AMERICAN UNIVERSITY OF BEIRUT

THE CONSTRUCTION OF A SIMPLICIAL RESOLUTION OF I^2 WHERE I IS SQUARE-FREE MONOMIAL IDEAL

by

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AN ABSTRACT OF THE THESIS OF

Jana Jamal Attieh for Master of Science

Major: Mathematics

Title: The construction of a simplicial resolution of I^2 where I is a square-free monomial ideal .

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables, and $I = (m_1, \ldots, m_q)$ a square-free monomial ideal in R. We consider the ideal $I^2 = \langle \{m_i m_j : i, j\} \rangle$ to be the monomial ideal generated by at most $\binom{q+1}{2}$ generators. We study the construction of a simplicial complex labeled by the monomials of I^2 which supports a free resolution of I^2.

CHAPTER 1

INTRODUCTION

Let R=k$[x_1, x_2, ..., x_n]$ be the polynomial ring in n variables with maximal ideal $\mathfrak{M}$, and let I be an ideal of R. A free resolution of I is an exact sequence of free modules that describes relations on the generators of the ideal which has the following form:

$$0 \rightarrow F_r \rightarrow \ldots \rightarrow F_1 \rightarrow F_0 \rightarrow I$$

with r being the pdim (I).

Suppose I is a monomial ideal i.e generated by monomials. Finding the minimal free resolution of I known as the minimal monomial resolution, can be quite complex despite the combinatorial structure that monomial ideals have. An important tool in studying monomial resolutions is to find topological objects whose chain maps can be homogenized to obtain free resolutions of these ideals. This approach began with Diana Taylor in her thesis in 1966. It consists of labeling the vertices of the simplex by the monomials of the ideal and the faces by the lcm of the monomials. However, the Taylor's resolution is far from minimal. Many mathematicians tried to generalize Taylor's approach by considering smaller topological objects with the hope of obtaining minimal free resolutions.

In this thesis, we let $I = (m_1, \ldots, m_q)$ be a monomial square-free ideal, and we consider the monomial ideal I^2 generated by the $m_i m_j$ for all i, j. The number of generators of I^2 is at most $\binom{q+1}{2}$, and so the number of vertices on the number of vertices on the Taylor simplex is growing exponentially. We are interested in learning about a subcomplex of the Taylor simplex whose simplicial complex supports of a free resolution of I^2. We learn the construction of simplicial complex $\mathbb{L}_q^2$ on $\binom{q+1}{2}$ vertices with fewer faces, and we exhibit a subcomplex of the $\mathbb{L}_q^2$ called $L_2(I)$ which supports a free resolution.

We begin the thesis by introducing some background and definitions in chapter 2. Then in chapter 3, we explain what are minimal free resolutions of ideals I in a polynomial ring in several variables and discuss some properties. In chapter 4, we define monomial resolutions and exhibit techniques used to construct simplicial resolutions. Finally, in our last chapter, we tackle our problem and explore the simplicial complex $\mathbb{L}_q^2$ and its sub-complex $L^2(I)$.

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CHAPTER 2

PRELIMINARIES

Let R be a commutative unitary ring. Here are some useful definitions on elements of the ring R.

2.1. Notions on Commutative Ring and Maximal Ideals

Definition 2.1. A **zero divisor** in R is an element x for which $\exists\ y \neq 0$ such that $xy = 0$.

Example 2.2. *In $M_2(\mathbb{R})$, consider A and B to be the following matrices*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

their product AB is the zero matrix while A and B are not, so A and B are two zero divisors.

A ring with no zero divisors (and in which $1 \neq 0$) is called an integral domain, just like $\mathbb{Z}$, $k[x_1, x_2, \ldots, x_n]$, where k is a field and $n \in \mathbb{N}$, are integral domains.

Definition 2.3. A **unit** in R is an element x which "divides 1", i.e an element x such that $xy = 1$ for some y in R.

Now, there will be a definition that we will use it later as an application.

Definition 2.4. Let R be a ring. Let M be an R-module. A sequence of elements $r_1, r_2, \ldots, r_n \in R$ is called a **regular sequence on M (or M-sequence)** if

1. $(r_1, r_2, \ldots, r_n)M \neq M$ and

2. for $i = 1, \ldots, n$, r_i is a non zero divisor on $M/(r_1, r_2, \ldots, r_{i-1})M$.

Definition 2.5. A **field** is a ring R in which $1 \neq 0$ and every non-zero element is a unit.

Example 2.6. $\mathbb{R}$, $\mathbb{C}$ *are fields* .

We note that every field is an integral domain but not conversely.

Definition 2.7. An ideal $\mathbf{m}$ in R is maximal if $\mathbf{m} \neq (1)$ and if there is no ideal A in R such that $\mathbf{m} \subseteq A \subseteq (1)$.

Example 2.8. *$p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$ where p is a prime number.*

Proof. let I be an ideal of $\mathbb{Z}$ such that $p\mathbb{Z} \subset I \subset \mathbb{Z}$ then I has a form of $d\mathbb{Z}$ with d is the smallest positive integer number in I. Hence $p\mathbb{Z} \subset d\mathbb{Z} \subset \mathbb{Z}$ which gives that $d \mid p$ and as a result $d = 1$ or $d = p$ (because p is prime). Thus, $I = \mathbb{Z}$ or $I = p\mathbb{Z}$. □

Proposition 2.9. *$\boldsymbol{m}$ is a maximal ideal iff $R/\boldsymbol{m}$ is a field.*

Proof. $\Rightarrow$ We have to prove that the only ideals of R$/m$ are $\{0\}$ and R/m. Indeed, let $\bar{I}$ be an ideal of R/m. Then I is an ideal of R such that $m \subseteq I$, but $m \subseteq I \subseteq R$; by maximality of m we get $m = I$ or $I = R$. Hence, $\bar{I} = \{\bar{0}\}$ and $\bar{I} = R/m$. So, R/m is field.

$\Leftarrow$) Let I be an ideal of R/m such that $m \subseteq I \subseteq R$; since $m \subset I$ we get $\bar{I}$ ideal of R/m. So, $\bar{I} = \{\bar{0}\}$ or R/m hence, $I = m$ or $I = R$. $\square$

Proposition 2.10. *Every ring $R \neq 0$ has at least one maximal ideal.*

Proof. Let Σ be the set of all ideals $\neq (1)$ in R, order Σ by the inclusion. Σ is a non-empty set because $(0) \in \Sigma$. We apply Zorn's lemma. In order to do so, we prove that there exists an upper bound for every chain of ideals in Σ. We know, that for any i,j $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Let $\mathcal{A} = \bigcup A_i$, $\mathcal{A}$ is an ideal (using inclusion) such that $1 \notin \mathcal{A}$. Hence, $\mathcal{A} \in \Sigma$ and $\mathcal{A}$ is the upper bound of the chain. By Zorn's lemma Σ has a maximal element. $\square$

Definition 2.11. A local ring R is a commutative ring with identity which has a unique maximal ideal $\mathbf{m}$.

Example 2.12.

- *Any field F is a local ring with unique maximal ideal $\{0\}$.*
- $\mathbf{Z}_p$ *is a field where p prime number as $p\mathbb{Z}$ is maximal ideal by example 2.8.*
- $R = \left\{ \begin{bmatrix} a & 0 \\ b & a \end{bmatrix} \in M_{2\times 2} / a, b \in \mathbf{Z_2} \right\}$ *is a local ring with* $\boldsymbol{m} = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

2.2. Noetherian Rings

Theorem 2.13. *We call a ring A to be* ***Noetherian*** *if it satisfies one of the three equivalent conditions:*

i. *Every non-empty set of ideals in A has a maximal element.*

ii. *Every ascending chain of ideals in A is stationary.*

iii. *Every ideal in A is finitely generated.*

Proof. $i \Rightarrow ii$) Suppose A satisfies the maximal condition of ideals of A. Let $A_1 \subseteq A_2 \subseteq \ldots$ be a strictly increasing chain of ideals that doesn't stop. So, the set $\{A_k, k \in \mathbf{N}\} \neq \varnothing$ of A does not have a maximal element which gives contradiction. Hence, the chain is stationary.

$ii \Rightarrow i$) Suppose that every ascending chain of A is stationary. If A does not satisfy maximal condition, then there is a non-empty set Σ of A with no maximal element. Let $A_0 \in \Sigma$ and let $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_k$ strictly increasing chain of ideals in Σ . Since Σ has no maximal element, so there exists $A_{k+1} \in \Sigma$ such that $A_k \subseteq A_{k+1}$; then $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_k \subseteq A_{k+1}$ strictly increasing chain of ideals in Σ. Proceeding in this way, we obtain an infinite ascending chain of ideals, this contradicts the assumption that A is stationary. Hence, A has a maximal element.

$ii \Rightarrow iii$) Let $\mathcal{N}$ be an ideal of A and Σ all finitely generated sub-ideals of $\mathcal{N}$. Σ is non-empty since $0 \in \Sigma$. Hence, it has a maximal element, say $\mathcal{N}_0$. If $\mathcal{N}_0 \neq \mathcal{N}$, then we consider the ideal $\mathcal{N}_0 + Ax$ with $x \in \mathcal{N}$ and $x \notin \mathcal{N}_0$. Thus, we have $\mathcal{N} \subseteq \mathcal{N}_0 + Ax$ and $\mathcal{N}_0 + xA$ is finitely generated; which contradicts with the maximality of $\mathcal{N}$. So, $\mathcal{N} = N_0$ and $\mathcal{N}$ is finitely generated.

$iii \Rightarrow ii$) Suppose every ideal of A is finitely generated. Let $A_1 \subseteq A_2 \subseteq \ldots$ be an infinite increasing chain of ideals of A and $B = \bigcup_{k \in \mathbb{N}} A_k$. Hence, B is an ideal in A. Using our assumption, B is finitely generated by a finite subset $\{x_1, \ldots, x_r\}$.

Then, there exists natural numbers $n_1, \ldots, n_r$ such that $x_i \in A_{n_i}, \forall i = 1, \ldots, r$. Let $k_0 = \max\{n_1, \ldots, n_r\}$ so $x_1, \ldots, x_r \in A_{k_0}$. Hence, $B \subseteq A_{k_0}$ but $A_{k_0} \subseteq B$ for every $k \geqslant k_0$ hence $B = A_{k_0}$ and the chain stops. $\square$

Example 2.14. *1. The ring $\mathbb{Z}$ is Noetherian since $m\mathbf{Z} \subseteq n\mathbf{Z} \Leftrightarrow n \mid m$. hence, it satisfies the second statement of the above theorem.*

2. Every field is Noetherian since it has no proper ideals.

Theorem 2.15. *(Hilbert Basis Theorem) If R is **Noetherian**, then the polynomial ring $R[x]$ is also **Noetherian**.*

Proof. Let $I \subseteq R[x]$ be an ideal. We need to show that I is finitely generated . The Elements are the polynomials in $R[X]$ with coefficients in R :

$\{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 + a_0; a_i \in R, n \gneqq 0\}$. If $I = (0)$ then it is a trivial case. Now, we assume that $I \neq (0)$ then, choose $f_1 \neq 0$ be a polynomial in I which has least degree among all non-zero elements in I. This means that if $f \in I$, $f \neq 0$, and $degree(f) \geqslant degree(f_1)$. Clearly, $(f_1) \subseteq I$. Now, if $(f_1) = I$ then we have done, but if $(f_1) \neq I$ so I contains other elements than (f_1) and then $I/(f_1) \neq \varnothing$. Now,choose f_2 to be a least degree polynomial in $I/(f_1)$ (all the polynomials in $I/(f_1)$ such that f_2 has least degree). Again, if $(f_1, f_2) = I$ then we are done. Otherwise, we continue f_3 to be the least degree polynomial in $I/(f_1, f_2)$. Let $a_i = lc(f_i)$ (the leading coefficient of f_i such that $a_i \in R$) and let $J = (a_1, a_2, \ldots) \subseteq R$. Since R is Noetherian then J is finitely generated. Hence, $J = (a_1, \ldots, a_n)$ for some n.

<u>*Claim*</u> : I is finitely generated by $f_1, \ldots, f_n$; i,e $I = (f_1, \ldots, f_n)$.

Proof. Suppose that $I \neq (f_1, \ldots, f_n)$, we would have chosen f_{n+1} to be the polynomial of least degree among all the elements of $I/(f_1, \ldots, f_n)$; $a_{n+1} \in J = (a_1, \ldots, a_n)$ then $a_{n+1} = \sum b_i a_i; i = 1, \ldots, n$. Now, consider $g = \sum b_i f_i x^{m_i}$ for $i = 1, \ldots, n$ such that $m_i = deg(f_{n+1}) - deg(f_i)$, by construction $g \in (f_1, \ldots, f_n)$ and $lc(g) = a_{n+1} = lc(f_{n+1})$ also $deg(f_{n+1} - g) < deg(f_{n+1})$ then $f_{n+1} - g \in (f_1, \ldots, f_n)$ and $g \in (f_1, \ldots, f_n)$. So, $f_{n+1} = (f_{n+1} - g) + g \in (f_1, \ldots, f_n)$ which contradicts. Therefore, I if finitely generated and $R = [x]$ is Noetherian. □

□

Theorem 2.16. *If R is **Noetherian**, then the polynmial ring $R[x_1, \ldots, x_n]$ is also **Noetherian**.*

Proof. Using the induction on n. □

2.3. Tensor product

Definition 2.17. let M,N and P be R-modules, we define a bilinear map from $M \times N$ to P by a map $\Phi : M \times N \to P$ such that

$$\Phi((am+a'm')\times(bn+b'n')) = ab\Phi(m\times n)+a'b\Phi(m'\times n)+ab'\Phi(m\times n')+a'b'\Phi(m'\times n')$$

where $m, m' \in M$ and $n, n' \in N$.

Definition 2.18. Define a **tensor product** $M \otimes_R N$ to be the module with generators $\{m \otimes n; m \in M, n \in N\}$ with relations

$$(am + a'm') \otimes (bn + b'n') = ab(m \otimes n) + a'b(m' \otimes n) + ab'(m \otimes n') + a'b'(m' \otimes n').$$

Definition 2.19. (**Wedge Product**) The wedge product or " exterior product" is a multiplication operator obtained form the wedge product by factoring out the product $m \otimes m$. It is also denoted by $\wedge$ such that $m \wedge n = m \otimes n - n \otimes m$. It has many properties such as:

- Associative; $(m \wedge n) \wedge l = m \wedge (n \wedge l)$.
- Anti-commutative, $m \wedge n = -n \wedge m$.
- Distributive under addition operation.

2.4. Complexes and Exact sequences

Let R be a commutative ring.

Definition 2.20. A finite **complex** E is a sequence of homomorphisms of R-modules of the form: $0 \xrightarrow{d_0} E^0 \to \ldots \xrightarrow{d_n} E^{n+1} \to 0$. Where $d_i : E^i \to E^{i+1}$ such that $d_{i+1} \circ d_i = 0$ for all i . Thus, $Im(d_i) \subseteq ker(d_{i+1})$.

Definition 2.21. The **Homology** H_i of the complex is defined to be $H_i = ker(d_{i+1})/Im(d_i)$. By definition, $H_0 = E_0$ and $H_n = E_n/Im(d_n)$.

Definition 2.22. Let E and F be two complexes. A homomorphism $f : E \to F$ is a sequence of homomorphisms $d_i : E_i \to F_i$ making the following diagram commutative for all i.

$$\begin{array}{ccccccc} \cdots \longrightarrow & E^i & \xrightarrow{d_E^i} & E^{i+1} & \longrightarrow \cdots \\ & f^i \big\downarrow & & \big\downarrow f^{i+1} & \\ \cdots \longrightarrow & F^i & \xrightarrow{d_F^i} & F^{i+1} & \longrightarrow \cdots \end{array} .$$

2.4.1. *Exact Sequences*

Most important kind of a complex is the exact sequence.

Definition 2.23. Sequence of R-modules and R-homomorphisms

$$\cdots \xrightarrow{f_{i-2}} M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \rightarrow \cdots$$

.

is said to be **exact** at M_i if $Im(f_i) = Ker(f_{i+1})$. The sequence is **exact** if it is exact at each M_i.

In Particular,

1. $0 \longrightarrow M' \xrightarrow{f} M$. is exact $\iff$ f is injective.

2. $M \xrightarrow{g} M'' \longrightarrow 0$ is exact $\iff$ g is surjective.

3. $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is exact $\iff$ f is injective and g is surjective.

also, g induces an isomorphism of $coker(f) = M/f(M') = M/Ker(g)$ onto M''. A sequence of last type is called **short exact sequence**.

2.5. Graded Rings and Modules

Definition 2.24. A **graded ring** is a ring R together with a family $(R_n)_{n\geq 0}$ subgroups of the additive group R, such that $R = \bigoplus_{n=0}^{\infty} R_n$ and $R_n R_m \subseteq R_{n+m}$ for every n and m.

Definition 2.25. "Graded polynomial Ring"

Let $R = k[x_1, \ldots, x_n]$ be the ring polynomial over the field k. Define R_n by $R_n = \{\sum_{m\in\mathbb{N}^d} r_m x^m / r_m \in k, m_1 + \ldots + m_d = n\}$ where $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$ and $x^m = {x_1}^{m_1} \ldots {x_d}^{m_d}$. The polynomial ring R is graded using 2.24 and let $deg(x_i) = 1$ for all i and

$R_0 = k$; then R is standard grading. A monomial ${x_1}^{m_1} \dots {x_d}^{m_d}$ has a degree $m_1 + \dots + m_d$. Denote by R_i the k-vector space spanned by all monomials of degree i.

Definition 2.26. Let R be a graded ring, a **graded R-module** is an R-module M together with the family $(M_n)_{n \geq 0}$ where of subgroups M such that $M = \overset{\infty}{\underset{n=0}{\oplus}} M_n$ for n positive and $R_m M_n \subseteq M_{n+m}$ for every positive numbers m and n.

Example 2.27. *Graded vector space is a graded K-module over a field K.*

Definition 2.28. A polynomial $h \in R$ is called **homogeneous** if $h \in R_i$ for some i. In this case, h has degree i denoted by $deg(h) = i$.

Definition 2.29. An element $m \in M$ is called **homogeneous** if $m \in M_n$ for some n, where n represents the degree of the element m.

Example 2.30. *Let $R = k[x, y, z]$, $R_0 = k$, and $R_1 = \{linear\ forms\}$. The polynomial $f = x^3 + yz^2$ is homogeneous because all of its term has degree 3; however, the polynomial $g = x^2 - y$ is not homogeneous since every element has a different degree.*

Definition 2.31. A proper ideal J ($J \neq \emptyset$ and $J \neq R$) is called **graded** or **homogeneous** if J is generated by homogeneous elements.

Remark 2.32. 0 *is homogeneous for any degree.*

Suppose that I is graded ideal in R (standard graded) then the quotient ring R/I inherits the grading by R_i/I_i.

We have seen that the polynomial ring is a graded ring, and the ideals are homogeneous ideals. This polynomial ring can be considered as a local ring. In fact, maximal ideals of $R = k[x_1, \dots, x_n]$ are of the form $m = (x_1 - a_1, \dots x_n - a_n)$ since $R/m \simeq k$, since m is homogeneous this gives that $a_1 = a_2 = ... = a_n = 0$. Therefore, $a_i = 0$ for all i, and $m = (x_1, \dots, x_n)$ is considered to be the homogeneous maximal ideal of R.

Definition 2.33. Irrelevant maximal ideal is the one which generated by all polynomials of positive degree and denoted by $m = (x_1, \dots, x_n)$.

Definition 2.34. Let $M = \bigoplus M_d$ with $d \in \mathbb{Z}$ be a finitely generated graded R-module with d-th graded component m_d. Let $M(a)$ the module M shifted by a: $(M_a)_d = M_{a+d}$.

Example 2.35. *x^2 has a degree in $R[x]$ since $x^2 \in R_2$. Hence, x^2 has a degree 0 in $R(-2)$ since if x^2 has degree 2 in R, then x^2 has degree $(2-2=0)$ in $R(-2)$.*

Definition 2.36. Let Φ be a homomorphism between two graded modules $\Phi : M \to N$ maps $x \to \Phi(x)$. Φ is said to be homogeneous if $deg(\Phi(x)) = deg(x)$ for every $x \in M$. These maps are also called degree-0 maps.

Example 2.37. *let $\Phi : R \to R(-2)$ that maps $1 \to x^2$ and $z \to zx^2$. For x and $z \in R$, $deg(\Phi(x)) = deg(zx^2) = 3$ in R, so it has degree $1 = 3-2$ in $R(-2)$. Hence, $deg(\Phi(z)) = deg(z)$.*

CHAPTER 3

GRADED FREE RESOLUTION

Let $R = k[x_1, x_2, \ldots, x_n]$ be a the graded polynomial ring in n variables and m its homogeneous maximal ideal in R.

3.1. Graded Free Resolution

Definition 3.1. A free resolution of a finitely graded R-module M is a sequence of homogeneous R-modules

$$\mathbb{F} : \ldots F_i \xrightarrow{\delta_i} F_{i-1} \xrightarrow{\delta_{i-1}} F_{i-2} \rightarrow \ldots \rightarrow F_1 \xrightarrow{\delta_1} F_0$$

such that:

1. $\mathbb{F}$ is a complex of finitely generated R-module F_i.

2. $\mathbb{F}$ is exact i.e $Im(\delta_{i-1}) = Ker(\delta_i)$

3. $M \cong F_0/Im(\delta_1)$

For convenience, we write

$\mathbb{F} : \ldots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \longrightarrow 0.$

3.1.1. *Construction of Free Resolutions*

In this subsection, we exhibit a manual construction of a free resolution of module M. This construction can be done through Macaulay 2.

1. Let M be an R-module and $\{m_i\}_{i=1,\ldots,n}$ be a finite set of generator of M, then we define a map from a free module F_0 to M by sending the i^{th} generator of F_0 to m_i for every i. Let

$M_1 \subset F_0$ be the kernel of F_0. Since R is Noetherian, by the Hilbert Basis theorem, M_1 is finitely generated and the elements of M_1 are called the syzygies on m_i.

2. Choosing finitely many homogeneous syzygies that generate M_1, we define a map $\delta_1 : F_1 \to F_0$ with $Im(\delta_1) = M_1 = \ker(\delta_0)$.

3. Continuing this way, we construct an exact sequence of free modules, called a free resolution of M: $\ldots \to F_i \xrightarrow{\delta_i} F_{i-1} \to \ldots \to F_1 \xrightarrow{\delta_1} F_0$.

Next we put a grading on the constructed free resolution. We recall that a resolution is **graded** if M is graded, $\mathbb{F}$ is a graded complex, and the maps are degree-preserving maps i.e. δ_i have degree 0 for all i.

Construction 3.2. *Given homogeneous elements $m_i \in M$ of degree a_i that generate M as an R-module, we will construct a graded free resolution of M by induction on homological degree. First, set $M_0 = M$ and Choose homogeneous generators $m_1, \ldots, m_r$ of M_0. Let $a_1, \ldots, a_r$ be their degrees respectively. Now set $F_0 = \underset{1 \leq i \leq r}{\oplus} R(-a_i)$. The map defined from the graded free module F_0 onto M sends the $i^t h$ generator f_i of $R(-a_i)$ to m_i. After constructing the F_i by taking the* $\ker(\delta_i)$, *we obtain finitely generated graded R-module F_i's and put a degree on them. So we write F_i as $\underset{p \in \mathbb{Z}}{\oplus} R(-p)^{\beta_{i,p}}$ to make the degree of the map equal to zero. Therefore, a graded complex of free finitely generated modules has the form*

$$\ldots \longrightarrow \underset{p \in \mathbb{Z}}{\oplus} R(-p)^{\beta_{i,p}} \xrightarrow{\delta_i} \underset{p \in \mathbb{Z}}{\oplus} R(-p)^{\beta_{i-1,p}} \longrightarrow \ldots \longrightarrow R.$$

It is an exact sequence of degree-0 maps between graded free modules such that the cokernel of δ_1 is M. Note that the numbers $\beta_{i,p}$ are the ***graded Betti numbers*** *of the complex.*

The following table represents all the graded betti numbers. The entry in the i_{th} column and p_{th} row is $\beta_{i,i+p}$ and the i_{th} column contains the data at the i_{th} step of the minimal graded free resolution.

Example 3.3. *One of the simplest family of graded free resolutions are called* ***Koszul complexes****. They resolve an ideal generated by a regular sequence. Let*

	β_0	β_1	β_2	$\dots$
0	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\dots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\dots$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\dots$
3	$\beta_{0,3}$	$\beta_{1,4}$	$\beta_{2,5}$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Table 3.1: The Graded Betti-numbers

$I = (x_1, x_2, x_3) \in k[x_1, x_2, x_3]$

$$0 \longrightarrow R(-3) \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} R^3(-2) \xrightarrow{\begin{pmatrix} 0 & x_3 & -x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}} R^3(-1) \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}} R$$

Example 3.4. *Let us introduce another resolution for the ideal $I = (x^2, xy, z^3)$ in the polynomial ring $R = [x, y, z]$.*

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} x^2 \\ xy \\ z^3 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -y & -z^3 & 0 \\ x & 0 & -z^3 \\ 0 & x^2 & xy \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} z^3 \\ -y \\ x \end{pmatrix}} R$$

where the betti-numbers are shown in the following table :

.	0	1	2	3
total	1	3	3	1
0	1	.	.	.
1	.	2	1	.
2	.	1	.	.
3	.	.	2	1

Table 3.2: Betti Table

Theorem 3.5. *(Hilbert Syzygy Theorem)*

let $R = k[x_1, \dots, x_n]$. Any finitely generated graded R-module M has a finite graded free resolution :

$$0 \longrightarrow F_m \xrightarrow{\delta_m} F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\delta_1} F_0$$

such that $m \leq n$, the number of variables in R.

3.1.2. *Examples from Macaulay 2*

In this subsection, we exhibit a few examples by using the software Macaulay 2.

Example 3.6. *The below is an example of a minimal resolution of the regular sequence $\{x^2, y^3, z^3\}$*

```
    i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : ideal(x^2,y^3, z^3)

                2   3   3
o2 = ideal (x , y , z )

o2 : Ideal of R

i3 : res o2

      1      3      3      1
o3 = R  <-- R  <-- R  <-- R  <-- 0

     0      1      2      3      4

o3 : ChainComplex
```

```
i4 : betti res o2

            0 1 2 3
o4 = total: 1 3 3 1
         0: 1 . . .
         1: . 1 . .
         2: . 2 . .
         3: . . 2 .
         4: . . 1 .
         5: . . . 1

o4 : BettiTally

i5 : o3.dd_1

o5 = | x2 y3 z3 |

             1       3
o5 : Matrix R  <--- R

i6 : o3.dd_2

o6 = {2} | -y3 -z3 0   |
     {3} | x2  0   -z3 |
     {3} | 0   x2  y3  |
```

```
             3       3
o6 : Matrix R  <--- R

i7 : o3.dd_3

o7 = {5} | z3  |
     {5} | -y3 |
     {6} | x2  |

             3       1
o7 : Matrix R  <--- R
```

Example 3.7. *Again in the polynomial ring in three variables, we resolve the ideal* $I = (x^2y, y^2z, z^3, z^2x^2)$

```
  i8 : ideal (x^2*y, y^2*z, z^3, z^2*x^2)

             2    2    3   2 2
o8 = ideal (x y, y z, z , x z )

o8 : Ideal of R

i9 : res o8

      1      4      4      1
o9 = R  <-- R  <-- R  <-- R  <-- 0

     0      1      2      3      4
```

```
o9 : ChainComplex

i10 : betti res o8

             0 1 2 3
o10 = total: 1 4 4 1
          0: 1 . . .
          1: . . . .
          2: . 3 . .
          3: . 1 4 .
          4: . . . 1

o10 : BettiTally

i11 : o9.dd_1

o11 = | x2y y2z z3 x2z2 |

              1       4
o11 : Matrix R  <--- R

i12 : o9.dd_2

o12 = {3} | -yz -z2 0   0   |
      {3} | x2  0   0   -z2 |
      {3} | 0   0   -x2 y2  |
      {4} | 0   y   z   0   |
```

```
               4        4
o12 : Matrix R  <--- R


i13 : o9.dd_3


o13 = {5} | -z2 |
      {5} | yz  |
      {5} | -y2 |
      {5} | -x2 |


               4        1
o13 : Matrix R  <--- R
```

3.2. Minimal Graded Free Resolution

Let $R = k[x_1, \cdots, x_n]$ be a polynomial ring with M a R-module. In this section, we will define graded minimal free resolutions.

Definition 3.8. A complex of graded R-modules

$$\cdots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \cdots$$

is called minimal if for each i, $\delta_i(F_i) \subset mF_{i-1}$.

Example 3.9. *Let $R = [x]$ and $I = (x^2, x^3)$. A resolution of R/I is :*

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -x \\ 1 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} -x & 1 \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

is not minimal element since $1 \in \delta_2 = \begin{pmatrix} -x \\ \\ 1 \end{pmatrix}$ *then* $\delta_2(\mathbb{R}) \subsetneq m(\mathbb{R}^{\not\vDash})$.

Construction 3.10. *The construction of graded minimal free resolutions is done by doing the same steps as the construction of the free resolution by taking* **minimal** *sets of generators for* M_0 *and then a minimal set of generators for every* $\ker \delta_i$ *for all* i.

Next, we will illustrate the construction via an example

Example 3.11. *Given the polynomial ring* $R = k[x, y, z, w]$ *and the ideal* $I = (xy, yz, zw)$. *The main goal is to construct the minimal graded free resolution of* R/I.

Step 1: *let* $F_0 = R$ *be a graded* k*-module and* $\delta_0 : R \longrightarrow R/I$.

Step 2: *The element* xy*,*yz *and* zw *are homogeneous generators of ker(*δ_0*), each of degree* 2. *Let* $F_1 = R^3(-2)$, *denote by* f_1 *the* 1*-generator of each* $R(-2)$ *such that* $i = 1, 2, 3$. *Now, let* $\delta_1 : F_1 \longrightarrow F_0$ *such that* $Im(\delta_1) = ker(\delta_0) = I$, *so*

$$R^3(-2) \overset{\begin{pmatrix} xy & yz & zw \end{pmatrix}}{\longrightarrow} R \longrightarrow R/I$$

step 3: *We have to find the homogeneous generators of* $ker(\delta_1)$ *in order to construct* δ_2 .

$$\begin{pmatrix} xy & yz & zw \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

such that $c_1, c_2, c_3 \in R$. *By calculation, We can see that* $R_1 = \begin{pmatrix} -z & x & 0 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & -w & y \end{pmatrix}$, *and* $R_3 = \begin{pmatrix} -wz & 0 & xy \end{pmatrix}$ *are three generators, but* $xR_2 + R_3 = wR_1$ *so, minimal generators of the solution* $\begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$ *is* $\{R_1, R_2\}$. *Therefore,* $-zf_1 + xf_2$ *and* $-wf_2 + yf_3$ *are homogeneous generators of ker(*δ_1*). Furthermore, the grading is done as follows:*

$deg(-wf_1 + xf_2) = deg(-z) + deg(f_1) = 1 + 2 = 3$ *same for* $-wf_2 + yf_3$. *Now, let* $F_2 = R^2(-3)$

and g_1,g_2 be the 1-generators of $R(-3)$; $deg(g_1) = deg(g_2) = 3$. Define

$$\delta_2 : R^2(-3) \longrightarrow R^3(-2)$$

that maps $g_1 \longrightarrow -zf_1 + xf_2$ and $g_2 \longrightarrow -wf_2 + yf_3$ such that $Im(\delta_2) = ker(\delta_1)$; then the differential matrix

$$\begin{pmatrix} -z & 0 \\ x & -w \\ 0 & y \end{pmatrix}$$

***Step 4**: We know, $Im(\delta_3) = ker(\delta_2)$ has no non-trivial solution. Hence, $F_3 = 0$ and $\delta_3 : 0 \longrightarrow R^2(-3)$.*

So, a minimal graded free resolution :

$$0 \longrightarrow R^2(-3) \xrightarrow{\begin{pmatrix} -z & 0 \\ x & -w \\ 0 & y \end{pmatrix}} R^3(-2) \xrightarrow{\begin{pmatrix} xy & yz & zw \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

Lemma 3.12. *(Nakayama). Suppose that M is a finitely generated graded R-module and $m_1, \cdots, m_n \in M$ generate $M/\mathfrak{m}M$. Then $m_1, \cdots, m_n$ generate M.*

Proof. Let $\overline{M} = M/\Sigma R_{m_i}$. If the $m_i's$ generate $M/\mathfrak{m}M$ then $\overline{M}/\mathfrak{m}\overline{M} = 0$ and $\overline{M} = \mathfrak{m}\overline{M}$. Now, if $\overline{M} \neq 0$, since $\overline{M}$ is finitely generated there would be a non-zero element of at least degree in $\overline{M}$; this element could not be in $\mathfrak{m}\overline{M}$. Thus, $\overline{M} = 0$ and M is finitely generated by the $m_i's$. □

Corollary 3.13. *A graded free resolution $\mathbb{F} : \ldots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \ldots$ is minimal as a complex if and only if for each i the map δ_i takes a basis of F_i to a minimal set of generators of the image of δ_i.*

Proof. $\Rightarrow$ Consider the right exact sequence $F_{i+1} \longrightarrow F_i \longrightarrow Im(\delta_i) \longrightarrow 0$. The above resolution is minimal $\Longleftrightarrow \delta_{i+1}(F_{i+1}) \subset \mathfrak{m}F_i$ for each $i \Longleftrightarrow \delta_{i+1} : F_{i+1} \longrightarrow F_i/\mathfrak{m}F_i$ is the zero map $\Longleftrightarrow \overline{\delta_{i+1}} : F_{i+1}/\mathfrak{m}F_{i+1} \longrightarrow F_i/\mathfrak{m}F_i$ is a zero map $\longrightarrow \delta_i : F_i/\mathfrak{m}F_i \Longrightarrow Im(\delta_i)/\mathfrak{m}Im(\delta_i)$ is an isomorphism. Now, suppose $\{f_1, \dots, f_n\}$ is a basis (minimal set of generators) of F_i, then $\{\overline{f_1}, \dots, \overline{f_n}\}$ set of generators of $F_i/\mathfrak{m}F_i$ and minimal by Nakayama's lemma. Hence, $\overline{\delta_i}(\overline{f_i}) = \overline{m_i}$ is a minimal set of generators of $Im(\delta_i)/\mathfrak{m}Im(\delta_i)$ and by Nakayama's lemma m_i is a minimal set of generators of $Im(\delta_i)$.

$\Leftarrow$ Suppose δ_i takes basis of F_i to minimal set of generators of $Im(\delta_i)$; by Nakayama's lemma , $\{\overline{f_1}, \dots, \overline{f_n}\}$ minimal set of generators of $F_i/\mathfrak{m}F_i$ and $\{m_i\}$ basis of $Im(\delta_i)/\mathfrak{m}Im(\delta_i)$ of same dimension as $F_i/\mathfrak{m}F_i$. Then, there is an isomorphism between $F_i/\mathfrak{m}F_i$ and $Im(\delta_i)/\mathfrak{m}Im(\delta_i)$. Again, by Nakayama's lemma this occurs if and only if a basis of F_i maps to a minimal set of generators of $Im(\delta_i)$. $\square$

The most interesting thing is the uniqueness of the minimality as we will see below.

Theorem 3.14. *Let M be a finitely generated graded R-module. If F and G are minimal graded free resolution of M, then there is a graded isomorphism of complexes $F \longrightarrow G$ inducing map on M.*

Proof.

$$\begin{array}{c} \mathbb{F} : \dots F_1 \longrightarrow F_0 \xrightarrow{d_0} M \longrightarrow 0 \\ \qquad\qquad\qquad \downarrow id_M \\ \mathbb{G} : \dots G_1 \longrightarrow G_0 \xrightarrow{\delta_0} M \longrightarrow 0 \end{array}$$

We start by constructing the identity map on M, We have $id_M \circ d_0 : F_0 \longrightarrow M$ then δ_0 is surjective by the exactness and since F_0 is free hence it is projective. So, there exists a map $f_0 : F_0 \longrightarrow G_0$ such that the diagram commutes and then $id_M \circ d_0 = \delta_0 \circ f_0$. We have to show that f_0 is isomorphism. To do so, we tensor both $\mathbb{F}$ and $\mathbb{G}$ by $K = R/\mathfrak{m}$ and show that $f_0 \circ id$ is isomorphism.

$$\begin{array}{c} \mathbb{F} : \dots F_1 \otimes K \longrightarrow F_0 \otimes K \xrightarrow{d_0 \otimes id} M \otimes K \longrightarrow 0 \\ \qquad\qquad\qquad\qquad \downarrow id_M \otimes id \\ \mathbb{G} : \cdots G_1 \otimes K \longrightarrow G_0 \otimes K \xrightarrow{\delta_0 \otimes id} M \otimes K \longrightarrow 0 \end{array}$$

since $\mathbb{F}$ and $\mathbb{G}$ are minimal, $F_0 \otimes K = F_0/\mathfrak{m}F_0$ and $G_0 \otimes K = G_0/\mathfrak{m}G_0$ which are k-vector spaces. Then, $d_0 \otimes id$ and $\delta_0 \otimes id$ are isomorphisms, using the corollary 3.13 and $f_0 \otimes id$ is isomorphism.

In order to show that f_0 is an isomorphism, let $f_0 = (a_{ij})$ then $f_0 \otimes Id = a_{ij} \otimes 1 = (\bar{a_{ij}})$ is invertible. Thus, $det(a_{ij})$ is unit in k and $det(a_{ij})$ is not in M which implies that $det(a_{ij})$ unit in R and the matrix is invertible. So, f_0 is isomorphism. We follow by the same procedure to construct f_1 as f_0 induces an isomorphism between $ker(d_0)$ and $ker(\delta_0)$. □

Definition 3.15. If M is finitely generated Graded R-module. We define the **projective dimension** of M to be the minimal length of a projective resolution of M which is equal to the length of the minimal graded free resolution and denote by $pd_R(M)$.

Example 3.16. *let $R = k[x, y, z]$ and let $I = (yz, xy)$. The projective dimension $pd_R(I)$ is equal to 1 in the minimal free resolution of I:*

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -x \\ z \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} yz & xy \end{pmatrix}} I \longrightarrow 0.$$

while the projective dimension is 2 in the minimal free resolution of R/I

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -x \\ z \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} yz & xy \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

Remark 3.17. $pd_R(R/I) = pd_R(I) + 1.$

CHAPTER 4

MONOMIAL RESOLUTIONS

In this part, we are interested in discussing free resolutions of monomial ideals which by definiton, the ideals that are generated by monomials. The structure of these such resolution is more complicated than of the previous resolution. However, the excitement work here is the involvement of the combinatorial techniques (as we will see later) to make it obvious.

4.1. i-grading

With all of the above, we can say that R is $\mathbb{N}^n$**-graded** (or simply multi-graded) by $mdeg(x_i)$=the $i'th$ standard vector in $\mathbb{N}^n$ where $mdeg$ represents the multi-degree. Given $a = (a_1 \ldots, a_n) \in \mathbb{N}^n$, there exists a unique monomial of $\mathbb{N}^n$ of degree a namely $x^a = x_1^{a_1} \ldots x_n^{a_n}$ and a its exponent vector. Hence, in this case, $R = \underset{m}{\oplus} R_m$ such that m is a monomial and R_m is a k-vector space spanned by m, with $R_m R'_m = R_{mm'}$ for all monomials m and m'. Also, an R-module T is called **multi-graded** if it can be written as $\underset{m}{\oplus} T_m$ as a k-vector space and $R_m T_{m'} \subseteq T_{mm'}$ for all monomials m and m'.

Notation 4.1. $R(x^a)$ stands for the free R-module with one generator in multi-degree x^a, it can be denoted also by R_a.

4.2. Multi-graded Free Resolutions

Referring to what we have defined before, we noticed that any monomial ideal is **homogeneous** with respect to the multi-grading. Hence, the constructed graded resolution in 3.2 works .

Denote by $\mathbb{F}_{\mathbb{M}}$ the multi-graded free resolution of R/M over R which has a form like

$$\cdots \longrightarrow \bigoplus_{m} R^{c_{i,p}} \xrightarrow{\delta_i} \bigoplus_{m} R^{c_{i-1,p}} \longrightarrow \cdots \longrightarrow R \tag{4.1}$$

where δ_i represents the differential matrices and m the monomials in R.

Example 4.2. *Let $R = k[x, y]$ be the polynomial ring and let I be an ideal of R generated by x^2 and xy, so $I = (x^2, xy)$. The minimal free resolution of R/I which is multi-graded is :*

$$\longrightarrow R(x^2y) \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R(x^2) \oplus R(xy) \xrightarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} R$$

4.2.1. *The Taylor Resolution*

One significant resolution is the **"Taylor resolution"** as it resolves all R/M whenever M is a monomial ideal. This resolution was discovered by Diana Taylor in her thesis in [8], and resolves the monomial ideals M by using the exterior algebra. However, this type of resolution is highly non-minimal although it has a simple structure. We first define the notion of exterior algebra:

Definition 4.3. Let $h_1, \ldots, h_q$ be elements in R. Define E to be the exterior algebra over K on the canonical basis $e_1, \ldots, e_q$. Thus,

$E = k(e_1, \ldots, e_q)/(\{e_i^2/1 \leq i \leq q\}, \{e_ie_j + e_je_i/1 \leq i \leq j \leq q\})$ which is the quotient of free algebra.

Definition 4.4. (Taylor Resolution) Denote $\mathbb{T}_{\mathbb{M}}$ (the taylor resolution) the R-module $R \bigotimes E$ graded homologically by $hdeg(e_{j_1} \wedge e_{j_2} \ldots \wedge e_{j_i}) = i$ and equipped with the differential:

$d(e_{j_1} \wedge \ldots \wedge e_{j_i}) = \sum\limits_{1 \leq p \leq i} (-1)^{p-1} \frac{lcm\{m_{j_1}, \ldots, m_{j_i}\}}{lcm\{m_{j_1}, \ldots, \hat{m}_{j_p}, \ldots m_{j_i}\}} e_{j_1} \wedge \ldots \wedge \hat{e}_{j_p} \wedge \ldots \wedge e_{j_i}$ where $\hat{e}_{j_p}$ and $\hat{m}_{j_p}$ mean that e_{j_p} and m_{j_p} are omitted respectively.

The standard grading of $\mathbb{T}_{\mathbb{M}}$ is represented by

$deg(e_{j_1}, \ldots, e_{j_i}) = deg(lcm(m_{j_1}, \ldots, m_{j_i}))$ where multi-grading is given by

$mdeg(e_{j_1} \wedge \ldots \wedge e_{j_p}) = lcm(m_{j_1}, \ldots, m_{j_i})$.

Example 4.5. *Let R= $k[x, y]$ and the ideal $M = (x^3, xy, y^2)$, the Taylor resolution of $R/(x^3, xy, y^2)$ is:*

$$\mathbb{T}_{\mathbb{M}} : 0 \longrightarrow R(x^3y^2) \xrightarrow{\begin{pmatrix} y \\ x^2 \\ 1 \end{pmatrix}} R(x^3y) \oplus R(xy^2) \oplus R(x^3y^2) \xrightarrow{\begin{pmatrix} -y & 0 & y^2 \\ x^2 & -y & 0 \\ 0 & x & -x^3 \end{pmatrix}} R(x^3) \oplus R(xy) \oplus R(y^2) \xrightarrow{\begin{pmatrix} x^3 \\ xy \\ y^2 \end{pmatrix}} R$$

4.3. Homogenization

In this section, we will use notations from "Peeva's Book". M represents a monomial ideal in R generated by $m_1, \ldots, m_r$. We denote by L_M the set of the least common multiples of subsets of $\{m_1, \ldots, m_r\}$. By convention, $1 \in L_M$ where $1 = lcm(\emptyset)$.

Definition 4.6. A **frame** (or an r-frame) U is a complex of finite K-vector spaces with differential δ and a fixed basis that satisfies the following conditions:

1. $U_i = 0$ for $i \leq -1$ and $i >>$
2. $U_0 = K$.
3. $U_1 = K^r$.
4. $\delta(w_j) = 1$ for each basis vector w_j in $U_1 = K^r$.

Definition 4.7. An M-complex G is a **multi-graded complex** of finitely generated free multi-graded R-modules with differential d and a fixed multi-homogeneous basis with multi-degrees in L_M that satisfies the following conditions:

1. $G_i = 0$ for $i \leq -1$ and i big enough.

2. $G_0 = R$.

3. $G_1 = R(m_1) \oplus \cdots \oplus R(m_r)$.

4. $d(w_j) = m_j$ for each basis element w_j of G_1.

Based on the above, the homogenization concept connects complexes of vector spaces and complexes of R-modules.

Definition 4.8. Let U be an **r-frame**. A M-complex G of free R-modules with differential d is called a **homogenization** of U if it is a sequence of free R-modules constructed by induction as follows.

- Let $G_0 = R$ and $G_1 = R(m_1) \oplus \ldots \oplus R(m_r)$
- Let $\bar{v_1}, \ldots, \bar{v_p}$ be a given basis for U_i.
- Let $\bar{u_1}, \ldots, \bar{u_q}$ be a given basis for U_{i-1}.
- Let $u_1, \ldots, u_q$ be the basis of $G_{i-1} = R^q$ chosen on the previous step by induction.

Now, introduce $v_1, \ldots, v_p$ the will be a basis of $G_i = R^p$. If $\delta(\bar{v_j}) = \sum\limits_{1 \leq s \leq j} \alpha_{s,j} \bar{u_s}$ such that $\alpha_{s,j} \in K$ then set:

- $mdeg(v_j) = Lcm(mdeg(v_s)/\alpha_{s,j} \neq 0)$ ($(\emptyset) = 1$ by convention)
- $G_i = \bigoplus\limits_{1 \leq j \leq p} R(mdeg(v_j))$ ($mdeg$ stands for multi-degree).
- $d(v_j) = \sum\limits_{1 \leq s \leq q} \alpha_{s,j} \frac{mdeg(v_j)}{mdeg(v_s)} u_s$.

Clearly, $coker(d_1) = R/M$ and the differentials are homogeneous by construction.

Our next target is to show that G is an M-complex of free R-modules and call it the complex G obtained from U by **M-homogenization**. We

Example 4.9. *Let $R = k[x, y]$ be the polynomial ring and let I be the ideal generated by x^2, xy and x^3, so $I = (x^2, xy, x^3)$.*

Consider its 3-frame

$$0 \longrightarrow k \xrightarrow{\begin{pmatrix}1\\1\\1\end{pmatrix}} k^3 \xrightarrow{\begin{pmatrix}-1 & 0 & 1\\ 1 & -1 & 0\\ 0 & 1 & -1\end{pmatrix}} k^3 \xrightarrow{\begin{pmatrix}1 & 1 & 1\end{pmatrix}} k$$

*The **I-homogenization** of the frame is :*

$$G: 0 \longrightarrow R(x^2y^3) \xrightarrow{\begin{pmatrix}y^2\\x\\1\end{pmatrix}} R(x^2y) \oplus R(xy^3) \oplus R(x^2y^3) \xrightarrow{\begin{pmatrix}-y & 0 & y^3\\ x & -y^2 & 0\\ 0 & x & -x^2\end{pmatrix}}$$
$$R(x^2) \oplus R(xy) \oplus R(y^3) \xrightarrow{\begin{pmatrix}x^2 & xy & y^3\end{pmatrix}} R$$

Proposition 4.10. *If **G** is the **M-homogenization** of a frame **U**, then **G** is an **M-complex**.*

Proof. Let $\overline{v_1}, \ldots, \overline{v_p}$ and $\overline{u_1}, \ldots . \overline{u_q}$ and $\overline{w_1}, \ldots, \overline{w_t}$ be the given bases of U_i, U_{i-1} and U_{i-2} respectively. Let $v_1, \ldots, v_p$ and $u_1, \ldots, u_q$ and $w_1, \ldots, w_t$ be the corresponding bases of G_i, G_{i-1} and G_{i-2} respectively. Let $1 \leq j \leq p$ be a fixed parameter, we know that U is complex then:

$$0 = \delta^2(\overline{v_j}) = \delta(\sum_{1 \leq s \leq q} \alpha_{s,j}\overline{u_s}) = \sum_{1 \leq s \leq q} \alpha_{s,j}(\sum_{1 \leq l \leq t} \beta_{l,s}\overline{w_l}) = \sum_{1 \leq l \leq t}(\sum_{1 \leq s \leq q} \alpha_{s,j}\beta_{l,s})\overline{w_l}$$

with $\alpha_{s,j}, \beta_{l,s} \in k$. Hence, $\sum_{1 \leq s \leq q} \alpha_{s,j}\beta_{l,s} = 0 \forall 1 \leq l \leq t$.

Moreover,

$$d^2(v_j) = d(\sum_{1 \leq s \leq q} \alpha_{s,j}\frac{mdeg(v_j)}{mdeg(u_s)}u_s) = \sum_{1 \leq s \leq q} \alpha_{s,j}\frac{mdeg(v_j)}{mdeg(u_s)}(\sum_{1 \leq l \leq t} \beta_{l,s}\frac{mdeg(u_s)}{mdeg(w_l)}w_l) \tag{4.2}$$

$$= \sum_{1 \leq l \leq t}(\sum_{1 \leq s \leq q} \alpha_{s,j}\beta_{l,s}\frac{mdeg(v_j)mdeg(u_s)}{mdeg(u_s)mdeg(w_l)})w_l = \sum_{1 \leq l \leq t}(\sum_{1 \leq s \leq q} \alpha_{s,j}\beta_{l,s})\frac{mdeg(v_j)}{mdeg(w_l)}w_l = 0. \tag{4.3}$$

Therefore, **G** is M-complex. ☐

Definition 4.11. Suppose **G** is a complex. Then, we can **dehomogenize** by setting $U = G \otimes R/(x_1 - 1, \ldots, x_n - 1)$. U is a finite complex of k-vector spaces with fixed basis and its differential matrices are obtained by setting $x_1 = 1 \ldots, x_n = 1$ in the differential matrices of $\mathbf{G}$.

Remark 4.12. *U is called the **frame** or the **dehomogenization** of G.*

4.4. Simplicial Resolution

An important tool in studying **monomial resolutions** is to find toplogical objects whose chain maps can be homogenized to obtain free resolutions. It starts by labeling the vertices of the simplical complex by the monomials of M and the faces of higher dimension by the lcm of the monomials of M then homogenize it into a simplcial resolution. We first begin by explaining smplicial complexes.

4.4.1. *Simplicial Complex*

Definition 4.13. A **simplicial complex** Δ over a vertex set $V = \{v_1 \ldots, v_p\}$ is a set of subsets of V such that if $\mathbf{F} \in \Delta$ and $\mathbf{G} \subset \mathbf{F}$ then $\mathbf{G} \in \Delta$. An element σ of Δ is called a face, and maximal faces under inclusion are called **facets**.

Definition 4.14. We say that Δ is a **simplex** if it has only one facet. That is, every subset of Δ is a face ($\{v_1, \ldots, v_p\}$ is a facet).

Remark 4.15.

- *A simplicial complex is called **void** if it has no faces.*
- *A simplicial complex is called **irrelevant** if the only face is $\emptyset$.*

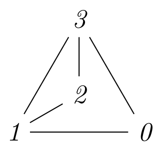
Definition 4.16. The dimension of a face σ is $\mid \sigma \mid -1$. The dimension of Δ is the maximum of the dimensions of its faces. Also, $-\infty$ if Δ is **void** and -1 which is the dimension of $\emptyset$ (by convention) for **irrelevant** complex. A simplicial complex is called **pure** if all of its facets have same dimension.

Example 4.17. *The simplicial complex on the set of vertices $\{v_1, v_2, v_3\}$ is*

$$\Delta = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}\}$$

.

Example 4.18. *However the below example is not a simplicial complex*



(4.4)

$\Delta = \{\{0,1,2,3\},\{1,2,3\},\{2,3,0\},\{1,2,0\},\{1,2\},\{1,0\},\{2,3\},\{1,3\},\{0,3\},$

$$\{0\},\{1\},\{2\},\{3\},\phi\}$$

misses $\{2,0\}$.

Example 4.19. *Let* $\{a,b,c\}$ *be the set of nodes of* Δ *and the sets of faces be* $\{\{a,b,c\},\{a,b\},\{a,c\},\emptyset\}$,*is a non-simplicial complex as* Δ *doesn't contain the face* $\{c,b\}$ *which is a subset of the face* $\{a,b,c\}$.

4.4.2. *Simplicial Resolution*

As we know, finding free resolutions of an ideal has been of interest to many mathematicians in the field. Diane Taylor introduced a method to resolve R/M by using combinatorial techniques that depend on labeling the faces of a simplex $\triangle$ with monomials then homogenizing it into a free resolution called the **simplicial resolution**. In order to find the complex of vector spaces coming from the simplicial complex, we introduce an orientation in the faces of Δ.

Definition 4.20. let τ' be a facet of τ, an orientation function is $[\tau,\tau'] := (-1)^i$ if $\tau \setminus \tau'$ is the $(i+1)$'st element in the sequence of the vertices of τ written in increasing order.

Example 4.21. *Let* m_1, m_2 *and* m_3 *represent the vertices of a simplicial complex* Δ. *Let* $\tau = \{m_1, m_2, m_3\}$ *and take* τ' *to be the facet of the edge* $\{m_1, m_2\}$. *So,* $[\tau,\tau'] = (-1)^2 = 1$ *as* $\tau \setminus \tau'$ *is equal to* m_3, *the third vertex* $(i = 2)$.

Definition 4.22. The **augmented oriented simplicial chain complex of** $\triangle$ **over** k is

$\tilde{C}(\Delta;k) = \underset{\tau\in\Delta}{\oplus} ke_\tau$, where e_τ denotes the basis element corresponding to the face τ, and the differential δ acts as $\delta(e_\tau) = \sum_{\tau' \ is \ a \ facet \ of \ \tau} [\tau,\tau']e'_\tau$

Remark 4.23. *We say that Δ supports a free resolution of I when a simplicial chain complex of Δ homogenized (using the monomial labels on the faces) to obtain a simplicial resolution of I which is generated by the monomials.*

Definition 4.24. $\tilde{C}(\Delta;k)[-1]$ is a frame after shifting $\tilde{C}(\Delta;k)$ in homological degree. Denote by F_Δ the $M-$homogenization of $\tilde{C}(\Delta;k)[-1]$. In this case, we say that F_Δ is **supported on** Δ, or Δ **supports** F_Δ.

For each vertex m_i , we set that m_i has multi-gedree mdeg(m_i)=m_i. We define that a face τ has multi-degree $mdeg(\tau) = lcm(m_i/m_i \in \tau)$. Note that $mdeg(\emptyset) = 1$ (by convention).

Theorem 4.25. *For each face τ of dimension i the complex F_Δ has the generator e_τ in homological degree $i+1$.*

1. *$mdeg(e_\tau) = mdeg(\tau)$.*
2. *The differential in F_Δ is $\delta(e_\tau)= \sum_{\tau' is \ a \ facet \ of \tau} [\tau,\tau']\frac{mdeg(\tau)}{mdeg(\tau')}e_{\tau'}$*
   $= \sum_{\tau' is \ a \ facet \ of \tau} [\tau,\tau']\frac{lcm(m_i|m_i\in\tau)}{lcm(m_i|m_i\in\tau')}e_{\tau'}$.

Proof. The second is direct from the first using the fact that the differential is multi-homogeneous.

$2 \Rightarrow 1)$: It will be proved by induction on homological degree. clearly, $mdeg(e_{m_i}) = m_i$ holds for each vertex m_i of Δ.

Since $\delta(e_\tau)= \sum_{\tau' is \ a \ facet \ of \tau} [\tau,\tau']e_{\tau'}$, by definition 4.8 it follows that:
mdeg(e_τ)= lcm{ mdeg($e_{\tau'}$)| τ' is a facet of τ}

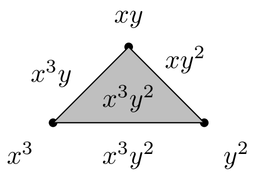
= lcm{ mdeg(τ')| τ' is a facet of τ}

= lcm{lcm{$m_i \mid m_i \in \tau'$} | τ' is a facet of τ}

= lcm{$m_i \mid m_i \in \tau$}= mdeg(τ).

□

Example 4.26. *Let $I = (x^3, xy, y^2)$, let us take the Taylor complex that is supported on the whole* *simplex. Consider the* Δ *with vertices* $x^3, xy,$ *and* y^2 *that are monomials generating the ideal* I *itself. We label each edge by the Lcm of it vertices. Hence, we get* x^3y, xy^2, x^3y^2 *on the edges.*



Now, the chain complex of k*-vector spaces is :*

$$0 \longrightarrow k \xrightarrow{\begin{pmatrix}1\\1\\1\end{pmatrix}} k^3 \xrightarrow{\begin{pmatrix}1 & 0 & -1\\-1 & 1 & 0\\0 & -1 & 1\end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix}1 & 1 & 1\end{pmatrix}} k$$

Where the sign of entries comes from the orientation. Due to homogenization of the above complex, we obtain the Taylor resolution of R/I *as follow:*

$$\mathbb{T}_{\mathbb{M}}\colon 0 \longrightarrow R(x^3y^2) \xrightarrow{\begin{pmatrix}y\\x^2\\1\end{pmatrix}} R(x^3y) \oplus R(xy^2) \oplus R(x^3y^2) \xrightarrow{\begin{pmatrix}-y & 0 & y^2\\x^2 & -y & 0\\0 & x & -x^3\end{pmatrix}}$$

$$R(x^3) \oplus R(xy) \oplus R(y^2) \xrightarrow{\begin{pmatrix}x^3 & xy & y^2\end{pmatrix}} R.$$

Example 4.27. *Let* $I = (xy, yz, zu)$ *in* $R = [x, y, z, u]$*. The labeled simplicial complex of vertices* xy, yz *and* zu *and edges* xyz, yzu *supports a free resolution of* I*. The chain complex of* Δ *is:*

$$0 \to k^2 \xrightarrow{\begin{pmatrix}1 & 0\\-1 & 1\\0 & -1\end{pmatrix}} k^3 \xrightarrow{\begin{pmatrix}1 & 1 & 1\end{pmatrix}} k$$

and homogenization is the simplicial resolution :

$$0 \to R(xyz) \bigoplus R(yzu) \xrightarrow{\begin{pmatrix}z & 0\\-x & u\\0 & -y\end{pmatrix}} R(xy) \bigoplus R(yz) \bigoplus R(zu) \to I \to 0$$

Another type of simplicial resolutions are called Lyubeznik Resolutions. We have seen above that the "Taylor Resolution" is not minimal in most cases, however the "Lyubeznik Resolution" is a resolution that is smaller than Taylor's.

Definition 4.28. Let I be an ideal generated by $M = \{m_1, \cdots, m_s\}$ and fix an ordering τ (does not depend on any property) to the monomial set in M such that $m_i \ \tau \ m_j$ for $i < j$. consider $\Delta_{I,\tau}$ be its simplex and μ be a monomial of I. Let $min(\mu) = min\{m_i, m_i \text{ divides } \mu\}$; for any face $F \in \Delta_{I,\tau}$, $min(F) = min(mdeg(F))$ but it is not always in F, instance:

Example 4.29. *Let $I = (a^2, ab, b^3)$ be an ideal and Let $F = \{a^2, b^3\}$ be a face. Set the order to be $ab\tau a^2\tau b^3$. We could have $min(F) = min(a^2, b^3) = ab, a$ or b.*

We say that a face F is rooted if every non-empty sub-face $G \subset F$ satisfies $min(G) \in G$.

Example 4.30. *By referring to 4.29, F is not rooted since $ab \notin F$.*

By construction, the set $\Lambda_{I,\tau} = \{\text{F} \in \Delta_I, \text{ F is rooted}\}$ is a simplicial comlpex which forms the "Lyubeznik Resolution".

Example 4.31. *Let $I = (a^2, ab, b^3)$ be a ideal , the lyubeznik resolution of I arises from the order $ab \ \tau \ a^2 \ \tau \ b^3$ is the following one :*

$$\mathbb{R}_{\mathbb{I}} : 0 \longrightarrow R(a^2, ab) \bigoplus R(ab, b^3) \xrightarrow{\begin{pmatrix} -b & 0 \\ a & -b^2 \\ 0 & a \end{pmatrix}} R(a^2) \bigoplus R(ab) \bigoplus R(b^3) \xrightarrow{\begin{pmatrix} a^2 & ab & b^3 \end{pmatrix}} I \quad (4.5)$$

note that $lcm(a^2, ab, b^3) = lcm(a^2, b^3) = ab$ which is not in $\{a^2, b^3\}$, hence (a^2, ab, b^3) and (a^2, b^3) removed as they are not rooted.

CHAPTER 5

SIMPLICIAL RESOLUTION OF I^2

5.0.1. *Quasi-Tree*

A simplicial complex can be uniquely determined by its facets, and we use the notation $\Delta = \{F_0, \cdots, F_q\}$ in order to describe a simplicial complex whose facets are $F_0, \cdots, F_q$.

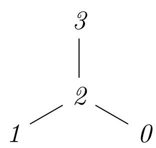
Definition 5.1. Suppose $G \subset V$ where V is a vertex set, we define the **induced subcomplex** of Δ on G denoted by Δ_G as follow: $\Delta_G = \{F \in \Delta / F \subset G\}$. Δ_G is the simplicial complex on G.

Remark 5.2. *A* ***subcollection*** *of* Δ_G *is a simplicial complex whose facets are also facets of* Δ.

Definition 5.3. The **dimension of a simplicial complex** Δ is dim(Δ)= max$\{dim(F)/F \in \Delta\}$. Thus, the set of vertices of Δ has dimension 0 while $\emptyset$ has dimension -1 (by convention).

Definition 5.4. A **leaf** of Δ is either the only facet F of Δ or the facet F such that if G is another facet G of Δ, which called the **joint** of G such that $F \cap H \subseteq G$ for every facet $H \neq F$.

Example 5.5.



(5.1)

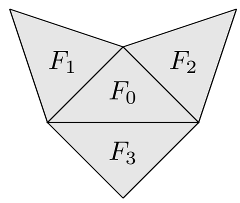
The facets are $F_1 = \{1, 2\}, F_2 = \{2, 3\}, F_3 = \{0, 2\}$. *Here every facet is a leaf with any other facet can be a joint, because the intersection is the vertex* 2 *which is common in all facets.*

Definition 5.6. We say a simplicial complex Δ is a **simplicial forest** if every non-empty sub-collection of Δ has a **leaf**. Also, it can be **connected** if $\forall\ v_i, v_j \in V$, $\exists$ a sequence of faces $F_0, \ldots, F_k$ such that $v_i \in F_k$ and $F_i \cap F_{i+1} \neq \emptyset\ \forall i = 0, \ldots, k-1$. A connected simplicial forest is called **a simplicial tree**.

Remark 5.7. *In case of simplicial tree, we can order the facets $F_1, \dots, F_q$ of Δ in a way that every F_i is a leaf of the induced sub-collection $(F_1, \cdots, F_i)$. This ordering is called a* ***leaf order****.*

Definition 5.8. We say a simplicial complex Δ is a **quasi-forest** if it has a **leaf order**. A connected quasi-forest is called a **quasi-tree**.

Example 5.9. *The simplicial complex below is a quasi-tree , with leaf order: F_0, F_1, F_2, F_3 meaning that each F_i is a leaf of $\langle F_0, \dots, F_i \rangle$. Thus, F_0 is the joint of F_i for all $i \geq 1$.*



5.0.2. *Simplicial Resolution of* I^2

As we mentioned before, **Taylor resolution** is usually far from minimal. However, if I is a monomial ideal with a free resolution supported on a simplicial complex Δ, then Δ is a sub-complex of Taylor(I).

Definition 5.10. Let Δ be I-complex and $m \in I$ be a monomial. We denote by $\Delta(\leq m)$ or Δ_m the sub-complex of Δ that is generated by the homogeneous elements of multi-degree dividing m. Recall that we let $lcm(I)$ denote the set of monomials that are least common multiples of arbitrary subsets of the minimal monomial generating set of I.

We introduce this theorem without proof.

Theorem 5.11. *(Criterion for quasi-trees supporting resolution)*

Let Δ be a quasi-tree whose vertices are labeled with the monomial generating set of a monomial ideal I in the polynomial ring R over a field K. Then Δ supports resolution of I if and only if for every monomial $\mathbf{m}$ in $lcm(I)$. Δ_m is empty or connected.

From now on, our goal is to study the free resolution of I^2 which was inspired by Lyubeznik. Suppose that I is an ideal minimally generated by q monomials, then I^2 is minimally generated by at most $\binom{q+1}{2}$ monomials. Thus, our goal, is to find a smaller sub-complex of the $\binom{q+1}{2}$- simplex which produces a free resolution of I^2 and depends only on q.

Definition 5.12. For an integer $q \geq 3$, the simplicial comlpex ${L_q}^2$ which is an induced sub-complex of the $\binom{q+1}{2}$-simplex over the vertex set $\{\ell_{i,j}, 1 \leq i \leq j \leq q\}$ is defined by the facets as ${L_q}^2 = (\{\ell_{i,j}, 1 \leq j \leq q\}_{1 \leq i \leq q}, \{\ell_{i,j}; 1 \leq i \lneq j \leq q\})$ where $\ell_{j,i} = \ell_{i,j}$ for $j \geq i$. for $q = 1$, $\{\ell_{i,j}; 1 \leq i \leq j \leq q\}$ is empty and is a face for $q = 2$ (but not a facet). Thus, ${L_1}^2$ is a point and ${L_2}^2$ is a complex with only 2 facets.

Example 5.13. *For $q = 3$, the simplicial complex L_3^2 is defined by the facets $\langle F_0, F_1, F_2, F_3 \rangle$ where $F_1 = \{\ell_{1,1}, \ell_{1,2}, \ell_{1,3}\}$, $F_2 = \{\ell_{2,2}, \ell_{1,2}, \ell_{3,2}\}$, $F_3 = \{\ell_{3,3}, \ell_{1,3}, \ell_{2,3}\}$, and the joint $F_0 = \{\ell_{1,2}, \ell_{1,3}, \ell_{2,3}\}$.*

Remark 5.14. *${L_q}^2$ has $\binom{q+1}{2}$ vertices (the same number of vertices of the $\binom{q+1}{2}$ simplex), and $q + 1$ facets when $q \geq 2$. The dimension of facets is $q - 1$ except only one for facet which has dimension $\binom{q}{2} - 1$.*

Example 5.15. *Referring to 5.13, L_3^2 is of 4 facets and each of dimension 3.*

Proposition 5.16. *For $q \geq 1$, ${L_q}^2$ is a quasi-tree.*

Proof. For $q = 1$, ${L_1}^2$ is a point thus it is trivial a quasi-tree. For $q = 2$, there are two facets F_1 and F_2, where F_2 is a leaf of (F_1, F_2) and F_1 joint; so ${L_2}^2$ is a quasi-tree. Now, for $q \geq 3$, order the facets of ${L_q}^2$ by $F_0 = \{\ell_{i,j}, 1 \leq i \lneq j \leq q\}$ and $F_i = \{\ell_{i,j}; 1 \leq j \leq q\}_{1 \leq i \leq q}$. For $i \neq k$, fix $F_k = \{\ell_{i,k}\} \subset F_0$ by definition. Hence, each F_i is a leaf of $(F_0, \ldots, F_i)$ with joint F_0. Therefore, ${L_q}^2$ is a quasi-tree. □

Given a square-free monomial ideal I, we define a labeled induced sub-complex of ${L_q}^2$, denoted by $L^2(I)$, which is obtained by deleting vertices from ${L_q}^2$.

Definition 5.17. For an ideal I minimally generated by the square-free monomials $m_1, \ldots, m_q$. Define $L^2(I)$ to be a labeled induced sub-complex of ${L_q}^2$ formed by the following rules:

1. Label each vertex of $\ell_{i,j}$ of ${L_q}^2$ with the monomial $m_i m_j$.

2. If for any indices $i, j, u, v \in \{1, \ldots, q\}$ with $i, j \neq u, v$. We have $m_i m_j \mid m_u m_v$, then

   - If $m_i m_j = m_u m_v$ and $i = min\{i, j, u, v\}$then delete the vertex $\ell_{i,j}$.
   - If $m_i m_j \neq m_u m_v$, then delete the vertex $\ell_{u,v}$.

3. Label each of the remaining faces with the least common multiple of the labels of its vertices.

The remaining labeled sub-complex of ${L_q}^2$ is called $L^2(I)$ and is a complex of **Taylor(I^2)**.

Before stating our main result, we exhibit the following two proposition in [2] that are essential to our main theorem. We omit the proof as both propositions are technical.

Proposition 5.18. *Suppose I is an ideal with minimal square-free generators $m_1, \ldots, m_q$. It says that for any positive number r, ${m_i}^r$ does not divide other monomial (by its minimality). That is; if ${m_i}^r \mid m_{w_1} \ldots m_{w_r}$ or $m_{w_1} \ldots m_{w_q} \mid {m_i}^r$ for some $i \in \{1, \ldots, q\}$ and $1 \leq w_1 \leq \ldots \leq w_r \leq q$. Then, $w_1 = w_2 = \ldots = w_r = i$ which means that $\ell_{i,i}$ belong to $L^2(I)$.*

Proposition 5.19. *Let I be an ideal with minimal generators $m_1, \ldots, m_q$ for $q \geq 2$. For any $i \in \{1, \ldots, q\}$ there is $j \neq i \in \{1, \ldots, q\}$ such that $m_u m_v$ does ot divide $m_i m_j$ for all $\{u, v\} \neq \{i, j\}$ in $\{1, \ldots, q\}$.*

We now get to the main result.

Theorem 5.20. *$L^2(I)$ is a free resolution of I^2 where I is a square -free monomial ideal.*

Proof. Let $m_1, \ldots, m_q$ be the minimal square free generators of an ideal I. As we mentioned above, ${L_q}^2$ is a quasi-tree and since $L^2(I)$ is an induced sub-complex by definition, hence it is a quasi-forest itself. Let W represents the set of vertices of $L^2(I)$. If we prove that ${L_m}^2(I)$ is connected for any monomial m in lcm(I^2), then directly by proposition 5.16 $L^2(I)$ is a free resolution of I^2.

Note that ${L_m}^2(I)$ is the induced sub-complex of the complex $L^2(I)$ on the set $W_m = \{\ell_{i,j} \in W; m_i m_j \mid m\}$.

If $q = 1$ and $m \in$ lcm(I^2), so ${L_m}^2(I)$ is either a point or empty, hence connected.

If $q = 2$ then $I^2 = ({m_1}^2, m_1m_2, {m_2}^2)$, in this case $L^2(I)$ has only two facets joined by the vertex $\ell_{1,2}$, hence connected. If $m \in \{{m_1}^2, {m_2}^2\}$, then ${L_m}^2(I)$ is a point and then connected. Otherwise, $m_1m_2 \mid m$ and then $\ell_{1,2}$ will be in ${L_m}^2(I)$. In all cases, $\ell_{1,2}$ will connect the vertices.

Now, for $q \geq 3$, order the facets of ${L_q}^2$ by $F_0, \dots F_q$. Thus, the maximal sets among the sets $F_0 \cap W_m, \dots, F_q \cap W_m$ are the facets of ${L_m}^2(I)$. If $m = {m_i}^2$ for some $i \in \{1, \dots, q\}$ by proposition 4.27, ${L_m}^2(I)$ is a point and hence connected. If $m \neq {m_i}^2$ for all $i \in \{1, \dots, q\}$, it means that $F_0 \cap W_m \neq \emptyset$ as F_0 is a joint. We need to show that ${L_m}^2(I)$ is connected that is there is an intersection between $F_0 \cap W_m$ and $F_i \cap W_m$ for all $i \in \{1, \dots, q\}$. We know that any vertex in $F_i \cap W_m$ other than $\ell_{i,i}$ is in $F_0 \cap W_m$. Thus, we need to prove that if $\ell_{i,i}$ belong to W_m for some $i \in \{1, \dots, q\}$, there is $b \neq i$ belong to $\{1, \dots, q\}$ such that $\ell_{i,b} \in W_m$.

To prove this, we will follow proposition 5.19 . Suppose $\ell_{i,i} \in W_m$ thus ${m_i}^2 \mid m$. Consider $A = \{j \in [q]; m_j \mid m\}$; by construction $i \in A$ and $m \neq {m_i}^2$, thus by proposition 5.19 there is $b \neq i$ belong to A such that m_um_v does not divide m_im_j for all $u, v \in A \mid \{i, b\}$. since $b \in A$ then $m_b \mid m$. Our goal is to show that $m_im_b \mid m$ and $\ell_{i,b} \in W$. set $m = {m_i}^2n$

- $m_b \mid m$ then $m_b \mid {m_i}^2n$ then $m_b \mid m_in \to m_im_b \mid {m_i}^2n \to m_im_b \mid m$
- $\ell_{i,b} \in W$, thus we should have $m_um_v \mid m_im_b$ for some $u, v \in [q]$ except $\{i, b\}$. Since $m_im_b \mid m$ so $m_um_v \mid m_im_b \mid m$ thus $m_u \mid m$ and $m_v \mid m$ so $u, v \in A$ which contradicts with above.

□

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