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DISCRETE OPTIMAL TRANSPORT

by
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Abstract
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We introduce the Assignment Problem, which involves minimizing the cost of transporting goods from a finite number of sources to a finite number of targets. Due to the discrete nature of the assignment problem, a solution might be computed using a brute-force numerical approach; however, these are not efficient. In this thesis, we relax the assignment problem, and connect it to the infamous Kantorovich Problem and its Dual. Theoretically, the problem consists of maximizing concave functional under linear inequality constraints. We develop the needed theoretical background from Functional and Convex analysis to solve the assignment problem which allowed us to present more efficient numerical methods such as the Bertsekas' auction algorithm, and the network simplex algorithm.

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Chapter 1

Introduction

Transportation costs are a significant challenge for many production companies, as they strive to transport their goods to sales markets in the most cost-effective strategies. The discrete transport problem is a mathematical approach to this challenge, seeking to find the optimal way to transport goods from a finite number of sources to a finite number of targets, while minimizing transportation costs and satisfying supply and demand limits. This problem was introduced first by the French Mathematician Gaspard Monge in 1781 and later mathematically studied by Tolstoi in 1920. Since then, it has been extensively researched, and various algorithms have been proposed to solve it efficiently.

One approach is the simplex algorithm [4] introduced by George B. Dantzig, which is an iterative method that moves along the edges of the feasible region to find the optimal solution. Another popular algorithm is the interior-point method [11], suggested by the Soviet mathematician I. I. Dikin in 1967. In addition to these methods, several other algorithms have been developed to solve discrete optimal transport such as Sinkhorn-Knopp's algorithm [7], Bertsekas' Auction Algorithm [2] and network simplex algorithm [1].

In this thesis, we first consider the ideal case where the number of sources and targets are the same as well as the supply and demand quantity at each of them is uniform, this is known as the Assignment Problem. The goal is to find a bijection σ that assigns all the quantity at a certain source x to one and only one target place y. However, we will relax the assignment problem by splitting the mass at x to several locations. This relaxation can be achieved by replacing the transport map with a transport measure resulting in the infamous Kantorovich problem. In Chapter 2, we delve into these two problems and give the exact formulation in Section 2.1. We will show in Section 2.3 that the optimal transportation of goods is best achieved without splitting, by proving the equality between the Assignment Problem and its relaxation. This result has significant implications for production companies, as it provides insight into the most cost-effective way to transport goods from sources to targets.

Next, in Chapter 3 we offer a fresh perspective on the discrete transport problem by framing it as a maximization problem rather than a minimization one. If a worker wants to transfer an amount of coal from N mines to N factories such that the maximum cost he is willing to pay for this transportation is $c(x, y)$, then one could solve his problem by setting a price $\phi(x)$ for loading material at source x, and a price $-\psi(y)$ for unloading it at destination y such that the total profit $\phi(x) - \psi(y)$ is less than or equal to the cost $c(x, y)$ to attract worker's attention. This setting is the dual to the Kantorovich Problem which is elaborated in Section 3.1. We show that the Dual problem is also equivalent to the previous two problems in Section 3.3.

Through the theoretical part, we leveraged concepts from linear algebra and functional analysis to prove the equivalence between the Assignment and Kantorovich Problem. On the other hand, The dual problem is stated as a maximal of a functional that we proved to be concave. We used then some concepts from convex analysis, introduced the notion of c-transform and c-cyclically monotone [7], to compute its superdifferential [10], [6], and prove properties of the maximizing function.

Besides the theoretical part of the discrete transport problem, we are interested in finding numerical ways and algorithms to find the optimal solution. As we are working in the discrete case, one could compute the cost of all the finite transport maps and get the optimal one. However, such a method is inefficient and takes a lot of time, for that we think of better methods and algorithms. In chapter 4, we present the Bertsekas Auction Algorithm [8], a coordinate ascent method, that modifies the price ψ in order to reach a maximizer for the dual problem. The bidding strategy or the amount needed to be added at each iteration on the price function is given in Section 4.2. However, in some cases, such as Example 4.2.5, this strategy may get stuck in an infinite loop and fail to converge. In order to overcome this issue, we propose a modification in Section 4.3 that introduces the concept of $\epsilon-$complementary slackness. The modified algorithm generates a transport plan within ϵ of being optimal. The complexity analysis and the pseudo-code presented in Section 4.5 show that its number of iterations is much better than that of the brute-force approach.

In practice, it is often the case that the number of sources is not equal to the number of destinations. In such cases, we cannot apply the assignment problem formulation. In Chapter 5, we introduce the Network Simplex Algorithm [3]. Through this algorithm, each transport plan is represented as a network graph. In particular, we show in Section 5.1 that extremal transport plans have a notable property in their corresponding graphs: they are acyclic, meaning that there contain no cycles or loops. Through this algorithm, we start with an initial feasible solution and iteratively improve it by exploring the graph and finding cycles that can be used to reduce the cost of the current solution as seen in Section 4.2.

Chapter 2

Discrete Optimal Transport Problem

2.1 Formulation

Let X, Y be two finite metric spaces of cardinality N, μ, and ν be two discrete probability measures on X and Y respectively defined as follows:

$$\mu = \sum_{x \in X} \mu_x \delta_x, \qquad \nu = \sum_{y \in Y} \nu_y \delta_y,$$

where μ_x and ν_y are the masses assigned to $x \in X$ and $y \in Y$. Define $c : X \times Y \to \mathbb{R}$ with $c(x_i, y_j)$ corresponds to the cost of moving x_i in X to y_j in Y for every $1 \leq i, j \leq N$. The goal is to find a transport map $T : X \to Y$ such that $T_{\#}\mu = \nu$, i.e. $\nu_{y_j} = \sum_{T(x_i)=y_j} \mu_{x_i}$, that minimizes the transport total cost. This problem is called the Monge Problem and can be written as follows

$$(MP) = \min_T \left\{ \sum_i c(x_i, T(x_i)) : T_{\#}\mu = \nu \right\}.$$

We consider the special case when the two measures μ and ν represent uniform probability measures i.e.

$$\mu = \frac{1}{N} \sum_{x \in X} \delta_x. \qquad \nu = \frac{1}{N} \sum_{y \in Y} \delta_y.$$

In such case, $T_{\#}\mu = \nu$ if and only T is bijective, and the Monge problem reduces to the linear assignment problem and given as follows

$$(AP) = \min \left\{ \frac{1}{N} \sum_{i=1}^{N} c(x_i, y_{\sigma(i)}) : \sigma \text{ is a permutation} \right\}.$$

Notice that the set of permutations is finite, then AP could be computed by comparing the total cost of all permutations. However, since the number of permutations is $N!$, such a method is inefficient.

We can then relax the idea of assigning source x_i to only one target $T(x_i)$ by splitting the mass at x_i to several locations. This relaxation can be achieved by replacing the map T or the permutation σ with the set of transport plans on $X \times Y$ which is given by

$$\Gamma(\mu, \nu) = \{\gamma = \sum_{x,y} \gamma_{x,y}\delta_{(x,y)} : \gamma_{x,y} \geq 0, \sum_{y \in Y} \gamma_{x,y} = \mu_x, \sum_{x \in X} \gamma_{x,y} = \nu_y\},$$

where $\gamma_{x,y}$ describes the amount of mass transported from point x to point y. The first constraint $\sum_{y \in Y} \gamma_{x,y} = \mu_x$ tells us that all masses from source x are transported to Y and similarly the second constraint tells us that all masses at y are transported from somewhere in X. The Kantorovich problem could be then formulated as follows

$$(KP) = \min \left\{ \sum_{x \in X, y \in Y} c(x, y)\gamma_{x,y} : \gamma \in \Gamma(\mu, \nu) \right\}.$$

We prove in this chapter that the Assignment Problem is equal to the Kantorovich Problem, that is

$$(AP) = (KP) \tag{2.1}$$

For this we need to prove some results related to permutation matrices.

2.2 Theoretical Background

We denote by $Perm(N)$ the set of permutations on $\{1, 2, \cdots, N\}$.

Definition 2.2.1 (Permutation Matrix). *A permutation matrix is a square matrix that has exactly one entry with value 1 in each row and column and 0's elsewhere.*

To every permutation $\sigma \in Perm(N)$, we associate the permutation matrix P_σ given by

$$(P_\sigma)_{i,j} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{otherwise} \end{cases}.$$

The set of permutation matrices is denoted by $\mathcal{P}_N = \{P_\sigma : \sigma \in Perm(N)\}$.

Remark 2.2.2. *The set of permutation matrices is a non convex set since its is discrete. As the assignment problem corresponds to the minimization of the total cost over this set, then it is equivalent to a non convex optimization problem.*

Definition 2.2.3. *(Bistochastic matrix). A bistochastic matrix (also called doubly stochastic matrix) is a square matrix $B = (b_{ij})$ with non negative real coefficients such that the sum of each row and column is equal to 1, i.e.*

$$\sum_i b_{ij} = \sum_j b_{ij} = 1.$$

The set of $N \times N$ bistochastic matrices is denoted by $\mathcal{B}_N$.

Example 2.2.4. *The matrix* $\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 0.6 & 0 & 0.4 \end{bmatrix}$ *is a bistochastic matrix.*

We denote by $M_N(\mathbb{R})$ the space of $N \times N$ matrices equipped with the standard metric.

Proposition 2.2.5. *The set of* $N \times N$ *bistochastic matrices* $\mathcal{B}_N$ *is convex, and compact.*

Proof. Let A and B be two bistochastic matrices and $0 \le \lambda \le 1$, then

$$\begin{aligned} C = \lambda A + (1-\lambda)B &= \begin{bmatrix} \lambda a_{11} & \dots & \lambda a_{1n} \\ \vdots & & \vdots \\ \lambda a_{n1} & \dots & \lambda a_{nn} \end{bmatrix} + \begin{bmatrix} (1-\lambda)b_{11} & \dots & (1-\lambda)b_{1n} \\ \vdots & & \vdots \\ (1-\lambda)b_{n1} & \dots & (1-\lambda)b_{nn} \end{bmatrix} \\ &= \begin{bmatrix} \lambda a_{11} + (1-\lambda)b_{11} & \dots & \lambda a_{1n} + (1-\lambda)b_{1n} \\ \vdots & & \vdots \\ \lambda a_{n1} + (1-\lambda)b_{n1} & \dots & \lambda a_{nn} + (1-\lambda)b_{nn} \end{bmatrix} \end{aligned}$$

So $\sum_i c_{ij} = \sum_i (\lambda a_{ij} + (1-\lambda) b_{ij}) = \lambda \sum_i a_{ij} + (1-\lambda) \sum_i b_{ij}$. Since A and B are two bistochastic matrices then $\sum_i a_{ij} = \sum_i b_{ij} = 1$ and thus $\sum_i c_{ij} = 1$. Similarly $\sum_j c_{ij} = 1$, therefore $C \in \mathcal{B}_N$. Then $\mathcal{B}_N$ is a convex set.

Let (B^n) be a sequence in $\mathcal{B}_N$. Every component (B^n_{ij}) is in $[0,1]$, so we can construct a subsequence $(B^{n_k}_{ij})$ that converges to $B = (B_{ij})$. We have that $B_{ij} \in [0,1]$ and

$$\sum_i B_{ij} = \sum_i \lim_{k \to +\infty} B^{n_k}_{ij} = \lim_{k \to +\infty} \sum_i B^{n_k}_{ij} = \lim_{k \to +\infty} 1 = 1.$$

Similarly, $\sum_j B_{ij} = 1$ and thus matrix $B \in \mathcal{B}_N$. Therefore, set $\mathcal{B}_N$ is a compact set. □

2.3 Proof of (2.1)

We reformulate the (KP) problem as a minimzation problem over the space of bistochastic matrices $\mathcal{B}_N$. To each $\gamma \in \Gamma(\mu, \nu)$ we associate the matrix $B = (b_{ij})$ with $b_{ij} = N\gamma_{x_i y_j}$, thus we obtain

$$(KP) = \min \left\{ \frac{1}{N} \sum_{x_i \in X, y_j \in Y} c(x_i, y_j) b_{ij} : B = (b_{ij}) \in \mathcal{B}_N \right\}.$$

Definition 2.3.1 (Extreme point). *An extreme point or a vertex of a convex set* S *in a vector space* X *is a point in* S *which doesn't lie in the interior of any segment in this set, i.e.* P *is an extreme point in* S *if there doesn't exist* $x, y \in S$ *and* $0 < t < 1$ *such that* $x \neq y$ *and* $P = tx + (1-t)y$*. The set of all extreme points of a set* S *is denoted by extreme(S).*

Example 2.3.2. *a and b are the extreme points of the closed interval $[a, b]$, and the extreme points of the closed unit ball in $\mathbb{R}^2$ is the unit circle.*

Definition 2.3.3. *Let S be a non-empty convex subset of a vector space X. A set $F \subseteq S$ is called a face of S if whenever $\lambda x + (1-\lambda)y \in F$ for $x, y \in S$ and $0 < \lambda < 1$ we have $x, y \in F$.*

Proposition 2.3.4. *Let S be a non-empty compact convex subset of a normed vector space X and $\mathcal{L}$ be any continuous linear functional on X, then the set $F_{\mathcal{L}} = \{x \in S : \mathcal{L}(x) = \max_{z \in S} \mathcal{L}(z)\}$ is a face of S.*

Proof. Denote by $\alpha = \max_{z \in S} \mathcal{L}(z)$. Let $\lambda x + (1-\lambda)y \in F_{\mathcal{L}}$ with $x, y \in S$ and $0 < \lambda < 1$ then $\mathcal{L}(\lambda x + (1-\lambda)y) = \alpha$, which by linearity implies that $\lambda\mathcal{L}(x) + (1-\lambda)\mathcal{L}(y) = \alpha$. Suppose that $\mathcal{L}(x) < \alpha$ or $\mathcal{L}(y) < \alpha$, then $\lambda\mathcal{L}(x) + (1-\lambda)\mathcal{L}(y) < \lambda\alpha + \alpha - \lambda\alpha < \alpha$ which is a contradiction. So $\mathcal{L}(y) = \mathcal{L}(x) = \alpha$. Thus $x, y \in F_{\mathcal{L}}$ and $F_{\mathcal{L}}$ is a face of S. $\square$

We list now the well-known Theorem of Hahn Banach and its geometrical implications [5].

Theorem 2.3.5. *(Hahn Banach). Let X be a vector space, q a semi norm defined on X, i.e. $q : X \to \mathbb{R}$ satisfying*

1. *$q(x + y) \leq q(x) + q(y)$*
2. *$q(\alpha x) \leq \alpha q(x)$ for all $\alpha > 0$ and $x \in X$*

and K a subspace of X. Suppose there exists a linear map $\Phi : K \to \mathbb{R}$ such that $\Phi(x) \leq q(x)$ for all $x \in K$, then the linear map Φ admits a linear extension $\Psi : X \to \mathbb{R}$ such that:

$$\Psi(x) = \Phi(x) \text{ and } \Psi(x) \leq q(x) \text{for all } x \text{ in } K$$

Lemma 2.3.6 (First Geometric form of Hahn Banach)**.** *. Let X be a vector space and C be a convex open subset of X containing 0. Suppose there exists $x_0 \notin C$, then there exists a linear map $\Psi : X \to \mathbb{R}$ such that*

$$\Psi(x_0) = 1 \text{ and } \Psi(v) < 1$$

for all $v \in C$.

Theorem 2.3.7. *(Hahn-Banach Separation Theorem). Let X be a vector space and A and B be two open convex disjoint subsets of X. Then there exists a linear map $\Phi : X \to \mathbb{R}$ such that:*

$$\Phi(a) < \Phi(b) \text{ for all } a \in A, b \in B.$$

Proof. Let $a_0 \in A$, and $b_0 \in B$ and define the set $C = \{a - b + b_0 - a_0 : a \in A, b \in B\}$. Since A and B are open, convex subsets of X then C is open and convex set, and $0 \in C$. Let $x_0 = b_0 - a_0$, since $A \cap B = \emptyset$ then $x_0 \notin C$ and so by Lemma 2.3.6 there exists a linear map $\Phi : X \to \mathbb{R}$ such that $\Phi(x_0) = 1$ and $\Phi(v) < 1$ for all $v \in C$. By linearity, since $\Phi(a - b + b_0 - a_0) < 1$ we obtain that

$$\Phi(a) < \Phi(b) + 1 - \Phi(x_0) = \Phi(b).$$

□

Definition 2.3.8. *(Convex hull). The convex hull of a set A is the smallest convex set containing A.*

Theorem 2.3.9. *(Krein-Milman theorem). Let S be a non-empty convex compact set of a normed vector space X. Then*

1. *The set of extreme points is non empty.*
2. *S is the closure of the convex hull of its extreme points.*

Proof. Let $\mathcal{F}$ be the set of all compact faces of S. As $S \in \mathcal{F}$ then $\mathcal{F} \neq \phi$. Let$(\mathcal{F}, \leq)$ be partially ordered such that

$$F_1 \leq F_2 \iff F_2 \subseteq F_1.$$

Let $C = \{F_i, i \in I\}$ be a chain in $\mathcal{F}$ and take $G = \bigcap_{i \in I} F_i$. Let $x, y \in S, 0 < t < 1$ and $tx + (1 - t)y \in G$ then $tx + (1 - t)y \in F_i$ for all $i \in I$ which implies that $x, y \in F_i$ for all $i \in I$ which in its turn gives us that $x, y \in G$. So G is a compact face of S. Obviously, $F_i \leq G$. It follows that G is an upper bound of C. Hence, using Zorn's lemma the set $(\mathcal{F}, <)$ contains a maximal element $\mathcal{M}$. We claim that $\mathcal{M}$ contains 1 element. Suppose to the contrary that it contains two distinct elements x and y. Then by Theorem 2.3.7 there exists a continuous linear map $\mathcal{L}$ such that $\mathcal{L}(x) < \mathcal{L}(y)$. By Proposition 2.3.4, the set $F_{\mathcal{L}} = \{x \in \mathcal{M} : \mathcal{L}(x) = \max_{z \in \mathcal{M}} \mathcal{L}(z)\}$ is a face of $\mathcal{M}$. Thus $F_{\mathcal{L}}$ is a face of S with $M < F_{\mathcal{L}}$ which is a contradiction. So $\mathcal{M} = \{a\}$. We show that $a \in extreme(S)$, in fact if $a = tx + (1 - t)y$ with $x, y \in S$ and $0 < t < 1$ then $x, y \in \mathcal{M}$ since $\mathcal{M}$ is a face which implies that $x = y = a$, concluding the proof of 1.

We next show 2. Since S is compact, then $extreme(S) \subseteq S$, and so by convexity

$$\overline{conv}[extreme(S)] \subseteq S.$$

Suppose the inclusion is proper then there exists an element $b \in S$ and $b \notin \overline{conv}[extreme(S)]$. Again, applying Theorem 2.3.7 there exists a continuous linear map $\mathcal{L}$ such that $\max \mathcal{L}(z) < \mathcal{L}(b)$ where the maximum is taken over all z in the compact convex set $\overline{conv}[extreme(S)]$. As a is an extreme point then $a \in \overline{conv}[extreme(S)]$ which implies that $\mathcal{L}(a) < \mathcal{L}(b)$. On the other hand, $F_{\mathcal{L}} = \{x \in S : \mathcal{L}(x) = \max_{z \in S} \mathcal{L}(z)\}$ is a face of S containing an extreme point a since $a \in \bigcap_{i \in I} F_i$, then we get $\mathcal{L}(a) = \max \mathcal{L}(z) \geq \mathcal{L}(b)$ which is a contradiction. □

Proposition 2.3.10. *Let S be a non-empty convex set of a normed vector space X. Then the following are equivalent*

1. *P is an extreme point of S.*
2. *If $P = \frac{A+B}{2}$ with $A, B \in S$ then A=B.*

Proof. (1) $\implies$ (2) from the definition of the extreme points. For the converse, assume (2) and $0 < t < 1$ such that $P = tx + (1-t)y$ for some $x, y \in S$ distinct. Let $m = \min(\|P - x\|, \|P - y\|)$. $A = P - \frac{m}{2}\frac{x-y}{\|x-y\|}$, and $B = P + \frac{m}{2}\frac{x-y}{\|x-y\|}$. Since $P = \frac{A+B}{2}$ then from 2. $A = B$, implying that $m = 0$ and so $x = y = P$ a contradiction. $\square$

Theorem 2.3.11. *(Birkhoff and von Neumann). The extreme points of $\mathcal{B}_N$ are the permutation matrices $\mathcal{P}_N$. In particular, $\mathcal{B}_N = \overline{conv}\{\mathcal{P}_N\}$.*

Proof. By Krein-Milman theorem, and Proposition 2.2.5 $extreme(\mathcal{B}_N) \neq \phi$.

We first prove that $\mathcal{P}_N \subseteq extreme(\mathcal{B}_N)$. Let $P \in \mathcal{P}_N$ and $P = \frac{A+B}{2}$ where $A, B \in \mathcal{B}_N$. We can notice that as $P_{ij} \in \{0,1\}$ then $A_{ij}, B_{ij} \in \{0,1\}$ which implies that $A = B$. It follows from proposition 2.3.10 that $P \in extreme(\mathcal{B}_N)$.

We prove the second inclusion. Let $P \in extreme(\mathcal{B}_N)$ and suppose that $P \notin \mathcal{P}_N$. Then there exists an entry $P_{i_1 j_1}$ such that $P_{i_1 j_1} \in (0,1)$. As the sum of rows and columns of P is an integer then there is another non integer entry $P_{i_1 j_2}$ and this in its turn gives another non integer entry $P_{i_2 j_2}$. We repeat this process $N^2 + 1$ by the Pigeonhole principle one of the entries $P_{i_m j_m}$ repeats. Let A be a matrix obtained from P by adding ϵ to all the entries $P_{i_k j_k}$ and subtracting ϵ from all the entries $P_{i_k j_{k+1}}$, while B be the matrix obtained from P by subtracting ϵ from all the entries $P_{i_k j_k}$ and adding $\epsilon > 0$ to all the entries $P_{i_k j_{k+1}}$. We choose ϵ small enough so that $A, B \in \mathcal{B}_N$ and $P = \frac{A+B}{2}$ with $A \neq B$. So by proposition 2.3.10, P is not an extreme point which is a contradiction. We conclude using the Krein-Milman theorem again that $\mathcal{B}_N = \overline{conv}\{\mathcal{P}_N\}$

Theorem 2.3.12. *Given X and Y two finite sets with same cardinal N, $c : X \times Y \to \mathbb{R}$ and μ and ν two uniform probability measures over these sets then $(AP) = (KP)$.*

Proof. The set of permutation is strictly included in the set of bistochastic matrices so $(KP) \leq (AP)$. As $(KP) = \min_{P \in \mathcal{B}_N} \sum_{i,j} c(x_i, y_j) P_{ij}$ then this minimum is attained at an extreme point of the set of bistochastic matrices. So it follows from the Birkhoff theorem that this minimum is attained at a permutation matrix. Therefore $(AP) = (KP)$. $\square$

Chapter 3

Kantorovich Functional

In this chapter we will go through computing the super-differential of Kantorovich functional that will be used to construct algorithms to solve the discrete optimal transport.

3.1 The Dual Problem

Suppose there is a worker who needs to transfer an amount of coal from N mines to N factories such that the maximum cost he is willing to pay for this transportation is $c(x, y)$. A vendor could solve his problem as follows, he sets a price $\phi(x)$ for loading one unit of coal at position x, and a price $-\psi(y)$ for unloading it at destination y. In order for the vendor to attract the worker's attention to his offer he should make the sum $\phi(x) - \psi(y)$ always less than or equal the cost $c(x, y)$. But as the vendor wants to maximize the total sum of $\phi(x) - \psi(y)$.

Definition 3.1.1. *(The Dual Problem). Let X and Y be two finite spaces, $c : X \times Y \to \mathbb{R}$ and μ, ν be two uniform probability measures. We then define the dual problem as follows*

$$(DP) = \sup \left\{ \sum_{x \in X} \phi(x)\mu_x - \sum_{y \in Y} \psi(y)\nu_y : \phi \in C(X), \psi \in C(Y), \text{ and } \phi(x) - \psi(y) \leq c(x, y) \right\}$$

To study this problem we introduce the notions of $c-$ and $\bar{c}$ transforms which are generalization of the known Legendre transform, [5, Section 1.4].

Definition 3.1.2. *Given compact sets X and Y and $c : X \times Y \mapsto \mathbb{R}$:*

- *the $c-$transform of a function $\psi : Y \to \mathbb{R}$ is given by*

$$\psi^c : X \to \mathbb{R} : \psi^c(x) = \inf_{y \in Y} (c(x, y) + \psi(y))$$

- *The $\bar{c}$ transform of a function $\phi : X \to \mathbb{R}$ is given by*

$$\phi^{\bar{c}} : Y \to \mathbb{R} : \phi^{\bar{c}}(x) = \sup_{x \in X} (-c(x, y) + \phi(x)),$$

Let

$$(DP_c) = \max_{\psi \in C(Y)} \left\{ \sum_{x \in X} \psi^c(x)\mu_x - \sum_{y \in Y} \psi(y)\nu_y \right\}$$

One can notice that

$$\psi^c(x) = \inf_{y \in Y} (c(x,y) + \psi(y)) \leq c(x,y) + \psi(y)$$

which gives $(DP_c) \leq (DP)$. But the DP problem is over all functions ϕ and ψ such that for every $x \in X$, $y \in Y$

$$\phi(x) \leq c(x,y) + \psi(y)$$

for which implies that $\phi(x) \leq \psi^c(x)$, for all admissible functions ϕ, and ψ. So

$$\sum_{x \in X} \phi(x)\mu_x - \sum_{y \in Y} \psi(y)\nu_y \leq \sum_{x \in X} \psi^c(x)\mu_x - \sum_{y \in Y} \psi(y)\nu_y,$$

implying that $(DP) \leq (DP_c)$, concluding that one can reformulate the dual problem (DP) as an unconstrained maximization problem with only one variable ψ.

$$(DP) = \max_{\psi \in C^(Y)} \left\{ \sum_{x \in X} \psi^c(x)\mu_x - \sum_{y \in Y} \psi(y)\nu_y \right\}$$

Definition 3.1.3. *(Kantorovich functional) The Kantorovich functional is defined on $C(Y)$ by*

$$\mathcal{K}(\psi) = \sum_{x \in X} \psi^c(x)\mu_x - \sum_{y \in Y} \psi(y)\nu_y$$

Notice that the Dual problem reduces then to maximize the Kantorovich functional over the space of continuous maps over Y, which requires in the case when $\mathcal{K}$ is differentiable to find its gradient, however in our case $\mathcal{K}$ might not be differentiable requiring then introducing the notion of superdifferential.

3.2 Gradient of the Kantorovich Functional

Definition 3.2.1. *(Superdifferential) Let $F : \mathbb{R}^N \mapsto \mathbb{R} \cup \{-\infty\}$. The superdifferential of the function F at the point $x \in \mathbb{R}^N$ is*

$$\partial^+ F(x) = \{v \in \mathbb{R}^N : F(y) \leq F(x) + v \cdot (y - x) \,\forall y \in Y\}$$

We say that v is a supergradient of f at the point x.

Proposition 3.2.2. *Let $F : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ then the following holds:*

1. *F is concave if and only if $\partial^+ F(x) \neq \emptyset \forall x \in \mathbb{R}^N$*

2. *For concave functions F, $\partial^+ F(x) = conv\{\lim_{n\to\infty} \nabla F(x_n) : x_n$ converges to x, $\nabla F(x_n)$ exists and admits a limit$\}$*

Proof. 1. if F is concave, then at each point there exists a plane tangent to the graph of F such that the graph of F is below the plane. Letting p be the normal vector to the tangent plain at the point $(x, F(x))$ given by $\mathcal{L}(z) = F(x) + p \cdot (z - x)$. We then Have for every y, $F(y) \leq L(y) = F(x) + p \cdot (y - x)$, implying that $p \in \partial^+ F(x)$.

Conversely, assume that the superdifferential is not empty. Let $x, y \in \mathbb{R}^n$, and $z = \lambda x + (1 - \lambda)y$, then as $\partial^+ F(z) \neq \phi$ there exists p such that

$$F(x) \leq F(z) + p \cdot .(x - z) \text{ and } F(y) \leq F(z) + p \cdot (y - z)$$

multiplying the first inequality by λ and the second one by $1 - \lambda$ and then adding we get

$$\lambda F(x) + (1-\lambda)F(y) \leq F(z) + p \cdot [\lambda x + (1-\lambda)y - z] = F(z) = F(\lambda x + (1-\lambda)y)$$

Which implies that F is concave function.

For the proof of 2, notice that in the argument of 1 if F is differentiable at F then $\partial^+ F(x) = \nabla F(x)$. In the more general case, the result follows from the fact that concave functions are differentiable almost everywhere (with respect to the Lebesgue measure) and the set of differentiability is dense in $\mathbb{R}^N$, see [9, Theorems, 25.5, and 25.6]

□

Our goal next is to calculate the superdifferential of the operation $\mathcal{K}$. We require the following definitions.

Definition 3.2.3. *Given $c : X \times Y \mapsto \mathbb{R}$, and $\psi : Y \mapsto \mathbb{R}$, the $c-$ subdifferential, $\partial^c \psi$ is subset of $X \times Y$ defined by*

$$\partial^c \psi = \{(x, y) \in X \times Y : \psi^c(x) - \psi(y) = c(x, y)\}$$

Definition 3.2.4. *With the setting of the previous definition, and μ probability measure on X, we denote by $\Gamma_\psi(\mu)$ the set of probability measures on $X \times Y$ such that its first marginal is μ and supported on the c-subdifferential $\partial^c \psi$, i.e*

$$\Gamma_\psi(\mu) = \{\gamma \in P(X \times Y) : \sum_{y \in Y} \gamma_{xy} = \mu_x \text{ and } spt(\gamma) \subseteq \partial^c \psi\}$$

Recall that in our case X and Y are finite, and we want to calculate the superdifferential of $\mathcal{K}$ defined in this case on $\mathbb{R}^Y$ (which is simply the space of functions on $\mathbb{R}^N$). We obtain the following theorem.

Theorem 3.2.5. *Given finite spaces X and Y, μ, ν probability measures on X and Y and $c : X \times Y \mapsto \mathbb{R}$. Then $\mathcal{K}$ is concave, more particularly for every $\psi_0 \in \mathbb{R}^Y$ we have*

$$\partial^+ \mathcal{K}(\psi_0) = \left\{ \sum_{x \in X} \gamma_{xy} - \nu_y : \gamma \in \Gamma_{\psi_0}(\mu) \right\}$$

Proof. Fix $\gamma \in \Gamma_{\psi_0}(\mu)$. We have for every $\psi \in \mathbb{R}^Y$

$$\begin{aligned}\mathcal{K}(\psi) = \sum_{x\in X} \psi^c(x)\mu_x - \sum_{y\in Y}\psi(y)\nu_y &= \sum_{x\in X} \min_{y\in Y}(c(x,y)+\psi(y))\mu_x - \sum_{y\in Y}\psi(y)\nu_y \\ &= \sum_{y\in Y}\sum_{x\in X}\min_{y\in Y}(c(x,y)+\psi(y))\gamma_{xy} - \sum_{y\in Y}\psi(y)\nu_y \\ &\leq \sum_{y\in Y}\sum_{x\in X}(c(x,y)+\psi(y))\gamma_{x,y} - \sum_{y\in Y}\psi(y)\nu_y\end{aligned}$$

where the second equality is because $\sum_{y\in Y}\gamma_{xy} = \mu_x$.

On the other hand, for $\psi = \psi_0$, We have that $\psi_0^c(x)\gamma_{xy} = (c(x,y)+\psi_0(y))\gamma_{xy}$ since $\gamma_{xy} = 0$ if $(x,y) \notin \partial^c\psi_0$. We then obtain that

$$\mathcal{K}(\psi_0) = \sum_{y\in Y}\sum_{x\in X}(c(x,y)+\psi_0(y))\gamma_{xy} - \sum_{y\in Y}\psi_0(y)\nu_y.$$

Following that we have

$$\mathcal{K}(\psi) - \mathcal{K}(\psi_0) \leq \sum_{y\in Y}(\psi-\psi_0)\left(\sum_{x\in X}\gamma_{xy} - \nu_y\right).$$

Hence

$$\left\{\sum_{x\in X}\gamma_{xy} - \nu_y | \gamma \in \Gamma_{\psi_0}(\mu)\} \subseteq \partial^+\mathcal{K}(\psi_0)\right\}$$

Having obtained that $\partial^+\mathcal{K}(\psi) \neq \emptyset$, we deduce from Proposition 3.2.2 that $\mathcal{K}$ is convex.

We now use the second part of the proposition to show the second inclusion. Let $v = \lim_{n\to\infty}\nabla\mathcal{K}(\psi^n)$ where $(\psi^n)_{n\in\mathbb{N}}$ converges to ψ_0, its gradient exists and admits a limit. For each $n \in N$, there exists $\gamma^n \in \Gamma_{\psi^n}(\mu)$ such that $\nabla\mathcal{K}(\psi^n) = \sum_{x\in X}\gamma^n_{xy} - \nu_y$. By the compactness of $P(X\times Y)$, γ^n has a convergent subsequence that converges weakly to γ.

Now $\sum_{y\in Y}\gamma_{xy} = \sum_{y\in Y}\lim_{k\to\infty}\gamma^k_{xy} = \lim_{k\to\infty}\sum_{y\in Y}\gamma^k_{xy} = \lim_{k\to\infty}\mu_x = \mu_x$. Similarly, γ is supported on the c-subdifferential $\partial^c\psi_0$. So we get that $\gamma \in \Gamma_{\psi_0}(\mu)$. It follows then that

$$\left\{\lim_{n\to\infty}\nabla\mathcal{K}(\psi^n) : \psi^n \to \psi_0,\ \nabla\psi_n \text{ exists and converges}\right\} \subseteq \left\{\sum_{x\in X}\gamma_{xy} - \nu_y : \gamma \in \Gamma_{\psi_0}(\mu)\right\}.$$

By taking the convex hall of the first set we conclude that

$$\partial^+\mathcal{K}(\psi_0) \subseteq \left\{\sum_{x\in X}\gamma_{xy} - \nu_y : \gamma \in \Gamma_{\psi_0}(\mu)\right\}$$

□

Motivated by our previous result, we define the Laguerre cells as follows

Definition 3.2.6. *(Laguerre Cell). Let ψ be a function on Y and $c : X \times Y \mapsto \mathbb{R}$, then for each point $y \in Y$ we define the Laguerre cell associated to y to be the subdifferential of ψ at this point, that is*

$$Lag_y(\psi) = \{x \in X : (x, y) \in \partial^c \psi\} = \{x \in X : \psi^c(x) - \psi(y) = c(x, y)\}$$

As well we define the strict Laguerre cell associated to y

$$SLag_y(\psi) = \{x \in X : \forall z \in Y, c(x, y) + \psi(y) < c(x, z) + \psi(z)\}$$

To clarify, given $x \in X$, $x \in Lag_y(\psi)$ for some $y \in Y$ if and only if $\min_{z \in Y} c(x, z) + \psi(x)$ is attained at $z = y$. A priori, the Laguerre cell might be empty or contains more than one element. The strict Laguerre cell however is defined in a way that it contains at most one element.

Corollary 3.2.7. *Given X and Y finite. Let $\psi \in \mathbb{R}^Y$ and $y_0 \in Y$. We define $\kappa(t) = \mathcal{K}(\psi^t)$ where $\psi^t = \psi + t1_{y_0}$. Then, $\kappa(t)$ is concave and*

$$\partial \kappa^+(t) = [\mu(SLag_{y_0}(\psi^t)) - \nu(y_0), \mu(Lag_{y_0}(\psi^t)) - \nu(y_0)]$$

This implies that $\mathcal{K}$ is differentiable at ψ if and only if $\mu(Lag_y(\psi) \setminus SLag_y(\psi)) = 0$ for all $y \in Y$ and we get in this case that

$$\nabla \mathcal{K}(\psi) = (\mu(Lag_y(\psi)) - \nu(y))_{y \in Y}$$

Proof. We have $\partial^+ \kappa(t) = \{r1_{y_0} : r \in \partial^+ \mathcal{K}(\psi^t)\} = \{\sum_{x \in X} \gamma_{xy_0} - \nu_{y_0} | \gamma \in \Gamma_{\psi^t}(\mu)\}$. So we need to show two things

$$\max\{\sum_{x \in X} \gamma_{xy_0} | \gamma \in \Gamma_{\psi^t}(\mu)\} = \mu(Lag_{y_0}(\psi^t))\}$$

and

$$\min\{\sum_{x \in X} \gamma_{xy_0} | \gamma \in \Gamma_{\psi^t}(\mu)\} = \mu(SLag_{y_0}(\psi^t))\}.$$

Let's consider $Z = X \setminus Lag_{y_0}(\psi^t)$ then we have that

$$\sum_{x \in X} \gamma_{xy_0} = \sum_{x \in Z} \gamma_{xy_0} + \sum_{x \in Lag_{y_0}(\psi^t)} \gamma_{xy_0}$$

But $spt(\gamma) \subseteq \partial^c(\psi^t)$ and $Z \times \{y_0\} \cap Lag_{y_0}(\psi^t) = 0$ which gives us that $\sum_{x \in Z} \gamma_{xy_0} = 0$ So this implies that

$$\sum_{x \in X} \gamma_{xy_0} = \sum_{x \in Lag_{y_0}(\psi^t)} \gamma_{xy_0} \leq \sum_{y \in Y} \sum_{x \in Lag_{y_0}(\psi^t)} \gamma_{xy} = \sum_{x \in Lag_{\psi_0}(\psi^t)} \mu_x = \mu(Lag_{y_0}(\psi^t))$$

So

$$\max\{\sum_{x \in X} \gamma_{xy_0} : \gamma \in \Gamma_{\psi^t}(\mu)\} \leq \mu(Lag_{y_0}(\psi^t)).$$

We still need to show equality. Consider the following map $T : X \to Y$ such that $Tx = y_0$ if $x \in Lag_{y_0}(\psi^t)$, and otherwise $Tx = z$, with z chosen arbitrarily so that $x \in Lag_z(\psi^t)$. Associate to T the following transport plan $\gamma^T = (id, T)_{\#}\mu$, i.e $\gamma^T(A \times B) = \mu(A \cap T^{-1}B)$. Notice that if $(x, y) \in spt(\gamma^T)$ then $\gamma^T_{xy} = \mu(\{x\} \cap \{T^{-1}(y)\}) \neq 0$ which gives that $Tx = y$. Then $x \in Lag_y(\psi^t)$, $(x, y) \in \partial^c\psi^t$. This gives us that $spt(\gamma^T) \subseteq \partial^c\psi^t$.

Notice that

$$\sum_{y\in Y} \gamma_{xy} = \sum_{y\in Y} \gamma(\{x\} \times \{y\}) = \sum_{y\in Y} \mu(\{x\} \cap T^{-1}(\{y\})) = \mu_x,$$

which implies that $\gamma^T \in \Gamma_{\psi^t}(\mu)$. Now by the construction of γ we have that $\sum_{x\in X} \gamma^T_{xy_0} = \mu(X \cap T^{-1}(y_0)) = \mu(Lag_{y_0}\psi^t)$.

For the other equality, we will proceed in the same way. Let $Z = X \setminus SLag_{y_0}(\psi^t)$ then we have that that for every $\gamma \in \Gamma_{\psi^t}(\mu)$

$$\begin{aligned}\sum_{x\in X} \gamma_{xy_0} = \sum_{x\in Lag_{y_0}(\psi^t)} \gamma_{xy_0} \geq \gamma(SLag_{y_0}(\psi^t) \times \{y_0\}) &= \sum_{x\in SLag_{y_0}(\psi^t)} \gamma_{xy_0} \\ &= \sum_{x\in SLag_{y_0}(\psi^t)} \sum_{y\in Y} \gamma_{xy}\delta_x \\ &= \sum_{x\in SLag_{y_0}} \mu_x = \mu(SLag_{y_0}(\psi^t)).\end{aligned}$$

So we get that $\min\{\sum_{x\in X} \gamma_{xy_0} : \gamma \in \Gamma_{\psi^t}(\mu)\} \geq \mu(SLag_{y_0}(\psi^t))\}$ By taking the same map T as before and plan γ^T and proceeding with the same argument we get equality.

Now back to the Kantorovich functional, in order for it to be differentiable, $\kappa(t)$ should be differentiable for all y in Y. So in other words, by concavity, $\partial\kappa^+(t)$ should only contain one element. This implies that $\mu(Lag_y(\psi^t)) = \mu(Slag_y(\psi^t))$ for all $y \in Y$ so $\mu(Lag_y(\psi) \setminus SLag_y(\psi)) = 0$ for all $y \in Y$. Thus we get that in this case

$$\nabla\mathcal{K}(\psi) = \{\mu(Lag_y(\psi)) - \nu(\{y\}) : y \in Y\}$$

$\square$

Remark 3.2.8. *For the discrete uniform case with $\mu = \frac{1}{N}\sum_{x\in X} \delta_x$ and $\nu = \frac{1}{N}\sum_{y\in Y} \delta_y$ being as follows, the gradient of the Kantorovich functional, when it exists, is given by*

$$\nabla\mathcal{K}(\psi) = \left\{\frac{1}{N}(card(Lag_y(\psi)) - 1)|y \in Y\right\}$$

3.3 $(KP) = (DP)$

We prove in this section that dual and Kantorovich problem are equivalent.

Definition 3.3.1. *Let X and Y be two metric space, and $c : X \times Y \to \mathbb{R}$. We say that a set $\Gamma \subseteq X \times Y$ is c-cyclically monotone (briefly c-CM) if for every $k \in \mathbb{N}$, permutation σ of $\{1, 2, \cdots, k\}$ we have*

$$\sum_{i=1}^{k} c(x_i, y_i) \leq \sum_{i=1}^{k} c(x_i, y_{\sigma(i)}).$$

for all $(x_1, y_1), \cdots, (x_k, y_k) \in \Gamma$.

Theorem 3.3.2. *Let γ be an optimal transport plan in the case when X and Y are finite, then $spt(\gamma)$ is a c-CM set.*

Proof. Suppose that there exist $k \in \mathbb{N}$, a permutation σ, and $(x_1, y_1), ..., (x_k, y_k) \in spt(\gamma)$ such that

$$\sum_{i=1}^{k} c(x_i, y_i) > \sum_{i=1}^{k} c(x_i, \sigma(x_i))$$

Take now $0 < \epsilon < \frac{1}{2k}(\sum_{i=1}^{k} c(x_i, y_i) - \sum_{i=1}^{k} c(x_i, \sigma(x_i)))$. We will construct a measure $\tilde{\gamma}$ such that $\sum_{x,y} c(x, y)\gamma_{x,y} > \sum_{x,y} c(x, y)\tilde{\gamma}_{x,y}$ which contradicts the fact that γ is an optimal transport plan. Define the measure $\gamma_i \in P(X \times Y)$ as follows

$$\gamma_i(x, y) = \delta_{(x_i,y_i)}$$

Let $\mu_i = (\pi_1)_{\#\gamma_i} = \delta_{x_i}$ and $\nu_i = (\pi_2)_{\#\gamma_i} = \delta_{y_i}$. Take also the product measure $\tilde{\gamma_i} = \delta_{x_i}(x)\delta_{\sigma(x_i)}(y)$. We then take

$$\tilde{\gamma} = \gamma - \frac{\min \gamma(x_i, y_i)}{2k} \sum_{i=1}^{k} (\gamma_i - \tilde{\gamma_i})$$

Taking any set $A \subset X \times Y$ we have

$$\begin{aligned}
\tilde{\gamma}(A) &\geq \gamma(A) - \frac{\min \gamma(x_i, y_i)}{2k} \sum_{i=1}^{k} (\gamma_i(A)) \\
&\geq \gamma(A)(1 - \frac{\min \gamma(x_i, y_i)}{2}) \\
&\geq \frac{1}{2}\gamma(A) \geq 0
\end{aligned}$$

and $\tilde{\gamma}(X \times Y) = 1$, then $\tilde{\gamma} \in P(X \times Y)$. Moreover, since $\gamma \in \Gamma(\mu, \nu), \gamma_i \in \Gamma(\mu_i, \nu_i), \tilde{\gamma_i} \in \Gamma(\mu_i, \nu_{\sigma(i)})$, then using the fact that σ is a permutation

$$(\pi_1)_{\#\tilde{\gamma}} = \mu - \frac{\min \gamma(x_i, y_i)}{2k} \sum_{i=1}^{k} (\mu_i - \mu_i) = \mu$$

$$(\pi_2)_{\#\tilde{\gamma}} = \nu - \frac{\min \gamma(x_i, y_i)}{2k} \sum_{i=1}^{k} (\nu_i - \nu_{\sigma(i)}) = \nu - \frac{\min \gamma(x_i, y_i)}{2k} \left(\sum_{i=1}^{k} \nu_i - \sum_{i=1}^{k} \nu_{\sigma(i)} \right) = \nu.$$

So we get that $\tilde{\gamma} \in \Gamma(\mu, \nu)$. Now,

$$\begin{aligned}
\sum_{x,y} c(x,y)\gamma_{x,y} - \sum_{x,y} c(x,y)\tilde{\gamma}_{x,y} &= \frac{\min \gamma(x_i, y_i)}{2N} \sum_{i=1}^{N} c(x,y)[(\delta_{(x_i,y_i)} - \delta_{(x_i,\sigma(x_i))})(x,y)] \\
&= \frac{\min \gamma(x_i, y_i)}{2k} \sum_{i=1}^{k} [c(x_i, y_i) - c(x_i, \sigma(x_i))] \\
&> \frac{\min \gamma(x_i, y_i)}{2k} 2k\epsilon \\
&= \epsilon \min \gamma(x_i, y_i) > 0
\end{aligned}$$

Hence, a contradiction so $spt(\gamma)$ is a c-CM set. □

The next theorem proof appears in [10] and we include it for completeness.

Theorem 3.3.3. *Let $c : X \times Y \to \mathbb{R}$. If $A \neq \emptyset$ is a c-CM set in $X \times Y$, then there exists a function $\psi : Y \to \mathbb{R}$ such that*

$$A \subseteq \{(x, y) \in X \times Y : \psi^c(x) - \psi(y) = c(x, y)\}$$

Proof. Fix $(x_0, y_0) \in A$. Define the following function on X.

$\phi(x) = \min\{c(x, y_n) - c(x_n, y_n) + c(x_n, y_{n-1}) - c(x_{n-1}, y_{n-1}) + ... + c(x_1, y_0) - c(x_0, y_0) : n \in \mathbb{N}, (x_i, y_i) \in A$ for all $i = 1, 2..., n\}$.

Define also the following function on Y

$\psi(y) = \min\{-c(x_n, y) + c(x_n, y_{n-1}) - c(x_{n-1}, y_{n-1}) + ... + c(x_1, y_0) - c(x_0, y_0) : n \in \mathbb{N}, (x_i, y_i) \in A$ for all $i = 1, 2..., n; y_n = y\}$.

The goal is to show that $\phi(x) = \psi^c(x)$. For $y \in \pi_2(A)$, we have that for every $n \in \mathbb{N}$ and $(x_i, y_i) \in A$ for all $i = 1, ..., n$ with $y_n = y$,

$$\begin{aligned}
\phi(x) &\leq c(x, y) - c(x_n, y) + c(x_n, y_{n-1}) - c(x_{n-1}, y_{n-1}) + ... + c(x_1, y_0) - c(x_0, y_0) \\
&\leq c(x, y) + (-c(x_n, y) + c(x_n, y_{n-1}) - c(x_{n-1}, y_{n-1}) + ... + c(x_1, y_0) - c(x_0, y_0)) \\
&\leq c(x, y) + \psi(y)
\end{aligned}$$

This gives us that $\phi(x) \leq c(x, y) + \psi(y)$ for every $y \in \pi_2(A)$. The inequality also holds for $y \notin \pi_2(A)$ since in this case the right hand side is equal to $+\infty$. Taking the minimum over all $y \in Y$ we get that $\phi(x) \leq \psi^c(x)$ for every $x \in X$.

Now we prove that $\phi(x) \geq \psi^c(x)$. Notice that for every $n \in \mathbb{N}$ and for all $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n) \in A$ we have:

$$\begin{aligned}
\psi^c(x) &= \min_{y \in Y}(c(x, y) + \psi(y)) \\
&\leq c(x, y_n) + \psi(y_n) \\
&\leq c(x, y_n) - c(x_n, y_n) + c(x_n, y_{n-1}) - c(x_{n-1}, y_{n-1}) + ... + c(x_1, y_0) - c(x_0, y_0)
\end{aligned}$$

Taking the minimum over all $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n) \in A$ gives that $\psi^c(x) \leq \phi(x)$. Hence we get $\psi^c(x) = \phi(x)$.

Then we have, $\phi(x) - \psi(y) = \min_{y \in Y}(c(x, y) + \psi(y)) - \psi(y) \leq c(x, y)$

For the other inequality, for $n \in \mathbb{N}$ and $(x, y), (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) \in A$ we have

$c(x, y_n) - c(x_n, y_n) + c(x_n, y_{n-1}) - c(x_{n-1}, y_{n-1}) + ... + c(x_1, y_0) - c(x_0, y_0)$
$= c(x, y) + [-c(x, y) + c(x, y_n) - c(x_n, y_n) + c(x_n, y_{n-1}) - c(x_{n-1}, y_{n-1}) + ... + c(x_1, y_0) - c(x_0, y_0)]$
$\geq c(x, y) + \psi(y)$

Now taking the minimum over all $n \in \mathbb{N}$ and $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n) \in A$, we get that $\phi(x) - \psi(y) \geq c(x, y)$. Hence

$$A \subseteq \{(x, y) \in X \times Y : \psi^c(x) - \psi(y) = c(x, y)\}$$

□

Remark 3.3.4. *If γ is an optimal transport plan, then by Theorems 3.3.2 and 3.3.3 there exists a function ψ such that*

$$spt(\gamma) \subset \partial^c \psi = \{(x, y) \in X \times Y : \psi^c(x) - \psi(y) = c(x, y)\}$$

Theorem 3.3.5. *Let X and Y be two finite sets with cardinal N, $c : X \times Y \to \mathbb{R}$, γ a probability measure on $X \times Y$ solution to the (KP) problem and μ and ν be the corresponding marginals of γ. let ψ_0 the solution corresponding to the (DP) problem then*

$$(KP) = (DP)$$

Proof. Notice that for any price function ψ we have

$$\begin{aligned}
\sum_{i=1}^{N} \psi^c(x_i)\mu_i - \sum_{j=1}^{N} \psi(y_j)\nu_j &= \sum_{i=1}^{N} \psi^c(x_i) \sum_{j=1}^{N} \gamma_{ij} - \sum_{j=1}^{N} \psi(y_j) \sum_{i=1}^{N} \gamma_{ij} \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}(\psi^c(x_i) - \psi(y_j)) \\
&\leq \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} c(x_i, y_j) = (KP)
\end{aligned}$$

This gives that $(DP) \leq (KP)$.

For the other inequality, Suppose that γ is a solution for (KP), then by Remark 3.3.4 there exists a function ψ_0 such that

$$spt(\gamma) \subset \partial^c \psi_0 = \{(x, y) \in X \times Y : \psi_0^c(x) - \psi_0(y) = c(x, y)\}$$

then we have the following

$$
\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}c(x_i,y_j) &= \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}(\psi^c(x_i)-\psi(y_j)) \\
&= \sum_{i=1}^{N}\psi^c(x_i)\sum_{j=1}^{N}\gamma_{ij} - \sum_{j=1}^{N}\psi(y_j)\sum_{i=1}^{N}\gamma_{ij} \\
&= \sum_{i=1}^{N}\psi^c(x_i)\mu_i - \sum_{j=1}^{N}\psi(y_j)\nu_j \\
&\leq \sum_{i=1}^{N}\psi_0^c(x_i)\mu_i - \sum_{j=1}^{N}\psi_0(y_j)\nu_j
\end{aligned}
$$

Then $(KP) \leq (DP)$. □

We connect our result now to the original Assignment Propblem (AP), and prove a relation between the maximizer ψ of the Kantorovich potential and the bijection σ constructed in Theorem 2.3.12.

Proposition 3.3.6. *Let $\psi : Y \to \mathbb{R}$ then the following are equivalent*

- *ψ is a global maximizer of the Kantorovich functional.*
- *There exists a bijection $\sigma : X \mapsto Y$ such that*

$$c(x,\sigma(x)) + \psi(\sigma(x)) = \min_{y\in Y} c(x,y) + \psi(y).$$

Moreover, this bijection σ is the solution for the linear (AP) problem.

Proof. For the first direction, Suppose that ψ is a solution for the Dual Problem. Then since $(KP) = (DP)$

$$\sum_{j=1}^{N}\sum_{i=1}^{N}c_{ij}\gamma_{ij} = \sum_{i=1}^{N}\psi^c(x_i)\mu_i - \sum_{j=1}^{N}\psi(y_j)\nu_j$$

where γ_{ij} is the optimal solution for the (KP) problem. But $\mu_i = \sum_{j=1}^{N}\gamma_{ij}$ and $\nu_j = \sum_{i=1}^{N}\gamma_{ij}$. We get then

$$\sum_{i=1}^{N}\psi^c(x_i)\mu_i - \sum_{j=1}^{N}\psi(y_j)\nu_j = \sum_{i=1}^{N}\psi^c(x_i)\sum_{j=1}^{N}\gamma_{ij} - \sum_{j=1}^{N}\psi(y_j)\sum_{i=1}^{N}\gamma_{ij}$$

this gives the following

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}(c_{ij}+\psi^c(x_i)-\psi(y_j)) = 0$$

Now as γ is a solution for the Kantorovich Problem, let P be the corresponding permutation matrix corresponding to the (AP) solution by Theorem 2.3.12, and σ the corresponding bijection. Then $\gamma_{ij} > 0$ if and only if $y_j = \sigma(x_i)$ which is equivalent to

$$c_{i\sigma_i} + \psi^c(x_i) - \psi(\sigma(x_i)) = 0$$

For the backward direction, suppose that there exists a bijection $\sigma : X \mapsto Y$ such that $c(x, \sigma(x)) + \psi(\sigma(x)) = \min_{y \in Y} c(x, y) + \psi(y)$ then for this ψ we have the following

$$\begin{aligned}
\mathcal{K}(\psi) &= \sum_{x \in X} \psi^c \mu_x - \sum_{y \in Y} \psi(y) \nu_y \\
&= \sum_{x \in X} (\min_{y \in Y} c(x, y) + \psi(y)) \mu_x - \sum_{y \in Y} \psi(y) \nu_y \\
&= \sum_{x \in X} (c(x, \sigma(x)) + \psi(\sigma(x))) \mu_x - \sum_{y \in Y} \psi(y) \nu_y \\
&= \sum_{x \in X} (c(x, \sigma(x)) + \psi(\sigma(x))) \mu_x - \sum_{x \in X} \nu_{\sigma(x)} \psi(\sigma(x))
\end{aligned}$$

On the other hand, let ψ' be any functional on Y, we have the following

$$\begin{aligned}
\mathcal{K}(\psi') &= \sum_{x \in X} \psi'^c(x) \mu_x - \sum_{y \in Y} \psi'(y) \nu_y \\
&= \sum_{x \in X} \min_{y \in Y} (c(x, y) + \psi'(y)) \mu_x - \sum_{y \in Y} \psi'(y) \nu_y \\
&\leq \sum_{x \in X} c(x, \sigma(x)) + \psi'(\sigma(x))) \mu_x - \sum_{x \in X} \psi'(\sigma(x)) \nu_{\sigma(x)} \\
&= \mathcal{K}(\psi)
\end{aligned}$$

thus, ψ is a global maximizer of the Kantorovich functional. $\square$

Chapter 4

Bertsekas Auction Algorithm

We can see from Proposition 3.3.6 that if ψ is a solution for the dual problem, we can construct our bijection σ, by assigning for each $y \in Y$ an element $x \in Lag_y(\psi)$. A priori, $Lag_y(\psi)$ can contain more than one x or even could be empty if ψ was not our optimal solution for the dual problem. In the next Chapter, we will introduce the Bertsekas auction algorithm that keeps modifying ψ until we reach the maximizer one which satisfies the property that $lag_y(\psi)$ is a singleton set.

4.1 Motivation for the Bertsekas Algorithm

Bertsekas Algorithm is a coordinate ascent method that modifies the price $\psi \in \mathbb{R}^Y$ in order to reach a maximizer for the Kantorovich functional $\mathcal{K}$.

As we have seen in the previous chapter, in order to get a transport map, we need to find an optimal price function ψ that reduces $Lag_y(\psi))$ to a singleton set for all $y \in Y$. This can be reached by iteratively increasing the price ψ at each y. The question is: How much we can we increase the price by keeping at least one source x interested in y?

The answer to this question is given in the following section by defining the bidding increment.

4.2 Bidding Increment

In this section, we calculate the optimal increment, which is known as the bidding increment.

Definition 4.2.1. *Let $\psi \in \mathbb{R}^N$ and $y_0 \in Y$ such that $Lag_{y_0}(\psi) \neq \varnothing$. For each $x \in Lag_{y_0}(\psi)$ we define the following:*

$$Bid_{y_0}(\psi, x) = (\min_{y \in Y \setminus \{y_0\}} c(x, y) + \psi(y)) - (c(x, y_0) + \psi(y_0))$$

and

$$Bid_{y_0}(\psi) = max\{Bid_{y_0}(\psi, x), x \in Lag_{y_0}(\psi)\}$$

Remark 4.2.2. ***(Economic interpretation of the bidding increment).*** *Assume that there is a worker who want to transfer a certain amount of coal from N mines to N factories. Given a set prices $\psi(y)$ that corresponds to unloading the coal in position y, the best mines x where the coal is to be taken is the ones such that $x \in Lag_y(\psi)$. In other words, the worker would choose a mine x that minimizes $c(x,y)+\psi(y)$. However, the worker would like to increase his profit $\psi(y)$ as much as possible while keeping one of the mines $x \in Lag_y(\psi)$. Thus $Bid_y(\psi, x)$ tells us how much it is possible to increase the profit $\psi(y)$, and $Bid_y(\psi)$ is the maximum bidding that the working can increase, i.e. it is the best choice of increasing $\psi(y)$ such that at least one mine $x \in Lag_y(\psi)$.*

Proposition 4.2.3. *Let $\psi \in \mathbb{R}^N$ and $y_0 \in Y$ such that $Lag_{y_0}(\psi) \neq \varnothing$. Then the maximum for the function $\kappa(t) = \mathcal{K}(\psi + ty_0)$ is attained at $Bid_{y_0}(\psi)$.*

Proof. Let $\psi^t = \psi + t1_{y_0}$. For $t > 0$ we have $Lag_{y_0}(\psi^t) \subseteq Lag_{y_0}(\psi)$. Indeed, Let $x \in Lag_{y_0}(\psi^t)$ then

$$c(x,y_0)+\psi(y_0) \leq c(x,y_0)+\psi(y_0)+t \leq c(x,z)+\psi^t(z) \leq c(x,z)+\psi(z)$$

for all $z \neq y_0$. Now for each $x \in X$ we have the following

$$\begin{aligned} x \in Lag_{y_0}(\psi^t) &\Leftrightarrow \forall z \neq y_0 \quad c(x,y_0)+\psi(y_0)+t \leq c(x,z)+\psi(z) \\ &\Leftrightarrow t \leq \min_{z \in Y \setminus y_0} (c(x,z)+\psi(z)) - c(x,y_0)+\psi(y_0) \\ &\Leftrightarrow t \leq bid_{y_0}(\psi, x) \end{aligned}$$

where the last equivalence results from the fact that $x \in Lag_{y_0}(\psi)$.

From Corollary 3.2.7, the superdifferential of $\kappa(t)$ is given by $\partial^+\kappa(t) = \mu(Lag_{y_0}(\psi^t) - \frac{1}{N})$. So, for $t \leq bid_{y_0}(\psi, x)$ we have $\partial^+\kappa(t) \geq 0$ and for $t \geq bid_{y_0}(\psi, x)$, we have $\partial^+\kappa(t) < 0$. Thus, $0 \in \partial^+\kappa(bid_{y_0}(\psi))$ □

Remark 4.2.4. *The coordinate ascent method is to choose at every step $y \in Y$ such that its laguerre cell is not empty and then to increase the $\psi(y)$ by the bidding increment $bid_y(\psi)$. Then to assign for every mine x the factory y which is optimal. This is known as the complementary slackness condition. However, sometimes the bidding increment turns to be zero and vanishes. In this case, such an algorithm gets stuck and doesn't converge to a maximizer. So in order to tackle such problem, Bertsekas and Eckstein [7] introduced the relaxation of the complementary slackness, which is also called the $\epsilon-$complementary slackness. Through this method, the bids are at least $\epsilon > 0$ and every mine x is assigned to the factory y which is nearly optimal, i.e. within ϵ of attaining $\min_{y \in Y}(c(x,y)+\psi(y))$*

Example 4.2.5. ***(Case when** $\epsilon = 0$)*
In this example we will see the importance of the tolerance ϵ in the Bertsekas auction algorithm. Consider the set $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ where,

$$x_1 = (2,0) \qquad x_2 = (3,0) \qquad x_3 = (4,0)$$

$$y_1 = (0,2) \qquad y_2 = (0,-2) \qquad y_3 = (-12,0)$$

and let the cost function be the euclidean distance, $c(x,y) = \|x-y\|$. Then we can notice the following:
As the distance between x_1 and y_1 is the least, then y_1 is the first best choice for x_1. However, the second best choice is y_2 where $c(x_1,y_2) = c(x_1,y_1)$. Similarly, the case with x_2 and x_3. Then the bidding in this case is zero. This keeps the prices at y_1 and y_2 unchanged as well as the laguerre cell. In this case, this algorithm will get stuck and will not converge to the solution.

4.3 $\epsilon-$Complementary Slackness

Definition 4.3.1. *Let $S \subseteq X$ and $\sigma : S \to Y$ be an injective map. Then the couple (S,σ) is said to be partial assignment.*

Definition 4.3.2. *($\epsilon-$Complementary Slackness). Let (S,σ) be a partial assignment and $\psi : Y \to \mathbb{R}$ be the price function. Then (S,σ) with ψ verify the $\epsilon-$complementary slackness if for every $x \in S$ the following holds:*

$$c(x,\sigma(x)) + \psi(\sigma(x)) \leq \min_{y\in Y} c(x,y) + \psi(y) + \epsilon$$

Lemma 4.3.3. *Let $\sigma : X \to Y$ be a bijection and $\psi \in \mathbb{R}^Y$ that satisfy together the complementary slackness condition. Then we have the following:*

$$(KP) \leq \frac{1}{N}\sum_{x\in X} c(x,\sigma(x)) \leq (KP) + \epsilon$$

Proof. The first inequality can be seen directly from the definition of (KP) where

$$(KP) = \frac{1}{N}\min_{x\in X, y\in Y} \sum_{x\in X, y\in Y} c(x,y) \leq \frac{1}{N}\sum_{x\in X, y\in Y} c(x,\sigma(x))$$

For the second inequality, we have from the complementary slackness condition that

$$\frac{1}{N}(c(x,\sigma(x)) + \psi(\sigma(x))) \leq \frac{1}{N}\min_{y\in Y}(c(x,y) + \psi(y)) + \epsilon$$

Now, by summing over $x \in X$ both sides we get:

$$\frac{1}{N}\sum_{x\in X}(c(x,\sigma(x)) + \psi(\sigma(x))) \leq \frac{1}{N}\sum_{x\in X}\min_{y\in Y}(c(x,y) + \psi(y)) + \epsilon$$

then,

$$\frac{1}{N}\sum_{x\in X}(c(x,\sigma(x)) \leq \frac{1}{N}\sum_{x\in X}(\min_{y\in Y}(c(x,y)+\psi(y))-\psi(\sigma(x)))+\epsilon \leq \mathcal{K}(\psi)+\epsilon \leq (DP)+\epsilon = (KP)+\epsilon$$

□

4.4 Bertsekas Auction Algorithm

We fix $\epsilon > 0$ and we start with an assignment $S \subset X$(usually S is taken to be the empty set) and a price function $\psi \in \mathbb{R}^N$ such that they satisfy $\epsilon - CS$. In each iteration, we have two phases:

1. **Bidding phase:**
   In this phase, we take $x \notin S$ and compute the value $y_1 = \min_{y \in Y}(c(x,y)+\psi(y))$ which is the best factory for this mine. Then we compute also the value of the second best factory after y_1 by: $y_2 = \min_{y \in Y \setminus y_1}(c(x,y) + \psi(y))$. After that, we find the bidding increment and increase $\psi(y_1)$ by this value, i.e. $\psi_{new}(y_1) = \psi(y_1) + c(x,y_2) + \psi(y_2) - (c(x,y_1) + \psi(y_1)) + \epsilon$.

2. **Assignment Phase:**
   We assign for the mine x the factory y_1, i.e. $\sigma(x) = y_1$. If $\sigma(x') = y_1$ then we remove x' from S and add x to it.

After these two phases, we end up with updated assignment S with bijection σ and price function ϕ satisfying also $\epsilon - CS$. In fact, this algorithm preserves $\epsilon - CS$. We then proceed iteratively until we reach a complete assignment S with a bijection $\sigma : X \to Y$ and price function ψ with $\epsilon - CS$ that solves the (KP) by Lemma 4.3.3.

Proposition 4.4.1. *If the assignment S and the price function ψ chosen at the beginning of the algorithm verifies the $\epsilon-$CS then the obtained ones after each iteration in the above algorithm also verifies this condition.*

Proof. Assume that (S, ψ) verifies the $\epsilon-$CS. Then for the iteration after that, $S' = S \bigcup \{x_0\}$ with $\sigma(x_0) = y_0$. Let $\psi_{new}(y_0)$ be the new price at y_0. So we only need to show that (S', ψ) verifies the $\epsilon-$CS at the source x_0 since from the assumption we know that it verifies it for all the other sources in S.

$$\begin{aligned}
c(x_0, y_0) + \psi_{new}(y_0) &= c(x_0, y_0) + \psi_(y_0) + Bid_{y_0}(\psi) + \epsilon \\
&= c(x_0, y_0) + \psi(y_0) + \min_{y \in Y \setminus y_0}(c(x_0, y) + \psi(y)) - c(x_0, y_0) - \psi(y_0) + \epsilon \\
&\leq \min_{y \in Y \setminus y_0}(c(x_0, y) + \psi(y)) + \epsilon
\end{aligned}$$

But since the inequality is also trivial for y_0 then

$$c(x, y_0) + \psi(y_0) \leq \min_{y \in Y}(c(x, y) + \psi(y)) + \epsilon$$

□

4.5 Algorithm Complexity

Consider below the Matlab pseudo code for the Bertsekas Auction Algorithm.

Algorithm 1 Bertsekas Auction Algorithm

function AUCTION($X, Y, \epsilon, c, \psi = 0$)
  $S = \emptyset$
  if $x \notin S$ **then**
    $y_1 = \arg\min_{y \in Y}(c(x,y) + \psi(y))$ ▷ This is the first best option for x
    $y_2 = \arg\min_{y \in Y \setminus y_1}(c(x,y) + \psi(y))$ ▷ This is the second best option for x
    $Bid = c(x, y_2) + \psi(y_2) - (c(x, y_1) + \psi(y_1)) + \epsilon$ ▷ Bidding increment
    $\psi(y_1) = \psi(y_1) + Bid$ ▷ increasing price of y_1
    if $\exists x' \in S$ **such that** $\sigma(x') = y_1$ **then**
      $S = S \setminus \{x'\}$ ▷ x' is removed from S
      $S = S \bigcup \{x\}$ ▷ x is added to S
      $\sigma(x) = y_1$ ▷ x is assigned to y_1
    end if
  end if
  return σ, ψ
end function

Remark 4.5.1. *We can notice the size of the set S can only increase at each iteration and one of the prices $\psi(y_i)$ increases by at least ϵ.*

Remark 4.5.2. *Lower bound of the number of iterations If we consider the same example 4.2.5 with tolerance ϵ then each time x_1, x_2 or x_3 chooses y_1 or y_2 their price will increase by the bidding 0 or ϵ. So after n iterations the prices of y_1 and y_2 will be at most $2n\epsilon$. Then in order for y_3 to be chosen by one of the sources, their price should exceed $c(x_i, y_3) + \psi(y_3) = \|x_i - y_3\|$ for at least one of x_i with $i \in \{1,2,3\}$. Thus, $2n\epsilon > C$ where $C = \min_{i \in \{1,2,3\}} \|x_i - y_3\|$. Hence, $n > \frac{C}{2\epsilon}$.*

Proposition 4.5.3. *The number of steps in the auction algorithm is at most $T = N(\frac{C}{\epsilon}+1)$ and the number of operation $S = N^2(\frac{C}{\epsilon}+1)$ where $C = \max_{(x,y) \in X \times Y} c(x,y)$*

Proof. Suppose to the contrary that the algorithm has not stopped after T steps. Then there exists y_0 such that $y_0 \notin \sigma(S)$ and its price has not increased from the beginning of the algorithm ($\psi(y_0) = 0$). Now suppose that there is y_1 such that the price $\psi(y_1)$ has been increased n times such that $n > \frac{C}{\epsilon} + 1$ then we get the following:

$$\psi(y_0) + c(x, y_0) = c(x, y_0) \leq C \leq n\epsilon - \epsilon \leq \psi(y_1) - \epsilon \leq \psi(y_1) + c(x, y_1) - \epsilon$$

However, this contradicts the $\epsilon - CS$ that is satisfied at y_1. Thus the price of each of the N objects should increase at most by $\frac{C}{\epsilon} + 1$ steps. Therefore, the number of steps is at most $N(\frac{C}{\epsilon} + 1)$. As in every step we are finding the minimum over N objects, then the number of operations is at most $S = N^2(\frac{C}{\epsilon} + 1)$ □

Remark 4.5.4. *According to Proposition 4.5.3 this method will terminate at a final number of steps and the obtained bijection σ satisfies the $\epsilon - CS$ based on Proposition*

4.4.1. This implies that this bijection is within ϵ of the optimal solution according to Lemma 4.3.3. So choosing ϵ small enough makes this method converges to the optimal solution

Chapter 5

Network Simplex Algorithm

In the previous chapters we considered the assignment case where the cardinal of the source and target are both equal to N. But in the real life, we are interested in the case where $card(X) = n$ and $card(Y) = m$ with $n \neq m$ and the unit masses μ_i at each source x_i and ν_i at target y_i are taken not to be uniformly equal. Similarly as in the case of $n = m$, we define the matrix $P \in M_{n\times m}(\mathbb{R})$ associated to the transport plan such as $P_{ij} = \gamma_{ij}$ which is the mass transported from x_i to y_j with the following constraints:$\sum_{j=1}^{m} P_{ij} = \mu_i$ and $\sum_{i=1}^{n} P_{ij} = \nu_j$. Let $\mathcal{U}(X, Y)$ denotes the set of feasible matrices for the transport plan.

5.1 Graph Construction of $\mathcal{U}(X, Y)$

For each transport plan P, the graph representation corresponding to it is $G(V \bigcup V', E(P))$ where the set of vertices $V \bigcup V'$ be the set of sources X with the set of targets Y and the set of edges $E(P)$ be the route connecting x_i to y_j, i.e $E(P) = \{(x_i, y_j), P_{ij} > 0\}$. Each edge has a cost associated with it, which corresponds of transporting one unit of resources along the route

Recall that the solution of the Kantorovich problem is attained at an extremal point of $\mathcal{U}(X, Y)$. However, extremal matrices share a special graph structure that will be presented in the following proposition.

Definition 5.1.1. *A cycle in graph G is a path that starts from a given vertex and end in the same vertex. If a graph G has no cycles then it is said to be acyclic and each connected component of it is said to be a tree.*

Proposition 5.1.2. *Let P be an extremal point of the feasible set $\mathcal{U}(X, Y)$. Then the graph $G(P) = (V \bigcup V', E)$ is acyclic*

Proof. Suppose to the contrary that graph $G(P)$ has a cycle

$$C = \{(x_1, y_1), (y_1, x_2), (x_2, y_2), ..., (x_k, y_k), (y_k, x_1)\}$$

Let A be a matrix obtained from P by adding ϵ to each entry $(x_i, y_j) \in C$ such that $x_i \to y_j$ and subtracting ϵ from each entry $(x_i, y_j) \in C$ such that $y_j \to x_i$, while B be a matrix obtained from P by subtracting ϵ from each entry $(x_i, y_j) \in C$

such that $x_i \to y_j$ and adding ϵ to each entry $(x_i, y_j) \in C$ such that $y_j \to x_i$ where $\epsilon < \min_{(x_i,y_j)\in E(P)} P_{ij}$. It is then clear that $A, B \in \mathcal{U}(X, Y)$ and $P = \frac{A+B}{2}$. However from Proposition 2.3.10, this contradicts that P is an extremal matrix in $\mathcal{U}(X, Y)$. □

Remark 5.1.3. *As the graph of any extremal point P has no cycles, then card (E(P)) is at most $n + m - 1$*

Remark 5.1.4. *As $G(P)$ has no cycles, then it is either a tree or a forest (union of disjoint trees).*

5.2 Network Simplex Algorithm

The idea behind the network simplex algorithm is to start with a vertex or an extremal point from the feasible set $\mathcal{U}(X, Y)$ and proceed iteratively by replacing this vertex with a better neighborhood vertex that improves the objective until we reach our optimal transport plan P. So first of all we are concerned about getting the initial vertex. There are several initialization schemes, in this section we will introduce a simple one which is the north-west corner rule.

5.2.1 *North-West Corner Rule*

In order to find the initial vertex P, we start by finding P_{11} to be the highest possible value. Suppose that μ_1 is the quantity that is ready to be transferred from the source x_1 and ν_1 to be the quantity needed to be displaced at target y_1, then we choose $P_{11} = \min(\mu_1, \nu_1)$. If $\min(\mu_1, \nu_1) = \mu_1$, then there is no more quantity to be taken from x_1 and the quantity still needed at y_1 is $r = \nu_1 - \mu_1$. So we go to source x_2 with mass μ_2 and let $P_{21} = \min(\mu_2, r)$. So we can notice that at each step the entry P_{ij} is chosen to either saturate the row constraint i or the column constraint j or even both constraints. We proceed in this way until we reach our last entry P_{nm}.

Proposition 5.2.1. *The generated matrix P by the North-West corner rule is an extremal point for the feasible set $\mathcal{U}(X, Y)$.*

Proof. Suppose that $G(P)$ has a cycle $\{(i_1, j_1), (j_1, i_2), ..., (j_q, i_1)\}$, then we can notice that either the quantity at y_{j_1} is received from two sources x_{i_1} and x_{i_2} or the quantity at x_{i_1} is splitted into two targets y_{j_1} and y_{j_q}. But this contradicts the way that P was constructed. Thus $G(P)$ has no cycles.
Now, suppose that P is not extremal matrix , then $P = \frac{A+B}{2}$ where A and B are two matrices in the feasible set $\mathcal{U}(X, Y)$ and $A \neq B \neq P$. Then there exists $P_{i_1 j_1} > 0$ such that $A_{i_1 j_1} \neq B_{i_1 j_1} \neq P_{i_1 j_1}$. As the three matrices verify the supply demand conditions i.e. $\sum_{i=1}^n P_{ij_1} = \sum_{i=1}^n A_{ij_1} = \sum_{i=1}^n B_{ij_1} = \nu_{j_1}$, then there exists $0 < P_{i_2,j_1} \neq A_{i_2,j_1} \neq B_{i_2,j_1}$. We repeat this process $nm + 1$ times by the Pigeon hole principle until we repeat one of the entries $P_{i_m j_k} > 0$. This contradicts the fact that P has no cycles. Then P is an extremal matrix. □

Proposition 5.2.2. *Let X and Y be two finite sets of cardinal n and m respectively. Then the following two statements are equivalent:*

1. *P is the optimal solution for (KP) and (ϕ, ψ) are the optimal ones for the dual problem.*

2. *$\phi(x_i) - \psi(y_j) = c(x_i, y_j)$ for all (x_i, y_j) such that $P_{ij} > 0$*

Proof. For the first direction, Suppose that P is the optimal solution for (KP) and (ϕ, ψ) are the optimal one for the dual problem. then by strong duality we have

$$\sum_{j=1}^{m}\sum_{i=1}^{n} c_{ij}P_{ij} = \sum_{i=1}^{n} \phi\mu_i - \sum_{j=1}^{m} \psi\nu_j$$

But $\mu_i = \sum_{j=1}^{m} P_{ij}$ and $\nu_j = \sum_{i=1}^{n} P_{ij}$. We get then

$$\sum_{i=1}^{n} \phi(x_i)\mu_i - \sum_{j=1}^{m} \psi(y_j)\nu_j = \sum_{i=1}^{n} \phi(x_i) \sum_{j=1}^{m} P_{ij} - \sum_{j=1}^{m} \psi(y_j) \sum_{i=1}^{n} P_{ij}$$

this gives the following:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} P_{ij}(c_{ij} + \phi(x_i) - \psi(y_j)) = 0$$

Then if $P_{ij} > 0$

$$\phi(x_i) - \psi(y_j) = c(x_i, y_j)$$

For the other direction, suppose that the equality is attained for all (x_i, y_j) such that $P_{ij} > 0$ then

$$\begin{aligned}(KP) \leq \sum_{i=1}^{n}\sum_{j=1}^{m} c(x_i, y_j)P_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{m} (\phi(x_i) - \psi(y_j))P_{ij} &= \sum_{i=1}^{n} \phi(x_i)\mu_i - \sum_{j=1}^{m} \psi(y_j)\nu_j \\ &\leq (DP) = (KP)\end{aligned}$$

This gives us that P, ψ, and ϕ are the optimal solutions for (KP) and (DP). □

Example 5.2.3. *Consider a source set $X = \{x_1, x_2, x_3\}$ and a target set $Y = \{y_1, y_2, y_3, y_4\}$ with the following supply and demand quantities:*

$$\mu = (0.1, 0.6, 0.3) \text{ and } \nu = (0.5, 0.3, 0.1, 0.1)$$

For the first entry $P_{11} = \min(0.1, 0.5) = 0.1$ then all the quantity at x_1 are distributed. Now for x_2 , the needed quantity left for y_1 is 0.4, so $P_{21} = \min(0.6, 0.4) = 0.4$. And so we complete in this pattern until we reach our transport plan P.

$$\begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0.4 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0.4 & 0.2 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \end{bmatrix} \longrightarrow$$
$$\begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0.4 & 0.2 & 0 & 0 \\ 0 & 0.1 & 0.1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0.4 & 0.2 & 0 & 0 \\ 0 & 0.1 & 0.1 & 0.1 \end{bmatrix}$$

5.2.2 ***Dual Pair Complementary to P***

After starting with a vertex P calculated using the North-West corner, we check if this transport plan P is the optimal one, if not we replace it by a better vertex. In order to check that we generate the dual variables (ϕ, ψ) relative to transport plan P in the following way: for each edge $(x_i, y_j) \in E(P)$ we have the equation $\phi(x_i) - \psi(y_j) = c(x_i, y_j)$. Then we get a system of $s = card(E(P))$ equations with n+m variables:

$$\begin{cases} \phi_{i_1} - \psi_{j_1} = c_{i_1,j_1} \\ \phi_{i_2} - \psi_{j_2} = c_{i_2,j_2} \\ \qquad . = . \\ \qquad . = . \\ \qquad . = . \\ \phi_{i_s} - \psi_{j_s} = c_{i_s,j_s} \end{cases} \tag{5.1}$$

Note that in such construction, we have no guarantee that $\phi(x_i) - \psi(y_j) \leq c(x_i, y_j)$ for all $x_i \in X$ and $y_j \in Y$. However if this was true then the obtained P is the optimal solution according to proposition 5.2.2 for (KP).

Notice that in the above system, we have $s \leq n + m - 1$ equations with $n + m$ variables. So such system is always undetermined. In order to fix this problem we go through each tree τ in G(P).

Consider a tree $\tau \in G(P)$ with $\{x_{i_1}, x_{i_2}, x_{i_3}, ..., x_{i_k}\}$ source nodes and $\{y_{j_1}, y_{j_2}, y_{j_3}, ..., y_{j_l}\}$ target nodes resulting in exactly $k + l - 1$ edges. This reduces the above system to $k+l-1$ equations with $k+l$ unknowns. We fix one of the dual price at any node to be zero and then find the others. Doing this to each tree gives as the dual variables corresponding to the transport plan P.

5.2.3 ***Network simplex Update***

After obtaining the dual variables corresponding to the transport plan P, we check if $\phi_i - \psi_j \leq c_{ij}$ for all $(x_i, y_j) \in G(P)$. If this was the case, then P is the optimal solution if not then there exists (x_i, y_j) such that $\phi_i - \psi_j > c_{ij}$ and the network simplex algorithm kicks in. We consider a new graph G' which is the same graph G adding to it the "entering" edge (i, j). Then we will have two cases:

1. In the first case, G' is still forest even after adding this edge. We then again compute the new dual variables as in 5.1. In this case transport plan is remained unchanged.

2. In the second case, G' has a cycle say $\{(x_{i_1}, y_{j_1}), (y_{j_1}, x_{i_2}), (x_{i_2}, y_{j_2}), ..., (x_{i_k}, y_{j_k}), (y_{j_k}, x_{i_1})\}$ where $(x_{i_1}, y_{j_1}) = (x_i, y_j)$ is the entering edge. In that case we need to remove an edge from G' to make sure it is forest again and modifying P with keeping it feasible. To do that we are going to rearrange the distribution of the quantities transported in this cycle by increasing the amount transported from x_i to y_j and changing the others. For simplicity, we call the edges of the form (x_{i_s}, y_{j_s}) as "positive edges" and the ones of the form $(y_{j_s}, x_{i_{s+1}})$ as "negative

edges".
We start by computing $\theta = \min_{1\leq s\leq k} P_{i_{s+1},j_s}$ which represents the maximum quantity that can be transported from x_i to y_j. Then we increase the flow in each positive edge and decrease that in each negative edge to obtain an updated transport plan P^n as follows:

$$P^n_{i_s,j_s} = P_{i_s,j_s} + \theta \text{ and } P^n_{i_{s+1},j_s} = P_{i_{s+1},j_s} - \theta$$

for all $1 \leq s \leq k$

Example 5.2.4. *Consider below the network graph of the transport plan obtained in example 5.2.3.*

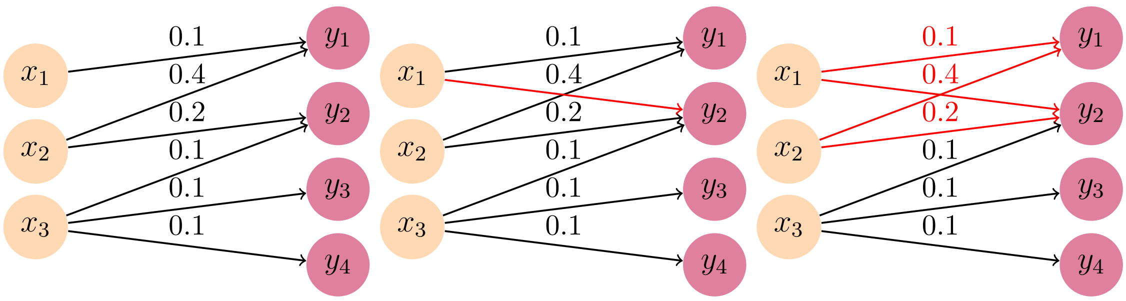

Figure 5.1: Graph of the initial transport plan P before and after adding the entering edge (x_1, y_2)

Suppose that $\phi_1 - \psi_2 > c_{1,2}$. Then we add the entering edge (x_1, y_2) to the graph of P. The new graph contains now a cycle $\{(x_1, y_2), (y_2, x_2), (x_2, y_1), (y_1, x_1)\}$ as shown in the figure 5.2. Then we classify the edges between positive ones $\{(x_1, y_2), (x_2, y_1)\}$ and negative ones $\{(y_2, x_2), (y_1, x_1)\}$ and obtain our $\theta = \min(P_{2,2}, P_{1,1}) = 0.1$. After adding this number to the positive edges and subtracting it from the negative edges we end up by removing the edge (x_1, y_1). This gives our new transport plan P'

$$P' = \begin{bmatrix} 0 & 0.1 & 0 & 0 \\ 0.5 & 0.1 & 0 & 0 \\ 0 & 0.1 & 0.1 & 0.1 \end{bmatrix}$$

Below is the updated network graph G(P')=G':

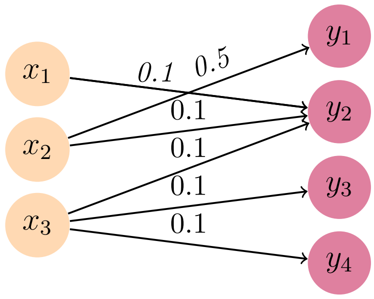

Figure 5.2: Graph of the transport plan P'

Proposition 5.2.5. *After each iteration, the updated transport plan P' has a reduced cost than transport plan P and the method then converges.*

Proof. We have that the two transport plans are equal except for the edges in the cycle $\{(x_{i_1}, y_{j_1}), (y_{j_1}, x_{i_2}), (x_{i_2}, y_{j_2}), ..., (x_{i_k}, y_{j_k}), (y_k, x_{i_1})\}$. So we get that:

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{m} P'_{ij}c_{i,j} - \sum_{i=1}^{n}\sum_{j=1}^{m} P_{ij}c_{i,j} &= \sum_{s=1}^{k} c_{i_s,j_s}(P'_{i_sj_s} - P_{i_sj_s}) + \sum_{s=1}^{k} C_{i_{s+1},j_s}(P'_{i_{s+1}j_s} - P_{i_{s+1}j_s}) \\
&= \sum_{s=1}^{k} c_{i_s,j_s}\theta + \sum_{s=1}^{k} c_{i_{s+1},j_s}(-\theta) \\
&= \sum_{s=1}^{k} c_{i_s,j_s}\theta + \sum_{s=1}^{k} c_{i_{s+1},j_s}(-\theta) \\
&= \theta\left(\sum_{s=1}^{k}(c_{i_sj_s} - c_{i_{s+1}j_s})\right)
\end{aligned}$$

However, using the dual variables computed at the previous iteration we have:

$$\begin{aligned}
\sum_{s=1}^{k}(c_{i_sj_s} - c_{i_{s+1}j_s}) &= c_{i_1j_1} + \sum_{s=2}^{k} c_{i_sj_s} - \sum_{s=1}^{k} c_{i_{s+1}j_s} \\
&= c_{ij} + \sum_{s=2}^{k}(\phi_{i_s} - \psi_{j_s}) - \sum_{s=1}^{k}(\phi_{i_{s+1}} - \psi_{j_s}) \\
&= c_{ij} - (\phi_{i_{k+1}} - \psi_{j_1}) \\
&= c_{ij} - (\phi_i - \psi_j)
\end{aligned}$$

where as c_{ij} is the entering edge then $\phi_i - \psi_j > c_{ij}$. This implies that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} P'_{ij}c_{i,j} - \sum_{i=1}^{n}\sum_{j=1}^{m} P_{ij}c_{i,j} < 0$$

As the obtained transport plan after each iteration has a reduced cost, then the feasible trees generated by the simplex algorithm are distinct. But as the number of all feasible trees finite, then this algorithm will eventually converge. □

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