AMERICAN UNIVERSITY OF BEIRUT

# FREQUENCY ANALYSIS AND APPLICATIONS

by
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# Abstract
# of the Thesis of

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The goal of this thesis is to study signals that have a regularity property defined in the frequency space, such as a decay on average of the amplitude of their Fourier transform, by using techniques from frequency analysis. Frequency analysis is a set of techniques that involve an analysis in the Fourier domain. We review some of these techniques and some principles. More precisely we will decompose a signal into countable sums of functions of which the Fourier transform is compactly supported in a ball or an annulus by performing a Littlewood–Paley decomposition. We will apply this technique to study the properties of functions having a specific regularity. Over two hundred years ago, Fourier studied problems related to the series expansions of periodic signals using elementary trigonometric polynomials. The theory was extended to non-periodic signals by using the Fourier transform and forms the core of harmonic analysis. Harmonic analysis is used in various fields such as signal processing and partial differential equations (PDEs).

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# Chapter 1

# The Fourier Transform

In this section we recall the definition of the Fourier transform and some of its properties in the classical setting and also in the Lebesgue setting.

## 1.1 The Fourier transform in the classical setting

In this subsection we follow [6]. The classical Fourier transform is defined for functions that are moderately decreasing or Schwartz functions.
Recall that the set of moderately decreasing functions (denoted by $\mathcal{M}(\mathbb{R}^d)$) is the set of continuous functions such that there exists $A > 0$ such that $|f(x)| \leq \frac{A}{1+|x|^{2d}}$.
Let $f \in \mathcal{M}(\mathbb{R}^d)$. Then $\hat{f}$ denotes the Fourier transform of $f$, and it is defined for all $\xi \in \mathbb{R}^d$ by

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2i\pi\xi\cdot x}\, dx \tag{1.1}$$

Let $\mathcal{S}(\mathbb{R}^d)$ be the set of Schwartz functions, i.e the set for functions $f$ such that $\sup_{x\in\mathbb{R}^d} (1 + |x|)^N |\partial^\alpha f(x)| < \infty$ for all $\alpha := (\alpha_1, .., \alpha_d) \in \mathbb{N}^d$ and $N \in \mathbb{N}$ (Recall that $\partial^\alpha := \partial^{\alpha_1} ... \partial^{\alpha_d}$). Recall (see [2]) that if $f \in \mathcal{S}(\mathbb{R}^d)$ then $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$. Moreover the Fourier inversion formula holds

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2i\pi\xi\cdot x}\, d\xi \tag{1.2}$$

Notice that the proof to establish (1.2) for Schwartz functions also works for functions $(f, \hat{f}) \in \mathcal{M}(\mathbb{R}^d)^2$.

Let $(x_0, \lambda) \in \mathbb{R}^d \times \mathbb{R}^+$. Let $T_{x_0}$ (resp. $D_\lambda$) denote the translation of vector $x_0$ (resp. the dilation operator of factor $\lambda$), i.e $T_{x_0}(g)(x) := g(x - x_0)$ ( resp. $D_\lambda(g)(x) := g(\lambda x)$. Let $(x_0, \xi_0, \lambda) \in (\mathbb{R}^d)^2 \times \mathbb{R}$.Then elementary changes of variables show that the equalities below hold:

$$\widehat{T_{x_0}f}(\xi) = e^{-2i\pi\xi\cdot x_0}\hat{f}(\xi),\ \widehat{D_\lambda f}(\xi) = \frac{1}{\lambda^d}\hat{f}\left(\frac{\xi}{\lambda}\right), \text{ and}$$
$$T_{\xi_0}\hat{f}(\xi) = \widehat{e^{2i\pi\xi_0\cdot x}f}(\xi). \tag{1.3}$$

## 1.2 The Fourier transform in the Lebesgue setting

In this subsection we follow [2]. The Fourier transform in the Lebesgue setting for functions in $L^1(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$, and $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

Let $f \in L^1(\mathbb{R}^d)$. Then the Fourier transform is again defined by (1.1). Moreover, assuming also that $\hat{f} \in L^1(\mathbb{R}^d)$, then $f$ is (a.e) equal to a continuous function $g$, more precisely

$$f(x) = g(x) := \int_{\mathbb{R}^d} \hat{f}(\xi)e^{2i\pi\xi\cdot x}\,d\xi \tag{1.4}$$

Let $f \in L^2(\mathbb{R}^d)$. Then there exists a sequence $\{f_n\}_{n\geq 1}$ such that $f_n$ and $\hat{f}_n \in L^1(\mathbb{R}^d)$ [1], and $\hat{f} = \lim_{n\to\infty} \hat{f}_n$ in $L^2(\mathbb{R}^d)$. Moreover the Plancherel theorem holds, i.e

$$\int_{\mathbb{R}^d} |f(x)|^2\,dx = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2\,dx \tag{1.5}$$

Assume also that $\hat{f} \in L^1(\mathbb{R}^d)$. Then (1.4) also holds [2].

Let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then the Fourier transform of $f$ in $L^1$ is (a.e) equal to its Fourier transform in $L^2$. Consequently (1.4) and (1.5) also holds.

[1]Observe that this implies that $\hat{f}_n \in L^2(\mathbb{R}^d)$: see [2].

[2]It suffices to use the same arguments in [2], more precisely in the proof of the Fourier inversion formula for $f$ and $\hat{f}$ in $L^1(\mathbb{R}^d)$, once we have proved that the transfer formula $\int_{\mathbb{R}^d} \hat{f}\phi = \int_{\mathbb{R}^d} f\hat{\phi}$ holds for $f$, with $\phi$ a Schwartz function. But the transfer formula clearly holds for all functions $h$ in $L^1(\mathbb{R}^d)$, so it also holds for $f$ by a limit process

# Chapter 2

# Properties of a Signal Localized in the Fourier Domain

In this section we study the properties of a signal that is localized in the Fourier domain, i.e a signal of which the amplitude is essentially concentrated around a point $\xi_0 \in \mathbb{R}^d$. More precisely we say that a function $f$ is localized in the frequency space around a point $\xi_0 \in \mathbb{R}^d$ if and only if for all $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$\xi \in \mathbb{R}^d : |\hat{f}(\xi)| \leq \tfrac{C_N}{(1+|\xi-\xi_0|)^N}. \tag{2.1}$$

The proposition below and its proof show that if a function $f$ has a Fourier transform that is localized around some point $\xi_0 \in \mathbb{R}^d$, then $f$ is smooth.

**Proposition 1.** *Let $\xi_0 \in \mathbb{R}^d$ and $f \in \mathcal{M}(\mathbb{R}^d)$ that is localized at $\xi_0$. Then $f \in C^\infty(\mathbb{R}^d)$.*

*Proof.* We may assume WLOG that $\xi_0 = 0$: indeed, we may use (1.3) to find $g$ such that $\hat{g}(\xi) := \hat{f}(\xi + \xi_0)$ and then apply Proposition 1 to $g$ at $\xi_0 = 0$.
Observe from (2.1) that for all $\alpha \in \mathbb{N}^d$, the map $\xi \to \partial_x^\alpha \left(\hat{f}(\xi)e^{2i\pi\xi\cdot x}\right)$ is bounded by an $L^1$ function that does not depend on $x$. Hence we can use the differentiation rule under the integral sign of the Fourier inversion formula (1.2) to come to the conclusion that $f \in C^\infty(\mathbb{R}^d)$ and, moreover,

$$\alpha \in \mathbb{N}^d : \ \partial^\alpha f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)(2\pi i\xi)^\alpha e^{2\pi i\xi\cdot x} d\xi$$

□

**Remark 1.** *The analogue of Proposition 1 in Lebesgue spaces is the following:*

**Proposition 1'.** *Let $\xi_0 \in \mathbb{R}^d$. Assume that $f \in L^1(\mathbb{R}^d)$ and that $\hat{f}$ is localized around $\xi_0$. Then $f$ is a.e equal to a $\mathcal{C}^\infty(\mathbb{R}^d)$ function.*

*Proof.* Again we may assume WLOG that $\xi_0 = 0$. We see from (2.1) that $\hat{f} \in L^1(\mathbb{R})$. Hence we get from (1.2) $f = g$ a.e with $g(x) := \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2i\pi\xi\cdot x}\, d\xi$. Then we replace '$\partial^\alpha f$" with"$\partial^\alpha g$" in the proof of Proposition 1 and we follow verbatim the same steps.

□

The next proposition shows that if $\hat{f}$ is very well-localized (in the sense that $\hat{f} \in \mathcal{C}_c(\mathbb{R}^d)$ [1]) then $f$ cannot be equal to zero on a ball centered at $x_0$ on a radius $R > 0$, even if $R$ is very small. This implies in particular that $f$ cannot be localized (in the sense that $f \notin \mathcal{C}_c(\mathbb{R}^d)$ )

**Proposition 2.** *Let $E := \{\ h$ of moderate decrease : $\hat{h} \in C_c(\mathbb{R}^d)\}$. Let $(x_0, R) \in \mathbb{R}^d \times (0, \infty]$. There exists no $f \in E$, $f \neq 0$ such that $f = 0$ on a ball $B(x_0, R)$.*

*Proof.* Suppose that $f = 0$ on a ball $B(x_0, R)$ where $x_o \in \mathbb{R}^d$ and $R > 0$. We may assume WLOG that $x_0 = 0$: indeed, if $x_0 \neq 0$, then we can apply (1.3) and Proposition 2 to $x \to f(x_0 + x)$.
Expanding $e^{2\pi i x\cdot\xi}$ in its Maclaurin series we get [2]

$$\begin{aligned} f(x) &= \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi \\ &= \int_{\mathbb{R}^d} \hat{f}(\xi) \sum_{k=0}^{\infty} \frac{1}{k!} (2\pi i x\cdot\xi)^k d\xi \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^d} (2\pi i x\cdot\xi)^k \hat{f}(\xi) d\xi \\ &= \sum_{\alpha\in\mathbb{N}^d} \frac{1}{\alpha!} x^\alpha \int_{\mathbb{R}^d} (2\pi i\xi)^\alpha \hat{f}(\xi) d\xi, \end{aligned} \tag{2.2}$$

using at the third line the normal convergence of the series in $\xi$ and at the fourth line the multinomial Newton formula. Now, recall from the proof of Proposition 1 that $\int_{\mathbb{R}^d} (2\pi i\xi)^\alpha \hat{f}(\xi) d\xi = \partial^\alpha f(0) = 0$ for all $\alpha$. Hence $f = 0$, which is impossible.

□

**Remark 2.** *The analogue of Proposition 2 in Lebesgue spaces is the following*

**Proposition 2'.** *Let $E' := \{h \in L^1(\mathbb{R}^d) : \hat{h} \in C_c(\mathbb{R}^d)\}$. Let $(x_0, R) \in \mathbb{R}^d \times (0, \infty)$. There is no $f \neq 0 \in E'$ such that $f = 0$ a.e on $B(x_0, R)$.*

[1] Recall that $\mathcal{C}_c(\mathbb{R}^d)$ is the set of continuous and compactly supported functions

[2] Recall that if $\alpha := (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ then $\alpha! := \alpha_1!...\alpha_d!$ and $x^\alpha := x_1^{\alpha_1}...x_d^{\alpha_d}$

*Proof.* Again we assume that the conclusion does not hold. We may assume WLOG the $x_0 = 0$. Let $g(x) := \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2i\pi x\cdot\xi}\, d\xi$. Then from (2.1) and a standard rule of continuity we see that $g$ is continuous. Hence $g = 0$ on $B(x_0, R)$ and, repeating the same argument as in (2.2), we get $g = 0$. Hence $f = 0$ a.e, which is a contradiction. □

In fact a stronger result holds:

**Proposition 3.** *Let $\xi_0 \in \mathbb{R}^d$. Let $\bar{E} = \{$ h of moderate decrease : $\forall N \in \mathbb{N}$, $\sup\limits_{\xi\in\mathbb{R}^d} (1 + |\xi - \xi_0|)^N |\hat{h}(\xi)| < \infty\}$. Then there does not exist $f \neq 0 \in \bar{E}$ such that $f = 0$ on $B(x_0, R)$.*

*Proof.* We only prove the proposition for $d = 1$.

Again we may assume WLOG that $\xi_0 = 0$.

Assume that there exists $f \neq 0$ such that $f = 0$ on $B(x_0, R)$. Let $g(z) := \int_{\mathbb{R}} \hat{f}(\xi) e^{2i\pi\xi z} d\xi$.

<u>Claim</u>: $g$ is an analytic function.

*Proof:*

Indeed, from the fundamental theorem of calculus

$$e^{2i\pi\xi(z+\triangle z)} - e^{2i\pi\xi z} = \triangle z \int_0^1 \frac{\partial e^{2i\pi\xi z'}}{\partial z'}(z + t\triangle z)\, dt$$

Moreover from $e^{z'+z''} = e^{z'} e^{z''}$ and the elementary estimate $|e^{z'} - 1| \lesssim |z'|$ for $|z'| \leq 1$ we get

$$\left| e^{2i\pi\xi(z+\triangle z)} - e^{2i\pi\xi z} \right| \lesssim |\xi||\triangle z|$$

Hence $g$ is analytic and

$$\lim_{\triangle z\to 0} \frac{g(z+\triangle z)-g(z)}{\triangle z} = \int_{\mathbb{R}} 2i\pi\xi \hat{f}(\xi) e^{2i\pi\xi\cdot x}\, d\xi\cdot$$

Alternatively one can use the Morera theorem to prove that $g$ is analytic. Let $R$ be a rectangle. Denoting by $\partial R$ its boundary we have, by Fubini theorem

$$\int_{\partial_R} g(z) dz = \int_{\mathbb{R}} \hat{f}(\xi) \int_{\partial R} e^{2i\pi\xi z}\, dz\, d\xi = 0,$$

where at the last equality we use the Cauchy theorem to the analytic function $z \to e^{2i\pi\xi z}$.
Moreover from (1.4) we see that $g$ has a zero that is not isolated. Hence $g = 0$ and $f = 0$. □

## 2.1 The Heisenberg uncertainty principle

We showed in Proposition 1' and more generally in Proposition 3 that if $\hat{f}$ is localized in the frequency space then $f$ cannot be localized in the physical space. In the next proposition, we prove a quantitative formulation of this phenomenon: the Heisenberg inequality.

**Proposition 4.** *Let $f \in \mathcal{S}\left(\mathbb{R}^d\right)$. Let $(x_0, \xi_0) \in (\mathbb{R}^d)^2$. Then*

$$\left(\int_{\mathbb{R}^d} |x-x_0|^2 |f(x)|^2 \, dx\right) \left(\int_{\mathbb{R}^d} |\xi-\xi_0|^2 |\hat{f}(\xi)|^2 \, d\xi\right) \geq \frac{1}{16\pi^2} \left(\int_{\mathbb{R}^d} |f(x)|^2 \, dx\right)^2 \quad (2.3)$$

*The same conclusion holds if $f \in L^2\left(\mathbb{R}^d\right)$ satisfies $\int_{\mathbb{R}^d} |x|^2 |f(x)|^2 \, dx < \infty$, and $\int_{\mathbb{R}^d} |\xi|^2 |\hat{f}(\xi)|^2 \, d\xi < \infty$.*

*Proof.* First let us assume that $f \in \mathcal{S}(\mathbb{R}^d)$.
By using the translation rules of (1.3) if necessary we may assume WLOG that $\xi_0 = x_0 = 0$.
By using the dilation rule of (1.3) if necessary we may assume WLOG that $(*)$ holds with $(*) : \int_{\mathbb{R}^d} |f(x)|^2 \, dx = 1$. Indeed an elementary change of variable shows that $\int_{\mathbb{R}^d} |D_\lambda f(x)|^2 \, dx = 1$ iff $\lambda^d = \|f\|_{L^2}^2$. Applying (2.3) to $f_\lambda$ we get

$$\left(\int_{\mathbb{R}^d} |x|^2 |f_\lambda(x)|^2 \, dx\right) \left(\int_{\mathbb{R}^d} |\xi|^2 |\widehat{f_\lambda}(\xi)|^2 \, d\xi\right) \gtrsim 1 \quad (2.4)$$

Hence, an elementary change of variable combined with ( 1.3) show that

$$\left(\int_{\mathbb{R}^d} |x|^2 |f_\lambda(x)|^2 \, dx\right) \left(\int_{\mathbb{R}^d} |\xi|^2 |\widehat{f_\lambda}(\xi)|^2 \, d\xi\right) \gtrsim \lambda^{-2d} \gtrsim \|f\|_{L^2}^2$$

We first prove (2.3) for $d = 1$. We follow e.g [6]. Integrating by parts and recalling that $f$ decays fast for $|x|$ large we get

$$\begin{array}{ll} 1 & = \int_{\mathbb{R}^d} x' |f(x)|^2 \, dx \\ & = -\int_{\mathbb{R}^d} x \frac{d}{dx}\left(|f(x)|^2\right) \, dx \\ & \lesssim \left(\int_{\mathbb{R}^d} x^2 |f(x)|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |f'(x)|^2 \, dx\right)^{\frac{1}{2}} \\ & \lesssim \left(\int_{\mathbb{R}^d} x^2 |f(x)|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \xi^2 |\hat{f}(\xi)|^2 \, d\xi\right)^{\frac{1}{2}}, \end{array}$$

using the Cauchy-Schwartz inequality at the third row and $\widehat{f'}(\xi) = 2i\pi\xi\hat{f}(\xi)$ at the fourth row.
Then we prove (2.3) for $d > 1$. Let $i \in \{1, ..., d\}$. Then from Fubini theorem

$$\begin{array}{ll} 1 & = \int_{\mathbb{R}^d} |f(x)|^2 \, dx_1 ... dx_d \\ & = \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} |f(x)|^2 \, dx_i\right) dx_1 ... dx_{i-1} dx_{i+1} ... dx_d \\ & \lesssim \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} x_i^2 |f(x)|^2 \, dx_i\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\partial_{x_i} f(x)|^2 \, dx_i\right)^{\frac{1}{2}} dx_1 ... dx_{i-1} dx_{i+1} ... dx_d, \\ & \lesssim \left(\int_{\mathbb{R}^d} x_i^2 |f(x)|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |\partial_{x_i} f(x)|^2 dx\right)^{\frac{1}{2}} \end{array}$$

using at the last row $(d-1)-$ times the Cauchy-Schwarz inequality. By summing over $i$ the inequality above we get, using also the Cauchy-Schwartz inequality for finite sequences

$$\begin{aligned} 1 &\lesssim \left(\int_{\mathbb{R}^d} |x|^2 |f(x)|^2 \, dx\right) \left(\int_{\mathbb{R}^d} |\nabla f(x)|^2 \, dx\right) \\ &\lesssim \left(\int_{\mathbb{R}^d} |x|^2 |f(x)|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |\xi|^2 |\hat{f}(\xi)|^2 \, d\xi\right)^{\frac{1}{2}} \end{aligned}$$

Now assume that $f \in L^2(\mathbb{R}^d)$ such that $|x| f \in L^2(\mathbb{R}^d)$ and $|\xi| \hat{f} \in L^2(\mathbb{R}^d)$. Let $R_n > 0$ be such that $\int_{|x| \geq R_n} |x|^2 |f(x)|^2 \, dx \to 0$ as $n \to \infty$. We slightly modify an argument in e.g [1]. Let $\chi$ be a smooth function such that $\chi(x) = 1$ if $|x| \leq 1$ and $\chi(x) = 0$ if $|x| \geq 2$. Let $f_n := \chi_n(\rho_n * f)$ with $\chi_n(x) := \chi\left(\frac{x}{R_n}\right)$ and $\{\rho_n\}_{n \geq 1}$ a sequence of mollifiers. Recall that $\rho_n * f \to f$ in $L^2(\mathbb{R}^d)$. Hence using also the decomposition $f_n = \chi_n(\rho_n * f - f) + \chi_n f$ and the dominated convergence theorem, we get $(1 + |x|) f_n \to (1 + |x|) f$ as $n \to \infty$. From $\nabla f_n = \nabla \chi_n \, (\rho_n * f) + \chi_n (\rho_n * \nabla f)$ we see that [3] $\nabla f_n \to \nabla f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Hence by Plancherel theorem we have $\int_{\mathbb{R}^d} |\xi|^2 \left| \widehat{f_n - f}(\xi) \right|^2 \, d\xi \to 0$ as $n \to \infty$.

□

A consequence of the inequality (2.3) is the following result that we can relate to Proposition 2' :

**Proposition 5.** *Let $(R, x_0, \xi_0) \in (0, \infty) \times (\mathbb{R}^d)^2$. There exists a constant $c > 0$ such that there is no function $f \in L^2(\mathbb{R}^d)$ for which $\hat{f} = 0$ outside the ball $B(\xi_0, R)$ and $f = 0$ outside the ball $B\left(x_0, \frac{c}{R}\right)$.*

*Proof.* We see from (2.3) and Plancherel theorem that $\int_{\mathbb{R}^d} |x - x_0|^2 |f(x)|^2 \, dx \leq \frac{c^2}{R^2} \int_{\mathbb{R}^d} |f(x)|^2 \, dx$, $\int_{\mathbb{R}^d} |\xi - \xi_0|^2 |\hat{f}(\xi)|^2 \, d\xi \leq R^2 \int_{\mathbb{R}^d} |f(x)|^2 \, dx$. This contradicts (2.3) for $c > 0$ small enough.

□

## 2.2 Benedick non-localization principle

We showed in the previous section a simultaneous non localization property for $f$ and $\hat{f}$. More precisely we proved in Proposition 2' that if $f \neq 0 \in L^1(\mathbb{R})$ has a smooth Fourier transform with bounded support (i.e there exists $\bar{R} > 0$ such that $\{\hat{f} \neq 0\} \subset B(O, \bar{R})$ ) then for all $(x_0, R) \in \mathbb{R}^d \times (0, \infty)$ we have $|\{f \neq 0\} \cap B(x_0, R)| > 0$ [4].

[3] Here $\nabla g := (\partial_{x_1} g, ...., \partial_{x_d} g)$, with $\partial_{x_i} g$ the $i - th$ distributional partial derivative (see [2] ) of $g \in \{f_n, f\}$.

[4] Here $|E|$ denotes the measure of a set $E$

We now prove a stronger proposition on simultaneous non-localization due to Benedicks: it says that if a function $f \neq 0 \in L^1(\mathbb{R}^d)$ has its Fourier transform $\hat{f}$ finitely supported ( i.e $\{\hat{f} \neq 0\}| < \infty$ ) then $f$ cannot be finitely supported (i.e $|\{f \neq 0\}| = \infty$ ). Notice that $\{\hat{f} \neq 0\}| < \infty$ ( resp. $|\{f \neq 0\}| = \infty$ ) may be unbounded and that $|\{f \neq 0\} \cap B(x_0, R)| > 0$ does not necessarily imply that $|\{f \neq 0\}| = \infty$.
We now state the Benedick non-localization principle:

**Proposition 6.** *Assume that $f \in L^1(\mathbb{R}^d)$ satisfies $|\{\hat{f} \neq 0\}| < \infty$ and $|\{f \neq 0\} < \infty|$. Then $f = 0$.*

*Proof.* We follow the exposition of the proof in [5].

Let $X := \{f \neq 0\}$ and $\hat{X} := \{\hat{f} \neq 0\}$. By using a scaling transformation $x \to ax$ if necessary, we may assume that $|X| < 1$. Consider the periodization of the indicator function of $\{\hat{f} \neq 0\}$,i.e the function $\xi \to h(\xi) := \sum_{k \in \mathbb{Z}^d} \mathbb{1}_{\{\hat{f} \neq 0\}}(\xi - k)$.

Claim: $h < \infty$ a.e.
Indeed it suffices to prove that $h < \infty$ a.e on the rectangle $[0, 1] \times ...[0, 1]$. Given $k := (k_1, ..., k_d) \in \mathbb{Z}^d$ let $R_k$ denote the rectangle $R_k := [k_1, k_1 + 1] \times ... \times [k_d, k_d + 1]$. From the monotone convergence theorem and an elementary change of variable we see that

$$\infty > \int_{\mathbb{R}^d} \mathbb{1}_{\{\hat{f} \neq 0\}} \, d\xi = \sum_{k \in \mathbb{Z}^d} \int_{R_k} \mathbb{1}_{\{\hat{f} \neq 0\}}(\xi) \, d\xi = \sum_{k \in \mathbb{Z}^d} \int_{R_{(0,\dots,0)}} \mathbb{1}_{\{\hat{f} \neq 0\}}(\xi - k) \, d\xi = \int_{R_{(0,\dots,0)}} h(\xi) \, d\xi$$

Hence the claim follows.

This implies that the property $(P)$ holds with
$(P):$ a.e $\xi$ there exists only a finite number of ks such that $\hat{f}(\xi - k) \neq 0$.
Let $f_\xi(x) := e^{2i\pi\xi \cdot x} f(x)$. Let $\widetilde{f_\xi} := \sum_{k \in \mathbb{Z}^d} f_\xi(x - k)$ be the periodic extension of $f_\xi$. Since $f_\xi \in L^1(\mathbb{R}^d)$, we can apply the Poisson formula and get that the Fourier series of $\widetilde{f_\xi}$ (denoted by $FS[\widetilde{f_\xi}]$) is equal to

$$FS[\widetilde{f_\xi}] = \sum_{k \in \mathbb{Z}^d} \hat{f}(\xi - k) e^{2i\pi k \cdot x}$$

(Here $FS[\widetilde{f_\xi}]$ denotes the Fourier series of $\widetilde{f_\xi}$). It is also clear from $(P)$ that $FS[\widetilde{f_\xi}]$ converges absolutely. Hence we get a.e

$$\widetilde{f_\xi} = FS[\widetilde{f_\xi}] = \sum_{k \in \mathbb{Z}^d} \hat{f}(\xi - k) e^{2i\pi k \cdot x} \tag{2.5}$$

Moreover from the definition of $\widetilde{f_\xi}$ we see that

$$\left\{x \in \mathbb{T}^d : \ \widetilde{f_\xi} \neq 0\right\} \subset \bigcup_{k \in \mathbb{Z}^d} \left\{x \in \mathbb{T}^d : \ f(x - k) \neq 0\right\} \subset \{f \neq 0\}$$

Hence $\left|\left\{x \in \mathbb{T}^d : \widetilde{f_\xi} \neq 0\right\}\right| < 1$. We also see from (2.5) that $\widetilde{f_\xi}$ is a.e equal to a trigonometric polynomial. Hence $\widetilde{f_\xi} = 0$ a.e and, since $\hat{f}(\xi - k)$ is the $k^{th}-$ Fourier coefficient of $\widetilde{f_\xi}$ we get (for a.e $\xi$) $\hat{f}(\xi - k) = 0$ for all $k \in \mathbb{Z}^d$. Hence $f = 0$.

□

## 2.3 Hardy uncertainty principle

We showed previously that $\hat{f}$ and $f$ cannot be simultaneously localized. A natural question is: given $f \neq 0$, how fast $f$ and $\hat{f}$ can decay simultaneously? We already know that if $f$ is the Gaussian function ( i.e $f(x) := e^{-\pi a x^2}$ with $a > 0$ ) then $\hat{f}(\xi) = \frac{1}{\sqrt{a}} e^{-\pi \xi^2 / a}$. The next question is: can we find a function $f \neq 0$ such that $f$ and $\hat{f}$ decays faster than a gaussian? The Hardy uncertainty principle gives a negative answer. It says that $f$ and $\hat{f}$ can't decay simultaneously faster than a gaussian. More precisely

**Proposition 7.** *Let $C$, $C'$ be two nonnegative constants and let $a$ be a positive constant Suppose that $f := \mathbb{R} \rightarrow \mathbb{C}$ is a (measurable) function such that $|f(x)| \leq Ce^{-\pi a x^2}$ and $|\hat{f}(\xi)| \leq C' e^{-\pi \xi^2 / a}$. Then there exists $\bar{C} \geq 0$ such that*

$$f(x) = \bar{C} e^{-\pi a x^2}. \tag{2.6}$$

*Proof.* We follow the exposition of [7]. By using (1.3) if necessary we may assume WLOG that $a = 1$. By considering $g = cf$ with $c > 0$ a small constant if necessary we may also assume WLOG the $C = C^{'} = 1$.
Observe that in view of the decay of the function $f$ as $|x| \rightarrow \infty$, we may extend the definition of the Fourier transform to the complex plane by letting $\hat{f}(z) := \int_{\mathbb{R}} e^{-2i\pi z x} f(x) dx$. Observe that $\hat{f}$ is analytic: this follows again from Morera theorem and Fubini theorem (see also proof of Proposition 3). By completing the square we get

$$\begin{aligned} \left|\hat{f}(\xi + i\eta)\right| & \lesssim \int_{\mathbb{R}} e^{2\pi \eta x} |f(x)| \, dx \\ & \leq e^{\pi \eta^2} \int_{\mathbb{R}} e^{-\pi(\eta - x)^2} \, dx \\ & \leq e^{\pi \eta^2} \end{aligned}$$

Hence if $F(z) := e^{\pi z^2} \hat{f}(z)$ then $F$ is bounded by one on the imaginary axis. Let $0 < \theta < \frac{\pi}{2}$ be close enough to $\frac{\pi}{2}$, $0 < \delta, \epsilon \ll 1$ be small parameters, and $R > 0$ a large number such that all the statements below are correct. Let $\Gamma_\theta$ denote the sector

$$\Gamma_\theta := \{re^{i\alpha}, 0 < r \leq R, 0 \leq \alpha \leq \theta\}$$

We would like to apply the maximum principle to the function $F$ on the upper right quadrant $\mathcal{H} \cap B(O, R)$ ( Here $B(O, R)$ is the closed ball with radius $R$ and center the

origin, and $\mathcal{H}$ the upper plane) with $R$ that can be arbitrarily large. Nevertheless observe that $F$ is not even bounded on $\mathcal{H}$. In order to circumvent the difficulty, we proceed as follows. We multiply $F$ by the analytic functions $e^{i\epsilon e^{i\epsilon}z^{2+\epsilon}}$, $e^{i\delta z^2}$ so that the resulting function $G_{\alpha,\epsilon}(z) := e^{i\delta z^2}e^{i\epsilon e^{i\epsilon}z^{2+\epsilon}}F(z)$ is bounded by one on the boundary of the bounded domain $B(O,R)\cap\Gamma_\theta$. Hence by the maximum modulus theorem we see that $G_{\alpha,\epsilon}$ is bounded by one on $\Gamma_{R,\theta}$. By letting $\epsilon\to 0$, $\theta\to\frac{\pi}{2}$, $R\to\infty$, and $\delta\to 0$, we see that $G$ is bounded by one on $\mathcal{H}$.
Similar arguments work for the other quadrants.
Hence we may apply Liouville theorem to conclude that $F$ is constant.

□

# Chapter 3

# Local constancy of signal localized in Fourier domain around 0 and consequences

## 3.1 Local constancy of signal localized in Fourier domain around 0 and consequences

We now explain informally why a signal localized in the Fourier domain around 0 is essentially constant at scales $\ll \frac{1}{R}$.
Indeed let $f \in \mathcal{M}(\mathbb{R}^d)$ such that $\hat{f} = 0$ outside $B(0, R)$. Then from the inversion Fourier formula (1.2) we get for $|x - x_0| \ll \frac{1}{R}$

$$\begin{aligned} f(x) &= \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2i\pi \cdot \xi_0} e^{2i\pi\xi \cdot (x - x_0)} d\xi \\ &\approx \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2i\pi\xi \cdot x_0}\, d\xi \\ &= f(x_0) \end{aligned}$$

since $|\xi \cdot (x - x_0)| \ll 1$.
We now draw consequences from this informal statement.

## 3.2 Shannon Sampling Theorem

Since a signal $f$ is essentially constant at scales $\ll \frac{1}{R}$ it is natural to ask oneself whether one can recover a signal localized in the Fourier domain from a sampling of its values at points separated from a distance roughly equal to $\frac{1}{R}$. The Shannon sampling theorem gives a positive answer to this question. It can be formulated as follows:

**Theorem 1.** *Let $R > 0$ and $f \in L^1(\mathbb{R})$ such that $\hat{f}$, the Fourier transform of $f$, is supported on the interval $[-R, R]$.*

*Then*

$$f(x) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2R}\right) sinc\left(2R(x - \frac{n}{2R})\right) \tag{3.1}$$

*in the $L^\infty-$ (resp. $L^2$ sense), i.e the series in the RHS of (3.1) converges to $f$ in $L^\infty(\mathbb{R})$ (resp. $L^2(\mathbb{R})$).*
*The same conclusions hold if we replace the assumption "$f \in L^1(\mathbb{R})$" with "$f \in L^2(\mathbb{R})$ ".*

**Remark 3.** *In other words, the sampling theorem, which is often named after Shannon, says that if an absolutely integrable function is band-limited, i.e it contains no frequencies higher than $R > 0$ hertz, then it is completely determined by its samples at a uniform grid spaced at distances $\frac{1}{2R}$ apart via the above formula.*
*Sampling theory is a tool used for functions to be reconstructed from sampled data, usually from the values of either the functions themselves or some transformations at a discrete set of points.*

**Remark 4.** *Let $f \in L^1(\mathbb{R})$ such that $supp(\hat{f}) \subset [-R, R]$. Then the triangle inequality applied to (1.1) shows that $\hat{f} \in L^\infty(\mathbb{R})$. Hence we see from Hölder inequality that $\hat{f} \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Hence by (1.4) $f$ is almost equal to the continuous function $g$ defined by $g(x) := \int_{\mathbb{R}} \hat{f}(\xi)e^{2i\pi\xi\cdot x}\, d\xi$. So we may abuse notation in (3.1) by writing " $f\left(\frac{n}{2R}\right)$" for " $g\left(\frac{n}{2R}\right)$".*

*Proof.* We see from Remark 4 that the Fourier series of $\hat{f}$ converges to $\hat{f}$ in $L^2([-R, R])$ : see e.g [6]. We can write

$$\hat{f}(\xi) = \sum_{n\in\mathbb{Z}} c_n e^{\frac{in\pi\xi}{R}},$$

where the equality holds in the $L^2([-R, R])-$ sense. In the expression above $c_n$ is the $n-$ the Fourier coefficient of $\hat{f}$, i.e

$$c_n := \frac{1}{2R}\int_{-R}^{R} \hat{f}(\xi)e^{-\frac{in\pi\xi}{R}}\, d\xi \cdot$$

Hence, using also (1.4), we get $c_n = \frac{1}{2R} f\left(-\frac{n}{2R}\right)$. So

$$\hat{f}(\xi) = \mathbb{1}_{[-R,R]} \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2R}\right) e^{\frac{-in\pi\xi}{R}}, \tag{3.2}$$

where the equality holds in the $L^2(\mathbb{R})-$ sense. Hence we get from (1.4) and the Cauchy-Schwartz inequality

$$\begin{aligned} f(x) &= \frac{1}{2R}\sum_{n\in\mathbb{Z}} f\left(\frac{n}{2R}\right) \int_{-R}^{R} e^{2i\pi\xi\left(x - \frac{n}{2R}\right)}\, d\xi \\ &= \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2R}\right) sinc\left(2R(x - \frac{n}{2R})\right), \end{aligned} \tag{3.3}$$

where the equality holds in the $L^\infty-$ sense. Observe that $f \in L^2(\mathbb{R}^d)$ since $f = \hat{h}$ with $h(\xi) = \hat{f}(-\xi)$ and we can use the results of Section 1 to $h \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

The equality (3.3) also holds in the $L^2-$ sense: it follows from the Plancherel theorem , (3.2), and (3.3).
In the case where $f \in L^2(\mathbb{R})$, we have $\hat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by Hölder inequality. From Section 1 we see that (1.4) holds. Hence, by following the same steps above, we infer that (3.1) holds in the $L^\infty-$ sense and in the $L^2-$ sense.

□

## 3.3 Bernstein Inequality

Since $f$ is essentially constant at scales $\ll \frac{1}{R}$, then (formally)

$$f \approx \sum_{\vec{i}:=(i_1,...,i_d)\in \mathcal{Z}^d} f\left(\frac{i_1}{R},...,\frac{i_d}{R}\right) \mathbb{1}_{Q_{\vec{i}}}$$

with $Q_{\vec{i}} := \left[\frac{i_1}{R}, \frac{i_1+1}{R}\right] \times .... \left[\frac{i_d}{R}, \frac{i_d+1}{R}\right]$, and therefore $\|f\|_{L^\infty(\mathbb{R}^d)} \lesssim R^{\frac{d}{p}} \|f\|_{L^p(\mathbb{R}^d)}$.
So it should be possible to control the $L^\infty(\mathbb{R}^d)-$ norm (and more generally the highest $L^q(\mathbb{R}^d)-$ norms) by lower $L^p(\mathbb{R}^d)-$ norms. The Bernstein inequality shows that this is indeed the case:

**Proposition 8.** *Let $1 \leq p \leq q \leq \infty$, $R > 0$, and $s \in \mathbb{R}$. Let $f \in L^p(\mathbb{R}^d)$. The following hold:*

1. *Assume that $\hat{f}$ is compactly supported on $\left\{\xi \in \mathbb{R}^d : \, |\xi| \leq R\right\}$. Then*

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim R^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)} \tag{3.4}$$

2. *Assume that $\hat{f}$ is compactly supported on the annulus $\left\{\xi \in \mathbb{R}^d : \, \frac{R}{2} \leq |\xi| \leq R\right\}$.*

*Then*

$$\|f\|_{L^p(\mathbb{R}^d)} \lesssim R^{-s} \|D^s f\|_{L^p(\mathbb{R}^d)} \tag{3.5}$$

**Remark 5.** *Observe that (3.4) does not hold for all $f \in L^p(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Indeed, arguing by contradiction, let $f_n(x) := 1$ if $|x| \leq \frac{1}{2^n}$ and $f_n(x) := 0$ if $|x| > \frac{1}{2^n}$. Then $\|f_n\|_{L^p(\mathbb{R}^d)} = \frac{1}{2^{\frac{nd}{p}}}$ and $\|f_n\|_{L^\infty(\mathbb{R}^d)} = 1$. This contradicts (3.4) as $n \to \infty$.*

*Proof.* Let $\phi$ be a smooth function such that $\phi(\xi) = 1$ for $|\xi| \leq 1$ and $\phi(\xi) = 0$ for $|\xi| \geq 2$. Let $\tilde{\phi} := |\xi|^{-s} \left(\phi(\xi) - \phi(4\xi)\right)$. If $h \in \{\phi, \tilde{\phi}\}$ then let $\check{h}(x) := \int_{\mathbb{R}^d} h(\xi) e^{2i\pi\xi\cdot x} \, d\xi$.

We first prove (3.4). By using the dilation rule of (1.3) if necessary we may assume WLOG that $R = 1$. Indeed let $D_{\frac{1}{R}} f$ satisfies is compactly supported on $\left\{\xi \in \mathbb{R}^d : \ |\xi| \leq 1\right\}$. Hence

$$\|f\|_{L^q(\mathbb{R}^d)} = R^{-\frac{d}{q}} \|D_{\frac{1}{R}} f\|_{L^q(\mathbb{R}^d)} \lesssim R^{\frac{d}{q}} \|D_{\frac{1}{R}} f\|_{L^p(\mathbb{R}^d)} \lesssim R^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)}$$

Hence $\hat{f} = \phi \hat{f}$, we get $f = f * \check{\phi}$; furthermore, an application of the Young inequality (with $\bar{p}$ such that $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{\bar{p}}$ ) shows that

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \|\check{\phi}\|_{L^{\bar{p}}(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

We then prove (3.5). Again, by using the dilation rule of (1.3) if necessary we may assume WLOG that $R = 1$. Now write $\hat{f} = \tilde{\phi} |\xi|^s \hat{f}$. Hence $f = \check{\tilde{\phi}} * D^s f$ and, taking into account that $\tilde{\phi} \in \mathcal{S}(\mathbb{R}^d)$, an application of the Young inequality yields

$$\|f\|_{L^p(\mathbb{R}^d)} \lesssim \|D^s f\|_{L^p(\mathbb{R}^d)} \|\check{\tilde{\phi}}\|_{L^1(\mathbb{R})} \lesssim \|D^s f\|_{L^p(\mathbb{R}^d)}$$

□

**Remark 6.** *Alternatively one can use Hölder inequality to conclude. Indeed from* $f = f * \phi$, *the definition of the convolution, and Hölder inequality we get* $\|f\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \|\check{\phi}\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$. *Hence by interpolation*

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|^{\frac{p}{q}}_{L^p(\mathbb{R}^d)} \|f\|^{1-\frac{p}{q}}_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

**Remark 7.** *We claim that the estimates (3.4) and (3.5) are sharp.*
*Indeed let $h$ be a smooth function such that $h(\xi) = 1$ for $|\xi| < \frac{3}{4}$ and $h(\xi) = 0$ for $|\xi| \geq 1$. Then $\check{h} \in \mathcal{S}(\mathbb{R}^d)$ (with $\check{h}(x) := \int_{\mathbb{R}^d} h(\xi) e^{2i\pi\xi \cdot x} \, d\xi$) since $\check{h}(x) := \hat{h}(-x)$.*
*Let $q_R(x) := R^d \check{q}(Rx)$. We prove (3.4). Let $q := h$ with $\phi$ defined in Subsection 4.1. Then $\widehat{q_R}(\xi) = \phi\left(\frac{\xi}{R}\right)$ is supported on $|\xi| \leq R$. Moreover an elementary change of variable shows that for $r \in \{p, q\}$ $\|q_R\|_{L^r(\mathbb{R}^d)} \approx R^{d\left(1-\frac{1}{r}\right)}$ ( Observe that $\check{q} \neq 0$ on a ball with center $O$ and small positive radius by Proposition 2 ). Hence $\|q_R\|_{L^q(\mathbb{R}^d)} \approx R^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \|q_R\|_{L^p(\mathbb{R}^d)}$. We now prove (3.5). Let $q$ be such that $q(\xi) := h(\xi) - h\left(\frac{5\xi}{4}\right)$. Then we have $D^s q_R(x) = R^{d+s} D^s \check{q}(Rx)$ and $\|D^s q_R\|_{L^p(\mathbb{R}^d)} \approx R^s \|q_R\|_{L^p(\mathbb{R}^s)}$ (again, observe that $D^s \check{q} \neq 0$ on a ball with center $O$ and small and positive radius by Proposition 2).Hence $\|D^s q_R\|_{L^p(\mathbb{R}^d)} \approx R^s \|q_R\|_{L^p(\mathbb{R}^d)}$.*

# Chapter 4

# Littlewood-Paley decomposition

So far we have studied properties of signals that are band-limited, i.e localized in the Fourier domain. It is unfortunate that most of the signals (solutions of PDES, signals in image processing, etc.) are not usually band-limited. So we perform a Littlewood-Paley decomposition. Roughly speaking, a Littlewood-Paley decomposition is a particular way of decomposing the phase plane which takes a function and writes it as a superposition of a countably infinite family of functions of varying frequencies. The Littlewood-Paley decomposition is of interest in multiple areas of mathematics and forms the basis for the so-called Littlewood-Paley theory.
More precisely we first define for $N \in 2^{\mathbb{Z}}$ a Littlewood-Paley projector $P_N$: this map, when applied to a function $f$, yields a Littlewood-Paley piece of $f$, i.e a function $P_N f$ that is localized in the Fourier domain in an annulus $|\xi| \approx N$. By defining appropriately $P_N$ we can prove that we can decompose $f$ into its Littlewood-Paley pieces $P_N f$, i.e $f = \sum_{N \in 2^{\mathbb{Z}}} P_N f$ $(*)$.
Then we establish decay estimates on average in the physical space for $P_N f$.
Finally we use these estimates in the decomposition $(*)$: we get general ( by "general" we mean estimates that hold for all functions $f$ and not only functions $f$ that are band-limited) estimates for $f$ by summing over $N \in 2^{\mathbb{Z}}$.

## 4.1 Setting

Let $\phi(\xi)$ be a radial bump function supported on $\left\{\xi \in \mathbb{R}^d : |\xi| \leq 2\right\}$ which equals 1 on $\left\{\xi \in \mathbb{R}^d : |\xi| \leq 1\right\}$. Let $\psi(\xi)$ be the function

$$\psi(\xi) := \phi(\xi) - \phi(2\xi).$$

Thus $\psi$ is a bump function supported on the annulus $\left\{\xi \in \mathbb{R}^d : \frac{1}{2} \leq |\xi| \leq 2\right\}$. By construction we have

$$\sum_{N \in 2^{\mathbb{Z}}} \psi\left(\frac{\xi}{N}\right) = 1 \tag{4.1}$$

for all $\xi \in \mathbb{R}^d$. Thus we can partition the unity into functions $\psi\left(\frac{\xi}{N}\right)$, with $N \in 2^{\mathbb{Z}}$ (the set of dyadic numbers), each of those is supported on the annulus of the form $|\xi| \sim N$. We now introduce the Littlewood-Paley decomposition operators $P_N$, $P_{\leq N}$ defined by

$$\widehat{P_N f}(\xi) = \psi\left(\frac{\xi}{N}\right)\hat{f}(\xi)$$

$$\widehat{P_{\leq N} f}(\xi) = \phi\left(\frac{\xi}{N}\right)\hat{f}(\xi)$$

Hence, from the equality $\widehat{p * q} = \hat{p}\hat{q}$ we get

$$P_N f(x) = N^d \int_{\mathbb{R}^d} f(y)\check{\psi}\left(N(x-y)\right)\, dy \text{ and}$$

$$\widehat{P_{\leq N} f}(\xi) = N^d \int_{\mathbb{R}^d} f(y)\check{\phi}\left(N(x-y)\right)\, dy.$$

We see from (4.1) that we have the Littlewood-Paley decomposition

$$f = \sum_{N \in 2^{\mathbb{Z}}} P_N f.$$

This decomposition takes a single function and writes it as a superposition of a countably infinite family of functions $P_N f$, where each one has a frequency of magnitude roughly $N$. Lower values of $N$ represent lower frequency components of f, while higher values represent high frequency components.

What does $P_N f$ look like? Since $P_N f = P_{\leq 4N} P_N f$, we see from that we have the self-reproducing formula

$$P_N f(x) = (4N)^d \int_{\mathbb{R}^d} P_N f(y)\check{\phi}\left(4N(x-y)\right)\, dy \tag{4.2}$$

Hence $P_N f(x) \approx N^d \int_{N|x-y|\lesssim 1} P_N f(y)\, dy$ since $\check{\phi}$ is essentially concentrated in a neighborhood of size $o(1)$ around the origin, which means that $P_N f$ is essentially constant at scales $\ll \frac{1}{N}$. On the other hand $P_{\leq \frac{N}{4}} P_N f = 0$. Hence

$$\int_{\mathbb{R}^d} P_N f(y)\check{\phi}\left(\frac{x-y}{\frac{N}{4}}\right)\, dy = 0$$

Hence $\int_{N|x-y|\lesssim 1} P_N f(y)\, dy \approx 0$, which implies that $P_N f$ has essentially a mean equal to zero at scales $\lesssim \frac{1}{N}$; in other words $P_N f$ has $O(1)$ oscillations at these scales.
To reconcile these two properties, we see that on each ball of radius $O(N^{-1})$, the function $P_N f$ is roughly constant at scales $O(N^{-1})$ and it contains about $O(1)$ oscillations.

What does $P_{\leq N} f$ look like? Since $P_{\leq N} f = P \leq_{4N} P_{\leq N} f$, we see by using similar arguments as before that $P_{\leq N}$ is essentially constant at scales $\ll N^{-1}$.

What does $P_{>N} f$ look like? Since $P \leq_{\frac{N}{4}} P_{>N} f = 0$, we see that $P_{>N}$ has $O(1)$ oscillations at scales $O(N^{-1})$.

## 4.2 Estimates for Littlewood-Paley projector

In this subsection we prove some quantitative estimates for functions that are localized in the Fourier domain within the annulus $\{\xi \in \mathbb{R}^d : |\xi| \approx N\}$ with $N \in 2^{\mathbb{Z}}$.

**Proposition 9.** *Let $f \in \mathcal{S}(\mathbb{R}^d)$. Let $N \in 2^{\mathbb{Z}}$. Then the following hold:*

1.
$$\|P_N f\|_{L^p(\mathbb{R}^d)} \lesssim N^{-1} \|\nabla f\|_{L^p(\mathbb{R}^d)}. \tag{4.3}$$

2. *Let $s \in \mathbb{R}$. Then*
$$\|P_N f\|_{L^p(\mathbb{R}^d)} \lesssim N^{-s} \|D^s f\|_{L^p(\mathbb{R}^d)}. \tag{4.4}$$

*Proof.* First we prove (4.3).

From the formula $\widehat{\partial_{x_j} f}(\xi) = 2i\pi \xi_j \hat{f}(\xi)$ we get $P_N f = \frac{1}{N} \sum_{j=1}^{d} K_{j,N} * \partial_{x_j} f$ with $K_{j,N}(x) := \frac{1}{2i\pi} \int_{\mathbb{R}^d} \frac{\xi_j}{N} \left( \frac{N}{|\xi|} \right)^2 \psi\left(\frac{\xi}{N}\right) e^{2i\pi \xi \cdot x} \, d\xi$. We claim that $(*)$ holds with $(*)$ : $\|K_{j,N}\|_{L^1(\mathbb{R}^d)} \lesssim 1$. Assuming that the claim holds applying the Young inequality we see that (4.3) also holds. It remains to prove $(*)$. The triangle inequality and an elementary change of variable show that $|K_{j,N}(x)| \lesssim N^d$. Moreover by integration by parts using the formula $e^{2i\pi\xi\cdot x} = \frac{1}{2i\pi} \frac{\nabla\left(e^{2i\pi\xi\cdot x}\right)\cdot x}{|x|^2}$ we get $|K_{j,N}(x)| \lesssim \frac{N^d}{|Nx|^{100d}}$. Hence

$$|K_{j,N}(x)| \lesssim N^d \min\left(1, \frac{1}{|Nx|^{100d}}\right). \tag{4.5}$$

Observe that the R.H.S of (4.5) is integrable. Hence $(*)$ holds.

Then we prove (4.4). Let $\bar{\psi}(\xi) := |\xi|^{-s} \psi(\xi)$. From $\widehat{P_N f}(\xi) = N^{-s} \bar{\psi}\left(\frac{\xi}{N}\right) |\xi|^s \hat{f}(\xi)$ we get

$$P_N f = N^{-s} N^d \int_{\mathbb{R}^d} \check{\bar{\psi}}\left(N(x-y)\right) D^s f(y) \, dy$$

Hence $\|P_N f\|_{L^p(\mathbb{R}^d)} \lesssim N^{-s} \|D^s f\|_{L^p(\mathbb{R}^d)}$.

□

Next we state and prove the proposition below stronger for functions such that their Fourier transform is localized on $|\xi| \approx N$.

**Proposition 10.** *Let $s \in \mathbb{R}$. Let $f \in \mathcal{S}(\mathbb{R}^d)$ such that supp$(\hat{f}) \subset \left\{\xi \in \mathbb{R}^d : |\xi| \approx N\right\}$. Let $p \geq 1$. Then*

$$\|\nabla f\|_{L^p(\mathbb{R}^d)} \approx N\|f\|_{L^p(\mathbb{R}^d)}, \textit{ and } \|D^s f\|_{L^p(\mathbb{R}^d)} \approx N^s\|f\|_{L^p(\mathbb{R}^d)}$$

*Proof.* Since *supp*$(\hat{f}) \subset \left\{\xi \in \mathbb{R}^d : |\xi| \approx N\right\}$, we can write $f = \tilde{P}_N f$ with $\tilde{P}_N$ a map defined by $\widehat{\tilde{P}_N}(f) := \psi\left(\frac{\xi}{CN}\right)\hat{f}(\xi)$ and $\psi$ defined in Subsection 4.1 and $C \gg 1$ a large positive constant.
We see from Proposition 9 that

$$\|f\|_{L^p(\mathbb{R}^d)} \lesssim N^{-1}\|\nabla f\|_{L^p(\mathbb{R}^d)}, \text{ and } \|f\|_{L^p(\mathbb{R}^d)} \lesssim N^{-s}\|D^s f\|_{L^p(\mathbb{R}^d)}.$$

Let $\bar{\psi}(\xi) := |\xi|^s\psi\left(\frac{\xi}{C}\right)$. We have

$$D^s f = N^s \int_{\mathbb{R}^d} \check{\bar{\psi}}\left(N(x-y)\right) f(y)\, dy$$

Hence from Young inequality we get $\|D^s f\|_{L^p(\mathbb{R}^d)} \lesssim N^s\|f\|_{L^p(\mathbb{R}^d)}$.
From the elementary estimate $\|\nabla f\|_{L^p(\mathbb{R}^d)} \lesssim \sum_{j=1}^{d} \left\|\partial_{x_j} f\right\|_{L^p(\mathbb{R}^d)}$, we see that it suffices to estimate $\left\|\partial_{x_j} f\right\|_{L^p(\mathbb{R}^d)}$. From $f = \tilde{P}_N f$ we get, after differentiation with respect to $x_j$

$$\partial_{x_j} f(x) = (CN)^{d+1} \int_{\mathbb{R}^d} \partial_{x_j}\check{\psi}(CN(x-y))f(y)\, dy.$$

Hence $\|\partial_{x_j} f\|_{L^p(\mathbb{R}^d)} \lesssim N\|f\|_{L^p(\mathbb{R}^d)}$.

□

# Chapter 5

# Applications

In this section we use the Littlewood-Paley decomposition to prove estimates of the form

$$\begin{aligned} \|f\|_{L^r(\mathbb{R}^d)} &\leq A\left(\|f\|_{L^p(\mathbb{R}^d)} + \|\nabla f\|_{L^p(\mathbb{R}^d)}\right), \text{ and more generally} \\ \|f\|_{L^r(\mathbb{R}^d)} &\leq A\left(\|f\|_{L^p(\mathbb{R}^d)} + \|D^s f\|_{L^p(\mathbb{R}^d)}\right), \end{aligned} \tag{5.1}$$

$f \in \mathcal{S}(\mathbb{R}^d)$, $(p, r)$ and $A > 0$ positive constants that do not depend on $f$ and to be determined. The main interest of these estimates is their robustness: they can be applied to a large class of functions. Hence they are useful in

- *signal processing.*
  For example, assume that for some $B > 0$ the signals $f$ satisfy the regularity property $\|f\|_{L^p(\mathbb{R}^d)} + \|D^s f\|_{L^p(\mathbb{R}^d)} \leq B$; then we see from (5.1) that they also have the decay property $\|f\|_{L^r(\mathbb{R}^d)} \lesssim B$
- *PDEs or functional analysis.*
  The existence of a solution of a PDE (or a function that satisfies some properties) is often related to the existence of a limit of a sequence of functions. For example, let $\{u_n\}_{n\geq 1}$ be a Cauchy sequence of Schwartz functions with respect to the norm $N(f) := \|f\|_{L^p(\mathbb{R}^d)} + \|\nabla f\|_{L^p(\mathbb{R}^d)}$. Then we see from (5.1) that $\{u_n\}_{n\geq 1}$ is also a Cauchy sequence in $L^r(\mathbb{R}^d)$. Since $L^r(\mathbb{R}^d)$ is a Banach space, there exists $u \in L^r(\mathbb{R}^d)$ such that $u_n \to u$ in $L^r(\mathbb{R}^d)$.

## 5.1 An application: the non-endpoint of the Sobolev embedding

In this subsection we show the non-endpoint of the Sobolev embedding.

**Proposition 11.** *Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then the following hold:*

1. *Let $1 \leq p < d$ and $\bar{p}$ such that $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{1}{d}$. Let $q < \bar{p}$. Then*

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} + \|\nabla f\|_{L^p(\mathbb{R}^d)}. \tag{5.2}$$

2. *More generally let $s > 0$, $1 \leq p < \frac{d}{s}$, and $\bar{p}$ such that $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{s}{d}$. Let $q < \bar{p}$. Then*

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} + \|D^s f\|_{L^p(\mathbb{R}^d)}. \tag{5.3}$$

**Remark 8.** *Proposition 11 says that the estimate (5.2) holds for $q$ lying in the interval $[p, \bar{p})$ that does not contain its endpoint $\bar{p}$. This is why this estimate is called the non-endpoint of the Sobolev embedding. Later we will see that (5.2) also holds for $q = \bar{p}$: this is the endpoint of (5.2).*
*Recall from Proposition 15 (see Appendix C) that if $1 < p < \infty$ then $\|Df\|_{L^p(\mathbb{R}^d)} \approx \|\nabla f\|_{L^p(\mathbb{R}^d)}$. Hence, if $1 < p < \frac{s}{d}$ then (5.3) is a generalization of (5.2).*

*Proof.* We only prove (5.2), since the proof of (5.3) is similar and therefore left to the reader.
The conclusion clearly holds if $q = p$. So we may assume WLOG that $q > p$.
We use a Paley-Littewood decomposition. We have

$$f = \sum_{N \in 2^{\mathbb{Z}}} P_N f = \sum_{N \leq 1} P_N f + \sum_{N > 1} P_M f = I + J$$

We have

$$\|I\|_{L^q(\mathbb{R}^d)} \lesssim \sum_{N \leq 1} \|P_N f\|_{L^q(\mathbb{R}^d)} \lesssim \sum_{N \leq 1} N^{d\left(\frac{1}{p} - \frac{1}{q}\right)} \|P_N f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

Hence $\|I\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$ and

$$\|J\|_{L^q(\mathbb{R}^d)} \lesssim \sum_{N > 1} \|P_N f\|_{L^q(\mathbb{R}^d)} \lesssim \sum_{N > 1} N^{-1 + d\left(\frac{1}{p} - \frac{1}{q}\right)} \|\nabla f\|_{L^p(\mathbb{R}^d)} \lesssim \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

Hence (5.2) holds.

□

## 5.2 An application: the endpoint of the Sobolev embedding

In this subsection we prove the endpoint of the Sobolev embedding. More precisely

**Proposition 12.** *Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then the following hold*

1. *Let $1 \leq p < d$ and $\bar{p}$ be such that $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{1}{d}$.*

$$\|f\|_{L^{\bar{p}}(\mathbb{R}^d)} \lesssim \|\nabla f\|_{L^p(\mathbb{R}^d)} \tag{5.4}$$

2. *More generally let* $s > 0$, $1 < p < \frac{d}{s}$, *and* $\bar{p}$ *such that* $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{s}{d}$. *Then*

$$\|f\|_{L^{\bar{p}}(\mathbb{R}^d)} \lesssim \|D^s f\|_{L^p(\mathbb{R}^d)}. \tag{5.5}$$

**Remark 9.** *We see from the arguments used in Remark 8 that if* $1 < p < \frac{d}{s}$ *then (5.5) is a generalization of (5.4).*

We now write three different proofs of these estimates.

### 5.2.1 *The proof of the endpoint Sobolev embedding for* $p > 1$

We see from Remark 9 that it suffices to prove (5.5).
We follow an exposition in [8].
By using (1.3) if necessary (and choosing appropriately the value of $\lambda$ ) we may assume WLOG that $\|D^s f\|_{L^p(\mathbb{R}^d)} = 1$.
We first prove the weak-type estimate

$$\|f\|_{L^{\bar{p},\infty}(\mathbb{R}^d)} \lesssim \|D^s f\|_{L^p(\mathbb{R}^d)} \tag{5.6}$$

( Here $\|f\|^{\bar{p}}_{L^{\bar{p},\infty}(\mathbb{R}^d)} := \sup_{\lambda > 0} \lambda^{\bar{p}} \,|\, |f| > \lambda \,|$ ). Let $\lambda > 0$ be a fixed positive number. Let $N \in 2^{\mathbb{Z}}$. From

$$\|P_N f\|_{L^\infty(\mathbb{R}^d)} \lesssim N^{\frac{d}{p} - s} \|D^s f\|_{L^p(\mathbb{R}^d)} \lesssim N^{\frac{d}{p} - s}. \tag{5.7}$$

we see that if $N_\lambda$ is such that $N_\lambda^{\frac{d}{p} - s} \approx \lambda$ then by summing (5.7) over $N \ll N_\lambda$ we get

$$\|P_{\ll N_\lambda} f\|_{L^\infty} \ll \lambda$$

On the other hand we see from Proposition 9 that

$$\|P_{\gtrsim N_\lambda} f\|_{L^p(\mathbb{R}^d)} \lesssim N_\lambda^{-s} \|D^s f\|_{L^p(\mathbb{R}^d)} \lesssim N_\lambda^{-s}$$

Therefore using also Chebyshev inequality we get

$$\lambda^{\bar{p}} \,| |f| > \lambda | \lesssim \lambda^{\bar{p}} \,| |P_{\gtrsim N_\lambda} f| > \lambda | \lesssim \lambda^{\bar{p} - p} \|P_{\gtrsim N_\lambda}\|^p_{L^p(\mathbb{R}^d)} \lesssim \lambda^{\bar{p} - p} N_\lambda^{-ps} \lesssim 1,$$

where at the last line we use the definition of $\bar{p}$. Hence (5.6) holds.

It remains to upgrade the weak-type estimates to strong-type estimates. For his purpose we recall the Marcinkiewicz interpolation theorem (see e.g [2])

**Theorem 2.** *Let* $(p_0, p_1, q_0, q_1) \in [1, \infty]^4$. *Let* $\theta \in (0, 1)$. *Let* $(p, q)$ *such that*

$$\tfrac{1}{p} = \tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1}, \text{ and } \tfrac{1}{q} = \tfrac{1-\theta}{q_0} + \tfrac{\theta}{q_1} \cdot$$

*Assume that* $T$ *is a linear map on* $L^{p_0}$ *weak-type* $(p_0, q_0)$ *and* $(p_1, q_1)$ *i.e*

$$\left( \sup_{\lambda > 0} \lambda^{q_0} \left| |Tf| > \lambda \right| \right)^{\frac{1}{q_0}} \lesssim \|f\|_{L^{p_0}(\mathbb{R}^d)}, \text{ and } \left( \sup_{\lambda > 0} \lambda^{q_1} \left| |Tf| > \lambda \right| \right)^{\frac{1}{q_1}} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^d)} \cdot \cdot$$

*Then* $T$ *is strong-type* $(p, q)$, *i.e*

$$\|Tf\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

By applying the Marcinkiewicz interpolation theorem to $T = D^{-s}$ we see from the previous estimates that $T$ is weak type $(p, \bar{p})$. Hence $T$ is strong type $(p, \bar{p})$ and we can conclude that (5.5) holds.

**Remark 10.** *Assume that we want to prove (5.4) without using Remark 9 and by using similar arguments as those above.*
*Similar arguments show that* $(*)$ : $\|\triangle^{-1}\nabla \cdot \vec{E}\|_{L^{\bar{p}}(\mathbb{R}^d)} \lesssim \|\vec{E}\|_{L^p(\mathbb{R}^d)}$ *holds ( Here* $\vec{E} := (E_1, ... E_d)$, $\nabla \cdot \vec{E} := \sum_{j=1}^{d} \frac{\partial E_j}{\partial x_j}$ *and* $\widehat{\triangle^{-1} f}(\xi) := |\xi|^{-2} \hat{f}(\xi)$ *); then, applying* $(*)$ *to* $\vec{E} := \nabla f$ *yields (5.4).*

### 5.2.2 *Another proof of the endpoint Sobolev embedding for* $p > 1$*: fractional integration*

In this subsection we write down another proof based upon a physical representation of the fractional integration of a function $f$.
Observe that (5.5) holds if and only if $\|D^{-s} f\|_{L^{\bar{p}}(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$. From Appendix $A$ (see (10)) we can write $D^{-s} f(x) = \int_{\mathbb{R}^d} f(x-y)|y|^{s-d}\, dy = K_1(x) + K_2(x)$ with

$$K_1(x) := \int_{|x-y|<R} f(y)|x-y|^{s-d}\, dy, \text{ and } K_2(x) := \int_{|x-y| \geq R} f(y)|x-y|^{s-d}\, dy$$

We have

$$|K_1(x)| \lesssim \sum_{M \in 2^{\mathbb{Z}}: M \lesssim R} M^s \tfrac{1}{|B(x,R)|} \int_{|x-y| \approx M} |f(y)|\, dy \lesssim R^s M f(x)$$

Moreover the Hölder inequality shows that

$$|K_2(x)| \lesssim \left( \int_{|y| \leq R} |y|^{(s-d)p'}\, dy \right)^{\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}^d)} \lesssim R^{-\frac{d}{q}} \|f\|_{L^p(\mathbb{R}^d)}$$

Therefore

$$I_s(f)(x) \lesssim R^s M(f)(x) + R^{-\frac{d}{q}} \|f\|_{L^p(\mathbb{R}^d)} \tag{5.8}$$

(Here $M(f)(x) := \sup_{R>0} \int_{B(x,R)} |f(y)| \, dy$ is the well-known maximal function: see e.g [2]). By choosing $R$ such that $R^s M(f)(x) \approx R^{-\frac{d}{q}} \|f\|_{L^p(\mathbb{R}^d)}$ in order to minimize the R.H.S of (5.8), we get

$$I_s(f) \lesssim (M(f)(x))^{\frac{p}{q}} \|f\|_{L^p(\mathbb{R}^d)}^{1-\frac{1}{q}}$$

Hence, by taking the $L^{\bar{p}}$ norm of $I_s(f)$ and using the well-known estimate $\|M(f)\|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^r(\mathbb{R}^d)}$ for $1 < r \leq \infty$ (see e.g [2]) we get

$$\|I_s(f)\|_{L^{\bar{p}}(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

#### 5.2.3 *A proof of (5.4) for* $p = 1$

In this subsection we give a proof of (5.4 ) for $p = 1$ by working in the physical space. The techniques used also yield another proof of (5.4) for $p > 1$.
We first state and prove a technical lemma that we will use later in the proof of (5.4 ) for $p = 1$.

**Lemma 1.** *Let* $f_1, ..., f_d \in L^{d-1}(\mathbb{R}^{d-1})$*. Given* $x \in \mathbb{R}^d$ *and* $1 \leq i \leq d$*, let* $\bar{x}_i := (x_1, ..., x_{i-1}, x_{i+1}, ..., x_d)$*. Let* $f(x) := f_1(\bar{x}_1)...f_d(\bar{x}_d)$*. Then* $f \in L^1(\mathbb{R}^d)$ *and*

$$\|f\|_{L^1(\mathbb{R}^d)} \lesssim \prod_{i=1}^{d} \|f_i\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

*Proof.* The case $d = 2$ follows from Fubini-Tonelli theorem.
Let $d = 3$. The Cauchy-Schwartz inequality yields

$$\int_{\mathbb{R}} |f(x)| \, dx_3 \lesssim |f_3(x_1, x_2)| \left( \int_{\mathbb{R}} |f_1(x_2, x_3)|^2 \, dx_3 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} |f_2(x_1, x_3)|^2 \, dx_3 \right)^{\frac{1}{2}}$$

Integrating the above estimate with respect to $x_2$ and $x_1$ and applying the Cauchy-Schwarz inequality again we see that

$$\int_{\mathbb{R}^3} |f(x)| \, dx \lesssim \left( \int_{\mathbb{R}^3} |f_3(x_1, x_2)|^2 \, dx_2 dx_1 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} |f_1(x_2, x_3)|^2 \, dx_3 dx_2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} |f_2(x_1, x_3)|^2 \, dx_3 dx_1 \right)^{\frac{1}{2}}$$

The general case $d$ is proved by induction. Assuming that Lemma 1 holds for $p$ let us prove it for $p + 1$. The Hölder inequality yields

$$\int_{\mathbb{R}^d} |f(x)| \, dx \lesssim \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} |f_1 ..., f_d|^{d'} dx_1 ... dx_d \right)^{\frac{1}{d'}},$$

with $d'$ such that $\frac{1}{d} + \frac{1}{d'} = 1$. Applying the induction assumption to the functions $|f_1|^{d'}$,..., $|f_d|^{d'}$, we obtain

$$\int_{\mathbb{R}^d} |f_1|^{d'}...|f_d|^{d'} \, dx_1...dx_d \lesssim \prod_{j=1}^{d} \|f_j\|_{L^d(\mathbb{R}^d)}^{d'}$$

Hence

$$\int_{\mathbb{R}^d} |f(x)| dx_1...dx_d \lesssim \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{j=1}^{d} \|f_j\|_{L^d(\mathbb{R}^{d-1})}$$

We now integrate the above estimate with respect to $x_{d+1}$. Applying again the Hölder inequality we get

$$\int_{\mathbb{R}^{d+1}} |f(x)| dx_1...dx_{d+1} \lesssim \prod_{j=1}^{d+1} \|f_j\|_{L^d(\mathbb{R}^d)}$$

□

Let us prove (5.4) for $p = 1$. The fundamental theorem of calculus shows that

$$|u(x_1, ..., x_d)| \quad = \left| \int_{-\infty}^{x_1} \frac{\partial u}{\partial x_1}(t, x_2, ..., x_d) \, dt \right|$$

In fact we have for $1 \leq i \leq d$

$$|u(x_1, ..., x_d)| \lesssim \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_i}(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_d) \right| \, dt$$

Let $\bar{x}_i := (x_1, ..., x_{i-1}, x_{i+1}, ..., x_d)$. Let $f_i(\bar{x}_i) := \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_i}(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_d) \right| \, dt$. The above estimates show that

$$|u(x)|^d \lesssim \prod_{i=1}^{d} f_i(\bar{x}_i)$$

Hence, recalling that $\bar{p} = \frac{d}{d-1}$, we get from Lemma 1

$$\int_{\mathbb{R}^d} |u(x)|^{\bar{p}} \, dx \lesssim \prod_{j=1}^{d} \|f_i\|_{L^1(\mathbb{R}^{d-1})}^{\frac{1}{d-1}} \lesssim \prod_{j=1}^{d} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d-1}}$$

Hence

$$\|u\|_{L^{\bar{p}}(\mathbb{R}^d)} \lesssim \prod_{j=1}^{d} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d}} \lesssim \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

We now turn to the case $1 < p < d$. We show that a modification of the argument for $p = 1$ yields (5.4). Let $m \geq 1$ to be chosen shortly. An application of (5.4) for $p = 1$ to the function $|u|^{m-1}u$ combined with Hölder inequality shows that

$$\begin{aligned}\|u\|^m_{L^{\frac{md}{d-1}}(\mathbb{R}^d)} &\lesssim \prod_{j=1}^{d} \left\| |u|^{m-1} u \frac{\partial u}{\partial x_j} \right\|_{L^1(\mathbb{R}^d)} \\ &\lesssim \|u\|^{m-1}_{L^{p'(m-1)}(\mathbb{R}^d)} \prod_{j=1}^{d} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^p(\mathbb{R}^d)}\end{aligned}$$

(Here $\frac{1}{p'} := 1 - \frac{1}{p}$ ). Choosing $m$ such that $p^{'}(m-1) = \frac{pd}{d-1}$ we get

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \lesssim \prod_{j=1}^{d} \left\| \frac{\partial u}{\partial x_j} \right\|^{\frac{1}{d}}_{L^p(\mathbb{R}^d)} \lesssim \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

## 5.3 An application: the Gagliardo-Nirenberg inequality

In this subsection we give another application of the Littlewood-Paley decomposition: the Gagliardo-Nirenberg inequality.

**Proposition 13.** *Let $f \in \mathcal{S}(\mathbb{R}^d)$. Let $\infty > p \geq 1$, $s > 0$ and $\theta \in (0,1)$. Let $\frac{1}{q} = \frac{1}{p} - \frac{\theta s}{d}$. Then*

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|^{\theta}_{L^p(\mathbb{R}^d)} \|D^s f\|^{1-\theta}_{L^p(\mathbb{R}^d)} \tag{5.9}$$

**Remark 11.** *The Gagliardo-Nirenberg inequality is usually better than the non-endpoint of the Sobolev embedding in situations where one controls (additionally) the $L^p-$ norm of the functions $f$. Indeed if the functions have an $L^p-$ norm that is fixed (i.e there exists $B > 0$ such that $\|f\|_{L^p(\mathbb{R}^d)} = B$)* [1] *then (5.9) implies that $\|f\|_{L^q(\mathbb{R}^d)} \lesssim B^{\theta} \|D^s f\|^{1-\theta}_{L^p(\mathbb{R}^d)}$: this estimate is sharper than (5.3) if $\|D^s f\|_{L^p(\mathbb{R}^d)} \gg 1$.*

*Proof.* We write $f = A + B$ with $A := \sum_{N \leq M} P_N f$ and $B := \sum_{N > M} P_N f$. We have

$$\begin{aligned}\|A\|_{L^q(\mathbb{R}^d)} &\lesssim \sum_{N \leq M} \|P_N f\|_{L^q(\mathbb{R}^d)} \\ &\lesssim \sum_{N \leq M} N^{d\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)} \\ &\lesssim M^{s\theta} \|f\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

We have

$$\begin{aligned}\|B\|_{L^q(\mathbb{R}^d)} &\lesssim \sum_{N > M} \|P_N f\|_{L^q(\mathbb{R}^d)} \\ &\lesssim \sum_{N > M} M^{d\left(\frac{1}{p} - \frac{1}{q}\right) - s} \|D^s f\|_{L^p(\mathbb{R}^d)} \\ &\lesssim M^{d\left(\frac{1}{p} - \frac{1}{q}\right) - s} \|D^s f\|_{L^p(\mathbb{R}^d)} \\ &\lesssim M^{s(\theta - 1)} \|D^s f\|_{L^p(\mathbb{R}^d)}\end{aligned}$$

[1] This situation happens quite often with $p = 2$ for solutions $u$ of PDEs such as the Schrödinger equations: indeed their mass $\|u(t)\|_{L^2(\mathbb{R}^d)}$ is conserved for all time.

Hence, by choosing $M$ such that $\|f\|_{L^p(\mathbb{R}^d)} \approx M^{-s}\|D^s f\|_{L^p(\mathbb{R}^d)}$ we get (5.9).

$\square$

# Appendix A

In this subsection we compute the Fourier inverse transform of a particular function.

**Proposition 14.** *Let* $f \in \mathcal{S}(\mathbb{R}^d)$*. Let* $0 < \alpha < d$*. Then*

$$\mathcal{F}^{-1}\left(|\xi|^{-\alpha}\hat{f}\right) = \int_{\mathbb{R}^d} f(x-y)|y|^{\alpha-d}\, dy \tag{10}$$

*with* $\mathcal{F}^{-1}(h)(x) := \int_{\mathbb{R}^d} h(\xi)e^{2i\pi\xi\cdot x}\, d\xi$*.*

*Proof.* An elementary change of variable shows that there exists $c_\alpha > 0$ such that

$$c_\alpha|\xi|^{-\alpha} = \int_0^\infty \; e^{-\pi|\xi|^2\lambda}\lambda^{\frac{\alpha}{2}-1}\, d\lambda \tag{11}$$

Hence, using also Fubini theorem, we get

$$\begin{aligned}
\mathcal{F}^{-1}\left(|\xi|^{-\alpha}\hat{f}(\xi)\right)(x) &= \int_{\mathbb{R}^d} |\xi|^{-\alpha}\hat{f}(\xi)e^{2i\pi\xi\cdot x} d\,\xi \\
&= \int_0^\infty \lambda^{\frac{\alpha}{2}-1}\int_{\mathbb{R}^d} e^{-\pi\lambda|\xi|^2\lambda}\hat{f}(\xi)e^{2i\pi\xi\cdot x}\, d\,\xi d\lambda \\
&= \int_0^\infty \lambda^{\frac{\alpha}{2}-1-\frac{d}{2}}\int_{\mathbb{R}^d} f(y)e^{-\pi\frac{|x-y|^2}{\lambda}}\, dy \\
&= \int_{\mathbb{R}^d} f(y)\int_0^\infty \lambda^{\frac{\alpha}{2}-1-\frac{d}{2}}e^{-\pi\frac{|x-y|^2}{\lambda}}\, d\lambda\, dy \\
&= c_{d-\alpha}I_\alpha(f)(x)
\end{aligned}$$

Hence (10) holds.

□

# Appendix B

In this subsection we prove that the quantities $\|Df\|_{L^p(\mathbb{R}^d)}$ and $\|\nabla f\|_{L^p(\mathbb{R}^d)}$ are comparable if $1 < p < \infty$.

**Proposition 15.** *Let $f \in \mathcal{S}(\mathbb{R}^d)$. Let $1 < p < \infty$. Then*

$$\|Df\|_{L^p(\mathbb{R}^d)} \approx \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

*Proof.* Let $1 \leq j \leq d$ and $R_j$ be the $j^{th}$ Riesz transform defined in the Fourier domain by $\widehat{R_j f}(\xi) := \frac{\xi_j}{|\xi|}\hat{f}(\xi)$. Recall that $R_j$ is bounded on $L^p(\mathbb{R}^d)$ (see Appendix $C$) i.e

$$\|R_j h\|_{L^p(\mathbb{R}^d)} \lesssim \|h\|_{L^p(\mathbb{R}^d)}. \tag{12}$$

Hence

$$\|\nabla f\|_{L^p(\mathbb{R}^d)} \lesssim \sum_{j=1}^{d} \|\partial_{x_j} f\|_{L^p(\mathbb{R}^d)} \lesssim \sum_{j=1}^{d} \|R_j Df\|_{L^p(\mathbb{R}^d)} \lesssim \|Df\|_{L^p(\mathbb{R}^d)}.$$

We then show that $\|Df\|_{L^p(\mathbb{R}^d)} \lesssim \|\nabla f\|_{L^p(\mathbb{R}^d)}$. Observe that this estimate is equivalent to show $(*)$ : $\|f\|_{L^p(\mathbb{R}^d)} \lesssim \|\nabla D^{-1} f\|_{L^p(\mathbb{R}^d)}$. Observe that $f = -i \sum_{j=1}^{d} R_j \partial_{x_j}(D^{-1} f)$. Hence, using again (12), we get $(*)$.

□

# Appendix C

In this subsection we recall the definition of the Riesz transforms $R_j$. Next we prove their boundedness.

**Definition 1.** *Let $1 \leq j \leq d$. The $j^{th}-$ Riesz transform $R_j$ is the operator acting on functions $\mathcal{S}(\mathbb{R}^d)$ defined by*

$$R_j f(x) := \int_{\mathbb{R}^d} \frac{\xi_j}{|\xi|} \hat{f}(\xi) e^{2i\pi\xi\cdot x} \, d\xi \tag{13}$$

**Remark 12.** *Observe that $R_j f$ is well-defined since $\frac{\xi_j}{|\xi|}\hat{f}(\xi) \in L^1(\mathbb{R}^d)$.*

Next we prove that $R_j f$ can be written as the limit of convolution of $f$ with truncated kernels.

**Proposition 16.** *Let $1 \leq j \leq d$. Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then there exists $\tilde{C} \in \mathbb{R}$ such that $R_j f(x) = \tilde{C} \lim\limits_{\epsilon \to 0} R_j^{\epsilon} f(x)$ with*

$$R_j^{\epsilon} f(x) := \tilde{C} \int_{\mathbb{R}^d} k_j^{\epsilon}(x-y) f(y) \, dy, \text{ and } k_j^{\epsilon}(z) := \frac{z_j}{|z|^{d+1}} \mathbf{1}_{|z| \geq \epsilon}.$$

**Remark 13.** *Let $h(y) := \frac{f(y)-f(x)}{y-x}$ if $y \neq x$ and $h(y) := 1$ if $y = x$. Then $h \in L^1(\mathbb{R}^d)$ and $R_j^{\epsilon} f(x) = \int_{|x-y|\geq\epsilon} h(y) \, dy$. Hence $\lim\limits_{\epsilon\to 0} R_{j,\epsilon} f(x)$ exists.*

*Proof.* Let $d > \alpha > 0$. Let $\delta > 0$. Let $\bar{C}$ be a positive constant of which the value may change and such that all the statements blow are true. Observe from the identity (11) and the Fubini-Tonelli theorem that

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{1}{|x|^{d-\alpha}} e^{-\pi\delta|x|^2} e^{2i\pi\xi\cdot x} \, dx &= \bar{C} \int_{\mathbb{R}^d} e^{2i\pi\xi\cdot x} \int_0^\infty t^{\frac{d-\alpha}{2}-1} e^{-\pi(t+\delta)|x|^2} \, dt \, dx \\
&= \bar{C} \int_0^\infty t^{\frac{d-\alpha}{2}-1} \int_{\mathbb{R}^d} e^{2i\pi\xi\cdot x} e^{-\pi(t+\delta)|x|^2} \, dx \, dt \\
&= \bar{C} \int_0^\infty t^{\frac{d-\alpha}{2}-1} (t+\delta)^{-\frac{d}{2}} e^{-\frac{\pi|\xi|^2}{t+\delta}} \, dt
\end{aligned}$$

Let $1 < \alpha < d$. By integrating by parts

$$\int_{\mathbb{R}^d} \partial_{x_j} \left( \frac{1}{|x|^{d-\alpha}} e^{-\pi\delta|x|^2} \right) e^{2i\pi\xi\cdot x} \, dx = -2\bar{C} i\pi\xi_j \int_0^\infty t^{\frac{d-\alpha}{2}-1} (t+\delta)^{-\frac{d}{2}} e^{-\frac{\pi|\xi|^2}{t+\delta}} \, dt$$

Hence $I_{\alpha,\delta}(x) + J_{\alpha,\delta}(x) = K_{\alpha,\delta}(x)$ with

$$I_{\alpha,\delta}(\xi) := (\alpha - d)\int_{\mathbb{R}^d} \frac{x_j}{|x|^{d+2-\alpha}} e^{-\pi\delta|x|^2} e^{2i\pi\xi\cdot x}\, dx,$$

$$J_{\alpha,\delta}(\xi) := -2\pi\delta \int_{\mathbb{R}^d} x_j \frac{1}{|x|^{d-\alpha}} e^{-\pi\delta|x|^2} e^{2i\pi\xi\cdot x}\, dx, \text{ and}$$

$$K_{\alpha,\delta}(\xi) := -2\bar{C}\pi i\xi_j \int_0^\infty t^{\frac{d-\alpha}{2}-1}(t+\delta)^{-\frac{d}{2}} e^{-\frac{\pi|\xi|^2}{t+\delta}}\, dt\cdot$$

Elementary considerations show that $\lim_{\delta\to 0} J_{\alpha,\delta}(\xi) = 0$. Using again (11) we get

$$\lim_{\delta\to 0} I_{\alpha,\delta}(\xi) = \bar{C} i\xi_j \int_0^\infty t^{-\frac{\alpha}{2}-1} e^{-\frac{\pi|\xi|^2}{t}}\, dt = \frac{\bar{C}\xi_j}{|\xi|^\alpha}\cdot$$

Hence from the dominated convergence theorem and the equality $\widehat{h_1 * h_2}(\xi) = \widehat{h_1}(\xi)\widehat{h_2}(\xi)$ we get

$$\begin{aligned} R_j f(x) &= \bar{C} \lim_{\substack{\alpha\to 1\\ \alpha>1}} \lim_{\delta\to 0} (d-\alpha)\int_{\mathbb{R}^d} I_{\alpha,\delta}(x-y) f(y)\, dy \\ &= \bar{C} \lim_{\epsilon\to 0} \int_{\mathbb{R}^d} k_j^\epsilon(x-y) f(y)\, dy\cdot \end{aligned} \tag{14}$$

Indeed, from $\int_{R_2\geq|x-y|\geq R_1} \frac{x_j-y_j}{|x-y|^\alpha}\, dy = 0$, we get $\int_{|x-y|\geq\epsilon} \frac{x_j-y_j}{|x-y|^{d+2-\alpha}} f(y)\, dy - \int_{|x-y|\geq\epsilon} \frac{x_j-y_j}{|x-y|^{d+1}} f(y)\, dy = A_1 + A_2$ with

$$\begin{aligned} A_1 &:= \int_{1\geq|x-y|\geq\epsilon} \left(\frac{x_j-y_j}{|x-y|^{d+2-\alpha}} - \frac{x_j-y_j}{|x-y|^{d+1}}\right)(f(y) - f(x))\, dy, \text{ and} \\ A_2 &:= \int_{|x-y|\geq 1} \left(\frac{x_j-y_j}{|x-y|^{d+2-\alpha}} - \frac{x_j-y_j}{|x-y|^{d+1}}\right) f(y)\, dy\cdot \end{aligned}$$

The mean value theorem shows that

$$\begin{aligned} |A_1| &\lesssim (\alpha-1)\int_{1\geq|x-y|\geq\epsilon} \frac{|\ln(|x-y|)|}{|x-y|^{d-1}}\, dy, \text{ and} \\ |A_2| &\lesssim (\alpha-1)\int_{|x-y|\geq 1} |f(y)|\, dy\cdot \end{aligned}$$

We also have $\lim_{\epsilon\to 0} \frac{x_j-y_j}{|x-y|^{d+1-\alpha}} f(y)\, dy = 0$. This explains the last equality in (14).

□

Next we turn to the boundedness of the Riesz transforms. To this end we first recall the Calderon-Zygmund kernel (see e.g [3]):

**Definition 2.** *$K$ is called a Calderon-Zygmund kernel if $K \in L^1_{loc}(\mathbb{R}^d - \{0\})$ and if there exists $B > 0$ such that $K$ satisfies the following conclusions*

- *Decay property:*

$$|K(x)| \leq B|x|^{-d} \tag{15}$$

- *Regularity property: for all $y \in \mathbb{R}^d$*

$$\int_{|x| \geq 2|y|} |K(x+y) - K(x)| \, dx \leq B \cdot \tag{16}$$

- *Cancellation property: for all $R_1 \leq R_2$ we have*

$$\int_{R_1 \leq |x| \leq R_2} K(y) \, dy = 0 \cdot \cdot \tag{17}$$

**Remark 14.** *Observe that if there exists $C > 0$ such that*

$$x \neq 0 : \; |\nabla K(x)| \leq \frac{C}{|x|^{d+1}}, \tag{18}$$

*then (16) holds for some $B := B(C)$.*
*Indeed the fundamental theorem of calculus yields*

$$\begin{aligned} \int_{|x| \geq 2|y|} |K(x+y) - K(x)| \, dx & \leq \int_{|x| \geq 2|y|} \int_0^1 |\nabla K(x+ty)||y| \, dt \, dx \\ & \lesssim |y| \int_0^1 \int_{|x| \geq 2|y|} \frac{1}{|x+ty|^{d+1}} \, dx \, dt \\ & \lesssim |y| \int_0^1 \int_{|x| \geq 2|y|} \frac{1}{\left(\frac{|x|}{2}\right)^{d+1}} \, dx \, dt \\ & \lesssim |y| \int_{2|y|}^{\infty} \frac{1}{r^2} \, dr \\ & \lesssim 1 \end{aligned}$$

We then prove that a kernel related to the $j^{th}-$ Riesz transform in a Calderon-Zygmund kernel.

**Proposition 17.** *Let $1 \leq j \leq d$. Let $K_j$ defined by $K_j(x) := \frac{x_j}{|x|}$. Then $K_j$ is a Calderon-Zygmund kernel.*

*Proof.* The proof is short. It is clear that the decay property (15) and the cancellation property (17) hold. The regularity property (16) follows from (18).

□

Next we prove the boundedness of the limit of operators associated to truncated Calderon-Zygmund kernels in $L^p(\mathbb{R}^d)$, $1 < p < \infty$.

**Proposition 18.** *Suppose that $K$ is Calderon-Zygmund kernel. Let $1 < p < \infty$. Given $\epsilon > 0$ let*

$$T_\epsilon(f)(x) := \int_{|y| \geq \epsilon} f(x-y) K(y) \, dy \cdot$$

*Then there exists $A_p > 0$ such that*

$$\|T_\epsilon f\|_{L^p(\mathbb{R}^d)} \leq A_p \|f\|_{L^p(\mathbb{R}^d)} \cdot \tag{19}$$

*Moreover $\lim_{\epsilon \rightarrow 0} T_\epsilon(f)$ exists in the sense of the $L^p$ norm, and if $Tf := \lim_{\epsilon \rightarrow 0} T_\epsilon(f)$ then*

$$\|Tf\|_{L^p(\mathbb{R}^d)} \leq A_p \|f\|_{L^p(\mathbb{R}^d)}. \tag{20}$$

*Proof.* We prove that $\lim_{\epsilon \to 0} T_\epsilon(f)$ exists in $L^p$. We have

$$\begin{aligned} T_\epsilon(f)(x) - T_\eta(f)(x) &= \int_{|y|\geq \epsilon} K(y) f(x-y)\, dy - \int_{|y|\geq \eta} K(y) f(x-y)\, dy \\ &= sgn(\eta - \epsilon) \int_{\min(\epsilon,\eta)\leq |y| \leq \max(\epsilon,\eta)} K(y) \left(f(x-y) - f(x)\right)\, dy : \end{aligned}$$

this comes from the cancellation property (17). Hence the fundamental theorem of calculus and the Mikowski inequality show that

$$\begin{aligned} \|T_\epsilon(f) - T_\eta(f)\|_{L^p(\mathbb{R}^d)} &\lesssim \left\| \int_{\min(\epsilon,\eta)\leq |y| \leq \max(\epsilon,\eta)} |K(y)| |\nabla f(x-y)|\, |y|\, dy \right\|_{L^p(\mathbb{R}^d)} \\ &\lesssim \int_{\min(\epsilon,\eta)\leq |y| \leq \max(\epsilon,\eta)} |K(y)| \|\nabla f(x-y)\|_{L^p(\mathbb{R}^d)}\, |y|\, dy \\ &\lesssim \int_{\min(\epsilon,\eta)\leq |y| \leq \max(\epsilon,\eta)} |y|^{-d+1}\, dy \\ &\lesssim \int_{\min(\xi,\eta)}^{\max(\xi,\eta)} dr \\ &\lesssim |\eta - \epsilon| \end{aligned}$$

Hence, if $x_n \to 0$ as $n \to \infty$ then $(T_{x_n})_{n\geq 1}$ is a Cauchy sequence so it converges. So $T(f) := \lim_{\epsilon \to 0} T_\epsilon(f)$ exists.

The proof of (19) relies upon Calderon-Zygmund theory and can be found in e.g [3, 9].

By letting $\epsilon \to 0$ in (19) we get (20).

□

**Proposition 19.** *Let $1 \leq j \leq d$ and $1 < p < \infty$. Then $R_j$ defined by (13) is bounded on $L^p(\mathbb{R}^d)$, i.e there exists $A_p > 0$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$*

$$\|R_j f\|_{L^p(\mathbb{R}^d)} \leq A_p \|f\|_{L^p(\mathbb{R}^d)}.$$

*Proof.* It follows from Proposition 16, Proposition 17, and Proposition 18.

□

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