AMERICAN UNIVERSITY OF BEIRUT

NEWTONIAN AND POST-NEWTONIAN ASPECTS OF MIMETIC GRAVITY

by
LEONID ADEL SARIEDDINE

A thesis
submitted in partial fulfillment of the requirements
for the degree of Master of Science
to the Department of Physics
of the Faculty of Arts and Sciences
at the American University of Beirut

Beirut, Lebanon
November 2023

AMERICAN UNIVERSITY OF BEIRUT

NEWTONIAN AND POST-NEWTONIAN ASPECTS OF MIMETIC GRAVITY

by

LEONID ADEL SARIEDDINE

Approved by: Ali Chamseddine

Dr. Ali Chamseddine, Professor — Advisor
Physics

LKlushin

Dr. Leonid Klushin, Professor — Member of Committee
Physics

Dr. Ola Hosseiky Malaeb, Lecturer — Member of Committee
Physics

Dr. Sara Najem, Assistant Professor — Member of Committee
Physics

Date of thesis defense: November 21, 2023

AMERICAN UNIVERSITY OF BEIRUT

THESIS RELEASE FORM

Student Name: Sarieddine Leonid Adel

Last First Middle

I authorize the American University of Beirut, to: (a) reproduce hard or electronic copies of my thesis; (b) include such copies in the archives and digital repositories of the University; and (c) make freely available such copies to third parties for research or educational purposes

✓ **As of the date of submission of my thesis**

___ **After 1 year from the date of submission of my thesis .**

___ **After 2 years from the date of submission of my thesis .**

___ **After 3 years from the date of submission of my thesis .**

Signature Date 14/12/2023

Acknowledgements

First and foremost, I would like to thank my adviser Dr. Ali Chamseddine for accepting me as his student and for all the productive discussions that we had. He was always available to help and provide constructive feedback for my work. I would also like to thank my other committee members, Dr. Leonid Klushin, Dr. Ola Malaeb, and Dr. Sara Najem all of whom provided valuable feedback.

Abstract of the Thesis of

Leonid Adel Sarieddine for Master of Science
Major: Physics

Title: Newtonian and Post-Newtonian Aspects of Mimetic Gravity

Mimetic gravity is a modified theory of gravity which is able to incorporate dark matter into the underlying geometry of space-time by isolating the conformal degree of freedom. The theory has been studied extensively in the cosmological regime, as such, in this thesis, we set out to study the implications of the theory at the solar system and galactic scales. To that end, we carry out the post-Newtonian expansion of mimetic gravity to lowest post-Newtonian order. We interpret the equations in the Newtonian limit and study some of the implications of the theory at the astrophysical scale. We solve the associated equations in several special cases. Then by establishing some bounds on the asymptotic behavior of the fields we prove that any static spherically symmetric space-time with a non trivial mimetic contribution cannot be asymptotically flat. Finally, we study static spherically symmetric solutions. To explain the rotation curves, one needs a logarithmic term in the potential, we show that even though the mimetic fluid can't reconstruct an exact logarithmic term, it is able to contribute a quasi-logarithmic term which recovers the basic qualitative features of galactic rotation curves.

Table of Contents

Illustrations

Chapter 1

Brief Introduction to General Relativity

1.1 Special Relativity

One of the most crucial steps in developing a geometric theory of gravity was the unification of the concepts of space and time. While space and time were absolute notions in the Newtonian theory, this seemingly intuitive view of the world quickly changed with the arrival of special relativity which postulates that the speed of light is constant for all inertial observers. One immediate consequence of this postulate is that two different observers may no longer agree on the length of their rods or the time that has passed on their clocks. The constancy of the speed of light, however, ensures that there is one quantity, quadratic in the space and time increments, which remains the same for all observers:

$$-c^2\Delta t^2 + \Delta x^2 = -c^2\Delta t'^2 + \Delta x'^2 \tag{1.1}$$

In other words, if we have two space-time events P_1 and P_2, and an observer measures a spatial distance of Δx and a time difference of Δt between the two events, while another observer in an another inertial frame measures $\Delta x'$, $\Delta t'$ corresponding to those same two events; then even though $\Delta x \neq \Delta x'$ and $\Delta t \neq \Delta t'$, they would still agree on this quadratic combination of space and time increments written in (1).

Eq (1) could be derived from the postulates of special relativity and its physical content is basically that the speed of light is constant to all inertial observers. To see this, consider a hypothetical event P_1 associated with the release of a photon and P_2 the absorption of a photon in the rest frame of an observer, then we know that this observer will measure an associated Δx (spatial length travelled by the photon) and a Δt (time of travel). Since the speed of light is c for that observer then we know

$$\frac{\Delta x}{\Delta t} = c$$

Then (1) immediately implies

$$\frac{\Delta x'}{\Delta t'} = c$$

Which means that the speed of light is also c for any other inertial observer. The constancy of the speed of light leads to a direct relationship between space and time coordinates which was otherwise absent in a Newtonian setting. If a light-like event satisfies $\Delta x > \Delta x'$ (such as the release/absorption of a photon in the primed frame which is moving w.r.t the unprimed) then the constancy of the speed of light would force upon you $\Delta t > \Delta t'$ in order that $\dfrac{\Delta x}{\Delta t} = \dfrac{\Delta x'}{\Delta t'} = c$ be satisfied.

From a mathematical point of view, (1) leads to a rich mathematical structure since it endows space-time with a pseudo-Riemannian metric, the so-called Minkowski metric. A Newtonian space-time can only ever be equipped with a Riemannian metric on constant time slices (i.e. only on the spatial 3-dimensional slices of space-time) hence it is neither a Riemannian or a pseudo-Riemannian manifold.

1.2 Metric Theory of Gravity

Until now, nothing is mentioned about gravity, and indeed incorporating gravity into the geometry of space-time has little to do with special relativity or the constancy of the speed of light but rather all to do with gravity's unique feature of accelerating all objects in the same manner regardless of their matter composition. This curious fact, that gravity acts like a fictitious force (at least locally) gives us the first hint that gravity is geometric in nature. Newton's first law tells us that a body will move in a straight line unless acted upon by forces. If gravity is not a force one might hope to incorporate gravity into the geometry of the underlying manifold by mimicking its effects in terms of the curvature of the manifold; in this way, Newton's second law applied to gravity becomes Newton's first law but in a curved manifold. Indeed, this can be done by postulating that the equation of motion for a test particle is the geodesic equation:

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\alpha}\dot{x}^{\nu}\dot{x}^{\alpha} = 0$$

parametrized in terms of some parameter and where the Γ's are the connection coefficients. The key idea, however, is that the underlying manifold should be the space-time manifold. It is impossible to incorporate gravity into geometry without a space-time formulation; in other words, gravity can be modelled as a curvature of space-time but not of purely space. This leads to an entirely novel formulation of the dynamics of particles: instead of thinking of particles as moving on paths through three-dimensional space which deviate from straight lines due to the force of gravity pushing on them, one thinks of particles as moving on the straightest possible paths in a curved space-time.

If we want to incorporate gravity into geometry in a relativistic fashion we must take into account special relativity as well. This is ensured by postulating that at every space-time point one can create a locally inertial frame where the laws of special relativity hold. Mathematically, this implies that space-time is equipped with a space-time metric g and at each point one can cancel the effects of gravity at that point (but only at that one point in general) and reduce g to η at that point. This is physically saying that any observer at the space-time point p would observe the speed of light to be c at p. The existence of a space-time metric, imposed by local Lorentz invariance, allows us to describe the geometry of space-time through the metric g. The fundamental theorem of Riemannian geometry (or pseudo-Riemannian) tells us that the connection is uniquely determined by the metric given that the connection is metric-compatible (i.e. straightest possible paths are also shortest paths) and torsion-free. While metric-compatibilty is very natural to assume, the fact that the connection is torsion-free follows from the existence of local inertial frames and goes as follows. In a coordinate basis, the torsion tensor is given by

$$T^{\mu}_{\beta\alpha} = \Gamma^{\mu}_{\beta\alpha} - \Gamma^{\mu}_{\beta\alpha}$$

Hence the existence of a local inertial frame, which implies that one can always make the connection coefficients vanish at one point, directly imply that the torsion tensor should vanish at this point, then the tensorial character of the torsion tensor implies that it is zero in any other coordinate system as well. Since one can make such a construction at each point in the manifold (which is based on physics i.e. the equivalence principle) one is directly lead to the fact that the connection is torsion free.

Now that we have established that the space-time is equipped with a metric tensor and a metric-compatible torsion-free connection, the fundamental theorem mentioned above allows us to compute the connection coefficients entirely in terms of the metric:

$$\Gamma^{\mu}_{\beta\alpha} = \frac{1}{2} g^{\mu\nu}(g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\beta\alpha,\nu})$$

where the inverse metric is defined by

$$g^{\mu\nu} g_{\nu\alpha} = \delta^{\mu}_{\alpha}$$

It follows that all the other geometric objects of interest, most importantly the curvature tensors and scalars, can be written in terms of the metric as well. The Riemann curvature tensor is defined to be a (3,1) tensor field and is given by the formula:

$$R(U,V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W$$

which represents the change in W after it is parallel transported around an infinitesimal loop determined by tangent vectors U and V. In coordinates it reads

$$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\nu\beta,\mu} - \Gamma^{\alpha}_{\mu\beta,\nu} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\beta\nu} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\beta\mu}$$

One can then define the Ricci tensor to be

$$R_{\beta\nu} = R^{\mu}_{\beta\mu\nu}$$

and the Ricci scalar

$$R = R_{\beta\nu}g^{\beta\nu}$$

The basic idea of GR is that energy and matter are the sources of space-time curvature. In Newtonian gravity the mass density ρ was the source of gravity, but in a relativistic theory this can't be the case because ρ is not a lorentz scalar, but rather one component of a rank-2 tensor called the energy-momentum or stress-energy tensor $T^{\mu\nu}$. So the form of the field equations must be

$$\text{spacetime curvature} \propto T^{\mu\nu}$$

Local conservation of energy and momentum demand that the divergence of the stress energy be zero.

$$\nabla_{\mu}T^{\mu\nu} = 0$$

Hence whatever is on the left-hand side must be divergence free. The unique symmetric, divergence-free tensor which is made from second derivatives of the metric and/or quadratic in first derivatives (this should be true by dimensional analysis) is the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

The field equations then become (proportionality constant determined by Newtonian limit).

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

By taking the trace one can also write the equation in the following so-called trace-reversed form

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})$$

One important implication of the divergence-free condition of the Einstein tensor (called the Bianchi identity) is that it guarantees the automatic conservation of energy and momentum of the sources. Hence the equations of motion for the matter fields are automatically built in the Einstein field equations. The above equation could also be derived from the so-called Einstein-Hilbert action:

$$S = \int_M \sqrt{-g}\ (R\ + L_{\text{matter}}) d^4x$$

When one varies the first term in S with respect to $g_{\mu\nu}$ one gets the Einstein tensor, and the left hand side is obtained by variation of the second term with respect to $g_{\mu\nu}$.

Chapter 2

Post-Newtonian Analysis in General Relativity

2.1 Newtonian Limit of The Geodesic Equation

The Newtonian limit in GR is determined by 3 conditions:

1) Weak fields i.e. metric is approximately Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where

$$h_{\mu\nu} <<< 1$$

2) Slow velocities of the sources and particles involved

3) Stationary fields i.e. $h_{\mu\nu}$ is independent of time

If all the above are satisfied then when we write the equations of motion of a particle.

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\alpha}\dot{x}^{\nu}\dot{x}^{\alpha} = 0$$

we notice that since the particle has low velocity then

$$\dot{t} \approx 1$$

$$\dot{x^i} \approx 0$$

$$\frac{d}{d\tau} \approx \frac{d}{dt}$$

So the only relevant term in the geodesic equation is the 00 term so the geodesic equation becomes

$$\ddot{x}^\mu + \Gamma^\mu_{00} = 0$$

Now using the weak field and static assumptions we find

$$\Gamma^0_{00} = -\frac{1}{2}\eta^{0\alpha}(2\partial_0 h_{\alpha 0} - \partial_\alpha h_{00}) = 0$$

$$\Gamma^i_{00} = -\frac{1}{2}\eta^{i\alpha}(2\partial_0 h_{\alpha 0} - \partial_\alpha h_{00}) = -\frac{1}{2}\partial_i h_{00}$$

So the 0-component equation is trivially satisfied and the space component equations become

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}$$

Hence we can clearly see that we can identify h_{00} with $-2U$ where U is the Newtonian gravitational potential (and indeed as we will see later it is compatible with the Newtonian limit of the field equation).

2.2 Newtonian Limit of The Field Equations

Turning our attention to the Newtonian limit of the field equations, we first compute the Ricci tensor

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} = \frac{1}{2}(\partial_\mu \partial_\alpha h^\alpha_\nu - \Delta h_{\mu\nu} - \partial_\mu \partial_\nu h + \partial_\nu \partial_\alpha h^\alpha_\mu)$$

where $h := \eta^{\mu\nu} h_{\mu\nu}$ and $h^\alpha_\mu := \eta^{\alpha\nu} h_{\nu\mu}$

If we take a simple energy momentum tensor of the form $T_{\mu\nu} = \rho v_\mu v_\nu$ where ρ is the matter density and v is its four velocity, then in the Newtonian limit only the 00 component contributes since the other velocity components are negligible (this is condition 2 which states that the sources have low velocities). If we note that

$$R_{00} = -\frac{1}{2}\Delta h_{00}$$

and use the trace reversed form of EFE

$$R_{\mu\nu} = 8\pi G\big(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}\big)$$

then we get

$$-\Delta h_{00} = 8\pi G \rho$$

for the 00 component, hence we see again that if $h_{00} = -2U$ then this reduces to the desired Poisson equation, the field equation in classical Newtonian gravity.

2.3 Motivation for Post-Newtonian Expansion

To motivate the post-Newtonian expansion we first note that if we look at the Newtonian limit we notice that h_{00} is of the order $\frac{GM}{rc^2}$ (since we were working in units where $c = 1$ they were suppressed), which is of the order of 10^{-9} on the surface of the Earth and decreases further if you go inside or farther away (assuming Earth has an approximately constant density). Note that $\frac{GM}{rc^2}$ is also of the order $\frac{v^2}{c^2}$ for a typical planet/particle in the solar system by virial theorem, so we hope to develop a systematic expansion in powers of such quantities. We know h_{00} is of the order of $\frac{v^2}{c^2}$ as shown, but what about the other components of the metric? To get a hint, we note that in the linearized regime the trace reversed EFE to lowest order read

$$R_{00} \propto \text{second derivatives of } h_{00} \propto \frac{G\rho}{c^2}$$

$$R_{0i} \propto \text{second derivatives of } h_{0i} \propto \frac{G\rho v_i}{c^3}$$

$$R_{ij} \propto \text{second derivatives of } h_{00} \text{ and } h_{ij} \propto \frac{G\rho}{c^2}$$

Hence we can conclude that the lowest order expansion for g_{00} starts at order $\frac{v^2}{c^2}$, g_{0i} starts at $\frac{v^3}{c^3}$ and g_{ij} starts at $\frac{v^2}{c^2}$. This can be seen more rigorously by imposing the harmonic gauge which we will get into shortly.

Another point that needs to be addressed is the issue of time dependence. To recover the Newtonian limit we needed to impose time independence conditions on the metric components, but our hope with the post-Newtonian formalism is to include time-dependent phenomena, hence we must allow time derivatives to enter the picture. Again by analyzing the linear regime but this time allowing time dependence we get the following relations when we focus on the space-time component:

$$R_{0i} \propto \text{space-space derivatives of } h_{0i} + \text{space-time derivatives of } h_{00} \propto \frac{G\rho v_i}{c^3}$$

Now we know that h_{00} is of the order of $\frac{v^2}{c^2}$ and that h_{0i} is of the order of $\frac{v^3}{c^3}$ hence the above relation forces us to conclude the following.

$$\frac{\partial_0 h_{0i}}{\partial_j h_{0i}} \propto \frac{v}{c}$$

So time derivatives introduce a higher order term in the expansion. Another way of saying this is that if we introduce the characteristic lengthscale R then we have

$$\frac{\partial}{\partial x^i} \propto \frac{1}{R}$$

$$\frac{\partial}{\partial t} \propto \frac{v}{cR}$$

So if a certain quantity is of the order $\frac{v^2}{c^2}$ (like h_{00} for example) then

$$\partial_i h_{00} \propto \frac{v^2}{c^2 R}$$

$$\partial_0 h_{00} \propto \frac{v^3}{c^3 R}$$

With the above facts established we now proceed with the systematic expansion in powers of $\frac{v}{c}$. To avoid clutter in the notation from hereon out we just refer to this as an expansion in powers of v^2.

2.4 Post-Newtonian Expansion

Now that we have established the basic rules of expansion we begin by expanding the metric as follows (see also [1]):

$$g_{00} = -1 + \overset{(2)}{g}_{00} + \overset{(4)}{g}_{00} + \overset{(6)}{g}_{00} + ...$$

$$g_{ij} = \delta_{ij} + \overset{(2)}{g}_{ij} + \overset{(4)}{g}_{ij} + \overset{(6)}{g}_{ij} + ...$$

$$g_{0i} = \overset{(3)}{g}_{0i} + \overset{(5)}{g}_{ij} + ...$$

where $\overset{(n)}{g}_{\mu\nu}$ means that it is a quantity of order v^n. From the following condition

$$g_{\mu\beta} g^{\beta\nu} = \delta^{\nu}{}_{\mu}$$

we obtain the expansion for the inverse metric:

$$g^{00} = -1 + \overset{(2)}{g}{}^{00} + \overset{(4)}{g}{}^{00} + \overset{(6)}{g}{}^{00} + ...$$

$$g^{ij} = \delta^{ij} + \overset{(2)}{g}{}^{ij} + \overset{(4)}{g}{}^{ij} + \overset{(6)}{g}{}^{ij} + ...$$

$$g^{0i} = \overset{(3)}{g}{}^{0i} + \overset{(5)}{g}{}^{0i} + ...$$

Where

$$\overset{(2)}{g}{}^{00} = -g_{00}$$

and similar expressions for the other components.

Similarly, the Christoffel symbols are expanded as follows:

$$\Gamma^{\mu}{}_{\nu\lambda} = \overset{(2)}{\Gamma}{}^{\mu}{}_{\nu\lambda} + \overset{(4)}{\Gamma}{}^{\mu}{}_{\nu\lambda} + ...,$$

for $\Gamma^{i}{}_{00}, \Gamma^{i}{}_{jk}, \Gamma^{0}{}_{i0}$.

$$\Gamma^{\mu}{}_{\nu\lambda} = \overset{(3)}{\Gamma}{}^{\mu}{}_{\nu\lambda} + \overset{(5)}{\Gamma}{}^{\mu}{}_{\nu\lambda} + ...,$$

for $\Gamma^{i}{}_{0j}, \Gamma^{0}{}_{00}, \Gamma^{0}{}_{ij}$, and where $\overset{(n)}{\Gamma}{}^{\mu}{}_{\nu\lambda}$ are quantities of order $\frac{v^n}{R}$.

The Ricci tensor defined as:

$$R_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu,\lambda} - \Gamma^{\lambda}{}_{\mu\lambda,\nu} + \Gamma^{\lambda}{}_{\mu\nu}\Gamma^{\alpha}{}_{\lambda\alpha} - \Gamma^{\lambda}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\lambda\nu}$$

is expanded in powers of $\frac{v^n}{R^2}$ as follows:

$$R_{00} = \overset{(2)}{R}_{00} + \overset{(4)}{R}_{00} + \overset{(6)}{R}_{00} + ...$$

$$R_{ij} = \overset{(2)}{R}_{ij} + \overset{(4)}{R}_{ij} + \overset{(6)}{R}_{ij} + ...$$

$$R_{0i} = \overset{(3)}{R}_{0i} + \overset{(5)}{R}_{ij} + ...$$

Explicitly we can write Ricci tensor expansion in terms of the metric expansion:

$$\overset{(2)}{R}_{00} = -\frac{1}{2}\Delta \overset{(2)}{g}_{00}$$

$$\overset{(3)}{R}_{0i} = \frac{1}{2}(-\partial_0\partial_i \overset{(2)}{g}_{jj} + \partial_i\partial_j \overset{(3)}{g}_{j0} + \partial_0\partial_j \overset{(2)}{g}_{ij} - \Delta \overset{(3)}{g}_{0i})$$

$$\overset{(2)}{R}_{ij} = \frac{1}{2}(\partial_i\partial_j \overset{(2)}{g}_{00} - \partial_i\partial_j \overset{(2)}{g}_{kk} + \partial_i\partial_k \overset{(2)}{g}_{kj} + \partial_k\partial_j \overset{(2)}{g}_{ik} - \Delta \overset{(2)}{g}_{ij})$$

Now we will impose the following gauge condition to simplify the expressions.

$$\Gamma^{\lambda}{}_{\mu\nu} g^{\mu\nu} = 0$$

The motivation for imposing such a gauge has a long and interesting history. Fock advocated the use of harmonic coordinates based on his assertion that harmonic coordinates are the analogues of inertial coordinates in GR. He was able to show in fact, but under rather strict conditions, that the harmonic coordinates are unique up to a Lorentz transformation [2]; in this way, he argued, harmonic coordinates are the direct analogues of Cartesian coordinates in curved space-time. In any case, if we expand the above equation we find that we can write the Ricci tensor in a simplified form

$$\overset{(2)}{R}_{00} = -\frac{1}{2}\Delta \overset{(2)}{g}_{00}$$

$$\overset{(3)}{R}_{0i} = -\frac{1}{2}\Delta \overset{(3)}{g}_{0i}$$

$$\overset{(2)}{R}_{ij} = -\frac{1}{2}\Delta \overset{(2)}{g}_{ij}$$

the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ is expanded as

$$R = \overset{(2)}{R} + \overset{(4)}{R} + \overset{(6)}{R} + ...$$

where $\overset{(n)}{R}$ is of order $\frac{v^n}{R^2}$. Thus to lowest order we have:

$$\overset{(2)}{R} = -\overset{(2)}{R}_{00} + \overset{(2)}{R}_{jj}$$

where

$$\overset{(2)}{R}_{jj} := \overset{(2)}{R}_{11} + \overset{(2)}{R}_{22} + \overset{(2)}{R}_{33}$$

Finally, we expand the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}g_{\mu\nu}$

$$G_{00} = \overset{(2)}{G}_{00} + \overset{(4)}{G}_{00} + \overset{(6)}{G}_{00} + ...$$

$$G_{ij} = \overset{(2)}{G}_{ij} + \overset{(4)}{G}_{ij} + \overset{(6)}{G}_{ij} + ...$$

$$G_{0i} = \overset{(3)}{G}_{0i} + \overset{(5)}{G}_{ij} + ...$$

Where again $\overset{(n)}{G}_{\mu\nu}$ indicates it is of order $\frac{v^n}{R^2}$.

Explicitly to lowest order we have

$$\overset{(2)}{G}_{00} = \frac{1}{2}\overset{(2)}{R}_{00} + \frac{1}{2}\overset{(2)}{R}_{jj}$$

$$\overset{(3)}{G}_{0i} = \overset{(3)}{R}_{0i}$$

$$\overset{(2)}{G}_{ij} = \overset{(2)}{R}_{ij} + \frac{1}{2}(\overset{(2)}{R}_{00} - \overset{(2)}{R}_{kk})\delta_{ij}$$

We will use the above equations for the Einstein tensor for mimetic gravity. Meanwhile, we focus on GR and rewrite the equations to lowest post-Newtonian order

$$\overset{(2)}{R}_{00} = -\frac{1}{2}\Delta\overset{(2)}{g}_{00}$$

$$\overset{(4)}{R}_{00} = \frac{1}{2}\left(-\Delta\overset{(4)}{g}_{00} + \partial_0\partial_0\overset{(2)}{g}_{00} + (\partial_j\partial_i\overset{(2)}{g}_{00})\overset{(2)}{g}_{ij} - (\nabla\overset{(2)}{g}_{00})^2 \right)$$

$$\overset{(3)}{R}_{0i} = -\frac{1}{2}\Delta\overset{(3)}{g}_{0i}$$

$$\overset{(2)}{R}_{ij} = -\frac{1}{2}\Delta\overset{(2)}{g}_{ij}$$

The expansion of the Energy-Momentum tensor proceeds in the same vein and we see that if we define

$$S_{\mu\nu} = T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}$$

then we can expand as follows

$$S_{00} = \overset{(0)}{S}_{00} + \overset{(2)}{S}_{00} + ...$$

$$S_{i0} = \overset{(1)}{S}_{i0} + \overset{(3)}{S}_{i0} + ...$$

$$S_{ij} = \overset{(0)}{S}_{ij} + \overset{(2)}{S}_{ij} + ...$$

where $\overset{(n)}{S}_{\mu\nu}$ are quantities of order $\frac{v^n M}{R}$. The introduction of a characteristic mass parameter M is actually not needed since the Energy-Momentum tensor is always multiplied by G and $\frac{GM}{R}$ is of order v^2 as we established prior. Explicitly in terms of Energy-Momentum tensor the components of S read

$$\overset{(0)}{S}_{00} = \frac{1}{2}\overset{(0)}{T}^{00}$$

$$\overset{(2)}{S}_{00} = \frac{1}{2}\left(\overset{(2)}{T}{}^{00} - 2\, \overset{(2)}{g}_{00} \overset{(0)}{T}{}^{00} + \overset{(2)}{T}{}^{ii} \right)$$

$$\overset{(1)}{S}_{0i} = -\frac{1}{2} \overset{(1)}{T}{}^{0i}$$

$$\overset{(0)}{S}_{00} = \frac{1}{2}\delta_{ij} \overset{(0)}{T}{}^{00}$$

Of course the order of the corresponding T components are the same as those of S. We can now apply the trace reversed EFE

$$\Delta \overset{(2)}{g}_{00} = -8\pi G \overset{(0)}{T}{}^{00}$$

$$\Delta \overset{(4)}{g}_{00} = \partial_0\partial_0 \overset{(2)}{g}_{00} + (\partial_j\partial_i \overset{(2)}{g}_{00}) \overset{(2)}{g}_{ij} - (\nabla \overset{(2)}{g}_{00})^2 - 8\pi G\left(\overset{(2)}{T}{}^{00} - 2\, \overset{(2)}{g}_{00} \overset{(0)}{T}{}^{00} + \overset{(2)}{T}{}^{ii} \right)$$

$$\Delta \overset{(3)}{g}_{i0} = 16\pi G \overset{(1)}{T}{}^{i0}$$

$$\Delta \overset{(2)}{g}_{ij} = -8\pi G \delta_{ij} \overset{(0)}{T}{}^{00}$$

2.5 Solving the Equations

We proceed to solve these equations. The first equation is well known since we see the identification

$$\overset{(2)}{g}_{00} = -2U$$

$$U(x,t) = -G \int d^3x' \frac{T^{00}(x',t)}{\|x - x'\|}$$

Similarly we can clearly see that

$$\overset{(2)}{g}_{ij} = -2\delta_{ij} U$$

Meanwhile $\overset{(3)}{g}_{i0}$ is a vector potential and its solution is

$$\overset{(3)}{g}_{i0}(x,t) = -4G \int d^3x' \frac{T^{00}(x',t)}{\|x - x'\|}$$

The fourth order equation is the trickiest. To solve it we first rewrite it in a more convenient form:

$$\Delta \overset{(4)}{g}_{00} = \left((\partial_j\partial_i \overset{(2)}{g}_{00}) \overset{(2)}{g}_{ij} - (\nabla \overset{(2)}{g}_{00})^2 + 16\pi G\, \overset{(2)}{g}_{00} \overset{(0)}{T}{}^{00} \right) - \left(8\pi G \overset{(2)}{T}{}^{00} + 8\pi G \overset{(2)}{T}{}^{ii} - \partial_0\partial_0 \overset{(2)}{g}_{00} \right)$$

Focusing on the first term, and plugging in the solutions we found for g_{00} and g_{ij} in terms of U we get

$$(\partial_j \partial_i \overset{(2)}{g}_{00}) \overset{(2)}{g}_{ij} - (\nabla \overset{(2)}{g}_{00})^2 + 16\pi G \overset{(2)}{g}_{00} \overset{(0)}{T^{00}} = -4U\Delta U - 4(\nabla U)^2$$

Now using the identity

$$2\Delta(U^2) = 4(\nabla U)^2 + 4U\nabla U$$

We see that we can rewrite the fourth order equation as

$$\Delta \overset{(4)}{g}_{00} = -2\Delta(U^2) + \Big(-8\pi G \overset{(2)}{T^{00}} - 8\pi G \overset{(2)}{T^{ii}} + \partial_0 \partial_0 \overset{(2)}{g}_{00} \Big)$$

The other second part is linear in all the quantities involved, hence a closed form solution for $\overset{(4)}{g}_{00}$ can be obtained

$$\overset{(4)}{g}_{00} = -2U^2 - 2V$$

Where

$$V(x,t) = -\int \frac{d^3x'}{\|x - x'\|} \Big(G \overset{(2)}{T^{00}}(x', t) + G \overset{(2)}{T^{ii}}(x', t) + \frac{1}{4\pi} \partial_0 \partial_0 U(x', t) \Big)$$

Chapter 3

Mimetic Gravity

3.1 Basic Idea

The original motivation for mimetic gravity was to isolate the conformal mode. More explicitly, one begins by parametrizing the physical metric in terms of an auxilliary metric and a scalar field [3]:

$$g_{\mu\nu} = g'_{\mu\nu} g'^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi$$

$g'^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ acts as a conformal mode of the metric and one can immediately see that the physical metric is invariant under conformal transformations of the auxilliary metric. Intuitively this means we split the 10 components of the physical metric into a conformal mode + other non conformal degrees of freedom akind to how we can write a vector in terms of its magnitude multiplied by the unit vector indicating its direction.

Now write the Einstein-Hilbert action in terms of the auxilliary metric and the scalar field.

$$S = \int_M \sqrt{-g(g_{\mu\nu}(g'_{\mu\nu}, \phi))} \left(R(g_{\mu\nu}(g'_{\mu\nu}, \phi)) + L_{\text{matter}} \right) d^4x$$

Then one varies the action with respect to $g_{\mu\nu}$ which can be written in terms of variations of $g'_{\mu\nu}$ and ϕ subject to the parametrization above. The variation with respect to $g'_{\mu\nu}$ yields the modified Einstein equation of motion

$$G_{\mu\nu} = T_{\mu\nu} - (G - T)\partial_\mu\phi\partial_\nu\phi$$

Meanwhile variations with respect to ϕ yield a continuity equation

$$\nabla^\mu((G - T)\partial_\mu\phi) = 0$$

the contra-variant version of the parametrization condition also implies

$$g'^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi g^{\mu\nu} = g'^{\mu\nu}$$

This immediately leads to the constraint equation.

$$g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = -1$$

It was later shown in [4] that an equivalent way to arrive at the above set of equations is to start from the following action.

$$S = \int_M \sqrt{-g}\,\left(R + \lambda(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + 1) + L_{\text{matter}}\right)\,d^4x$$

where λ is a lagrange multiplier. Varying with respect to $g_{\mu\nu}$ one gets the Einstein equation. A variation with respect to λ yields the constraint equation, and finally variation with respect to ϕ yields its continuity equation. Finally one can identify the Lagrange multiplier with $G - T$ to finally yield the equations.

3.2 Mimicking Dark Matter

If we look at the governing equations we note that constraint equation implies that the gradient of the mimetic field is timelike. Also note that $G - T \neq 0$ in general hence one could potentially find solutions more general than the ones in GR. In principle one could solve for ϕ and $G - T$ in terms of metric components using the continuity and constraint equations, then plug them back into Einstein equations to solve for the metric. To understand the physical significance of these equations let us start by making the following identifications

$$\rho := G - T$$

$$\partial_\mu\phi = v_\mu$$

then we can immediately see that the gradient of the scalar field can play the role of the fluid velocity and the $G - T$ can play the role of the energy density of the fluid. Clearly, $(G - T)\partial_\mu\phi\partial_\nu\phi$ plays the role of the stress energy tensor of the fluid, and it has the exact form of the stress energy tensor of dust (ideal pressure-less fluid), which is the standard model for cold dark matter. Also, the fact that the gradient of the scalar field is timelike directly leads to the normalization of the velocity field

$$v^\mu v_\mu = 1$$

One very important feature of General Relativity is that the stress-energy tensor is conserved on shell (local conservation of energy and momentum). In GR, this is automatically built in via the Bianchi identity, i.e. due to the fact that

$$\nabla^{\mu} G_{\mu\nu} = 0$$

which would automatically imply that

$$\nabla^{\mu} T_{\mu\nu} = 0$$

In mimetic gravity the bianchi identity merely implies that

$$\nabla^{\mu}(T_{\mu\nu} + (G - T)\partial_{\mu}\phi\partial_{\nu}\phi) = 0$$

which by itself doesn't imply that the matter stress energy tensor is conserved. However we can prove that the mimetic stress energy tensor is conserved which would also imply that the matter stress energy tensor is conserved.

$$\nabla^{\mu}((G - T)\partial_{\mu}\phi\partial_{\nu}\phi) = \nabla^{\mu}((G - T)\partial_{\mu}\phi)\partial_{\nu}\phi + (G - T)\partial_{\mu}\phi\nabla^{\mu}(\partial_{\nu}\phi)$$

the first term is zero because of the continuity equation. The second term is zero because the constraint equation implies that.

$$g^{\alpha\beta}\partial_{\alpha}\phi\nabla_{\mu}\partial_{\beta}\phi = 0$$

and

$$\nabla_{\mu}\partial_{\beta}\phi = \nabla_{\beta}\partial_{\mu}\phi$$

3.3 Cosmology

Applying this theory to cosmology, we start by working in the synchronous gauge.

$$ds^2 = dt^2 - g_{ij}dx^i dx^j$$

taking

$$\phi = t$$

we automatically satisfy the constrain equation. The continuity equation then becomes

$$\partial_0(\sqrt{g}(G - t)) = 0$$

which implies that

$$G - T = \frac{C(x)}{\sqrt{g}}$$

with $C(x)$ an integration constant that determines the amount of mimetic matter. For a Friedmann universe we obtain

$$G - T = \frac{C}{a^3}$$

which, as expected, has the exact required density profile as dust. In this sense $G - T$ is able to mimic the energy density profile of cold dark matter on cosmological scales.

It should also be noted that similar theories to mimetic gravity were also considered before (see in particular [5]). Using the Lagrange multiplier approach, it becomes easy to see that mimetic gravity is a form of scalar Einstein aether theory [6]. By modifying the mimetic gravity action to include a potential which depends solely on the mimetic field one can show that it can mimic inflationary periods, quintessence, bouncing universe, and essentially any background cosmology that one desires [7]. By introducing further terms in the action one can also resolve cosmological singularities [8] as well as black hole singularities [9]. Static spherically symmetric space-times in mimetic gravity were studied in [10] and [11], where they derived solutions with complex scalar fields and/or assumed a space-like constraint for the gradient of the mimetic field.

Chapter 4

Post-Newtonian Expansion in Mimetic Gravity

4.1 Deriving the Equations

The equations of mimetic gravity are:

$$G_{\mu\nu} + (G - T)\partial_\mu\phi\partial_\nu\phi = T_{\mu\nu}$$

$$\partial_\mu(\sqrt{-g}(G - T)g^{\mu\nu}\partial_\nu\phi) = 0$$

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -1$$

Obviously we expect that the $G - T$ term to be expanded as:

$$G - T = \overset{(2)}{G - T} + \overset{(4)}{G - T} + \overset{(6)}{G - T} + ...$$

where $\overset{(n)}{G - T}$ is of the order of $\dfrac{v^n}{R^2}$.

As for ϕ, it is to be expanded in the following manner:

$$\phi = t + \overset{(1)}{\phi} + \overset{(3)}{\phi} + \overset{(5)}{\phi} + ...$$

where $\overset{(n)}{\phi}$ is of the order of Rv^n. Perhaps a more suggestive way to look at the above expansion is to note that ϕ only ever appears in terms of derivatives and hence it's more suggestive to write the expansion of ϕ in this case:

$$\partial_i\phi = \overset{(1)}{\partial_i\phi} + \overset{(3)}{\partial_i\phi} + \overset{(5)}{\partial_i\phi} + ...$$

where now $\overset{(n)}{\partial_i\phi}$ is of order v^n, and

$$\partial_0\phi = 1 + \overset{(2)}{\partial_0\phi} + \overset{(4)}{\partial_0\phi} + \overset{(6)}{\partial_0\phi}$$

where $\overset{(n)}{\partial_0\phi}$ is of order v^n.

Now we begin by expanding the constraint equation. Using the expansion of the inverse metric and ϕ we compute:

$$g^{00}\partial_0\phi\partial_0\phi + 2g^{0i}\partial_0\phi\partial_i\phi + g^{ij}\partial_i\phi\partial_j\phi = -1$$

$$\begin{aligned}
&(-1 + \overset{(2)}{g^{00}} + \overset{(4)}{g^{00}} + ...)(1 + \overset{(2)}{\partial_0\phi} + \overset{(4)}{\partial_0\phi} + ...)(1 + \overset{(2)}{\partial_0\phi} + \overset{(4)}{\partial_0\phi} + ...)\\
&+2(\overset{(3)}{g_{0i}} + \overset{(5)}{g_{0i}} + ...)(1 + \overset{(2)}{\partial_0\phi} + \overset{(4)}{\partial_0\phi} + ...)(\overset{(1)}{\partial_i\phi} + \overset{(3)}{\partial_i\phi} + \overset{(5)}{\partial_i\phi} + ...)\\
&+(\delta^{ij} + \overset{(2)}{g^{ij}} + \overset{(4)}{g^{ij}})(\overset{(1)}{\partial_i\phi} + \overset{(3)}{\partial_i\phi} + \overset{(5)}{\partial_i\phi} + ...)(\overset{(1)}{\partial_j\phi} + \overset{(3)}{\partial_j\phi} + \overset{(5)}{\partial_j\phi} + ...) = -1
\end{aligned}$$

We will expand the above equation to second order only, hence we obtain:

$$-1 + \overset{(2)}{g^{00}} - 2\overset{(2)}{\partial_0\phi} + \delta^{ij}\overset{(1)}{\partial_i\phi}\overset{(1)}{\partial_i\phi} = -1$$

Hence the final equation which needs to be satisfied at second order is:

$$-\frac{1}{2}\overset{(2)}{g_{00}} + \frac{1}{2}|\overset{(1)}{\nabla\phi}|^2 = \overset{(2)}{\partial_0\phi}$$

Where we have used the fact that $\overset{(2)}{g_{00}} = -\overset{(2)}{g^{00}}$

Now we expand the second equation as follows:

$$\partial_0(\sqrt{-g}(G-T)g^{0\nu}\partial_\nu\phi) + \partial_i(\sqrt{-g}(G-T)g^{i\nu}\partial_\nu\phi) = 0$$

Note that the determinant can be expanded as:

$$-g = 1 + \overset{(2)}{g} + ...$$

where $\overset{(2)}{g}$ is of order v^2 and so on. Hence the square root can be expanded as:

$$\sqrt{-g} = \sqrt{1 + \overset{(2)}{g} + ...} = 1 + \frac{1}{2}\overset{(2)}{g} + ...$$

Meanwhile the term $g^{0\nu}\partial_\nu\phi$ can be expanded as:

$$g^{0\nu}\partial_\nu\phi = \Big(-1+ \overset{(2)}{g}{}^{00} + \overset{(4)}{g}{}^{00} + ...\Big)\Big(1+ \overset{(2)}{\partial_0\phi} + \overset{(4)}{\partial_0\phi} + ...\Big) + \Big(\overset{(3)}{g}{}^{0i}$$
$$+ \overset{(5)}{g}{}^{0i} + ...\Big)\Big(\overset{(1)}{\partial_i\phi} + \overset{(3)}{\partial_i\phi} + \overset{(5)}{\partial_i\phi} + ...\Big)$$

We only expand this to order 2:

$$g^{0\nu}\partial_\nu\phi = -1+ \overset{(2)}{g}{}^{00} + \overset{(2)}{\partial_0\phi} + ...$$

Meanwhile as for the $g^{i\nu}\partial_\nu\phi$ term:

$$g^{i\nu}\partial_\nu\phi = \Big(\overset{(3)}{g}{}^{0i} + \overset{(5)}{g}{}^{0i} + ...\Big)\Big(1+ \overset{(2)}{\partial_0\phi} + \overset{(4)}{\partial_0\phi} + ...\Big) + \Big(\delta^{ij}+$$
$$\overset{(2)}{g}{}^{ij} + \overset{(4)}{g}{}^{ij} + ...\Big)\Big(\overset{(1)}{\partial_j\phi} + \overset{(3)}{\partial_j\phi} + \overset{(5)}{\partial_j\phi} + ...\Big)$$

Finally we obtain:

$$g^{i\nu}\partial_\nu\phi = \overset{(1)}{\partial_i\phi} + (\overset{(3)}{\partial_i\phi} + \overset{(2)}{g}{}^{ij}\overset{(1)}{\partial_j\phi} + \overset{(3)}{g}{}^{i0}) + ...$$

Hence the full expansion is:

$$\partial_0\left(\Big(1+\frac{1}{2}\overset{(2)}{g} + ...\Big)\Big(\overset{(2)}{G-T} + ...\Big)\Big(-1+ \overset{(2)}{g}{}^{00} + \overset{(2)}{\partial_0\phi} + ...\Big)\right)$$
$$+\partial_i\left(\Big(1+\frac{1}{2}\overset{(2)}{g} + ...\Big)\Big(\overset{(2)}{G-T} + ...\Big)\Big(\overset{(1)}{\partial_i\phi} + (\overset{(3)}{\partial_i\phi} + \overset{(2)}{g}{}^{ij}\overset{(1)}{\partial_j\phi} + \overset{(3)}{g}{}^{i0}) + ...\Big)\right) = 0$$

To extract the lowest order term recall that spatial derivatives are weighed differently than time derivatives. The term $\partial_0\big(\overset{(2)}{G-T}\big)$ is clearly of order $\dfrac{v^3}{R^3}$. The product of $(\overset{(2)}{G-T})\overset{(1)}{\partial_i\phi}$ is of order $\dfrac{v^3}{R^2}$ and a spatial derivative introduces an extra factor of $1/R$, hence these two terms are of the same order; so to lowest order this equation reads:

$$\partial_0\,(\overset{(2)}{G-T}) = \partial_i\big((\overset{(2)}{G-T})\overset{(1)}{\partial_i\phi}\big)$$

This almost looks like a continuity equation and in fact will be whenever we interpret ϕ properly later on.

Now we expand the modified Einstein equation. To do that we first expand the $(G-T)\partial_\mu\phi\partial_\nu\phi$ term:

$$(G-T)\partial_0\phi\partial_0\phi = (\overset{(2)}{G-T} + \overset{(4)}{G-T})(1+ \overset{(2)}{\partial_0\phi} + \overset{(4)}{\partial_0\phi})(1+ \overset{(2)}{\partial_0\phi} + \overset{(4)}{\partial_0\phi})$$

hence it's easily seen that to lowest order

$$(G-T)\partial_0\phi\partial_0\phi = (\overset{(2)}{G-T}) + ...$$

Similarly the other terms to lowest order are:

$$(G-T)\partial_0\phi\partial_i\phi = (\overset{(2)}{G-T})\overset{(1)}{\partial_i\phi} +...$$

$$(G-T)\partial_i\phi\partial_j\phi = (\overset{(2)}{G-T})\overset{(1)}{\partial_i\phi}\overset{(1)}{\partial_j\phi} +...$$

Note that the space-space term starts at order 4 and hence won't contribute to lowest order field equations. Hence finally we can write down the field equations to lowest order; plugging in the expansion of the Einstein tensor previously derived we get the following:

$$\frac{1}{2}\overset{(2)}{R_{00}} + \frac{1}{2}\overset{(2)}{R_{jj}} + (\overset{(2)}{G-T}) = \overset{(2)}{T_{00}}$$

$$\overset{(3)}{R_{0i}} + (\overset{(2)}{G-T})\overset{(1)}{\partial_i\phi} = \overset{(3)}{T_{0i}}$$

$$\overset{(2)}{R_{ij}} + \frac{1}{2}(\overset{(2)}{R_{00}} - \overset{(2)}{R_{kk}})\delta_{ij} = \overset{(2)}{T_{ij}}$$

If we write these explicitly in terms of the metric (and note that $\overset{(2)}{T_{ij}} = 0$ from standard post-Newtonian theory) then:

$$-\frac{1}{4}\Delta\overset{(2)}{g_{00}} - \frac{1}{4}\Delta\overset{(2)}{g_{jj}} = \overset{(2)}{T_{00}} - (\overset{(2)}{G-T})$$

$$-\frac{1}{2}\Delta\overset{(3)}{g_{0i}} = \overset{(3)}{T_{0i}} - (G-T)\overset{(1)}{\partial_i\phi}$$

$$-\frac{1}{2}\Delta\overset{(2)}{g_{ij}} + \frac{1}{4}(-\Delta\overset{(2)}{g_{00}} + \Delta\overset{(2)}{g_{kk}})\delta_{ij} = 0$$

It is possible to further simplify the equations; taking the 3-trace of the last equation we obtain:

$$\frac{3}{4}\Delta \overset{(2)}{g}_{00} = \frac{1}{4}\Delta \overset{(2)}{g}_{kk}$$

plugging this back into the first equation we obtain:

$$\Delta \overset{(2)}{g}_{00} = - \overset{(2)}{T}_{00} + (G \overset{(2)}{-} T)$$

Finally, we arrive at the Post-Newtonian Mimetic equations as:

$$\Delta \overset{(2)}{g}_{00} = - \overset{(2)}{T}_{00} + (G \overset{(2)}{-} T)$$

$$-\frac{1}{2}\Delta \overset{(3)}{g}_{0i} = \overset{(3)}{T}_{0i} - (G - T)\, \overset{(1)}{\partial_i \phi}$$

$$\partial_0 \, (G \overset{(2)}{-} T) \; = \partial_i \Big(\, (G \overset{(2)}{-} T) \overset{(1)}{\partial_i \phi} \, \Big)$$

$$-\frac{1}{2} \overset{(2)}{g}_{00} + \frac{1}{2} | \overset{(1)}{\nabla \phi} |^2 = \overset{(2)}{\partial_0 \phi}$$

4.2 Interpreting The Equations

We will be working with the 4 equations above and hence to simplify notation we define $g_{00} := -2U$ $E := \frac{(G-T)}{2}$, and $\psi := -\phi$, all evaluated to lowest order.
Notice that if we take the gradient of the last equation we obtain:

$$\nabla\nabla\psi \cdot \nabla\psi + \partial_0 \nabla\psi = -\nabla U$$

Where $\nabla\nabla\psi$ is the Hessian of ψ and $\cdot$ is the vector tensor dot product. Notice that this is exactly the Euler equation for a presureless fluid with $\nabla\psi$ playing the role of a velocity field. The left hand side represents the convective derivative of the velocity field and the RHS is the gravitational force. Of course this could have also been derived by the conservation of energy momentum tensor (and expanding it to lowest order). We also have a continuity equation:

$$\partial_0 E = -\nabla \cdot (E \nabla \psi)$$

We see the somewhat expected final result that mimetic gravity is equivalent to Newtonian gravity coupled to an Eulerian pressureless fluid whose velocity field comes from a potential function. This fluid, however, is of course not part of any regular matter but may be thought of as an intrinsic component of gravity.

4.3 Some Interesting Implications

Let us re-write the 3 main equations (we drop the space-time Einstein equation since it plays no role in what follows) where we also reinstate relevant constant:

$$\Delta U = 4\pi G\rho - E \tag{4.1}$$

$$\partial_0 E = -\nabla \cdot (E\nabla\psi) \tag{4.2}$$

$$\frac{U}{c^2} + \frac{1}{2}\|\nabla\psi\|^2 = -\partial_0\psi \tag{4.3}$$

Notice that this is a system of 3 non-linear PDE's for 3 unknowns provided that ρ is given; if not then we would couple it to the standard hydrodynamic matter. So even if the matter density is given, one can't in general write a closed form solution for the potential U since it is also determined by E which in turn is coupled to ψ which in turn is coupled back to the potential U. We notice that if $E = 0$ then (4.2) is identically satisfied and the solution for U agrees with the standard GR/Newtonian solution. Then ψ is determined in terms of U via (4.3). In this case, ψ doesn't directly affect U neither the dynamics of particles in the Newtonian limit (which are only sensitive to U; this can be seen from the expansion of geodesic equation). Also the fact that E is subtracted from ρ is not particularly significant since in practice its sign is determined by an integration constant.

To obtain solutions not equivalent to GR we must have $E \neq 0$. Let us begin by assuming that E is time independent, then (4.2) implies that a general class of solutions for $E\nabla\psi$ is given by:

$$E\nabla\psi = \nabla \times \vec{A} \tag{4.4}$$

$\vec{A}$ is a free parameter of the theory and may be thought as representing the amount of mimetic matter as well as its direction (in the same spirit as when it was originally conceived to mimic cold dark matter in the original paper). Hence $\vec{A}$ immediately implies a preferred direction in space or a preferred frame. This is of course not surprising because there is a well known link between Mimetic gravity and Einstein aether theories as mentioned in the introduction. We will eventually be interested in applying the theory to explain the effects of dark matter on astrophysical scales. (4.4) also allows us to immediately write down the solution to g_{0i}:

$$\overset{(3)}{g}_{0i} = -2\int \overset{(3)}{T}_{0i} - (\nabla \times \vec{A})_i$$

Note that mimetic gravity's original success was attributed to it being able to incorporate cosmological dark matter into the geometry of space-time, so the natural question is whether it is able to do so on astrophysical scales. After all, the original evidence for dark matter came from analyzing the so called rotation curves of galaxies and it was realized that the visible matter couldn't account for the observations [12], hence the standard solution to the problem is to postulate the existence of an extra component of matter called dark matter (the other being to modify the gravitational theory itself, as for example is done in MOND). Mimetic gravity is able to incorporate dark matter into the geometry of space-time on cosmological scales where it is assumed that space-time (and hence, by extension, matter) is homogeneous and isotropic. This is a very special setting and possesses a very high degree of symmetry; when we go down to astrophysical scales (where Newtonian limit still applies), we observe that mimetic gravity is unable to incorporate the effects of dark matter into the mimetic fluid. To see this we first note that to obtain the required rotation curve on galactic scale it is required that the density profile of E to satisfy

$$E \propto \frac{1}{r^2}$$

outside the regular matter (where $\rho = 0$). From the continuity equation we see that

$$E\psi' \propto \frac{1}{r^2}$$

Hence we see that to get the required profile we must have $\psi' =$ constant. But (4.3) then implies that U is also a constant (assuming a static case). On the other hand, using (4.1) we get

$$U = -GM/r + \text{term linear in } r$$

Clearly, contradicting the existence of such a profile. So we see that on astrophysical scales mimetic gravity can't mimic the exact profile dark matter.

A natural question at this point would be the following: What kind of constraints can we put on the mimetic matter/fluid such that it agrees with solar system tests? The general method to answer such a question would be to compute the PPN parameters of the theory and relate them to the experimentally known values. Unfortunately, mimetic gravity doesn't seem to fit in the parametrized post-Newtonian framework as established, say, in [13], hence more direct methods are required. We can, however, say something about a static space-time. As mentioned in the introduction, previous work on static space-times in mimetic gravity assumed that the mimetic field is independent of time. In our work, we expanded ϕ around the solution $\phi = t$, hence it is not time-independent, but this is of no harm since one notices that only gradients of ϕ enter the equations, hence we can still study static space-times.

We find that any static solution would necessarily lead to a different Newtonian limit than standard GR. To see this, a static space-time implies the solution to (4.2) is given by (4.4) which implies that:

$$E^2\|\nabla\psi\|^2 = \|\nabla \times \vec{A}\|^2$$

Then (4.3) implies that (assuming $U < 0$ which it is in regular Newtonian case).

$$\|\nabla\psi\|^2 = -\frac{2U}{c^2} \tag{4.5}$$

Then assuming $E > 0$ (energy density of mimetic fluid is positive) we have

$$E = \frac{c\|\nabla \times \vec{A}\|}{\sqrt{-2U}}$$

Plugging this back in (4.1) we obtain:

$$\Delta U = 4\pi G\rho - \frac{c\|\nabla \times \vec{A}\|}{\sqrt{-2U}}$$

Clearly this will lead to a solution quite different than the standard Newtonian case, and the deviation could be thought of as being measured by the free parameter $\vec{A}$ (if $\vec{A}$ or $\nabla \times \vec{A} = 0$ we recover GR). Hence we conclude that all non-GR static solutions (with $E \neq 0$) necessarily lead to noticeable deviations from the standard Newtonian potential. This is to be contrasted to the Einstein vector theory where one can obtain a solution in which the aether field is in the direction of the killing vector ∂_t and the result would just be a modification of the effective gravitational constant [14]. But the proportionality constant between geometry and matter is determined by the Newtonian limit hence there's no contradiction with Newtonian theory. Such a thing is not possible in mimetic gravity because, among other things, a mimetic fluid being in the direction of the time-like Killing vector contradicts (4.3).

4.4 Lack of Asymptotic Flatness

Here we shall show that any static spherically space-time cannot be asymptotically flat . Let us take our domain to be $D = \{x \in R^3 : \|x\| \geq R\}$ for some radius R. We will prove the claim by a contradiction. Suppose that space-time is asymptotically flat; this implies

$U \to 0$ as $r \to \infty$; then (4.5) implies $\|\nabla\psi\| \to 0$ as well. So there exist a constant b and a radius R' such that $\frac{1}{\|\nabla\psi\|} \geq b$ for all $r \geq R'$[1].
Static space-time implies E is time-independent and hence

$$\nabla \cdot (E\nabla\psi) = 0$$

Using the divergence theorem, this implies that for any spherical surface of radius r (denoted by S_r) we have:

$$\oint_{S_r} E\nabla\psi \cdot ds = \oint_{S_R} E\nabla\psi \cdot ds = L$$

where L is a constant. If $r \geq R'$ and if E doesn't change sign (which is reasonable since E is an energy density analogue) then we have

$$|L| = \left| \oint_{S_r} E\nabla\psi \cdot ds \right| = \left|4\pi r^2 \psi' E\right|$$

Hence we have the following bound, for $r \geq R'$:

$$b|L| \leq \left|4\pi r^2 E\right|$$

Now we turn to U. By using Gauss law we have

$$U'(r)4\pi r^2 = \int_{\|x'\| \leq R'} (\rho - E)\, d^3x' - \int_{r \geq \|x'\| \geq R'} E\, d^3x'$$

$$U'(r) = \frac{1}{4\pi r^2}\left(\int_{\|x'\| \leq R'} (\rho - E)\, d^3x' - \int_{r \geq \|x'\| \geq R'} E\, d^3x' \right)$$

The first integral clearly leads to a contribution to U (not U') that vanishes at ∞ hence we concentrate on the second one:

$$\frac{1}{4\pi r^2}\left| \int_{r \geq \|x'\| \geq R'} E\, d^3x' \right| = \frac{1}{4\pi r^2}\left| \int_{R'}^{r} E 4\pi r'^2\, dr' \right| \geq \frac{b|L|}{4\pi r^2}(r - R') = \frac{b|L|}{4\pi r} - \frac{b|L|R'}{4\pi r^2}$$

Hence by integrating the above expression we see that as $r \to \infty$ we have:

$$|U(r)| \geq \frac{b|L|}{4\pi} \ln(r) \pm O(\frac{1}{r})$$

[1]Without loss of generality we assume that the matter density vanishes exterior to R'

leading to a contradiction. Essentially this says that the right hand side term E in (4.1) doesn't decay fast enough to have an asymptotically flat space-time. Or in pure Newtonian terms, the mimetic energy density E doesn't decay fast enough to have a vanishing potential at infinity.

This would imply that mimetic (static) black holes can't be asymptotically flat, since if they were, then in the asymptotic region the post Newtonian expansion would be valid. The argument above, however, shows that in the asymptotic region the contribution of the mimetic fluid is always significant enough to guarantee a non-asymptotically flat solution, which leads to a contradiction. This is in agreement with [6] [15] (although it should be noted that neither is working within the exact same framework as we are).

4.5 Static Spherically Symmetric Case: Approximate Solutions

Let us now study the physically most interesting situation where we assume static spherically symmetric space-time. As pointed out before, spherically symmetric space-times were already studied, but only with space-like constraints or complex scalar field. We study real solutions and seek to qualitatively explain rotation curves. Even though we already showed that an exact $1/r^2$ profile can't be obtained, the basic qualitative features can be obtained in the mimetic framework. Note that [11] [6] explained rotation curves through an additional potential term (as well as complex scalar field and/or space-like constraint), we seek to do so without any extra potential terms with a real mimetic field. A static space-time immediately implies that E should be time-independent and a function of r only, and that ψ is a function of r and possibly linear in t. We start by assuming it's independent of t. Then (4.2) implies:

$$\nabla \cdot (E\nabla\psi) = 0$$

If we assume $E\nabla\psi$ vanishes at infinity then we obtain:

$$E\nabla\psi = \frac{\alpha\vec{r}}{r^3} \tag{4.6}$$

where α is the free parameter of our theory. If we set it to zero this will immediately imply that E is zero (otherwise (4.3) would be inconsistent), hence we assume it's not zero and proceed. In that case (4.3) reads:

$$\frac{U}{c^2} + \frac{\psi'(r)^2}{2} = 0$$

Hence assuming $-U > 0$ we obtain:

$$\psi'(r) = \pm\frac{1}{c}\sqrt{-2U}$$

Plugging that in (4.4) we obtain:

$$E = -\frac{\alpha c}{r^2\sqrt{-2U}}$$

where the sign ambiguity has been absorbed into α (the sign of α is chosen so that $-E$ comes out positive because this is the only physically relevant case). Plugging this back into (4.1) we arrive at the equation for the potential U:

$$\Delta U = 4\pi G\rho + \frac{\alpha c}{r^2\sqrt{-2U}} \tag{4.7}$$

In general this has no closed form solution; hence we will obtain the solution numerically, but before we do so we can attempt to solve it approximately. To do this we first choose a characteristic length-scale R of the solar system (say 1AU). Then we notice that α has dimensions of v^2; intuitively it is an energy/mass flux free parameter which represents the amount of mimetic contribution to the background. Hence by dimensional analysis we can compare it to a quantity made up of characteristic mass and lengths of the system: $\frac{GM}{R}$. If we assume that

$$\alpha \ll \left(\frac{GM}{R}\right)^{\frac{3}{2}}\frac{1}{c} \tag{4.8}$$

then it is apparent that we can solve the equation perturbatively. It will also turn out that this approximate solution will exhibit the basic qualitative features of the more precise numerical solution. Hence we solve the equations perturbatively. At zeroth order we have the well known solution:

$$U = -\frac{GM}{r}$$

where

$$M = \int \rho(x) d^3x$$

Then we plug this solution back in (4.7) to obtain:

$$\Delta U = 4\pi G\rho + \frac{\alpha c}{r^2\sqrt{2\frac{GM}{r}}}$$

Since we are interested in the behavior far away from the sources (where $\rho = 0$) we obtain:

$$\Delta U = \frac{\alpha c}{r^2\sqrt{2\frac{GM}{r}}} = \frac{\alpha c}{\sqrt{2GMr^3}}$$

Since U is only a function of r we have:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) = \frac{\alpha c}{\sqrt{2GMr^3}}$$

Integrating twice we finally obtain the full solution as

$$U = -GM/r + \alpha c\sqrt{\frac{8r}{9GM}} \tag{4.9}$$

Clearly if α satisfies (4.8) then the second term is negligible on the solar system scale. In this way the mimetic fluid wouldn't affect the predictions on the solar system scale. However the hope is that it could play a role on galactic scales and explain the flat rotation curves.

If we now apply the theory to the galactic scale and attempt (at least qualitatively) to reconstruct the dark matter potential, then M would be the mass of the galaxy, and R would be the characteristic length of the galaxy (say, the radius of visible matter). Then the second term must be of the same order as the first term (of course, then the perturbative analysis would break down but we'll do a numerical simulation shortly which will reveal the same qualitative features) at this radius R, from this one can determine α:

$$\alpha \approx \left(\frac{GM}{R}\right)^{\frac{3}{2}}\frac{1}{c}$$

Then from (4.9) we can see that the second term will dominate beyond the radius R. To recover this result in a more rigorous manner one needs to perform a numerical simulation. Recall, however, that if ψ is time independent, then (4.3) implies that U can't be positive. However one needs U to have positive values to recover the correct galactic potential (with dark matter contribution). One can do this by assuming ψ linearly depends on t. This would still lead to a static space-time, since only derivatives of ψ appear in the equations. Hence if we assume that ψ is of the form $\psi(t,r) = -kt + f(r)$ then (4.3) would imply

$$f'(r) = \psi'(r) = \sqrt{\frac{-2U}{c^2} + 2k}$$

then the modified Poisson equation to be solved is

$$\Delta U = 4\pi G\rho + \frac{\alpha}{r^2\sqrt{\frac{-2U}{c^2} + 2k}} \tag{4.10}$$

k is dimensionless and must be of order $\frac{v^2}{c^2}$ to be consistent with the expansion. To get rid of c let us redefine the variables $\alpha \to \alpha c$, so now α has dimensions of v^3 and $k \to kc^2$ so now k has dimensions of speed squared and must be of order v^2, then the above equation becomes

$$\Delta U = 4\pi G\rho + \frac{\alpha}{r^2\sqrt{-2U + 2k}} \tag{4.11}$$

We will solve this equation numerically. To do so we make the following further assumption: We assume that inside the radius R only the first term contributes and so the second term is actually multiplied by a step function. Another way of saying this is as follows: We choose units in which $GM = 1$ and $R = 1$. Then we will simulate the equation

$$\Delta U = \frac{\alpha}{r^2\sqrt{-2U + 2k}} \tag{4.12}$$

with the boundary condition $U(1) = -1$, $U'(1) = 1$

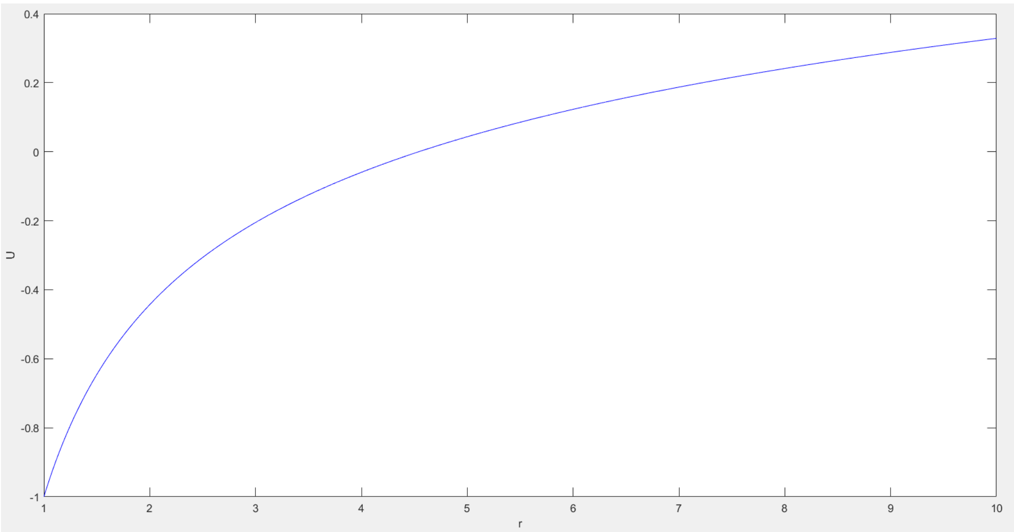

Figure 4.1: Numerical method 4th order Runge Kutta was used, with max step size 0.0001. The domain was taken to be [1,10] and the boundary conditions were $U(1) = -1$, $U'(1) = 1$ in accordance with the standard Newtonian solution. $k = 5$, $\alpha = 1$.

Note that when we choose $k = 5$ what we actually mean is $k = 5\frac{GM}{R}$. We can also plot the energy density and compare it with a $\frac{1}{r^2}$ density.

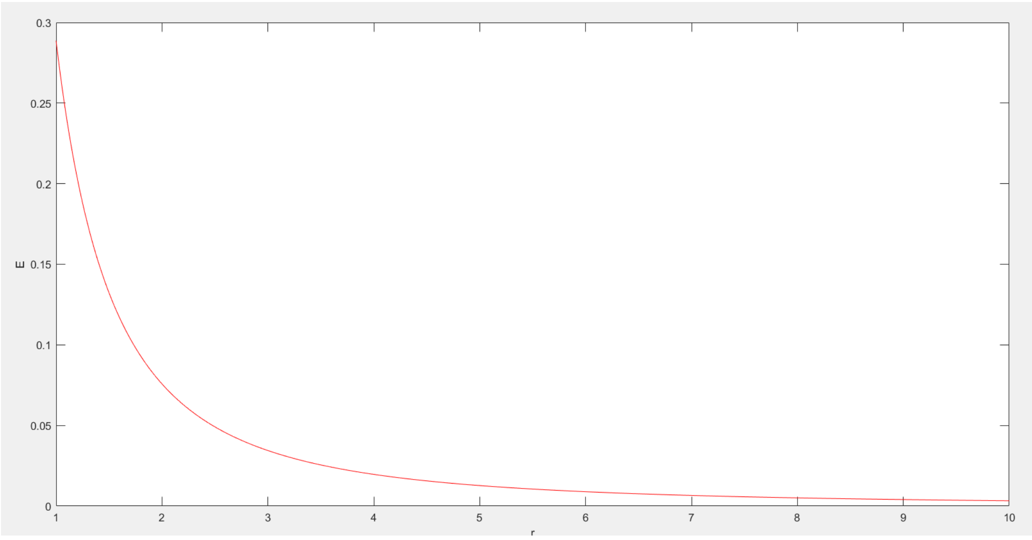

Figure 4.2: Plot of E vs r with domain size [1,10]. Same Boundary conditions as above.

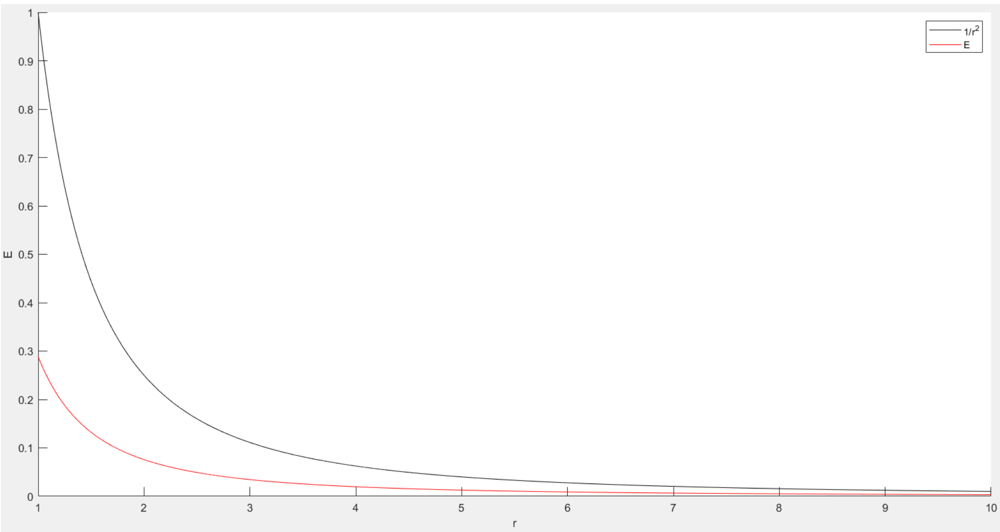

Figure 4.3: Comparison of E with an inverse square density on a [1,10] domain.

If we choose a bigger domain then the comparison would be as follows:

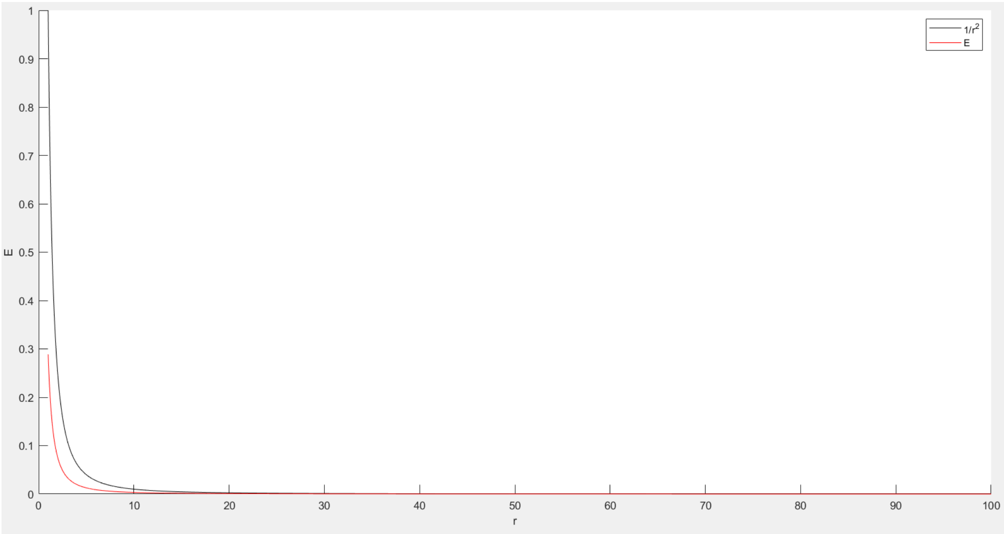

Figure 4.4: Comparison of E with an inverse square density with [1,100] domain.

As for the potential, choosing an even bigger domain reveals that this extra potential term has essentially a logarithmic dependence at large r:

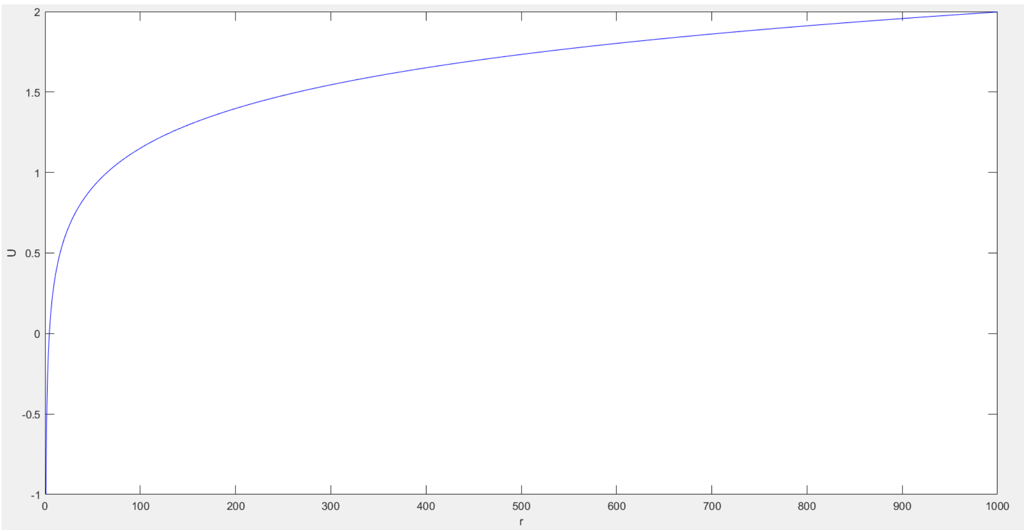

Figure 4.5: The domain was taken to be [1,1000] and the boundary conditions were $U(1) = -1$, $U'(1) = 1$. k=5, $\alpha = 1$.

A direct comparison shows that it actually varies slower than a logarithm but faster than the square root of a logarithm. A logarithmic term is exactly what one needs to explain flat rotation curves. Once the potential is computed, one can solve for the velocity profile by equating the centrifugal force/acceleration to the gradient of the potential.

Hence, we recover the basic qualitative features of the rotation curves by fixing particular boundary conditions.

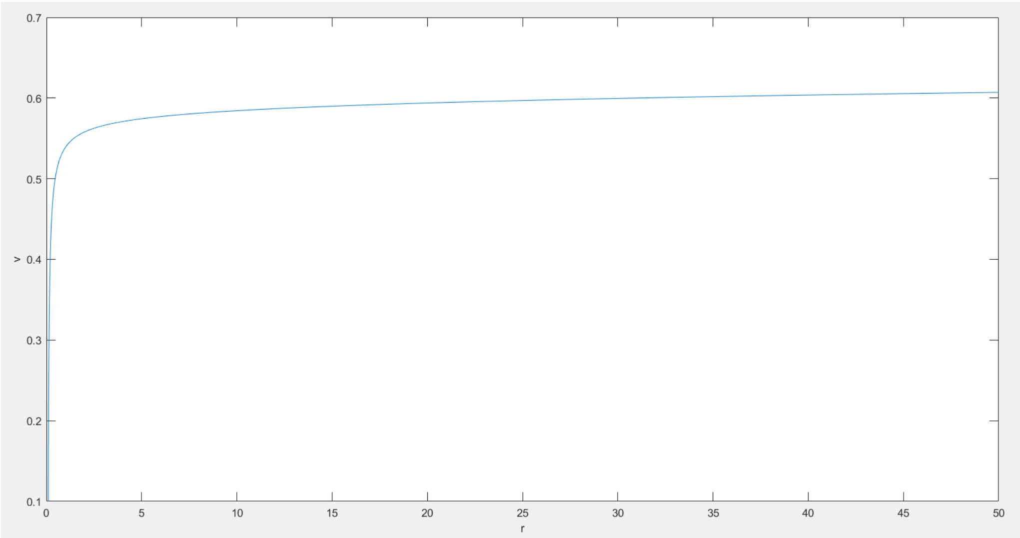

Figure 4.6: galaxy rotation curves. The domain was taken to be [0.1,50], the boundary conditions used are $U(0.1) = 0.1$, $U'(0.1) = -0.1$.

Fixing the boundary conditions amounts to fixing the amount of mimetic contributions inside the radius $r = 0.1$. Here is the plot of the same boundary conditions with various values for α.

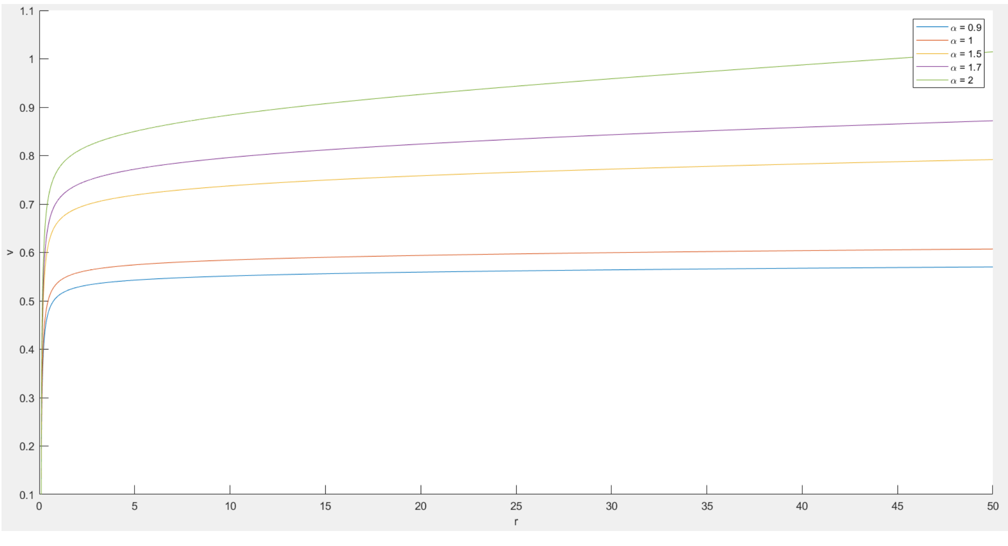

Figure 4.7: galaxy rotation curves. The domain was taken to be [0.1,50], the boundary conditions used are $U(0.1) = 0.1$, $U'(0.1) = -0.1$.

Chapter 5

Conclusion and Future Work

We carried out the post-Newtonian expansion in Mimetic gravity. We were then able to derive the Newtonian limit of the theory. The mimetic contribution is characterized by an energy flux parameter; using this parameter we sought to mimic the effects of dark matter on astrophysical scales. We are able to recover the Newtonian solution as well as qualitatively explain flat rotation curves on galactic scales. To account precisely for the effects of dark matter one needs a logarithmic term in the potential; we show that even though such a term can't be obtained as an exact solution to the equations, by using numerical simulations one can obtain a quasi-logarithmic term in the potential. In this way, we are able to recover the flat rotation curves. We also prove a theorem that all static spherically symmetric solutions with non trivial mimetic contributions are not asymptotically flat.

To get an even better picture of where mimetic gravity stands regarding the galactic rotation curves, one needs to compare the numerical simulation curves to the most recent dark matter profiles/velocity curves coming from observational data. Note that one also has to solve the equations of mimetic gravity starting from inside the visible galactic matter (we only did so starting from the outside).

Bibliography

[1] N. Straumann, *General relativity*. Springer Science & Business Media, 2012.

[2] V. Fock, *The theory of space, time and gravitation*. Elsevier, 2015.

[3] A. H. Chamseddine and V. Mukhanov, "Mimetic dark matter," *Journal of High Energy Physics*, vol. 2013, no. 11, pp. 1–5, 2013.

[4] A. Golovnev, "On the recently proposed mimetic dark matter," *Physics letters B*, vol. 728, pp. 39–40, 2014.

[5] E. A. Lim, I. Sawicki, and A. Vikman, "Dust of dark energy," *Journal of Cosmology and Astroparticle Physics*, vol. 2010, no. 05, p. 012, 2010.

[6] L. Sebastiani, S. Vagnozzi, R. Myrzakulov, *et al.*, "Mimetic gravity: A review of recent developments and applications to cosmology and astrophysics," *Advances in High Energy Physics*, vol. 2017, 2017.

[7] A. H. Chamseddine, V. Mukhanov, and A. Vikman, "Cosmology with mimetic matter," *Journal of Cosmology and Astroparticle Physics*, vol. 2014, no. 06, p. 017, 2014.

[8] A. H. Chamseddine and V. Mukhanov, "Resolving cosmological singularities," *Journal of Cosmology and Astroparticle Physics*, vol. 2017, no. 03, p. 009, 2017.

[9] A. H. Chamseddine and V. Mukhanov, "Nonsingular black hole," *The European Physical Journal C*, vol. 77, pp. 1–9, 2017.

[10] R. Myrzakulov and L. Sebastiani, "Spherically symmetric static vacuum solutions in mimetic gravity," *General Relativity and Gravitation*, vol. 47, pp. 1–12, 2015.

[11] R. Myrzakulov, L. Sebastiani, S. Vagnozzi, and S. Zerbini, "Static spherically symmetric solutions in mimetic gravity: Rotation curves and wormholes," *Classical and quantum gravity*, vol. 33, no. 12, p. 125 005, 2016.

[12] A. Arbey and F. Mahmoudi, "Dark matter and the early universe: A review," *Progress in Particle and Nuclear Physics*, vol. 119, p. 103 865, 2021.

[13] C. M. Will, *Theory and experiment in gravitational physics*. Cambridge university press, 2018.

[14] S. M. Carroll and E. A. Lim, "Lorentz-violating vector fields slow the universe down," *Physical Review D*, vol. 70, no. 12, p. 123 525, 2004.

[15] M. A. Gorji, A. Allahyari, M. Khodadi, and H. Firouzjahi, "Mimetic black holes," *Physical Review D*, vol. 101, no. 12, p. 124 060, 2020.