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ECONOMIC ORDERED QUANTITY MODEL WITH PARTIAL DELAY IN PAYMENTS

by

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AN ABSTRACT OF THE THESIS OF

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The classical economic order quantity model assumes that the supplier is paid for the items at the moment the order is received. However, often in practice, the supplier provides the purchasers a permissible delay in payment for the items supplied. In this research, we develop an economic order quantity model in which the supplier allows partial delay in payment, whereby a fraction of the order cost is paid upon receipt, and the rest after a fixed period of time, and subject to interest. An extension is developed, to reflect the real-life business situations, where suppliers offer the credit terms in conjunction with a cash discount. We derive the optimal ordering policy of both models. Furthermore, a comparison between the classical and our partial trade credit EOQ models is done to identify the ideal interest rates that favor adopting delay in payment for the retailer. Finally, numerical examples are presented, the results are discussed, and managerial insights are presented.

Keywords. EOQ, trade credit, inventory, cash discount, permissible delay in payment.

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CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

Nowadays, inventory management emerges to be an essential part and a core task for the smooth functioning of any contemporary business (2020). Perfectly managed inventory results in long-standing relationships with satisfied customers and monetary profits. For the purpose of reaching these various organizational goals, and for addressing the fundamental questions of when and how much to order, inventory theories are implemented. One the basic models of inventory theory is the economic order quantity (EOQ) model, which was introduced by Harris (1913). This model applies typically to fast-moving consumer goods with steady demand.

The competitiveness of modern markets encourages many suppliers to provide different schemes and adopt various promotional policies to increase their sales, attract new retailers and satisfy their needs. One of the most popular promotional policies is the trade credit facility, which is characterized by a delay in payments. In the classical EOQ model, retailers are asked to pay for the purchased items at the time when items are received. However, within the trade credit facility framework, suppliers offer the retailer a permissible delay in payments. Such a policy can, in a way, be considered as a price discount since delaying a payment can reduce the cost of holding and motivate the retailer to increase their order quantity.

Over the years, considerable research work has been conducted in the area of inventory management to treat different versions of inventory models under the trade credit facility. Many researchers elaborated on this topic with a lot of extensions. For example, deterministic inventory models under conditions of permissible delay in payments took a lot of attention and Goyal (1985) was among the first authors to develop such models. He assumed that the entire order cost is to be settled after a fixed period and considered that the unit purchase cost and the selling price per unit are the same and concluded that the order quantity increases and the annual cost decreases as a result delays in payment. Later, Chand and Ward (1987) examined Goyal's model under the capital cost concept, yielding various results. In other words, their model differed from Goyal's model by the way the cost of funds tied up in the inventory is modeled. Shah (1993), Aggarwal and Jaggi (1995) and Chu et al. (1998) then extended Goyal's model to consider items that degrade in quality and utility with time and known as deteriorating items. Then, Jamal et al. (1997) extended the deterioration criteria to generalize the inventory model to allow for both deterioration and shortages. Furthermore, such models were extended in other directions. For example, Hwang and Shinn (1997) extended the same model to account for price-dependent demand and presented a model to determine the retailer's optimal price and order quantity. Abbad and Jaggi (2003) determined the optimal item price under a credit period by developing a supplier-retailer model under the trade credit facility, which considers the demand rate as a function of the selling price. Other extensions assume that the selling price is not the same as the purchasing price (2002). Additionally, Goyal et al. (2007) developed an EOQ model for a retailer that is based on charging a progressive interest by the supplier every time the credit period is not respected, and they generated an efficient solution procedure for such cases. Moreover, Chung and

Huang (2003) analyzed trade credit facility within the economic production quantity (EPQ) framework and derived an efficient procedure to identify the ideal production and replenishment cycles.

Research considering economic order quantity under permissible delays in payments mostly assumed that the wholesaler offers the retailer full payments delays. However, in reality this is considered an extreme case. Suppliers may require a payment when the purchase occurs unless they set some conditions linked to order quantity as discussed by Ouyang et al. (2003), Chung et al. (2005), and others. In other words, the supplier will accept a full delay in payment if the retailer orders a quantity that is greater than or equal to a predetermined amount. In practice, however, most of the time the supplier relaxes this requirement by offering the retailer partial payments delay. That is, the retailer must pay part of the purchase amount when the order is received, then the remaining balances must be paid by the end of the permissible delay period. This option has received some attention in the literature. For example, Wang et al. (2009) investigated retailers' optimal lot-sizing decisions under partial delays in payments within the EOQ framework by assuming that the unit purchasing and selling price are not necessarily equal and shortages are not allowed. Taleizadeh et al. (2013) assumed that shortages can be permitted and determined the order and shortage quantities of an EOQ model that allows both partial delays in payment and partial backordering. Annadurai and Uthayakumar (2012) analyzed the economic order quantity model based on the partial trade credit financing for decaying items with shortages. They assumed that the supplier may accept a partial delay in payments even if the ordered quantity is less than an agreed quantity. Vandana and Sharma (2016) had similar conditions to Annadurai and Uthayakumar's (2012) research except that it was for non-decaying items.

Many other recent articles consider partial trade credit, e.g., Mahata (2012), Taleizadeh and Nematollahi (2014), Ahmadi et al. (2015).

Based on the literature reviewed, little research has addressed charging interest by the supplier over the delay period. In this thesis, we develop an economic order quantity model in which the retailer has the option to delay part of the payment for an agreed period where an interest will be charged by the supplier continuously over it. Another model, reflecting a real business situation, is extended from the first where the supplier offers the retailer the chance of obtaining a certain discount on the fraction of amount paid upon the receipt of items. From the supplier's perspective, the supplier hopes to receive payment from the retailer as soon as possible to avoid the possibility of bad debt occurrence. As a result, in most business transactions, the supplier will offer the retailer credit terms that combine a cash discount and trade credit. Such combination was the interest of some researchers as well. Lieber and Orgler (1975) developed a model where they considered credit period and cash discount as variables. Hill and Riener (1979) developed a model in which the cash discount in the company's credit facility can be determined.

The main purpose of this study is to investigate the optimal replenishment policy that minimizes the total relevant cost. Furthermore, this study helps in identifying the maximum value of the interest rate charged by the supplier, below which the retailer should adopt trade credit model, and paying for all orders upon receipt. In addition, our research helps in determining the optimal fraction of money to be paid upon receipt of the order, when the supplier interest rate is fixed. This thesis can contribute to the business and inventory fields by being a reference for retailers regarding when to accept the interest charged by the supplier under such conditions.

The remainder of the thesis is structured as follows: Chapter 2 presents our model formulation methodology with an explanation of assumptions, notations, logic used in deriving the model, and a comparison with the classical EOQ model. Chapter 3 presents the same procedure to develop a model which consists of a combination of trade credit and cash discount at the same time. Then Chapter 4 presents numerical examples and some motivating numerical results and insights. Finally, Chapter 5 presents our conclusions and ideas for future research.

CHAPTER 2

MODEL AND ASSUMPTIONS

Before starting the development of the retailer's inventory model under partial permissible delay in payments, some terminologies and assumptions that will be used throughout this thesis will be defined.

2.1. Terminology

Parameters	Definitions
α	The fraction of the purchasing cost to be paid at the time of the receipt ($0 \leq \alpha \leq 1$)
K	Fixed Setup Cost per order
β	Demand quantity per unit time
c	Unit purchase cost
p	Unit selling price
i	Inventory holding cost rate
r	Interest earned by the retailer per year
j	Interest charged by the supplier per year
t_0	Length of an inventory cycle (time between orders)
T	Permissible delay in settling the second payment

Decision Variable	Definitions
y	Order quantity

2.2. Assumptions

When deriving the model, the following assumptions are made:

1. The demand rate, β, for the items is deterministic.
2. Replenishments are instantaneous, i.e. the order quantity y to replenish inventory arrives all at once when the inventory level drops to zero.
3. Shortages are not allowed.
4. Planning horizon is infinite.
5. The inventory system is only one type of inventory, single item inventory.

2.3. The Model

The model of this study is defined as a constrained optimization problem. It consists on finding the optimal replenishmenet policy that minimizes the total cost per unit time of the retailer. The constraints are the non negativity and the boundaries set for the decision variable y based on the chosen case. Two cases will be considered, (1) $T \leq t_0 = \frac{y}{\beta}$ (Figure 1), and (2) $T \geq t_0 = \frac{y}{\beta}$ (Figure 2).

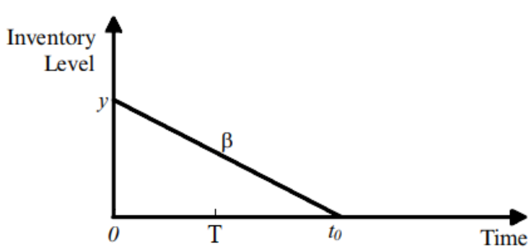

Figure 1: Cycle of the instantaneous replenishments when $T \leq t_0$

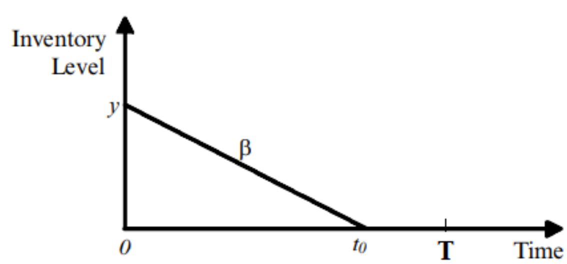

Figure 2: Cycle of the instantaneous replenishments when $T \geq t_0$

Based on the defined assumptions, the total inventory cost is the sum of the procurement and inventory holding cost.

The procurement costs consist of a variable component that depends on the number of units purchased, and a fixed component which is independent of the purchased units. The former is known by the “purchasing cost” and represents the amount of money paid to the supplier from which the unit is purchased. However, the fixed cost known by the “setup cost” or “ordering cost” is the cost associated with the procurement process.

The inventory holding costs consist of all the costs associated with the storage of the inventory until it is used or sold. Such costs consist also of two components: the first one associated with holding the items purchased in an inventory system, and the second one associated with the funds tied in the inventory and known by the “opportunity cost”.

Thus, the general total inventory cost per cycle adopted in this study is obtained as follows:

$$Total\ Inventory\ Cost = Purchasing\ Cost + Setup\ Cost + Holding\ Cost$$

2.3.1. Cash Flow per Period

Let’s simplify the total inventory cost and get into the cash flow per period of each component.

Purchasing Cost

In a partial delay in payment inventory model, the retailer pays the purchasing amount partially:

- At the moment the retailer receives the order quantity y, a partial payment of the purchasing cost takes place. So,

  At time 0 $\alpha c y$

- After a period of T, the remaining amount is paid with an additional amount which is the interest charged by the supplier continuously all over the delayed period. So,

  At time T $(1 - \alpha)\ c y e^{jT}$

  Setup Cost

- The setup cost takes place at the beginning of each cycle whenever an order is placed. So,

  At time 0 K

At this phase, purchasing and setup costs are represented by the cash flow diagram below:

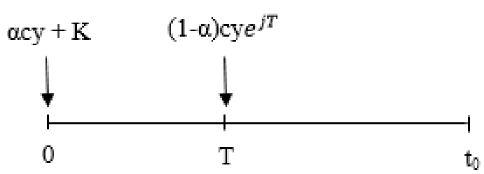

Figure 3: Cash Flow Diagram for $T \leq t_0$

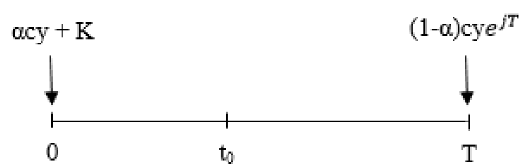

Figure 4: Cash Flow Diagram for $T \geq t_0$

Holding Cost

As mentioned previously, holding cost accounts for storage, insurance, tax, opportunity cost, etc. For this model, the unit holding cost, h, will mainly be accounting for the opportunity cost of capital tied up in inventory (i.e. the financing cost) where $h = ic$. At this moment, the inventory per one cycle is considered, and to be more accurate, the inventory stored per cycle will be equivalent to the average inventory per cycle.

To simplify the computation of this cost, two scenarios were generated. In the first scenario, the holding cost is accounted for the partial payment paid at the moment the items are received, as shown in Figure 5. In other words, the opportunity cost is accounted for the fraction of amount paid for the whole cycle, and it's the same for case (1) and (2) where $T \leq t_0$ and $T \geq t_0$.

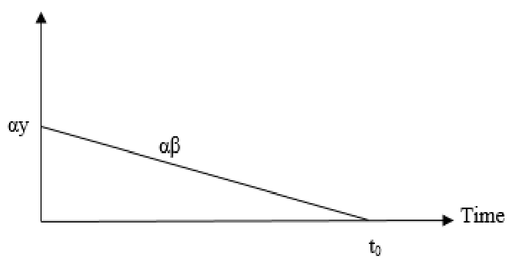

Figure 5: Opportunity Cost of the Partial Payment Paid at t = 0

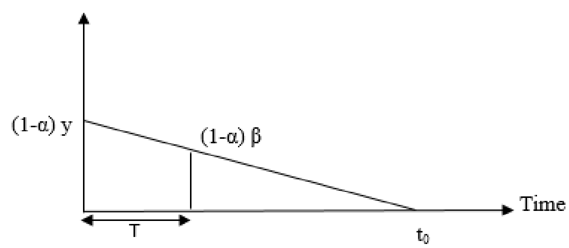

Figure 6: Opportunity Cost of the Remaining Payment

In this scenario, the average inventory represents the whole area per cycle and is defined as follows:

$$Average\ Inventory\ = \alpha \frac{y}{2}$$

So,

$$Holding\ cost\ per\ cycle\ = ic\left(\frac{\alpha y}{2}\right)t_0$$

However, the second scenario covers the opportunity cost tied when the remaining balance is paid after a delayed period T. For case (1) where $T \leq t_0$, the focus will be on the area after the delayed period T, as shown in Figure 6. However, for case (2) where $T \geq t_0$, the opportunity cost of the delayed payment is null as it is not paid during the cycle, the remaining balance was invested somewhere else.

So, the average inventory for the case where $T \leq t_0$ is defined as follows:

$$Average\ Inventory = \frac{\left[(1-\alpha)y-(1-\alpha)\beta T\right](t_0-T)}{2t_0}$$

And,

$$Holding\ cost = ic\left(\frac{\left[(1-\alpha)y-(1-\alpha)\beta T\right](t_0-T)}{2t_0}\right)t_0$$

Therefore, the total inventory cost per cycle for each period can be summarized as follows:

$$TCU_{per\ cycle} = \begin{cases} K+\alpha cy + (1-\alpha)cye^{jT}+ic\left(\frac{\alpha y}{2}\right)t_0 + ic\frac{\left[(1-\alpha)y-(1-\alpha)\beta T\right]\left(t_0-T\right)}{2} & if\ T\leq t_0 \\ K+\alpha cy + (1-\alpha)cye^{jT}+ic\left(\frac{\alpha y}{2}\right)t_0 & if\ T\geq t_0 \end{cases}$$

Once all the elements of total inventory cost are obtained, it’s time to develop the mathematical models for determining the optimal order quantities for the two cases.

2.3.2. Determination of the Optimal Order Quantity when $T \leq t_0$

The first step of solving our model consists of deriving the total cost per unit time. Since the total cost per unit time is required, then the total cost per cycle expression must be divided by the length of an inventory cycle t_0.

$$TCU\ (y_1) = \frac{K}{t_0}+\frac{\alpha cy_1 + (1-\alpha)cy_1e^{jT}}{t_0}+ic\left(\frac{\alpha y_1}{2}\right)+\frac{ic\left[(1-\alpha)y_1-(1-\alpha)\beta T\right]\left(t_0-T\right)}{2t_0}$$

$$= \frac{K}{\frac{y_1}{\beta}} + \frac{\alpha c y_1 + (1-\alpha) c y_1 \; e^{jT}}{\frac{y_1}{\beta}} + ic \; \frac{\alpha y_1}{2} + \frac{ic \; \left[(1-\alpha) y_1 - (1-\alpha)\beta T\right]\left(\frac{y_1}{\beta} - T\right)}{2\frac{y_1}{\beta}}$$

$$= \frac{\beta}{y_1} \; K + \alpha c \beta + (1-\alpha) c \beta \; e^{\,jT} + ic \; \frac{\alpha y_1}{2} + \frac{ic \; \left[(1-\alpha) y_1 - (1-\alpha)\beta T\right]\left(\frac{y_1}{\beta} - T\right)}{2\frac{y_1}{\beta}}$$

Hence,

$$TCU\,(y_1) = \frac{\beta}{y_1} \; K + \alpha c \beta + \; (1-\alpha) c \beta \; e^{\,jT} + ic \left[\frac{\alpha y_1}{2} + \frac{(1-\alpha) y_1}{2}\left(1 - \frac{\beta T}{y_1}\right)^2\right] \tag{1}$$

The main target of this model is to find the optimum value of the order quantity y_{CF}^* resulting in the minimum cost under some specific conditions. For this reason, the first-order optimality conditions are applied where the *TCU* expression (1) is derived according to the decision variable y_1 and set equal to zero.

$$\frac{d(TCU)}{dy_1} = -\frac{\beta}{y_1^2}.K + \frac{ic\alpha}{2} + \frac{ic.(1-\alpha)}{2} \cdot \frac{d}{dy_1}\left[y_1 .\left(1 - \frac{\beta T}{y_1}\right)^2\right]$$

After a series of calculations, the first derivative of $TCU(y_1)$ is obtained as:

$$\frac{d(\,TCU)}{dy_1} = -\,\frac{\beta}{y_1^2}\;\; K + \,ic\left[\frac{\alpha}{2} + \frac{(\,1-\alpha)}{2}\;\left(1 - \frac{\beta^2 T^2}{y_1^2}\right)\right] \tag{2}$$

The next step in deriving the optimal solution is to set the right hand of expression (2) equal to zero when it’s the case of a convex function. To prove that the function is convex, the second derivative of the *TCU (y_1)* with respect to y_1 must be positive.

The second derivative is defined as follows:

$$\frac{d^2(TCU)}{dy_1^2} = \frac{d}{dy_1}\left(\frac{d(TCU)}{dy_1}\right) = 2\frac{\beta}{y_1^3}.K + \frac{i.c.(1-\alpha)\,\beta^2T^2}{y_1^3}$$

$$\frac{d^2(TCU)}{dy_1^2} = \frac{2\beta K + i.c.(1-\alpha)\,\beta^2T^2}{y_1^3} \qquad \geq 0$$

Having all the parameters positive, the second derivative is found to be positive and so the derived solution is the minimum of this model.

After proving the convexity of the function and by assuming α is constant, the optimal solution was found by setting $\frac{dTCU}{dy_1} = 0$ and hence,

$$y_1^* = \sqrt{\frac{2\beta K + ic(1-\alpha)\,.\beta^2T^2}{ic}}$$

Since all analyses in this research will be based on the classical EOQ model, the expressions of the optimal quantity and TCU will be derived with the same form of the classical ones.

Starting by the optimal quantity expression,

$$y_1^* = \sqrt{\frac{2\beta K + ic(1-\alpha)\,.\beta^2T^2}{ic}} = \sqrt{\frac{2\beta}{ic}\left[K + \frac{ic(1-\alpha)\times\beta T^2}{2}\right]}$$

Hence,

$$y_1^* = \sqrt{\frac{2\beta K'}{ic}} \; where \; K' = K + \frac{ic.(1-\alpha)\times\beta T^2}{2} \tag{3}$$

Since we're dealing with the case where $T \leq t_0$, then the optimal solution y_1^* must always be higher than βT, then:

$$y^*_{CF,1} = max\left(\beta T,\ \sqrt{\frac{2K'\beta}{ic}}\right) \tag{4}$$

Moving forward to the optimal total cost per unit time expression, $y^*_{CF,1}$ is substituted in equation (1) as follows:

$$TCU^*_{CF,1} = \frac{\beta}{y^*_{CF,1}} K + \alpha c\beta + (1-\alpha)c\beta\, e^{jT} + ic\left[\frac{\alpha\, y^*_{CF,1}}{2} + \frac{(1-\alpha)\, y^*_{CF,1}}{2}\left(1 - \frac{\beta T}{y^*_{CF,1}}\right)^2\right]$$

Where

If $y^*_{CF,1} = \beta T$

$$TCU^*_{CF,1} = \frac{K}{T} + \alpha c\beta + (1-\alpha)\, c\beta e^{jT} + ic\left(\frac{\alpha\beta T}{2}\right) \tag{5}$$

If $y^*_{CF,1} = \sqrt{\frac{2K'\beta}{ic}}$

$$TCU^*_{CF,1} = \sqrt{2K'\beta ic} + \alpha c\beta + (1-\alpha)c\beta\,(e^{jT} - iT) \tag{6}$$

Theorem 1: If $T \leq t_0$, the annual cost function $TCU(y_1)$ is convex in y_1 with a unique global minimum $y^*_{CF,1}$ and its corresponding policy and expression is given by

$$y^*_{CF,1} = max\left(\beta\ T, \sqrt{\frac{2K'\beta}{ic}}\right)\ units$$

And,

$$TCU^*_{CF,1} = \begin{cases} \frac{K}{T} + \alpha c\beta + (1-\alpha)\, c\beta\ e^{jT} + \frac{ic\ \alpha\ \beta T}{2} & if\ y^*_{CF,1} = \beta\ T \\ \sqrt{2K'\beta ic} + \alpha c\beta + (1-\alpha)\, c\beta\ (e^{jT} - iT) & if\ y^*_{CF,1} = \sqrt{\frac{2K'\beta}{ic}} \end{cases}$$

Check the proof of Theorem 1 attached in Appendix.

2.3.3. Determination of the Optimal Order Quantity when $T \geq t_0$

As mentioned before, the first step of solving this model is generating the total cost per unit time. Hence,

$$TCU\left(y_2\right) = \frac{K}{t_0} + \frac{\alpha c y_2 + (1-\alpha) c y_2 e^{jT}}{t_0} + ic\,\frac{\alpha y_2}{2}$$

Substituting t_0 by $\frac{y_2}{\beta}$,

$$TCU\left(y_2\right) = \frac{\beta}{y_2} K + \alpha c \beta + (1-\alpha)\, c \beta\, e^{jT} + ic\left(\frac{\alpha y_2}{2}\right) \tag{7}$$

Just as explained before, the optimal quantity y_2^* resulting in the minimum cost under some specific conditions is obtained by deriving *TCU* expression (7) according to the decision variable y_2 and setting it to zero. So the first-order optimality equality is obtained as

$$\frac{d(TCU)}{dy_2} = -\frac{\beta}{y_2^2}.K + \frac{ic\alpha}{2} \tag{8}$$

And the optimal quantity expression is found to be

$$y_2^* = \sqrt{\frac{2K''\beta}{ic}} \quad where\ K'' = \frac{K}{\alpha} \tag{9}$$

Since we're dealing with the case where $T \geq t_0$, then the order size y must be less than the demand during the delayed period ($y \leq \beta T$). In other words, when the delayed period is higher than the length of the cycle, the amount on hand will be depleted to zero, at an annual demand rate β, before the due date T of the remaining payment of the purchasing cost payment as shown in Figure 2.

So, the optimal solution y_2^* must always be smaller than βT. Therefore,

$$y^*_{CF,2} = min\left(\beta.T, \sqrt{\frac{2K''\beta}{ic}}\right) \tag{10}$$

Moving forward to the optimal total cost per unit time expression, $y^*_{CF,2}$ is substituted in equation 7 as previously done and so the optimal total cost per unit time for each value of y is obtained as follows.

If $y^*_{CF,2} = \beta.T$

$$TCU^*_{CF,2} = \frac{K}{T} + \alpha c\beta + (1-\alpha)\, c\beta e^{jT} + \frac{ic\alpha\beta T}{2} \tag{11}$$

If $y^*_{CF,2} = \sqrt{\frac{2K''\beta}{ic}}$

$$TCU^*_{CF,2} = \alpha\sqrt{2K''\beta ic} + \alpha c\beta + (1-\alpha)c\beta\, e^{jT} \tag{12}$$

Theorem 2: If $T \geq t_0$, the annual cost function $TCU(y_2)$ is convex in y_2 with a unique global minimum $y^*_{CF,2}$ and corresponding annual cost is given by:

$$y^*_{CF,2} = min\left(\beta\ T, \sqrt{\frac{2K''\beta}{ic}}\right) units$$

And,

$$TCU^*_{CF,2} = \begin{cases} \frac{K}{T} + \alpha c\beta + (1-\alpha)\, c\beta\, e^{jT} + \frac{ic\, \alpha\, \beta T}{2} & if\ y^*_{CF,2} = \beta\, T \\ \alpha\sqrt{2K''\beta ic} + \alpha c\beta + (1-\alpha)\, c\beta\, e^{jT} & if\ y^*_{CF,2} = \sqrt{\frac{2K''\beta}{ic}} \end{cases}$$

Check the proof of Theorem 2 attached in Appendix.

2.4. Model Verification

To verify if the model is well formulated, the classical economic order quantity model can be used.

The classical EOQ model is based on the same assumptions adopted in the derivation of this model. However, it does not allow a partial delay in payment ($T = 0$), i.e. the full purchasing amount is paid at the moment the items are received ($\alpha = 1$).

The total cost per unit time of the Classical EOQ inventory model is given by

$$TCU_{EOQ} = \frac{\beta}{y}K + c\beta + h\frac{y}{2} \tag{13}$$

Substituting the value of alpha (i.e. $\alpha = 1$) in the derived equations (1) and (7) of total cost per, the following equation is obtained:

$$TCU(y) = \frac{\beta}{y}\ K + c\beta + i\ c\ \frac{y}{2}$$

By comparing these two equations, it’s noticeable that the latter equation is same as the classical EOQ equation. Hence, the formulated model is well derived.

Another way to verify the formulated model, is to have the expressions of total cost per unit time in case (1) and (2) equal when $y = \beta.T$. Based on expressions (5) and (11), it is found that

$$TCU^*_{CF,1} = TCU^*_{CF,2} = \frac{K}{T} + \alpha c\beta + (1-\alpha)c\beta\ e^{jT} + ic\ \frac{\alpha\beta T}{2}$$

2.5. Solution Algorithm

After deriving the optimal policy required when selecting the credit facility approach, it's time to identify how a retailer can use it. For every retailer who wants to adapt a partial delay in payment, the following steps represent the right track to follow.

Step 1: Determine the optimal order quantity $y^*_{CF,1}$ from expression (4), then calculate its corresponding $TCU^*_{CF,1}$ from equations (5) or (6).

Step 2: Determine the optimal order quantity $y^*_{CF,2}$ from expression (10), then calculate its corresponding $TCU^*_{CF,2}$ from equations (11) or (12).

Step 3: Compare $TCU^*_{CF,1}$ and $TCU^*_{CF,2}$. Select the order quantity associated with smaller total cost per unit time evaluated in step 1 and 2.

2.6. Situations Favoring Credit Facility over Classical EOQ Model

In the previous part of this chapter, the optimal policy when selecting the credit facility approach was identified, but this approach is not always the best one from a retailer perspective. Sometimes, the classical approach might be the most attractive.

To help the retailer in deciding whether to choose to pay immediately at the receipt of items, or delay part of the purchasing cost and accept the interest charges at the rate j, we will use the developed model for each case to generate an expression for the maximum value of j. In this part of the study, we will start from the idea that the retailer must accept the interest charged by the supplier as long as the total inventory cost per unit time of the credit facility (TCU^*_{CF}) is less than the one of the classical EOQ (TCU^*_{EOQ}).

2.6.1. For case (1) where $T < t_0$

Lemma 1:

$$TCU^*_{CF,1} < TCU^*_{EOQ} \qquad\qquad j < g(\alpha) \tag{14}$$

where $g(\alpha) = \frac{1}{T} \times \ln\left(iT + \frac{i}{(1-\alpha)\beta}\left(EOQ - y^*_{CF,1}\right) + 1\right)$

Note that for this expression to be acceptable, the argument of the log must be positive, i.e.:

$$iT + \frac{i}{(1-\alpha)\beta}\left(EOQ - y^*_{CF,1}\right) + 1 > 0$$

Once this expression satisfies the positive condition, we can look to the value of j for a specific value of alpha to identify where the credit facility is better than the classical EOQ. However, if this expression is negative, then automatically the classical EOQ will be better than the CF.

2.6.2. For case (2) where $T > t_0$

Lemma 2:

$$TCU^*_{CF,2} < TCU^*_{EOQ} \qquad\qquad j < g(\alpha) \qquad\qquad (15)$$

where $g(\alpha) = \frac{1}{T} \times \ln\left(\frac{1}{(1+\sqrt{\alpha})}\sqrt{\frac{2Ki}{c\beta}} + 1\right)$

2.6.3. For case (1) & (2) where $T = t_0$

Lemma 3:

$$TCU^*_{CF,1} = TCU^*_{CF,2} < TCU^*_{EOQ} \qquad\qquad j < g(\alpha) \qquad\qquad (16)$$

where $g(\alpha) = \frac{1}{T} \times \ln\left(\frac{1}{(1-\alpha)}\left[\sqrt{\frac{2Ki}{c\beta}} - \frac{K}{c\beta T} - \frac{i\alpha T}{2}\right] + 1\right)$

Note that for this expression to be acceptable, the argument of the log must be positive, i.e.:

$$\left[\sqrt{\frac{2Ki}{c\beta}} - \frac{K}{c\beta T} - \frac{i.\alpha.T}{2}\right] + 1 > 0$$

So, for a fixed value of α, the $g(\alpha)$ expression represents the maximum value of j that the retailer must accept when choosing credit facility over EOQ model. In other words, if the supplier charges an interest higher than $g(\alpha)$, the retailer must reject the trade credit facility and choose to pay the purchase cost at the moment of the receipt and adopt the classic EOQ model.

Furthermore, this expression can be used for another reason. If the supplier charges the retailer a certain interest rate j, the later can refer to $g(\alpha)$ expression to identify the best proportion of money to be paid at the beginning of the cycle (α) and which makes CF better than EOQ.

2.7. Model Formulation with Interest Earned during the Credit Period

Based on the literature review, some authors in their articles such as Goyal (1985) accounted for the interested generated when depositing the generated sales revenues in an interest-bearing account. In the model developed in the previous section, we assumed that retailers do not deposit the revenues in an interest-bearing account. However, in this special case, a sample of the model in which interest will be subtracted from the total cost per unit time will be developed briefly in this section by following the same method as before.

For this special case, the general total inventory cost per cycle adopted is obtained as follows.

$$Total\ Inventory\ Cost = Purchas\text{ing}\ Cost + Setup\ Cost + Holding\ Cost - Interest\ Earned$$

Purchasing, Setup, and Holding cost for both cases ($T \leq t_0\ and\ T \geq t_0$ remains the same. However, a new component enters the model which is interest earned.

Interest Earned

All over the cycle, the retailer can deposit all the revenues generated from sales in an interest-bearing account, and the interest earned is going to be considered as a negative cost.

- **For case (1):** ***T*** **Ò** $\boldsymbol{t_0}$

The maximum number of units sold during the cycle, is the ordered quantity *y* required in a cycle, i.e. βt_0. Hence, the interest earned on the total revenues generated from sales is obtained as follows.

$$Interest\ earned\ per\ cycle\ = Revenues\ \times\ Interest$$

$$Interest\ earned\ per\ cycle\ = \frac{\beta\, t_0^2\, pr}{2}$$

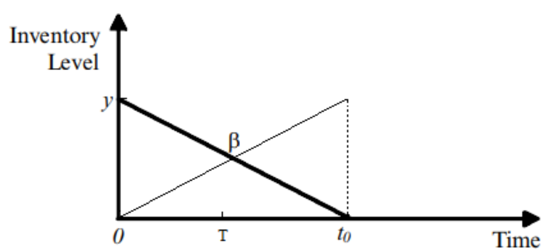

Figure 7: Variation of Inventory Level and Revenues over Time

- **For case (2):** $\boldsymbol{T \geq t_0}$

The maximum number of units sold during the cycle and delay period *T*, is the ordered quantity *y* required in a cycle, i.e. βt_0. Hence, the interest earned on the total revenues generated from sales during the permissible settlement period will be divided into two parts: the first part representing the interest earned on the average number of items sold during one cycle, and the second part representing the interest earned on the revenues deposited in the account for a period of *(T- t_0)* as shown in Figure 8. Note that the dotted line in Figure 7 and 8, represents the variation of revenues all over time.

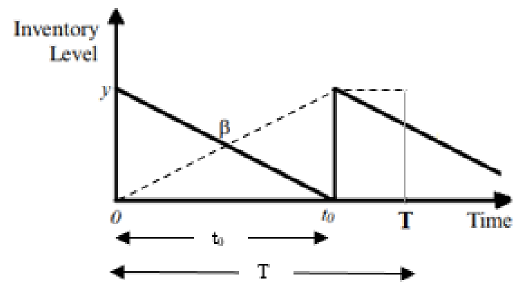

Figure 8: Variation of Inventory Level and Revenues for $T \geq t_0$

$$\text{Interest earned per cycle } = \left(\frac{\beta\ t_0^{\,2}\ p}{2} + \beta\ t_0\ p\ (T - t_0) \right) r = \beta\ t_0\ p\ r \left(T - \frac{t_0}{2} \right)$$

Once all the elements of total inventory cost are obtained, it's time to develop the mathematical models for determining the economic order quantities for the two cases similar to the previous section.

Following the same procedure, the total cost per unit time and the optimal quantity for each case was found and their respective theorem was generated and proved.

2.7.1. Determination of the Optimal Order Quantity when $T \leq t_0$

As mentioned before, the first step of solving our model consists of deriving the total cost per unit time.

$$TCU\left(y_1\right) = \frac{K}{t_0} + \frac{\alpha cy + (1-\alpha)cy\, e^{jT}}{t_0} + ic\,\frac{\alpha y}{2} + \frac{ic\,\left[(1-\alpha)y - (1-\alpha)\beta T\right](t_0 - T)}{2t_0} - \frac{\beta\, t_0^2\, p\, r}{2t_0}$$

Hence,

$$TCU\left(y_1\right) = \frac{\beta}{y_1}\,K + \alpha c\beta + (1-\alpha)c\beta\, e^{jT} + \left[(ic\alpha - p\,r) + ic\,(1-\alpha)\left(1 - \frac{\beta T}{y_1}\right)^2\right]\frac{y_1}{2} \tag{17}$$

After deriving the optimality equation and set it equal to zero, the following expressions were generated.

$$\frac{d(TCU)}{dy_1} = -\frac{\beta}{y_1^2}\,K + \frac{ic\alpha - p\,r}{2} + \frac{ic\,(1-\alpha)}{2}\left(1 - \frac{\beta^2 T^2}{y_1^2}\right) \tag{18}$$

Therefore,

$$y_1^* = \sqrt{\frac{2K'\beta}{h'}}\ \text{where}\ K' = K + \frac{ic\,(1-\alpha)\,\beta T^2}{2}\ \text{and}\ h' = ic - p\,r \tag{19}$$

Since we're dealing with the case where $T \le t_0$, then the optimal solution y_1^* must always be higher than $\beta.T$, then

$$y^*_{CF,1} = max\left(\beta\, T,\ \sqrt{\frac{2K'\beta}{h'}}\right) \tag{20}$$

Moving forward to the optimal total cost per unit time expression, $y^*_{CF,1}$ is replaced in equation (17) and the following expressions were obtained.

$$TCU^*_{CF,1} = \begin{cases} \frac{K}{T} + \alpha c\beta + (1-\alpha)\, c\beta e^{jT} + (ic\alpha - pr)\,\frac{\beta T}{2} & if\ y^*_{CF,1} = \beta T \quad (21) \\ \sqrt{2K'\beta h'} + \alpha c\beta + (1-\alpha)\, c\beta(e^{jT} - iT) & if\ y^*_{CF,1} = \sqrt{\frac{2K'\beta}{h'}} \quad (22) \end{cases}$$

Theorem 3: The annual cost function $TCU(y_1)$ was found to be convex in y_1 with a unique global minimum $y^*_{CF,1}$ and the optimum inventory policy for the proposed model is

$$y^*_{CF,1} = max\left(\beta T, \sqrt{\frac{2K'\beta}{h'}}\right) \ units$$

Check the proof of Theorem 3 in the Appendix.

2.7.2. Determination of the Optimal Order Quantity when $T \geq t_0$

For this case, the total variable cost per unit time is obtained as follows.

$$TCU(y_2) = \frac{K}{t_0} + \frac{\alpha cy + (1-\alpha)cy\ e^{jT}}{t_0} + ic\ \frac{\alpha y}{2} - \beta\ p\ r\left(T - \frac{t_0}{2}\right)$$

Hence,

$$TCU(y_2) = \frac{\beta}{y_2}\ K + \alpha c\beta + (1-\alpha)c\beta\ e^{jT} + (ic\ \alpha + p\ r)\ \frac{y_2}{2} - \beta T\ p\ r \tag{23}$$

After deriving the optimality equation and set it equal to zero, the following expressions were generated.

$$\frac{d(TCU)}{dy_2} = -\frac{\beta}{y_2^2}\ K + \frac{ic\ \alpha + p\ r}{2} \tag{24}$$

Therefore,

$$y_2^* = \sqrt{\frac{2K\beta}{h''}}\ where\ h'' = ic\ \alpha + p\ r \tag{25}$$

Since we're dealing with the case where T ≥ t_0, then the optimal solution $y^*_{CF,2}$ must always be less than $\beta.T$, then

$$y^*_{CF,2} = min\left(\beta T,\ \sqrt{\frac{2K\beta}{h''}}\right) \tag{26}$$

Moving forward to the optimal total cost per unit time expression, $y^*_{CF,2}$ is replaced in equation (23) and the following expressions were obtained.

(27)

$$TCU^*_{CF,2} = \begin{cases} \dfrac{K}{T} + \alpha c\beta + (1-\alpha)\, c\beta e^{jT} + (ic\alpha - pr)\, \dfrac{\beta T}{2} & if\ \ y^*_{CF,2} = \beta T \\ \sqrt{2K\beta h''} + \alpha c\beta + (1-\alpha)\, c\beta e^{jT} - \beta T pr & if\ \ y^*_{CF,2} = \sqrt{\dfrac{2K\beta}{h''}} \end{cases}$$

(28)

Theorem 4: The annual cost function $TCU(y_2)$ is convex in y_2 with a unique global minimum $y^*_{CF,2}$ and the optimum inventory policy for the proposed model is

$$y^*_{CF,2} = min\left(\beta T,\ \sqrt{\frac{2K\beta}{h''}}\right) units$$

Check the proof of Theorem 4 attached in the appendix.

2.7.3. Model with Earned Interest Verification

To verify if the model is well formulated, the model derived in the previous section can be used. The previous model is based on the same assumptions adopted in the derivation of this model. However, it does not account for the interest earned on the generated sales revenues (*r = 0).*

By substituting the value of r in equation (17) and (23), the following equations are obtained and are similar to the ones generated in the previous section, i.e. expressions (1) and (7).

2.7.4. Situations Favoring Credit Facility over Classical EOQ Model

Same procedure was followed to identify when the credit facility approach is better than EOQ model, and so the following expressions were generated.

2.7.4.1. For case (1) where $T < t_0$

Lemma 4:

$TCU_{CF,1}^{*} < TCU_{EOQ}^{*}$ $\qquad j < g(\alpha)$

Where

$$j < \frac{1}{T}\ \ln\left(iT + \frac{i}{(1-\alpha)\ \beta}\left(EOQ - \frac{\sqrt{2K'\beta h'}}{ic} \right) + 1 \right) \tag{29}$$

Note that for this expression to be acceptable, the argument of the log must be positive, i.e.

$$iT + \frac{i}{(1-\alpha)\ \beta}\left(EOQ - \frac{\sqrt{2K'\beta h'}}{ic} \right) + 1 > 0$$

2.7.4.2. For case (2) where $T > t_0$

$TCU^*_{CF,1} < TCU^*_{EOQ}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $j < g(\alpha)$

where

$$j < \frac{1}{T} \times ln\left(\frac{\sqrt{2K\beta}}{(1-\alpha)c\beta}(\sqrt{ic} - \sqrt{h''}) + 1 + \frac{T\,p\,r}{(1-\alpha)c} \right) \tag{30}$$

2.7.4.3. <u>For case (1) and (2) where $T = t_0$</u>

$TCU^*_{CF,1} < TCU^*_{EOQ}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $j < g(\alpha)$

where

$$j < \frac{1}{T} \ \ln\left(\frac{1}{1-\alpha}\left[\sqrt{\frac{2Ki}{c\beta}} - \frac{K}{c\beta T} - \frac{i\alpha T}{2} + \frac{p\ r\ T}{2c} \right] + 1 \right) \tag{31}$$

CHAPTER 3

EOQ MODEL FORMULATION UNDER CASH DISCOUNT AND TRADE CREDIT

As mentioned before, to reflect the real-life business situations, an extension of our model is derived. In the previous chapter, the supplier offers the retailer a fixed delay period, which is the trade credit period when settling the remaining accounts, with an additional interest charged all over this period. However, from the supplier's perspective, the supplier prefers that the retailer pays the payment as soon as possible to avoid the possibility of resulting in bad debt. For that reason, in most business transactions, the supplier offers both a trade credit and a cash discount to the retailer. So, the retailer obtains the cash discount on the partial payment paid at the receipt of items. Otherwise, the retailer will pay the desired partial payment without any discount. This chapter deals with determining the optimal ordering quantity under conditions of trade credit and cash discount and then comparing it to the previous model to help the retailer in identifying the best scenario.

3.1. Model Formulation

Before starting the development of the retailer's inventory model under conditions of cash discount and partial permissible delay in payments, same terminologies and assumptions that were defined in the previous chapter will be used throughout this thesis in addition to the discount cash discount rate γ where $0 \leq \gamma \leq 1$.

So as mentioned previously, the model of this study is defined as a constrained optimization problem. It consists on finding the optimal replenishmenet policy that minimizes the total cost per unit time of the retailer. The constraints are the non negativity and the boundaries set for the decision variable y based on the chosen case. Two cases will be considered, (1) $T \leq t_0 = \frac{y}{\beta}$ (Figure 1), and (2) $T \geq t_0 = \frac{y}{\beta}$ (Figure 2).

Following the same procedure as before, it is found that the cash flow per period of setup and purchasing cost is respresented as follows.

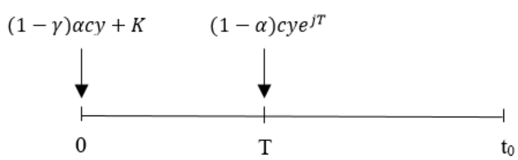

Figure 9: Cycle of the instantaneous replenishments when $T \leq t_0$

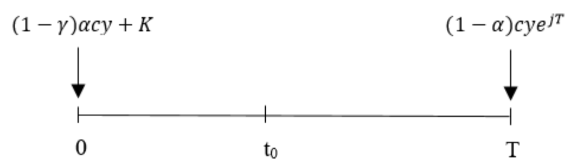

Figure 10: Cycle of the instantaneous replenishments when $T \geq t_0$

Moreover, the holding cost was derived the same way as before, by projecting the opportunity cost through two scenarios. The holding cost generated from the first scenario representing the fraction of purchasing amount paid at the beginning of the cycle is derived to be

$$Holding\ cost\ per\ cycle = ic\ \frac{(1-\gamma)\,\alpha y}{2}\ t_0$$

And the one of the second scenario representing the remaining balance paid after a certain delayed period is derived to be

$$Holding\ cost = ic\ \frac{\left[(1-\alpha)y-(1-\alpha)\beta T\right](t_0-T)}{2t_0}\ t_0$$

Therefore, the total inventory cost per cycle for each period can be summarized as follows:

$$TCU_{per\ cycle} = \begin{cases} K+(1-\gamma)\alpha cy + (1-\alpha)cye^{jT} + ic\frac{(1-\gamma)\alpha y}{2}t_0 + ic\frac{\left[(1-\alpha)y-(1-\alpha)\beta T\right]\left(t_0-T\right)}{2} & if\ T\le t_0 \\ K+(1-\gamma)\alpha cy + (1-\alpha)cye^{jT} + ic\frac{(1-\gamma)\alpha y}{2}t_0 & if\ T\ge t_0 \end{cases}$$

Once all the elements of total inventory cost are obtained, it’s time to develop the mathematical models for determining the optimal order quantities for the two cases.

3.1.1. *Determination of the Optimal Order Quantity when* $T \le t_0$

As mentioned before, the first step of solving our model consists of deriving the total cost per unit time.

To distinguish the two developed models, let’s represent our decision variable by y_{D1}.

$$TCU\,(y_{D1}) = \frac{K}{t_0} + \frac{(1-\gamma)\alpha cy_{D1} + (1-\alpha)cy_{D1}\ e^{jT}}{t_0} + ic\ \frac{(1-\gamma)\alpha y_{D1}}{2} + \frac{ic\ \left[(1-\alpha)y_{D1}-(1-\alpha)\beta T\right](t_0-T)}{2t_0}$$

Hence,

$$TCU\left(y_{D1}\right) = \frac{\beta}{y_{D1}}K + (1-\gamma)\alpha c\beta + (1-\alpha)c\beta e^{jT} + ic\left[\frac{(1-\gamma)\alpha y_{D1}}{2} + \frac{(1-\alpha)y_{D1}}{2}\left(1-\frac{\beta T}{y_{D1}}\right)^2\right] \quad (32)$$

This model is related to the partial delay in period model developed in Chapter 1 as follows:

$$TCU\left(y_{D1}\right)=TCU\left(y_{1}\right)-\left(\gamma.\alpha c\beta+\frac{ic.\gamma.\alpha y_{D1}}{2}\right) \tag{33}$$

After deriving the optimality equation and set it equal to zero, the following expressions were generated.

$$\frac{d(TCU)}{dy_{D1}}=-\frac{\beta}{y_{D1}^{2}}K+\frac{ic\ (1-\gamma)\ \alpha}{2}+\frac{ic\ (1-\alpha)}{2}\left(1-\frac{\beta^{2}T^{2}}{y_{D1}^{2}}\right) \tag{34}$$

Therefore,

$$y_{D1}^{*}=\sqrt{\frac{2\beta K'}{h'}}\ where\ K'=K+\frac{ic\ (1-\alpha)\ \beta T^{2}}{2}\ and\ h'=ic\ (1-\gamma\ \alpha) \tag{35}$$

Since we’re dealing with the case where $T \leq t_0$, then the optimal solution y_{D1}^{*} must always be higher than βT, then

$$y_{CF,D1}^{*}=max\left(\beta\ T,\sqrt{\frac{2\beta K'}{h'}}\right) \tag{36}$$

Moving forward to the optimal total cost per unit time expression, $y_{CF,D1}^{*}$ is replaced in equation (32) and the following expressions were obtained.

$$TCU_{CF,D1}^{*}=\begin{cases}\frac{K}{T}+(1-\gamma)\ \alpha c\beta+(1-\alpha)c\beta\ e^{jT}+\frac{ic\ (1-\gamma)\ \alpha\ \beta T}{2} & if\ y_{CF,D1}^{*}=\beta\ T \quad (37)\\ \sqrt{2K'\beta h'}+(1-\gamma)\ \alpha c\beta+(1-\alpha)c\beta\left(e^{jT}-iT\right) & if\ y_{CF,D1}^{*}=\sqrt{\frac{2K'\beta}{h'}} \quad (38)\end{cases}$$

Theorem 5: The annual cost function TCU is convex in y_{D1} with a unique global minimum y_{D1}^{*} and the optimum inventory policy for the proposed model is

$$y^{*}_{CF,D1} = max\left(\beta\ T, \sqrt{\frac{2\beta K'}{h'}}\right)\ units$$

Check proof of Theorem 5 attached in the appendix.

3.1.2. Determination of the Optimal Order Quantity when $T \geq t_0$

For $T \geq t_0$, the total cost per unit time is derived as follows.

$$TCU\ (y_{D2}) = \frac{K}{t_0} + \frac{(1-\gamma)\alpha c y_{D2} + (1-\alpha) c y_{D2}\ e^{jT}}{t_0} + ic\ \frac{(1-\gamma)\alpha y_2}{2}$$

Hence,

$$TCU(y_{D2}) = \frac{\beta}{y_{D2}} K + (1-\gamma)\ \alpha c\beta + (1-\alpha)\ c\beta e^{jT} + ic\frac{(1-\gamma)\ \alpha y_{D2}}{2} \tag{39}$$

In other terms,

$$TCU(y_{D2}) = TCU(y_2) - \left(\gamma\ \alpha c\beta + \frac{ic\ \gamma\ \alpha y_{D2}}{2}\right) \tag{40}$$

The *TCU* expression (41) is derived according to the decision variable y_{D2} as follows.

$$\frac{d(\ TCU)}{dy_{D2}} = -\ \frac{\beta}{y_{D2}^{2}}\ K + \frac{ic\ (1-\gamma)\ \ \alpha}{2} \tag{41}$$

Therefore,

$$y^*_{D2} = \sqrt{\frac{2K\beta}{h''}} \; where\; h'' = ic\;(1-\gamma)\;\alpha \tag{42}$$

Since we're dealing with the case where $T \geq t_0$, then the order size y must be less than the demand during the delayed period ($y \leq \beta T$).

$$y^*_{CF,D2} = min\left(\beta\; T, \sqrt{\frac{2K\beta}{h''}}\right) \tag{43}$$

Moving forward to the optimal total cost per unit time expression, $y^*_{CF,\mathrm{D2}}$ is replaced in equation as follows.

$$TCU^*_{CF,D2} = \begin{cases} \frac{K}{T} + (1-\gamma)\;\alpha c\beta + (1-\alpha)\,c\beta\; e^{jT} + ic\;\frac{(1-\gamma)\;\alpha\;\beta T}{2} & if\; y^*_{CF,D2} = \beta\; T \quad (44) \\ \sqrt{2K\beta h''} + (1-\gamma)\;\alpha c\beta + (1-\alpha)\,c\beta\; e^{jT} & if\; y^*_{CF,D2} = \sqrt{\frac{2K\beta}{h''}} \quad (45) \end{cases}$$

Theorem 6: Another theorem can be generated from this study when $T \geq t_0$. The annual cost function $TCU(y_{D2})$ is convex in y_{D2} with a unique global minimum y^*_{D2} and the optimum inventory policy for the proposed model is

$$Order\; y^*_{CF,\,D2} = min\left(\beta\; T,\; \sqrt{\frac{2K\beta}{h''}}\right) units \qquad every\; t_0 = \frac{y^*_{CF,\,D2}}{\beta}\; time\; units$$

Check proof of Theorem 5 attached in the appendix.

3.2. Model Verification

As mentioned before, our developed model can be verified by using the classical economic order quantity model.

The total cost per unit time of the Classical EOQ inventory model is given by

$$TCU_{EOQ} = \frac{\beta}{y} K + c\beta + h\frac{y}{2}$$

Substituting the value of alpha (i.e. $\alpha = 1$) and gamma $\gamma = 0$ in the derived equations (32) and (39) of total cost per, the following equation is obtained.

$$TCU\,(y) = \frac{\beta}{y}.K + c\beta + i.c.\frac{y}{2}$$

By comparing these two equations, it’s noticeable that the latter equation is same as the classical EOQ equation. Hence, the formulated model is well derived.

Another way to verify the formulated model, is to have the expressions of total cost per unit time in case (1) and (2) equal when $y = \beta T$. Comparing both expressions (37) and (44), it is found that these two are similar. Hence, our model is well formulated.

3.3. Model Algorithm

Similar to the algorithm derived in the previous section, this model requires retailers to follow to following track to be able to identify the optimal order quantity when the supplier offers both the trade credit and cash discount options.

Step 1: Determine the optimal order quantity $y^*_{CF,D1}$ from expression (36), then calculate its corresponding $TCU^*_{CF,D1}$ from equations (37) or (38).

Step 2: Determine the optimal order quantity $y^*_{CF,D2}$ from expression (43), then calculate its corresponding $TCU^*_{CF,D2}$ from equations (44) or (45).

Step 3: Compare $TCU^*_{CF,D1}$ and $TCU^*_{CF,D2}$. Select the order quantity associated with the smallest total cost per unit time evaluated in step 1 and 2.

3.4. Situations Favoring Credit Facility over Classical EOQ Model

3.4.1. For case (1) where $T < t_0$

Lemma 7:

$$TCU^*_{CF,D1} < TCU^*_{EOQ} \qquad\qquad j < g(\alpha) \tag{46}$$

$$Where\ g(\alpha) = \frac{1}{T} \times \ln\left(iT + \frac{i}{(1-\alpha)\beta}\left[EOQ - (1-\gamma\alpha)\, y^*_{D1}\right] + 1 + \frac{\gamma\alpha}{(1-\alpha)} \right)$$

3.4.2. For case (2) where $T > t_0$

Lemma 8:

$$TCU^*_{CF,D2} < TCU^*_{EOQ} \qquad\qquad j < g(\alpha) \tag{47}$$

where

$$j < \frac{1}{T} \times \ln\left(\frac{1}{(1-\alpha)} \times \sqrt{\frac{2Ki}{c\beta}} \left[1 - (1-\gamma)\,\alpha \right] + 1 + \frac{\gamma\alpha}{(1-\alpha)} \right)$$

3.4.3. *For case where* $T = t_0$

Lemma 9:

$$TCU^{*}_{CF,D1} = TCU^{*}_{CF,D2} < TCU^{*}_{EOQ} \qquad j < g(\alpha) \qquad (48)$$

where $g(\alpha) = \frac{1}{T} \times \ln\left(\frac{1}{(1-\alpha)}\left[\sqrt{\frac{2Ki}{c\beta}} - \frac{K}{c\beta T} + \text{ɣ}\alpha - \frac{i(1-\text{ɣ})\alpha T}{2}\right] + 1\right)$

Note that for this expression to be acceptable, the argument of the log must be positive, i.e.:

$$\frac{1}{(1-\alpha)}\left[\sqrt{\frac{2Ki}{c\beta}} - \frac{K}{c\beta T} + \text{ɣ}\alpha - \frac{i(1-\text{ɣ})\alpha T}{2}\right] + 1 > 0$$

CHAPTER 4

NUMERICAL ANALYSIS

4.1. Numerical Analysis: Base Model 1

Consider the following situation: a purchased item is consumed at a fairly steady rate of β = 1,200 units per year. The retailer pays c = \$25 per unit and estimates that fixed cost for placing and receiving orders amount to about K = \$100. Holding costs are based on an annual interest rate of i = 10% and shortages are not allowed. The supplier charges a continuous interest rate of j = 8% for a credit facility of 55 days (T = 0.15 year).

To help the retailer in choosing between credit facility or classical EOQ model, the following steps of the defined algorithm in Chapter II will be followed. First, the three cases described in credit facility approach must be evaluated to check whether the retailer must delay the settlement for a period smaller, higher, or equal to the cycle length. Then, the result of the latter approach must be compared to the classical EOQ model to decide whether to pay immediately or delay the payment.

Let's start first by the credit facility approach. If the retailer decides to have the delay period less than cycle length, i.e. $T \leq t_0^*$, then this is equivalent to the constraint $y_{CF,1}^* \geq 180$. For this numerical example, if the retailer is allowed to pay half of the amount at the receipt of items ($\alpha = 0.5$), and the other half after time T = 55 days (0.15 years), then the optimal policy would be ordering a quantity of units 334.963 every 0.28 years with a total cost of \$30,793.49 per year ($TCU_{CF,1}^*$).

However, if the retailer decides to have the delay period bigger than cycle length, i.e. $T \geq t_0^*$, then this is equivalent to the constraint $y_{CF,2}^* \leq 180$. If the retailer is allowed to pay half of the amount at the receipt of items ($\alpha = 0.5$), and the other half after T = 55 days, then the optimal policy would be ordering a quantity of 438.18 units every 0.37 years with a total cost of \$ 30,728.81 per year. However, since the delayed period is extended, y_{CF}^* must not exceed its maximum value of 180. Based on this constraint, the optimal policy when selecting credit facility approach is ordering an optimal quantity of 180 units every 0.15 years with a total cost of \$30,960.25 per year ($TCU_{CF,2}^*$).

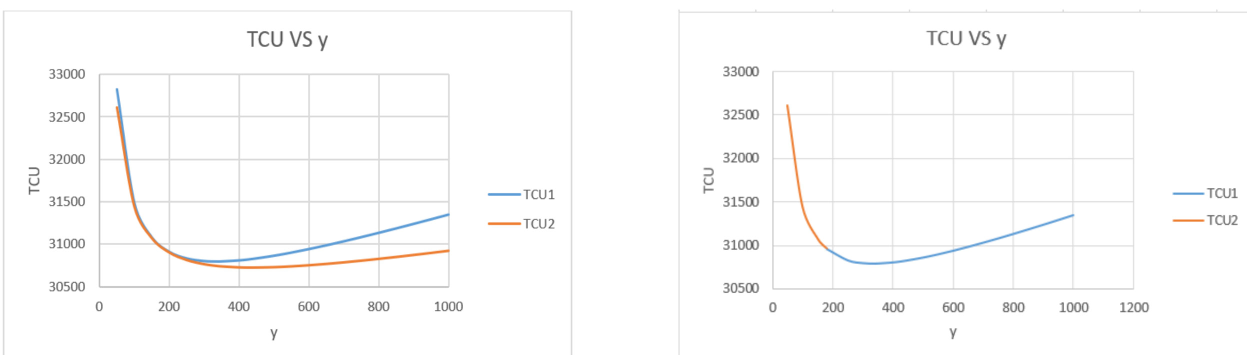

Figure 11: Variation of TCUs with y

To be able to compare these two illustrations of total cost per unit time, we combined the two curves in the first figure to result in one curve which represent TCU when y is less and more than 180.

So, if we compare $TCU^*_{CF,1}$ and $TCU^*_{CF,2}$ it is found that the retailer must choose to delay the remaining settlement for a period less than t_0 since it has the smaller total variable cost per unit time.

Now it is time to compare the CF with the classical EOQ model to check whether the retailer must pay the purchasing amount immediately or delay it to a period less than t_0.

In this case, if the retailer decides to follow the classical economic order quantity model, then the optimal policy would be ordering a quantity of 309.84 units every 0.26 years with a total cost of $30,774.60 per year. However, if the retailer is allowed to pay half of the amount at the receipt of items ($\alpha = 0.5$), and the other half after time T = 55 days, then the optimal policy would be ordering a quantity 334.963 every 0.28 years with a total cost of $30,793.49 per year. As we notice, the optimal quantity obtained when choosing credit facility, y^*_{CF}, is significantly higher than the one obtained by the classical EOQ model. This can be explained by the fact that under credit facility the inventory holding (opportunity) cost is less. This is a well-known result.

In this case, the classical EOQ model seems to be economically attractive more than the credit facility model due to smaller total cost. So, under these conditions, the retailer is advised to pay the purchasing amount immediately at the receipt of items.

If the retailer insists on delaying part or the full amount of payment, the developed model is used to derive a value of the interest rate *j* that makes the retailer prefers the credit facility model over the classical EOQ model. To understand how the critical interest rate, below which the retailer adopts credit facility, is affected by the partial credit terms, we plot

$g(\alpha)$ and try to deduce the desired range of alpha. For this numerical example, $g(\alpha)$ plot was found to be a decreasing function as shown in Figure 10. For example, if the supplier charges the retailer an annual interest rate of j = 7.15%, the retailer must refer to $g(\alpha)$ plot and deduce that any fraction of money paid at the receipt of items less than 0.5801, makes the credit facility economically attractive.

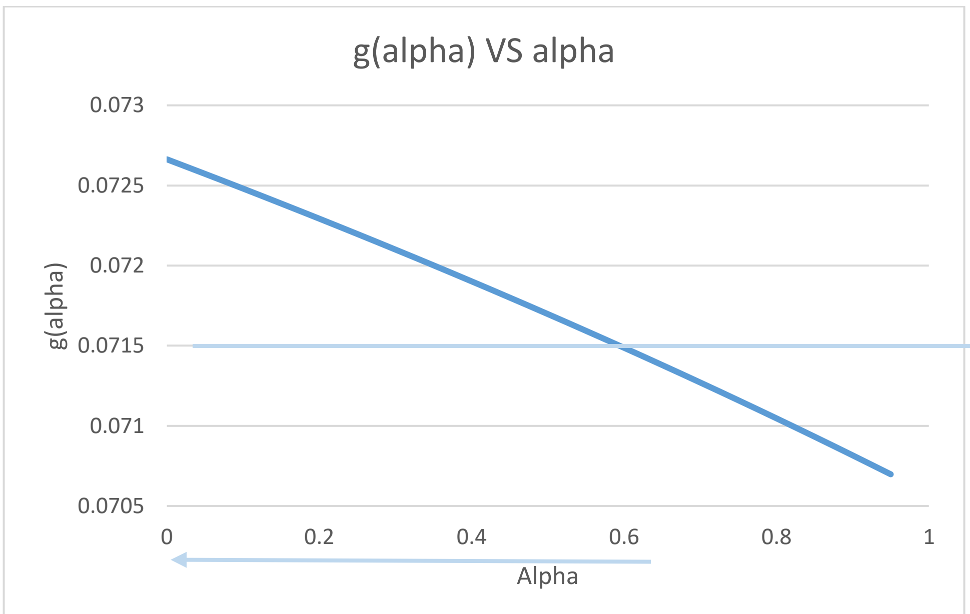

Figure12: Variation of $g(\alpha)$ expression in function of α

4.1.1. Sensitivity Analysis

To analyze the significant influence of different factors on the optimal policy, several variants to the model input were studied.

The first variable to be varied in this model is the interest rate j charged by the supplier. By varying this value, we are able to observe its impact on the optimal order policy and on the retailer's decision.

It is known that, for a specific value of *alpha*, as *j* increases, y_{CF}^{*} remains the same since it is independent of j, and TCU_{CF}^{*} increases. In other words, if the supplier increases the interest charged for the late payments, then the retailer has to pay higher cost, and the credit facility model becomes less attractive. This variation is shown in Table 1. Table 1 indicates that in both cases (i.e. $T \leq t_0$ and $T \geq t_0$), an increase in the total variable cost takes place. However, it is shown that an increase in the value of the interest rate *j* does not affect the retailer's decision when choosing to pay half of the amount at the receipt of items in a credit facility approach. As we notice, as long as *j* increases, $TCU_{CF,1}^{*}$ remains smaller than $TCU_{CF,2}^{*}$ which indicates that the retailer must always choose to delay the payment for a period less than the cycle length. In addition to that, in the case where the retailer chooses the first case as a credit facility approach, it is shown that for a value of *j* less than 7.169%, the credit facility is more economically attractive than the classical EOQ model. This conclusion can be used as a proof to validate what we noticed in Figure 9.

Table 1: Effect of j on the optimal quantity and its TCU for $\alpha = 0.5$

j	$\boldsymbol{y^*_{CF,1}}$	$\boldsymbol{TCU^*_{CF,1}}$	$\boldsymbol{y^*_{CF,2}}$	$\boldsymbol{TCU^*_{CF,2}}$
0.01	334.963	30,634.92	180	30,801.68
0.03	334.963	30,680.06	180	30,846.82
0.05	334.963	30,725.33	180	30,892.09
0.07	334.963	30,770.74	180	30,937.50
0.07169	*334.963*	*30,774.6*	*180*	*30,941.36*
0.09	334.963	30,816.28	180	30,983.04
0.1	334.963	30,839.10	180	31,005.86
0.3	334.963	31,302.82	180	31,469.58
0.5	334.963	31,780.67	180	31,947.43
0.7	334.963	32,273.07	180	32,439.83
0.9	334.963	32,780.46	180	32,947.22
1	334.963	33,039.92	180	33,206.68

Moreover, to study the effect of another important input to the model, another analysis is performed. Usually, retailers have, at each time placing an order, decide how much to pay at the receipt of items. It is known that higher the amount paid at the receipt is, lower the optimal ordered quantity is and smaller the total cost per unit time is because of ordering smaller quantities. In other words, when retailers pay a high fraction of the purchasing amount at the beginning of the cycle, they tend to order less in order to prevent paying high amount of opportunity cost. This is illustrated in Table 2 where we considered j = 8% and we varied the value of alpha to study its impact on y^*_{CF}, TCU^*_{CF} and the optimal policy.

Table 2: Effect of α on the optimal quantity and its TCU for j = 8%

$\boldsymbol{\alpha}$	$\boldsymbol{y^*_{CF,1}}$	$\boldsymbol{TCU^*_{CF,1}}$	$\boldsymbol{y^*_{CF,2}}$	$\boldsymbol{TCU^*_{CF,2}}$
0	358.33	30,807.99	-	31,051.34
0.1	353.78	30,805.40	180	31,015.12
0.2	349.17	30,802.66	180	31,001.4
0.3	344.5	30,799.77	180	30,987.68
0.4	339.76	30,796.71	180	30,973.97
0.5	334.96	30,793.49	180	30,960.25
0.6	330.09	30,790.09	180	30,946.53
0.7	325.15	30,786.52	180	30,932.82
0.8	320.12	30,782.75	180	30,919.10
0.9	315.02	30,778.78	180	30,905.38
1.0	309.84	30,774.6	309.84	30,774.6

For this numerical example, we can say that for an annual interest rate of 8%, the retailer must always choose to delay the payment for a period smaller than the cycle length due to having the total variable cost $TCU^*_{CF,1}$ always smaller than $TCU^*_{CF,2}$ regardless of the fraction of money the retailer desire to pay at the beginning of the cycle. In addition to that, another conclusion can be observed from Table 2 which is that for an interest rate of 8%, the retailer finds that the credit facility approach is not encouraged for any value of alpha since for different values of *alpha*, the total cost per unit time is always higher than the one for the classical EOQ model.

Note that when the retailer decides to pay the full amount at the receipt of items, there is no more the option to choose whether the delay payment is smaller or higher than the cycle length. Therefore, the constraints on the optimal order quantity are not considered anymore and such case is equivalent to the classical EOQ model.

Investigating more in identifying the optimal value of *alpha* for a specific value of j, we decided to link the three variables *j, i, and alpha*. So, we tried two different values of *j*, other than *j* = 8%, one when $j < i$ and another when $j > i$ where i = 10% as illustrated in Table 3.

Table 3: Effect of α for j = 7.17% and j = 17.17 %

$\boldsymbol{\alpha}$	$\boldsymbol{y^*_{CF,1}}$	$\boldsymbol{TCU^*_{CF,1}}$
0	358.33	30,770.20
0.1	353.78	30,771.39
0.3	344.50	30,773.32
0.5	334.96	30,774.60
0.7	325.15	30,775.18
0.9	315.02	30,775.00
1.0	309.84	30,774.60

$\boldsymbol{\alpha}$	$\boldsymbol{y^*_{CF,1}}$	$\boldsymbol{TCU^*_{CF,1}}$
0	358.33	31,228.50
0.1	353.78	31,183.86
0.3	344.50	31,094.13
0.5	334.96	31,003.75
0.7	325.15	30,912.67
0.9	315.02	30,820.83
1.0	309.84	30,774.60

As illustrated in Table 3, for *j* =7.17% the credit facility approach is better than the classical EOQ model as long as the retailer decides to pay an amount that range between 0 and 50 % of the purchasing amount at the receipt of items. However, the optimal value to be paid when choosing the trade credit approach and for this specific value of *j*, is 0%. In other words, for $j < i$, when retailers decide to delay all the purchase amount for a period T, they are ordering the highest amount of items in the least cost compared to other values of alpha.

Regarding the values illustrated in the second table where $j > i$, the optimal value to be paid at the receipt of items is the full amount since it has the smallest cost compared to other values of alpha. It is a well-known result since for a high value of interest charged by the supplier, it is better to pay the opportunity cost instead of paying extra interest on each delayed day.

To better observe the variation of y and *TCU*, a two-ways sensitivity analysis was applied where the variables *alpha* and *j* were studied simultaneously as illustrated in Table 4.

Table 4: Effect of α and j on the optimal policy

		$\boldsymbol{\alpha = 0.1}$				$\boldsymbol{\alpha = 0.3}$		
$\boldsymbol{j}$	$\boldsymbol{y^*_{CF,1}}$	$\boldsymbol{TCU^*_{CF,1}}$	$\boldsymbol{y^*_{CF,2}}$	$\boldsymbol{TCU^*_{CF,2}}$	$\boldsymbol{y^*_{CF,1}}$	$\boldsymbol{TCU^*_{CF,1}}$	$\boldsymbol{y^*_{CF,2}}$	$\boldsymbol{TCU^*_{CF,2}}$
0.01	353.78	30,519.98	180	30,729.70	344.5	30,577.77	180	30,765.69
0.03	353.78	30,601.22	180	30,810.94	344.5	30,640.96	180	30,828.88
0.05	353.78	30,682.71	180	30,892.43	344.5	30,704.34	180	30,892.26
0.08	353.78	30,805.40	180	31,015.12	344.5	30,799.77	180	30,987.68
0.1	353.78	30,887.50	180	31,097.22	344.5	30,863.62	180	31,051.54
0.3	353.78	31,722.20	180	31,931.92	344.5	31,512.83	180	31,700.75
0.5	353.78	32,582.32	180	32,792.04	344.5	32,1841.82	180	32,369.73
		$\boldsymbol{\alpha = 0.8}$				$\boldsymbol{\alpha = 1}$		
0.01	320.13	30,719.32	180	30,855.67	309.84	30774.6	309.84	30774.6
0.03	320.13	30,737.37	180	30,864.73	309.84	30774.6	309.84	30774.6
0.05	320.13	30,755.48	180	30,891.84	309.84	30774.6	309.84	30774.6
0.08	320.13	30,782.75	180	30,919.10	309.84	30774.6	309.84	30774.6
0.1	320.13	30,800.99	180	30.937.35	309.84	30774.6	309.84	30774.6

0.3	320.13	30,986.48	180	31,122.83	309.84	30774.6	309.84	30774.6
0.5	320.13	31,177.62	180	31,313.97	309.84	30774.6	309.84	30774.6

As a conclusion from this analysis, we can notice that for this numerical example, the variation of interest charged by the supplier or alpha has no impact on the optimal policy. For this reason, other variables were changed to study their effect on retailer's decision.

Let's start by the setup cost *K*.

Table 5: Impact of Setup Cost K on the Optimal Policy

$K = \$15$					$K = \$50$				
j	$y^*_{CF,1}$	$TCU^*_{CF,1}$	$y^*_{CF,2}$	$TCU^*_{CF,2}$	j	$y^*_{CF,1}$	$TCU^*_{CF,1}$	$y^*_{CF,2}$	$TCU^*_{CF,2}$
0.01	180	30,235.02	169.71	30,234.65	0.01	253.38	30,430.96	180	30,468.35
0.03	180	30,280.15	169.71	30,101.59	0.03	253.38	30,476.10	180	30,513.49
0.05	180	30,325.42	169.71	30,155.35	0.05	253.38	30,521.37	180	30,558.76
0.08	180	30,393.58	169.71	30,244.72	0.08	253.38	30,589.53	180	30,626.92
0.1	180	30,439.20	169.71	30,226.70	0.1	253.38	30,635.14	180	30,672.53
0.3	180	30,902.92	169.71	30,690.42	0.3	253.38	31,098.86	180	31,136.25
0.5	180	31,380.76	169.71	183,160.38	0.5	253.38	31,576.71	180	31,614.10
α	$y^*_{CF,1}$	$TCU^*_{CF,1}$	$y^*_{CF,2}$	$TCU^*_{CF,2}$	α	$y^*_{CF,1}$	$TCU^*_{CF,1}$	$y^*_{CF,2}$	$TCU^*_{CF,2}$
0	216.33	30,453.00	-	-	0	283.55	30,621.04	-	-
0.1	208.71	30,442.73	180	30,448.45	0.1	277.78	30,615.39	180	40,448.45
0.3	192.56	30,419.92	180	30,155.35	0.3	265.86	30,603.16	180	30,421.02
0.5	180	30,393.58	169.71	30,393.22	0.5	253.38	30,589.53	180	30,393.58
0.8	180	30,352.43	134.16	30,340.76	0.8	233.41	30,565.96	180	30,352.43

1	180	30,325.00	120	30,300	1	219.09	30,547.72	180	30,325

From the numbers illustrated in Table 5, different points were concluded. Setup cost K was found to have a great impact on the optimal policy. As we notice in the first two quarters of the table, as setup cost K increases, the retailer decisions might change, i.e. the retailer might decide to change the delayed period from a large one to a smaller one compared to the cycle length. This illustration can be explained as follows, for a small setup cost, the retailer order smaller quantity due to the fact that the retailer can order frequently without paying so much on each and every order. This fact reduces the holding cost and thus makes the retailer afford a high cost generated from the interest charged by the supplier when choosing to delay the payment after the cycle length.

The same point was being concluded when K increases at the same time where alpha increases as well.

The second variable studied was the cost of capital i. As we know, an increase in the cost of capital does not encourage retailers to order higher amount of items to prevent the generation of high costs. This fact can be noticed in Table 6.

Table 6: Impact of Setup Cost K on the Optimal Policy

$\boldsymbol{i}$	$\boldsymbol{y^*_{CF,1}}$	$\boldsymbol{TCU^*_{CF,1}}$	$\boldsymbol{y^*_{CF,2}}$	$\boldsymbol{TCU^*_{CF,2}}$
0.05	456.29	30,638.95	180	30,904.00
0.1	334.96	30,793.49	180	30,960.25
0.3	219.55	31,152.67	180	31,185.25

0.5	188.15	31,407.95	180	31,410.25
0.7	180	31,635.25	165.62	31,630.22
0.9	180	31,860.25	146.06	31,824.25
1	180	31,972.75	138.56	31,913.14

However, if we observe the values in Table 6, we notice that for values of *i* up to 0.5, the first policy, which consists of delaying the payment for a period less than cycle length, is better. For higher values of *i*, the optimal policy consists of delaying the payment for a period higher than the cycle length. This can be illustrated in Figure 11. Therefore, we can deduce that *i* have a good impact on retailers' decision while managing their inventory.

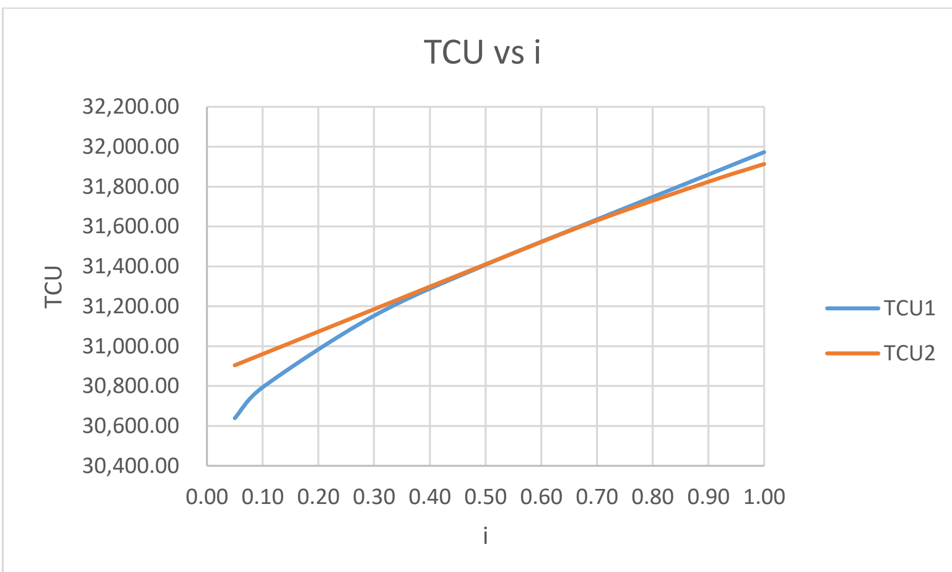

Figure 13: Variation of TCU in function of i

We decided as well to observe the variation of j in function of alpha. So as illustrated in the Table 7 and Figure 14, we observe that as alpha increases, it means as the retailer settles larger payments, a smaller percentage of j must be charged by the supplier since the retailer is paying a big portion of the payment and the supplier is safe, and his risk of losing money is decreasing.

Table 7: Variation of j with α

α	j
0	0.07266
0.1	0.07248
0.3	0.07210
0.5	0.07169
0.7	0.07127
0.9	0.07082

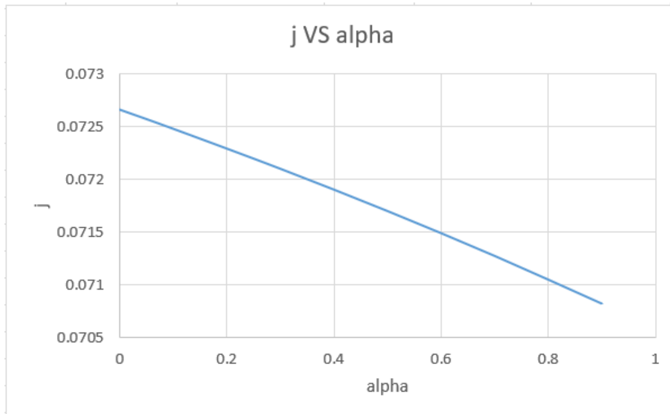

Figure14: Variation of j with α

4.2. Numerical Example of Model with Earned Interest

In this example, we're dealing with the same conditions as the first example except that in this case, the interest earned when revenues generated from sales are invested in an Interest-bearing account at a rate of $r = 2\%$. Note that the selling price of units is $p = \$45$.

To help the retailer in choosing between credit facility or classical EOQ model, this example will be treated the same way the previous one was solved. Let's start first by the credit facility approach. If the retailer decides to have the delay period less than cycle length, i.e. $T \leq t_0^*$, then the optimal policy would be ordering a quantity of 418.703 units every 0.35 years with a total cost of \$30,626.01 per year ($TCU_{CF,1}^*$).

However, if the retailer decides to have the delay period bigger than cycle length, i.e. $T \geq t_0^*$, then the optimal policy would be ordering a quantity of 334.11 units every 0.28 years with a total cost of $ 30,737.42 per year. However, since the delayed period is extended, y_{CF}^* must not exceed its maximum value of 180. Based on this constraint, the optimal policy when selecting credit facility approach is ordering an optimal quantity of 180 units every 0.15 years with a total cost of $30,879.251 per year ($TCU_{CF,2}^*$).

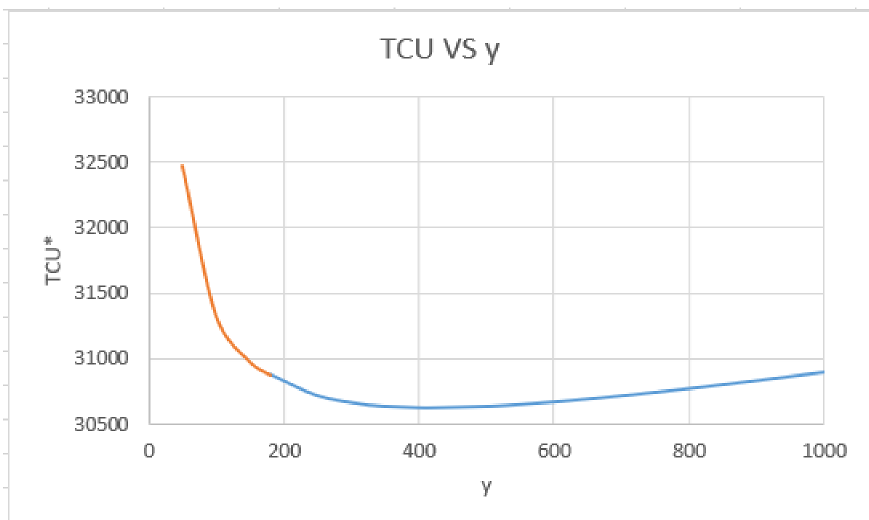

Figure 15: Variation of TCU vs y when interest earned on revenues is accounted

So, if we compare $TCU_{CF,1}^*$ and $TCU_{CF,2}^*$ it is found that the retailer must choose to delay the remaining settlement for a period less than t_0 since it has the smaller total variable cost per unit time.

When comparing the CF with the classical EOQ model, it is found that if the retailer decides to follow the classical EOQ model, then the optimal policy would be ordering a quantity of 309.84 units every 0.26 years with a total cost of \$30,774.60 per year. However, if the retailer is allowed to pay half of the amount at the receipt of items ($\alpha = 0.5$), and the other half after time T = 55 days, then the optimal policy would be ordering a quantity of 418.703 units every 0.35 years with a total cost of \$30,626.01 per year. In this case, the credit facility seems to be economically attractive more than the classical EOQ model due to smaller total cost. So, under these conditions, the retailer is advised to pay the purchasing amount partially.

As an observation, it is found that using the revenues generated from sales as an investment makes the transactions more profitable where the retailer is ordering higher amount at a small cost.

To help the retailer in identifying when a shift to the classical EOQ model must take place, it is necessary to identify the maximum value of the interest rate j that makes the retailer prefers the credit facility model over the classical EOQ model. To reach this aim, $g(\alpha)$ of this special case was plotted where the maximum value of j and the desired range of alpha were identified. For this numerical example, $g(\alpha)$ plot was found to be an increasing function as shown in Figure 13. For example, if the supplier decides to pay half the amount at the receipt of items, the maximum value of interest the retailer must accept the supplier to charge for the delay in payment is 14.5 % as shown in the figure below.

Another way to read this curve, for an interest rate $j = 14.5\%$, the credit facility approach is better than EOQ model when the retailer decides to pay half the amount or more at the receipt of items.

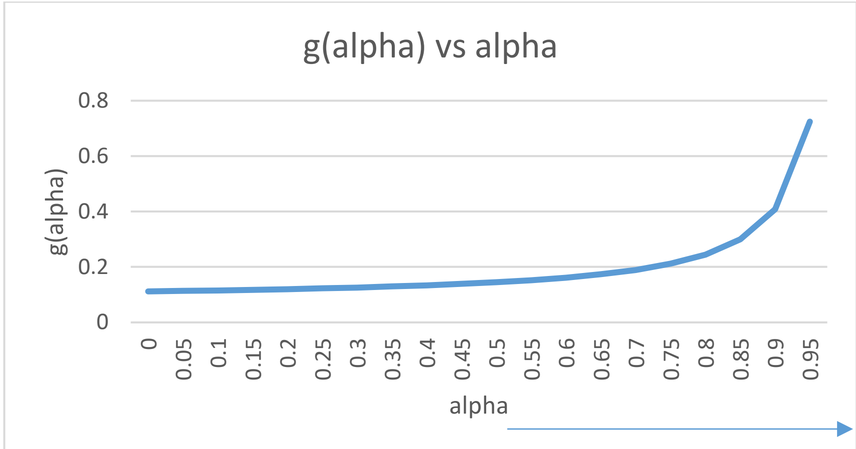

Figure 16: Variation of g(α) vs alpha when interest earned on revenues is accounted

4.2.1. Sensitivity Analysis of Model with Interest

In this example, the same variables were changed to study their impact on the optimal policy. As illustrated in Table 8, the variations of *alpha* and *j* has no impact on the optimal policy. It means that whatever the supplier charges an interest to the retailer, or whatever fraction the retailer decides to pay upon the receipt of items, the retailer would prefer delaying the remaining balance for a period smaller than the cycle length.

Table 8: Effect of j and alpha on the optimal quantity and its TCU when interest earned on revenues is accounted

$\alpha = 0.5$				
j	$y^*_{CF,1}$	$TCU^*_{CF,1}$	$y^*_{CF,2}$	$TCU^*_{CF,2}$
0.01	418.703	30,467.44	180	30,720.68
0.03	418.703	30,512.58	180	30,765.82
0.08	418.703	30,626.01	180	30,879.25
0.1	418.703	30,671.62	180	30,924.86
0.3	418.703	31,135.34	180	31,388.58
0.5	418.703	31,613.19	180	31,866.43
1	418.703	32,872.44	180	33,125.68

$j = 0.08$				
α	$y^*_{CF,1}$	$TCU^*_{CF,1}$	$y^*_{CF,2}$	$TCU^*_{CF,2}$
0	447.92	30,628.83	180	30,947.84
0.1	442.22	30,628.31	180	30,934.12
0.3	430.63	30,627.52	180	30,906.68
0.5	418.703	30,626.01	180	30,879.25
0.7	406.433	30,623.94	180	30,851.82
0.9	393.780	30,621.26	180	30,824.38
1.0	387.298	30,619.68	180	30,810.67

In this special case, a new variable was studied which is the value of interest earned on the generated revenues deposited in an interest-bearing account. As illustrated in Table 9, an increase in the value of interest *r* motivates the retailer to order more items. However, this increase does not affect the retailer decision when choosing the period to delay the payment of the remaining balance.

Table 9: Effect of r on the optimal quantity and its TCU

r	$y^*_{CF,1}$	$TCU^*_{CF,1}$	$y^*_{CF,2}$	$TCU^*_{CF,2}$
0	334.96	30,793.49	180	30,960.25
0.01	369.90	30,714.39	180	30,919.75
0.02	418.70	30,626.01	180	30,879.25
0.04	633.02	30,399.20	180	30,798.25
0.06	-	-	180	30,717.25
0.08	-	-	180	30,636.25
0.1	-	-	180	30,555.251

As we can observe from Table 9, as *r* increases it is found it is better to choose the policy where the delayed period is smaller than the cycle length. Note that the invalid values in the first case is due to having a negative denominator which makes the earned revenues higher than opportunity cost. The following equation is the one which explains these invalid values when *r* increases.

$$y_1^* = \sqrt{\frac{2K'\beta}{h'}} \text{ where } K' = K + \frac{ic\,(1-\alpha)\,\beta T^2}{2} \text{ and } h' = ic - p\,r$$

4.3. Numerical Example on Model with Mixture of Trade Credit and Discounts

Consider the same situation as the base example: numerical example 1, in which we add a new parameter known by the discount rate. For this example, suppliers charge a continuous interest rate of j = 8% for a credit facility of 55 days (T = 0.15 year) and offers a discount rate of γ = 5% to convince retailers to pay a high fraction of the purchase amount at the receipt of items.

To help the retailer in choosing between credit facility or classical EOQ model, the following steps of the defined algorithm in Chapter IV will be followed. If the retailer decides to have the delay, then the optimal policy would be ordering a quantity of units 339.23 every 0.28 years with a total cost of $30,032.96 per year ($TCU^*_{CF,D1}$). However, if the retailer decides to have the delay period bigger than cycle length, then the optimal policy would be ordering a quantity of 449.56 units every 0.37 years with a total cost of $ 29,964.94 per year. However, since the delayed period is extended, y^*_{CF} must not exceed its maximum value of 180. Based on this constraint, the optimal policy when selecting credit facility approach is ordering an optimal quantity of 180 units every 0.15 years with a total cost of $30,204.63 per year ($TCU^*_{CF,D2}$).

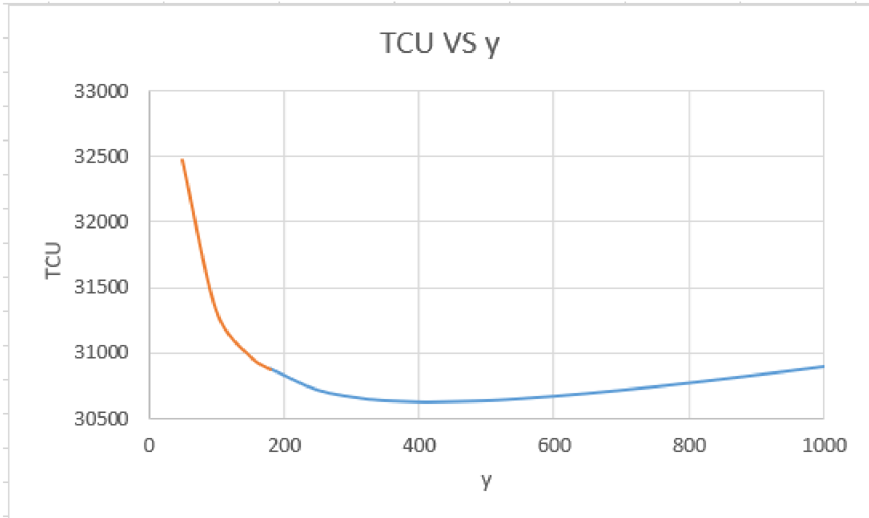

Figure 17: Variation of TCU VS y

So, if we compare $TCU^*_{CF,D1}$ and $TCU^*_{CF,D2}$ it is found that the retailer must choose to delay the remaining settlement for a period less than t_0 since it has the smaller total variable cost per unit time.

Now it is time to compare the CF with the classical EOQ model to check whether the retailer must pay the purchasing amount immediately or delay it to a period less than t_0.

In this case, if the retailer decides to follow the classical economic order quantity model, then the optimal policy would be ordering a quantity of 309.84 units every 0.26 years with a total cost of $30,774.60 per year. As we notice, the optimal quantity obtained when choosing the mix of discount and credit facility, $y^*_{CF,D1}$ is significantly higher than the one obtained by the classical EOQ model. This can be explained by the fact that under this mix of discount and credit facility model, the inventory holding (opportunity) cost is less. This is a well-known result.

In this case, the model under cash discount and trade credit seems to be economically attractive more than the classical EOQ model due to smaller total cost. So, under these conditions, the retailer is advised to pay half the purchasing amount immediately at the receipt of items and delay the other half for a period of 0.15 years.

It is a must to compare results of the trade credit model alone with the model accounting for cash discount and delay payment for the same numerical example and investigate the impact on the retailer decision.

Table 10: Comparison of the Theorems' Results

	Theorem 1	***Theorem 2***	***Theorem 5***	***Theorem 6***
y^*(units)	334.963	180	339.23	180
$TCU^*(\$\ per\ year)$	30,793.49	30,960.25	30,032.96	30,204.63

As illustrated in Table 10, the best theorem used when the retailer is deciding on the optimal quantity to be ordered, is theorem 5 having the lowest total cost per unit time and hence resulting in the highest quantity to be ordered by the retailer. This is a very well knowns result, since a discount is applied on the fraction paid at the receipt of items and the inventory holding (opportunity) cost is less than the usual which motivates retailers to order more.

4.3.1. Sensitivity Analysis on Trade and Discount Mix Model

To analyze the significant influence of different factors on the optimal policy, several variants to the model input were studied.

The first variable to be varied in this model is the discount rate **ɔ** offered by the supplier. By varying this value, we can observe its impact on the optimal order policy and on the retailer's decision.

It is known that, for a specific value of *alpha* and j, as the discount rate γ increases**,** y_{CF}^{*} increases since it is dependent of **ɔ**, and TCU_{CF}^{*} decreases. In other words, if the supplier increases the discount rate applied on the cash paid at the receipt of items, then the retailer must pay lower cost, hence order more, and the credit facility model becomes less attractive and the model under cash discount and trade credit becomes preferable. This variation is shown in Table 11. Table 11 indicates that in both cases (i.e. $T \leq t_0$ and $T \geq t_0$), a decrease in the total variable cost takes place. However, it is shown that an increase in the discount rate **ɔ** does not affect the retailer's decision when choosing to pay half of the amount at the receipt of items in a combination of cash discount and credit facility approach. As we notice, as long as **ɔ** increases, $TCU_{CF,D1}^{*}$ remains smaller than $TCU_{CF,D2}^{*}$ which indicates that the retailer must always choose to delay the payment for a period less than the cycle length. In addition to that, in the case where the retailer chooses the first case, it is shown that for any value of γ, this model which is under the cash discount and the trade credit, is more economically attractive than the classical EOQ model having a total cost less than $30,774.6 per year.

Table 11: Effect of ɣ on the Optimal Policy

ɔ	$y^*_{CF,D1}$	$TCU^*_{CF,D1}$	$y^*_{CF,D2}$	$TCU^*_{CF,D2}$
0.01	335.803	30,641.39	180	30,809.13
0.03	337.504	30,337.19	180	30,506.88
0.05	339.230	30,032.96	180	30,204.63
0.08	341.870	29,576.57	180	29,751.25
0.1	343.664	29,272.29	180	29,449.00
0.3	363.318	26,228.25	180	26,426.50
0.5	386.782	23,181.30	180	23,404.00

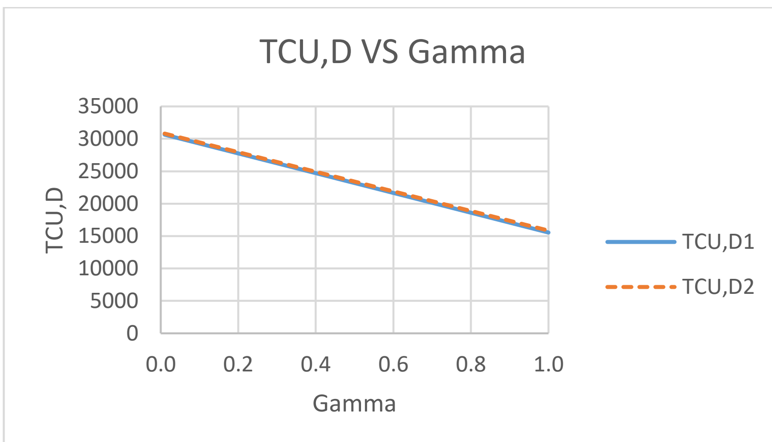

Figure 18: Variation of TCU VS ɣ

Moreover, to study the effect of another important input to the model, another analysis is performed. It is known that higher the amount paid at the receipt is, lower the optimal ordered quantity is and smaller the total cost per unit time is because of ordering smaller quantities. In this model, a discount is added as an extra variable to the model which makes us expect that the variable illustrated in the table below will be smaller than ones already illustrated in Table 12.

Table 12: Effect of α on the optimal policy for j = 8% and γ =5%

$\boldsymbol{\alpha}$	$\boldsymbol{y^*_{CF,1}}$	$\boldsymbol{TCU^*_{CF,1}}$	$\boldsymbol{y^*_{CF,2}}$	$\boldsymbol{TCU^*_{CF,2}}$
0	358.33	30,807.99	180	31,028.84
0.1	354.67	30,653.19	180	30,863.99
0.3	347.11	30,343.28	180	30,534.31
0.5	339.23	30,032.96	180	30,204.63
1.0	317.89	29,254.98	--	29,380.417

Another variable was studied in this analysis which is the interest rate j. As illustrated in Table 13, as j increases, the same as the previous model occurs, the retailer's decision is not affected when choosing to pay half of the amount at the receipt of items in a credit facility approach. However, $TCU^*_{CF,Ds}$ decreases and $TCU^*_{CF,D1}$remains smaller than $TCU^*_{CF,D2}$ which indicates that the retailer must always choose to delay the payment for a period less than the cycle length. In addition to that, in the case where the retailer chooses the first case of this model, it is shown that there exists a critical value j which makes the model under cash discount and trade credit more economically attractive than the classical EOQ model. This conclusion can be used as a proof to validate what we noticed in Figure 2.

Table 13: Effect of j on the optimal policy for α = 0.5 and γ =5%

$\boldsymbol{j}$	$\boldsymbol{y^*_{CF,1}}$	$\boldsymbol{TCU^*_{CF,1}}$	$\boldsymbol{y^*_{CF,2}}$	$\boldsymbol{TCU^*_{CF,2}}$
0.01	339.23	29,874.39	180	30,046.06
0.03	339.23	29,919.53	180	30,091.19
0.05	339.23	29,964.80	180	30,136.47
0.08	339.23	30,032.96	180	30,204.63
0.1	339.23	30,078.57	180	30,250.24
0.3	339.23	30,542.29	180	30,713.96
0.5	339.23	31,020.14	180	31,191.80
1.0	339.23	32,279.39	180	32,451.06

For such model, a search for the optimal fraction to be paid at the beginning of the cycle (i.e. α) was held. For $\gamma = 0.05$, it is found that the optimal value of alpha is 1 which is a well-known result since retailers are motivated to pay the highest amount of money when a discount is offered. We tried different values of discount rate, smaller and higher than the interest j charged by the supplier, therefore the result did not change, i.e. the optimal value of alpha remained 1. Hence, paying the full amount, when both trade credit and cash discount are offered to the retailer, is the optimal policy a retailer can have to minimize the total cost per unit time.

CHAPTER 5

CONCLUSIONS AND FUTURE RESEARCH

Due to the intense competition on today's market, many suppliers offer credit terms to retailers to gain larger market shares and promote commodities. Such motivational policies were considered as alternative incentive policies to quantity discounts. Many researchers were interested in the impact of allowable payment delays on inventory policies. In this thesis, we developed an EOQ model with a permissible partial delay in payments which is subjected to an interest rate charged by the supplier continuously all over the delayed period. Such models were developed to help retailers in identifying how much to order every cycle in a way to result in the minimum total cost per unit time, in addition to be a reference for retailers to know what's the optimal value of interest i that they must accept or what's the optimal fraction of purchasing amount to be paid at the beginning of each cycle when choosing trade credit approach. We also extended our derived model by deriving two other models: one which takes into consideration the interest earned on revenues generated from sales, and another one which consist of offering two approaches for retailers, i.e. trade credit and cash discount of the fraction of amount paid at the beginning. A detailed sensitivity analysis was done for all models to see the impact of each parameter on the optimal ordering policy and retailer's decision as well.

For the future work, it is recommended to include cost components of the supplier in our model which helps us in identifying a win-win situation for both retailers and suppliers.

In other words, an extension of our model must be derived to try to understand how things can go between two parties, and what's the optimal policy that makes retailers order the highest quantity in the least cost, and that makes suppliers earn the highest profit in the least risk of bad debt. Furthermore, our model can be extended by changing some characteristics such as stochastic demand, multi-item, and multiple suppliers. These changes can be reflected in the constraints of our optimization model, e.g. multi-item requires the addition of many constraints such as storage space allocation constraint, budget constraint, etc. However, multiple suppliers could add to our model the possibility of having different interest rates charged by the supplier over the delayed period.

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APPENDIX

Proof of Theorem 1

$$TCU^*_{CF,1} = \frac{\beta}{y^*_{CF,1}} K + \alpha c\beta + (1-\alpha)c\beta\, e^{jT} + ic\left[\frac{\alpha\, y^*_{CF,1}}{2} + \frac{(1-\alpha)\, y^*_{CF,1}}{2}\left(1 - \frac{\beta T}{y^*_{CF,1}}\right)^2\right]$$

If $y^*_{CF,1} = \sqrt{\frac{2K'\beta}{ic}}$

$$y^*_{CF} = \sqrt{\frac{2K'\beta}{ic}} \quad where\ K' = K + \frac{ic\ (1-\alpha) \times \beta T^2}{2}$$

$$\Rightarrow \frac{\beta}{y^*} K = \frac{\beta\sqrt{ic}}{\sqrt{2K'\beta}} K = \frac{\sqrt{\beta ic}}{\sqrt{2K'}} K$$

$$\Rightarrow ic\alpha\frac{y^*}{2} = \frac{ic\alpha}{2}\sqrt{\frac{2K'\beta}{ic}} = \alpha\frac{\sqrt{ic}}{\sqrt{2}}\sqrt{K'\beta} = \alpha\sqrt{\frac{K'\beta ic}{2}}$$

$$\Rightarrow ic\frac{(1-\alpha)\, y^*}{2}\left(1 - \frac{\beta T}{y^*}\right)^2 = \frac{(1-\alpha)\, ic}{2}\sqrt{\frac{2K'\beta}{ic}} \times \left(1 - \frac{\beta T\sqrt{ic}}{\sqrt{2K'\beta}}\right)^2$$

$$= \frac{(1-\alpha)\sqrt{ic}}{\sqrt{2}}\sqrt{K'\beta} \times \left(\frac{\sqrt{2K'\beta} - \beta T\sqrt{ic}}{\sqrt{2K'\beta}}\right)^2$$

$$= \frac{(1-\alpha)\sqrt{ic}}{2\sqrt{2}\sqrt{K'\beta}} \times \left(\sqrt{2K'\beta} - \beta T . \sqrt{ic}\right)^2$$

$$= \frac{(1-\alpha)\sqrt{ic}}{2\sqrt{2}\sqrt{K'\beta}} \times \left(2K'\beta - 2 . \sqrt{2K'\beta}\ \beta T\sqrt{ic} + \beta^2 T^2 ic\right)$$

So,

$$TCU^*_{CF} = \frac{\sqrt{\beta ic}}{\sqrt{2K'}} K + \alpha c\beta + (1-\alpha)c\beta\, e^{jT} + \alpha\sqrt{\frac{K'\beta ic}{2}} + \frac{(1-\alpha)\sqrt{ic}}{2\sqrt{2}\sqrt{K'\beta}} \times \left(2K'\beta - 2\sqrt{2K'\beta}\ \beta T\sqrt{ic} + \beta^2 T^2 ic\right)$$

For a series of calculations,

$$TCU^*{}_{CF} = \frac{\sqrt{\beta ic}}{\sqrt{2K'}}\ K + \alpha c\beta + (1-\alpha)c\beta\ e^{jT} + \alpha\sqrt{\frac{K'\beta ic}{2}} + \frac{(1-\alpha)\sqrt{ic}}{2\sqrt{2}\ \sqrt{K'\beta}} \times \left(2K'\beta - 2\sqrt{2K'\beta}\ \beta T\ \sqrt{ic} + \beta^2 T^2 ic\right)$$

$$= \frac{\sqrt{\beta ic}}{\sqrt{2K'}}\ K + \alpha c\beta + \left(1-\alpha\right)c\beta\ e^{jT} + \alpha\sqrt{\frac{K'\beta ic}{2}} + \frac{(1-\alpha)\sqrt{K'\beta ic}}{\sqrt{2}} - \left(1-\alpha\right) ic\ \beta T + \frac{(1-\alpha)\ ic\sqrt{ic}}{2\sqrt{2}\ \sqrt{K'\beta}}\ \beta^2 T^2$$

$$= \frac{\sqrt{\beta ic}}{\sqrt{2K'}}\ K + \sqrt{\frac{K'\beta ic}{2}} + \frac{(1-\alpha)\ ic\sqrt{ic}}{2\sqrt{2}\ \sqrt{K'}}\ \beta\sqrt{\beta}\,T^2 + \alpha c\beta + \left(1-\alpha\right)c\beta\ e^{jT} - \left(1-\alpha\right) ic\ \beta T$$

$$= \frac{\sqrt{\beta ic}}{\sqrt{2}}\left[\frac{K}{\sqrt{K'}} + \sqrt{K'} + \frac{(1-\alpha)\ ic}{2\sqrt{K'}}\ \beta T^2\right] + \alpha c\beta + \left(1-\alpha\right)c\beta\ e^{jT} - \left(1-\alpha\right) ic\ \beta T$$

$$= \frac{\sqrt{\beta ic}}{\sqrt{2}}\left[\sqrt{K'} + \frac{K + \frac{(1-\alpha)\ ic\ \beta T^2}{2}}{\sqrt{K'}}\right] + \alpha c\beta + \left(1-\alpha\right)c\beta\ e^{jT} - \left(1-\alpha\right) ic\ \beta T$$

$$= \frac{\sqrt{\beta ic}}{\sqrt{2}}\left[2\sqrt{K'}\right] + \alpha c\beta + \left(1-\alpha\right)c\beta\ e^{jT} - \left(1-\alpha\right) ic\ \beta T$$

$$TCU^*_{CF,1} = \sqrt{2K'\beta ic} + \alpha c\beta + \left(1-\alpha\right)c\beta\ \left(e^{jT} - iT\right)$$

Proof of Theorem 2

If $y^*_{CF,2} = \sqrt{\frac{2K''\beta}{ic}}$

$$TCU^*_{CF,2} = \frac{\sqrt{\beta ic}}{\sqrt{2K''}}\ K + \alpha c\beta + (1-\alpha)c\beta\ e^{jT} + \alpha\sqrt{\frac{K''\beta ic}{2}}$$

$$= \frac{\sqrt{\beta ic}}{\sqrt{2K''}}\ K''\ \alpha + \alpha c\beta + (1-\alpha)c\beta\ e^{jT} + \alpha\sqrt{\frac{K''\beta ic}{2}}$$

$$= \alpha\sqrt{\frac{K''\beta ic}{2}} + \alpha c\beta + (1-\alpha)c\beta\ e^{jT} + \alpha\sqrt{\frac{K''\beta ic}{2}}$$

$$TCU^*_{CF,2} = \alpha\sqrt{2K''\beta ic} + \alpha c\beta + (1-\alpha)c\beta\, e^{jT}$$

The second derivative of the *TCU (y₂)* with respect to y_2 is defined as follows:

$$\frac{d^2(TCU)}{dy_2^2} = \frac{d}{dy_2}\left(\frac{d(TCU)}{dy_2}\right) = 2\frac{\beta}{y_2^3}\ K \qquad \geq 0$$

Having all the parameters positive, the second derivative is found to be positive and so the derived solution is the minimum of this model.

Proof of Lemma 1

$$TCU^*_{CF,1} < TCU^*_{EOQ}$$

$$\sqrt{2K'\beta ic} + \alpha c\beta + (1-\alpha)\ c\beta\ (e^{jT} - iT) < \sqrt{2K\beta ic} + c\ \beta$$

$$(1-\alpha)\ c\beta\ (e^{jT} - iT) < \sqrt{2K\beta ic} + c\ \beta - \sqrt{2K'\beta ic} - \alpha c\beta$$

$$(1-\alpha)\ ic\beta\left(\frac{e^{jT}}{i} - T\right) < \sqrt{2K\beta ic} + c\ \beta - \sqrt{2K'\beta ic} - \alpha c\beta$$

$$(1-\alpha)\ ic\beta\left(\frac{e^{jT}}{i} - T\right) < \sqrt{2K\beta ic} + c\ \beta - \sqrt{2K'\beta ic} - \alpha c\beta$$

$$\frac{e^{jT}}{i} < T + \frac{\sqrt{2K\beta ic}}{(1-\alpha)\ ic\beta} + \frac{(1-\alpha)\ c\ \beta}{(1-\alpha)\ ic\beta} - \frac{\sqrt{2K'\beta ic}}{(1-\alpha)\ ic\beta}$$

$$e^{jT} < iT + \frac{i}{(1-\alpha)\ \beta}(EOQ - y^*_{CF,1}) + 1$$

By taking the natural logarithm of both sides, the inequality becomes

$$j < \frac{1}{T} \times ln\left(iT + \frac{i}{(1-\alpha)\beta}(EOQ - y^*_{CF,1}) + 1\right)$$

Where $g(\alpha) = \frac{1}{T} \times \ln\left(iT + \frac{i}{(1-\alpha)\beta}\left(EOQ - y^*_{CF,1}\right) + 1\right)$

Proof of Lemma 2

$$TCU_{CF,2}^{*} < TCU_{EOQ}^{*}$$

$$\alpha\ \sqrt{2K''\beta ic} + \alpha c\beta + (1-\alpha)c\beta\ e^{\,jT} < \sqrt{2K\beta ic} + c\beta$$

$$(1-\alpha)c\beta\ e^{\,jT} < \sqrt{2K\beta ic} + c\beta - \alpha\ \sqrt{2K''\beta ic} - \alpha c\beta$$

$$(1-\alpha)c\beta\ e^{\,jT} < \sqrt{2K\beta ic} - \alpha\ \sqrt{\frac{2K}{\alpha}\ \beta ic} + (1-\alpha)c\beta$$

$$(1-\alpha)c\beta\ e^{\,jT} < \sqrt{2K\beta ic} - \sqrt{\alpha}\ \sqrt{2K\beta ic} + (1-\alpha)c\beta$$

$$(1-\alpha)c\beta\ e^{\,jT} < \left(1-\sqrt{\alpha}\right)\ \sqrt{2K\beta ic} + (1-\sqrt{\alpha})\ (1+\sqrt{\alpha})c\beta$$

$$e^{\,jT} < \frac{\left(1-\sqrt{\alpha}\right)\ \sqrt{2K\beta ic}}{(1-\sqrt{\alpha})(1+\sqrt{\alpha})\ c\beta} + \frac{\left(1-\sqrt{\alpha}\right)\ (1+\sqrt{\alpha})c\beta}{(1-\sqrt{\alpha})(1+\sqrt{\alpha})\ c\beta}$$

$$e^{\,jT} < \frac{}{(1+\sqrt{\alpha})}\ \sqrt{\frac{2Ki}{c\beta}} + 1$$

By taking the natural logarithm of both sides, the inequality becomes

$$j < \frac{1}{T}\ \ln\left(\frac{1}{(1+\sqrt{\alpha})}\ \sqrt{\frac{2Ki}{c\beta}} + 1\right)$$

Proof of Lemma 3

$$TCU_{CF,1}^{*} = TCU_{CF,2}^{*} < TCU_{EOQ}^{*}$$

$$\frac{K}{T} + \alpha c\beta + (1-\alpha)c\beta\ e^{jT} + ic\ \frac{\alpha\beta T}{2} < \sqrt{2K\beta ic} + c\beta$$

$$(1-\alpha)c\beta\ e^{jT} < \sqrt{2K\beta ic} + c\beta - \frac{K}{T} - \alpha c\beta - ic\ \frac{\alpha\beta T}{2}$$

$$e^{jT} < \frac{\sqrt{2K\beta ic}}{(1-\alpha)c\beta} + \frac{(1-\alpha)c\beta}{(1-\alpha)c\beta} - \frac{K}{(1-\alpha)c\beta T} - ic\ \frac{\alpha\beta T}{2(1-\alpha)c\beta}$$

$$e^{jT} < \frac{1}{(1-\alpha)}\ \sqrt{\frac{2Ki}{c\beta}} - \frac{1}{(1-\alpha)}\left[\frac{K}{c\beta T} + \frac{i\ \alpha\ T}{2}\right] + 1$$

$$e^{jT} < \frac{1}{(1-\alpha)}\left[\sqrt{\frac{2Ki}{c\beta}} - \frac{K}{c\beta T} - \frac{i\ \alpha\ T}{2}\right] + 1$$

By taking the natural logarithm of both sides, the inequality becomes

$$j < \frac{1}{T}\ ln\left(\frac{1}{(1-\alpha)}\left[\sqrt{\frac{2Ki}{c\beta}} - \frac{K}{c\beta T} - \frac{i\ \alpha\ T}{2}\right] + 1\right)$$

Proof of Theorem 3

To prove that the generated solution is the minimum, the second derivative of the *TCU* with respect to *y* must be positive. The second derivative is defined as follows:

$$\frac{d^2(\,TCU)}{dy_1^2} = \frac{d}{dy_1}\left(\frac{d(\,TCU)}{dy_1}\right) = 2\,\frac{\beta}{y_1^3}.K + \frac{i.c.(\,1-\alpha)\,\beta^2T^2}{y_1^3}$$

$$\frac{d^2(\,TCU)}{dy_1^2} = \frac{2\beta K + i.c.(\,1-\alpha)\,\beta^2T^2}{y_1^3} \qquad \geq 0$$

Having all the parameters positive, the second derivative is found to be positive and so the derived solution is the minimum of this model.

Proof of Theorem 4

For the case where $T \geq t_0$, the second derivative of the *TCU* (y_2) with respect to y_2 is defined as follows.

$$\frac{d^2(TCU)}{dy_2^2} = \frac{d}{dy_2}\left(\frac{d(TCU)}{dy_2}\right) = 2\frac{\beta}{y_2^3}\ K \qquad \geq 0$$

Having all the parameters positive, the second derivative is found to be positive and so the derived solution is the minimum of this model.

Proof of Lemma 4

$$TCU^*_{CF,1} < TCU^*_{EOQ}$$

$$\sqrt{2K'\beta h'} + \alpha c\beta + (1-\alpha)\, c\beta\, \left(e^{jT} - iT\right) < \sqrt{2K\beta ic} + c\beta$$

$$(1-\alpha)\, ic\beta\, \left(\frac{e^{jT}}{i} - T\right) < \sqrt{2K\beta ic} + c\beta - \sqrt{2K'\beta h'} - \alpha c\beta$$

$$\frac{e^{jT}}{i} < T + \frac{\sqrt{2K\beta ic}}{(1-\alpha)\, ic\beta} + \frac{(1-\alpha)\, c\beta}{(1-\alpha)\, ic\beta} - \frac{\sqrt{2K'\beta h'}}{(1-\alpha)\, ic\beta}$$

$$e^{jT} < iT + \frac{i}{(1-\alpha)\ \ \beta}\left(EOQ - \frac{\sqrt{2K'\beta h'}}{ic}\right) + 1$$

By taking the natural logarithm of both sides, the inequality becomes

$$j < \frac{1}{T}\ \ln\left(iT + \frac{i}{(1-\alpha)\ \ \beta}\left(EOQ - \frac{\sqrt{2K'\beta h'}}{ic}\right) + 1\right)$$

Proof of Lemma 5

$$TCU^*_{CF,2} < TCU^*_{EOQ}$$

$$\sqrt{2K\beta h''} + \alpha c\beta + (1-\alpha)c\beta\ e^{jT} - \beta\ T\ p\ r < \sqrt{2K\beta ic} + c\beta$$

$$(1-\alpha)c\beta\ e^{jT} < \sqrt{2K\beta ic} + c\beta - \sqrt{2K\beta h''} - \alpha c\beta + \beta\ T\ p\ r$$

$$e^{jT} < \frac{\sqrt{2K\beta ic}}{(1-\alpha)c\beta} + \frac{(1-\alpha)c\beta}{(1-\alpha)c\beta} - \frac{\sqrt{2K\beta h''}}{(1-\alpha)c\beta} + \frac{\beta\ T\ p\ r}{(1-\alpha)c\beta}$$

$$e^{jT} < \frac{\sqrt{2K\beta}}{(1-\alpha)c\beta}(\sqrt{ic} - \sqrt{h''}) + 1 + \frac{T\ p\ r}{(1-\alpha)c}$$

Proof of Lemma 6

$$TCU^*_{CF,1} = TCU^*_{CF,2} < TCU^*_{EOQ}$$

$$\frac{K}{T} + \alpha c\beta + (1-\alpha)c\beta\ e^{jT} + (ic\ \alpha - p\ r)\ \frac{\beta T}{2} < \sqrt{2K\beta ic} + c\beta$$

$$(1-\alpha)c\beta\ e^{jT} < \sqrt{2K\beta ic} + c\beta - \frac{K}{T} - \alpha c\beta - (ic\ \alpha - p\ r)\ \frac{\beta T}{2}$$

$$e^{jT} < \frac{\sqrt{2K\beta ic}}{(1-\alpha)c\beta} + \frac{(1-\alpha)c\beta}{(1-\alpha)c\beta} - \frac{K}{T\,(1-\alpha)c\beta} - \frac{(ic\ \alpha - p\ r)\ \beta T}{2(1-\alpha)c\beta}$$

$$e^{jT} < \frac{1}{(1-\alpha)}\left[\sqrt{\frac{2Ki}{c\beta}} - \frac{K}{c\beta T} - \frac{i\alpha T}{2} + \frac{p\ r\ T}{2c}\right] + 1$$

Hence,

$$j < \frac{1}{T}\ \ln\left(\frac{1}{1-\alpha}\left[\sqrt{\frac{2Ki}{c\beta}} - \frac{K}{c\beta T} - \frac{i\alpha T}{2} + \frac{p\ r\ T}{2c}\right] + 1\right)$$

Proof of Theorem 5

To prove that the generated solution is the minimum, the second derivative of the *TCU* (y_{D1}) with respect to y_{D1} must be positive.

The second derivative is defined as follows:

$$\frac{d^2(TCU)}{dy_{D1}^2} = \frac{d}{dy_{D1}}\left(\frac{d(TCU)}{dy_{D1}}\right) = 2\frac{\beta}{y_{D1}^3}.K + \frac{ic.(1-\alpha)\beta^2T^2}{y_{D1}^3}$$

$$\frac{d^2(TCU)}{dy_{D1}^2} = \frac{2\beta K + ic.(1-\alpha)\beta^2T^2}{y_{D1}^3} \geq 0$$

Having all the parameters positive, the second derivative is found to be positive and so the derived solution is the minimum of this model.

Proof of Theorem 6

For the case where $T \geq t_0$, the second derivative of the *TCU* (y_{D2}) with respect to y_{D2} is defined as follows:

$$\frac{d^2(TCU)}{dy_{D2}^2} = \frac{d}{dy_{D2}}\left(\frac{d(TCU)}{dy_{D2}}\right) = 2\frac{\beta}{y_{D2}^3}.K \qquad \geq 0$$

Having all the parameters positive, the second derivative is found to be positive and so the derived solution is the minimum of this model.

Proof of Lemma 7

$$TCU^*_{CF,D1} < TCU^*_{EOQ}$$

$$\sqrt{2K'\beta h'} + (1-\gamma)\,\alpha c\beta + (1-\alpha)\,c\beta(e^{jT} - iT) < \sqrt{2K\beta ic} + c\beta$$

$$(1-\alpha)\,c\beta(e^{jT} - iT) < \sqrt{2K\beta ic} + c\beta - \sqrt{2K'\beta h'} - \alpha c\beta + \gamma\alpha c\beta$$

$$(1-\alpha)\,ic\beta\left(\frac{e^{jT}}{i} - T\right) < \sqrt{2K\beta ic} + (1-\alpha)\,c\beta - \sqrt{2K'\beta h'} + \gamma\alpha c\beta$$

$$\frac{e^{jT}}{i} < T + \frac{\sqrt{2K\beta ic}}{(1-\alpha)\,ic\beta} + \frac{(1-\alpha)\,c\beta}{(1-\alpha)\,ic\beta} - \frac{\sqrt{2K'\beta h'}}{(1-\alpha)\,ic\beta} + \frac{\gamma\alpha c\beta}{(1-\alpha)\,ic\beta}$$

$$e^{jT} < iT + \frac{i}{(1-\alpha)\,\beta}\left[\sqrt{\frac{2K\beta}{ic}} - \frac{\sqrt{2K'\beta ic(1-\gamma\alpha)} \times (1-\gamma\alpha)}{ic(1-\gamma\alpha)}\right] + 1 + \frac{\gamma\alpha}{(1-\alpha)}$$

$$e^{jT} < iT + \frac{i}{(1-\alpha)\,\beta}\left[EOQ - (1-\gamma\alpha)\,y^*_{D1}\right] + 1 + \frac{\gamma\alpha}{(1-\alpha)}$$

Hence,

$$j < \frac{1}{T}\ln\left(iT + \frac{i}{(1-\alpha)\,\beta}\left[EOQ - (1-\gamma\alpha)\,y^*_{D1}\right] + 1 + \frac{\gamma\alpha}{(1-\alpha)}\right)$$

Proof of Lemma 8

$$TCU^*_{CF,D2} < TCU^*_{EOQ}$$

$$\sqrt{2K\beta h''} + (1-\gamma)\,\alpha c\beta + (1-\alpha)\,c\beta . e^{jT} < \sqrt{2K\beta ic} + c\beta$$

$$(1-\alpha)\,c\beta . e^{jT} < \sqrt{2K\beta ic} - \sqrt{2K\beta h''} - (1-\alpha)\,c\beta + \gamma\alpha c\beta$$

$$e^{jT} < \frac{\sqrt{2K\beta ic}}{(1-\alpha)\,c\beta} - \frac{\sqrt{2K\beta h''}}{(1-\alpha)\,c\beta} - \frac{(1-\alpha)\,c\beta}{(1-\alpha)\,c\beta} + \frac{\gamma\alpha c\beta}{(1-\alpha)\,c\beta}$$

$$e^{jT} < \frac{1}{(1-\alpha)} \times \sqrt{\frac{2Ki}{c\beta}}\left[1-(1-\gamma)\,\alpha\right] + 1 + \frac{\gamma\alpha}{(1-\alpha)}$$

$$j < \frac{1}{T} \times \ln\left(\frac{1}{(1-\alpha)} \times \sqrt{\frac{2Ki}{c\beta}}\left[1-(1-\gamma)\,\alpha\right] + 1 + \frac{\gamma\alpha}{(1-\alpha)}\right)$$

Proof of Lemma 9

$$TCU^*_{CF,D1} = TCU^*_{CF,D2} < TCU^*_{EOQ}$$

$$\frac{K}{T} + (1-\gamma)\alpha c\beta + (1-\alpha)c\beta e^{jT} + ic\frac{(1-\gamma)\alpha\beta T}{2} < \sqrt{2K\beta ic} + c\beta$$

$$(1-\alpha)c\beta e^{jT} < \sqrt{2K\beta ic} + c\beta - \frac{K}{T} - (1-\gamma)\alpha c\beta - ic\frac{(1-\gamma)\alpha\beta T}{2}$$

$$e^{jT} < \frac{\sqrt{2K\beta ic}}{(1-\alpha)c\beta} + \frac{(1-\alpha)c\beta}{(1-\alpha)c\beta} - \frac{K}{(1-\alpha)c\beta T} + \frac{\gamma\alpha c\beta}{(1-\alpha)c\beta} - ic\frac{(1-\gamma)\alpha\beta T}{2(1-\alpha)c\beta}$$

$$e^{jT} < \frac{1}{(1-\alpha)}\left[\sqrt{\frac{2Ki}{c\beta}} - \frac{K}{c\beta T} + \gamma\alpha - \frac{i(1-\gamma)\alpha T}{2}\right] + 1$$

By taking the natural logarithm of both sides, the inequality becomes

$$j < \frac{1}{T} \times \ln\left(\frac{1}{(1-\alpha)}\left[\sqrt{\frac{2Ki}{c\beta}} - \frac{K}{c\beta T} + \gamma\alpha - \frac{i(1-\gamma)\alpha T}{2}\right] + 1\right)$$