AMERICAN UNIVERSITY OF BEIRUT

A METHOD FOR STUDYING ZEROS OF PARTIAL SUMS OF SOME ENTIRE FUNCTIONS

by

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ABSTRACT
OF THE THESIS

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We present in this paper a method of Edrei for studying the distribution of zeros of partial sums of entire power series. We then apply it in a class of L-functions of order less than one, with prescribed asymptotic behavior of the maximum modulus of the function.

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1. Introduction

Let F(z)=$\sum_{n=0}^{\infty} a_n z^n$ be a given entire function and let $S_m(z) = \sum_{k=0}^{m} a_k z^k$ be a partial sum or section .
In this thesis , we study the distribution of zeros of the normalized sections $S_m(R_m w)$ for large values of m (w is a complex number such that $|w| < 1$ and $\{R_m\}_m$ is a sequence of positive real numbers to be determined later).

Historically, the first such study involved the partial sums of the exponential function by Szego, Iverson, and later by Neumann and Rivlin who showed the existence of a parabolic region in the complex plane $\mathbb{C}$, free of all zeros of all the sections of exp(z).

In this thesis, we follow a method of Edrei to study normalized sections of L-functions, of order less than one , satisfying the asymptotic condition :

$$\log M(r) \sim B_1 r^{\lambda} \log r$$

with M(r)=$max_{|z|=r}|F(z)|$ and B_1 a positive number.

The method involves obtaining a connection between the normalized sections and an error function using a major result of Hayman on the so called "admissible functions". The L-functions under consideration are first shown to be admissible. This, in its turn gives precise information about the Maclaurin coefficients a_n, which in turn leads to information about the sections, and ultimately about the zero distribution of the normalized sections.

The main result in this thesis is the following theorem due to Edrei, Saff and Varga :

$$(1 + (\tfrac{2}{\lambda m})^{\frac{1}{2}}\zeta)^{-m}\{F(R_m)\}^{-1} S_m(R_m(1 + (\tfrac{2}{\lambda m})^{\frac{1}{2}}\zeta)) \xrightarrow{m\to\infty} \tfrac{1}{2} e^{\zeta^2} erfc(\zeta)$$

uniformly on every compact set of the $\zeta-$ plane.

Because of the technical nature of this result, we shall break up the proof into sections, where each section will carry the main result of that part, with the technical details left to a later appendix.

1.2. Definitions

1.2.1. *Definition 1*

Let F be an entire function. For $r > 0$, the maximum modulus M(r) of F, is defined by:

$$M(r) = max_{|z|=r}|F(z)|.$$

It can be shown that the function $M(r)$ is analytic except at isolated points so that we can define the two functions a(r) and b(r) by:

$$a(r) = r\frac{M'(r)}{M(r)} \ , b(r) = ra'(r).$$

The order of F is a non negative real number λ given by:

$$\lambda = limsup \ _{r\to\infty}(\ \frac{\log(\log M(r))}{\log r}).$$

Later on, it will turn out that M(r)=F(r) and this will make it possible to define $a(z)$ and $b(z)$ as functions of the complex variable z by setting:

$$a(z) = z\frac{F'(z)}{F(z)} \ , b(z) = za'(z).$$

1.2.2. *Definition 2*

An entire function F is called an L-function of genus zero if :

1. Its order is less than 1.
2. All its zeros are real and negative, so that it can be expressed as:

   $$\text{F(z)=F(0)} \prod_{k=1}^{\infty}(1 + \frac{z}{x_k}) = \Sigma_{j=0}^{\infty}a_j z^j \ ,$$

   where $x_k > 0$,$F(0) > 0$,$\Sigma_{k=1}^{\infty}\frac{1}{x_k} < \infty$.
3. $\log M(r) \sim B_1 r^{\lambda} \log r$, with B_1 a positive constant.

Note that the asymptotic relation $\log M(r) \sim B_1 r^{\lambda} \log r$ can be replaced by a different asymptotic relation such as:

$$B_1 r^{\lambda}, B_1 r^{\lambda}(\log r)^{\alpha}(\alpha \text{ real }) \ , \ \ B_1 r^{\lambda}(\log r)^{\alpha_1}(\log_2 r)^{\alpha_2}...(\log_k r)^{\alpha_k}$$

where α_i's are real and $\log_j r$ denotes the iterated logarithm $\log_j r = \log(\log_{j-1} r)$.

We also define the counting function of the zeros of F by : $n(t) = \sum_{x_k \le t} 1 \ \ (t > 0)$.

1.2.3. *Definition 3*

[3] An entire function f is said to be **Admissible in the sense of Hayman** if :

1. f(r) is real and strictly positive for $(r > r_0 > 0)$.
2. b(r) $\rightarrow \infty$ as $r \rightarrow \infty$.
3. There exists some function $\delta(r)$ defined for $r > r_0$ such that $0 < \delta(r) < \pi$, and

   $$f(re^{i\theta}) \sim f(r)\ exp(i\theta a(r) - \frac{1}{2}\theta^2 b(r))\ \ (r \rightarrow \infty)$$

   uniformly for $|\theta| \leq \delta(r)$, and

   $$f(re^{i\theta}) = o(f(r))(b(r))^{-1/2}\ \ (r \rightarrow \infty)$$

   uniformly for $\delta(r) \leq |\theta| \leq \pi$.

We recall that the complementary error function $erfc$ is defined by:

$$erfc(\zeta) = \frac{2}{\sqrt{\pi}} \int_{\zeta}^{\infty} e^{-t^2} dt$$

where ζ is a complex number .

We note that throughout the thesis, $\eta(z)$ will denote a new error function each time. However, the reader will be informed when η is changed.

CHAPTER 2

ASYMPTOTICS

In this chapter, we study the asymptotic behavior of n(t) , $logF(z)$, a(z) and b(z) using only the information given about the asymptotic behavior of $\log M(r)$.
The functions a(z) and b(z) will be needed to approximate the Maclaurin coefficients of $S_n(z)$.
Through out our work , we will restrict ourselves to the sector Δ which is defined to be:

$$\Delta(\varepsilon_1) = \{z = re^{i\theta}; |\theta| \leq \pi - \varepsilon_1, r > 0\}$$

where $0 < \varepsilon_1 < \pi$ and ε_1 is otherwise arbitrary .
We notice that on Δ ,

$$|t+z| = |te^{-i\frac{\theta}{2}} + re^{i\frac{\theta}{2}}| \geq (t+r)cos(\tfrac{\theta}{2}) \geq (t+r)sin(\tfrac{\varepsilon_1}{2}) = (t+r)\gamma_1.$$

We shall need this inequality to bound some special integrals.

2.1. Approximating $\log F(z)$

Let F be an L-function, then by Valiron's formula [2]

$$\log F(z) = z\int_0^\infty \frac{n(t)}{t(t+z)},$$

Since n(t)=0 for $t < x_1$ where $-x_1$ is the first zero of F, then

$$\log F(z) = z\int_{x_1}^\infty \frac{n(t)}{t(t+z)}.$$

Since $a_j > 0$, $\log F(r) = \log M(r)$ and therefore $\log F(r) \sim B_1 r^\lambda \log r$.

It then follows from Valiron's tauberian theorem [5] that :

$$n(r) \sim \gamma r^\lambda \log r \ (r \to \infty) \ ; \ \gamma = \frac{B_1 sin(\pi\lambda)}{\pi}.$$

Therefore , given ε such that $0 < \varepsilon < \frac{1}{2}$, exists $r_0(\varepsilon) = r_0 > 1 + x_1$ such that $r > r_0$ implies

$$n(r) = \gamma r^\lambda \log(r)(1 + \eta_1(r)) \text{ with } |\eta_1(r)| < \varepsilon.$$

We will now prove that:

$$\log F(z) = B_1 z^\lambda \log z + E(z) = B_1 z^\lambda \log z\ (1 + \eta(z))$$

with $|\eta(z)| = |\frac{E(z)}{B_1 z^\lambda log(z)}| \to 0$ uniformly as $z \to \infty$.

Indeed, $\log F(z) = z \int_{x_1}^{\infty} \frac{n(t)-\gamma t^\lambda \log(t)+\gamma t^\lambda \log(t)}{t(t+z)} dt$

$= z \int_{x_1}^{\infty} \frac{n(t)-\gamma t^\lambda \log(t)}{t(t+z)} dt + \gamma\ [z \int_0^\infty \frac{t^{\lambda-1}\log(t)}{(t+z)} dt - z \int_0^{x_1} \frac{t^{\lambda-1}\log(t)}{(t+z)} dt] =$

$= z \int_{x_1}^{r_0} \frac{n(t)-\gamma t^\lambda \log(t)}{t(t+z)} dt + z \int_{r_0}^{\infty} \frac{\gamma t^\lambda \log(t)\eta_1(t)}{t(t+z)} dt + \gamma c_1 z^\lambda + B_1 z^\lambda \log z - \gamma z \int_0^{x_1} \frac{t^{\lambda-1}\log(t)}{(t+z)} dt$

$= B_1 z^\lambda \log z + E(z)$

with

$|E(z)| < |z \int_{x_1}^{r_0} \frac{n(t)}{t(t+z)} dt| + |\gamma z \int_{x_1}^{r_0} \frac{t^{\lambda-1}\log(t)}{(t+z)} dt| + |\gamma \varepsilon z \int_{r_0}^{\infty} \frac{t^{\lambda-1} log(t)}{(t+z)} dt| + \gamma c_1 |z^\lambda| + \gamma c_2$

$< n(r_0)\log(\frac{r_0}{x_1}) + n(r_0)|\log(\frac{x_1+z}{r_0+z})| + \gamma c_3 |z^\lambda| + \gamma\varepsilon |z \int_0^\infty \frac{t^{\lambda-1} log(t)}{(t+z)} dt| + \gamma c_1 |z^\lambda| + \gamma c_2$

(c_1, c_2 and c_3 are positive constants illustrated in the appendix)

As $z \to \infty$ and then as $\varepsilon \to 0$, $|\frac{E(z)}{B_1 z^\lambda log(z)}| \to 0$ uniformly.

2.2. Approximating a(z)

$a(z) = z\frac{F'(z)}{F(z)} = z\frac{d}{dz}(log(F(z)) = z\frac{d}{dz}[z \int_0^\infty \frac{n(t)}{t(t+z)} dt] = z \int_0^\infty \frac{n(t)}{(t+z)^2} dt$

$= z \int_0^\infty \frac{n(t)-\gamma t^\lambda \log(t)+\gamma t^\lambda \log(t)}{(t+z)^2} dt = z \int_0^{r_0} \frac{n(t)-\gamma t^\lambda \log(t)}{(t+z)^2} dt + z \int_{r_0}^{\infty} \frac{n(t)-\gamma t^\lambda \log(t)}{(t+z)^2} dt + z \int_0^\infty \frac{\gamma t^\lambda \log(t)}{(t+z)^2} dt$

However,

$|z \int_0^{r_0} \frac{n(t)-\gamma t^\lambda \log(t)}{(t+z)^2} dt| \le |z\gamma \int_0^{x_1} \frac{t^\lambda \log(t)}{(t+z)^2} dt| + |\frac{r}{\gamma_1{}^2} \int_{x_1}^{r_0} \frac{n(t)-\gamma t^\lambda \log(t)}{(t+r)^2} dt|$

$\le |z\gamma \int_0^{x_1} \frac{t^\lambda \log(t)}{(t+z)^2} dt| + |\frac{r}{\gamma_1{}^2}\ n(r_0) \int_{x_1}^{r_0} \frac{1}{(t+r)^2} dt| + |\frac{r}{\gamma_1{}^2}\gamma \int_{x_1}^{r_0} \frac{t^\lambda \log(t)}{(t+r)^2} dt|$

$\le \gamma c_4 + |\frac{r}{\gamma_1{}^2}\ n(r_0) \int_{x_1}^{r_0} \frac{1}{(t+r)^2} dt| + |\frac{r}{\gamma_1{}^2}\gamma \int_{x_1}^{r_0} \frac{t^\lambda \log(t)}{(t+r)^2} dt|$

where c_4 is a positive constant(illustrated in the appendix).

and

$|z \int_{r_0}^{\infty} \frac{n(t)-\gamma t^\lambda \log(t)}{(t+z)^2} dt| \le |\frac{r}{\gamma_1{}^2} \int_{r_0}^{\infty} \frac{n(t)-\gamma t^\lambda \log(t)}{(t+r)^2} dt| \le |\frac{r}{\gamma_1{}^2} \int_{r_0}^{\infty} \gamma\varepsilon t^{\lambda-2}\log(t) dt|$

$\le c_5 r$ where c_5 is a positive constant obtained from integration by parts and

$z\int_0^\infty \frac{\gamma t^\lambda \log(t)}{(t+z)^2}dt = B_1\lambda z^\lambda \log(z) + \gamma c_6 z^\lambda$

(c_6 is a positive constant illustrated in the appendix).

Therefore , $a(z) = B_1\lambda z^\lambda \log(z)(1+\eta(z))$; (new η)

with $|\eta(z)| \xrightarrow{z\to\infty} 0$ uniformly.

2.3. Approximating b(z)

$$a'(z) = (B_1\lambda^2 z^{\lambda-1}\log(z) + B_1\lambda z^{\lambda-1})(1+\eta(z)) + B_1\lambda z^\lambda \log(z)\ \eta'(z)$$
$$= B_1\lambda^2 z^{\lambda-1}\log(z)(1+\eta(z) + \frac{\lambda}{\log(z)} + \frac{\lambda\eta(z)}{\log(z))} + \lambda z\eta'(z))$$

However, $\eta(z) = \frac{1}{B_1\lambda z^\lambda \log(z)}(z\int_0^{r_0}\frac{n(t)-\gamma t^\lambda \log(t)}{(t+z)^2}dt + z\int_{r_0}^\infty \frac{n(t)-\gamma t^\lambda \log(t)}{(t+z)^2}dt + \gamma c_6 z^\lambda)$

Then, using basic calculation, we can deduce that $z\eta'(z) \to 0$ uniformly as $z\to\infty$.

Therefore, $b(z) = za'(z) = B_1\lambda^2 z^\lambda \log(z)(1+\eta(z))$ with $|\eta(z)|$ converges uniformly to zero as $z\to\infty$. Note that η here denotes a new error funciton.

CHAPTER 3

THE PARTIAL SUMS

In order to study the distribution of the zeros of the normalized partial sums $S_m(R_m w)$, we put :

$$Q_m(w) = \frac{S_m(R_m w)}{a_m {R_m}^m w^m} = \sum_{j=0}^{m} b_{-j}\,(m)\,w^{-j}$$

where w is a small complex number such that $\{w; |w| \leq 1 - \epsilon\ (0 < \varepsilon < 1)\}$ and $\{R_m\}_m$ is a sequence of postive real numbers that satisfy the equation $a(R_m) = m$.

We express Q_m as a difference,$Q_m(w) = U_m(w) - G_m(w)$ where

$$U_m\,(w) = \frac{F(R_m w)}{a_m {R_m}^m w^m} \quad w \neq 0 \qquad\qquad G_m\,(w) = \sum_{j=1}^{\infty} b_j\,(m)\,w^j$$

This chapter will be divided into two parts.In the first part,we estimate the radius R_m, and the coefficients a_j and b_j where $b_j(m) = \frac{a_{m+j}}{a_m} {R_m}^j\ (j \geq -m)$.
In the second part, we study the functions U_m and G_m.

3.1. Calculating $\{R_m\}_m$, a_j and b_j

3.1.1. *Calculating* $\{R_m\}_m$

By Hadamard Three Circle Theorem, $a(r)$ is a non decreasing function of r.Therefore, as $a(r)$ is non decreasing, non-negative, continuous and unbounded, there exists a uniquely defined, positive, increasing, unbounded sequence $\{R_m\}_m$ which satisfies the equation $a(R_m) = m$ with $m = 1, 2, ...$

$$a(R_m) = B_1\,\lambda\,{R_m}^{\lambda} \log(R_m)\,(\,1 + \eta\,(R_m)) = m$$

$$\Rightarrow e^{\log\left({R_m}^{\lambda}\right)} \log({R_m}^{\lambda}) = \frac{m}{B_1(\,1 + \eta(R_m))}$$

$$\Rightarrow R_m = \exp\left(\frac{1}{\lambda} W_0\left(\frac{m}{B_1(\,1 + \eta(R_m))}\right)\right),$$

where W_0 in the principal branch of the Lambert W function.

Also, we notice that $b(R_m) = \lambda\ m\ (1 + \eta_m\prime\,(R_m))$ where $\eta_m\prime(R_m) \xrightarrow{m\to\infty} 0$ uniformly.

3.1.2. *Calulating* a_j *and* b_j

Having obtained the asymptotic behavior of $a(z)$ and $b(z)$, we shall use these to prove that F is an admissible function, and thereby obtain the asymptotic behavior of a_j

3.1.2.1. Proposition 1

An L-function F of genus zero is admissible in the sense of Hayman.[4]

Proof. 1. F is entire and transcendental.

2. $F(r) = F(0)\prod_{k=1}^{\infty}(1+\frac{r}{x_k})$ is real and positive.

3. As$r \to \infty, m \to \infty$ since $m = B_1\ \lambda\ {R_m}^{\lambda} \log(R_m)(\ 1+\ \eta(R_m))$.
Therefore, $b(r) \to \infty$ as $r \to \infty$.

4. Proving $F(re^{i\theta}) \sim F(r)\ e^{(i\theta a(r)-\frac{1}{2}\theta^2 b(r))}$ is equivalent to proving that
$\log F(re^{i\theta}) \sim \log F(r) + i\theta a(r) - \frac{1}{2}\theta^2 b(r)$.
However, for all θ such that $|\theta| \le \pi$
$\log F(re^{i\theta}) \sim B_1 r^{\lambda} e^{i\lambda\theta} \log(re^{i\theta}) \sim B_1 r^{\lambda}(1 + i\lambda\theta - \frac{\lambda^2\theta^2}{2})(\log r + i\theta)$
$\sim B_1 r^{\lambda}\log r + iB_1 r^{\lambda}\lambda\theta\log r - \frac{\lambda^2}{2}\theta^2 B_1 r^{\lambda}\log r$
$\sim \log F(r) + i\theta a(r) - \frac{1}{2}\theta^2 b(r)$ uniformly $(r \to \infty)$

5. Proving $F(re^{i\theta}) = o(F(r))(b(r))^{-1/2}\ \ (r \to \infty)$ uniformly
is equivalent to proving that $\lim_{r\to\infty}\frac{F(re^{i\theta})(b(r))^{1/2}}{F(r)} = 0$
$\lim_{r\to\infty}\frac{F(re^{i\theta})(b(r))^{1/2}}{F(r)} = \lim_{r\to\infty}\frac{F(r)\ e^{(i\theta a(r)-\frac{1}{2}\theta^2 b(r))}}{F(r)} = 0$ uniformly for all θ such that $|\theta| \le \pi$

□

3.1.2.2. Theorem 1 (Hayman's fundamental Theorem) [4]

Let F(z)=$\sum_{n=0}^{\infty} a_n z^n$, be admissible and define $a_n = 0$ for $n < 0$.
Then as $r \to \infty$ we have uniformly for all integers n,

$$a_n r^n = \frac{F(r)}{2\pi b(r)^{\frac{1}{2}}}\{\ \exp[-\frac{(a(r)-n)^2}{2b(r)}] +\ \eta(r,n)\ \}\text{ with }\eta(r,n) \to 0.$$

Moreover , from this representation of $a_n r^n$, Hayman draws the following simple consequences which we will use later :[4]

$$\sum_{\ j\le a(r)} a_j r^j \sim \ \frac{1}{2}\ F(r)\text{ as }r \to \infty$$
$$\sum_{\ a(r)<j} a_j r^j \sim \ \frac{1}{2}\ F(r)\text{ as }r \to \infty.$$

Applying Hayman's theorem on our L-function F , and replacing r by R_m , we get that :

$$a_n {R_m}^n = \frac{F(R_m)}{\{2\pi b(R_m)\}^{\frac{1}{2}}}\{\exp\left[-\frac{(a(R_m)-n)^2}{2b(R_m\ )}\right]+\ \eta\,(R_m,n)\ \}.$$

However, noticing that $a(R_m) = m$ and $b(R_m) = \lambda m(1+\eta'(R_m))$,we get that :

$$b_j\,(m) = \frac{a_{m+j}{R_m}^{m+j}}{a_m {R_m}^m} = exp(\frac{-j^2}{2\lambda\ m}) + \tilde{\eta}_j\,(m) \quad \text{with} \quad \tilde{\eta}_j(m) \to 0 \text{ as } m \to \infty.$$

We also note that , $b_j\,(m) \to 1$ as $m \to \infty$ for every fixed j.

3.2. Studying U_m and G_m

3.2.1. *Studying* $G_m(w)$:

As $b_j\,(m) \to 1$, then $G_m(w)$ converges uniformly to $\frac{w}{1-w}$ $(m \to \infty)$.

Also,

$$|G_m\,(w)\,| \leq \frac{1}{a_m {R_m}^m} \sum_{j=m}^{\infty} a_j {R_m}^j$$

but

$$\sum_{a(r)<j} a_j r^j \sim\ \frac{1}{2}\ F(r) \text{ as } r \to \infty \qquad \text{and} \qquad a_m {R_m}^m \sim \frac{F(R_m)}{\{2\pi b(R_m)\}^{\frac{1}{2}}}$$

then,

$$|G_m(w)| \leq \{2\pi\lambda\ m\}^{\frac{1}{2}}\,(|w| \leq 1\ \ ,\ m > m_0\ ).$$

3.2.2. *Studying* $U_m(w)$

3.2.2.1. Step 1:

It can be proved that for a non-zero complex number z_0 such that $F(z) \neq 0$ throughout the disk $|z - z_0| \leq \frac{1}{2}|z_0|$,

$$log(F(z_0+s)) - log(F(z_0)) = s\ \frac{F'(z_0)}{F(z_0)} + \frac{s^2}{2}(\frac{F''(z_0)}{F(z_0)} - \frac{F'(z_0)}{[F(z_0)]^2}\ ) + E_3(z_0,s)$$

where s is a complex number such that $|s| \leq \frac{1}{4}|z_0|$ and

$|E_3(z_0, s)| \leq \frac{1}{2(1-2\eta)}\eta^{-2}|\frac{s}{z_0}|^3 max_\theta \ |a(z_0(1+2\eta e^{i\theta}))|$.

For our thesis, we set $z_0 = R_m$ and $s = (w-1)R_m$ .So,

$log(F(wR_m)) = log(F(R_m)) + (w-1)R_m \frac{F'(R_m)}{F(R_m)} + \frac{((w-1)R_m)^2}{2}(\frac{F''(R_m)}{F(R_m)} - [\frac{F'(R_m)}{F(R_m)}]^2).$

However, $b(R_m)$ is equal to $R_m a'(R_m) = \ R_m \ \frac{F'(R_m)}{F(R_m)} + R_m^2 \frac{F''(R_m)}{F(R_m)} - \ [R_m \frac{F'(R_m)}{F(R_m)}]^2$

on one hand, and to $\lambda \ m \ (1 + \eta_m' \ (R_m))$ on the other hand.

Therefore, $F(wR_m) = F(R_m)\, exp[(w-1)\, m + \frac{(w-1)^2}{2}(\lambda m(1+\eta_m'') - m) + E_3\,(R_m, (w-1)\,R_m)].$

with $|E_3(R_m, (w-1)R_m)| \leq c|\frac{(w-1)R_m}{R_m}|^3 \frac{2H(m)+1}{1+log(H(m))+1} log(2(\frac{2H(m)+1}{1+log(H(m)+1)})^{\frac{1}{\lambda}})$

where $c = \frac{B_1 \ \lambda \ 2^\lambda \eta^{-2}}{(1-2\eta)}$ $\qquad H(m) = \frac{m}{B_1(1+\ \eta(R_m))}.$

Taking $w = 1 + (\frac{2}{\lambda m})^{\frac{1}{2}}\zeta$, where ζ is a complex number such that $|\zeta| < B < \infty$ we get that $|E_3(R_m, (w-1)R_m)| \to 0$ uniformly as $m \to \infty$.

3.2.2.2. Step 2:

Using Taylor expansion on $\log(w)$

$$\frac{1}{w^m} = exp[-m \ log(w)] = exp[m\,(1-w) + m\frac{1}{2}\,(1-w)^2 - m\omega A\,(1-w)^3].$$

where ω is a complex number that satisfies $|\omega| < 1$, and A is a positive contant.

Moreover,

$$a_m {R_m}^m = \frac{F(R_m)}{[2\pi b(R_m)]^{\frac{1}{2}}}(1 + \ \eta\,(R_m, m)) = \frac{F(R_m)}{[2\pi\lambda m(1+\eta_m(R_m))]^{\frac{1}{2}}}(1 + \ \eta\,(R_m, m)).$$

The results of the above two steps give that :

$$U_m(w) = \{2\pi\lambda m\}^{\frac{1}{2}}(1+\eta_m)\, exp[\frac{(w-1)^2}{2}(\lambda m\left(1 + \eta_m''\right)) - m\omega A\,(1-w)^3 + E_3\,(R_m, (w-1)\,R_m)]$$

$$= \{2\pi\lambda m\}^{\frac{1}{2}}\,(1+\eta_m)\ exp[\frac{(w-1)^2}{2}\ (\lambda m\left(1 + \eta_m''\right))\ + E_4\,(R_m, (w-1)\,R_m)].$$

Taking $w = 1 + (\frac{2}{\lambda m})^{\frac{1}{2}}\zeta$, $|E_4(R_m, (w-1)R_m)| \to 0$ uniformly as $m \to \infty$.

Then , finally , $U_m(1 + (\frac{2}{\lambda m})^{\frac{1}{2}}\zeta) \to \{2\pi\lambda m\}^{\frac{1}{2}} exp[\zeta^2]$ as $m \to \infty$.

Note that to prove our main theorem, we will only be concerned with w of the form $1 + (\frac{2}{\lambda m})^{\frac{1}{2}}\zeta$.

However, a straightforward consequence of the Enestrom-Kakeya theorem tells that $Q_m(w)$ has no zeros in the annulus $|w| \geq 1$ .Therefore, we are allowed to choose ζ such that $|w| = |1+(\frac{2}{\lambda m})^{\frac{1}{2}}\zeta| < 1$. Also note that we can choose ζ such that $|\zeta| < B$ (B a positive arbitrary bound) .

CHAPTER 4

THE MAIN THEOREM

This theorem describes a more precise behavior of our normalized partial section $S_m(R_m w)$ in terms of the complementary error function $erfc(\zeta)$.

4.1. The Main Theorem

Given an L-function F , such that $log(M(r)) \sim B_1 r^{\lambda} log(r)$, the behavior for large values of m of the normalized section S_m , is described as follows :

$$(1 + (\tfrac{2}{\lambda m})^{\frac{1}{2}}\zeta)^{-m}\{F(R_m)\}^{-1} S_m(R_m(1 + (\tfrac{2}{\lambda m})^{\frac{1}{2}}\zeta)) \xrightarrow{m\to\infty} \tfrac{1}{2} e^{\zeta^2} erfc(\zeta)$$

uniformly on every compact set of the $\zeta-$ plane .

Proof. For easier representation , we define $\Omega_m(\zeta)$ to be :

$$\Omega_m(\zeta) = (1 + (\tfrac{2}{\lambda m})^{\frac{1}{2}}\zeta)^{-m}\{F(R_m)\}^{-1} S_m(R_m(1 + (\tfrac{2}{\lambda m})^{\frac{1}{2}}\zeta))$$

In view of Vitali's theorem it will be sufficient to establish the value of the limit for ζ real, positive and $1 \le \zeta = x \le B$.

4.1.1. *Step 1:*

In the previous chapter,we proved that $|G_m(z)| \le (\pi\lambda m)^{\frac{1}{2}} \quad (|w| \le 1, m > m_0)$

Since $Q_m(z) = U_m(z) - G_m(z)$, then :

$$(2\pi\lambda m)^{-\frac{1}{2}}|Q_m(1+(\tfrac{2}{\lambda m})^{\frac{1}{2}}\zeta)| \le (2\pi\lambda m)^{-\frac{1}{2}}[|U_m(1+(\tfrac{2}{\lambda m})^{\frac{1}{2}}\zeta)|+|G_m(1+(\tfrac{2}{\lambda m})^{\frac{1}{2}}\zeta)|]$$

$$\le 2e^{|\zeta^2|} + \tfrac{\sqrt{2}}{2} \le 2(exp(B^2) + 1)$$

Noticing that

$$(2\pi\lambda m)^{-\frac{1}{2}} Q_m(1 + (\tfrac{2}{\lambda m})^{\frac{1}{2}}\zeta) = \Omega_m(\zeta)$$

we can deduce that

$\Omega_m(\zeta)$ are uniformly bounded on every compact set of the ζ_- plane .

4.1.2. *Step 2:*

$(2\pi\lambda m)^{-\frac{1}{2}}[e^{x^2}\sum_{j=0}^{L(m)} exp\{-(x+\frac{j}{(2\lambda m)^{\frac{1}{2}}})^2\} - Q_m(1+(\frac{2}{\lambda m})^{\frac{1}{2}}x)] \xrightarrow{m\to\infty} 0$

a) We introduce the integer

$$L(m) = [\frac{m^{\frac{1}{2}}}{|\eta_m'|^{\frac{1}{3}}+log(m)^{-1}}]$$

where η_m' is defined in the subsection 3.1.1

We observe that

$$L(m)m^{\frac{-1}{2}} = \frac{1}{|\eta_m'|^{\frac{1}{3}}+log(m)^{-1}} = H_m \longrightarrow \infty \quad (m\to\infty)$$

and

$$L^2(m)|\eta_m''| = [\frac{m}{(|\eta_m''|^{\frac{1}{3}}+log(m)^{-1})^2}]|\eta_m''| \le [\frac{m}{|\eta_m''|^{\frac{2}{3}}}]|\eta_m''| = m|\eta_m''|^{\frac{1}{3}}.$$

b) We now estimate with some precision the formula of the coefficients $b_{-j}(m)$ and their behavior as $m\to\infty$.

Using the approximation of b_j in chapter 3, we obtain that

$$b_{-j}(m) = [exp(\frac{-j^2}{2\lambda\ m}) + \tilde{\eta}_{-j}(m) + \ \eta(R_m, m-j)][1+\eta(R_m,m)]^{-1}$$

with $|\tilde{\eta}_{-j}(m)| < |exp(\frac{j^2\eta''(R_m)}{2\lambda m\ \ (1+\eta''(R_m))}) - 1|$.

It can be proved now that for $|j| < L(m)$

$$b_{-j}(m) = exp(\frac{-j^2}{2\lambda m}) + \beta_m(-j)$$

where uniformly in j ,

$$|\beta_m(-j)| < \beta_m' \ ,\quad \beta_m' \xrightarrow{m\to\infty} 0.$$

and for $|j| > L(m)$,

$$0 < b_{-j}(m) < [exp(\frac{-H_m^2}{2\lambda}) + \tilde{\eta}_{-j}(m) + \eta(R_m, m-j)][1+\eta(R_m,m)]^{-1} < \beta_m{''}$$

$$\text{with } \beta_m{''} \xrightarrow{m\to\infty} 0.$$

From the above two cases , and noticing that $L(m) < m$ for large values of m , we can deduce that :

$$|Q_m(w) - \sum_{j=0}^{L(m)} exp(\frac{-j^2}{2\lambda m})w^{-j}| = |\sum_{j=0}^{m} b_{-j}(m)w^{-j} - \sum_{j=0}^{L(m)} exp(\frac{-j^2}{2\lambda m})w^{-j}|$$

$$= |\sum_{j=0}^{L(m)} [b_{-j}(m) - exp(\frac{-j^2}{2\lambda m})]\ w^{-j} + \sum_{j=L(m)}^{m}\ b_{-j}(m)\ w^{-j}|$$

$$\leq |\sum_{j=0}^{L(m)} \beta_m{'} + \sum_{j=L(m)}^{m} \beta_m{''}| \leq (\beta_m{'} + \beta_m{''})\frac{|w|}{|w|-1} \qquad (|w| > 1)\ .$$

c) Letting $w = 1 + (\frac{2}{\lambda m})^{\frac{1}{2}}x > 1$, we get that :

$$(2\pi\lambda m)^{\frac{-1}{2}} Q_m(1 + (\frac{2}{\lambda m})^{\frac{1}{2}}x) = (2\pi\lambda m)^{\frac{-1}{2}} \sum_{j=0}^{L(m)} exp(\frac{-j^2}{2\lambda m})(1+(\frac{2}{\lambda m})^{\frac{1}{2}}x)^{-j}$$

$$+\omega(\beta_m{'} + \beta_m{''})B\pi^{\frac{-1}{2}} \qquad (m > m_0)$$

However, using Taylor expansion on $(1+(\frac{2}{\lambda m})^{\frac{1}{2}}x)^{-j}$, we can deduce that

$$(2\pi\lambda m)^{\frac{-1}{2}} Q_m(1 + (\frac{2}{\lambda m})^{\frac{1}{2}}x)$$

$$= (2\pi\lambda m)^{-\frac{1}{2}} e^{x^2} \sum_{j=0}^{L(m)} (1 + \frac{2j\omega x^2}{\lambda m})\ exp\{-(x + \frac{j}{(2\lambda m)^{-\frac{1}{2}}})^2\} + \omega(\beta_m{'} + \beta_m{''})B\pi^{\frac{-1}{2}}.$$

d) Let $\Lambda_m(x)$ be :

$$\Lambda_m(x) = (2\pi\lambda m)^{-\frac{1}{2}} e^{x^2} \sum_{j=0}^{L(m)} exp\{-(x + \frac{j}{(2\lambda m)^{\frac{1}{2}}})^2\}$$

We can show that

$$\{\Lambda_m(x) - (2\pi\lambda m)^{-\frac{1}{2}} Q_m(1 + (\frac{2}{\lambda m})^{\frac{1}{2}}x)\} \longrightarrow 0 \text{ as } m \to \infty$$

uniformly on the interval $1 < x < B$.

4.1.3. *Step 3:*

Obviously , $\sum_{j=0}^{L(m)} exp(-\frac{j^2}{2\lambda m}) \leq 1 + \int_0^{\infty} exp(-\frac{t^2}{2\lambda m})dt = 1 + (\frac{\pi\lambda m}{2})^{\frac{1}{2}}$

So, using the following elementary inequalities :

$exp\{-(x + \frac{j+1}{(2\lambda m)^{\frac{1}{2}}})^2\} < \int_j^{j+1} exp\{-(x + \frac{t}{(2\lambda m)^{\frac{1}{2}}})^2\}dt < exp\{-(x + \frac{j}{(2\lambda m)^{\frac{1}{2}}})^2\}$

we conclude that:

$$0 < \sum_{j=0}^{L(m)} exp\{-(x + \frac{j}{(2\lambda m)^{\frac{1}{2}}})^2\} - \int_0^{L(m)+1} exp\{-(x + \frac{t}{(2\lambda m)^{\frac{1}{2}}})^2\}dt < e^{-x^2}$$

However, since $L(m) = H_m m^{\frac{-1}{2}}$

$$|\Lambda_m(x) - e^{x^2}(2\pi\lambda m)^{-\frac{1}{2}} \int_0^{\infty} exp\{-(x + \frac{t}{(2\lambda m)^{\frac{1}{2}}})^2\}dt\ |$$

$$< (2\pi\lambda m)^{-\frac{1}{2}} + e^{x^2}(2\pi\lambda m)^{-\frac{1}{2}} \int_{H_m m^{\frac{-1}{2}}}^{\infty} exp\{-(x + \frac{t}{(2\lambda m)^{\frac{1}{2}}})^2\}dt.$$

So, using the change of variable $\sigma = x + \frac{t}{(2\lambda m)^{\frac{1}{2}}}$, we show that :

$$|\Lambda_m(x) - e^{x^2}\pi^{-\frac{1}{2}} \int_x^{\infty} e^{-\sigma^2}d\sigma\ | < (2\pi\lambda m)^{-\frac{1}{2}} + e^{x^2}\pi^{-\frac{1}{2}} \int_{\frac{H_m}{(2\lambda)^{\frac{1}{2}}}}^{\infty} e^{-\sigma^2}d\sigma$$

However, $H_m \xrightarrow{m\to\infty} \infty$, then :

$$(2\pi\lambda m)^{-\frac{1}{2}}Q_m(1 + (\frac{2}{\lambda m})^{\frac{1}{2}}x) \longrightarrow e^{x^2}\pi^{-\frac{1}{2}} \int_x^{\infty} e^{-\sigma}d\sigma = \frac{e^{x^2}}{2}\ erfc(x)$$

uniformly for all $x \in [1, B]$.

□

APPENDIX 1: ASYMPTOTICS

1.1. Some Special Integrals

1. **Calculating** $z\int_0^\infty \frac{t^{\lambda-1}\log(t)}{(t+z)}dt$

We start by calculating $\int_0^\infty r\frac{t^{\lambda-1}\log(t)}{(t+r)}dt,\ r>0.$

Make the change of variable t=sr where s is real positive number, to get :

$\int_0^\infty r\frac{t^{\lambda-1}\log(t)}{(t+r)}dt = r^\lambda\int_0^\infty \frac{s^{\lambda-1}\log(s)}{(s+1)}ds + r^\lambda\int_0^\infty \frac{s^{\lambda-1}\log(r)}{(s+1)}ds$

$= c_1r^\lambda + r^\lambda\log(r)\int_0^\infty \frac{s^{\lambda-1}}{(s+1)}ds = c_1r^\lambda + \frac{\pi}{sin(\pi\lambda)}r^\lambda\log(r)$

with $c_1 = \int_0^\infty \frac{s^{\lambda-1}\log(s)}{(s+1)}ds.$

Confining z to Δ, we observe that $h(z) = z\int_0^\infty \frac{t^{\lambda-1}\log(t)}{(t+z)}dt$ is analytic.

On the other hand , $g(z) = c_1z^\lambda + \frac{\pi}{sin(\pi\lambda)}z^\lambda\log(z)$ is also analytic in the same region as that of h.

It follows that h(z)=g(z) by analytic continuation i.e.$z\int_0^\infty \frac{t^{\lambda-1}\log(t)}{(t+z)}dt$
$= c_1z^\lambda + \frac{\pi}{sin(\pi\lambda)}z^\lambda\log(z) = c_1z^\lambda + \frac{B_1}{\gamma}z^\lambda\log(z).$

2. **Bounding** $\int_0^{x_1} z\frac{t^{\lambda-1}\log(t)}{(t+z)}dt, r>0$

Confining z to Δ, $|\int_0^{x_1} z\frac{t^{\lambda-1}\log(t)}{(t+z)}dt| \leq \frac{r}{\gamma_1}\int_0^{x_1} \frac{|t^{\lambda-1}\log(t)|}{(t+r)}dt$

Suppose $x_1>1$, then

$\int_0^{x_1} r\frac{|t^{\lambda-1}\log(t)|}{(t+r)}dt = -\int_0^1 r\frac{t^{\lambda-1}\log(t)}{(t+r)}dt + \int_1^{x_1} r\frac{t^{\lambda-1}\log(t)}{(t+r)}dt$

$= r\int_0^1 (t^{\lambda-1}\log(t)[\frac{1}{r} - \frac{1}{(t+r)}])dt - \int_0^1 t^{\lambda-1}\log(t)dt + \int_1^{x_1} r\frac{t^{\lambda-1}\log(t)}{(t+r)}dt$

$= \int_0^1 \frac{t^\lambda \log(t)}{(t+r)}dt - \int_0^1 t^{\lambda-1}\log(t)dt + \int_1^{x_1} r\frac{t^{\lambda-1}\log(t)}{(t+r)}dt.$

So, $\int_0^{x_1} r\frac{|t^{\lambda-1}\log(t)|}{(t+r)}dt \leq 2|\int_0^1 t^{\lambda-1}\log(t)dt| + \int_1^{x_1} r\frac{t^{\lambda-1}\log(t)}{(t+r)}dt$

$\leq |\frac{t^\lambda}{\lambda}\log t\Big|_0^1 - \int_0^1 \frac{t^\lambda}{\lambda t}dt| + \int_1^{x_1} r\frac{t^{\lambda-1}\log(t)}{(t+r)}dt = c_2.$

So, $|\int_0^{x_1} z\frac{t^{\lambda-1}\log(t)}{(t+z)}dt|$ is bounded from above by a positive constant c_2.

Similar argument if $x_1 \leq 1$.

3. **Bounding** $z\int_{x_1}^{r_0} \frac{n(t)}{t(t+z)}dt$

$z\int_{x_1}^{r_0} \frac{n(t)}{t(t+z)}dt \leq n(r_0)\int_{x_1}^{r_0} \frac{z}{t(t+z)}dt = n(r_0)\int_{x_1}^{r_0}[\frac{1}{t} - \frac{1}{(t+z)}]dt$

$=n(r_0)log(\frac{r_0}{x_1}) + n(r_0)\log(\frac{x_1+z}{r_0+z}).$

4. **Bounding** $z\int_{x_1}^{r_0} \frac{t^{\lambda-1}\log(t)}{(t+z)}dt$

$|z\int_{x_1}^{r_0} \frac{t^{\lambda-1}\log(t)}{(t+z)}dt| \leq max\{|\log(x_1)|, |\log(r_0)|\}|z\int_{x_1}^{r_0} \frac{t^{\lambda-1}}{(t+z)}dt|$

$\leq max\{|\log(x_1)|, |\log(r_0)|\}|\frac{\pi}{sin(\lambda\pi)}z^\lambda| = c_3|z^\lambda|.$

5. **Bounding** $\int_0^{x_1} z\frac{t^\lambda \log(t)}{(t+z)^2}dt, r > 0$

Suppose $x_1 > 1$,

$|\int_0^{x_1} z\frac{t^\lambda \log(t)}{(t+z)^2}dt| \leq \frac{1}{\gamma_1}\int_0^{x_1} r\frac{t^\lambda|\log(t)|}{(t+r)^2}dt$

$\int_0^{x_1} r\frac{t^\lambda|\log(t)|}{(t+r)^2}dt \leq \int_0^1 r\frac{t^\lambda \log(t)}{(t+r)^2}dt + \int_1^{x_1} r\frac{t^\lambda \log(t)}{(t+r)^2}dt$

$\leq \int_0^1 r\frac{t^\lambda \log(t)}{(t+r)^2}dt - \int_0^1 r\frac{t^\lambda \log(t)}{2tr}dt + \int_0^1 r\frac{t^\lambda \log(t)}{2tr}dt + \int_1^{x_1} r\frac{t^\lambda \log(t)}{(t+r)^2}dt$

$\leq \int_0^1 \frac{-r^2-t^2}{2t(t+r)^2}t^\lambda \log(t)dt + \int_0^1 \frac{1}{2}t^{\lambda-1}\log(t)dt + \int_1^{x_1} r\frac{t^\lambda \log(t)}{(t+r)^2}dt$

$\int_0^1 \frac{-r^2-t^2}{2t(t+r)^2} t^\lambda \log(t)dt \le \int_0^1 \frac{-2}{2t} t^\lambda \log(t)dt$

So, $|\int_0^{x_1} z\frac{t^\lambda \log(t)}{(t+z)^2}dt|$ is bounded from above by a positive constant c_4.

Similar argument if $x_1 \le 1$

6. Calculating $z\int_0^\infty \frac{t^\lambda \log(t)}{(t+z)^2}dt$

We start by calculating $r\int_0^\infty \frac{t^\lambda \log(t)}{(t+r)^2}dt$

Make the change of variable t=sr where s is real positive number, to get :

$r\int_0^\infty \frac{t^\lambda \log(t)}{(t+r)^2}dt = \int_0^\infty \frac{r^\lambda s^\lambda \log(s)}{(1+s)^2}ds + \int_0^\infty \frac{r^\lambda s^\lambda \log(r)}{(1+s)^2}ds.$

$\int_0^\infty \frac{s^{\lambda-1}}{(1+s)}ds = \left(\frac{1}{1+s}.\frac{s^\lambda}{\lambda}\right)\Bigg|_0^\infty + \int_0^\infty \frac{s^\lambda}{\lambda}\frac{1}{(1+s)^2}ds = \frac{1}{\lambda}\int_0^\infty \frac{s^\lambda}{(1+s)^2}ds.$

Then , $r\int_0^\infty \frac{t^\lambda \log(t)}{(t+r)^2}dt = r^\lambda \int_0^\infty \frac{s^\lambda \log(s)}{(1+s)^2}ds + \frac{\lambda\pi}{sin(\lambda\pi)}r^\lambda \log(r).$

Away from the negative real axis,we observe that:

$h(z) = \int_0^\infty z\frac{t^\lambda \log(t)}{(t+z)^2}dt$ is analytic.

On the other hand , $g(z) = z^\lambda \int_0^\infty \frac{s^\lambda \log(s)}{(1+s)^2}ds + \frac{\lambda\pi}{sin(\lambda\pi)}z^\lambda \log(z)$

is also analytic in the same region as that of h.

It follows that h(z)=g(z) by analytic continuation i.e.

$\int_0^\infty z\frac{t^\lambda \log(t)}{(t+z)^2}dt = z^\lambda \int_0^\infty \frac{s^\lambda \log(s)}{(1+s)^2}ds + \frac{\lambda\pi}{sin(\lambda\pi)}z^\lambda \log(z) = c_6 z^\lambda + \frac{\lambda B_1}{\gamma}z^\lambda \log(z).$

APPENDIX 2 :THE PARTIAL SUMS

1.2. Maclaurin Coefficients of the Partial Sum

1.2.1. *a(r) is an increasing function*

Theorem 2. *(Hadamard Three-Circle Theorem)*
Let f(z) be an entire function and let M(r) be the maximum of $|f(z)|$ *on the circle* $|z| = r$*. Then,* $\log M(r)$ *is a convex function of the logarithm* $\log(r)$*.*

Applying the theorem on our L-function F , we get that $(M(e^x))$is a convex function of x. However, $M(r) = F(r)$ since F is an entire function with positive coefficients .

$\Rightarrow \log(F(e^x))$is a convex function of x

$\Rightarrow (log(F(e^x)))'' > 0$

$\Rightarrow a'(e^x) = (e^x \frac{F'(e^x)}{F(e^x)})' = (\log(F(e^x)))'' > 0$

$\Rightarrow a(r)$ is increasing .

1.2.2. *Calulating Maclurian coefficients* a_j *then deducing* $b_j = \frac{a_{m+j}}{a_m} {R_m}^j$

Applying Hayman's theorem on our L-function F, we get that :

$$a_n {R_m}^n = \frac{F(R_m)}{\{2\pi b(R_m)\}^{\frac{1}{2}}}[\exp\left[-\frac{(a(R_m)-n)^2}{2b(R_m\ )}\right] + \ \eta\,(R_m, n)\ ]$$

However, noticing $a(R_m) = m$, we get that :

for n=m , $a_m R^m = \frac{F(R_m)}{\{2\pi b(R_m)\}^{\frac{1}{2}}}(1 + \ \eta(R_m, m))$

and for n=m+j , $a_{m+j} R^{m+j} = \frac{F(R_m)}{\{2\pi b(R_m)\}^{\frac{1}{2}}}(exp(\frac{-j^2}{2b(R-m)}) + \ \eta(R_m, m+j))$

Therefore, $b_j\,(m) = \frac{a_{m+j}}{am} {R_m}^j = exp(\frac{-j^2}{2b(R_m)}) + \ \eta(R_m, m+j)[1 + \eta(R_m, m)]^{-1}$

but $b(R_m) = \lambda m(1 + \eta''(R_m))$, then

$$b_j\,(m) = exp(\frac{-j^2}{2\lambda\ m}) + \tilde{\eta}_j\,(m) + \ \eta(R_m, m+j)[1 + \eta(R_m, m)]^{-1}$$

with $\tilde{\eta}_j(m) \to 0$ as $m \to \infty$.

Indeed , $exp(\frac{-j^2}{2b(R_m)}) = exp\ (\frac{-j^2}{2\lambda m(1+\ \eta''(R_m))}) = exp(\frac{-j^2}{2\lambda\ m}) + \tilde{\eta}_j\,(m)$

with $\tilde{\eta}_j\,(m) = \ exp\ (\frac{-j^2}{2\lambda\ m\ (1+\ \eta''(R_m))}) \ - exp(\frac{-j^2}{2\lambda\ m})$

So, $exp(\frac{-j^2}{2b(R_m)}) = exp\ (\frac{-j^2+j^2\eta''(R_m)-j^2\eta''(R_m)}{2\lambda\ m\ (1+\ \eta''(R_m))}) \ - exp(\frac{-j^2}{2\lambda\ m})$

$$= exp(\frac{-j^2}{2\lambda\ m})exp(\frac{j^2\eta''(R_m)}{2\lambda m\ (1+\ \eta''(R_m))}) - exp(\frac{-j^2}{2\lambda\ m})$$

$$= exp(\frac{-j^2}{2\lambda\ m})[\ exp(\frac{j^2\eta''(R_m)}{2\lambda m\ (1+\ \eta''(R_m))}) - 1].$$

then $|\tilde{\eta}_j\,(m)\,| < |exp(\frac{j^2\eta''(R_m)}{2\lambda m\ (1+\ \eta''(R_m))}) - 1| \to 0$ as $m \to \infty$ (for fixed j)

Therefore , $b_j\,(m) \to 1$ as $m \to \infty$.

1.3. Studying $U_m(w)$

1.3.1. *Calculating* $F(R_m w)$

Let η be a real number such that $0 < \eta < \frac{1}{2}$, and z_0 a non-zero complex number. η and z_0 have to be chosen such that $F(z) \neq 0$ throughout the disk $|z - z_0| \leq 2\eta|z_0|$ Since all the zeros of F are real and negative , we choose z_0 to be real positive and η to be $\frac{1}{4}$.

Let s be a complex number that satisfies $|s| \leq \eta|z_0| = \frac{1}{4}|z_0|$,
(therefore $(z_0 + s) \in disk\ |z - z_0| \leq 2\eta|z_0| = \frac{1}{2}|z_0|$).

Then, $log(F(z_0 + s)) - log(F(z_0)) = s\ \frac{F'(z_0)}{F(z_0)} + \frac{s^2}{2}(\frac{F''(z_0)}{F(z_0)} - \frac{F'(z_0)}{[F(z_0)]^2}\ ) + E_3(z_0, s)$

where $|E_3(z_0, s)| \leq \frac{1}{2(1-2\eta)}\eta^{-2}|\frac{s}{z_0}|^3 max_\theta\ |a(z_0(1 + 2\eta e^{i\theta}))|$.

Proof:

let $g(s) = log(\frac{F(z_0+s)}{F(z_0)}), g(0) = 0$

$g(s) = g(0) + (s-0)g'(0) + \frac{(s-0)^2}{2}g''(0) + \frac{(s-0)^3}{6}g'''(\delta)$ with $\delta \in (0,s)$

so $log(F(z_0+s)) - log(F(z_0)) = s\frac{F'(z_0)}{F(z_0)} + \frac{s^2}{2}(\frac{F''(z_0)}{F(z_0)} - [\frac{F'(z_0)}{F(z_0)}]^2) + \frac{(s)^3}{6}g'''(\delta)$,

$E_3(z_0,s) = \frac{(s)^3}{6}g'''(s) = \frac{s^3}{2\pi i}\int_C \frac{g(\zeta)}{\zeta^3(\zeta-s)}d\zeta$

where the contour C is chosen to be $C : \zeta = 2\eta z_0 e^{i\theta}(0 \le \theta \le 2\pi)$

so $E_3(z_0,s) = \frac{s^3}{2\pi i}\int_0^{2\pi} \frac{g\left(2\eta z_0 e^{i\theta}\right)}{(2\eta z_0 e^{i\theta})^3(2\eta z_0 e^{i\theta}-s)}2\eta z_0 e^{i\theta}d\theta$

$|E_3(z_0,s)| \le \frac{|s|^3}{2\pi}\int_0^{2\pi} |\log(\frac{F(z_0+2\eta z_0 e^{i\theta})}{F(z_0)})| \frac{|2\eta z_0|}{|2\eta z_0|^3.\ ||2\eta z_0|-|s||}d\theta$

$\le \frac{|s|^3}{2\pi}max_\theta\ |log(\frac{F(z_0(1+2\eta e^{i\theta}))}{F(z_0)})| \int_0^{2\pi} \frac{|2\eta z_0|}{|2\eta z_0|^3.\ |2\eta z_0|-|s||}d\theta$

$\le \frac{1}{8\pi}\left(\frac{|s|}{\eta|z_0|}\right)^3 max_\theta\ |\log\left(\frac{F(z_0(1+2\eta e^{i\theta}))}{F(z_0)}\right)| \int_0^{2\pi}\frac{|\eta z_0|}{||2\eta z_0|-|s||}d\theta$, $|s| \le \eta|z_0|$

$\le \frac{1}{8\pi}(\frac{|s|}{\eta|z_0|})^3 max_\theta |\log(\frac{F(z_0(1+2\eta e^{i\theta}))}{F(z_0)})| \int_0^{2\pi}d\theta$

Then $|E_3(z_0,s)| \le \frac{1}{4}\left(\frac{|s|}{\eta|z_0|}\right)^3 max_\theta\ |log\left(\frac{F\left(z_0\left(1+2\eta e^{i\theta}\right)\right)}{F(z_0)}\right)|$

But, $|log(\frac{F(z_0(1+2\eta e^{i\psi}))}{F(z_0)})| = |\int_{z_0}^{z_0\left(1+2\eta e^{i\psi}\right)}\frac{F'(t)}{F(t)}dt| = |\int_{z_0}^{z_0(1+2\eta e^{i\psi})}\frac{a(t)}{t}dt|$

$\le \frac{2\eta}{1-2\eta}max_\theta\ |a(z_0(1+2\eta e^{i\theta}))|$

Indeed,
$a(z)$ is a non-constant harmonic function so by maximum principle
$a(z)$ attains its maximum on boundaries of disk of center z_0 , radius $2\eta|z_0|$

and $log(z_0\left(1+2\eta e^{i\psi}\right)) - \log(z_0) = \log(1+2\eta e^{i\psi}) = \log(1+2\eta\cos\psi + i2\eta\sin\psi)$

$= \log(\sqrt{(1+2\eta\cos\psi)^2+(2\eta\sin\psi)^2}) + i\ arg(1+2\eta\cos\psi + i2\eta\sin\psi)$

$= \log(\sqrt{1+4\eta^2+4\eta\cos\psi}) + i\frac{2\eta\sin\psi}{1+2\eta\cos\psi}$.

$|\log(\sqrt{1+4\eta^2+4\eta\cos\psi}) + i\frac{2\eta\sin\psi}{1+2\eta\cos\psi}|$

$= \sqrt{\log^2(\sqrt{1+4\eta^2+4\eta\cos\psi}) + \left(\frac{2\eta\sin\psi}{1+2\eta\cos\psi}\right)^2}$

$\leq \sqrt{\log^2(\sqrt{1+4\eta^2+4\eta\cos\psi})+\left(\frac{2\eta\sin\psi}{1+2\eta\cos\psi}\right)^2}\Big|\psi=\pi \qquad (\text{for } \eta=\frac{1}{4})$

$= \sqrt{\log^2(\sqrt{1+4\eta^2-4\eta})} = \sqrt{\log^2(1-2\eta)} = log(1-2\eta) \leq \frac{2\eta}{1-2\eta}.$

It's time to set $z_0 = R_m$ and $s = (w-1)R_m$ with w a complex number such that $w \in \{w; |w-1| \leq 1 \text{ and } |w| < 1\}$.

Replacing z_0 & s by their values we get that:

$$log(F(wR_m)) = log(F(R_m)) + (w-1)R_m\frac{F'(R_m)}{F(R_m)} + \frac{((w-1)R_m)^2}{2}(\frac{F''(R_m)}{F(R_m)} - [\frac{F'(R_m)}{F(R_m)}]^2)$$
$$+E_3(R_m,(w-1)R_m)$$

$$F(wR_m) = F(R_m)\ exp[(w-1)a(R_m) + \frac{(w-1)^2}{2}(R_{m^2}\frac{F''(R_m)}{F(R_m)} - [R_m\frac{F'(R_m)}{F(R_m)}]^2)$$
$$+E_3(R_m,(w-1)R_m)]$$

But $b(R_m) = R_m a'(R_m) = R_m(R_m\frac{F'(R_m)}{F(R_m)})' = R_m(\frac{F'(R_m)}{F(R_m)} + R_m\frac{F''(R_m)}{F(R_m)} - R_m[\frac{F'(R_m)}{F(R_m)}]^2)$

$$= R_m\frac{F'(R_m)}{F(R_m)} + R_m^2\frac{F''(R_m)}{F(R_m)} - [R_m\frac{F'(R_m)}{F(R_m)}]^2$$

On the other hand , $b(R_m) = \lambda m(1+\eta''_m)$

Then , by comparison , $R_m^2\frac{F''(R_m)}{F(R_m)} - [R_m\frac{F'(R_m)}{F(R_m)}]^2 = \lambda m\left(1+\eta''_m\right) - R_m\frac{F'(R_m)}{F(R_m)}$ $= \lambda m\left(1+\eta''_m\right) - m$

So $F(wR_m) = F(R_m)\ exp[(w-1)\,m + \frac{(w-1)^2}{2}(\lambda m(1+\eta''_m) - m) + E_3\,(R_m,(w-1)\,R_m)]$

$|E_3(R_m,(w-1)R_m)| \leq c|\frac{(w-1)R_m}{R_m}|^3\frac{2H(m)+1}{1+log(H(m))+1)}log(2(\frac{2H(m)+1}{1+log(H(m)+1)})^{\frac{1}{\lambda}})$

where $c = \frac{B_1\ \lambda\ 2^{\lambda}\eta^{-2}}{(1-2\eta)}$ $\qquad H(m) = \frac{m}{B_1(1+\ \eta(R_m))}$

1.3.2. *Deducing* $U_m(w)$

Step 1 :

$$a_m {R_m}^m = \frac{F(R_m)}{2\pi b(R_m)^{\frac{1}{2}}}(1+\ \eta(R,m)) = \frac{F(R_m)}{2\pi[\lambda m(1+\eta_m(R_m))]^{\frac{1}{2}}}(1+\ \eta(R,m)).$$

Step 2:

$log(w) = log(1-1+w) = log(1-[1-w]) = -(1-w) - \frac{1}{2}(1-w)^2 + \omega A(1-w)^3$ $(|1-w| < \frac{1}{2})$
then $\frac{1}{w^m} = exp[-m\ log(w)] = exp[m(1-w) + m\frac{1}{2}(1-w)^2 - m\omega A(1-w)^3]$.

Step 3:

The restrictions $(1-2\eta)R_m \leq |z| \leq 2R_m$, $z \in \Delta$
together with the facts that :

1) $a(R_m) = B_1\ \lambda\ {R_m}^\lambda \log(R_m)(\ 1 + \eta(R_m))$

2) ${R_m}^\lambda = \exp(W_0(\frac{m}{B_1(\ 1+\ \ \eta(R_m))}))$

3) $W_0(\frac{m}{B_1(\ 1+\ \ \eta(R_m))}) \leq log(\frac{\frac{2m}{B_1(\ 1+\ \ \eta(R_m))}+1}{1+log(\frac{m}{B_1(\ 1+\ \ \eta(R_m))}+1)})$ since $\frac{m}{B_1(\ 1+\ \ \eta(R_m))} = \lambda {R_m}^\lambda log(R_m)$

and $\lambda {R_m}^\lambda log(R_m)$ is real and positive for large m

imply that :

$$|a(z)| \leq |a(2R_m)| = B_1\ \lambda\ 2^\lambda {R_m}^\lambda \log(2R_m)(\ 1+\eta(2R_m))$$
$$\leq B_1\ \lambda\ 2^{\lambda+1} {R_m}^\lambda \log(2R_m)$$
$$\leq B_1 \lambda\ 2^{\lambda+1} \frac{\frac{2m}{B_1(\ 1+\ \ \eta(R_m))}+1}{1+log(\frac{m}{B_1(\ 1+\ \ \eta(R_m))}+1)} log(2(\frac{\frac{2m}{B_1(\ 1+\ \ \eta(R_m))}+1}{1+log(\frac{m}{B_1(\ 1+\ \ \eta(R_m))}+1)})^{\frac{1}{\lambda}})\ \text{with}\ \ m > m_0$$

So $|E_3(z_0,s)| \leq \frac{1}{2(1-2\eta)}\eta^{-2}|\frac{s}{z_0}|^3 max_\theta\ |a(z_0(1+2\eta e^{i\theta}))|$

$$\leq \frac{1}{2(1-2\eta)}\eta^{-2}|\frac{s}{z_0}|^3\ B_1\lambda\ 2^{\lambda+1}\frac{\frac{2m}{B_1(\ 1+\ \ \eta(R_m))}+1}{1+log(\frac{m}{B_1(\ 1+\ \ \eta(R_m))}+1)} log(2(\frac{\frac{2m}{B_1(\ 1+\ \ \eta(R_m))}+1}{1+log(\frac{m}{B_1(\ 1+\ \ \eta(R_m))}+1)})^{\frac{1}{\lambda}})$$

The results of the above steps all together give that :

$$U_m(w) = \frac{F(R_m)\ exp[(w-1)m + \frac{(w-1)^2}{2}\ (\lambda m\left(1+\eta''_m\right)-m) + E_3(R_m,(w-1)R_m)]}{\frac{F(R_m)(1+\ \eta(R_m,m))\ w^m}{[\ 2\pi\lambda m\ \ (1+\eta_m(R_m))\ ]^{\frac{1}{2}}}}$$

$$= \frac{1}{w^m} \{2\pi\lambda m\}^{\frac{1}{2}} (1+\eta_m)\, exp[\, (w-1)\, m + \frac{(w-1)^2}{2} \left(\lambda m \left(1+\eta''_m\right) - m\right) + E_3 \left(R_m, (w-1)\, R_m\right)]$$

$$= \{2\pi\lambda m\}^{\frac{1}{2}} (1+\eta_m)\ exp[\frac{(w-1)^2}{2}\ (\lambda m(1+\eta''_m)) + (w-1)m\ - m\ \frac{(w-1)^2}{2}$$

$$+m(1-w) + \frac{m}{2}(1-w)^2 - m\omega A(1-w)^3\ + E_3(R_m, (w-1)R_m)]$$

$$= \{2\pi\lambda m\}^{\frac{1}{2}} (1+\eta_m)\, exp[\ \frac{(w-1)^2}{2} \left(\lambda m \left(1+\eta''_m\right)\right) - m\omega A\, (1-w)^3 + E_3 \left(R_m, (w-1)\, R_m\right)]$$

$$= \{2\pi\lambda m\}^{\frac{1}{2}} (1+\eta_m)\ exp[\frac{(w-1)^2}{2}\ \left(\lambda m \left(1+\eta''_m\right)\right)\ + E_4 \left(R_m, (w-1)\, R_m\right)].$$

$$|E_4 \left(R_m, (w-1)\, R_m\right)| \le |m\omega A\, (1-w)^3| + |E_3 \left(R_m, (w-1)\, R_m\right)|$$

$$\le |m\omega A\, (1-w)^3| + |c\, (|w-1|)^3 \left(\frac{2H(m)+1}{1+log(H(m)+1)}\right) log(2(\frac{2H(m)+1}{1+log(H(m)+1)})^{\frac{1}{\lambda}})|$$

$$\sim |m\omega A\, (1-w)^3| + |m\, (1-w)^3|.$$

APPENDIX 3 :THE MAIN THEOREM

1.4. Step 2 :Approximating $\Omega_m(\zeta)$ by a summation

1.4.1. *Approximating* $b_{-j}(m)$

Case 1: $|j| < L(m)$

$$|\tilde{\eta}_{-j}(m)| < |exp(\frac{[L(m)]^2\eta''(R_m)}{2\lambda m\ (1+\eta''(R_m))} - 1| < |exp(\frac{m|\eta_m''|^{\frac{1}{3}}}{2\lambda m\ (1+\eta''(R_m))})) - 1|$$
$$< |exp(\frac{|\eta_m''|^{\frac{1}{3}}}{\lambda(1+\eta''(R_m))})) - 1|$$

So ,

$$b_{-j}(m) = exp(\frac{-j^2}{2\lambda m}) + \beta_m(-j)$$

where uniformly in j ,

$$|\beta_m(-j)| < \beta_m' \ , \quad \beta_m' \xrightarrow{m\to\infty} 0.$$

Case 2: $|j| > L(m)$

$$0 < b_{-j}(m) = [exp(\frac{-j^2}{2\lambda\ m}) + \tilde{\eta}_{-j}(m) + \ \eta(R, m-j)][1+\eta(R,m)]^{-1}$$
$$< [exp(\frac{-H_m^2}{2\lambda}) + \tilde{\eta}_{-j}(m) + \ \eta(R, m-j)][1+\eta(R,m)]^{-1} < \beta_m''$$

with $\beta_m'' \xrightarrow{m\to\infty} 0$.

1.4.2. *Special value of* w

Letting $w = 1 + (\frac{2}{\lambda m})^{\frac{1}{2}}x > 1$, we get that :

$$(2\pi\lambda m)^{\frac{-1}{2}}Q_m(1+(\frac{2}{\lambda m})^{\frac{1}{2}}x) = (2\pi\lambda m)^{\frac{-1}{2}}\sum_{j=0}^{L(m)}(1+(\frac{2}{\lambda m})^{\frac{1}{2}}x)^{-j}exp(\frac{-j^2}{2\lambda m})$$

$$+\omega(\beta_m{}' + \beta_m{}'')B\pi^{\frac{-1}{2}} \qquad (m > m_0)$$

However, using Taylor expansion ,

$(1+(\frac{2}{\lambda m})^{\frac{1}{2}}x)^{-j} = exp\ (log[(1+(\frac{2}{\lambda m})^{\frac{1}{2}}x)^{-j}])$

$= exp\ (-j[(\frac{2}{\lambda m})^{\frac{1}{2}}x - \omega(\frac{2}{\lambda m})x^2]) = (1+\frac{2j\omega x^2}{\lambda m})exp(-j(\frac{2}{\lambda m})^{\frac{1}{2}}x)$

$\geq (1+\frac{2\omega B^2\log m}{\lambda m^{\frac{1}{2}}})exp(-j(\frac{2}{\lambda m})^{\frac{1}{2}}x) \qquad (|\omega| \leq 1)$

So,$(2\pi\lambda m)^{\frac{-1}{2}}Q_m(1+(\frac{2}{\lambda m})^{\frac{1}{2}}x)$

$\geq (2\pi\lambda m)^{\frac{-1}{2}}\sum_{j=0}^{L(m)}(1+\frac{2\omega B^2\log m}{\lambda m^{\frac{1}{2}}})exp(\frac{-j^2}{2\lambda m} - \frac{2jx}{(2\lambda m)^{\frac{1}{2}}}) + \omega(\beta_m{}' + \beta_m{}'')B\pi^{\frac{-1}{2}}$

$= (2\pi\lambda m)^{-\frac{1}{2}}exp(x^2)\sum_{j=0}^{L(m)}(1+\frac{2\omega B^2\log m}{\lambda m^{\frac{1}{2}}})exp(-(x+j(2\lambda m)^{-\frac{1}{2}})^2)+$

$\omega(\beta_m{}' + \beta_m{}'')B\pi^{\frac{-1}{2}}$.

Let $\Lambda_m(x)$ be :

$$\Lambda_m(x) = (2\pi\lambda m)^{-\frac{1}{2}}exp(x^2)\sum_{j=0}^{L(m)}exp(-(x+j(2\lambda m)^{-\frac{1}{2}})^2)$$

$|\Lambda_m(x) - (2\pi\lambda m)^{-\frac{1}{2}}Q_m(1+(\frac{2}{\lambda m})^{\frac{1}{2}}x)|$

$\leq |(2\pi\lambda m)^{-\frac{1}{2}}exp(x^2)\sum_{j=0}^{L(m)}(\frac{2\omega B^2\log m}{\lambda m^{\frac{1}{2}}})exp(-(x+j(2\lambda m)^{-\frac{1}{2}})^2)+\omega(\beta_m{}'+\beta_m{}'')B\pi^{\frac{-1}{2}}|$

$\leq km^{-1}log(m)\sum_{j=0}^{L(m)}exp(-\frac{j^2}{2\lambda m}) + (\beta_m{}' + \beta_m{}'')B\pi^{\frac{-1}{2}}$, where k is a constant .

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