AMERICAN UNIVERSITY OF BEIRUT

ENTIRE FUNCTION WITH TWO SEPARATED VALUES

by
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A thesis
submitted in partial fulfillment of the requirements
for the degree of Master of Science
to the Department of Mathematics
of Faculty of Arts and Sciences
at the American University of Beirut

Beirut, Lebanon
July 2023

AMERICAN UNIVERSITY OF BEIRUT

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Acknowledgements

I would like to express my sincere gratitude to all those who have contributed to the completion of this thesis.

First and foremost, I would like to extend my deepest appreciation to my advisor, Professor Farouk Abi Khuzam. Your guidance and support throughout this master's thesis have been invaluable. I am truly fortunate to have had the privilege of working under your supervision.

I am also grateful to the American University of Beirut for providing me with the necessary resources and a conducive research environment.

Finally, I would like to thank my family and friends for their constant support and encouragement. This accomplishment would not have been possible without their presence in my life.

Abstract
of the Thesis of

Batoul Mohamad Saab for Master of Science
Major: Pure Mathematics

Title: Entire Function With Two Separated Values

It's known that an entire function with zeros and 1-points lying on a finite number of rays must have finite order. This thesis considers a transcendental entire function with two separated values in disjoint sectors. Given that such function is of finite order, and under some conditions related to the sizes of the sectors, it's possible to determine the form of the function.[1]

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INTRODUCTION

Let f be a transcendental entire function, and let $S_a = f^{-1}(\{a\})$ be the set of a-values of f. If S_a is an infinite set, it must be an infinite sequence which tends to ∞, and this is the only condition on such a set, and any set with this condition is the set of a-points of an entire function by the Weierstrass factorization theorem. On the other hand, if a and b are two distinct complex values, then the positions of the sets S_a and S_b relative to each other may be reflected in the form of the function f. For example, if both S_0 and S_1 lie on a finite number of rays through the origin, the order of f must be finite, allowing the application of Hadamard's factorization theorem to study the possible forms of f.

In this thesis, we present a study of the relationship between a specific geometric condition placed on the sets S_0 and S_1 and the form of the function f. More precisely, if these sets lie in certain disjoint sectors in the plane, and if the function has finite order, then f must be of the form

$$f(z) = \int_0^z p(\zeta)\mathrm{e}^{q(\zeta)}\,\mathrm{d}\zeta + c$$

where p and q are polynomials.

The proof takes off from the given geometric condition on the two sets S_0 and S_1 to obtain precise information about the derivative f'. This is done following a preliminary step to show the genus must be ≥ 1, through some auxiliary subharmonic functions u and v, obtained as limits of other subharmonic functions ($u_k(z) = \frac{\log|f(r_k z)|}{\log M(r_k)}, v_k(z) = \frac{\log|f(r_k z)-1|}{\log M(r_k)}$) defined via Pólya peaks $\{r_k\}$ of the maximum modulus $M(r)$ of the given function.

The two functions u and v serve to define one entire function whose order is shown to be a positive integer. This is then used to obtain information about the derivative f', showing it to be of positive integer order. Differentiation of u and v then provides information about the zeros of f, which turn out to be finite in number. By Hadamard's theorem, f' must be of the form

$$f'(z) = p(z)\mathrm{e}^{q(z)}$$

and the result follows by integration.

Chapter 1

PRELIMINARIES

Subharmonic Functions

1.1 Definition of Subharmonic Function

A real function u defined on a region Ω of $\mathbb{C}$ is said to be subharmonic if for each $z_0 \in \Omega, u$ satisfies the following:

- $-\infty \leq u(z_0) < \infty$
- upper semi continuity

$$u(z_0) = \lim_{\epsilon \to 0} \sup_{|z - z_0| < \epsilon} u(z)$$

- mean value inequality

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta \qquad \text{for r small enough}$$

All harmonic function satisfy a mean-value inequality and so are subharmonic. The connection between the two comes through the majorant property.

1.2 Harmonic Majorants of Subharmonic Function

If u is subharmonic function in a region Ω of the complex plane, and h is harmonic on Ω, then h is a harmonic majorant of u in Ω if $u \leq h$ in Ω. Such an inequality can be viewed as a growth condition on u. In fact, if $u \leq h$ on $\partial\Omega$, then $u \leq h$ in Ω.

1.3 Maximum Principle

Let u be a subharmonic function on a region Ω in $\mathbb{C}$. If there exists $z_0 \in \Omega$ such that

$$u(z_0) = \max_{z \in \overline{\Omega}} u(z)$$

then u is constant. This means that the maximum of u is attained on the boundry of Ω.

1.4 Order of Subharmonic Function

Let u be a subharmonic in $\mathbb{C}$, we put $B(r,u) := \max_{|z|=r} u(z)$.
The order ρ of u is defined by

$$\rho := \limsup_{r \to \infty} \frac{\log B(r,u)}{\log r}$$

A sufficient condition for u to be subharmonic in Ω is that the Laplacian of u exists and is non-negative in Ω, i.e., $\Delta u \geq 0$. Indeed, if u is twice continuously differentiable in Ω and $z_0 \in \Omega$, then Green's theorem gives

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta - u(z_0) = \int_0^r \frac{1}{2\pi t}\int\int_{|z-z_0|\leq t} \Delta u(z)\,dA \quad \text{for small r}$$

from which the mean-value inequality follows if $\Delta u \geq 0$. This formula serves to define Δu as a distribution when u is not smooth enough.

An important example is $\log|f(z)|$ where f is holomorphic in Ω. In this case, the Laplacian is the counting measure of the zeros of f, i.e., $\Delta(u)(D(0,r)) := \phi_u(D(0,r)) = \#$ of zeros of f in the disk of radius r.

The precise connection between a subharmonic function and its "Laplacian" is presented in the "Riesz representation theorem."

1.5 The Fundamental Riesz Theorem

Let D be a domain in $\mathbb{C}$, and u be a subharmonic function in D. Then there exists a unique non-negative Borel measure μ in D such that, for each subdomain E compactly embedded into D, $\mu(E) < \infty$ and u can be represented as a sum of logarithmic potential of μ and a harmonic function h in E.

$$u(z) = \iint_E \log|z - \zeta|\,d\mu(\zeta) + h(z)$$

The measure μ is called the associated measure for the function u or the Riesz measure.

1.6 Jensen's Formula for Subharmonic Functions

Let u be a bounded subharmonic function in a disk $\mathbb{D}_R = \{z : |z| < R\}$ such that $u(0) \neq -\infty$, and let μ be the Riesz measure of u. Then,

$$u(0) + \int_0^R \frac{n(t)}{t}\, dt = \mathfrak{N}(R, 0; u)$$

where $n(t) := \mu(\{z : |z| \leq t\}$ and $\mathfrak{N}(r, z; u) := \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\, d\theta$.

This yields that

$$N(r) := \int_1^r \frac{n(t)}{t}\, dt \leq B(r, u) + \mathcal{O}(1) \tag{1.1}$$

If u is a subharmonic function of finite order, then there exits an integer $q \geq 0$ such that $q \leq \rho \leq q + 1$ and

$$\int_1^\infty \frac{B(r, u)}{r^{q+2}}\, dr < \infty$$

Putting this with (1.1) implies that

$$\int_1^\infty \frac{N(r)}{r^{q+2}}\, dr < \infty \tag{1.2}$$

Furthermore, (1.2) yields that

$$\int_1^\infty \frac{n(t)}{t^{q+2}}\, dt < \infty \tag{1.3}$$

Indeed

$$\begin{aligned} \int_1^r \frac{n(t)}{t^{q+2}}\, dt =& \frac{N(t)}{t^{q+1}}\Big|_1^r + (q+1)\int_1^r \frac{N(t)}{t^{q+2}}\, dt \\ &+ \frac{N(r)}{r^{q+1}} - N(1) + (q+1)\int_1^r \frac{N(t)}{t^{q+2}}\, dt \end{aligned} \tag{1.4}$$

We need to show that $lim_{r\to\infty}\frac{N(r)}{r^{q+1}}$ exits. Since the integral (1.2) is convergent, given $\epsilon > 0$, $\exists\, R > 0$ such that

$$\int_r^{2r} \frac{N(t)}{t^{q+2}}\, dt < \epsilon \quad \text{for all } r > R$$

In fact, N is increasing then $N(t) \geq N(r)$. Hence we have for $r > R$

$$\int_r^{2r} \frac{N(r)}{t^{q+2}}\, dt \leq \int_r^{2r} \frac{N(t)}{t^{q+2}}\, dt < \epsilon$$

Which leads

$$\frac{1}{q+1}\left(1 - \frac{1}{2^{q+1}}\right)\frac{N(r)}{r^{q+1}} < \epsilon$$

Therefore,

$$lim_{r\to\infty}\frac{N(r)}{r^{q+1}}=0$$

.

Letting $r\to\infty$ in (1.4) implies that (1.3) holds.

We refer in the following theorem to [2, Theorem 4.2]

1.7 Subharmonic Version of The Hadamard Factorization Theorem

Let u be a subharmonic function in $\mathbb{C}$, harmonic in a neighborhood of zero, with $u(0)=0$. Suppose that u is of finite order ρ. Then u can be written as a sum of an integral and a harmonic polynomial h of degree at most ρ.

$$u(z)=\int_{\mathbb{R}^2}\log|E_q(\frac{z}{\zeta})|\,d\mu(\zeta)+h(z)$$

where

$q=[\rho]$

μ is the Riesz measure of u

$$E_q\left(\frac{z}{\zeta}\right)=\left(1-\frac{z}{\zeta}\right)\exp\left(\frac{z}{\zeta}+\frac{1}{2}\left(\frac{z}{\zeta}\right)^2+\cdots+\frac{1}{q}\left(\frac{z}{\zeta}\right)^q\right)$$

More precisely u can be written in the form

$$u(z)=v(z)+w(z)+h(z)$$

where

$$v(z)=\int_{|\zeta|<R}\log|z-\zeta|\,d\mu(\zeta)$$

$$w(z)=\int_{|\zeta|\geq R}(\log|z-\zeta|+\sum_{j=1}^{q}\frac{1}{j}\operatorname{Re}(\frac{z}{\zeta}))\,d\mu(\zeta)$$

We are going to use this versions of Phragmén-Lindelöf in the proofs of the next chapter [3, Corollary 2.3.8]

1.8 Phragmén-Lindelöf

Let u be a subharmonic function in the right half-plane $H:=\{z:\operatorname{Re}z>0\}$. Assume that there exist constants $A,B\in\mathbb{R}$ such that

$$u(z)\leq A+B|z|,\quad \text{for all } z\in H \tag{1.5}$$

and

$$\limsup_{\zeta \to z} u(\zeta) \le 0, \quad \text{for all } z \in \partial H. \tag{1.6}$$

Define

$$L := \limsup_{x \to \infty} \frac{u(x)}{x} \tag{1.7}$$

Then, we have

$$u(z) \le L \operatorname{Re} z \quad \text{for all } z \in H. \tag{1.8}$$

1.9 Extension for Liouville Theorem

Let u be a subharmonic function on $\mathbb{C}$ such that

$$\limsup_{z \to \infty} \frac{u(z)}{\log |z|} \le 0 \tag{1.9}$$

Then u is constant on $\mathbb{C}$.

In particular, every subharmonic function on $\mathbb{C}$ which is bounded above must be constant. Indeed, if u is bounded then $B(r,u) \le M$ for M being a constant. Leading that

$$lim_{r \to \infty} \frac{B(r,u)}{\log r} = 0$$

Hence, (1.9) is satisfied and u is a constant.

Chapter 2

PRELIMINARIES RESULTS

Theorem[section]

Chapter 2 paves the way for the proofs of our main theorems in the subsequent chapter. In this chapter, we will define two subharmonic functions that are constructed using sequences of Pólya peaks. These functions possess important properties that will be essential for our future analysis. Additionally, we will prove a lemma for entire functions, and some lemmas related to subharmonic functions, each play a vital in the next chapter.

Let f be an entire function of finite order $\rho \geq 1$, with zeros and 1-points of f in disjoint sectors S_0 and S_1 respectively. We use the standard notation of the maximum modulus of f

$$M(r) = M(r, f) = \max_{|z| \leq r} |f(z)|.$$

Since f is of finite order then there exists a sequence (r_k) tending ∞ with the property

$$\log M(tr_k) = \mathcal{O}(\log M(r_k)) \quad \text{as } k \to \infty \quad \text{for } t > 1 \tag{2.1}$$

A sequence (r_k) is called a sequence of Pólya peaks of order $\lambda \in [0, \infty)$ for $\log M(r)$, if for every $\epsilon > 0$, $\epsilon \leq t \leq \epsilon^{-1}$ we have

$$\log M(tr_k) \leq (1 + \epsilon) t^{\lambda} \log M(r_k) \quad \text{for k large enough.}$$

Fixing a sequence (r_k) with the property (2.1), allows us to define the following sequences (u_k) and (v_k) of subharmonic functions

$$u_k(z) := \frac{\log |f(r_k z)|}{\log M(r_k)} \quad \text{and} \quad v_k(z) := \frac{\log |f(r_k z) - 1|}{\log M(r_k)}.$$

(2.1) gives that these sequences are bounded from above on every compact subset of $\mathbb{C}$. As we assumed $f(0) \notin \{0, 1\}$ $u_k(0)$ and $v_k(0)$ tends to 0. It follows from [4, Theorems 3.2.12, 3.1.12] that there exits a subsequence, will be also denoted (r_k) for simplicity, such that u_k and v_k converges in the Schwartz space $\mathscr{D}'$ to subharmonic functions u and v in $\mathbb{C}$.

$$u(z) := \lim_{k\to\infty} \frac{\log|f(r_k z)|}{\log M(r_k)} \qquad \text{and} \qquad v(z) := \lim_{k\to\infty} \frac{\log|f(r_k z) - 1|}{\log M(r_k)}. \tag{2.2}$$

This implies the convergence in L^1_{loc}.

This lemma explores the properties of functions u and v, as referenced in [5, p.97],

Lemma 2.0.1. *Let u and v be given as in (2.2) , then u and v satisfies the following.*

(a) $\max\{u(z), 0\} = \max\{v(z), 0\}$ *for all* $z \in \mathbb{C}$

(b) $\{z : u(z) < 0\} \cap \{z : v(z) < 0\} = \emptyset$

(c) u is harmonic in $\mathbb{C} \setminus S_0$ and v is harmonic in $\mathbb{C} \setminus S_1$

(d) $\max_{|z|=1} u(z) = \max_{|z|=1} v(z) = 1.$

If (r_k) is a sequence of Pólya peaks of order $\lambda > 0$, then we also have:

(e) $u(0) = v(0) = 0$

(f) $\max\{u(z), v(z)\} \leq |z|^\lambda$ *for all* $z \in \mathbb{C}$.

In preparation of the next chapter, we shall need the fact that if the real part of an entire function satisfies certain growth condition, then the function should be a polynomial.

Lemma 2.0.2. *Let g be a function defined on the complex plane $\mathbb{C}$. Suppose that g is entire, and there exits $\lambda \in \mathbb{R}$ such that*

$$\operatorname{Re} g(z) \leq |z|^\lambda \quad \text{for all } z \in \mathbb{C}.$$

Then, g is a polynomial of degree at most λ.

Proof. Let $A(r)$ denotes the maximum of $\operatorname{Re} g$ for $|z| = r$.
By Caratheodary theorem we have for $0 < r < R$

$$\begin{aligned} |g^{(n)}(z)| &\leq \frac{2^{n+2} n! R}{(R-r)^{n+1}} (A(R) + g(0)) \\ &\leq c \frac{2^{n+2} n! R}{(R-r)^{n+1}} (R^\lambda + g(0)) \end{aligned}$$

For $n > \lambda$. let $R \to \infty$, we get $|g^{(n)}(z)| = 0 \ \forall z \in z$. Which gives the desired result. ∎

2.1 Order and Sign of a Subharmonic Function

If u is subharmonic in a neighborhood of z_0 and $u(z_0) = 0$, then u cannot be negative completely on any circle centered at z_0 due to the mean-value inequality. In dealing with the subject of this thesis, we shall need to consider the preliminary question of how large can an arc of such a circle be on which $u < 0$. The answer comes through the use of the harmonic majorant property mentioned above.

Lemma 2.1.1. *let u be a subharmonic function in a neighborhood V of zero, with $u(0) = 0$. Suppose that $u(z) < 0$ in $S := \{z : |argz| < \alpha\}$ where $0 < \alpha \leq \pi$. Then, $\alpha \leq \frac{\pi}{2}$ and there exits $c > 0$ and $r_0 > 0$ such that*

$$\int_{-\alpha}^{\alpha} u(re^{it})\, dt \leq -cr^{\pi/2\alpha} \qquad \text{for } r \in (0, r_0) \tag{2.3}$$

Proof. u is subharmonic in V then there exists $r_1 > 0$ such that $D[0, r_1] \subseteq V$ and h is a harmonic majorant of u in $S_\alpha = \{z : |\arg z| < \alpha, |z| < r_1\}$, such that $h(re^{\pm i\alpha}) = 0 \; for \; 0 < r < r_1$.

Let $r_2 = r_1^{\pi/2\alpha}$ and $S_{\frac{\pi}{2}} = \{z : \operatorname{Re} z > 0, |z| < r_2\}$. Define,

$$v(z) = h(z^{\pi/2\alpha}) = h(e^{\frac{2\alpha}{\pi}\log z})$$

where the principle branch of the logarithmic function is used.
if $z \in S_{\frac{\pi}{2}}$ then $z^{\frac{2\alpha}{\pi}} \in S_\alpha$ and so v is harmonic and negative in $S_{\frac{\pi}{2}}$ because h is harmonic and negative in S_α. Also, $v(iy) = h(re^{\pm i\alpha}) = 0$ i.e. $v = 0$ on the vertical part of the boundary which gives that $v(0) = 0$.

Now, by the reflection principle, v extends to a harmonic function in $D(0, r_2)$, and $\nabla v \neq 0$. Then v can be written as $v(z) = \operatorname{Re} f(z)$ where f is analytic fuction.

$$\begin{aligned} v(z) &= \operatorname{Re}(a_0 + a_1 z + \mathcal{O}(z^2)) && \text{for small } z \\ &= \operatorname{Re}(a_0 z) + \operatorname{Re}(a_1 z) + \mathcal{O}(z^2) \\ &= \operatorname{Re}(a_1 z) + \mathcal{O}(z^2) && \text{since } v(0) = 0 \end{aligned}$$

Also, we know that $v(re^{i\theta}) < 0$ for $|\theta| < \frac{\pi}{2}$ which implies that $\operatorname{Re}(a_1 z) = -|a_1| \operatorname{Re} z$. Therefore, $v(z) = -c_0 \operatorname{Re}(z) + \mathcal{O}(z^2)$ near zero for some $c_0 > 0$. Thus,

$$\int_{-\pi/2}^{\pi/2} v(re^{it})\, dt \leq -\frac{1}{2} c_0 r \qquad \text{for } r \text{ small enough.}$$

Which implies that h satisfies (2.3). Indeed,

$$\int_{-\pi/2}^{\pi/2} v(re^{it})\, dt = \int_{-\alpha}^{\alpha} \frac{\pi}{2\alpha} h(r^{\frac{2\alpha}{\pi}} e^{i\theta})\, d\theta \leq -\frac{1}{2} c_0 r$$

Then

$$\int_{-\alpha}^{\alpha} h(re^{i\theta})\, d\theta \leq -\frac{\alpha}{\pi} c_0 r^{\pi/2\alpha}$$

As $u \leq h$ in S_α it follows that u satisfies (2.3).

It's left to show that $\alpha \leq \frac{\pi}{2}$. We consider $S'_\alpha = \{z : \alpha < \arg z < 2\pi - \alpha, |z| < r_1\}$, the complement of $\overline{S_\alpha}$ within $D(0, r_1)$. Let h_1 be a harmonic majorant of u in S'_α, satisfying $h_1(re^{\pm i\alpha}) = 0$ for $r \in (0, r_1)$. Following the same argument as above yields that there exists a positive constant c_1 such that

$$\int_{2\pi-\alpha}^{\alpha} u(re^{it})\, dt \leq c_1 r^{\frac{\pi}{2\beta}} \quad \text{for } r \in (0, r_0)$$

with $\beta := \pi - \alpha$. This, together with the expression (2.3) , implies the following inequality:

$$0 = u(0) \leq \int_{2\pi+\alpha}^{\alpha} u(re^{it})\, dt \leq c_1 r^{\frac{\pi}{2\beta}} - c r^{\frac{\pi}{2\alpha}} \quad \text{for } r \in (0, r_0).$$

Which means that $c_1 r^{\frac{\pi}{2\beta}} \leq c r^{\frac{\pi}{2\alpha}}$ for r small. Consequently, we can conclude that $\beta \geq \alpha$, Leading $\alpha \leq \frac{\pi}{2}$. □

Furthermore, if a subharmonic function is negative in some half plane, there is a restriction on its order. Indeed, the following Lemma will show that the order must be one.

Lemma 2.1.2. *Let u be a subharmonic function in $\mathbb{C}$ with $u(0) = 0$, suppose that there exits $\rho, K > 0$ such that*

$$u(z) \leq K|z|^\rho \qquad \forall z \in \mathbb{C}. \tag{2.4}$$

If u is negative in some half plane. Then, $\rho = 1$.

Proof. Without loss of generality we may assume that $u(z) < 0$ in the right-half plane. As u is subharmonic, mean value inequality at 0 for r small enough gives

$$0 = 2\pi u(0) \leq \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} u(re^{it})\, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(re^{it})\, dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} u(re^{it})\, dt.$$

Moreover, Lemma 2.1.1 with $\alpha = \frac{\pi}{2}$ yields for r small we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(re^{it}) \leq -cr,$$

and (2.4) yields

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} u(re^{it})\, dt \leq K r^\rho \pi.$$

Combining all together gives $-cr + Kr^\rho \pi \geq 0$ for r sufficiently small. Thus, $\rho \leq 1$.

Furthermore,

$$\begin{aligned} u(z) &\le K|z|^\rho \\ &\le K(1+|z|)^\rho \\ &\le K(1+|z|) \end{aligned}$$

Therefore, (1.5) holds with $A = B = K$ and Phragmén-Lindelöf (1.8) implies that $u(z) \le L\,\mathrm{Re}\, z$ for z in the left half plane with L given in (1.7). Since u is non-constant and hence by Liouville theorem (1.9) u is unbounded, we deduce that $L > 0$. Then we have

$$0 < \limsup_{x\to\infty} \frac{u(x)}{x} \le \limsup_{x\to\infty} \frac{k|x|^\rho}{x}.$$

Thus $\rho \ge 1$, but we deduced $\rho \le 1$. Therefore, $\rho = 1$. □

2.2 From Subharmonic to Harmonic

In this section we will explores the harmonicity of a subharmonic function defined on the complex plane. Specifically, when it has order at most 1, bounded on the imaginary axis, and harmonic within certain domains defined by radial and angular constraints.

Lemma 2.2.1. *let u be a subharmonic function on $\mathbb{C}$, with order at most 1. Suppose that u is bounded on the imaginary axis. If there exists $R, \epsilon > 0$ such that u is harmonic within the two domains $T_\pm := \{z : z > R, |argz \pm \frac{\pi}{2}| < \epsilon\}$, then u is harmonic on $\mathbb{C}$ and of the form $u = a\,\mathrm{Re}\, z + b$, where $a, b \in \mathbb{R}$.*

Proof. As u is subharmonic on $\mathbb{C}$, using Hadamard factorization theorem (1.7) we can write u as $u(z) = v(z) + w(z) + h(z)$ with $q = 0$ or 1, since the order of u is at most 1.

Assume to the contrary that u is not harmonic, then we can choose $R > 0$ such that $\mu(\{z/|z| < R\}) > 0$ and find that $v(z) \to \infty$ as $|z| \to \infty$. Let's consider

$$u(iy) + u(-iy) = v(iy) + v(-iy) + w(iy) + w(iy) + h(iy) + h(-iy)$$

h is a real part of a polynomial with a degree at most 1, i.e. $h(z) = Re(Az + B)$ with A and $B \in \mathbb{C}$. Therefore, we have $h(iy)+h(-iy) = Re(Aiy+B)+Re(-Aiy+B) = 2Re(B)$.

Furthermore, $v(iy) + v(-iy) \to \infty$, thus one can deduce that $Q(y) := w(iy) + w(-iy) \to -\infty$ as u is bounded on the imaginary axis.

We are going to consider both cases when $q = 0$ and $q = 1$.

case q=0 :
Define $Q'(z) = w(z) + w(\overline{z})$. Note that $Q'(iy) = Q(y)$, indicating that Q' is unbounded on the imaginary axis.

On the other hand, $B(r, Q') = o(r)$ as $r \to \infty$ since this holds for w, the application of Phragmén-Lindelöf (1.8) implies that $Q' \le 0$ and is therefore constant, which leads to a contradiction.

case q=1 :

$$
\begin{aligned}
Q(y) &= \int_{|\zeta|\geq R} \log|1-\frac{iy}{\zeta}| + \mathrm{Re}\,(\frac{iy}{\zeta})\,d\mu(\zeta) + \int_{|\zeta|\geq R} \log|1-\frac{-iy}{\zeta}| + \mathrm{Re}\,(\frac{-iy}{\zeta})\,d\mu(\zeta)\\
&= \int_{|\zeta|\geq R} \log|1-\frac{iy}{\zeta}| + \log|1-\frac{iy}{\overline{\zeta}}|\,d\mu(\zeta)\\
&= \int_{|\zeta|\geq R} \log\Big|1-i\frac{2y\,\mathrm{Re}\,(z)}{|\zeta|^2} - \frac{y}{|\zeta|^2}\Big|\,d\mu(\zeta)\\
&= \frac{1}{2}\int_{|\zeta|\geq R} \log\left((1-\frac{y^2}{|\zeta|^2}) + 4\frac{y^2(\mathrm{Re}\,\zeta^2}{|\zeta|^4}\right)\,d\mu(\zeta)
\end{aligned}
$$

For $|\arg\zeta \pm \frac{\pi}{2}| \geq \varepsilon$, put $\zeta = r(cos(\theta)+isin(\theta))$, we have $cos(\theta)\geq\alpha$ with $\alpha := \cos\left(\frac{\pi}{2}-\varepsilon\right) > 0$. Thus, $|\mathrm{Re}\,\zeta| \geq \alpha|\zeta|$ and we get

$$
\begin{aligned}
&\log\left(\left(1-\frac{y^2}{|\zeta|^2}\right)^2 + \frac{4y^2(\mathrm{Re}\,\zeta)^2}{|\zeta|^4}\right)\\
&\geq \log\left(\left(1-\frac{y^2}{|\zeta|^2}\right)^2 + 4\alpha^2\frac{y^2}{|\zeta|^2}\right) \qquad (2.5)\\
&= \log\left(1+(4\alpha^2-2)\frac{y^2}{|\zeta|^2}+\frac{y^4}{|\zeta|^4}\right)
\end{aligned}
$$

As u is harmonic in $T_{\pm}$ the minimum principle gives that (2.5) holds for every ζ in the support of μ which satisfies $|\zeta|\geq R$. Hence,

$$
\begin{aligned}
Q(y) &\geq \frac{1}{2}\int_{|\zeta|\geq R}\log\left(1+(4\alpha^2-2)\frac{y^2}{|\zeta|^2}+\frac{y^4}{|\zeta|^4}\right)\,d\mu(\zeta)\\
&= \int_R^{\infty}\log\left(1+(4\alpha^2-2)\frac{y^2}{t^2}+\frac{y^4}{t^4}\right)\,dn(t) \qquad (2.6)
\end{aligned}
$$

let $n_R(r) = \{z : R\leq|z|\leq r\}$, as $n_R(r) = n(r) - \mu(\{z : |z| < R\})$ and using integration by parts we can write (2.6) as

$$
Q(y) \geq 2\int_0^{\infty}\frac{n_R(t)}{t}f(\frac{y}{t})\,dt
$$

where

$$
f(x) = \frac{x^2(2\alpha^2-1+x^2)}{1+(4\alpha^2-2)x^2+x^4}
$$

Now, since $\lim_{y\to\infty}Q(y) = -\infty$ we get that $\exists y_0 > 0$ such that $Q(y)\leq 0$ for $y\geq y_0$. Take $0<\delta<1$, we can deduce that

$$
\int_{y_0}^{\infty}\frac{1}{y^{2+\delta}}Q(y)\,dy \leq \int_{y_0}^{\infty}\frac{1}{y^{2+\delta}}\int_0^{\infty}\frac{n_R(t)}{t}f(\frac{y}{t})\,dt\,dy
$$

In fact, $f(x)\leq\frac{1}{4\alpha\sqrt{1-\alpha^2}}x^2$ for $x\geq 0$ and $f(x)\leq\frac{4x^2}{2+x^4}$ for $x\geq 2$

This is to say that the above integrals are finite, so we can use Fubini-Tonelli theorem to interchange the order of the integral and taking $s = \frac{y}{t}$ we can write

$$\int_0^\infty n_R(t) \int_{y_0}^\infty \frac{1}{ty^{2+\delta}} f(\frac{y}{t})\, dy\, dt = \int_0^\infty n_R(t) \int_{\frac{y_0}{t}}^\infty \frac{1}{s^{2+\delta}} f(s)\, ds\, dt \leq 0.$$

Moreover, we have

$$\begin{aligned} \int_0^{\frac{y_0}{t}} \frac{f(s)}{s^{2+\delta}}\, ds &\leq \frac{1}{4\alpha\sqrt{1-\alpha^2}} \int_0^{\frac{y_0}{t}} \frac{ds}{s^\delta} \\ &= \frac{1}{4\alpha\sqrt{1-\alpha^2}} \cdot \frac{1}{1-\delta} \left(\frac{y_0}{t}\right)^{1-\delta}. \end{aligned} \tag{2.7}$$

(2.7) with (1.3) implies

$$\int_0^\infty \frac{n_R(t)}{t^{2+\delta}} \int_0^{\frac{y_0}{t}} \frac{f(s)}{s^{2+\delta}}\, ds\, dt \leq c \int_0^\infty \frac{n_R(t)}{t^3} < \infty$$

with $c = \frac{(y_0)^{1-\delta}}{4\alpha\sqrt{1-\alpha^2}(1-\delta)}$ a constant which is still bounded when $\delta \to 0$. Putting all together we get,

$$\int_0^\infty \frac{n_R(t)}{t^{2+\delta}}\, dt \int_0^\infty \frac{f(s)}{s^{2+\delta}}\, ds \leq c \tag{2.8}$$

On the other hand, we have

$$\int_0^\infty \frac{f(s)}{s^{2+\delta}} = \frac{1}{2} \int_0^\infty x^{-\gamma} \frac{x+\beta}{1+2\beta x + x^2}\, dx$$

where $\beta := 2\alpha^2 - 1$ and $\gamma := \frac{1+\delta}{2} < 1$.

Solving this integral in a classical way using residue theorem gives

$$\int_0^\infty \frac{f(s)}{s^{2+\delta}} = \frac{1}{2} \frac{\pi}{sin(\pi\gamma)} cos\left(\gamma(\pi - 2\epsilon)\right)$$

thus

$$\lim_{\delta \to 0} \int_0^\infty \frac{f(s)}{s^{2+\delta}} = \int_0^\infty \frac{f(s)}{s^{2+\delta}} = \frac{1}{2} \frac{\pi}{sin(\frac{\pi}{2})} cos(\frac{\pi}{2} - \epsilon) = \frac{\pi}{2}\alpha > 0$$

and we know that

$$\int_0^\infty \frac{n_R(t)}{t^2}\, dt = \infty \qquad \text{by (1.3).}$$

Hence

$$\lim_{\delta \to 0} \int_0^\infty \frac{n_R(t)}{t^{2+\delta}}\, dt \int_0^\infty \frac{f(s)}{s^{2+\delta}}\, ds = \infty$$

But, (2.8) gives that the above integral is bounded by a constant which leads to a contradiction.

Therefore, u is harmonic and of the form $u = a \operatorname{Re} z + b$ where $a, b \in \mathbb{R}$. □

Chapter 3

ENTIRE FUNCTIONS WITH TWO SEPARATED VALUES

If an entire function f has all its zeros and 1-points on a finite number of rays, it must be of finite order.

However, if f is entire and has all its zeros and 1-points in certain sectors we need to make the further assumption that it is of finite order to determine this function.

In this chapter we shall present the main result of this thesis, namely the classification of entire functions of finite order whose zeros and 1-points are restricted to certain sectors.

3.1 Accumulation of a-values of a Function

In order to gain insight into the distribution of a-points for a specific form of a function f, we introduce a Lemma as referenced in [6]. This lemma explores where a-points of f accumulate, which we shall need is the upcoming theorems.

Lemma 3.1.1. *Let f be a function of the form*

$$f(z) = \int_0^z p(\zeta)e^{q(\zeta)}\, d\zeta + C$$

where p and q are polynomial and $C \in \mathbb{C}$

Denote by d the degree of d and by Q the coefficient of the leading term of q. Define

$$\phi_K = \frac{(2k-1)\pi - \arg Q}{d} \tag{3.1}$$

for $k \in \{1, 2, ..., d\}$. Let

$$a_k := lim_{r\to\infty} f(re^{i\phi_k}). \tag{3.2}$$

In fact this limit exits, and we have $\forall \epsilon > 0$, as $|z| \to \infty$

$$f(z) \to a_k \quad for \quad \phi_k - \frac{\pi}{2d} + \varepsilon \leq \arg(z) \leq \phi_k + \frac{\pi}{2d} - \varepsilon$$

while

$$|f(z)| \to \infty \quad \textit{for } \phi_k + \frac{\pi}{2d} + \varepsilon \le \arg(z) \le \phi_{k+1} - \frac{\pi}{2d} - \varepsilon$$

where $\phi_d + 1 = \phi_1 + 2\pi$

Let $\phi_k - \frac{\pi}{d} \le \arg z \le \phi_k + \frac{\pi}{d}$*. Then*

$$f(z) = a_k + \frac{p(z)}{q'(z)} e^{q(z)} \left(1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right) \qquad \textit{as } |z| \to \infty.$$

A direct consequence of this Lemma is that for any $a \in \mathbb{C} \setminus \{a_k\}$ with $0 < \epsilon < \frac{\pi}{d}$ each of the sectors $\{z : |\arg z - \phi_k \pm \frac{\pi}{2d}| < \epsilon\}$ contains infinitely many a-points, but only finitely many a_k-points.

Therefore, for any $a \in \mathbb{C} \setminus \{a_k\}$ the a-points of f can only accumulate at the rays $argz = \phi_k \pm \frac{\pi}{2d}$. In fact, a-points accumulate on both rays $argz = \phi_k \pm \frac{\pi}{2d}$.

Theorem 3.1.2. *Let f be an entire transcendental function with finite order. Let S_0 and S_1 be two closed sectors of opening angle at most π with $S_0 \cap S_1 = \{0\}$. Suppose that all but finitely many zeros of f are in S_0, and all but finitely many 1-points of f are in S_1. Then, f has the form*

$$f(z) = \int_0^z p(\zeta) e^{q(\zeta)}\, d\zeta + C \tag{3.3}$$

where p and q are polynomials and $C \in \mathbb{C}$.

Proof. S_0 and S_1 are closed sectors of opening angle at most π with $S_0 \cap S_1 = \{0\}$, then either θ_0 or θ_1 is $< \pi$. Without loss of generality suppose that $\theta_0 < \pi$ and $f(0) \notin \{0, 1\}$.

Claim that the genus of f is at least 1. Suppose to the contrary that the genus of f is 0. Then, using Hadmard factorization theorem on f and $f - 1$ we can write

$$\begin{aligned} f(z) &= f(0) \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 - \frac{z}{a_k}\right) \\ &= 1 + (f(0) - 1) \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 - \frac{z}{b_k}\right) \end{aligned}$$

where (a_k) is the sequence of zeros of f, and (b_k) is the sequence of 1-points of f. Put

$$P_n(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{a_k}\right) \quad \text{and} \quad Q_n(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{b_k}\right)$$

Since the convergence is uniform we can write

$$f'(z) = f(0) \lim_{n\to\infty} P'(z) = (f(0) - 1) \lim_{n\to} \infty Q'(z)$$

Now, using Hurwitz's theorem we can deduce that the zeros of f' are limit points of the set of zeros of P_n' and Q_n'. Furthermore, Gauss-Lucas theorem gives that all zeros of P_n' lie within the convex hull of zeros of P_n and all zeros of Q_n' lie within the convex hull of zeros of Q_n. By applying Hurwitz's theorem again we get that the zeros of f are limit points of the set of zeros of P_n and the 1-points of f are the limit points of the set of zeros of $f-1$.

Therefore, f' has all its zeros within the convex hull of a_k-points and within the convex hull of b_k-points at the same time. However, this is impossible since a_k-points and b_k- points are separated.

Finally, we deduce that f' has only finitely many zeros. Together with the assumption that the genus of f is zero we can deduce that f is a polynomial, which contradicts the hypothesis that f is transcendental. Hence, the genus of f is at least one.

Let ρ be the order of f. As f has genus at least one we have that $\rho \geq 1$.

Since f is of finite order then there exists a sequence (r_k) tending ∞ with the property (2.1)

It's clear that a sequence of Pólya peaks satisfies (2.1). According to result of Darsin and Shea [7], sequence of Pólya peaks of order λ exists for all finite $\lambda \in [\rho_*, \rho^*]$ where

$$\rho^* := \sup \left\{ p \in \mathbb{R} : \limsup_{r,t\to\infty} \frac{\log M(tr)}{t^p \log M(r)} = \infty \right\}$$

and

$$\rho_* := \inf \left\{ p \in \mathbb{R} : \liminf_{r,t\to\infty} \frac{\log M(tr)}{t^p \log M(r)} = 0 \right\}.$$

Also we always have

$$0 \leq \rho_* \leq \rho \leq \rho^* \leq \infty. \tag{3.4}$$

So, when ρ is finite there exits a Pólya peaks of finite order λ. Fix such a sequence (r_k) of order λ, and let u and v be as defined in (2.2). Then, u and v are two subharmonic functions satisfying all the properties of Lemma 2.0.1.

Let $P = \{z \mid u(z) > 0 \text{ or } v(z) > 0\}$. In view of (a), P is the set where $u > 0$ and $v > 0$. We claim that P is open.

To prove the claim, let $z_0 \in P$. By property (e), we know that $z_0 \neq 0$. Furthermore, we know that $S_0 \cap S_1 = \{0\}$, so we can deduce from (c) that, either u or v is harmonic at z_0, implying its continuity at z_0. Without loss of generality, assume that u is harmonic and continuous at z_0.

Since u is continuous at z_0, there exists a small neighborhood V of z_0 such that $u(z) > 0$ for all $z \in V$. Therefore, $V \subset P$, as u is positive on V. This establishes that P is open.

Let $N = \{z \mid u(z) < 0 \text{ or } v(z) < 0\}$. We claim that N is open. To prove the claim, let $z_0 \in N$. Without loss of generality, assume that $u(z_0) < 0$. Since u is subharmonic, it is upper semi-continuous at z_0, which means $u(z_0) = \lim_{\epsilon \to 0} \sup_{|z-z_0|<\epsilon} u(z)$

Therefore, for a small neighborhood V of z_0, we have $u(z) \leq u(z_0) < 0$. Consequently, $u(z) < 0$ for all $z \in V$, implying that $V \subset N$. Hence, N is open.

Thus, the complement $E = \mathbb{C} \setminus (P \cup N) = \{z \mid u(z) = v(z) = 0\}$ is closed. We claim that E has an empty interior.

To prove the claim, suppose to the contrary, that there exists z_0 in the interior of E, i.e., there exists a neighborhood V of z_0 such that $V \subset E$. Without loss of generality, assume that $z_0 \notin S_1$.

By property (c), v is harmonic in $\mathbb{C} \setminus S_1$, particularly at z_0. Hence, we can find a neighborhood U in $\mathbb{C} \setminus S_1$ of z_0 such that $U \subset V$. Since $v = 0$ on U and is harmonic in $\mathbb{C} \setminus S_1$, it follows from the identity theorem that $v = 0$ in $\mathbb{C} \setminus S_1$. By property (a), we have $u(z) \leq 0$ for all $z \in \mathbb{C} \setminus S_1$.

Applying the maximum principle, if there exists z_1 in $\mathbb{C} \setminus S_1$ such that $u(z_1) < 0$, then $|z_1| \leq |z_0|$ since $u(z_0) = 0$. Thus, we have $u(z) = 0$ for all $z \in \mathbb{C} \setminus (S_1 \cup B[0, |z_0|])$. By property (c), u is harmonic in $\mathbb{C} \setminus S_0$. Therefore, by the identity theorem, $u = 0$ in $\mathbb{C} \setminus S_0$. Moreover, by applying the maximum principle on $B(0, 2|z_0|)$, we know that $u \leq 0$, so we find that u is the constant 0 since the maximum is attained at the origin. Consequently, $u = 0$ throughout the entire plane, and we reach a contradiction. Therefore, E cannot have an interior.

Our objective is to show that either $N \subset S_0 \cup S_1$, or $N \supset \mathbb{C} \setminus S_1$ which is only possible if $\theta_1 = \pi$.

To proceed, let Q be the component of N such that $u(z) < 0$ for $z \in Q$. In fact, Q is open since u is upper semi continuous. We aim to prove that $\partial Q \subset S_1$. Suppose that there exits $z_0 \in \partial Q \setminus S_1$, then v is harmonic in a neighborhood V of z_0 by (c). According to properties (a) and (b), $v(z) = 0$ for $z \in Q$, and in particular for $z \in V \cap Q$. As v is harmonic in V and $v = 0$ on an open set inside V, then $v = 0$ for $z \in V$.

Since $v(z_0) = 0$, it follows that $u(z_0) \leq 0$. However, as Q is open, $z_0 \notin Q$, which implies that $u(z_0) = 0$. By applying the mean value inequality, there exists $z_1 \in V$ such that $u(z_1) > 0$, then $v(z_1) > 0$ by (a), which leads to a contradiction. Hence, $\partial Q \subset S1$.

If $\theta_1 < \pi$, according to Lemma (2.1.1), Q cannot contain $\mathbb{C} \setminus S_1$, implying that $Q \subset S_1$. However, in the case where $\theta_1 = \pi$, it is possible for Q to include $\mathbb{C} \setminus S_1$.

Similarly, let R be the component of N where $v(z) < 0$ for $z \in R$, then $\partial R \subset S_0$. As we assumed $\theta_0 < \pi$, and using lemma (2.1.1) we conclude that $R \subset S_0$.

Therefore, $N = Q \cup R \subset S_0 \cup S_1$, or $N \supset \mathbb{C} \setminus S_1$. In the later case $u(z) < 0$ for $z \in \mathbb{C} \setminus S_1$.

Now, we are going two consider both cases, and to prove that in each case f satisfies the form (3.3).

case 1 : $N \subset S_0 \cup S_1$

In this case, we can deduce that $u \geq 0$ and $v \geq 0$ in $\mathbb{C} \setminus S_0 \cup S_1$. According to property (c), both u and v are harmonic in $\mathbb{C} \setminus S_0 \cup S_1$. Furthermore, the minimum principle implies that $u > 0$ and $v > 0$ in this region. Property (a) gives that $u = v$ for all $z \in \mathbb{C} \setminus S_0 \cup S_1$. Thus, we can define a harmonic function w on $\mathbb{C} \setminus \{0\}$ as follows:

$$w(z) := \begin{cases} u(z) & \text{if } z \in \mathbb{C} \setminus S_0, \\ v(z) & \text{if } z \in \mathbb{C} \setminus S_1. \end{cases} \tag{3.5}$$

By the removable singularity theorem, w is harmonic on $\mathbb{C}$. Consequently, w is real part of an entire function g, i.e. $w = \operatorname{Re} g$. Since $w(0) = 0$ we can choose g such that $g(0) = 0$.

As (r_k) is a sequence of Pólya peaks of order $\lambda > 0$, we can deduce from (f) that

$$w(z) = \operatorname{Re} g(z) \leq |z|^{\lambda}. \tag{3.6}$$

Lemma 2.0.2 implies that g is a polynomial of degree at most λ.

Moreover, λ is a positive integer and we have

$$g(z) = c|z|^{\lambda} \tag{3.7}$$

for $c \in \mathbb{C}$. Also, we can deduce from (d) that $|c| = 1$.

In fact, for any $\lambda \in [\rho_*, \rho^*] \cap (0, \infty)$, there is a sequence (r_k) of Pólya peaks of order λ but, we deduced above that λ is a positive integer which means that we have only finitely many possibilities of λ. Thus $\rho_* = \rho^*$ and referring to (3.4) we get

$$\rho_* = \rho = \rho^* \in \mathbb{N} \tag{3.8}$$

Therefore, ρ^* is finite and the only possibility of λ is $\lambda = \rho$. Furthermore, (3.8) implies that for any $\delta > 0$ there exist r_0, $t_0 > 0$ such that

$$t^{\rho-\delta} \log M(r) \leq \log M(tr) \leq t^{\rho+\delta} \log M(r) \quad \text{for } r \geq r_0 \text{ and } t \geq t_0 \tag{3.9}$$

.

This implies that equation (2.1) holds for any sequence $r_k \to \infty$. Consequently, we can drop the assumption that (r_k) must be a sequence of Pólya peaks and instead consider a sequence that tends to infinity. Moreover, we still retain the properties from (a) to (d). However, instead of (f), we can deduce (f') from (3.9) as follows:

$$\max\{u(z), v(z)\} \leq \begin{cases} |z|^{\rho+\delta} & \text{for } |z| \geq t_0, \\ |z|^{\rho-\delta} & \text{for } |z| \leq 1/t_0. \end{cases}$$

This deduction still provides property (e).

Once again, we observe that the function w defined in (3.5) is harmonic and of the form $w = \operatorname{Re} g$ with g entire function. Instead of (3.6), which was derived from (f), we now deduce from (f') the following inequality:

$$w(z) = \operatorname{Re} g(z) \leq \begin{cases} |z|^{\rho+\delta} & \text{for } |z| \geq t_0, \\ |z|^{\rho-\delta} & \text{for } |z| \leq 1/t_0. \end{cases}$$

This inequality implies that g is a polynomial of degree at most $\rho + \delta$, with a zero of multiplicity at least $\rho - \delta$ at the origin. By choosing $\delta < 1$, we find again that g has the form (3.7) with $\lambda = \rho$,

$$g(z) = c|z|^{\rho}$$

As a conclusion, every sequence tending to ∞ has a subsequence (r_k) such that

$$\lim_{k\to\infty} \frac{\log|f(r_k z)|}{\log M(r_k)} = \operatorname{Re}(cz^\rho) \quad \text{for } z \in \mathbb{C} \setminus S_0, \tag{3.10}$$

and

$$\lim_{k\to\infty} \frac{\log|f(r_k z) - 1|}{\log M(r_k)} = \operatorname{Re}(cz^\rho) \text{ for } z \in \mathbb{C} \setminus S_1. \tag{3.11}$$

Using the fact that $h' = \frac{\partial \operatorname{Re} h}{\partial x} - i\frac{\partial \operatorname{Re} h}{\partial y}$, for h holomorphic function. We can deduce from (3.10) and (3.11) that

$$\lim_{k\to\infty} \frac{r_k f'(r_k z)}{f(r_k z)\log M(r_k)} = c\rho z^{\rho-1} \quad \text{for } z \in \mathbb{C} \setminus S_0.$$

and

$$\lim_{k\to\infty} \frac{r_k f'(r_k z)}{(f(r_k z) - 1)\log M(r_k)} = c\rho z^{\rho-1} \quad \text{for } z \in \mathbb{C} \setminus S_1.$$

Therefore, if T_0 is a closed sector in $\mathbb{C} \setminus S_0$, then f' has only finitely many zeros in T_0. Moreover, if T_1 is a closed sector in $\mathbb{C} \setminus S_1$ then f' has only finitely many zeros in T_1. Choosing T_0 and T_1 such that $T_0 \cup T_1 = \mathbb{C}$, implies that f' has finite number of zeros on $\mathbb{C}$. In addition, we have that f' is of finite order, since f is of finite order. So, we can deduce that f' has the form $f' = pe^q$ with p and q polynomials. Finally, f has the form (3.3).

Now, let's consider the second case where $N \supset \mathbb{C} \setminus S_1$.

Case 2: $N \supset \mathbb{C} \setminus S_1$**.**

This situation can only occur when $\theta_1 = \pi$. Without loss of generality, let's assume that S_1 corresponds to the left half plane. This allows us to express S_0 as, $S_0 = \{z : |\arg(z)| \leq \frac{\pi}{2} - \epsilon\}$ for some $\epsilon > 0$.

We know that $u(z) < 0$ for $z \in \mathbb{C} \setminus S_1$. From property (e), we have $u(0) = 0$, and according to property (f), $u(z) \leq |z|^\lambda$. We see that the hypothesis of Lemma 2.1.2 is satisfied. Applying this Lemma yields that $\rho = 1$. Since u is harmonic in $\mathbb{C} \setminus S_0$, we can further apply Lemma 2.2.1 which implies that u is harmonic and of the form $u(z) = a \operatorname{Re} z + b$ for $a, b \in \mathbb{R}$. From $u(0) = 0$, it follows that $b = 0$. Property (d) gives $|a| = 1$, but since u is negative in the right half plane, we have $a = -1$. Therefore, we obtain that

$$u_k(z) = \frac{\log|f(r_k z)|}{\log M(r_k)} \to -\operatorname{Re}(z). \tag{3.12}$$

Again, following the same procedure as in the previous case, we can drop the assumption that (r_k) is a sequence of Pólya peaks and assume instead that (r_k) is a sequence tending to ∞.

We can deduce from (3.12) that there exists a curve γ near the imaginary axis from both directions such that $|f(z)| = 1$ for $z \in \gamma$.

Suppose that f has at least one zero. This implies that the function $\log|f|$ is not harmonic. Since u is of order zero, it follows that $\log|f|$ is also of order zero. Therefore, by the converse of Lemma 2.2.1, we can conclude that $\log|f|$ is not bounded on the imaginary axis. Additionally, all but finitely many zeros of f are in S_0 that does not include the imaginary axis, so we excludes the possibility of $\log|f|$ approaching negative infinity. Thus, there exists a real sequence (y_k) such that

$$T_k := |f(iy_k)| \to \infty \tag{3.13}$$

Without loss of generality assume that $y_k \to +\infty$. Assuming that $T_k > 1$, there exists $x_k > 0$ such that $z_k := x_k + iy_k$ lies on the curve γ. We have $x_k = o(y_k)$ as $k \to \infty$ by (3.12).

Applying the same reasoning as in the previous case, using (3.12) with $r_k = y_k$ and differentiating, we find that

$$\lim_{k\to\infty} \frac{y_k f'(y_k z)}{f(y_k z)\log M(y_k)} = -1 \quad \text{for} \quad z \in \mathbb{C} \setminus S_0. \tag{3.14}$$

It follows from (3.14) that

$$\frac{1}{2}\frac{\log M(y_k)}{y_k} \le \left|\frac{f'(z)}{f(z)}\right| \le 2\frac{\log M(y_k)}{y_k} \qquad \text{for } |z - iy_k| \le 2x_k \tag{3.15}$$

for k large enough. Thus,

$$\begin{aligned} \log T_k &= \log|f(iy_k)| - \log|f(x_k + iy_k)| \\ &= \operatorname{Re}\left(-\int_0^{x_k} \frac{f'(x+iy_k)}{f(x+iy_k)}\,dx\right) \\ &\le \int_0^{x_k} \frac{f'(x+iy_k)}{f(x+iy_k)}\,dx \\ &\le 2x_k \frac{\log M(y_k)}{y_k}. \end{aligned} \tag{3.16}$$

Consider γ_k, which represents the component of the intersection of γ with the disk $\{z : |z - z_k| \le x_k\}$ that contains z_k. It follows that $f \circ \gamma_k$ is a curve entirely contained within the unit circle. For sufficiently large values of k, we have $f'(z) \neq 0$ for $z \in \gamma_k$, by (3.15). Consequently, the argument of $f(z)$ monotonically increases as z traverses γ_k. Furthermore, we can establish that the length of γ_k is greater than or equal to $2x_k$. Therefore,

$$\operatorname{length}(f \circ \gamma_k) \ge 2x_k \inf_{z\in\gamma_k} |f'(z)| = 2x_k \inf_{z\in\gamma_k} \left|\frac{f'(z)}{f(z)}\right| \qquad \text{for } k \text{ large.}$$

Putting this with (3.13), (3.15) and (3.16) gives

$$\operatorname{length}(f \circ \gamma_k) \ge x_k \frac{\log M(y_k)}{y_k} \ge \frac{1}{2}\log T_k > 2\pi \qquad \text{for } k \text{ large enough.}$$

This means, $f \circ \gamma_k$ wraps around the unit circle at least once. Consequently, γ_k must contain a 1-point of f. However, this contradicts the hypothesis that all the 1-points of f are in the left half plane. Hence, we can conclude that f has no zeros, which implies that f has the form $f(z) = e^{az+b}$, and in particular f is of the form (3.3). □

Theorem 3.1.3. *Let S_0 and S_1 be closed sectors in $\mathbb{C}$ satisfying $S_0 \cap S_1 = \{0\}$. Denote the opening angle of S_j as θ_j, and suppose that*

$$\min\{\theta_0, \theta_1\} < \frac{\pi}{2} \quad \textit{and} \quad \max\{\theta_0, \theta_1\} < \pi.$$

Then there does not exist a transcendental entire function of finite order for which all but finitely many zeros are contained in S_0, while all but finitely many 1-points are contained in S_1.

Proof. Let f be a transcendental entire function of finite order such that all but finitely many zeros of f are in S_0, while all but finitely many 1-points of f are in S_1. By Theorem 3.1.2, we know that f has the form (3.3) with polynomials p and q. We will first show that the degree of q is even.

To proceed, let ϕ_k and a_k be as given in (3.1) and (3.2) respectively. It is important to observe that $a_k \in \{0, 1\}$ since otherwise the zeros and 1-points would accumulate at the same rays. Now, suppose that d is odd, i.e., $d = 2m - 1$ for some $m \in \mathbb{N}$, and fix $k \in \{1, \ldots, d\}$. Suppose that $a_k = 0$. In this case, the 1-points of f accumulate on the ray $\arg(z) = \phi_k + \frac{\pi}{2d}$. However, since the 1-points are separated in sector of opening angle less than π, they cannot accumulate on the ray $\arg(z) = \phi_k + \frac{\pi}{2d} + \pi$.

But note that $\arg z = \phi_k + \frac{\pi}{2d} + \pi = \phi_{k+m} - \frac{\pi}{2d}$. Thus, the 1-points do not accumulate at the ray $\arg z = \phi_{k+m} + \frac{\pi}{2d}$. Furthermore, the zeros accumulate at the rays $\arg z = \phi_{k+m} \pm \frac{\pi}{2d}$.

Also, we have that the zeros are separated in a sector of opening angle less that π. Following the same argument we can deduce that the zeros can not accumulate at the ray $\arg z = \phi_{k+m} + \frac{\pi}{2d} + \pi = \phi_{k+2m} - \frac{\pi}{2d}$. Then, the zeros can not accumulate on the ray $\arg z = \phi_{k+2m} + \frac{\pi}{2d}$.

It follows that the 1-points accumulate on the rays $\arg z = \phi_{k+2m} \pm \frac{\pi}{2d} = \phi_{k+1} \pm \frac{\pi}{2d}$. Here the index in ϕ_k is taken modulo d; i.e, $\phi_j = \phi_k$ if $j \equiv k \pmod{d}$.

Therefore, using induction we can prove that the 1-point accumulate at rays $\arg z = \phi_j \pm \frac{\pi}{2d}$ for all $j \in \{1, \ldots, d\}$, which contradicts our hypothesis that the one points are separated in a sector of opining angle less than π. Thus, d must be even.

According to the hypothesis, there exists an open sector T with an opening angle greater than $\frac{\pi}{4}$, in which f has only finitely many zeros and 1-points within T. This means that T does not contain any of the rays $\arg z = \phi_k \pm \frac{\pi}{2d}$ because the zeros or 1-points accumulate on these rays. It follows that, $\frac{\pi}{d} > \frac{\pi}{4}$, leading that $d < 4$.

Then the only possibility left is $d = 2$. Suppose that $d = 2$. Without loss of generality assume that $\theta_0 < \frac{\pi}{2}$, this indicates that all but finitely many zeros are contained within a sector of opening angle less than $\frac{\pi}{2}$. Then, the zeros can not accumulate on rays $\arg z = \phi_k \pm \frac{\pi}{4}$ for both $k = 0$ and $k = 1$. Consequently, the 1-point accumulate on rays $\arg z = \phi_1 \pm \frac{\pi}{4}$ and $\arg z = \phi_2 \pm \frac{\pi}{4}$. This contradicts the hypothesis that all but finitely many 1-points of f are contained within a sector of opening angle less than π.

Therefore, the existence of such an f is not possible.

□

Theorem 3.1.4. *Let f be a transcendental entire function of finite order. let S be a closed sector in $\mathbb{C}$ of opening angle less than $\frac{\pi}{3}$, and let H be a closed half plane such that $S \cap H = \{0\}$. Suppose that all but finitely many zeros of f are within S, and all but finitely many 1-points are within H. Then f has the form $f(z) = p(z)e^{az}$ where p is a polynomial and $a \in \mathbb{C}$.*

Proof. Theorem 3.1.2 implies that f has the form (3.3).
Let ϕ_k and a_k be as given in (3.1) and (3.2) respectively. We note that $a_k \in \{0, 1\}$ since otherwise the zeros and one points will accumulate on the same sectors. It follows from the hypothesis that there exits an open sector T of opening angle greater than $\frac{\pi}{3}$ that contains only finitely many zeros and 1-points of f. This implies that T does not intersect any of the rays $\arg z = \phi_k \pm \frac{\pi}{2d}$. In fact, the number of those rays is $2d$ and they are distributed equally at an angle $\frac{\pi}{d}$. Thus, we can conclude that $\frac{\pi}{d} > \frac{\pi}{3}$, leading to $d < 3$.

Assuming $d = 2$, as the 1-points are contained in a half-plane, we have $a_k = 0$ for some $k \in \{1, 2\}$. Consequently, the zeros accumulate at both rays $\arg z = \phi_k \pm \frac{\pi}{4}$. This implies the existence of infinitely many zeros not contained in S, which contradicts our hypothesis. Hence, we can deduce that $d = 1$, resulting that f have the form $f(z) = p(z)e^{az}$ where p is a polynomial and $a \in \mathbb{C}$. □

3.2 Examples that show the sharpness of the results

In this section we are going to consider some examples that shows that the conditions on the opening angles of the sectors of theorems of chapter 3 are sharp.

The following example shows that the condition $\min\{\theta_0, \theta_1\} < \frac{\pi}{2}$ in Theorem (3.1.3) can not be relaxed to $\min\{\theta_0, \theta_1\} \le \frac{\pi}{2}$.

Example 3.2.1. *Let*

$$f(z) = \frac{2}{\sqrt{\pi}} \int_0^z t^2 e^{-t^2} \, dt + \frac{1}{2}$$

f has the form (3.3) with $q(t) = -t^2$ indicating that $d := \deg(q) = 2$.
In fact

$$\int_0^\infty t^2 e^{-t^2} \, dt = \frac{\sqrt{\pi}}{4}$$

Using the result of Lemma (3.1.1), where $\phi_1 = 0$ $\phi_2 = \pi$, $a_1 = 1$, and $a_2 = 0$, we can conclude that the zeros of f accumulate at rays where $\arg z = \pm\frac{\pi}{4}$, and the 1-points of f accumulate at the rays where $\arg z = \pi \pm \frac{\pi}{4}$.

Therefore, all but finitely many zeros of f are in $S_0 = \{z : |\arg z| \le \frac{\pi}{4}\}$, while all but finitely many 1-points are in $S_1 = \{z : |\arg z - \pi| \le \frac{\pi}{4}\}$.

The next example shows that the condition $\max\{\theta_0, \theta_1\} < \pi$ can not be relaxed to $\max\{\theta_0.\theta_1\} \leq \pi$.

Example 3.2.2. *Let*

$$f(z) = e^z$$

f has no zeros, and the 1-points of f lie on the imaginary axis for $z = 2ni\pi$ with $n \in \mathbb{N}$. This indicates that the 1-points of f are within the closed right half-plane.

The below example shows that the condition on the opening angle of sector S being $< \frac{\pi}{3}$ in Theorem (3.1.4) can not be relaxed to $\leq \frac{\pi}{3}$.

Example 3.2.3. *Define a and b as following*

$$a \int_0^\infty t^3 e^{-t^3}\, dt = \frac{1}{3} \qquad \textit{and} \qquad b \int_0^\infty t e^{-t^3}\, dt = \frac{1}{3}$$

Let

$$f(z) = \int_0^z (at^3 + bt) e^{-t^3}\, dt + \frac{1}{3}$$

In this example f has the form (3.3) with $q(t) = -t^3$ indicating that $d = 3$. Here we have $\phi_1 = 0$, $\phi_2 = \frac{2\pi}{3}$, $\phi_3 = \frac{4\pi}{3}$, $a_1 = 1$, and $a_2 = a_3 = 0$.

Using the result of Lemma (3.1.1), we conclude that zeros of f accumulate at the rays given $\arg z = \pm\frac{\pi}{2}$,and the 1-points accumulate at the rays given by $\arg z = \pm\frac{5\pi}{6}$ and $\arg z = \pm\frac{\pi}{2}$.

Therefore, all but finitely many zeros of f are in the sector $S = \{z : |\arg z| \leq \frac{\pi}{6}\}$, while all but finitely many 1-points are in the closed left-half plane.

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