_1, Z(z_1))^2 + \left(\frac{1}{n_b} - \tau_b(z_1, Z(z_1))\right)^2$$

So (5.26) becomes

$$\mu_r(t, Z)^2\left[\mu_b(z_1, Z(z_1))^2 + \left(\frac{1}{n_b} - \tau_b(z_1, Z(z_1))\right)^2\right] = \mu_b(z_1, Z(z_1))^2\left[\mu_r(t, Z)^2 + \left(\frac{1}{n_r} - \tau_r(t, Z)\right)^2\right]$$

Then

$$\mu_r(t, Z)^2 \left(\frac{1}{n_b} - \tau_b(z_1, Z(z_1))\right)^2 = \mu_b(z_1, Z(z_1))^2 \left(\frac{1}{n_r} - \tau_r(t, Z)\right)^2 \tag{5.27}$$

From (5.20) proved in Step 3, and (5.24)

$$\tau_r(t, Z) = m_r(t) \cdot e \geq \frac{1}{n_r}, \qquad \tau_b(z_1, Z(z_1)) = m_b(z_1(t)) \cdot e \geq \frac{1}{n_b}$$

Also, by (5.25), since Λ_r and Λ_b are both positive, $\mu_r(t, Z)$ and $\mu_b(z_1, Z(z_1))$ have the same sign. Hence, (5.26) becomes

$$\mu_r(t, Z) \left(\tau_b(z_1, Z(z_1)) - \frac{1}{n_b}\right) = \mu_b(z_1, Z(z_1)) \left(\tau_r(t, Z) - \frac{1}{n_r}\right)$$

This means that the two vectors $(\mu_r(t, Z), \tau_r(t, Z) - \frac{1}{n_r})$ and $(\mu_b(z_1, Z(z_1)), \tau_b(z_1, Z(z_1)) - \frac{1}{n_b})$ are collinear. By (5.24), the result follows.

Step 8: We have for $t \in [-\delta, \delta]$, $f_{r2}(t) = f_{b2}(z_1(t))$.
We know that f_r and f_b have normal vector in this interval (step 6). By Snell's law at $f_r(t)$ and $f_b(t)$ we have $m_r(t) - \frac{1}{n_r}e$ and $m_b(z_1(t)) - \frac{1}{n_b}e$ are orthogonal to the tangent vectors $f_r'(t)$ and $f_b'(z_1(t))$ respectively. Hence, by the previous step, $f_r'(t)$ and $f_b'(z_1(t))$ are parallel. By steps 4 and 5 and Remark 5.2.1, we have

$$f_{r1}'(t) = z_1'(t) f_{b1}'(z_1(t))$$

By step 6, for $t \in [-\delta, \delta]$ and since $|z_1(t)| \leq |t|$, we have that $f_r'(t) \neq 0$ and $f_{b1}'(z_1(t)) \neq 0$, then

$$f_{r2}'(t) = z_1'(t) f_{b2}'(z_1(t))$$

Integrating the last identity we obtain $f_{r2}(t) = f_{b2}(z_1(t)) + c$. By (5.22) and (5.23),

$$f_{r2}(0) = \rho(0) + d_r(0) m_{r2}(0) = \rho_0 + d_0 = f_{b2}(0) = f_{b2}(z_1(0))$$

Hence, $c = 0$ and the result follows. □

5.4 Existence and uniqueness of solution to the dichromatic lens problem

We are now ready to prove the existence of a lens solving the dichromatic problem. Given $\rho_0, d_0 > 0$ and $H = (h_1, h_2, h_3)$ given in (5.15), (5.16), and (5.18), from Section 5.3, it is enough to prove that the system $P = H(0, 0, 0, P, P)$ has a solution $P = (p_1, p_2, p_3)$ with $0 < |p_1| < 1$, and to show that the system (5.6) has a solution. The first part will be studied in Section 5.4.1 and the second part will be analyzed in Section 5.4.2. We will highly rely on Theorem 3.4.1 and prove the following theorem

Theorem 5.4.1. *Given $\rho_0, d_0 > 0$, let $k_0 = \dfrac{\rho_0}{d_0}$.*

If $k_0 < \dfrac{(\Delta_r - \Delta_b)^2}{4\Delta_r\Delta_b}$, then a lens (σ_1, σ_2) refracting colors r and b into e exists in the sense of (5.1). Moreover this lens is the unique one such that the lower face passes through the point $(0, \rho_0)$, and the upper face passes through the point $(0, \rho_0 + d_0)$.

On the other hand, if $k_0 > \dfrac{(\Delta_r - \Delta_b)^2}{4\Delta_r\Delta_b}$, then no such lens exists. For the case where $k_0 = \dfrac{(\Delta_r - \Delta_b)^2}{4\Delta_r\Delta_b}$, the result is inconclusive.

5.4.1 *Solving the system* $P = H(0, \mathbf{0}, \mathbf{0}, P, P)$

Proposition 5.4.2. *The system $P = H(0, \mathbf{0}, \mathbf{0}, P, P)$ has a solution if and only if $k_0 \le \frac{(\Delta_r - \Delta_b)^2}{4\Delta_r\Delta_b}$.*

Proof. We have from (5.15), (5.16), (5.18), and List 3

$$p_1 = z_1'(0) = h_1(0, \mathbf{0}, \mathbf{0}, P, P) = \frac{\tilde{F}_{r1}(0, \mathbf{0}, \mathbf{0}, P, P)}{\tilde{F}_{b1}(0, \mathbf{0}, \mathbf{0}, P, P)} = \frac{\rho_0 + d_0(1 - \frac{1}{\Delta_r}\frac{p_3}{\rho_0})}{\rho_0 + d_0(1 - \frac{1}{\Delta_b}\frac{p_3}{\rho_0})} \tag{5.28}$$

and

$$p_2 = z_2'(0) = h_2(0, \mathbf{0}, \mathbf{0}, P, P) = 0,$$

Also, since $\tilde{A}_r(0, \mathbf{0}, P) = 0$

$$p_3 = z_3'(0) = h_3(0, \mathbf{0}, \mathbf{0}, P, P) = \frac{\rho_0}{1 - n_r}\left[-1 + 0 + n_r \frac{p_1\left[1 - \frac{1}{\Delta_b}\frac{p_3}{\rho_0}\right]\dfrac{1}{\Delta_r}}{\dfrac{1}{\Delta_b}}\right] + \rho_0 = p_1(p_3 - \rho_0\Delta_b) + \rho_0\Delta_r$$

Solving for p_1 we get

$$p_1 = \frac{\dfrac{p_3}{\rho_0} - \Delta_r}{\dfrac{p_3}{\rho_0} - \Delta_b} \tag{5.29}$$

(5.28) and (5.29) yields

$$\left(\rho_0 + d_0\left(1 - \frac{1}{\Delta_b}\frac{p_3}{\rho_0}\right)\right)\left(\frac{p_3}{\rho_0} - \Delta_b\right) = \left(\frac{p_3}{\rho_0} - \Delta_r\right)\left(\rho_0 + d_0\left(1 - \frac{1}{\Delta_r}\frac{p_3}{\rho_0}\right)\right)$$

Expanding

$$d_0\left(\frac{1}{\Delta_b} - \frac{1}{\Delta_r}\right)\left(\frac{p_3}{\rho_0}\right)^2 + d_0\left(\frac{\Delta_b}{\Delta_r} - \frac{\Delta_r}{\Delta_b}\right)\frac{p_3}{\rho_0} + (\rho_0 + d_0)(\Delta_r - \Delta_b) = 0$$

Simplifying, we get the following quadratic equation in $\dfrac{p_3}{\rho_0}$

$$\left(\frac{p_3}{\rho_0}\right)^2 - (\Delta_r + \Delta_b)\frac{p_3}{\rho_0} + (k_0 + 1)\,\Delta_b\Delta_r = 0 \tag{5.30}$$

The discriminant δ of (5.30) is

$$\delta = (\Delta_r + \Delta_b)^2 - 4(k_0 + 1)\Delta_r\Delta_b = (\Delta_r - \Delta_b)^2 - 4k_0\Delta_r\Delta_b \tag{5.31}$$

The quadratic equation (5.30) has real solutions if and only if $\delta \geq 0$ which is equivalent to

$$k_0 \leq \frac{(\Delta_r - \Delta_b)^2}{4\Delta_r\Delta_b} \tag{5.32}$$

□

Remark 5.4.3. The condition (5.32) is a necessary condition for the solvability of the dichromatic problem. If the dichromatic problem has a local solution, then the map

$$Z(t) = (\phi(t), \rho(t) - \rho_0, \rho'(t))$$

solves the system (5.6) for t in a neighborhood of zero. This means that, it should satisfy it at $t = 0$, i.e.

$$Z'(0) = H(0.\mathbf{0}, \mathbf{0}, Z'(0), Z'(0)).$$

Hence, $Z'(0)$ solves the system $P = H(0, \mathbf{0}, \mathbf{0}, P, P)$ which by Theorem 5.4.2 implies (5.32).

Proposition 5.4.4. *If* $k_0 < \dfrac{(\Delta_r - \Delta_b)^2}{4\Delta_r\Delta_b}$ *then the system* $P = H(0, \mathbf{0}, \mathbf{0}, P, P)$ *admits one solution* $P = (p_1, p_2, p_3)$ *satisfying* $0 < |p_1| < 1$. *In fact, we get*

$$P = \left(\frac{\Delta_b - \Delta_r + \sqrt{\delta}}{\Delta_r - \Delta_b + \sqrt{\delta}}, 0, \frac{\Delta_b + \Delta_r + \sqrt{\delta}}{2}\rho_0\right). \tag{5.33}$$

with δ *given in* (5.31),

Proof. From the assumption on k_0, we have $\delta > 0$. Then solutions to (5.30) are

$$\frac{p_3}{\rho_0} = \frac{(\Delta_b + \Delta_r) \pm \sqrt{\delta}}{2} \tag{5.34}$$

which implies from (5.29)

$$p_1 = \frac{\frac{(\Delta_b + \Delta_r) + \sqrt{\delta}}{2} - \Delta_r}{\frac{(\Delta_b + \Delta_r) + \sqrt{\delta}}{2} - \Delta_b} = \frac{\Delta_b - \Delta_r + \sqrt{\delta}}{\Delta_r - \Delta_b + \sqrt{\delta}}$$

and

$$p_1' = \frac{\frac{(\Delta_b+\Delta_r)-\sqrt{\delta}}{2} - \Delta_r}{\frac{(\Delta_b+\Delta_r)-\sqrt{\delta}}{2} - \Delta_b} = \frac{\Delta_b - \Delta_r - \sqrt{\delta}}{\Delta_r - \Delta_b - \sqrt{\delta}}$$

Since $\delta > 0$ in (5.31), $(\Delta_r - \Delta_b)^2 > \delta$ and so from (5.14) $\Delta_r - \Delta_b > \sqrt{\delta}$, hence

$$0 < |p_1| = \left| \frac{\Delta_b - \Delta_r + \sqrt{\delta}}{\Delta_r - \Delta_b + \sqrt{\delta}} \right| = \frac{\Delta_r - \Delta_b - \sqrt{\delta}}{\Delta_r - \Delta_b + \sqrt{\delta}} < 1$$

$$|p_1'| = \left| \frac{\Delta_b - \Delta_r - \sqrt{\delta}}{\Delta_r - \Delta_b - \sqrt{\delta}} \right| = \frac{\Delta_r - \Delta_b + \sqrt{\delta}}{\Delta_r - \Delta_b - \sqrt{\delta}} > 1$$

Notice that if $k_0 = \frac{(\Delta_r-\Delta_b)^2}{4\Delta_r\Delta_b}$, then we have $\delta = 0$ and then $|p_1| = |p_1'| = 1$. □

5.4.2 *Verifying the condition of Theorem 3.4.1*

We complete in this section the Proof of Theorem 5.4.1. Let $\mathcal{P} = (0, \mathbf{0}, \mathbf{0}, P, P)$ with P given in (5.33). We need $0 < |p_1| < 1$ and hence from the proof of Proposition 5.4.4 we will assume that $k_0 < \dfrac{(\Delta_r - \Delta_b)^2}{4\Delta_r\Delta_b}$.

Calculating $\nabla_{w^0} h_1(\mathcal{P})$ and $\nabla_{w^1} h_1(\mathcal{P})$. From (5.15),

$$h_1(t, v^0, v^1, w^0, w^1) = \frac{\tilde{F}_{r1}(t, v^0, w^0)}{\tilde{F}_{b1}(v_1^0, v^1, w^1)}.$$

Notice that the denominator is independent of w^0, so

$$\nabla_{w^0} h_1(\mathcal{P}) = \frac{\nabla_{w^0}\tilde{F}_{r1}(0, \mathbf{0}, P)}{\tilde{F}_{b1}(0, \mathbf{0}, P)}.$$

Using List 3, and that $p_2 = 0$ we have the following

$$\tilde{F}_{b1}(0, \mathbf{0}, P) = \rho_0 + d_0 \left(1 - \frac{1}{\Delta_b}\frac{p_3}{\rho_0}\right), \qquad \nabla_{w^0}\tilde{F}_{r1}(0, \mathbf{0}, P) = d_0 \nabla_{w^0}\tilde{\mu}_r(0, \mathbf{0}, P)$$

Again by List 3, we have

$$\nabla_{w^0}\tilde{\mu}_r(0, \mathbf{0}, P) = \left(0, 0, \frac{-1}{\rho_0\Delta_r}\right).$$

Therefore,

$$\nabla_{w^0} h_1(\mathcal{P}) = \left(0, 0, \frac{\frac{-1}{k_0\Delta_r}}{\rho_0 + d_0(1 - \frac{1}{\Delta_b}\frac{p_3}{\rho_0})}\right).$$

On the other hand, the numerator of h_1 is independent of w^1 then

$$\nabla_{w^1} h_1(\mathcal{P}) = \tilde{F}_{r1}(0,\mathbf{0},P)\frac{-\nabla_{w^1}\tilde{F}_{b1}(0,0,P)}{\tilde{F}_{b1}(0,0,P)^2} = \frac{-h_1(\mathcal{P})}{\tilde{F}_{b1}(0,\mathbf{0},P)}\nabla_{w^1}\tilde{F}_{b1}(0,\mathbf{0},P).$$

Since $h_1(\mathcal{P}) = z_1(0) = p_1$, proceeding as above we get

$$\nabla_{w^1} h_1(\mathcal{P}) = \frac{-p_1}{\rho_0 + d_0(1-\frac{1}{\Delta_b}\frac{p_3}{\rho_0})} d_0\left(0,0,\frac{-1}{\rho_0\Delta_r}\right) = \left(0,0,\frac{\frac{p_1}{k_0\Delta_b}}{\rho_0 + d_0(1-\frac{1}{\Delta_b}\frac{p_3}{\rho_0})}\right)$$

Calculating $\nabla_{w^0} h_2(\mathcal{P})$ and $\nabla_{w^1} h_2(\mathcal{P})$. From (5.16),

$$h_2(t,v^0,v^1,w^0,w^1) = v_3^0$$

which is independent of w^0 and of w^1. Therefore,

$$\nabla_{w^0} h_2(\mathcal{P}) = \nabla_{w^1} h_2(\mathcal{P}) = (0,0,0).$$

Calculating $\nabla_{w^0} h_3(\mathcal{P})$ and $\nabla_{w^1} h_3(\mathcal{P})$. From (5.18),

$$\begin{aligned} h_3(t,v^0,v^1,w^0,w^1) = \frac{1}{\cos t A_r(v^0)}\Big[& -\cos t + \tilde{A}_r(v^0,w^0)((v_2^0+\rho_0)\sin t - v_3^0\cos t) \\ & + n_r\frac{w_1^0[\tilde{\mu}_b(v_1^0,v^1,w^1)\Lambda_r(t,v^0) - \mu_r(t,v^0)\tilde{\Lambda}_b(v_1^0,v^1,w^1)] + \mu_b(v_1^0,v^1)\tilde{\Lambda}_r(t,v^0,w^0)}{\Lambda_b(v_1^0,v^1)}\Big] \\ & + \frac{w_2^0\sin t + (v_2^0+\rho_0)\cos t + v_3^0\sin t}{\cos t} \end{aligned}$$

Using List 3 and the fact that $p_2 = 0$ we have

$$\frac{\partial h_3}{\partial w_0^j} = \frac{1}{A_r(\mathbf{0})}\left[\frac{n_r}{\Lambda_b(0,\mathbf{0})}\delta_1^j(\tilde{\mu}_b(0,\mathbf{0},P)\Lambda_r(0,\mathbf{0}))\right] = -\rho_0\Delta_b\left(1-\frac{1}{\Delta_b}\frac{p_3}{\rho_0}\right)\delta_1^j$$

and so

$$\nabla_{w^0} h_3(\mathcal{P}) = \left(-\rho_0\Delta_b\left(1-\frac{1}{\Delta_b}\frac{p_3}{\rho_0}\right),0,0\right).$$

On the other hand and again using List 3, we have

$$\nabla_{w^1} h_3(\mathcal{P}) = \frac{1}{A_r(\mathbf{0})}\left[\frac{n_r w_1^0}{\Lambda_b(0,\mathbf{0})}\Lambda_r(0,\mathbf{0})\nabla_{w^1}\tilde{\mu}_b(0,\mathbf{0},P)\right] = -\rho_0\Delta_b\left(0,0,\frac{-p_1}{\rho_0\Delta_b}\right) = (0,0,p_1)$$

As a result we get the matrices

$$\nabla_{w^0} H(\mathcal{P}) = \begin{bmatrix} 0 & 0 & \frac{\frac{-1}{k_0\Delta_r}}{\rho_0+d_0(1-\frac{1}{\Delta_b}\frac{p_3}{\rho_0})} \\ 0 & 0 & 0 \\ -\rho_0\Delta_b(1-\frac{1}{\Delta_b}\frac{p_3}{\rho_0}) & 0 & 0 \end{bmatrix}, \quad \nabla_{w^1} H(\mathcal{P}) = \begin{bmatrix} 0 & 0 & \frac{\frac{p_1}{k_0\Delta_b}}{\rho_0+d_0(1-\frac{1}{\Delta_b}\frac{p_3}{\rho_0})} \\ 0 & 0 & 0 \\ 0 & 0 & p_1 \end{bmatrix} \tag{5.35}$$

We shall next prove that $Id - \nabla_{w^0} H(\mathcal{P})$ is invertible. We have

$$\det\left(Id - \nabla_{w^0} H(\mathcal{P})\right) = \left\| \begin{bmatrix} 1 & 0 & \frac{\frac{1}{k_0 \Delta_r}}{\rho_0 + d_0(1 - \frac{1}{\Delta_b}\frac{p_3}{\rho_0})} \\ 0 & 1 & 0 \\ \rho_0 \Delta_b (1 - \frac{1}{\Delta_b}\frac{p_3}{\rho_0}) & 0 & 1 \end{bmatrix} \right\| = 1 - \alpha\beta$$

where $\alpha = \frac{\frac{1}{k_0 \Delta_r}}{\rho_0 + d_0(1 - \frac{1}{\Delta_b}\frac{p_3}{\rho_0})}$ and $\beta = \rho_0 \Delta_b (1 - \frac{1}{\Delta_b}\frac{p_3}{\rho_0})$. Observe the following

$$\begin{aligned} \alpha\beta &= \frac{\frac{1}{k_0 \Delta_r}}{\rho_0 + d_0(1 - \frac{1}{\Delta_b}\frac{p_3}{\rho_0})} \rho_0 \Delta_b \left(1 - \frac{1}{\Delta_b}\frac{p_3}{\rho_0}\right) = \frac{\frac{1}{\Delta_r}}{k_0 + (1 - \frac{1}{\Delta_b}\frac{p_3}{\rho_0})} \left(\Delta_b - \frac{p_3}{\rho_0}\right) \\ &= \frac{\Delta_b (\Delta_b - \frac{p_3}{\rho_0})}{\Delta_r \Delta_b k_0 + \Delta_r (\Delta_b - \frac{p_3}{\rho_0})} \end{aligned}$$

Expanding the quadratic equation (5.30) gives

$$0 = \left(\frac{p_3}{\rho_0}\right)^2 - \Delta_r \frac{p_3}{\rho_0} - \Delta_b \frac{p_3}{\rho_0} + k_0 \Delta_b \Delta_r + \Delta_b \Delta_r = k_0 \Delta_r \Delta_b + \Delta_r \left(\Delta_b - \frac{p_3}{\rho_0}\right) + \left(\frac{p_3}{\rho_0}\right)^2 - \Delta_b \frac{p_3}{\rho_0}$$

Concluding that

$$\alpha\beta = \frac{\Delta_b (\Delta_b - \frac{p_3}{\rho_0})}{\Delta_b \frac{p_3}{\rho_0} - (\frac{p_3}{\rho_0})^2} = \frac{\Delta_b}{\frac{p_3}{\rho_0}}. \tag{5.36}$$

From (5.33) and inequality (5.14), we have $\frac{p_3}{\rho_0} > \Delta_b$ and hence we have

$$0 < \alpha\beta < 1$$

Therefore, the determinant $1 - \alpha\beta \neq 0$, and the matrix $Id - \nabla_{w^0} H(\mathcal{P})$ is invertible.

It remains to prove that the spectral radius of the matrix $[Id - \nabla_{w^0} H(\mathcal{P})]^{-1} \nabla_{w^1} H(\mathcal{P})$ is strictly less than 1. First

$$[Id - \nabla_{w^0} H(\mathcal{P})]^{-1} = \frac{1}{\alpha\beta - 1} \begin{bmatrix} -1 & 0 & \alpha \\ 0 & \alpha\beta - 1 & 0 \\ \beta & 0 & -1 \end{bmatrix}$$

Notice from (5.35) $\nabla_{w^1} H(\mathcal{P}) = p_1 \begin{bmatrix} 0 & 0 & \alpha \frac{\Delta_r}{\Delta_b} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and so

$$[Id - \nabla_{w^0} H(\mathcal{P})]^{-1} \nabla_{w^1} H(\mathcal{P}) = \frac{p_1}{\alpha\beta - 1} \begin{bmatrix} 0 & 0 & \alpha(1 - \frac{\Delta_r}{\Delta_b}) \\ 0 & 0 & 0 \\ 0 & 0 & \alpha\beta \frac{\Delta_r}{\Delta_b} - 1 \end{bmatrix}$$

Since the above matrix is upper triangulat, its eigenvalues are its diagonal entries

$$0, 0, \left(\alpha\beta\frac{\Delta_r}{\Delta_b} - 1\right)\frac{p_1}{\alpha\beta - 1}.$$

Notice from(5.36), and (5.29)that

$$\left(\alpha\beta\frac{\Delta_r}{\Delta_b} - 1\right)\frac{p_1}{\alpha\beta - 1} = \left(\frac{\Delta_b}{\frac{p_3}{\rho_0}}\frac{\Delta_r}{\Delta_b} - 1\right)\left(\frac{p_1}{\frac{\Delta_b}{\frac{p_3}{\rho_0}} - 1}\right) = p_1\frac{\Delta_r - \frac{p_3}{\rho_0}}{\Delta_b - \frac{p_3}{\rho_0}} = p_1^2.$$

Hence,

$$R_{[Id - \nabla_{w^0} H(\mathcal{P})]^{-1}\nabla_{w^1} H(\mathcal{P})} = \left|\left(\alpha\beta\frac{\Delta_r}{\Delta_b} - 1\right)\frac{p_1}{\alpha\beta - 1}\right| = |p_1^2| < 1,$$

concluding the proof of Theorem 5.4.1. □

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