AMERICAN UNIVERSITY OF BEIRUT

ANALYTIC DISKS AND MAPPING PROBLEMS FOR REAL HYPERSURFACES IN COMPLEX SPACES

by
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A thesis
submitted in partial fulfillment of the requirements
for the degree of Master of Science
to the Department of Mathematics
of the Faculty of Arts and Sciences
at the American University of Beirut

Beirut, Lebanon
January 2024

AMERICAN UNIVERSITY OF BEIRUT

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Date of thesis defense: January 22, 2024

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Acknowledgements

First, I would like to sincerely thank my advisor Dr. Giuseppe Della Sala. This thesis is the culmination of numerous meetings with and suggestions by Dr. Della Sala, and he played an essential role in breaking things down for me and handing me the appropriate tools for navigating this project. His continuous encouragement and insightfulness were salient to making this a success.
I would also like to thank Dr. Florian Bertrand for his illuminating role in my MS journey. He was always and without hesitation supportive, helpful, and reassuring.
I would like to thank Dr. Farouk Abi Khuzam as well for being a member of this thesis' committee.
My parents, and my siblings Dr. Fatima, Ali, and Hassan, my most dear and beloved, I dedicate this thesis to you. Thank you for continuing to bestow great meaning upon my life.

Abstract of the Thesis of

Ahmad Hisham Hussein for Master of Science
Major: Mathematics

Title: Analytic Disks and Mapping Problems for Real Hypersurfaces in Complex Spaces

The thesis will review the theory of certain invariant objects associated to CR submanifolds of $\mathbb{C}^n$, especially families of attached analytic disks, and their implications for the study of the properties of holomorphic maps. Particular attention will be given to the dynamics of those invariants on the sphere in $\mathbb{C}^2$.

Table of Contents

List of Illustrations

Chapter 1

Introduction

The domain of Complex Analysis is a rich and an ongoing field of research, primarily known and notable for its essentially different nature from Real Analysis. There is a flavour of rigidity to the former that is not present in the latter. This flavour is intriguingly present enough to force things like infinite differentiability (even being analytic) to be equivalent to mere (one-time) differentiability, but also permissive enough to allow for a variegated theory, especially when one considers more than one variable.

The theory of Several Complex Variables (SCV) is yet on a different level. Many results in one complex variable do not generalize to SCV in the manner one would expect initially, and much of the motivation for SCV is this point.

One famous example of this is the Riemann mapping theorem, which we discuss in Chapter 2. The inapplicability of this theorem in SCV opens questions as to what sort of weaker yet worthwhile rigidity results one might have in SCV. This, together with results such as Hartog's theorem, motivates introduction of notions such as domains of holomorphy and classifications such as pseudoconvexity.

CR Geometry, grounded in SCV, is most broadly the study of real hypersurfaces in complex spaces. More particularly, it concerns the general question of what can be said of functions defined on such hypersurfaces with the understanding that their ambient space is $\mathbb{C}^n$. Topics of CR Geometry are numerous, and our focus in this thesis are the notions of stationary disks and automorphisms of real hypersurfaces in the complex spaces $\mathbb{C}^n$.

Stationary disks historically proved useful in various contexts. In this thesis, this usefulness is reiterated in recovering particular CR automorphisms of hypersurfaces that "locally look like a sphere". To do that, we primarily rely on the dynamics between said automorphisms and the stationary disks which are attached to these hypersurfaces.

We consider the paraboloid $\Re w = |z|^2$ in $\mathbb{C}^2$ as our model, representing otherwise strongly pseudoconvex hypersurfaces. We consider the stationary disks attached to this paraboloid and passing through the origin at $\zeta = 1$. Our starting point is demonstrating the necessary and sufficient dependence of these disks on their second jets at $\zeta = 1$. Our next point is considering a random automorphism and pushing the disks forward by the 2-jets of this automorphism at the origin. By imposing holomorphy, we then parameterize the automorphism by said 2-jets. As such, we completely nail down the desired automorphisms and prove that they are equivalent to the ones already present in the literature by other means.

The content of Chapter 2 is more or less routine preliminaries, motivations for SCV, and familiarizing the reader with our notations. Chapter 3 is intended as a classical summary of basic Differential Geometry and CR Geometry, as well as motivation for our work and some background in similar problems.

Chapter 4 is mostly about studying the stationary disks attached to our model, and the first section of Chapter 5 is where we parameterize our automorphisms and conclude. In particular, Corollary 5.1.1 summarizes the results of the computations, and the two corollaries 5.1.2 and 5.1.3 showcase the back and forth between the obtained automorphisms and the ones already in the literature, specifically as listed on page 19 of [9] (with slight modifications to accommodate our notation).

In the second section of Chapter 5, we very briefly describe some suggestions for future work.

Chapter 2

Prelude

The aim of this chapter is to introduce the reader to the basics of Several Complex Variables and recall various notions that will be needed throughout this thesis. We will assume that the reader is fairly familiar with one-variable Complex Analysis, as well as the basics of Real Analysis, Topology, and Linear Algebra. **The reader who is familiar with Several Complex Variables can just skim through this chapter**.

2.1 Analysis and Calculus Notes and Notations

In this section, we review some basics of Metric Topology and Calculus, particularly on vector spaces. We lay out the definitions and notations as well as a few tools that we need.

2.1.1 *Metric, Normed, and Hermitian Spaces*

This subsection is intended as a quick refresher and a chance to introduce some notations.

Definition 2.1.1. *Let (X, d) be a metric space, $a \in X$ and $r > 0$. The open ball in X of center a and radius r is defined to be $B(a, r) = \{x \in X, d(x, a) < r\}$, and a set $O \subset X$ is called open if for each $a \in O$, there is some $\epsilon > 0$ such that $B(a, \epsilon) \subset O$. A closed set is a set whose complement is open.*

Definition 2.1.2. *Let X be a metric space. A neighborhood of a point $x \in X$ is an open set $O \subset X$ such that $x \in O$. The interior of a a subset A of X, denoted by A°, is the set of all $x \in A$ such that x has a neighborhood O with $O \subset A$.*

Remark 2.1.1. *A set C in a metric space (X, d) is closed if and only if for each $(x_n) \subset C$, if (x_n) is convergent in X, then $\lim x_n \in C$.*

Notation 2.1.1. *Given a metric space X and a subset A of X, we denote by A' the set of all limit points of A. In other words,*

$$A' := \{x \in X, \forall \epsilon > 0, B(x, \epsilon) \cap A \setminus \{x\} \neq \varnothing\}$$

Recall that, with this notation, the topological closure of A is simply $\overline{A} = A \cup A'$, and $\partial A = \overline{A} \setminus A^\circ$.

Definition 2.1.3. *Let (X, d) be a metric space. A subset A of X is called dense in X if $\overline{A} = X$. Equivalently, A is dense in X if for each open set $O \subset X$, we have $A \cap O \neq \varnothing$.*

Definition 2.1.4. *Let (X, d) be a metric space. A sequence $(x_n) \subset X$ is called a Cauchy sequence if $\lim_{n,m} d(x_n, x_m) = 0$. In other words, (x_n) is Cauchy if it holds that $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N, d(x_n, x_m) < \epsilon$. A metric space (X, d) in which every Cauchy sequence is convergent is called a complete metric space.*

If $(X, \|\cdot\|)$ is a normed vector space (nvs), then $\|\cdot\|$ induces a metric on X given by $d(x, y) = \|x - y\|$. This metric in turn induces a topology on X, the metric topology associated to the norm $\|\cdot\|$. This structure induced by the norm allows for notions of convergence, openness, continuity, etc. In particular, in this context, a subset U of X is open if and only if:

$$\forall x \in U, \exists \epsilon > 0, B(x, \epsilon) = \{y \in X : \|y - x\| < \epsilon\} \subset U$$

Recall that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space X are said to be equivalent if there are $a, b > 0$ such that for all $x \in X$:

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$$

Two equivalent norms generate the same topology on the given vector space.

Theorem 2.1.1. *All norms on a finite dimensional vector space are equivalent, and any two finite dimensional vector spaces of the same dimension are homeomorphic in the topology induced by any choice of a norm. In particular, the Euclidean topological spaces $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ are homeomorphic.*

Definition 2.1.5. *Let $(X, \|\cdot\|)$ be a nvs. We call $(X, \|\cdot\|)$ a Banach space if (X, d) is a complete metric space, where d is the metric induced by $\|\cdot\|$.*

Definition 2.1.6. *A Hermitian form on a given vector space V is a functional $h : V \times V \to \mathbb{C}$ which is $\mathbb{C}$-linear in the first component and satisfies conjugate symmetry, i.e. satisfies $h(x, y) = \overline{h(y, x)}$ for all $x, y \in V$. If V is a vector space and h is a Hermitian form on V, then (V, h) is called a Hermitian space.*

Definition 2.1.7. *An inner product $\langle \cdot, \cdot \rangle$ on a vector space V is a positive definite Hermitian form. In other words, an inner product on V is a Hermitian form h on V which satisfies $h(x, x) > 0$ for all $x \in V \setminus \{0\}$. If $\langle \cdot, \cdot \rangle$ is an inner product on V, then $(V, \langle \cdot, \cdot \rangle)$ is called an inner product space.*

Remark 2.1.2. *If $(X, \langle \cdot, \cdot \rangle)$ is an inner product space, then $\langle \cdot, \cdot \rangle$ induces a norm on X given by $\|x\| = \sqrt{\langle x, x \rangle}$.*

Definition 2.1.8. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space. We call $(H, \langle \cdot, \cdot \rangle)$ a Hilbert space if the nvs $(H, \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is the norm induced by $\langle \cdot, \cdot \rangle$.*

Definition 2.1.9. *Given a metric space X, we say that X is disconnected if there are two non-empty disjoint open sets $A, B \subset X$ such that $X = A \cup B$, and we say that X is connected if X is not disconnected.*

Definition 2.1.10. *Let X be a metric space. We say that X is compact if every open cover of X admits a finite subcover. Equivalently, X is compact if every sequence in X has a convergent subsequence.*

Theorem 2.1.2. *(Heine-Borel theorem)*
A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Theorem 2.1.3. *(Extreme Value theorem)*
Let X be a compact metric space, and let $f : X \to \mathbb{R}$ be continuous. Then, f admits both a maximum and a minimum on X, i.e. there are $x_0, x_1 \in X$ such that $f(x_0) = \inf_{x \in X} f(x)$ and $f(x_1) = \sup_{x \in X} f(x)$.

Definition 2.1.11. *Let X be any set and (Y, d) be a metric space. A sequence of functions (f_n) from X to Y is said to converge pointwise to a function $f : X \to Y$ if for each $x \in X$, the sequence $(f_n(x))$ converges to $f(x)$. Also, (f_n) is said to converge uniformly to f if $\forall \epsilon > 0, \exists N \geq 1, \forall n \geq N, \forall x \in X, d(f_n(x), f(x)) < \epsilon$. If Y is a nvs, we may talk of a series of functions from X to Y, $\sum_{n \geq 1} f_n$, and such a series is pointwise (resp. uniformly) convergent if its sequence of partial sums is pointwise (resp. uniformly) convergent.*

Definition 2.1.12. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be nvs. A linear map $L : X \to Y$ is said to be bounded if there exists some $M > 0$ such that for all $x \in X$, we have $\|L(x)\|_Y \leq M\|x\|_X$.*

Proposition 2.1.1. *Let X and Y be nvs and let $L : X \to Y$ be a linear map. Then, L is continuous on X if and only if L is bounded (and this is also equivalent to L being continuous at 0).*

Remark 2.1.3. *Given two vector spaces X and Y, the set of all linear functions from X to Y is a vector space under the usual addition of functions and scalar multiplication.*

Notation 2.1.2. *Given two nvs X and Y, we denote by $\mathcal{L}(X, Y)$ the set of all **bounded** (i.e. continuous) linear operators from X to Y.*

Proposition 2.1.2. *The set $\mathcal{L}(X, Y)$ with the usual addition of functions and scalar multiplication is a vector space. Furthermore, we may define $\|\cdot\|$ on $\mathcal{L}(X, Y)$ by:*

$$\|L\| := \sup_{x \in X} \frac{\|L(x)\|_Y}{\|x\|_X}$$

This $\|\cdot\|$ is a norm, called the operator norm, and so $(\mathcal{L}(X, Y), \|\cdot\|)$ is a nvs.

Definition 2.1.13. *Let $X_1, \ldots, X_n$ and Y be vector spaces. A map $\varphi : \prod_{i=1}^n X_i \to Y$ is called an n-linear map if φ is linear in each component.*

Remark 2.1.4. *Let $X_1, \ldots, X_n$ and Y be nvs. An n-linear map $\varphi : \prod_{i=1}^{n} X_i \to Y$ is called bounded if there is $M > 0$ such for all $x_i \in X_i$, we have:*

$$\|\varphi(x_1, \ldots, x_n)\|_Y \leq M \prod_{i=1}^{n} \|x_i\|_{X_i}$$

Also, the set of all n-linear maps from $\prod_{i=1}^{n} X_i$ to Y is a vector space with respect to the obvious operations and actions, and one denotes this vector space by $\mathcal{L}(X_1, \ldots, X_n; Y)$.
Finally, if one puts, for $\varphi \in \mathcal{L}(X_1, \ldots, X_n; Y)$,

$$\|\varphi\| := \sup_{x_i \in X_i} \frac{\|\varphi(x_1, \ldots, x_n)\|_Y}{\|x_1\|_{X_1} \cdots \|x_n\|_{X_n}}$$

then one gets a norm on $\mathcal{L}(X_1, \ldots, X_n; Y)$ (the operator norm).

2.1.2 *Differentiability: Main Definitions and Tools*

Derivatives as multilinear maps, relevant terminologies, Taylor expansion, and a few classical theorems.

Definition 2.1.14. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be nvs, let $U \subset X$ and let $f : U \to Y$ be a map. We say that f is (Fréchet) differentiable at a point $x_0 \in U$ if there exists an $L \in \mathcal{L}(X, Y)$ such that:*

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - L(x - x_0)\|_Y}{\|x - x_0\|_X} = 0$$

or, equivalently,

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - L(h)\|_Y}{\|h\|_X} = 0$$

Remark 2.1.5. *This condition can also be reformulated using the $\epsilon - \delta$ definition as: $\forall \epsilon > 0, \exists \delta > 0, \forall h \in X, \|h\|_X < \delta \implies \|f(x_0 + h) - f(x_0) - L(h)\|_Y < \epsilon \|h\|_X$. We can also write it in Landau notation as: $f(x_0 + h) = f(x_0) + L(h) + o(h)$.*

Remark 2.1.6. *In the above definition, if such an L exists, then it is unique.*

Definition 2.1.15. *With the same notation as above, this unique L is called the (Fréchet) derivative of f at x_0, and it is denoted by $Df(x_0)$.*

Definition 2.1.16. *Let X and Y be nvs, let $U \subset X$ be open, and let $f : U \to Y$. We say that f is differentiable on U if f is differentiable at each point in U. If this is the case, we denote by Df the map $U \to \mathcal{L}(X, Y)$ associating to each $x \in U$ the map $Df(x)$ defined above. This map Df is called the derivative map of f.*

Remark 2.1.7. *The map Df is defined on U and lands in $\mathcal{L}(X, Y)$ which is itself a nvs, and we can talk of the continuity and differentiability of Df with respect to these nvs.*

Definition 2.1.17. *With the same setting as above, and assuming that f is differentiable on U, so that the derivative map $Df : U \to \mathcal{L}(X,Y)$ is defined, we say that f is twice differentiable at a point $x_0 \in U$ (resp. on U) if Df is differentiable at x_0 (resp. on U). In this case, the second derivative of f at x_0 is then defined to be $D^2 f(x_0) = D(Df)(x_0)$. If the second derivative exists on all of U, we can then define the second derivative map $D^2 f : U \to \mathcal{L}(X, \mathcal{L}(X,Y))$. One can then inductively define the notion of n-times differentiability and the n-th derivative of f at x_0.*

Definition 2.1.18. *With the same preceding notation, we say that f is of class C^k at a point $x_0 \in U$ (resp. on U) if f is k-times differentiable on a neighborhood of x_0 (resp. on U) and $D^k f$ is continuous at x_0 (resp. on U). We say that f is of class C^∞ at x_0 (resp. on U) if for all $n \in \mathbb{N}$, the n-th derivative of f exists at x_0 (resp. on U).*

Terminology 2.1.1. *Let X and Y be nvs, and let $f : X \to Y$ be a map. We say that f is smooth on X if f is of class C^∞ on X.*

Remark 2.1.8. *Per our notations, $\mathcal{L}(X,Y;Z)$ denotes the vector space of all bounded bilinear maps $X \times Y \to Z$. We note that $\mathcal{L}(X, \mathcal{L}(Y,Z)) \cong \mathcal{L}(X,Y;Z)$ (through an isometric isomorphism), and so, one can look at the second derivative map as a map which outputs bilinear maps $X \times X \to Y$. More generally, the n-th derivative is a function that outputs multilinear maps $X^n \to Y$.*

Theorem 2.1.4. *(Taylor's theorem)*
Let X and Y be nvs, $U \subset X$ be open, $x_0 \in U$, and let $f : U \to Y$ be of class C^{n+1} on U. If the segment connecting x_0 to $x_0 + h$, $S(x_0, x_0+h)$, lies in U, then there is some $\xi \in S(x_0, x_0+h)$ such that:

$$f(x_0+h) = \sum_{k=0}^{n} \frac{1}{k!} D^k f(x_0) h^k + \frac{1}{(n+1)!} D^{n+1} f(\xi) h^{n+1}$$

where $D^k f(x_0) h^k$ denotes $D^k f(x_0)(h, h, \ldots, h)$ (with the multilinear notation).

Remark 2.1.9. *Another widely-recognized way to write Taylor's formula is in the integral form:*

$$f(x_0+h) = \sum_{k=0}^{n} \frac{1}{k!} D^k f(x_0) h^k + \int_0^1 \frac{(1-t)^n}{n!} D^{n+1} f(x_0 + th) h^{n+1} dt$$

Also, consequently, with the same assumptions, one may write:

$$f(x_0+h) = \sum_{k=0}^{n} \frac{1}{k!} D^k f(x_0) h^k + o(h^n)$$

Note as well that if one puts $x = x_0 + h$, then one may write:

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} D^k f(x_0)(x-x_0)^k + o\left((x-x_0)^n\right)$$

Terminology 2.1.2. *With the same assumptions as in Theorem 2.1.4 and the same notation as in Remark 2.1.9, the sum:*

$$\sum_{k=0}^{n} \frac{1}{k!} D^k f(x_0)(x-x_0)^k$$

is called the n-th ***Taylor polynomial*** *of f. Moreover, if f is smooth at x_0 and if the series of partial sums given by the Taylor polynomials of f at x_0 converges pointwise to f in a neighborhood of x_0, then one says that f is* ***analytic*** *at x_0, and the sum:*

$$\sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x_0)(x-x_0)^n$$

is called the ***Taylor series*** *(or expansion) of f at the point x_0.*

Theorem 2.1.5. *(Chain Rule)*
Let X, Y, Z be three nvs. Let $U \subset X$ be open, $V \subset Y$ be open, $f : U \to Y$ and $g : V \to Z$, and suppose that $f(U) \subset V$. Let $x_0 \in U$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0, and one has:

$$D(g \circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0)$$

Consequently, as well, if f is invertible onto $f(U)$ and differentiable at x_0, and if $f^{-1} : f(U) \to U \subset X$ is differentiable at $f(x_0)$, then $Df(x_0)$ is invertible, and one has:

$$Df^{-1}(f(x_0)) = [Df(x_0)]^{-1}$$

Definition 2.1.19. *Let X and Y be nvs and $U \subset X$ be open. Let $f : U \to Y$ be a map. We say that f is a diffeomorphism if f is differentiable, bijective, and $f^{-1} : Y \to U \subset X$ is differentiable. Similarly, f is a C^k-diffeomorphism, for $k \in \mathbb{N} \cup \{\infty\}$, if f is of class C^k, bijective, and f^{-1} is of class C^k.*

Proposition 2.1.3. *Let $\{(X_i, \|\cdot\|_i)\}_{i=1}^n$ be a collection of nvs and let $X = \prod_{i=1}^n X_i$. The functional $\|\cdot\|_\infty$ defined on X by $\|(x_1, \ldots, x_n)\|_\infty = \max_{1 \le i \le n} \|x_i\|_i$ is a norm on X, called the maximum norm.*

Proposition 2.1.4. *With the same set-up as the preceding proposition, let $(Y, \|\cdot\|)$ be a nvs, $U \subset X$ open, and $f : U \to Y$ be differentiable at $a = (a_1, \ldots, a_n) \in U$. There's a $\delta > 0$ such that $B_\infty(a, \delta) \subset U$. In particular, $\prod_{i=1}^n B_i(a_i, \delta) \subset U$. Given i, the map $e_i : \xi \in B_i(a_i, \delta) \mapsto f(a_1, \ldots, a_{i-1}, \xi, a_{i+1}, \ldots, a_n)$ is differentiable at a_i, and we introduce the notation: $\dfrac{\partial f}{\partial x_i}(a) := De_i(a_i)$, which is what we call the "partial derivative of f with respect to x_i". We also write $D_i f(a) = De_i(a_i)$.*

Proposition 2.1.5. *With the same preceding set-up, we have that for every point $(h_1, \ldots, h_n) \in X$, $Df(a)(h_1, \ldots, h_n) = \sum_{i=1}^n \dfrac{\partial f}{\partial x_i}(a) h_i$. Also, consequently, given i, we have for all $h \in X_i$, $\dfrac{\partial f}{\partial x_i}(a)h = Df(a)(0, \ldots, 0, h, 0, \ldots, 0)$ (where h is at the ith position).*

Proposition 2.1.6. *Let X, Y and U be as above and let $a \in U$. Suppose that the partial derivatives of f exist and are continuous on a neighborhood of a. Then, f is differentiable at a.*

Theorem 2.1.6. *(Inverse Function theorem)*
Let X and Y be Banach spaces, and let $U \subset X$ be open. Let $f : U \to Y$ be of class C^1. Let $x_0 \in U$ and suppose that $Df(x_0)$ is an isomorphism. Then, there exists an open neighborhood $V \subset U$ of x_0 and an open neighborhood W of $f(x_0)$ such that f is a C^1-diffeomorphism from V onto W.

Theorem 2.1.7. *(Implicit Function theorem)*
Let X, Y, Z be three Banach spaces, let $U \subset X \times Y$ be open, and let $f : U \to Z$ be a map of class C^1. Let $(a, b) \in U$ with $f(a, b) = 0$. Then, there exists an open neighborhood $V \subset U$ of (a, b), there exists an open neighborhood W of a in X, and there exists a map $g : W \to Y$ of class C^1 such that:

$$\{(x, y) \in W \times Y, y = g(x)\} = \{(x, y) \in V, f(x, y) = 0\}$$

In other words, $(x_0, y_0) \in V$ is a solution of $f(x, y) = 0$ if and only if $x_0 \in W$ and $y_0 = g(x_0)$.

2.2 Notions of Several Complex Variables

Here, we delve directly into the theory of several complex variables. We mention some of the major theorems, and we try to provide some motivation for this theory.

2.2.1 *Basic Definitions and Notes*

The complex space $\mathbb{C}$ is defined to be the $\mathbb{R}$-algebra $(\mathbb{R}^2, +, \cdot)$, with the addition operation given by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, and the multiplication given by $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$. We define $i = (0, 1)$ and $1 = 1_{\mathbb{C}} = (1, 0)$. Then, $\{1, i\}$ forms a basis for $\mathbb{C}$ as an algebra over $\mathbb{R}$, and a complex number $z \in \mathbb{C}$ has then a unique expression $z = x + iy$, $x, y \in \mathbb{R}$.

Notation 2.2.1. *We denote by $\mathbb{C}[z]$ the ring of all polynomials in z with complex number coefficients, and we denote by $\deg(P)$ the degree of a given non-zero polynomial P.*

Theorem 2.2.1. *(Fundamental Theorem of Algebra)*
Let $P \in \mathbb{C}[z]$ be non-constant. Then, P has exactly $\deg(P)$ roots (including the possibility of repeated roots) in $\mathbb{C}$.

Notation 2.2.2. *Given $z = x+iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, we denote the "complex conjugate" of z by $\overline{z} = x - iy$. Also, throughout this section and elsewhere, whenever we write $z_j = x_j + iy_j$ for a given $z \in \mathbb{C}^n = \underbrace{\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}}_{n \text{ times}}$, it is to be understood that $x_j, y_j \in \mathbb{R}$, and for a given $z = x + iy \in \mathbb{C}$, we will write $\Re z$ for the "real part" x of z, and $\Im z$ for the "imaginary part" y of z. Finally, the modulus of $z = x + iy \in \mathbb{C}$ is given by $|z| = \sqrt{x^2 + y^2}$.*

Remark 2.2.1. *The set $\mathbb{C}^n$ is a Hilbert space under the inner product given by:*

$$\langle z, z' \rangle_{\mathbb{C}^n} = \sum_{j=1}^{n} z_j \overline{z'_j}$$

We will drop this notation and simply write $\langle z, \overline{z'} \rangle$ for $\sum_{j=1}^{n} z_j \overline{z'_j}$.
This naturally gives $\mathbb{C}^n$ a metric topology induced by this inner product, its standard (Euclidean) topology.

Remark 2.2.2. *The $\mathbb{R}$-vector spaces $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ are isomorphic.*

Definition 2.2.1. *Let $U \subset \mathbb{C}^n$ be open, and let $f : U \to \mathbb{C}$ be a map. Let $a \in U$. We say that f is* ***complex-differentiable*** *at a if f, looked at as a map from an open subset of the $\mathbb{C}$-vector-space $\mathbb{C}^n$ to $\mathbb{C}$, is (Fréchet) differentiable at a. In other words, if there are $c_1, \dots, c_n \in \mathbb{C}$ such that:*

$$f(z) = f(a) + \sum_{j=1}^{n} c_j(z_j - a_j) + o(z - a)$$

or, equivalently and adapting the previous notation,

$$f(a + h) = f(a) + \sum_{j=1}^{n} c_j h_j + o(h)$$

If this is the case, the ***complex gradient*** *of f at a is given by:*

$$\partial f(a) := \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

We say that f is ***holomorphic*** *on U if f is differentiable at each point in U, and we may then define the complex gradient on U, from which we get ourselves a map $\partial f : U \to \mathbb{C}^n \equiv \mathcal{L}(\mathbb{C}^n, \mathbb{C})$ given at each point as defined above at a.*
We say that f is twice differentiable at $a \in U$ if f is holomorphic in a neighborhood $V \subset U$ of a and $\partial f : V \to \mathbb{C}^n$ is differentiable at a. The second derivative of f is often identified with the ***complex Hessian*** *of f, which, given $a \in U$, is the matrix representation of the bilinear map $D^2 f(a)$.*
We also call maps $U \subset \mathbb{C}^n \to \mathbb{C}^m$ complex-differentiable at a given point (resp. holomorphic) if each of their component functions is complex-differentiable at the given point (resp. holomorphic) according to the above definitions.

Note that the complex gradient of a complex-differentiable function $f : U \subset \mathbb{C}^n \to \mathbb{C}$ at a point a is the vector of partial derivatives of f at a in the sense of the definition in Proposition 2.1.4. In other words,

$$\partial f(a) = \begin{pmatrix} \dfrac{\partial f}{\partial z_1}(a) \\ \dots \\ \dfrac{\partial f}{\partial z_n}(a) \end{pmatrix}$$

Remark 2.2.3. *We also talk of complex gradients of maps which are not necessarily holomorphic but for which the partial derivatives with respect to the z_j's make sense (although might produce non-holomorphic expressions). Specifically, when we do this for a function $f(z_1, \ldots, z_n)$, we are looking at f as a function of the z_j* ***and*** *the $\overline{z_k}$, with the z_j and $\overline{z_k}$ variables looked at as* ***independent*** *variables. This will be the case with local defining functions for hypersurfaces, for instance. When we do this, we are meaning the vector of partial derivatives with respect to the z_j's as indicated above.*

Example 2.2.1. *Consider $\rho : \mathbb{C}^3 \to \mathbb{R}$, given by $\rho(z) = |z_1 z_2|^2 + 2\Re z_3$. This ρ is not holomorphic. Indeed, it cannot possibly be because the only holomorphic real-valued maps are the constant maps. However, we can write down its complex gradient. First, we re-write ρ as $\rho(z) = z_1\overline{z_1}z_2\overline{z_2} + z_3 + \overline{z_3}$. Then, differentiate ρ with respect to the z_j:*

$$\partial\rho = \begin{pmatrix} \overline{z_1}z_2\overline{z_2} \\ z_1\overline{z_1}\overline{z_2} \\ 1 \end{pmatrix}$$

Remark 2.2.4. *This operation of taking the complex gradient of a map (which is not necessarily holomorphic) can be looked at as a formal algebraic maneuver, and one can verify that taking derivatives in this formal manner preserves the basic properties we desire: sum/product rule, chain rule, etc. even when the other functions involved are in fact holomorphic.*

Notation 2.2.3. *Let $f : U \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic, and denote the coordinates in $\mathbb{C}^n$ by $(z_1, \ldots, z_n)$, writing for each j, $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$. We may regard f as a map $U \subset \mathbb{R}^{2n} \to \mathbb{C}$, and by differentiability, we may talk of $\dfrac{\partial f}{\partial x_j}$ and $\dfrac{\partial f}{\partial y_j}$, and we will define the following notations as well:*

$$\frac{\partial f}{\partial z_j} := \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right)$$

and

$$\frac{\partial f}{\partial \overline{z_j}} := \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right)$$

More generally, we talk of the Wirtinger operators:

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$$

and

$$\frac{\partial}{\partial \overline{z_j}} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

The first among the preceding notations actually coincides with the partial derivative of f with respect to z_j, and so no confusion arises from this usage. Also, note that we used here the fact that holomorphy implies real-differentiability, which holds because of Corollary 2.2.2, and we mention it as Corollary 2.2.3.

Theorem 2.2.2. *Let $f : U \subset \mathbb{C}^n \to \mathbb{C}$, where $U \subset \mathbb{C}^n$ is a domain (meaning open and connected). We may write $f(z) = u(x_1, y_1, \ldots, x_n, y_n) + iv(x_1, y_1, \ldots, x_n, y_n)$ where $u, v : U' \subset \mathbb{R}^{2n} \to \mathbb{R}$ are real-valued functions of real variables and U' is U sitting inside $\mathbb{R}^{2n}$. Then, f is holomorphic on U if and only if for every j, the maps:*

$$\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial y_j}, \frac{\partial v}{\partial x_j}, \frac{\partial v}{\partial y_j}$$

exist and are continuous, and for all j, $\dfrac{\partial f}{\partial \overline{z_j}} = 0$.

Remark 2.2.5. *These conditions are also known as the "Cauchy-Riemann equations". The one-variable case is doable by direct verification, starting from the definition of derivative, and one may prove the general case from the one-variable case.*

Definition 2.2.2. *Let $f : U \subset \mathbb{C}^n \to \mathbb{C}$ be continuous, and let $\gamma : [a, b] \subset \mathbb{R} \to \mathbb{C}^n$ be of class C^1. We define the integral of f over the curve $\gamma([a, b])$ to be:*

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt$$

Remark 2.2.6. *One can check that this is well-defined. In other words, regardless of how the image of γ is parameterized (by γ or any other C^1 map $[a, b] \to \mathbb{C}^n$), the integral is invariant.*

Notation 2.2.4. *Given an open set $U \subset \mathbb{C}^n$, with abuse of notation as to ignoring the dimension, we denote by $\mathcal{O}(U)$ the vector space of all holomorphic functions with domain U.*

Theorem 2.2.3. *(Cauchy's Integral theorem)*
Let $U \subset \mathbb{C}^n$ be a domain, and let $K = \prod_{j=1}^n K_j \subset U$ be a compact set. Let $f : U \to \mathbb{C}$, and assume that $f \in \mathcal{O}(U)$. Then, one has:

$$\int_{\partial K_1} \int_{\partial K_2} \cdots \int_{\partial K_n} f(z_1, \ldots, z_n)dz_n \cdots dz_1 = 0$$

Definition 2.2.3. *Let $a \in \mathbb{C}^n$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n$ with each $r_j > 0$. The open polydisk of center a and polyradius r is defined as:*

$$P(a, r) = \{z \in \mathbb{C}^n, \forall j, |z_j - a_j| < r_j\} = \prod_{j=1}^n B(a_j, r_j) = \prod_{j=1}^n P_j$$

where $B(x, \rho)$ is the open ball in $\mathbb{C}$ of center x and radius ρ. If for all j, $r_j = \delta$, we write $P(a, \delta)$ for $P(a, r)$. The open unit polydisk is simply $P(0, 1) = B(0, 1)^n$.

Notation 2.2.5. *Given a metric space X and a nvs Y, we denote by $C(X, Y)$ the vector space of all continuous functions from X to Y. When Y is clear from the context, we simply write $C(X)$ for $C(X, Y)$. When X is a nvs, for an open set $U \subset X$, we define $C^k(U, Y)$, where $k \in \mathbb{N} \cup \{\infty\}$, to be the vector space of all maps $U \to Y$ that are of class C^k, and we also write $C^k(U)$ when Y is understood.*

Theorem 2.2.4. *(Cauchy's Integral Formula)*
Let $U \subset \mathbb{C}^n$ be open. Let $P = P(a,r) = \prod_{j=1}^n P_j$ be an open polydisk centred at some $a \in \mathbb{C}^n$ and of polyradius r. Let $f : U \to \mathbb{C}$ and suppose that $f \in C(\overline{P}) \cap \mathcal{O}(P)$. Write $\Gamma = \prod_{j=1}^n \partial P_j$. Then, one has for all $z \in P$:

$$f(z) = f(z_1, \dots, z_n) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} d\zeta$$

Corollary 2.2.1. *Let U be a domain in $\mathbb{C}^n$, and let $f : U \to \mathbb{C}$ be a holomorphic map. Then, f is (complex) smooth on U.*

Corollary 2.2.2. *Let $U \subset \mathbb{C}$ be open and $f \in \mathcal{O}(U)$. Then, f is analytic in U, meaning that for any $a \in U$, there is a neighborhood $V \subset U$ of a such that f can be written in a power series on V - i.e. for some $(a_n) \subset \mathbb{C}$, we can write $f(z) = \sum_{n \geq 0} a_n (z-a)^n$ for all $z \in V$. But moreover, this series is uniformly convergent to f on the compact subsets of U, and one has for all $z \in U$:*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

Note that the same also holds for maps defined on open subsets of $\mathbb{C}^n$.

Corollary 2.2.3. *Let $f : U \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic. Then, f is real-analytic, meaning that if one views f as a map from the open subset U of the real vector space $\mathbb{C}^n$ to the real vector space $\mathbb{C}$, then f is still an analytic map.*

Corollary 2.2.4. *With the same setting as that of Theorem 2.2.4, and considering a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, one has for all $z \in P$:*

$$f^{(\alpha)}(z) = \frac{\alpha!}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{(\zeta_1 - z_1)^{\alpha_1+1}\cdots(\zeta_n - z_n)^{\alpha_n+1}} d\zeta$$

where $\alpha! := \prod_{j=1}^n \alpha_j!$ and:

$$f^{(\alpha)} := \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}$$

where $|\alpha| = \sum_{j=1}^n \alpha_j$ is the "norm" of the multi-index α.

Theorem 2.2.5. *(Identity theorem)*
Let $U \subset \mathbb{C}^n$ be a domain, and let $f : U \to \mathbb{C}$ be holomorphic. Let $A \subset U$ with $A' \cap U \neq \varnothing$. If $f \equiv 0$ on A, then $f \equiv 0$ on U.

Theorem 2.2.6. *(Maximum Modulus Principle)*
Let $U \subset \mathbb{C}^n$ be a domain, and let $f : U \to \mathbb{C}$ be holomorphic. If $|f| : U \to \mathbb{R}$ admits a maximum on U, then f is constant.

Corollary 2.2.5. *Let $U \subset \mathbb{C}^n$ be a bounded domain. Let $f : \overline{U} \to \mathbb{C}$ and suppose that $f \in \mathcal{O}(U) \cap C(\overline{U})$. Then, $|f| : \overline{U} \to \mathbb{R}$ attains its maximum over the compact set $\overline{U}$ on ∂U.*

Corollary 2.2.6. *Let $U \subset \mathbb{C}^n$ be a bounded domain. If $f \in \mathcal{O}(U) \cap C(\overline{U})$ vanishes everywhere on ∂U, then $f \equiv 0$ on $\overline{U}$.*

2.2.2 *Complications of Several Variables: Theorems and Examples*

Many of the problems of several complex variables concern appropriate generalizations or analogues to things we have seen in $\mathbb{C}$ and inspection of multivariate phenomena that drastically differ from the one-variable-case, such as Hartog's phenomenon or the failure of the Riemann Mapping theorem in the multivariable situation. This subsection serves to highlight these points.

Definition 2.2.4. *Let $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$ be open sets. Let $f : U \to V$ be a map. We say that f is a* ***biholomorphism*** *if f is bijective and holomorphic, and f^{-1} is holomorphic. We say that U and V are* ***biholomorphic*** *if there is a biholomorphism between them.*

Remark 2.2.7. *The relation "biholomorphic to" is an equivalence relation on the set of all subsets of $\sqcup_n \mathbb{C}^n$. Hence, the notion of "U and V being biholomorphic" is well-defined.*

Definition 2.2.5. *A property in the context of complex analysis is said to be an* ***invariant*** *if it is preserved under all biholomorphic transformations.*

Definition 2.2.6. *Let X be a metric space. A path in X from a point $x \in X$ to a point $y \in X$ is a continuous function $f : [0,1] \subset \mathbb{R} \to X$ such that $f(0) = x$ and $f(1) = y$.*

Remark 2.2.8. *We can talk of paths (and the few upcoming notions) in general topological spaces, and we don't really need a metric or a metrizable topology.*

Definition 2.2.7. *A metric space X is said to be* ***path-connected*** *if for any two points $x, y \in X$, there exists a path in X from x to y.*

Definition 2.2.8. *Let X and Y be metric spaces, and let $f, g : X \to Y$ be continuous maps. A* ***homotopy*** *between f and g is a continuous function $H : X \times [0,1] \to Y$ satisfying $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. If there is such a map between f and g, we say that f and g are homotopic. One should think of a homotopy as a way to continuously transform one embedding of an object into the embedding of another one. Note that "homotopic to" is an equivalence relation, and thus this notion as formulated is well-defined.*

Example 2.2.2. *The closed unit disk $B[0,1]$ in $\mathbb{R}^2$ is homotopic to the origin. In precise terms, let $f : B[0,1] \to B[0,1]$ be the identity map, and consider the map $\vec{0} : B[0,1] \to B[0,1]$ given by $\vec{0}(x,y) = (0,0)$. Then, $H : B[0,1] \times [0,1] \to B[0,1]$ given by $H(x,t) = (1-t)x$ is a homotopy between f and $\vec{0}$. Here, one should think of how the disk can be "shrunk" to the origin by "deforming" all the segments connecting the origin to a point on the boundary of the disk to the origin in a continuous manner.*

Definition 2.2.9. *Let X be a metric space. We say that X is* ***simply connected*** *if X is path-connected and all the paths in X with the same starting point and the same ending point are homotopic to each other.*

One should think of simple-connectedness as a way to formalize the notion of "having no holes" in the set. The requirement that we can deform any two paths with the same starting point and same ending point to each other functions to formalize the notion of "scanning for holes and finding none". Each path in the set is to be considered as a "first mark", and the homotopy sending it to another path (the "second mark") in the set is to be thought of as the "scanning machine". The result of the scanning is that there are no "missing points". See Figure 2.1 for an example of a set with "holes" or "missing points". Note that holes might simply refer to removed points and need not be about removed chunks. For instance, $\mathbb{C} \setminus \{0\}$ is not simply connected. Most formally, one would say that a set is simply connected if it's path-connected and its fundamental group at each point (a group of equivalence classes of loops based at the point, a way to formalize "the number of holes", where a "hole" comes out as a generator of the group) is trivial, which means that it's isomorphic to the additive group $(\{0\}, +)$.

Notation 2.2.6. *Here and elsewhere, we denote by Δ the open unit disk in $\mathbb{C}$, i.e. $\Delta := \{\zeta \in \mathbb{C}, |\zeta| < 1\}$, we denote by $\mathbb{H}_+$ the upper-half plane in $\mathbb{C}$, i.e. $\mathbb{H}_+ := \{\zeta \in \mathbb{C}, \Im\zeta > 0\}$, and we denote by $\mathbb{B}_n$ the open unit ball (i.e. the open unit disk) in $\mathbb{C}^n$, i.e. $\mathbb{B}_n = \left\{ z \in \mathbb{C}^n, \|z\| = \left(\sum_j |z_j|^2\right)^{1/2} < 1 \right\}$.*

Theorem 2.2.7. *(Riemann Mapping theorem)*
Every non-empty, open and simply connected subset of $\mathbb{C}$ that is not all of $\mathbb{C}$ is biholomorphic to the open unit disk in $\mathbb{C}$.

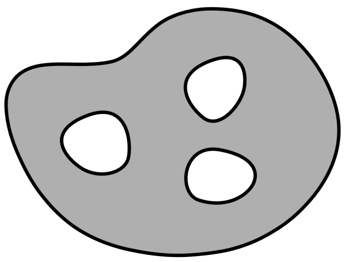

Figure 2.1: This set (the shaded region) is not simply connected. It has three "holes" (the white parts are not part of the set).

Example 2.2.3. *The set $\mathbb{H}_+$ can be proven to be simply connected, and it's surely non-empty and $\neq \mathbb{C}$. The Riemann Mapping theorem tells us that there exists a biholomorphism from Δ to $\mathbb{H}_+$ (or vice-versa). In fact, one can verify that the map $f : \Delta \to \mathbb{H}_+$ given by:*

$$f(z) = i\frac{1+z}{1-z}$$

is a biholomorphism.

Remark 2.2.9. *There is no $n \geq 2$ for which the statement of the Riemann Mapping theorem in $\mathbb{C}^n$ holds. We will show this after we prepare the necessary ingredients in the next few lines.*

Definition 2.2.10. *Let X and Y be metric spaces, and let $f : X \to Y$ be continuous. We say that f is a proper map if the inverse image under f of any compact subset of Y is a compact subset of X. In other words, f is proper if for every compact subset K of Y, we have that $f^{-1}(K)$ is compact.*

Definition 2.2.11. *Let (X, d) be a metric space and $(x_n) \subset X$. We say that a point $x \in X$ is an accumulation point of (x_n) if it is true that: $\forall \epsilon > 0, \forall n \in \mathbb{N}, \exists k > n$ such that $d(x_k, x) < \epsilon$. Equivalently, x is an accumulation point of (x_n) if x is the limit of some subsequence of (x_n).*

Lemma 2.2.1. *Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^d$ be bounded domains. Let $f : X \to Y$ be continuous. Then, f is a proper map if and only if for every sequence $(x_n) \subset X$ converging to some $x \in \partial X$, we have $A \subset \partial Y$, where A is the set of accumulation points of $(f(x_n))$.*

Proof. Suppose that f is proper, and let $(x_n) \subset X$ with $x_n \to x \in \partial X$. Let $y \in A$. Then, there is a subsequence $(f(x_{n_k}))$ of $(f(x_n))$ that converges to y. We know that $y \in \overline{Y}$. Suppose that $y \in Y^\circ$. Then, $\exists \delta > 0$ such that $B(y, \delta) \subset Y$. As $(f(x_{n_k}))$ converges to y, we know that there is $N \geq 1$ such that $f(x_{n_k}) \in B := B(y, \delta/2)$ for all $k \geq N$. In particular, $f(x_{n_k}) \in \overline{B}$ for all $k \geq N$. Then, $x_{n_k} \in f^{-1}(\overline{B})$ for all $k \geq N$. But $\overline{B}$ is compact, so since f is proper, $f^{-1}(\overline{B})$ is also compact. Hence, $f^{-1}(\overline{B})$ is closed in $\mathbb{R}^m$. So, $x = \lim x_{n_k} \in f^{-1}(\overline{B})$. As f is continuous and $B(y, \delta)$ is an open subset of Y, we know that $f^{-1}(B(y, \delta)) \subset X^\circ = X$. In particular, $f^{-1}(\overline{B}) \subset X$. Hence, $f^{-1}(\overline{B}) \cap \partial X = \varnothing$. But $x \in f^{-1}(\overline{B}) \cap \partial X$. A contradiction. Therefore, $y \notin Y^\circ$, so $y \in \partial Y$.
Now let's show the backward implication. Let $K \subset Y$ be compact. Then, K is closed in $\mathbb{R}^d$, and so $f^{-1}(K)$ is closed in X. Since X is bounded, so is $f^{-1}(K)$. So, it's enough to show that $f^{-1}(K)$ is closed in $\mathbb{R}^m$. If it's not, then there is a sequence $(x_n) \subset f^{-1}(K)$ such that $x_n \to x \in \partial X$ (since $f^{-1}(K)$ is closed in X). We know that $(f(x_n)) \subset K$ which is compact, so it has a convergent subsequence converging to some $y \in K$. By our assumption, and since y is an accumulation point of $(f(x_n))$, we must have $y \in \partial Y$. Thus, $y \in K \cap \partial Y$. But since Y is open, $Y \cap \partial Y = \varnothing$, in particular, $K \cap \partial Y = \varnothing$. This is a contradiction. Therefore, $f^{-1}(K)$ is closed in $\mathbb{R}^m$ and is thus compact. Therefore, f is proper. □

Remark 2.2.10. *It is not enough for f to be continuous for the previous lemma to hold. Indeed, take $X = (0, 3)$, $Y = (0, 5)$, and consider the function $f : X \to Y$ given by $f(x) = 4x - x^2$. Obviously, X and Y are bounded domains, and f is continuous. However, take $(x_n) \subset X$ to be the sequence given by $x_n = 3 - \dfrac{1}{n}$. Then, $x_n \to 3 \in \partial X$, but $f(x_n) = 3 + \dfrac{2}{n} - \dfrac{1}{n^2} \to 3 \notin \partial Y$.*

Theorem 2.2.8. *Let $U \subset \mathbb{C}$ be a domain, and let (f_n) be a sequence of holomorphic functions $U \to \mathbb{C}$. Suppose that $f_n \to f$ uniformly on compact subsets of U. Then, f is holomorphic, and $f_n' \to f'$ uniformly on compact subsets of U.*

Theorem 2.2.9. *(Montel's theorem)*
Let $U \subset \mathbb{C}$ be open and (f_n) be a sequence of holomorphic functions $U \to \mathbb{C}$. If (f_n)

is uniformly bounded on compact subsets of U *(meaning that for any given compact* $K \subset U$, $\exists c_K > 0$ *such that* $|f_k(z)| \leq c_K$ *for all* $z \in K$ *and all* $k \geq 1$*), then there exists a subsequence* (f_{n_k}) *that converges uniformly on the compact subsets of* U*.*

Theorem 2.2.10. *(Open Mapping theorem)*
Let $U \subset \mathbb{C}$ *be a domain and* $f : U \to \mathbb{C}$ *be a non-constant holomorphic function. Then,* f *is an open map, meaning that for all open sets* $O \subset U$, $f(O)$ *is open.*

Theorem 2.2.11. *(Rothstein)*
For every $n \geq 2$*, the open unit ball in* $\mathbb{C}^n$ *and the open unit polydisk in* $\mathbb{C}^n$ *are not biholomorphic. Hence, the statement of the Riemann Mapping theorem fails in* $\mathbb{C}^n$*, for all* $n \geq 2$*.*

The following proof is inspired by the one presented in section 1.4 of [3].

Proof. Suppose that they are biholomorphic, and let $f : P(0,1) = \Delta^n \to \mathbb{B}_n$ be a biholomorphism. In particular, f is a holomorphic, proper map. Fix some $\eta \in \partial\Delta^{n-1}$ and let (η_k) be a sequence in Δ^{n-1} such that $\eta_k \to \eta$. Consider the sequence of functions (g_k) defined by $g_k : \Delta \to \mathbb{B}_n$, $g_k(\zeta) = f(\zeta, \eta_k)$. Clearly, every g_k is holomorphic. Moreover, (g_k) is uniformly bounded on compact subsets of Δ: in fact, (g_k) is uniformly bounded on Δ, as each $g_k(\zeta)$ lives inside $\mathbb{B}_n$. So, by Montel's theorem, there is a subsequence (g_{k_j}) of (g_k) that converges uniformly on the compact subsets of Δ to some holomorphic function $g : \Delta \to B[0,1] = \overline{\mathbb{B}_n}$ (the closed unit ball in $\mathbb{C}^n$). By the lemma, we know that $g(\Delta) \subset \partial\mathbb{B}_n$, and by the Open Mapping theorem, this implies that g is constant. Now the sequence of derivatives (g'_{k_j}) converges (uniformly) to $g' \equiv 0$ (on compact subsets of Δ). But, one has for each j,

$$g'_{k_j}(\zeta) = \frac{\partial f}{\partial z_1}(\zeta, \eta_{k_j})$$

so that:

$$\lim_j \frac{\partial f}{\partial z_1}(\zeta, \eta_{k_j}) = 0$$

and this is true for all ζ. Now fix a ζ. The map $\xi \mapsto \frac{\partial f}{\partial z_1}(\zeta, \xi)$ is defined and holomorphic on Δ^{n-1}, and by the preceding reasoning, we may extend it continuously to $\overline{\Delta^{n-1}}$. By the Maximum Modulus Principle, one sees that $\frac{\partial f}{\partial z_1}(\zeta, \cdot) \equiv 0$, and this holds for all ζ. We conclude that $\frac{\partial f}{\partial z_1} \equiv 0$. We may repeat the same argument for the other variables to conclude that in fact $\partial f \equiv 0$. Thus, f is constant (by connectedness), and this contradicts the fact that f is proper. Indeed, if $f \equiv c$ is constant, then the inverse image of any compact set containing c is $P(0,1)$ itself, which is not compact. Alternatively, we can also conclude by saying that f being constant contradicts its bijectivity. $\square$

Remark 2.2.11. *It is a fact that if* X *and* Y *are nvs,* $U \subset X$ *is open and connected, and* $f : U \to Y$ *is differentiable on* U *with* $Df \equiv 0$*, then* f *is constant, and this is what we used at the end above.*

Definition 2.2.12. *Let $U \subset \mathbb{C}^n$ be a domain, and let $V \subset U$ be a domain as well. A domain V' is called an $\mathcal{O}(U)$-analytic-completion (or just* ***analytic completion****) of V if $V \subset V'$ and $\forall f \in \mathcal{O}(U)$, $f\mid_V$ extends holomorphically to V'.*

Definition 2.2.13. *A domain $U \subset \mathbb{C}^n$ is called a* ***domain of holomorphy*** *if for all $V \subset U$, U contains all the analytic completions of V.*

Remark 2.2.12. *Any domain $U \subset \mathbb{C}$ is a domain of holomorphy. In fact, the only analytic completions of a given subdomain $V \subset U$ are V itself and subsets of U that contain V. Indeed, suppose that V' is an analytic completion of V, and suppose that $V' \not\subset U$. Let $z_0 \in V' \setminus U$. Consider $f : U \to \mathbb{C}$ given by $f(z) = (z - z_0)^{-1}$. Then, f extends holomorphically to V', given by some g. We know that g agrees with $z \mapsto (z - z_0)^{-1}$ on the domain V, so by the Identity theorem, $g(z) = (z - z_0)^{-1}$ for all $z \in V'$, and this is impossible because $z_0 \in V'$. A contradiction. So, these concepts are quite useless unless we are in $\mathbb{C}^n$ for $n \geq 2$.*

Theorem 2.2.12. *(Hartog's theorem)*
Let $n \geq 2$ and $U \subset \mathbb{C}^n$ be a domain. Let $K \subset U$ be compact, and suppose that $U \setminus K$ is connected (so that it is also a domain). Then, every map $f \in \mathcal{O}(U \setminus K)$ extends uniquely to a map $f^* \in \mathcal{O}(U)$.

One consequence of Hartog's theorem is that $\mathbb{C}^n$, for $n \geq 2$, abundantly contains domains which have non-trivial analytic completions. Give me a ball $U = B(a, \epsilon)$ and denote by K its closure (so that K is compact). Let $U' = B(a, \epsilon + 1)$. Then, U' is a domain, $K \subset U'$ is compact and $U' \setminus K$ is the annulus of center a, smaller radius ϵ and larger radius $\epsilon + 1$, which is obviously connected. Let $f : U' \setminus K \to \mathbb{C}$ be any holomorphic map. Then, by Hartog's theorem, f extends holomorphically to U'. This shows that all annuli in $\mathbb{C}^n$ are not domains of holomorphy. One can see from this how the concept of a domain of holomorphy emerged: it is interesting to know what it is that makes certain sets (if any) immune to holomorphic extensions. The earliest form of the "Levi problem" concerned this question, and it now has a definitive answer: in Chapter 3, we define the notion of a "pseudoconvex" domain, and we note that being a domain of holomorphy is the same as being a pseudoconvex domain (see Theorem 3.1.3).

2.3 Further Complex-Analytic Notes

Some basic and requisite definitions for one-variable complex analysis are laid out in the first subsection. In the second subsection, we talk a bit about some known automorphisms in the context of $\mathbb{C}$ and we mention Cartan's uniqueness theorem.

2.3.1 *Useful One-Variable Definitions and Remarks*

Here, we talk about singularities, meromorphicity, Laurent series and a few other basic concepts.

Definition 2.3.1. *Let $U \subset \mathbb{C}$ be open. A map $f : V \subset \mathbb{C} \to \mathbb{C}$ is said to be meromorphic on U if for every $z_0 \in U$, there is a neighborhood W of z_0 in U such that on W, either f or $1/f$ is definable and holomorphic.*

Notation 2.3.1. *Given an open set $U \subset \mathbb{C}$, we denote by $\mathcal{M}(U)$ the vector space of all meromorphic functions on U.*

Definition 2.3.2. *Let $U \subset \mathbb{C}$ be open and let $f \in \mathcal{M}(U)$. A point $z_0 \in U$ is called a **pole** of f of order $n \geq 1$ if $z \mapsto (z - z_0)^n f(z)$ is definable as a holomorphic and nowhere-zero map in a neighborhood of z_0.*

Remark 2.3.1. *In the previous definition, it is clear that if such an n exists, then it is unique, and we talk of **the** order of the pole z_0 of f.*

Example 2.3.1. *Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be the map $f(z) = 1/z$. Then, $f \in \mathcal{M}(\mathbb{C})$ and 0 is a pole of f of order 1. Similarly, given $n \geq 2$, we have that 0 is a pole of $z \mapsto 1/z^n$ of order n.*

Definition 2.3.3. *A set X is called countable if there is an injective map from X to $\mathbb{N}$ (so that our notion of "countable" includes all the finite sets as well).*

Proposition 2.3.1. *Let $U \subset \mathbb{C}$ be open. If $f \in \mathcal{M}(U)$, then f is definable and differentiable except at countably many points in U. The converse also holds, so that meromorphic functions on an open set $U \subset \mathbb{C}$ are precisely those f for which there's a countable set A such that $f \in \mathcal{O}(U \setminus A)$.*

Definition 2.3.4. *Let $U \subset \mathbb{C}$ be open, $z_0 \in U$, and $f \in \mathcal{M}(U)$ such that f is not defined at z_0. The point z_0 is called a **removable singluarity** of f if $\lim_{z \to z_0} f(z)$ exists in $\mathbb{C}$. On the other hand, z_0 is called an **essential singularity** of f if z_0 is neither a removable singularity nor a pole of f.*

Theorem 2.3.1. *(Casorati-Weierstrass theorem)*
Let $U \subset \mathbb{C}$ be open, $z_0 \in U$, and $f : U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Suppose that z_0 is an essential singularity of f. Then, for every neighborhood $V \subset U$ of z_0, we have that $f(V \setminus \{z_0\})$ is dense in $\mathbb{C}$.

Proposition 2.3.2. *Let $f : \Delta \setminus \{0\} \to \mathbb{C}$ be a holomorphic, injective map. Then, 0 is not an essential singularity of f.*

Proof. Suppose that 0 is an essential singularity of f. Let $V_1 = B(0, 1/2)$ and $V_2 = B(3/4, 1/4)$. Then, V_1 and V_2 are open and disjoint. As $0 \in V_1 \subset U = \Delta$, we know by the Casorati-Weierstrass theorem that $f(V_1 \setminus \{0\})$ is dense. By the Open Mapping theorem, $f(V_2)$ is open. Hence, by density, $f(V_2) \cap f(V_1 \setminus \{0\}) \neq \varnothing$. In particular, by injectivity, $f((V_1 \setminus \{0\}) \cap V_2) = f(V_2) \cap f(V_1 \setminus \{0\}) \neq \varnothing$. However, $V_1 \cap V_2 = \varnothing$, so $f(V_1 \cap V_2) = \varnothing$. A contradiction. □

Proposition 2.3.3. *The properties of being a zero, a pole of a specific order, a removable singularity and of being an essential singularity of a given map are all invariants. In fact, let $U, V \subset \mathbb{C}$ be biholomorphic open sets by some biholomorphism*

$\varphi : U \to V$. Let $f \in \mathcal{M}(U)$ and let $z_0 \in U$ be a zero (or a pole of order $n \geq 1$, or removable singularity, or essential singularity) of f. Then, $f \circ \varphi^{-1} \in \mathcal{M}(V)$, and $\varphi(z_0)$ is a zero (resp. a pole of order n, a removable singularity, an essential singularity) of $f \circ \varphi^{-1}$.

Proof. That $f \circ \varphi^{-1} \in \mathcal{M}(V)$ is obvious, and if z_0 is a zero of f, i.e. $f(z_0) = 0$, then surely $(f \circ \varphi^{-1})(\varphi(z_0)) = f(z_0) = 0$. Let's show that poles are preserved (we leave the rest to the reader). Suppose that z_0 is a pole of f of order $n \geq 1$. Then, $g : z \mapsto (z - z_0)^n f(z)$ is definable on a neighborhood $W \subset U$ of z_0, $g \in \mathcal{O}(W)$ and $g(z) \neq 0$ for all $z \in W$. Let $W' = \varphi(W)$. Then, $W' \subset V$ is a neighborhood of $\varphi(z_0)$. Consider the map $h : z \mapsto (z - \varphi(z_0))^n (f \circ \varphi^{-1})(z)$. We have:

$$h(\varphi(z)) = (\varphi(z) - \varphi(z_0))^n f(z) = \left(\frac{\varphi(z) - \varphi(z_0)}{z - z_0} \right)^n g(z) = \eta(z) g(z)$$

The map η is defined on $W \setminus \{z_0\}$, and by holomorphy of φ, we know that as $z \to z_0$, we have $\eta(z) \to (\varphi'(z_0))^n \in \mathbb{C}$. Hence, z_0 is a removable singularity of η, and $\eta \in \mathcal{O}(W)$. Hence, $h \circ \varphi \in \mathcal{O}(W)$. As the composition of holomorphic maps is holomorphic, we have $h = (h \circ \varphi) \circ \varphi^{-1} \in \mathcal{O}(W')$. Moreover, for any $\xi \in W'$, there is $z \in W$ such that $\xi = \varphi(z)$, and then we may write $h(\xi) = h(\varphi(z)) = \eta(z)g(z)$. We know that $g(z) \neq 0$, so if $h(\xi) = 0$, then $\eta(z) = 0$, so that $\varphi(z) = \varphi(z_0)$. By injectivity, $z = z_0$, so $0 = \eta(z) = \eta(z_0) = (\varphi'(z_0))^n$, hence $\varphi'(z_0) = 0$, contradicting the injectivity of φ. This shows that $h(\xi) \neq 0$ for all $\xi \in W'$. And therefore, $\varphi(z_0)$ is a pole of $f \circ \varphi^{-1}$ of order n. □

Remark 2.3.2. *Let $U \subset \mathbb{C}$ be open and let $f \in \mathcal{M}(U)$. Let $z_0 \in U$ be a pole of f of order n. We know that there is a neighborhood W of z_0 such that $z \mapsto (z - z_0)^n f(z)$ is holomorphic on W, and by analyticity, we may then do a power series expansion at z_0:*

$$(z - z_0)^n f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

and then,

$$f(z) = \frac{c_0}{(z - z_0)^n} + \frac{c_1}{(z - z_0)^{n-1}} + \cdots + \frac{c_{n-1}}{z - z_0} + \sum_{k=0}^{\infty} c_{k+n} (z - z_0)^k$$

Definition 2.3.5. *The above expansion of f at z_0 is called the **Laurent series** of f at the pole z_0, and c_{n-1} is called the **residue** of f at z_0, and we use the notation* $\operatorname{Res}(f, z_0)$ *for the residue of f at z_0.*

Definition 2.3.6. *Given $z \in \mathbb{C} \setminus \{0\}$, let $w = \dfrac{z}{\|z\|}$. Then, $w \in \partial\Delta$, which is parameterizable by $\theta \in [-\pi, \pi) \mapsto \cos\theta + i \sin\theta$, and there is a unique $\theta \in [-\pi, \pi)$ such that $w = \cos\theta + i\sin\theta = e^{i\theta}$, and then $z = \|z\| e^{i\theta}$. This θ is called **the principal argument** of z, denoted by* $\operatorname{Arg}(z)$. *More generally, we talk of the set valued function* $\arg : \mathbb{C} \setminus \{0\} \to \mathcal{P}(\mathbb{R})$ *given by* $\arg(z) = \{\operatorname{Arg}(z) + 2k\pi, k \in \mathbb{Z}\}$*, and sometimes we talk of* $\arg$ *as a multi-valued correspondence, and then each value of* $\arg(z)$ *is called **an** argument of z.*

2.3.2 *A Few Words About Automorphisms*

Automorphisms in Complex Analysis are naturally the biholomorphisms from a set to itself. These provide a lot of insight into the objects at work, and this subsection aims at familiarizing the reader a bit with this concept.

Proposition 2.3.4. *Let $U \subset \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$ be a holomorphic and injective map. Then, $f'(z) \neq 0$ for all $z \in U$.*

Proposition 2.3.5. *The automorphisms of $\mathbb{C}$ are precisely the non-constant maps of the form $f(z) = az + b$ for some $a, b \in \mathbb{C}$.*

Proof. Let $f \in \operatorname{Aut}(\mathbb{C})$. By analyticity, there is some $(a_n) \subset \mathbb{C}$ such that for all $z \in \mathbb{C}$, we have $f(z) = \sum_{n\geq 0} a_n z^n$. Consider $g : \Delta \backslash \{0\} \to \mathbb{C}$ given by $g(z) = f(1/z)$. It's clear that g is holomorphic and injective. So, 0 is not an essential singularity of g. So, 0 is either a removable singularity of g or a pole of g of some order k. We have:

$$g(z) = \sum_{n\geq 0} \frac{a_n}{z^n}$$

so that if 0 is removable, then we must have $a_n = 0$ for all $n \geq 1$, and so f is constant, a contradiction to injectivity of f. Hence, 0 is a pole of g of order k, and we must have $a_n = 0$ for all $n \geq k + 1$. Thus, $f(z) = a_0 + a_1 z + \cdots + a_k z^k$, with $k \geq 1$. By the Fundamental Theorem of Algebra, f has k roots in $\mathbb{C}$, and by injectivity, these roots must all be equal. In other words, there are $\alpha, z_0 \in \mathbb{C}$ with $\alpha \neq 0$ such that $f(z) = \alpha(z - z_0)^k$. Suppose that $k \geq 2$. Then, for instance, the equation $f(z) = 1$ has k different solutions, and this contradicts injectivity. Hence, $k = 1$, and so, $f(z) = \alpha(z - z_0) = az + b$ with $a \neq 0$. □

Theorem 2.3.2. *(Schwarz lemma)*
Let $f \in \mathcal{O}(\Delta)$ with $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \Delta$. Then, $|f(z)| \leq |z|$ for all $z \in \Delta$, and $|f'(0)| \leq 1$. Moreover, if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$ or $|f'(0)| = 1$, then there is $a \in \partial\Delta$ such that $f(z) = az$ for all $z \in \Delta$.

Definition 2.3.7. *A **Blaschke factor** is a map of the form $B_a(z) = \dfrac{z - a}{1 - \overline{a}z}$ for some $a \in \Delta$.*

Remark 2.3.3. *Blaschke factors are automorphisms of Δ, and given $a \in \Delta$, we have $B_a^{-1}(z) = \dfrac{z + a}{1 + \overline{a}z}$.*

Proposition 2.3.6. *Let $f \in \operatorname{Aut}(\Delta)$. Then, there is some $a \in \partial\Delta$ such that for all $z \in \Delta$, $B_b(f(z)) = az$, where $b = f(0)$.*

Proof. We have $B_b \circ f \in \operatorname{Aut}(\Delta)$, and $B_b \circ f(0) = B_b(f(0)) = B_b(b) = 0$. So, by Schwarz lemma, $|B_b(f(z))| \leq |z|$ for all $z \in \Delta$. On the other hand, we know that $(B_b \circ f)^{-1} \in \operatorname{Aut}(\Delta)$, and $(B_b \circ f)^{-1}(0) = 0$, so again by Schwarz lemma, we have $|(B_b \circ f)^{-1}(z)| \leq |z|$ for all $z \in \Delta$. Now given $z \in \Delta$, write $\xi = B_b \circ f(z)$. Then, applying the latter inequality to ξ, we get $|z| \leq |B_b(f(z))|$. We conclude that $|B_b(f(z))| = |z|$ for all $z \in \Delta$. In particular, again by Schwarz lemma, there is $a \in \partial\Delta$ such that $B_b(f(z)) = az$ for all $z \in \Delta$. □

Corollary 2.3.1. *The automorphisms of* Δ *are precisely the maps of the form* $f(z) = \dfrac{az+b}{\overline{b}z+\overline{a}}$ *for some* $a, b \in \mathbb{C}$ *with* $|a| = 1$ *and* $|b| < 1$.

Proof. Let $f \in \mathrm{Aut}(\Delta)$. By above, there is $a \in \mathbb{C}$ with $|a| = 1$ such that for all z, $B_b(f(z)) = az$, with $b = f(0)$. Then,

$$f(z) = B_b^{-1}(az) = \frac{az+b}{1+\overline{b}az} = \frac{az+b}{\overline{b}z+\overline{a}}$$

and since $b = f(0)$, we have $|b| < 1$. $\square$

Proposition 2.3.7. *The automorphisms of a given domain* $U \subset \mathbb{C}^n$ *are invariants in the following sense: let* $V \subset \mathbb{C}^m$ *and suppose that* U *and* V *are biholomorphic through some biholomorphism* $\varphi : U \to V$*; we have* $\mathrm{Aut}(V) = \varphi \circ \mathrm{Aut}(U) \circ \varphi^{-1}$*, by which we mean that:* $H \in \mathrm{Aut}(U)$ *if and only if* $\varphi \circ H \circ \varphi^{-1} \in \mathrm{Aut}(V)$.

Definition 2.3.8. *We call a* ***Mobius transformation*** *any function* $T : U \subset \mathbb{C} \to \mathbb{C}$ *of the form* $T(z) = \dfrac{az+b}{cz+d}$*, where* U *is an appropriately chosen domain of* T*, such that* $a, b, c, d \in \mathbb{C}$ *with* $ad - bc \neq 0$.

Corollary 2.3.2. *Every automorphism of* $\mathbb{H}_+$ *(the upper-half plane) is a Mobius transformations of* $\mathbb{H}_+$ *with real coefficients. More particularly, every element of* $\mathrm{Aut}(\mathbb{H}_+)$ *is of the form* $T(z) = \dfrac{az+b}{cz+d}$ *with* $a, b, c, d \in \mathbb{R}$ *and* $ad - bc > 0$.

Proof. We already know the automorphisms of Δ, and Example 2.2.3 gives us a biholomorphism $f : \Delta \to \mathbb{H}_+$ which has the inverse $f^{-1}(z) = \dfrac{z-i}{z+i}$. Hence, the automorphisms of H_+ are precisely the maps $f \circ H \circ f^{-1}$, where $H \in \mathrm{Aut}(\Delta)$. If we consider some $H \in \mathrm{Aut}(\Delta)$, one of the form displayed in Corollary 2.3.1 with parameters $\alpha, \beta \in \mathbb{C}$ satisfying $|\alpha| = 1$ and $|\beta| < 1$, then one can verify the computation:

$$f \circ H \circ f^{-1}(z) = \frac{(\alpha + \overline{\alpha} + \beta + \overline{\beta})z - i(\alpha - \overline{\alpha}) + i(\beta - \overline{\beta})}{[i(\alpha - \overline{\alpha}) + i(\beta - \overline{\beta})]z + \alpha + \overline{\alpha} - (\beta + \overline{\beta})} = \frac{az+b}{cz+d}$$

where clearly $a, b, c, d \in \mathbb{R}$. Furthermore, one has $ad - bc = 4|\alpha|^2 - 4|\beta|^2 = 4(1 - |\beta|^2)$, and since $|\beta| < 1$, we get $ad - bc > 0$.
Note that here we actually precisely nailed down $\mathrm{Aut}(\mathbb{H}_+)$, but we do not display this in the statement of the corollary because it is a messy expression with many conditions, hence we decided to just state a necessary condition. $\square$

Definition 2.3.9. *Let* $U \subset \mathbb{C}^n$ *be a domain and let* $f \in \mathcal{O}(U)$*. The* m*-jet of* f *at a point* $z_0 \in U$ *is the collection* $\{f(z_0), Df(z_0), D^2 f(z_0), \ldots, D^m f(z_0)\}$*, and one says that a family of functions* $\mathcal{F}$ *has an* m*-jet determination at a point* $z_0 \in U$ *if for all* $f, g \in \mathcal{F}$*, if* f *and* g *have equal* m*-jets at* z_0*, then* $f = g$.

Theorem 2.3.3. *(Cartan's Uniqueness Theorem)*
Let $U \subset \mathbb{C}^n$ be a bounded domain, and let $z_0 \in U$ and $H \in \mathrm{Aut}(U)$. If $H(z_0) = z_0$ and $DH(z_0) = I$ (the identity map of U), then $H = I$.

Remark 2.3.4. *This theorem essentially says that automorphisms of bounded domains have a 1-jet determination at any given point in these domains, and this is a very powerful statement. Moreover, the reader should notice that Cartan's theorem is more or less a generalization of the Schwarz lemma.*

Automorphisms are biholomorphisms, and in this respect, they are as important as biholomorphisms are: a different way of looking at the structure and therefore new information and an ameliorated understanding, as well as a way to capture the invariants. The latter can be very helpful if one can move to a structure that is simpler, as the Riemann Mapping Theorem allows one to do. But automorphisms are essentially different machines. Automorphisms are particularly symmetries of objects, and this means that they are the essence of invariance and that they display invariant information in much better language and precision. For instance, if one has a certain family of invariant objects of a structure X that is not completely determined let's say, then having a few automorphisms of X might allow us to get more information about the family by giving us other members of it by passing through an automorphism of X. We will discuss this in the upcoming chapters and the reader will see how this approach applies to stationary disks. However, if one is generally considering a biholomorphism, no information is expected to be obtained about such a family by passing through the biholomorphism, as the latter mostly only serves to explain the role of a member of that family in the new structure: it is not expected to generate information about other members of the family.

Chapter 3

Background in Differential Geometry

In this chapter, we review the theory of Differential Geometry and we build up the topic of this thesis. Of the things we do here is go over basic Differential-Geometric concepts (manifolds, tangent bundles, the Levi form, convexity and related notions, etc.), then talk about analytic and stationary disks, and we conclude this chapter with explaining the motivation for and the goal of this thesis.

3.1 Manifolds, Complex Structures, and Real Hypersurfaces

In this section, we lay out some basics of Differential Geometry and CR Geometry. It is intended as a quick revision leading to the topic of our work.

3.1.1 *Smooth Real Manifolds and Complex Manifolds*

This is a summary of the classical basic notions of Differential Geometry in which we define smooth real manifolds and their tangent bundles, embedded submanifolds, smooth maps between manifolds and related ideas, as well as briefly mention complex manifolds.

Definition 3.1.1. *Let M be a topological space. We say that M is a* ***real topological manifold*** *of dimension n if M is Hausdorff and second countable, and for each $p \in M$, there is a neighborhood $U \subset M$ of p such that U is homeomorphic to an open subset of $\mathbb{R}^n$. If this is the case, we also say that M is a real n-manifold.*

Definition 3.1.2. *A chart for a real n-manifold M is a 2-tuple (U, φ) where U is an open subset of M and $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a homeomorphism, and the transition map from a chart (U, φ_U) to a chart (V, φ_V) is the map $\psi_{UV} = \varphi_V \circ \varphi_U^{-1}$ with domain $\varphi_U(U \cap V)$ and codomain $\varphi_V(U \cap V)$. Two such charts are called compatible if ψ_{UV} is a C^∞-diffeomorphism.*

Definition 3.1.3. *A (smooth or differentiable) atlas for a real n-manifold M is a collection $\mathcal{A} = \{(U_j, \varphi_j), j \in J\}$ of pairwise compatible charts such that $M = \cup_{j\in J} U_j$.*

And a maximal atlas for M is an atlas which contains every chart that is compatible with every member of $\mathcal{A}$.

Remark 3.1.1. *It is a fact that for any real manifold M and any atlas $\mathcal{A}$ for M, there is a unique maximal atlas for M that contains $\mathcal{A}$.*

Definition 3.1.4. *A maximal atlas on a real manifold M is called a smooth (or differentiable) structure on M, and a manifold M equipped with a maximal atlas is called a* ***smooth real manifold****.*

We may also talk of real **manifolds of class** C^k, which are real topological manifolds whose transition maps are C^k-diffeomorphisms rather than C^∞-diffeomorphisms. A smooth manifold is also called a manifold of class C^∞. Note that all the concepts we lay out here can be reformulated appropriately for manifolds of class C^k with $1 \le k < \infty$.

Example 3.1.1. *The Euclidean space $\mathbb{R}^n$ is naturally a smooth real n-manifold, $\mathbb{C}^n$ can be looked at as a smooth real $2n$-manifold, and the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ can be made into a smooth real n-manifold.*

Definition 3.1.5. *Let M be a smooth, real m-manifold with (maximal) atlas $\{(U_k, \varphi_k)\}_{k\in K}$, and N be an n-manifold with atlas $\{(Vj, \eta_j)\}_{j\in J}$. Let $F : M \to N$ be a continuous map. We say that F is differentiable if for every $k \in K$ and $j \in J$, the map $\eta_j \circ F \circ \varphi_k^{-1} : \varphi_k(U_k \cap F^{-1}(V_j)) \subset \mathbb{C}^m \to \mathbb{C}^n$ is differentiable. If, moreover, F is a bijection and F^{-1} is differentiable in the aforementioned sense, we say that F is a diffeomorphism, and M and N are then called diffeomorphic.*

The notion of a differentiable map between smooth manifolds defined above is the most natural way to generalize the concept of differentiability of a map on things which are not necessarily open subsets of vector spaces. In fact, differentiability is a local property - one originally talks of differentiability at a point and talks of perturbation of the function near the point. This naturally generalizes to zooming into coordinates near a point (which is what charts are about), with smoothness of transition maps between charts expressing the irrelevance as to which particular coordinates one chooses for the purposes of differentiability.

Example 3.1.2. *The space $\mathbb{S}^{n-1} \setminus \{N\} \subset \mathbb{R}^n$, where $N = (0, \ldots, 0, 1)$ is the "north pole", is diffeomorphic to $\mathbb{R}^{n-1}$, and one possible diffeomorphism between them is the classical stereographic projection.*

Definition 3.1.6. *Let M be a smooth, real n-manifold. We define $C^\infty(M)$ to be the $\mathbb{R}$-algebra of all maps $f : M \to \mathbb{R}$ which are smooth in the sense of manifolds. One also defines a* ***derivation*** *at a point $p \in M$ to be a linear map $D : C^\infty(M) \to \mathbb{R}$ that satisfies the "product rule for derivatives" (aka "Leibniz rule"): for all maps $f, g \in C^\infty(M)$, $D(fg) = D(f)g(p) + f(p)D(g)$. Notice that from this rule, it follows that $D(1) = 0$, and then that $D(f) = 0$ if f is constant.*

Definition 3.1.7. *Let M be a smooth, real n-manifold and let $p \in M$. We define the **tangent space** of M at p, denoted by T_pM, to be the (real) vector space of all derivations at p, with the natural addition of, and scalar multiplication on, linear maps.*

Remark 3.1.2. *A (smooth, real) n-manifold M comes naturally with a choice of a maximal atlas, and then given $p \in M$, one considers a chart (U, φ) such that $p \in U$, and then if one writes $\varphi = (x_1, \ldots, x_n)$ (i.e. if $(x_1, \ldots, x_n)$ are "local coordinates" at p), one may define (with abuse of notation) a basis for T_pM given by $\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}$, where given $1 \leq j \leq n$ and $f \in C^\infty(M)$,*

$$\frac{\partial}{\partial x_j}(f) := \left(\frac{\partial}{\partial x_j} \left(f \circ \varphi^{-1} \right) \right) (\varphi(p))$$

Hence, $\dim T_pM = \dim M = n$. Also, one should note well that such a basis depends on the point p and the choice of a chart containing p.

Proposition 3.1.1. *With the same setting as above and letting N be another smooth manifold, a smooth map $F : M \to N$ with $q = F(p)$ induces a linear map between tangent spaces, $dF_p : T_pM \to T_qM$, called the differential of F at p. This map is defined by the formula: $dF_p(X)(f) = X(f \circ F)$ for $X \in T_pM$ and $f \in C^\infty(N)$.*

Theorem 3.1.1. *Let M and N be two smooth manifolds and let $p \in M$. If a map $F : M \to N$ is local diffeomorphism at p (meaning that there is a neighborhood $U \subset M$ of p such that $F : U \to \varphi(U)$ is a diffeomorphism), then dF_p as defined above is an isomorphism. A partial converse also holds: if F is smooth and dF_p is an isomorphism, then F is a local diffeomorphism at p.*

Remark 3.1.3. *Since charts of smooth manifolds are by definition local diffeomorphisms everywhere, one sees that for any chart (U, φ), we have at every point $p \in M$ an induced isomorphism $d\varphi_p : T_pM \to T_{\varphi(p)}\varphi(U)$.*

Definition 3.1.8. *Given a smooth, real manifold M, we define the **tangent bundle** of M to be the disjoint union $TM := \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} \{p\} \times T_pM$ equipped with a particular smooth structure that we will unfold here. Define $\pi : TM \to M$ to be the projection map $\pi(p, v) = p$. Fix $p_0 \in M$ and let (U, φ) be a chart with $p_0 \in U$ and denote by $(x_1, \ldots, x_n)$ the associated coordinates. Define $\varphi' : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by:*

$$\varphi' \left(p, \sum_j v_j \frac{\partial}{\partial x_j} \right) = (\varphi(p), v_1, \ldots, v_n)$$

We can do this for every chart (U, φ), and we get ourselves a collection of the form $\{(U', \varphi')\}$ with $U' = \pi^{-1}(U)$ and φ' as defined above for a given (U, φ). We then equip TM with the following topology: a subset V of TM is open if for every such (U', φ') as defined above, we have that $\varphi'(V \cap \pi^{-1}(U))$ is open. And one can prove that with this topology and with the charts (U', φ'), TM is a $2n$ dimensional real topological manifold. Furthermore, these (U', φ') form compatible charts of TM and make it into a smooth manifold.

Definition 3.1.9. *Let M be a smooth real n-manifold. A subset S of M is called an* ***embedded submanifold*** *of M of dimension $d \leq n$ if for all $p \in S$, there is a neighborhood U of p in M and a chart $\varphi : U \to \varphi(U)$ such that:*

$$\varphi(S \cap U) = \varphi(U) \cap \{x \in \mathbb{R}^n, x_{d+1} = \cdots = x_n = 0\}$$

Remark 3.1.4. *As defined above, S can be shown to be a smooth real d-manifold (with appropriately defined charts). More elaborately, let us note that S itself is a topological manifold of dimension d, and it has a smooth structure making it a smooth d-manifold in such a way that the inclusion map $S \hookrightarrow M$ is an injective immersion (see Terminology 3.1.1) and a topological embedding (i.e. a homeomorphism onto its image in the subspace topology). Such a map as just described is usually referred to as a "smooth embedding".*

Remark 3.1.5. *Although there are other different notions of "submanifold", when we talk of a submanifold we will always be meaning "embedded submanifold". We will not be concerned with other types of submanifolds.*

Proposition 3.1.2. *Let M be a smooth real manifold and let $S \subset M$. Suppose that for every $p \in S$, there is a neighborhood U of p in M such that $S \cap U$ is a submanifold of U of dimension d. Then, S itself is a submanifold of M of dimension d.*

Proposition 3.1.3. *Let S be a submanifold of a smooth real manifold M, and let $p \in S$. Then, T_pN can be seen as (vector) subspace of T_pM, where it is given by:*

$$T_pN = \{X \in T_pM, X(f) = 0 \text{ for all } f \in C^\infty(M) \text{ with } f\mid_N \equiv 0\}$$

Also, a similar construction as outlined in Definition 3.1.8 may be carried out to define the smooth tangent bundle TS of S.

Definition 3.1.10. *Let M be a smooth real manifold. Let S be a submanifold of M. A* ***smooth vector field*** *on S is a smooth map $X : S \to TS$. It is common to write X_p for $X(p)$, $p \in S$.*

Definition 3.1.11. *Let $F : M \to N$ be a smooth map between two smooth real manifolds. The rank of F at a point $p \in M$, denoted by $\operatorname{rank}_p F$, is defined to be the rank of the linear map $dF_p : T_pM \to T_qN$, where $q = F(p)$.*

Terminology 3.1.1. *Such a map F as in the previous definition is called an* ***immersion at*** *p if $\operatorname{rank}_p F = \dim M$, and it is called a* ***submersion at*** *p if $\operatorname{rank}_p F = \dim N$. If F is an immersion at each point in M, then F is called an immersion. Similarly, F is called a submersion if it is a submersion at each point. Moreover, given $r \geq 0$, F is said to be of* ***constant rank*** *r if for all $p \in M$, we have $\operatorname{rank}_p F = r$, and in this case we write $\operatorname{rank} F = r$.*

Remark 3.1.6. *A submersion $M \to N$ is the same thing as a (smooth) constant rank map whose constant rank is $\dim N$. Similarly, an immersion $M \to N$ is the same as a constant rank map whose rank is $\dim M$.*

Theorem 3.1.2. *(Constant Rank theorem)*
Let M and N be smooth real manifolds of dimensions m and n respectively, and let $F : M \to N$ be a smooth map of constant rank r. Then, for every $p \in M$, there are charts (U, φ) around p and (V, η) around $F(p)$ such that $F(U) \subset V$ and the map $f := \eta \circ F \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \to \eta(V) \subset \mathbb{R}^n$, the expression of F in these coordinates, is given by the rule $\varphi(U) \ni x = (x_1, \dots, x_m) \mapsto f(x) = (x_1, \dots, x_r, 0, \dots, 0)$.

Definition 3.1.12. *Let X be any set. A **level set** of a function $f : X \to \mathbb{R}$ is a set of the form $f^{-1}(a)$ for some $a \in \mathbb{R}$. Note that, in general, for a map $f : X \to Y$ where $Y \neq \varnothing$, and for $a \in Y$, one defines $f^{-1}(a)$ to be the set $\{x \in X, f(x) = a\}$.*

Terminology 3.1.2. *If S is a d-dimensional submanifold of an n-manifold M, we say that S has **codimension** $n - d$, and we write $\operatorname{codim} S = n - d$.*

Corollary 3.1.1. *(Constant Rank Level Set)*
Let M and N be smooth real manifolds, and let $F : M \to N$ be a smooth map of constant rank r. Then, every level set of F is a closed submanifold of M of codimension r.

Corollary 3.1.2. *(Submersion theorem)*
Let $F : M \to N$ be a smooth map between smooth real manifolds M and N. If F is a submersion, then every level set S of F is a closed submanifold of M with $\operatorname{codim} S = \dim N$.

Definition 3.1.13. *A **complex topological manifold** M of dimension n is also a Hausdorff, second countable topological space, but with the charts (U, φ) of M consisting of homeomorphisms $\varphi : U \to \mathbb{B}_n \subset \mathbb{C}^n$ and with each U being homeomorphic to $\mathbb{B}_n$.*

Definition 3.1.14. *A complex-differentiable or **holomorphic manifold** M is the same as a smooth real manifold but with the transition maps being biholomorphisms. Atlases and related notions are defined in the same manner.*

Note that the notion of a holomorphic map between complex manifolds, the notions of tangent space and tangent bundles, and all that we discussed for real manifolds is defined in an analogous manner for complex manifolds with "holomorphic" replacing "smooth" and with charts mapping into $\mathbb{B}_n$.

3.1.2 *Complexification and Decomplexification of Vector Spaces*

Given a complex vector space V, one can "decomplexify" (or "realify") the space V by restricting the action of $\mathbb{C}$ to that of its subset $\mathbb{R}$. Conversely, given a real vector space V, one can "complexify" V by basically defining multiplication by i. Here and elsewhere, tensor products $\otimes$ are understood to be over $\mathbb{R}$.

Definition 3.1.15. *Let V be a complex vector space. If one looks only at the action of $\mathbb{R}$ on V, one obtains a real vector space $\mathbb{R} \otimes V$, the **decomplexification** or **realification** of V.*

Definition 3.1.16. *If V and W be complex vector spaces and $L : V \to W$ is a $\mathbb{C}$-linear map, then L induces an $\mathbb{R}$-linear map $T : \mathbb{R} \otimes V \to \mathbb{R} \otimes W$, given by $T(v) = L(v)$ for all $v \in V$. This map T is called the decomplexification of L.*

Proposition 3.1.4. *Let V be a complex vector space, and let $\{e_1, \ldots, e_n\}$ be a basis of V over $\mathbb{C}$. Then, $\mathcal{E} := \{e_1, \ldots, e_n, ie_1, \ldots, ie_n\}$ is a basis of V over $\mathbb{R}$, i.e. a basis of $\mathbb{R} \otimes V$. In particular, we have $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V$.*

Proof. Given $v \in V$, there are $c_j \in \mathbb{C}$ such that $v = \sum_{j=1}^{n} c_j e_j$. Then, writing $c_j = a_j + ib_j$ for each $j = 1, \ldots, n$, where $a_j, b_j \in \mathbb{R}$, we have:

$$v = \sum_{j=1}^{n} c_j e_j = \sum_{j=1}^{n} (a_j + ib_j) e_j = \sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j (ie_j)$$

Thus, the e_j and ie_j generate V over $\mathbb{R}$. Now, given $\alpha_j \in \mathbb{R}$ and $\beta_j \in \mathbb{R}$ such that:

$$\sum_{j=1}^{n} \alpha_j e_j + \sum_{j=1}^{n} \beta_j (ie_j) = 0$$

we have:

$$\sum_{j=1} (\alpha_j + i\beta_j) e_j = 0$$

and since the e_j are $\mathbb{C}$-linearly independent, we get $\alpha_j + i\beta_j = 0$ for each j. In particular, $\alpha_j = 0$ and $\beta_j = 0$ for all j. This shows that $\mathcal{E}$ is a basis for V over $\mathbb{R}$. In particular, $\dim_{\mathbb{R}} V = |\mathcal{E}| = 2\,|\{e_1, \ldots, e_n\}| = 2 \dim_{\mathbb{C}} V$. $\square$

Proposition 3.1.5. *Let V and W be complex vector spaces, and let $\mathcal{B} = \{e_1, \ldots, e_n\}$ and $\mathcal{B}' = \{e'_1, \ldots, e'_m\}$ be bases of V and W over $\mathbb{C}$ respectively. Let $L : V \to W$ be a $\mathbb{C}$-linear map, and let C be the matrix of L with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$. We may write $C = A + iB$, where A and B are real $m \times n$ matrices. Then, the matrix of the decomplexification T of L in the bases $\mathcal{E} = \{e_1, \ldots, e_n, ie_1, \ldots, ie_n\}$ and $\mathcal{E}' = \{e'_1, \ldots, e'_m, ie'_1, \ldots, ie'_m\}$ is given by the block matrix:*

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

Proof. To get the matrix of T with respect to the (ordered) bases $\mathcal{E}$ and $\mathcal{E}'$, we need to find the components of the $T(e_j)$ and the $T(ie_j)$ in terms of the e'_j and ie'_j. Let us observe that, for instance, $L(e_1)$ is given in $\mathcal{B}'$ by:

$$C \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (A + iB) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + B \cdot i \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So, if we write $C = (c_{st})$, $A = (a_{st})$ and $B = (b_{st})$, then we may write:

$$L(e_1) = \sum_{k=1}^{m} c_{k1} e'_k = \sum_{k=1}^{m} a_{k1} e'_k + \sum_{k=1}^{m} b_{k1} (ie'_k)$$

Thus, since T is just L, we have:

$$T(e_1) = \sum_{k=1}^{m} a_{k1} e'_k + \sum_{k=1}^{m} b_{k1}(ie'_k)$$

and this shows that the first column of the matrix of T with respect to $\mathcal{E}$ and $\mathcal{E}'$ is given as such: the first column of A gives us the first m entries, and the first column of B gives us the second m entries. The exact same reasoning applies to the $T(e_j)$ for $2 \leq j \leq n$. This gives us the first and third blocks of our desired matrix.
On the other hand, for all j, we have $T(ie_j) = L(ie_j) = iL(e_j) = iT(e_j)$. Hence,

$$T(ie_j) = i\left(\sum_{k=1}^{m} a_{kj} e'_k + \sum_{k=1}^{m} b_{kj}(ie'_k)\right) = \sum_{k=1}^{m}(-b_{kj})e'_k + \sum_{k=1}^{m} a_{kj}(ie'_k)$$

and this gives us the second and fourth blocks $-B$ and A. □

Definition 3.1.17. *Let V be an real vector space. A **complex structure** on V is a linear map $J : V \to V$ satisying $J^2 = -I$, i.e. $J \circ J = -I$. Here, I denotes the identity map of V.*

Proposition 3.1.6. *Let V be a real vector space, and let J be a complex structure on V. We may introduce the action of $\mathbb{C}$ on V given for $a, b \in \mathbb{R}$ and $v \in V$ as: $(a+ib)v = av + bJ(v)$. Then, V endowed with this action is a complex vector space, and the decomplexification of this latter complex vector space is the real vector space we started with.*

Corollary 3.1.3. *If V is a finite dimensional real vector space on which there is a defined complex structure J, then $\dim_{\mathbb{R}} V$ is even, and there is a basis of V over $\mathbb{R}$ such that the matrix of J in this basis is given by the block matrix:*

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix, 0 is the $n \times n$ zero matrix, and $n = \dfrac{\dim_{\mathbb{R}} V}{2}$.

Proposition 3.1.7. *Let V be a real vector space. Define on the external direct sum $V \oplus V$ the operator J given by the formula $J(v_1, v_2) = (-v_2, v_1)$. Then, J is a complex structure on $V \oplus V$.*

Definition 3.1.18. *Let V be a real vector space. The **complexification** of V is defined to be the complex vector space one obtains from the real vector space $V \oplus V$ through the action in Proposition 3.1.6 with J being the map defined just as in Proposition 3.1.7. One denotes the complexification of V by $\mathbb{C} \otimes V$.*

To view the real vector space V inside $\mathbb{C} \otimes V$, one identifies V with the subspace $\{(v, 0), v \in V\}$ of $V \otimes V$. Then, also note that one has $i(v, 0) = J(v, 0) = (0, v)$ for all $v \in V$. Hence, after viewing V as $V \otimes \{0\} \subset \mathbb{C} \otimes V$, one may write every $(v_1, v_2) \in \mathbb{C} \otimes V$ as $(v_1, v_2) = (v_1, 0) + (0, v_2) = (v_1, 0) + i(v_2, 0) = v_1 + iv_2$. Thus, we see that $\mathbb{C} \otimes V = V \oplus iV$.

Proposition 3.1.8. *Let V be a real vector space which has a complex structure J.*

1) *Every $\mathbb{R}$-basis of V is a $\mathbb{C}$-basis of $\mathbb{C}\otimes V$. In particular, $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} \mathbb{C}\otimes V$.*

2) *Let W be a real vector space with a complex structure, and let $L : V \to W$ be an $\mathbb{R}$-linear map. Then, the map $L' : \mathbb{C}\otimes V \to \mathbb{C}\otimes W$ given by the fomrula $L'(v_1, v_2) = (L(v_1), L(v_2))$ is a $\mathbb{C}$-linear map whose decomplexification is L.*

Remark 3.1.7. *The L' in the second point of the latter proposition is usually called the complexification of the linear map L.*

Proposition 3.1.9. *Let $p \in \mathbb{C}^n$, where $\mathbb{C}^n$ is viewed as a smooth real manifold of dimension $2n$. Then, in the same notation of Remark 3.1.2, the set:*

$$\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right\}$$

forms a basis for the real vector space $T_p\mathbb{C}^n$.

Remark 3.1.8. *One may then define by extension the linear operator J on $T_p\mathbb{C}^n$ given at the basis elements of $T_p\mathbb{C}^n$ by:*

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}$$

and:

$$J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$$

for each $1 \leq j \leq n$. It is easy to see that $J^2 = -I$. Hence, J is a complex structure on $T_p\mathbb{C}^n$. Also, J is well-defined because the definition of J is independent of the choice of holomorphic charts at p. Therefore, we obtain the complexification $\mathbb{C}\otimes T_p\mathbb{C}^n$ of $T_p\mathbb{C}^n$ as in Definition 3.1.18, which is also commonly denoted by $\mathbb{C}T_p\mathbb{C}^n$. Note that by Proposition 3.1.8, J extends to a $\mathbb{C}$-linear operator on $\mathbb{C}T_p\mathbb{C}^n$, which, with slight abuse of notation, we also denote by J.

3.1.3 *Real Hypersurfaces, The Levi Form, and Related Notions*

We aim in this subsection to revise a few things about real hypersurfaces: their properties and associated concepts such as the Levi form of a hypersurface and pseudocovexity. We also mention smooth domains and a characterization of domains of holomorphy.

Definition 3.1.19. *A subset M of $\mathbb{C}^n$ is called a smooth **real hypersurface** if for every $p \in M$, there is a neighborhood U of p in $\mathbb{C}^n$ and a smooth (with $\mathbb{C}^n$ looked at as an $\mathbb{R}$-vector-space) map $\rho : U \to \mathbb{R}$ such that for all $q \in U$, $D\rho(q) \neq 0$, and such that $M \cap U = \{z \in U, \rho(z, \overline{z}) = 0\}$. Such a ρ is called a **local defining function** for M near p. If there is real-smooth ρ defined on an open set containing M with nowhere vanishing derivative on M and with the property that for all z, $\rho(z) = 0 \iff z \in M$, then ρ is called a global defining function for M or just defining function for M.*

Two notes are due. First, by $D\rho$, we are referring to the real differential of ρ, i.e. its derivative as a map between real vector spaces. Hence, the statement that $D\rho(q) \neq 0$ for all $q \in U$ is the statement that at least one of the real partial derivatives $\partial\rho/\partial x_j$ for some j or $\partial\rho/\partial y_k$ for some k is non-zero at q. This, by the way, is the condition needed to apply the Implicit Function theorem (Theorem 2.1.7) which allows us to locally write the zero set $\{\rho = 0\}$ as a graph, and that's what makes M a submanifold (see Proposition 3.1.11 for a rigorous proof of this fact). Second, this definition includes some abuse of notation. The function ρ is indeed defined on U, and it makes no sense to speak of $\rho(z, \overline{z})$, as if ρ is taking an input in $\mathbb{C}^n \times \mathbb{C}^n$. Instead, one talks of $\rho(z)$ for $z \in \mathbb{C}^n$. This abuse of notation is standard, and it is done because it is good to look at ρ as a function of z and $\overline{z}$ especially in the context of holomorphic and anti-holomorphic tangent vectors and other notions, for instance those related to the Levi form, discussed below.

Example 3.1.3. *Given any smooth domain $U \subset \mathbb{C}^n$ (see Definition 3.1.26), we have that ∂U is a real hypersurface in $\mathbb{C}^n$. For instance, the unit sphere $\partial\mathbb{B}_n$ in $\mathbb{C}^n$ is a real hypersurface in $\mathbb{C}^n$.*

Example 3.1.4. *Consider $\mathbb{C}^{n+1}$ with coordinates $(w, z) \in \mathbb{C} \times \mathbb{C}^n$, and the subset M with local defining function $\rho = \Re w - \|z\|^2$. This is a hypersurface in $\mathbb{C}^{n+1}$, and most of our attention in this thesis will be on it.*

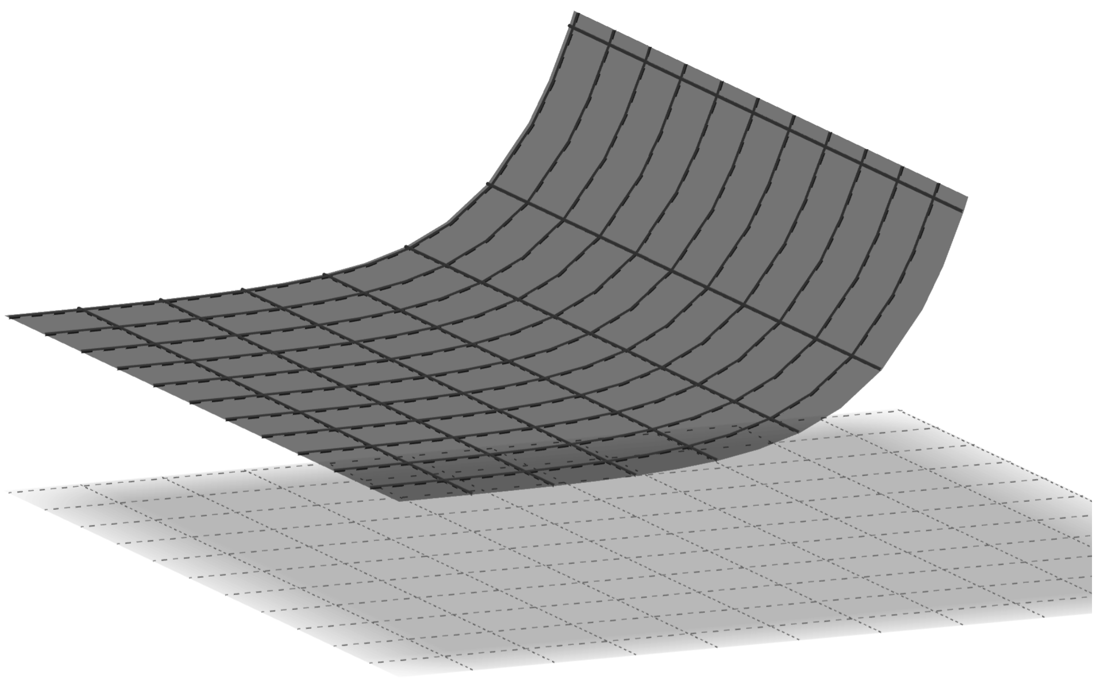

Figure 3.1: Real hypersurfaces in $\mathbb{C}^n$ are real submanifolds of $\mathbb{C}^n$ of codimension 1 (see Proposition 3.1.11). This figure illustrates how one would imagine such objects: what is shown here is a real submanifold of $\mathbb{R}^3$ of codimension 1.

Example 3.1.5. *The unit sphere $\mathbb{S}^{2n-1}$ in $\mathbb{C}^n$ is a hypersurface with local defining function $\rho = \|z\|^2 - 1$. Note that the hypersurface $\mathbb{S}^{2n-1} \setminus \{N\}$ is homeomorphic to M, the hypersurface in the previous example (this time in $\mathbb{C}^n$ and not $\mathbb{C}^{n+1}$), where $N = (0, \ldots, 0, 1) \in \mathbb{C}^n$ is the "north pole". Indeed, let $\varphi : \mathbb{S}^{2n-1} \setminus \{N\} \to \mathbb{C}^n$ be the map $\varphi = (\varphi_1, \ldots, \varphi_n)$ given by:*

$$\varphi_j(z) = \frac{iz_j}{1 - z_n} \text{ for } 1 \leq j \leq n-1, \text{ and } \varphi_n(z) = \frac{1 + z_n}{1 - z_n}$$

This φ lands in M. Indeed, for any $z \in \mathbb{S}^{2n-1} \setminus \{N\}$, and writing $z_n = x_n + iy_n$, we have:

$$\varphi_n(z) = \frac{(1 + z_n)(1 - \overline{z_n})}{|1 - z_n|^2} = \frac{(1 + x_n + iy_n)(1 - x_n + iy_n)}{|1 - z_n|^2}$$

so:

$$\Re\varphi_n(z) = \frac{1 - (x_n^2 + y_n^2)}{|1 - z_n|^2} = \frac{1 - |z_n|^2}{|1 - z_n|^2} = \frac{\sum_{j=1}^{n-1} |z_j|^2}{|1 - z_n|^2} = \sum_{j=1}^{n-1} \frac{|z_j|^2}{|1 - z_n|^2} = \sum_{j=1}^{n-1} |\varphi_j(z)|^2$$

Moreover, φ is a bijection onto M, and $\varphi^{-1} : M \to \mathbb{S}^{2n-1} \setminus \{N\}$ is given by $(\psi_1, \ldots, \psi_n)$, where:

$$\psi_j(z) = \frac{-2iz_j}{1 + z_n} \text{ for } 1 \leq j \leq n-1, \text{ and } \psi_n(z) = \frac{-1 + z_n}{1 + z_n}$$

It's obvious that φ and φ^{-1} are continuous maps, so that φ is a homeomorphism. Furthermore, note that the map φ is a biholomorphism (a biholomorphic change of coordinates) between the open sets $\{z_n \neq 1\}$ and $\{z_n \neq -1\}$.

Proposition 3.1.10. *Let M be a real hypersurface in $\mathbb{C}^n$. Let $p \in M$, and let ρ_1 and ρ_2 be two local defining functions for M near p with associated neighborhoods U_1 and U_2 respectively. Then, there is an open set $U \subset U_1 \cap U_2$ containing p and a map $g \in C^\infty(U)$ such that $\rho_2 = g\rho_1$ on U and g is nowhere zero on U. Moreover, if ρ_1 and ρ_2 are global defining functions for M with associated open sets U_1 and U_2 resepctively containing M, then there is an open set $U \subset U_1 \cap U_2$ and a $g \in C^\infty(U)$ such that $\rho_2 = g\rho_1$ and g is nowhere zero on U.*

Proposition 3.1.11. *A subset S of $\mathbb{C}^n$ is a real hypersurface in $\mathbb{C}^n$ if and only if S is a real submanifold of the smooth, real manifold $\mathbb{C}^n$, with* $\operatorname{codim} S = 1$.

Proof. Suppose that S is a (real) submanifold of (the smooth, real manifold) $\mathbb{C}^n$ and that S has codimension 1. Let $p \in S$. By Definition 3.1.9, there is a chart (U, φ) around p such that:

$$\varphi(S \cap U) = \varphi(U) \cap \left(\mathbb{R}^{2n-1} \times \{0\}\right)$$

In other words, if one writes $\varphi = (\varphi_1, \ldots, \varphi_{2n-1}, \varphi_{2n})$, then one may say:

$$\varphi(S \cap U) = \{(\varphi_1(z), \ldots, \varphi_{2n-1}(z), 0)\,, z \in U\}$$

Take $\rho = \varphi_{2n} : U \to \mathbb{R}$. If there is a point $q \in U$ such that $D\rho(q) = 0$, then $d\varphi(q)$ is non-invertible, and by Theorem 3.1.1, this implies that φ is not a local

diffeomorphism, a contradiction. Hence, $D\rho(q) \neq 0$ for all $q \in U$. Moreover, for $z \in S \cap U$, $\varphi(z) \in \varphi(S \cap U)$, so $\rho(z) = \varphi_{2n}(z) = 0$. Hence, $S \cap U \subset \{z \in U, \rho(z) = 0\}$. On the other hand, let $z \in U$ and suppose that $\rho(z) = 0$. As $z \in U$, $\varphi(z) \in \varphi(U)$, and by definition of ρ, we have $\varphi_{2n}(z) = 0$. Hence, $\varphi(z) \in \varphi(U) \cap (\mathbb{R}^{2n-1} \times \{0\})$, so that $\varphi(z) \in \varphi(S \cap U)$. As φ is injective, we have $z \in S \cap U$. This shows that $\{z \in U, \rho(z) = 0\} \subset S \cap U$. Therefore, we conclude that $S \cap U = \{z \in U, \rho(z) = 0\}$. We have thus shown that S is a real hypersurface in $\mathbb{C}^n$.
Now suppose that S is a real hypersurface in $\mathbb{C}^n$. Let $p \in S$. By definition, there is a neighborhood U of p in $\mathbb{C}^n$ and a smooth map $\rho : U \to \mathbb{R}$ such that $D\rho$ nowhere vanishes on U and $S \cap U = \{z \in U, \rho(z) = 0\}$. As $D\rho$ nowhere vanishes, we know that ρ has constant rank $1 = \dim \mathbb{R}$, so that ρ is a submersion. Moreover, $S \cap U = \rho^{-1}(0)$. Hence, by Corollary 3.1.2, we see that $S \cap U$ is submanifold of U of codimension 1. By Proposition 3.1.2, we deduce that S is a submanifold of $\mathbb{C}^n$ of codimension 1. $\square$

Definition 3.1.20. *Let $p \in \mathbb{C}^n$ and let $X \in T_p\mathbb{C}^n$, where $\mathbb{C}^n$ is looked at as a smooth real manifold of dimension $2n$. Then, by Proposition 3.1.9, we know that there are $a_j, b_j \in \mathbb{R}$ such that:*

$$X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial y_j}$$

*Given a hypersurface $M \subset \mathbb{C}^n$ with $p \in M$, we say that X is **tangent** to M at p if $X(\rho) = 0$, i.e. if:*

$$\sum_{j=1}^{n} a_j \frac{\partial \rho}{\partial x_j}(p, \overline{p}) + \sum_{j=1}^{n} b_j \frac{\partial \rho}{\partial y_j}(p, \overline{p}) = 0$$

where ρ is any local defining function of M near p.

Note that this definition is independent of the choice of a local defining function ρ near p. The tangent space at p of M as a manifold is the same as the tangent space of M defined as:

$$T_pM = \{X \in T_p\mathbb{C}^n,\ X \text{ is tangent to } M \text{ at } p\}$$

The **complexified tangent space** $\mathbb{C}T_pM$ of M at p and the complexified tangent space $\mathbb{C}T_p\mathbb{C}^n$ of $\mathbb{C}$ at p are both defined just as in the previous subsection 3.1.2, and we can also understand them as those spaces which emerge when allowing the a_j and b_j in the latter expressions to range over the complex numbers.
Using the Wirtinger notation as in Notation 2.2.3, which applies and works as it should in terms of corresponding holomorphic charts, we may write $\mathbb{C}T_p\mathbb{C}^n$ in the form:

$$\mathbb{C}T_p\mathbb{C}^n = \left\{ \sum_{j=1}^{n} a_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial \overline{z_j}}, a_j, b_j \in \mathbb{C} \right\}$$

Definition 3.1.21. *Following the latter discussion, given $X \in \mathbb{C}T_p\mathbb{C}^n$, we may write:*

$$X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial \overline{z_j}}$$

for some $a_j, b_j \in \mathbb{C}$. We then say that:

*a) X is a **holomorphic vector** at p if for all j, $b_j = 0$.*

*b) X is an **anti-holomorphic vector** at p if for all j, $a_j = 0$.*

Note that, by the chain rule, we can show that this definition is independent of the choice of holomorphic charts.

Notation 3.1.1. *We denote by $T_p^{1,0}\mathbb{C}^n$ the vector subspace of $\mathbb{C}T_p\mathbb{C}^n$ consisting of all holomorphic tangent vectors to $\mathbb{C}^n$ at p, and we denote by $T_p^{0,1}\mathbb{C}^n$ the subspace of all anti-holomorphic tangent vectors at p.*

Remark 3.1.9. *The vector space $T_p^{1,0}\mathbb{C}^n$ has the $\mathbb{C}$-basis $\{\partial/\partial z_1, \ldots, \partial/\partial z_n\}$. On the other hand, $T_p^{0,1}\mathbb{C}^n$ has the $\mathbb{C}$-basis $\{\partial/\partial\overline{z_1}, \ldots, \partial/\partial\overline{z_n}\}$. In particular, we have $\dim_{\mathbb{C}} T_p^{1,0}\mathbb{C}^n = \dim_{\mathbb{C}} T_p^{0,1}\mathbb{C}^n = n$. Moreover, note that $\mathbb{C}T_p\mathbb{C}^n = T_p^{1,0}\mathbb{C}^n \oplus T_p^{0,1}\mathbb{C}^n$, and that $\dim_{\mathbb{C}} \mathbb{C}T_p\mathbb{C}^n = 2n$.*

Notation 3.1.2. *Let M be a real hypersurface in $\mathbb{C}^n$, and let $p \in M$. We denote by $\mathcal{V}_p$ the space of all anti-holomorphic vectors tangent to M at p. In other words, we put $\mathcal{V}_p := T_p^{0,1}\mathbb{C}^n \cap \mathbb{C}T_pM$. We also write $T_p^{0,1}M$ for $\mathcal{V}_p$. On the other hand, we denote by $\overline{\mathcal{V}_p}$ the space of all holomorphic vectors tangent that are tangent to M at p, i.e. $\overline{\mathcal{V}_p} = T_p^{1,0}\mathbb{C}^n \cap \mathbb{C}T_pM$.*

Definition 3.1.22. *For a real hypersurface $M \subset \mathbb{C}^n$ and $p \in M$, define the **complex tangent space** of M at p, T_p^cM, to be $T_pM \cap J^{-1}(T_pM)$. Or, in other words, we define $T_p^cM := \{X \in T_pM, J(X) \in T_pM\}$, where J is as defined in Remark 3.1.8.*

Proposition 3.1.12. *In the same setting as above, J restricts to an operator on T_p^cM and so defines a complex structure on T_p^cM. Also, J extends to a $\mathbb{C}$-linear operator on $\mathbb{C}T_p^cM$.*

Proposition 3.1.13. *Let $M \subset \mathbb{C}^n$ be a real hypersurface and let $p \in M$. Then, we have that $\mathcal{V}_p = \{X \in \mathbb{C}T_pM, J(X) = -iX\}$. Moreover, $\mathcal{V}_p \oplus \overline{\mathcal{V}_p} = \mathbb{C}T_p^cM$, $\dim_{\mathbb{C}} \mathbb{C}T_p^cM = 2n-2$, and $\dim_{\mathbb{C}} \mathcal{V}_p = \dim_{\mathbb{C}} \overline{\mathcal{V}_p} = n-1$.*

Many of the above (and below) definitions can also be made for real submanifolds of $\mathbb{C}^n$ which are not necessarily of codimension 1, i.e. not necessarily real hypersurfaces (see Chapter 1 of [6] for explication). For instance, if one defines $\mathcal{V}_p$ for a general real submanifold M of $\mathbb{C}^n$, one says that M is a **CR submanifold** of $\mathbb{C}^n$ if $\dim_{\mathbb{C}} \mathcal{V}_p$ is constant for $p \in M$. This constant dimension is called the **CR dimension** of M. In particular, following the very latter proposition, every real hypersurface in $\mathbb{C}^n$ is CR submanifold of CR dimension $n-1$.

Definition 3.1.23. *A **smooth complex vector field** X on a hypersurface $M \subset \mathbb{C}^n$ is map which associates to each $p \in M$ an element X_p of $\mathbb{C}T_p^cM$ in a real-smooth manner. In other words, such an X may be written as:*

$$X = \sum_{j=1}^{n-1} a_j(\cdot)\frac{\partial}{\partial z_j} + \sum_{j=1}^{n-1} b_j(\cdot)\frac{\partial}{\partial \overline{z_j}}$$

where the $a_j, b_j : M \to \mathbb{C}$ are real-smooth maps, i.e. X acts as $p \mapsto X_p$ where:

$$X_p = \sum_{j=1}^{n-1} a_j(p)\frac{\partial}{\partial z_j} + \sum_{j=1}^{n-1} b_j(p)\frac{\partial}{\partial \overline{z_j}}$$

Moreover, X must satisfy $X_p(\rho) = 0$ for all pairs (p, ρ) with $p \in M$ and ρ a local defining function at p:

$$\sum_{j=1}^{n-1} a_j(p)\frac{\partial \rho}{\partial z_j} + \sum_{j=1}^{n-1} b_j(p)\frac{\partial \rho}{\partial \overline{z_j}} = 0$$

Definition 3.1.24. *Let $M \subset \mathbb{C}^n$ be a real hypersurface. A smooth complex vector field X on M is called a **CR vector field** on M if for all $p \in M$, we have $X_p \in \mathcal{V}_p$. Also, a function $f \in C^k(M)$ for some $k \geq 1$ is called a **CR function** if $Xf \equiv 0$ for every CR vector field on M.*

Stated otherwise, a CR vector field X on M is a map of the form:

$$X = \sum_{j=1}^{n-1} b_j(\cdot)\frac{\partial}{\partial \overline{z_j}}$$

where every $b_j : M \to \mathbb{C}$ is real-smooth. The condition that $Xf \equiv 0$ for such a CR vector field X on M and a map $f \in C^k(M)$ with $k \geq 1$ is then the condition that for all $p \in M$:

$$\sum_{j=1}^{n-1} b_j(p)\frac{\partial f}{\partial \overline{z_j}} = 0$$

Definition 3.1.25. *Let $f : \mathbb{C}^n \to \mathbb{R}$ be a real-smooth function on $\mathbb{C}^n$. We may define the **Levi form of the map** f at a point $p \in \mathbb{C}^n$ as the bilinear form on $\mathbb{C}^n \times \mathbb{C}^n$ given by the formula:*

$$L_p(f)(\xi, \eta) = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(p)\xi_j\overline{\eta_k}$$

where f is looked at as a function of the z_j and $\overline{z_j}$.

Note that it is also customary to define the Levi form as being instead the **quadratic form** associated to that bilinear form, i.e.

$$L_p(f)(\xi) = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(p)\xi_j\overline{\xi_k}$$

from which one can reconstruct that original bilinear form by standard techniques of bilinear algebra.
Also, note that instead of understanding f as a function of the z_j and $\overline{z_j}$, we may understand the z_j and $\overline{z_j}$ derivatives as being derivations (living in $\mathbb{C}T_p\mathbb{C}^n$) and acting on the element f of $C^\infty(\mathbb{C}^n)$.

Remark 3.1.10. *With f being as a above, $L_p(f)$ is in fact a Hermitian form. This will also be the case for the Levi form of a smooth domain in $\mathbb{C}^n$ and that of a real hypersurface in $\mathbb{C}^n$, mentioned below.*

Definition 3.1.26. *Let $U \subset \mathbb{C}^n$. U is called a **smooth domain** if U is given by $\{z \in \mathbb{C}^n, \rho(z) < 0\}$ i.e. $\{\rho < 0\}$ and if $\partial U = \{\rho = 0\}$ for some real-smooth map $\rho : \mathbb{C}^n \to \mathbb{R}$ such that $D\rho(z) \neq 0$ for all $z \in \partial U$. Such a ρ is called a defining function for U.*

Remark 3.1.11. *If $U \subset \mathbb{C}^n$ is a smooth domain, then ∂U is a real hypersurface.*

Let $U \subset \mathbb{C}^n$ be a smooth domain. Given a defining function ρ of U and a point $p \in \partial U$, we may look at the Levi form $L_p(\rho)$ of ρ at p. If σ is any other defining function of U, then $L_p(\rho)$ and $L_p(\sigma)$ have the same **signature**, i.e. the same number of positive, zero, and negative eigenvalues. Moreover, this is true even if one applies a holomorphic change of coordinates. Hence, $L_p(\rho)$ is positive semi-definite i.e. PSD (resp. positive definite i.e. PD) if and only if $L_p(\sigma)$ is PSD (resp. PD), and these properties are biholomorphic invariants.

Definition 3.1.27. *Given a smooth domain $U \subset \mathbb{C}^n$ and $p \in \partial U$, we say that U is **pseudoconvex** if for a given defining function ρ of U, we have that $L_p(\rho)$ is PSD. We say that U is **strongly pseudoconvex** if instead $L_p(\rho)$ is PD.*

The pseudoconvexity condition is very analogous to the "Hessian principle" in real analysis of several variables regarding convexity/concaveness. If the Levi form, which in essence is modeling a complex Hessian matrix of a sort, is PSD, then the shape is convex in a particular sense. In fact, if one does the Taylor expansion of a real-smooth map $f : \mathbb{C}^n \to \mathbb{R}$ and then one employs the Wirtinger notation to write things in terms of the z_j and $\overline{z_k}$ derivatives, then one obtains an expansion in which the Levi form appears naturally and suggests this role of a Hessian.

Definition 3.1.28. *A real hypersurface $M \subset \mathbb{C}^n$ is said to be pseudoconvex (resp. strongly pseudoconvex) if M is the boundary of some pseudoconvex (resp. strongly pseudoconvex) smooth domain U.*

Usually, one defines the Levi map of an abstract CR manifold M at a point $p \in M$ as a certain bilinear map $\mathcal{V}_p \times \mathcal{V}_p \to \mathbb{C}T_pM/(\mathcal{V}_p \oplus \overline{\mathcal{V}_p}) = \mathbb{C}T_pM/\mathbb{C}T_p^cM$. This map is generally not a Hermitian form. For a rigorous construction of the Levi map, one can refer to Chapters 1 and 2 of [6]. If M is a real hypersurface (and in more general contexts), this Levi map induces a Levi form on M at a given point p and coheres with the definitions we are making here. In particular, one may define what it means for a real hypersurface to be pseudoconvex or strongly pseudoconvex starting from this Levi map.

Theorem 3.1.3. *(Solution to the classical Levi problem)*
Let $U \subset \mathbb{C}^n$ be a smooth domain. Then, U is a pseudoconvex domain if and only if U is a domain of holomorphy.

Proposition 3.1.14. *Let $M = \{(w,z) \in \mathbb{C} \times \mathbb{C}^n, \rho(w,z) < 0\}$, where ρ is the map $\rho(w,z) = \|z\|^2 - \Re w$. We have that M is a smooth strongly pseudoconvex domain in $\mathbb{C}^{n+1}$ with boundary $\{\rho = 0\}$.*

Proof. It is clear that ρ is real-smooth. Let us write the coordinates as $w = x_0 + iy_0$ and $z_j = x_j + iy_j$ for all $1 \le j \le n$. We have:

$$\frac{\partial \rho}{\partial x_0} = \frac{\partial}{\partial x_0}\left(\sum_{j=1}^{n}(x_j^2 + y_j^2) - x_0\right) = -1$$

so the real differential $D\rho$ is never zero, because $D\rho$ is the column vector whose components are the $\partial\rho/\partial x_0, \dots, \partial\rho/\partial x_n, \partial\rho/\partial y_0, \dots, \partial\rho/\partial y_n$.
Let $(w,z) \in \mathbb{C} \times \mathbb{C}^n$ and suppose that $\rho(w,z) = 0$. Let $\{(w_n, z_n)\}_n \subset \mathbb{C}^{n+1}$ be the sequence given by $z_n = z$ and $w_n = w - \dfrac{1}{n}$ for all $n \ge 1$. Then, for all n,

$$\rho(w_n, z_n) = \|z_n\|^2 - \Re w_n = \|z\|^2 - \Re w - \frac{1}{n} = \rho(w,z) - \frac{1}{n} = -\frac{1}{n} < 0$$

so, $\{(w_n, z_n)\}_n$ is a sequence in M. Clearly $(w_n, z_n) \to (w,z)$, so that $(w,z) \in \partial M$. Conversely, let $(w,z) \in \partial M$. Then, there is a sequence in M converging to (w,z), and by continuity of ρ this implies that $\rho(w,z) \le 0$. Since $(w,z) \in \partial M$, and since M is an open set (by continuity of ρ), we necessarily have $(w,z) \notin M$, so that $\rho(w,z) = 0$. This show that $\partial M = \{\rho = 0\}$.
Let us now compute the (general) Levi form of ρ. We have:

$$L(\rho)(\xi,\eta) = \sum_{j=1}^{n} \frac{\partial^2 \rho}{\partial w \partial \overline{z_j}} \xi_0 \overline{\eta_j} + \sum_{j=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{w}} \xi_j \overline{\eta_0} + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}} \xi_j \overline{\eta_k}$$

But for all t:

$$\frac{\partial^2 \rho}{\partial w \partial \overline{z_t}} = \frac{\partial^2}{\partial w \partial \overline{z_t}}\left(\sum_{j=1}^{n} z_j \overline{z_j} - \frac{1}{2}(w + \overline{w})\right) = \frac{\partial}{\partial w}(z_t) = 0$$

and for all t:

$$\frac{\partial^2 \rho}{\partial z_t \partial \overline{w}} = \frac{\partial}{\partial z_t}\left(-\frac{1}{2}\right) = 0$$

Finally, for all j and k, we have:

$$\frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}} = \frac{\partial}{\partial z_j}(z_k) = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \ne k \end{cases}$$

Therefore, we have:

$$L(\rho)(\xi,\eta) = \sum_{j=1}^{n} \xi_j \overline{\eta_j}$$

In particular,

$$L(\rho)(\xi,\xi) = \sum_{j=1}^{n} \xi_j \overline{\xi_j} = \sum_{j=1}^{n} |\xi_j|^2$$

Hence, clearly $L(\rho)(\xi,\xi) > 0$ for all $\xi \ne 0$, i.e. $L(\rho)$ is PD. This shows that M is strongly pseudoconvex. ☐

3.2 Analytic and Stationary Disks

We define analytic and stationary disks, discuss some strong reasons why they are useful objects, and motivate our work in Chapters 4 and 5.

3.2.1 *Analytic Disks: a More General Viewpoint*

Analytic disks are holomorphic embeddings of Δ in $\mathbb{C}^n$ with a continuous boundary. It is common to understand that analytic disks are the complex-analytic analogue of segments in real analysis: a one-complex-dimensional bounded simply connected holomorphic complex-analytic object is what corresponds to the smoothest form of a one-real-dimensional bounded connected real-analytic object.

Definition 3.2.1. *An **analytic disk** in $\mathbb{C}^n$ is a function $f : \overline{\Delta} \to \mathbb{C}^n$ which is continuous on $\overline{\Delta}$ and holomorphic on Δ.*

Definition 3.2.2. *An analytic disk $f : \overline{\Delta} \to \mathbb{C}^n$ is called **attached to** a subset M of $\mathbb{C}^n$ if $f(\partial\Delta) \subset M$. Also, **the center** of such an analytic disk is the point $f(0)$.*

Notation 3.2.1. *For a subset M of $\mathbb{C}^n$, we denote by $\mathcal{A}(M)$ the set of all analytic disks in $\mathbb{C}^n$ attached to M.*

The following theorem is stated as Proposition 6.2.2 in [6] but for what are termed **generic** submanifolds of $\mathbb{C}^n$. A generic submanifold of $\mathbb{C}^n$ is a real submanifold M of $\mathbb{C}^n$ that admits at every $p \in M$ a local defining function whose components' complex gradients are linearly independent over $\mathbb{C}$. See Chapter 1 in [6]. Real hypersurfaces are by default a subfamily of generic submanifolds because their defining functions have only one component whose complex gradient does not vanish anywhere (if it vanishes somewhere, so would its real-differential $D\rho$, which we know doesn't happen). To stay consistent with the build-up we made, we state the theorem here for real hypersurfaces instead. We stress, though, that it applies more generally.

Theorem 3.2.1. *Let M be a real hypersurface in $\mathbb{C}^n$, and let $p \in M$. There is an open neighborhood $U \subset M$ of p such that for every CR function f on U of class C^1 there exists a continuous function F defined on W, where:*

$$W := \bigcup_{g \in \mathcal{A}(U)} g\left(\overline{\Delta}\right)$$

with the following properties:

1) For all $g \in \mathcal{A}(U)$, we have $F \circ g \in \mathcal{O}(\Delta)$.

2) The restriction of F to U is f.

3) $F \in \mathcal{O}\left(W^{\circ}\right)$.

This result says that for certain types of submanifolds (including hypersurfaces), every CR function locally admits a holomorphic extension to an open set generated by all sufficiently small analytic disks attached to the submanifold. This reveals a robust connection between analytic disks and questions of holomorphic extensions, and it is already a great reason for one to be interested in analytic disks in general questions of construction of holomorphic maps. It is one of the primary reasons we adapt our approach to finding our desired automorphisms to considerations of certain types analytic disks, the stationary disks, our next topic.

3.2.2 *Stationary Disks and their Relation to Automorphisms*

Stationary disks attached to real hypersurfaces in $\mathbb{C}^n$ are a subfamily of attached analytic disks that satisfy a certain lifting condition. Stationary disks proved very useful in general mapping problems, and they are biholomorphic invariants. These two facts, alongside results like Theorem 3.2.1, naturally push us to explore very basic questions such as the one we tackle in our thesis: **is it possible to better understand automorphisms of real hypersurfaces in $\mathbb{C}^n$ by observing how they interact with attached stationary disks?**
Specifically, we ask the question for hypersurfaces which have a "nice" enough shape locally, the strongly pseudoconvex ones, which are locally more or less a "perturbed" sphere. In our work, we study the paraboloid $\{(w, z) \in \mathbb{C} \times \mathbb{C}, \Re w = |z|^2\}$ from the suggested point of view, and we recover certain types of automorphisms of this paraboloid. Something which we rely on a lot in our work is that we do not need to worry about local issues: we have as many "slanted" disks as we want at our disposal because the paraboloid is a relatively simple model. Other models, though, might pose local issues, and one then needs to tweak their approach to accommodate for this necessity of staying close enough to the origin. Furthermore, other more general models, "perturbed" ones, might prove challenging in terms of nailing down the disks and thus add other layers of restrictions.

Definition 3.2.3. *Let $M \subset \mathbb{C}^m$ be a real hypersurface given by a global defining function $\sigma : \mathbb{C}^m \to \mathbb{R}$. A map $f : \overline{\Delta} \to \mathbb{C}^m$ is called a **stationary disk** attached to M if the following conditions hold:*

1. $f \in \mathcal{O}(\Delta) \cap C(\overline{\Delta})$
2. $f(\partial\Delta) \subset M$
3. *There exists a continuous map $c : \partial\Delta \to \mathbb{R}^*$, such that the map:*

$$\partial\Delta \ni \zeta \mapsto \zeta c(\zeta)\partial\sigma(f(\zeta))$$

extends holomorphically to Δ.

Note that by $\partial\sigma$, we are referring to the complex gradient of σ in the sense of Remark 2.2.3. Most definitely, the only possible holomorphic defining functions are the constant functions, for the same reason mentioned in Example 2.2.1.
The stationarity condition, expressed in the third point, can also be formulated as

being that the map $\partial\Delta \ni \zeta \mapsto c(\zeta)\partial\sigma(f(\zeta))$ admits a meromorphic extension to Δ, holomorphic on $\Delta \setminus \{0\}$, with a possible pole of order at most 1 at 0. In the formulation of Definition 3.2.3, the role of the ζ in $\zeta c(\zeta)\partial\sigma(f(\zeta))$ is to take care of that potential pole.
Geometrically, the stationarity condition is formulated in terms of cotangent and conormal bundles. See Definition 1.4 in [8] for instance, where this formulation is done in precise terms and adequate references are provided for further exploration. The reason we do not approach stationarity in this manner is that we do not wish to use any geometric language of that sort in our work. Our primary interest in these kinds of disks is, as mentioned above, two-fold. First, they are biholomorphic invariants (Proposition 3.2.2). Second, they are "abundant" enough in a sense which will become clear in Chapter 4 as we discuss further.

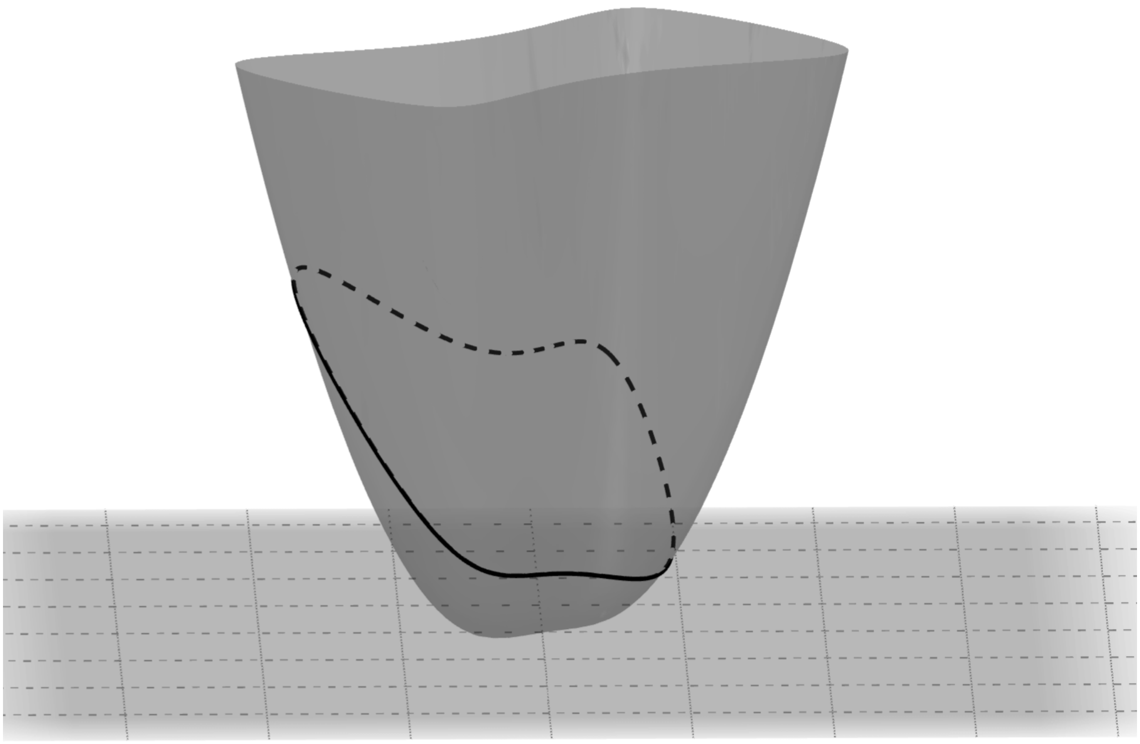

Figure 3.2: For a stationary disk f attached to a hypersurface M, the image $f(\partial\Delta)$ traces M and closes on itself, just as here the (black) curve traces this hypersurface in $\mathbb{R}^3$ and closes on itself.

Proposition 3.2.1. *The stationarity condition is independent of the global defining function for the considered real hypersurface M.*

Proof. Let M be a real hypersurface in $\mathbb{C}^m$ and let ρ and σ be two global defining functions for M. Let f be a stationary disk attached to M looked at as defined by ρ. By Proposition 3.1.10, there is an open set $U \subset \mathbb{C}^m$ containing M and a real-smooth map $g : U \to \mathbb{R}$ such that $\sigma = g\rho$ on U and g is nowhere zero on U.

Following Remark 2.2.4, we have $\partial\sigma = (\rho)\partial g + (g)\partial\rho$ on U. In particular, we have for all $\zeta \in \partial\Delta$:

$$\partial\sigma(f(\zeta)) = \rho(f(\zeta))\partial g(f(\zeta)) + g(f(\zeta))\partial\rho(f(\zeta))$$

and since $f(\zeta) \in M$ for all $\zeta \in \partial\Delta$, we have $\rho(f(\zeta)) = 0$ for all $\zeta \in \partial\Delta$. Hence, for all $\zeta \in \partial\Delta$:

$$\partial\sigma(f(\zeta)) = g(f(\zeta))\partial\rho(f(\zeta))$$

Define $c' : \partial\Delta \to \mathbb{R}^*$ by:

$$c'(\zeta) = \frac{c(\zeta)}{g(f(\zeta))}$$

This c' is well-defined because g is nowhere zero on U, in particular nowhere zero on M. As g is real-continuous (since it's real-smooth) and f is real-continuous on $\partial\Delta$ (Theorem 2.1.1), the map c' is real-continuous on $\partial\Delta$, hence complex-continuous (again, Theorem 2.1.1). In summary, c' as defined is a continuous maps between the topological spaces $\partial\Delta$ and $\mathbb{R}^*$. Now, observe that for all $\zeta \in \partial\Delta$:

$$\zeta c'(\zeta)\partial\sigma(f(\zeta)) = \zeta\frac{c(\zeta)}{g(f(\zeta))}g(f(\zeta))\partial\rho(f(\zeta)) = \zeta c(\zeta)\partial\rho(f(\zeta))$$

hence, since the map $\partial\Delta \ni \zeta \mapsto \zeta c(\zeta)\partial\rho(f(\zeta))$ extends holomorphically to Δ, so does the map $\partial\Delta \ni \zeta \mapsto \zeta c'(\zeta)\partial\sigma(f(\zeta))$. This shows that f satisfies the stationarity condition for M looked at as defined by σ, which is what we wanted to prove. □

Proposition 3.2.2. *(Biholomorphic invariance of stationary disks)*
Let $M, N \subset \mathbb{C}^n$ be two real hypersurfaces. Let $U, V \subset \mathbb{C}^n$ be open sets with $M \subset U$ and $N \subset V$. Let $H : U \to V$ be a biholomorphism with $H(M) = N$, and let f be a stationary disk attached to M. Then, the map $F : \overline{\Delta} \to \mathbb{C}^n$ given by $F = H \circ f$ is a stationary disk attached to N.

Proof. First, $F \in \mathcal{O}(\Delta)$ since $f \in \mathcal{O}(\Delta)$ and $H \in \mathcal{O}(U)$, and $F \in C(\overline{\Delta})$ since $f \in C(\overline{\Delta})$ and $H \in C(U)$. Second, $F(\partial\Delta) = H(f(\partial\Delta)) \subset H(M) = N$.
Now, let's show that F satisfies the stationarity condition. By Proposition 3.2.1, it is enough to show that the stationarity condition is satisfied for a choice of a defining function for N. Let ρ be a defining function for M, and let $\sigma : V \to \mathbb{R}$ be the map $\sigma = \rho \circ H^{-1}$. Let's first show that σ is a defining function for N.
As ρ is real-smooth and H^{-1} is holomorphic hence real-smooth (Corollary 2.2.3), we know that σ is real-smooth. Also, we have $D\sigma(z) = DH^{-1}(z)D\rho(H^{-1}(z))$ for all $z \in V$, where DH^{-1} is the differential of H^{-1} as a map between real vector spaces. Note that the biholomorphicity of H makes H a real-diffeomorphism, for the same reasons of Corollary 2.2.3. In particular, by the Chain Rule (Theorem 2.1.5), the real differential $DH(\eta)$ is an isomorphism for all $\eta \in U$, and we have:

$$[DH(\eta)]^{-1} = DH^{-1}(H(\eta))$$

Hence, for all $z \in V$, 0 is not an eigenvalue of the linear map $DH^{-1}(z)$. Hence, for all $z \in V$, if $DH^{-1}(z)D\rho(H^{-1}(z)) = 0$ then $D\rho(H^{-1}(z)) = 0$, which is never the case because $D\rho$ never vanishes. Thus, $D\sigma(z) \neq 0$ for all $z \in V$. Moreover, we have for all $z \in V$:

$$\sigma(z) = 0 \iff \rho(H^{-1}(z)) = 0 \iff H^{-1}(z) \in M \iff z \in N$$

Thus, we have shown that σ is a defining function for N.
We know that f satisfies the stationarity condition for M, so there is a continuous map $c : \partial\Delta \to \mathbb{R}^*$ such that $G : \Delta \to \mathbb{C}^n$ given by $G(\zeta) = \zeta c(\zeta)\partial\rho(f(\zeta))$ is holomorphic on Δ. Notice that, again by the Chain Rule, for all $\zeta \in \Delta$, we have:

$$\partial\sigma(F(\zeta)) = DH^{-1}(F(\zeta))\partial\rho(H^{-1}(F(\zeta)) = DH^{-1}(F(\zeta))\partial\rho(f(\zeta))$$

Hence, we have for all $\zeta \in \Delta$, $\zeta c(\zeta)\partial\sigma(F(\zeta)) = DH^{-1}(F(\zeta))G(\zeta)$.
We may identify $DH^{-1} \circ F$ as a matrix and G as a column vector, both of whose entries are holomorphic maps because of the holomorphy of H^{-1}, F and G. Hence, the map $\Delta \ni \zeta \mapsto \zeta c(\zeta)\partial\sigma(F(\zeta))$ is holomorphic, showing that F satisfies the stationarity condition. This completes the proof. $\square$

Chapter 4

Stationary Disks Attached to the Sphere

In this chapter, we focus all of our attention on the hyperquadric $Q \subset \mathbb{C}^{n+1}$ given by the zero set of ρ, where $\rho : \mathbb{C}^{n+1} \to \mathbb{R}$ is given by

$$\rho(z) = \rho(z_0, \dots, z_n) = \Re(z_0) - \sum_{j=1}^{n} |z_j|^2 = \frac{1}{2}(z_0 + \overline{z_0}) - \sum_{j=1}^{n} z_j \overline{z_j}$$

We let $M = \{z \in \mathbb{C}^{n+1}, \rho(z) > 0\}$, so that M is a smooth strongly pseudoconvex domain in $\mathbb{C}^{n+1}$ (see Proposition 3.1.14), and $Q = \partial M$. We will find the stationary disks attached to Q and study some of their properties and dynamics with the automorphisms of Q that fix 0.

4.1 Studying the Attached Stationary Disks

In this section, we characterize the stationary disks attached to Q, then list some of their properties, and then in 4.1.3, we provide the proofs and computations of the facts mentioned in 4.1.2.

4.1.1 *Their Characterization*

We state what the stationary disks attached to Q look like and provide a proof which details on the outline described in [7].

Theorem 4.1.1. *(Blanc-Centi)*
The stationary disks attached to Q are the functions $f : \overline{\Delta} \to \mathbb{C}^{n+1}$ of the form $f = (f_0, f_1, \dots, f_n)$, where:

$$f_0(\zeta) = \overline{v}^t v + 2\overline{v}^t w \frac{\zeta}{1 - \alpha\zeta} + \frac{\overline{w}^t w}{1 - |\alpha|^2} \frac{1 + \alpha\zeta}{1 - \alpha\zeta} + iy_0$$

and for all $j \geq 1$,

$$f_j(\zeta) = v_j + w_j \frac{\zeta}{1 - \alpha\zeta}$$

where $v, w \in \mathbb{C}^n, y_0 \in \mathbb{R}$, and $\alpha \in \Delta$.

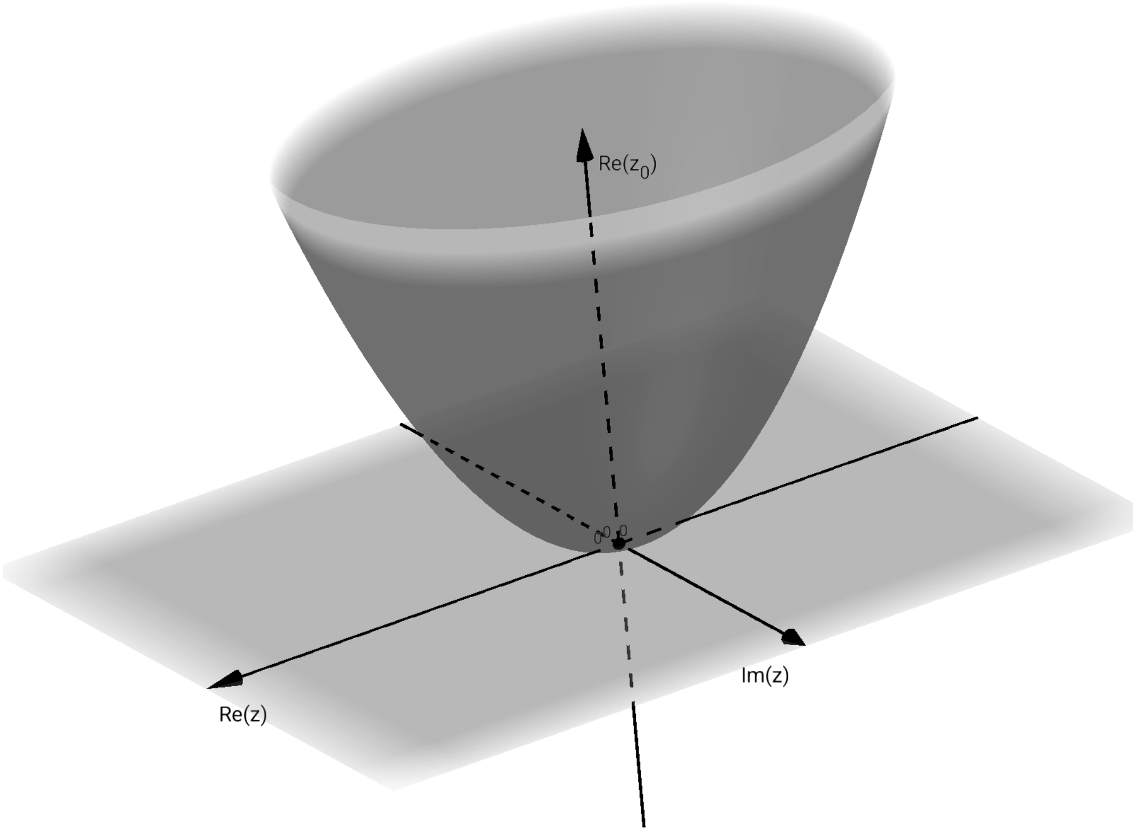

Figure 4.1: An oversimplified graphical respresentation of the domain $M \subset \mathbb{C}^{n+1}$ and its boundary Q (our hyperquadric). The plane is supposed to represent the hyperplane $\Re z_0 = 0$, the boundary of the paraboloid is Q, and the interior of the paraboloid is M.

Proof. Let $f : \overline{\Delta} \to \mathbb{C}^{n+1}$ be a stationary disk attached to Q, and let us write $f = (f_0, \ldots, f_n)$. By definition, we have a non-zero function $c : \partial\Delta \to \mathbb{R}_+$ such that the map $f^* : \partial\Delta \to \mathbb{C}^{n+1}$ given by $f^*(\zeta) = \zeta c(\zeta)\partial\rho(f(\zeta))$ extends holomorphically to Δ. We will also write $f^* = (f_0^*, \ldots, f_n^*)$.
We have that:

$$\partial\rho(z) = \partial\rho(z_0, z_1, \ldots, z_n) = \begin{pmatrix} \frac{1}{2} \\ -\overline{z_1} \\ -\overline{z_2} \\ \vdots \\ -\overline{z_n} \end{pmatrix}$$

Let's expand our c in Fourier series (c is continuous on $\partial\Delta$). We have for all $\zeta \in \partial\Delta$:

$$c(\zeta) = \sum_{k=-\infty}^{\infty} \gamma_k \zeta^k$$

where for all k, $\gamma_k \in \mathbb{C}$. But since c is real-valued, we must have that $\gamma_0 \in \mathbb{R}$, and that for all $k \geq 1$, $\gamma_{-k} = \overline{\gamma_k}$. Also, note that for $\zeta \in \partial\Delta$, one has $\zeta^{-1} = \overline{\zeta}$. Thus, we may write for all $\zeta \in \partial\Delta$:

$$c(\zeta) = \cdots + \overline{\gamma_2}\overline{\zeta}^2 + \overline{\gamma_1}\overline{\zeta} + \gamma_0 + \gamma_1\zeta + \gamma_2\zeta^2 + \cdots$$

and then,

$$\zeta c(\zeta) = \cdots + \overline{\gamma_2}\overline{\zeta} + \overline{\gamma_1} + \gamma_0\zeta + \gamma_1\zeta^2 + \cdots$$

From the expression of $\partial\rho$, we have:

$$f_0^*(\zeta) = \frac{1}{2}\zeta c(\zeta)$$

As f^* is holomorphic on Δ, so is every f_j^* for each $j \in \{0, 1, \ldots, n\}$. In particular, so is f_0^*, and this means that $\zeta c(\zeta)$ is holomorphic, forcing the $\overline{\gamma_k}$, for $k \geq 2$, to be 0. We conclude that:

$$\zeta c(\zeta) = \overline{\gamma_1} + \gamma_0\zeta + \gamma_1\zeta^2$$

We may re-write this as:

$$\zeta c(\zeta) = a + b\zeta + \overline{a}\zeta^2$$

where $a \in \mathbb{C}$ and $b \in \mathbb{R}$.
For each $1 \leq j \leq n$, we will write for $\zeta \in \partial\Delta$:

$$f_j(\zeta) = \sum_{k=0}^{\infty} A_k^{(j)}\zeta^k$$

Looking at $\partial\rho$ once again, we see that for all $j \geq 1$, we have:

$$f_j^*(\zeta) = \zeta c(\zeta)\left(-\overline{f_j(\zeta)}\right)$$

Hence, for $j \geq 1$,

$$\begin{aligned} -f_j^*(\zeta) &= \left(a + b\zeta + \overline{a}\zeta^2\right)\left(\overline{\sum_{k=0}^{\infty} A_k^{(j)}\zeta^k}\right) \\ &= \left(a + b\zeta + \overline{a}\zeta^2\right)\left(\overline{A_0^{(j)}} + \overline{A_1^{(j)}}\overline{\zeta} + \overline{A_2^{(j)}}\overline{\zeta}^2 + \cdots\right) \end{aligned}$$

Expanding, we get:

$$a\overline{A_0^{(j)}} + b\overline{A_1^{(j)}} + \overline{a}\overline{A_2^{(j)}} + \left(b\overline{A_0^{(j)}} + \overline{a}\overline{A_1^{(j)}}\right)\zeta + \overline{a}\overline{A_0^{(j)}}\zeta^2 + \sum_{k=1}^{\infty}\left(a\overline{A_k^{(j)}} + b\overline{A_{k+1}^{(j)}} + \overline{a}\overline{A_{k+2}^{(j)}}\right)\overline{\zeta}^k$$

Holomorphy forces the relation:

$$a\overline{A_k^{(j)}} + b\overline{A_{k+1}^{(j)}} + \overline{a}\overline{A_{k+2}^{(j)}} = 0, \forall j, k \geq 1 \tag{4.1}$$

If we already have $a = 0$, then, since c is not the zero map, we must have $b \neq 0$, and the relation (4.1) becomes just $A_{k+1}^{(j)} = 0, \forall j, k \geq 1$, i.e. $A_k^{(j)} = 0$ for all $j \geq 1$ and $k \geq 2$. This implies that for all $j \geq 1$, $f_j(\zeta) = \alpha_j + \beta_j \zeta$ for some $\alpha_j, \beta_j \in \mathbb{C}$ and all $\zeta \in \partial\Delta$. By Corollary 2.2.6, this holds for all $\zeta \in \overline{\Delta}$.
Now since $f(\partial\Delta) \subset Q$, we must have $\rho(f(\zeta)) = 0$ for all $\zeta \in \partial\Delta$. This means that for all $\zeta \in \partial\Delta$, one has:

$$f_0(\zeta) + \overline{f_0(\zeta)} = 2\sum_{j=1}^{n} f_j(\zeta)\overline{f_j(\zeta)} \tag{4.2}$$

But we have:

$$\begin{aligned}\sum_{j=1}^{n} f_j(\zeta)\overline{f_j(\zeta)} &= \sum_{j=1}^{n} (\alpha_j + \beta_j\zeta)\left(\overline{\alpha_j} + \overline{\beta_j}\cdot\overline{\zeta}\right) \\ &= \sum_{j=1}^{n} \left(|\alpha_j|^2 + \overline{\beta_j}\alpha_j\overline{\zeta} + \overline{\alpha_j}\beta_j\zeta + |\beta_j|^2\right)\end{aligned}$$

This shows that:

$$f_0(\zeta) = \sum_{j=1}^{n} \left(|\alpha_j|^2 + 2\overline{\alpha_j}\beta_j\zeta + |\beta_j|^2\right) + iy_0$$

for some $y_0 \in \mathbb{R}$.
Define $v, w \in \mathbb{C}^n$ as:

$$v = \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}, w = \begin{pmatrix}\beta_1\\ \vdots \\ \beta_n\end{pmatrix}$$

Then, we can re-write f_0 as:

$$f_0(\zeta) = \overline{v}^t v + 2\overline{v}^t w\zeta + \overline{w}^t w + iy_0$$

and then one gets:

$$f(\zeta) = \left(\overline{v}^t v + 2\overline{v}^t w\zeta + \overline{w}^t w + iy_0, v + w\zeta\right)$$

Now suppose that $a \neq 0$, and let us solve the recurrence relation:

$$aB_k + bB_{k+1} + \overline{a}B_{k+2} = 0, \forall k \geq 1 \tag{4.3}$$

There are many ways to do this. We will do it here using a classical "generating function" approach. In particular, let $g(x) = \sum_{k=1}^{\infty} B_k x^k$. In (4.3), multiplying through by x^k and summing over the values of k, we get:

$$a\sum_{k=1}^{\infty} B_k x^k + b\sum_{k=1}^{\infty} B_{k+1}x^k + \overline{a}\sum_{k=1}^{\infty} B_{k+2}x^k = 0$$

so that

$$ag(x) + \frac{b}{x}\sum_{k=1}^{\infty} B_{k+1}x^{k+1} + \frac{\overline{a}}{x^2}\sum_{k=1}^{\infty} B_{k+2}x^{k+2} = 0$$

hence,

$$ag(x) + \frac{b}{x}\left(g(x) - B_1 x\right) + \frac{\overline{a}}{x^2}\left(g(x) - B_1 x - B_2 x^2\right) = 0$$

which we can re-arrange to get:

$$\frac{g(x)}{x} = \frac{\overline{a}B_1 + (bB_1 + \overline{a}B_2)x}{\overline{a} + bx + ax^2}$$

Consider now the quadratic polynomial $\overline{a} + bx + ax^2$. Let a_1 and a_2 (possibly equal, but won't be equal as we will see now) be the roots of this quadratic. Then, $a_1 a_2 = \dfrac{\overline{a}}{a}$, so $|a_1 a_2| = |a_1||a_2| = 1$. Now if $|a_1| = |a_2|$, then $|a_1| = |a_2| = 1$, and this means that our quadratic has a root $\eta \in \partial\Delta$. But this implies that $\overline{\eta} = \dfrac{1}{\eta}$ is a root of $a + bx + \overline{a}x^2$ i.e. of $xc(x)$, so $\overline{\eta}c(\overline{\eta}) = 0$, hence $c(\overline{\eta}) = 0$, and this is impossible since $\overline{\eta} \in \partial\Delta$ where c never vanishes. Thus, we can't have $|a_1| = |a_2|$. Let's then WLOG say $0 < |a_1| < 1 < |a_2|$.

Let's do partial fraction decomposition for $\dfrac{g(x)}{x}$. We have:

$$\frac{g(x)}{x} = \frac{\overline{a}B_1 + (bB_1 + \overline{a}B_2)x}{a(a_1 - x)(a_2 - x)} = \frac{\lambda}{a_1 - x} + \frac{\mu}{a_2 - x}$$

where:

$$\lambda = \frac{\overline{a}B_1 + (bB_1 + \overline{a}B_2)a_1}{a(a_2 - a_1)}$$

and

$$\mu = \frac{\overline{a}B_1 + (bB_1 + \overline{a}B_2)a_2}{a(a_1 - a_2)}$$

Now, we expand in power series with the understanding that $|x| < |a_1|$ and that we do not care about x in any case. We have:

$$\begin{aligned}
\frac{g(x)}{x} &= \frac{\lambda}{a_1 - x} + \frac{\mu}{a_2 - x} \\
&= \frac{\lambda}{a_1} \cdot \frac{1}{1 - \frac{x}{a_1}} + \frac{\mu}{a_2} \cdot \frac{1}{1 - \frac{x}{a_2}} \\
&= \frac{\lambda}{a_1}\sum_{k=0}^{\infty}\left(\frac{x}{a_1}\right)^k + \frac{\mu}{a_2}\sum_{k=0}^{\infty}\left(\frac{x}{a_2}\right)^k \\
&= \sum_{k=0}^{\infty}\left(\frac{\lambda}{a_1^{k+1}} + \frac{\mu}{a_2^{k+1}}\right)x^k
\end{aligned}$$

But we know that:

$$\begin{aligned}\frac{g(x)}{x} &= \frac{1}{x}\left(B_1x + B_2x^2 + B_3x^3 + \cdots\right)\\ &= B_1 + B_2x + B_3x^2 + \cdots\\ &= \sum_{k=0}^{\infty} B_{k+1}x^k\end{aligned}$$

and therefore, we have for all $k \geq 0$,

$$B_{k+1} = \frac{\lambda}{a_1^{k+1}} + \frac{\mu}{a_2^{k+1}}$$

i.e.

$$B_k = \frac{\lambda}{a_1^k} + \frac{\mu}{a_2^k}, \forall k \geq 1 \tag{4.4}$$

As we would expect, plugging in $k = 1$ or $k = 2$ in (4.4) will not give us any new information. However, (4.4) will determine the B_k for $k \geq 3$ as a function of B_1 and B_2. In short, the solution to (4.3) is: $B_1, B_2 \in \mathbb{C}$ arbitrary, and B_k for $k \geq 3$ as displayed in (4.4).

We may then solve (4.1). We have that $A_1^{(j)}$ and $A_2^{(j)}$ are free for all $j \geq 1$, and for all $k \geq 3$ and $j \geq 1$ we have:

$$A_k^{(j)} = \lambda_j \frac{1}{\overline{a_1}^k} + \mu_j \frac{1}{\overline{a_2}^k}$$

where

$$\lambda_j = \frac{aA_1^{(j)} + \left(bA_1^{(j)} + aA_2^{(j)}\right)\overline{a_1}}{\overline{a}(\overline{a_2} - \overline{a_1})}$$

and

$$\mu_j = \frac{aA_1^{(j)} + \left(bA_1^{(j)} + aA_2^{(j)}\right)\overline{a_2}}{\overline{a}(\overline{a_1} - \overline{a_2})}$$

Then, for $j \geq 1$,

$$\begin{aligned}f_j(\zeta) - A_0^{(j)} &= \sum_{k=1}^{\infty} A_k^{j)}\zeta^k\\ &= \sum_{k=1}^{\infty}\left(\lambda_j \frac{1}{\overline{a_1}^k} + \mu_j \frac{1}{\overline{a_2}^k}\right)\zeta^k\\ &= \lambda_j \sum_{k=1}^{\infty}\left(\frac{\zeta}{\overline{a_1}}\right)^k + \mu_j \sum_{k=1}^{\infty}\left(\frac{\zeta}{\overline{a_2}}\right)^k\end{aligned}$$

Now the formula:

$$f_j(\zeta) = \sum_{k=0}^{\infty} A_k^{(j)}\zeta^k$$

is at least valid for $|\zeta| = 1$, i.e. this series converges for $|\zeta| = 1$, and so does the geometric series of common ratio $\frac{\zeta}{\overline{a_2}}$, but not the geometric series of common ratio $\frac{\zeta}{\overline{a_1}}$ (because of the assumptions on $|a_1|$ and $|a_2|$). This forces $\lambda_j = 0$ for all $j \geq 1$, and therefore

$$A_k^{(j)} = \mu_j \frac{1}{\overline{a_2}^k}, \forall j, k \geq 1$$

As $\lambda_j = 0$, the numerator of λ_j is also 0, and we can then write:

$$bA_1^{(j)} + aA_2^{(j)} = -\frac{aA_1^{(j)}}{\overline{a_1}}$$

in particular, the $A_2^{(j)}$ are no more free. Plugging into μ_j, one gets:

$$\mu_j = \frac{1}{\overline{a_1}a_1a_2} A_1^{(j)}$$

Write $\alpha = \overline{a_1}a_1a_2$. Then, $\mu_j = \frac{1}{\alpha} A_1^{(j)}$, so:

$$A_k^{(j)} = \frac{A_1^{(j)}}{\alpha} \frac{1}{\overline{a_2}^k}$$

and therefore,

$$\begin{aligned}
f_j(\zeta) &= A_0^{(j)} + \frac{A_1^{(j)}}{\alpha} \sum_{k=1}^{\infty} \left(\frac{\zeta}{\overline{a_2}} \right)^k \\
&= A_0^{(j)} + \frac{A_1^{(j)}}{\alpha} \cdot \frac{\zeta}{\overline{a_2}} \cdot \frac{1}{1 - \frac{\zeta}{\overline{a_2}}} \\
&= A_0^{(j)} + \frac{A_1^{(j)}}{\overline{a_1}a_1a_2} \cdot \frac{\zeta}{\overline{a_2} - \zeta} \\
&= A_0^{(j)} + A_1^{(j)} \frac{\zeta}{|a_1a_2|^2 - \overline{a_1}a_1a_2\zeta} \\
&= A_0^{(j)} + A_1^{(j)} \frac{\zeta}{1 - \alpha\zeta}
\end{aligned}$$

where $\alpha = a_1\overline{a_1}a_2$. Notice that $|\alpha| = |a_1a_2| \cdot |\overline{a_1}| = 1 \cdot |\overline{a_1}| = |a_1| < 1$.
Now once again, since $f(\partial\Delta) \subset Q$, we have that (4.2) holds for any $\zeta \in \partial\Delta$. Putting:

$$v = \begin{pmatrix} A_0^{(1)} \\ \vdots \\ A_0^{(n)} \end{pmatrix}, w = \begin{pmatrix} A_1^{(1)} \\ \vdots \\ A_1^{(n)} \end{pmatrix}$$

one has:

$$f_0(\zeta) + \overline{f_0(\zeta)} = 2\sum_{j=1}^{n}\left(v_j + w_j\frac{\zeta}{1-\alpha\zeta}\right)\left(\overline{v_j} + \overline{w_j}\frac{\overline{\zeta}}{1-\overline{\alpha}\overline{\zeta}}\right)$$
$$= 2\sum_{j=1}^{n}\left(|v_j|^2 + \overline{v_j}w_j\frac{\zeta}{1-\alpha\zeta} + v_j\overline{w_j}\frac{\overline{\zeta}}{1-\overline{\alpha}\overline{\zeta}} + |w_j|^2\frac{1}{(1-\alpha\zeta)(1-\overline{\alpha}\overline{\zeta})}\right)$$

To determine f_0, we will need to do the following decomposition:

$$\frac{1}{(1-\alpha\zeta)(1-\overline{\alpha}\overline{\zeta})} = \frac{A+B\zeta}{1-\alpha\zeta} + \frac{\overline{A}+\overline{B\zeta}}{1-\overline{\alpha}\overline{\zeta}}$$

Upon solving for A and B, one gets:

$$A = \frac{1}{2(1-|\alpha|^2)}, B = \frac{\alpha}{2(1-|\alpha|^2)}$$

and therefore,

$$\frac{1}{(1-\alpha\zeta)(1-\overline{\alpha}\overline{\zeta})} = \frac{1}{2(1-|\alpha|^2)}\left(\frac{1+\alpha\zeta}{1-\alpha\zeta} + \frac{1+\overline{\alpha}\overline{\zeta}}{1-\overline{\alpha}\overline{\zeta}}\right)$$

and then we deduce that:

$$f_0(\zeta) = \overline{v}^t v + 2\overline{v}^t w\frac{\zeta}{1-\alpha\zeta} + \frac{\overline{w}^t w}{1-|\alpha|^2}\frac{1+\alpha\zeta}{1-\alpha\zeta} + iy_0$$

and for all $j \geq 1$,

$$f_j(\zeta) = v_j + w_j\frac{\zeta}{1-\alpha\zeta}$$

where $v, w \in \mathbb{C}^n, y_0 \in \mathbb{R}$, and $\alpha \in \Delta$. This concludes the proof. $\square$

An essential step in the proof was the expansion of c into Fourier series on $\partial\Delta$. This comes from the general theory of Fourier series: it is well-known that for a function c which is continuous on the unit circle in $\mathbb{C}$ (i.e. our $\partial\Delta$), the sequence of Fourier polynomials of c will converge to c in the max norm $\|\cdot\|_\infty$ on $C(\partial\Delta)$ given by $\|a\|_\infty = \max_{\zeta\in\partial\Delta}|a(\zeta)|$, i.e. uniformly. We do not mention this in the preliminary pages because we assume the familiarity of the reader with this.

4.1.2 *Some Properties of these Disks*

Here, we merely list the properties we will be needing, and we postpone the statements and proofs to the next subsection. The purpose of this subsection is to keep an easy-to-read list. In this list of facts, $f = (f_0, g) : \overline{\Delta} \to \mathbb{C}\times\mathbb{C}^n$ is a stationary disk attached to Q with $f(1) = 0$. Also, $p = (p_0, p') = f(0)$. Note that if $p = 0$, then $f \equiv 0$, and in this list we assume $p \neq 0$. Finally, when $n = 2$, we write $f'(1) = (c_1, c_2)$ and $f''(1) = (c_3, c_4)$.

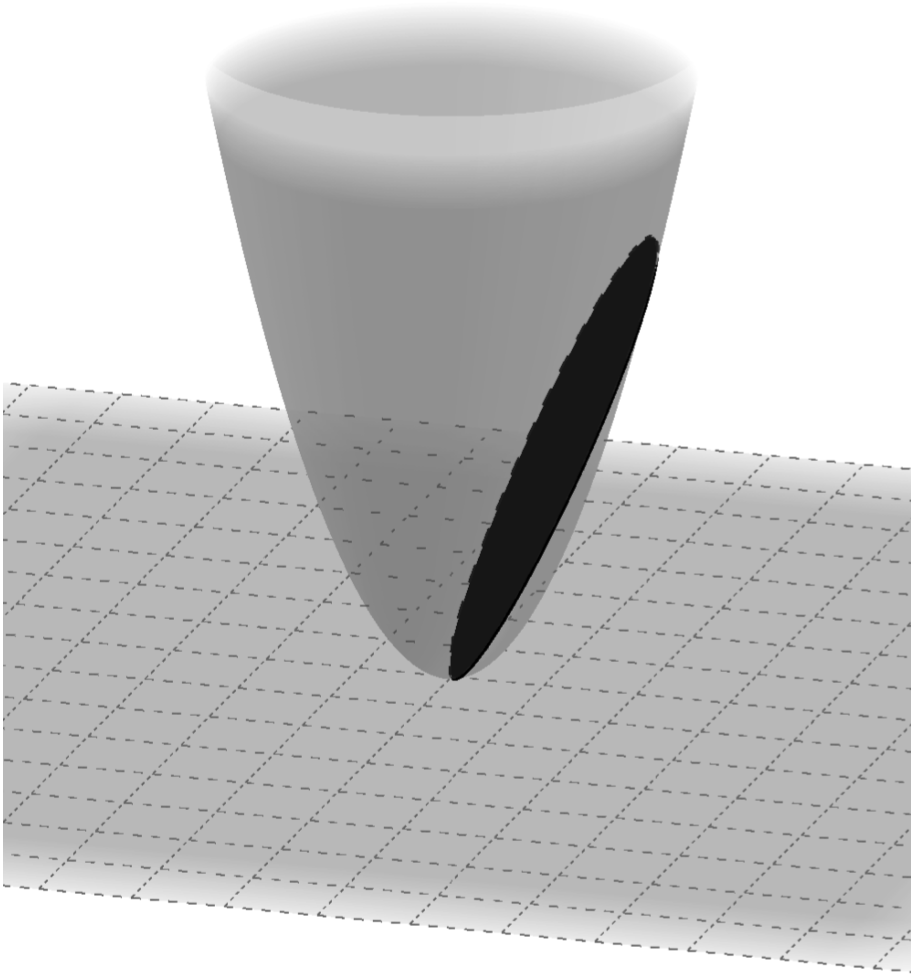

Figure 4.2: An intuitive illustration of our paraboloid. The slanted black disk represents the kind of disk we are dealing with here (attached to the origin).

Fact 4.1.1. *We can write f in terms of only α and v. We have:*

$$f_0(\zeta) = \frac{2(1-\alpha)\|v\|^2}{1-|\alpha|^2}\frac{1-\zeta}{1-\alpha\zeta}$$

and:

$$g(\zeta) = \frac{1-\zeta}{1-\alpha\zeta}v$$

Fact 4.1.2. *In terms of α and p, the 2-jet of f at any point ζ is given by:*

$$f'(\zeta) = \frac{\alpha-1}{(1-\alpha\zeta)^2}p$$

And:

$$f''(\zeta) = \frac{2\alpha(\alpha-1)}{(1-\alpha\zeta)^3}p$$

Fact 4.1.3. *The 2-jet of f at 1 is given in terms of α and v by:*

$$
\begin{aligned}
f_0'(1) &= \frac{2\|v\|^2}{|\alpha|^2 - 1} \\
f_0''(1) &= \frac{4\alpha\|v\|^2}{(|\alpha|^2 - 1)(1-\alpha)} \\
g'(1) &= \frac{1}{\alpha - 1} v \\
g''(1) &= \frac{-2\alpha}{(1-\alpha)^2} v
\end{aligned}
$$

Fact 4.1.4. *In terms of p, we have $v = p$ and:*

$$\alpha = 1 - \frac{2(\Re p_0 - \|p'\|^2)}{\overline{p_0}}$$

Fact 4.1.5. *In terms of p, we may write:*

$$f(\zeta) = \frac{1-\zeta}{1-\alpha\zeta} p$$

with α as displayed in the previous fact.

Fact 4.1.6. *In terms of p, the 2-jet of f at 1 is given by:*

$$f'(1) = \frac{\overline{p_0}}{2(\|p'\|^2 - \Re p_0)} p$$

$$f''(1) = \frac{(p_0 - 2\|p'\|^2)\overline{p_0}}{2(\|p'\|^2 - \Re p_0)^2} p$$

Fact 4.1.7. *The α, v, c_1 and c_3 are all functions of c_2 and c_4, and we have:*

$$\alpha = \frac{c_4}{2c_2 + c_4}$$

and:

$$v = -\frac{2c_2^2}{2c_2 + c_4}$$

and for the c_i,

$$c_1 = -\frac{2|c_2|^4}{|c_2|^2 + \Re(\overline{c_2}c_4)}$$

and:

$$c_3 = -\frac{2|c_2|^2\overline{c_2}c_4}{|c_2|^2 + \Re(\overline{c_2}c_4)}$$

Fact 4.1.8. *The expressions of c_1, c_2, c_3 and c_4 in terms of p are:*

$$c_1 = \frac{|p_0|^2}{2(|p'|^2 - \Re p_0)}$$

$$c_2 = \frac{\overline{p_0} p'}{2(|p'|^2 - \Re p_0)}$$

$$c_3 = \frac{(p_0 - 2|p'|^2)|p_0|^2}{2(|p'|^2 - \Re p_0)^2}$$

and:

$$c_4 = \frac{(p_0 - 2|p'|^2)\overline{p_0} p'}{2(|p'|^2 - \Re p_0)^2}$$

Fact 4.1.9. *Let us not take p to be $f(0)$, and let's suppose that the c_i are known. The solution to $f(0) = q$ is given by $q = (q_0, q')$, where $q = 0$ if $c_1 = 0$, and else:*

$$q_0 = \frac{2c_1^2}{\overline{c_3} + 4|c_2|^2}$$

and:

$$q' = \frac{2c_1 c_2}{\overline{c_3} + 4|c_2|^2}$$

4.1.3 *Proofs and Computations*

This subsection is more or less the proper mathematical treatment of the preceding list of facts.

Proposition 4.1.1. *The stationary disks f attached to Q satisfying $f(1) = 0$ are the ones of the form $f = (f_0, g) : \overline{\Delta} \to \mathbb{C} \times \mathbb{C}^n$ with:*

$$f_0(\zeta) = \frac{2(1-\alpha)\|v\|^2}{1 - |\alpha|^2} \frac{1-\zeta}{1-\alpha\zeta}$$

and:

$$g(\zeta) = \frac{1-\zeta}{1-\alpha\zeta} v$$

where $\alpha \in \Delta$ and $v \in \mathbb{C}^n$. Moreover, the 2-jets of these maps at 1 are given by:

$$f_0'(1) = \frac{2\|v\|^2}{|\alpha|^2 - 1}$$

$$f_0''(1) = \frac{4\alpha\|v\|^2}{(|\alpha|^2 - 1)(1-\alpha)}$$

$$g'(1) = \frac{1}{\alpha - 1} v$$

$$g''(1) = \frac{-2\alpha}{(1-\alpha)^2} v$$

Proof. Let f be a stationary disk attached to Q. Then, by Theorem 4.1.1, there are $v, u \in \mathbb{C}^n$, $\alpha \in \Delta$, and $y_0 \in \mathbb{R}$ such that $f = (f_0, g)$, where:

$$f_0(\zeta) = \|v\|^2 + 2\langle \overline{v}, u \rangle \frac{\zeta}{1-\alpha\zeta} + \frac{\|u\|^2}{1-|\alpha|^2} \frac{1+\alpha\zeta}{1-\alpha\zeta} + iy_0$$

and

$$g(\zeta) = v + u\frac{\zeta}{1-\alpha\zeta}$$

As $g(1) = 0$, we have that v and u are linearly dependent, in particular $u = (\alpha-1)v$. Hence, from $f_0(1) = 0$, we see that:

$$\|v\|^2 + 2(\alpha-1)\|v\|^2\frac{1}{1-\alpha} + \frac{|\alpha-1|^2\|v\|^2}{1-|\alpha|^2}\frac{1+\alpha}{1-\alpha} + iy_0 = 0$$

i.e.

$$\|v\|^2 - 2\|v\|^2 - \frac{(\alpha-1)(\overline{\alpha}-1)(\alpha+1)\|v\|^2}{(1-\alpha\overline{\alpha})(\alpha-1)} + iy_0 = 0$$

hence,

$$-\|v\|^2 + \frac{(1-\overline{\alpha})(1+\alpha)}{1-\alpha\overline{\alpha}}\|v\|^2 + iy_0 = 0$$

which we can rearrange to:

$$\left(\frac{\alpha-\overline{\alpha}}{1-|\alpha|^2}\right)\|v\|^2 + iy_0 = 0$$

i.e.

$$i\left(y_0 + \frac{2\Im\alpha}{1-|\alpha|^2}\|v\|^2\right) = 0$$

and therefore,

$$y_0 = \frac{2\Im\alpha}{|\alpha|^2-1}\|v\|^2$$

Let's now substitute back the u and y_0 into f. First,

$$g(\zeta) = v + (\alpha-1)v\frac{\zeta}{1-\alpha\zeta} = \left(1 + \frac{(\alpha-1)\zeta}{1-\alpha\zeta}\right)v = \frac{1-\zeta}{1-\alpha\zeta}v$$

Second,

$$\begin{aligned}
f_0(\zeta) &= \|v\|^2 + \frac{2(\alpha-1)\zeta}{1-\alpha\zeta}\|v\|^2 + \frac{|1-\alpha|^2}{1-|\alpha|^2}\frac{1+\alpha\zeta}{1-\alpha\zeta}\|v\|^2 + i\left(\frac{2\Im\alpha}{|\alpha|^2-1}\right)\|v\|^2 \\
&= \left(1 + \frac{2(\alpha-1)\zeta}{1-\alpha\zeta} + \frac{|1-\alpha|^2}{1-|\alpha|^2}\frac{1+\alpha\zeta}{1-\alpha\zeta} + \frac{2i\Im\alpha}{|\alpha|^2-1}\right)\|v\|^2 \\
&= \left(1 + \frac{2(\alpha-1)\zeta}{1-\alpha\zeta} + \frac{(1-\alpha)(1-\overline{\alpha})}{1-\alpha\overline{\alpha}}\frac{1+\alpha\zeta}{1-\alpha\zeta} + \frac{\alpha-\overline{\alpha}}{|\alpha|^2-1}\right)\|v\|^2
\end{aligned}$$

The factor of $\|v\|^2$ can be re-written after taking $(1-\alpha\zeta)(1-\alpha\overline{\alpha})$ as a common denominator as:

$$\frac{(1-\alpha\overline{\alpha})(1-\alpha\zeta) + 2(\alpha-1)(1-\alpha\overline{\alpha})\zeta + (1-\alpha)(1-\overline{\alpha})(1+\alpha\zeta) - (\alpha-\overline{\alpha})(1-\alpha\zeta)}{(1-|\alpha|^2)(1-\alpha\zeta)}$$

The numerator of this expression, fully expanded, is given by:

$$1 - \alpha\zeta - \alpha\overline{\alpha} + \alpha^2\overline{\alpha}\zeta - 2\zeta + 2\alpha\overline{\alpha}\zeta + 2\alpha\zeta - 2\alpha^2\overline{\alpha}\zeta + 1 + \alpha\zeta - \overline{\alpha}$$

$$-\alpha\overline{\alpha}\zeta - \alpha - \alpha^2\zeta + \alpha\overline{\alpha} + \alpha^2\overline{\alpha}\zeta + \overline{\alpha} - \alpha\overline{\alpha}\zeta - \alpha + \alpha^2\zeta$$

and this is just $2 + 2\alpha\zeta - 2\zeta - 2\alpha$, i.e. $2(1-\alpha)(1-\zeta)$. Therefore, we get:

$$f_0(\zeta) = \frac{2(1-\alpha)(1-\zeta)}{(1-|\alpha|^2)(1-\alpha\zeta)}\|v\|^2$$

Now, note that:

$$\frac{d}{d\zeta}\left(\frac{1-\zeta}{1-\alpha\zeta}\right) = \frac{\alpha-1}{(1-\alpha\zeta)^2}$$

and

$$\frac{d^2}{d\zeta^2}\left(\frac{1-\zeta}{1-\alpha\zeta}\right) = \frac{2\alpha(\alpha-1)}{(1-\alpha\zeta)^3}$$

so that f_0', g', f_0'' and g'' are given by:

$$f_0'(\zeta) = -\frac{2(1-\alpha)^2\|v\|^2}{(1-|\alpha|^2)(1-\alpha\zeta)^2}$$

$$g'(\zeta) = \frac{\alpha-1}{(1-\alpha\zeta)^2}v$$

$$f_0''(\zeta) = -\frac{4\alpha(1-\alpha)^2\|v\|^2}{(1-|\alpha|^2)(1-\alpha\zeta)^3}$$

and:

$$g''(\zeta) = \frac{2\alpha(\alpha-1)}{(1-\alpha\zeta)^3}v$$

and then one just substitutes $\zeta = 1$ to get the $f_0'(1), f_0''(1), g'(1)$ and $g''(1)$ as displayed above. $\square$

Proposition 4.1.2. *Let $f = (f_0, g)$ be a stationary disk attached to Q with $f(1) = 0$. Then, f is completely determined by its value at 0. In fact, if $f(0) = 0$, then $f \equiv 0$. Else, $p_0 \neq 0$ and $p' \neq 0$, and if we write $f(0)$ as $p = (p_0, p') \in \mathbb{C} \times \mathbb{C}^n$, then we have:*

$$f_0(\zeta) = \frac{1-\zeta}{1-\alpha\zeta}p_0$$

and

$$g(\zeta) = \frac{1-\zeta}{1-\alpha\zeta}p'$$

where:

$$\alpha = 1 - \frac{2(\Re p_0 - \|p'\|^2)}{\overline{p_0}}$$

In other words, f is just the map:

$$f(\zeta) = \frac{1-\zeta}{1-\alpha\zeta}p$$

Proof. Let $f(0) = p = (p_0, p') \in \mathbb{C} \times \mathbb{C}^n$. We know that p is in the region bounded by Q, so $\Re p_0 > \|p'\|^2$. The previous proposition gives us the expressions of f_0 and g in terms of v and α, and as $g(0) = p'$, we have $v = p'$. Now from $f_0(0) = p_0$, we have:

$$\frac{2(1-\alpha)\|p'\|^2}{1-|\alpha|^2} = p_0$$

If $p' = 0$, then f is the zero map and there is nothing to prove. Assume then that $p' \neq 0$. Then, we also have $p_0 \neq 0$, and:

$$1 - \alpha = \frac{1-|\alpha|^2}{2\|p'\|^2} p_0$$

so:

$$\alpha = 1 - \frac{1-|\alpha|^2}{2\|p'\|^2} p_0$$

Write $p_0 = x + iy$, so that:

$$\alpha = \left(1 - \frac{1-|\alpha|^2}{2\|p'\|^2} x\right) + i\left(-\frac{1-|\alpha|^2}{2\|p'\|^2} y\right)$$

hence:

$$\begin{aligned}
|\alpha|^2 &= \left(1 - \frac{1-|\alpha|^2}{2\|p'\|^2} x\right)^2 + \left(\frac{1-|\alpha|^2}{2\|p'\|^2} y\right)^2 \\
&= 1 - \frac{1-|\alpha|^2}{\|p'\|^2} x + \frac{(1-|\alpha|^2)^2}{4\|p'\|^4} |p_0|^2 \\
&= 1 + \mu(1-|\alpha|^2) + \lambda(1-|\alpha|^2)^2 \\
&= \lambda|\alpha|^4 - (2\lambda+\mu)|\alpha|^2 + \lambda + \mu + 1
\end{aligned}$$

with:

$$\lambda = \frac{|p_0|^2}{4\|p'\|^4} > 0 \text{ and } \mu = -\frac{\Re p_0}{\|p'\|^2}$$

Rearranging, we get:

$$\lambda t^2 - (2\lambda + \mu + 1)t + \lambda + \mu + 1 = 0$$

where $t = |\alpha|^2$, which is a quadratic in t with discriminant:

$$\begin{aligned}
\delta &= (2\lambda+\mu+1)^2 - 4\lambda(\lambda+\mu+1) \\
&= \mu^2 + 2\mu + 1 \\
&= (\mu+1)^2
\end{aligned}$$

Note that:

$$\mu + 1 = \frac{\|p'\|^2 - \Re p_0}{\|p'\|^2} < 0$$

so that $\sqrt{\delta} = -\mu - 1$, and then the solutions to the quadratic are:

$$t_1 = \frac{2\lambda + \mu + 1 + \mu + 1}{2\lambda} = 1 + \frac{\mu + 1}{\lambda}$$

and:

$$t_2 = \frac{2\lambda + \mu + 1 - \mu - 1}{2\lambda} = 1$$

But t_2 is not a viable solution because then $|\alpha|^2 = 1$, i.e. $|\alpha| = 1$, which contradicts the fact that $a \in \Delta$. So, t_1 is the only solution, so that:

$$|\alpha|^2 = 1 + \frac{\mu + 1}{\lambda} = 1 + \frac{4\|p'\|^2(\|p'\|^2 - \Re p_0)}{|p_0|^2}$$

hence,

$$1 - |\alpha|^2 = \frac{4\|p'\|^2(\Re p_0 - \|p'\|^2)}{|p_0|^2}$$

and thus,

$$\alpha = 1 - \frac{1 - |\alpha|^2}{2\|p'\|^2} p_0 = 1 - \frac{2(\Re p_0 - \|p'\|^2)}{|p_0|^2} p_0 = 1 - \frac{2(\Re p_0 - \|p'\|^2)}{\overline{p_0}}$$

Noting that:

$$1 - \alpha = \frac{1 - |\alpha|^2}{2\|p'\|^2} p_0$$

we may write:

$$f_0(\zeta) = \frac{2\left(\frac{1 - |\alpha|^2}{2\|p'\|^2} p_0\right)\|p'\|^2}{1 - |\alpha|^2} \frac{1 - \zeta}{1 - \alpha\zeta} = \frac{1 - \zeta}{1 - \alpha\zeta} p_0$$

and:

$$g(\zeta) = \frac{1 - \zeta}{1 - \alpha\zeta} p'$$

□

Corollary 4.1.1. *Let $f = (f_0, g)$ be a stationary disk attached to Q with $f(1) = 0$ and $f(0) = p = (p_0, p') \in \mathbb{C} \times \mathbb{C}^n$. Then, the 2-jet of f at any ζ is given by:*

$$f'(\zeta) = \frac{\alpha - 1}{(1 - \alpha\zeta)^2} p$$

And:

$$f''(\zeta) = \frac{2\alpha(\alpha - 1)}{(1 - \alpha\zeta)^3} p$$

In particular, the 2-jet of f at 1 is given by:

$$f'(1) = \frac{\overline{p_0}}{2(\|p'\|^2 - \Re p_0)} p$$

$$f''(1) = \frac{(p_0 - 2\|p'\|^2)\overline{p_0}}{2(\|p'\|^2 - \Re p_0)^2} p$$

Proof. From the preceding proposition, we have:

$$f(\zeta) = \frac{1-\zeta}{1-\alpha\zeta}p$$

with α as explicitly mentioned above in terms of p. Recall that:

$$\frac{d}{d\zeta}\left(\frac{1-\zeta}{1-\alpha\zeta}\right) = \frac{\alpha-1}{(1-\alpha\zeta)^2} \text{ and } \frac{d^2}{d\zeta^2}\left(\frac{1-\zeta}{1-\alpha\zeta}\right) = \frac{2\alpha(\alpha-1)}{(1-\alpha\zeta)^3}$$

so,

$$f'(\zeta) = \frac{\alpha-1}{(1-\alpha\zeta)^2}p$$

And:

$$f''(\zeta) = \frac{2\alpha(\alpha-1)}{(1-\alpha\zeta)^3}p$$

so that also in particular:

$$f'(1) = \frac{1}{\alpha-1}p = \frac{\overline{p_0}}{2(\|p'\|^2 - \Re p_0)}p$$

and:

$$\begin{aligned} f''(1) = -\frac{2\alpha}{(1-\alpha)^2}p &= -2\left(1 - \frac{2(\Re p_0 - \|p'\|^2)}{\overline{p_0}}\right)\frac{\overline{p_0}^2}{4(\|p'\|^2 - \Re p_0)^2}p \\ &= \frac{(2\Re p_0 - 2\|p'\|^2 - \overline{p_0})\overline{p_0}}{2(\|p'\|^2 - \Re p_0)^2}p \\ &= \frac{(p_0 - 2\|p'\|^2)\overline{p_0}}{2(\|p'\|^2 - \Re p_0)^2}p \end{aligned}$$

which is what we claimed. □

Corollary 4.1.2. *Fix $n = 2$, so that $Q = \{(w,z) \in \mathbb{C}^2, \Re w = |z|^2\}$. Let f be a stationary disk attached to Q with $f(1) = 0$. Write $f = (f_0, g)$, $g'(1) = c_2$, $g''(1) = c_4$, $f_0'(1) = c_1$ and $f_0''(1) = c_3$. Also, write $p = (p_0, p') = f(0)$. Then, we have:*

$$c_1 = \frac{|p_0|^2}{2(|p'|^2 - \Re p_0)}$$

$$c_2 = \frac{\overline{p_0}p'}{2(|p'|^2 - \Re p_0)}$$

$$c_3 = \frac{(p_0 - 2|p'|^2)|p_0|^2}{2(|p'|^2 - \Re p_0)^2}$$

and:

$$c_4 = \frac{(p_0 - 2|p'|^2)\overline{p_0}p'}{2(|p'|^2 - \Re p_0)^2}$$

Proof. The proof is merely a matter of writing down the definitions of the c_i and substituting the expressions in terms of p obtained in Corollary 4.1.1:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} f_0'(1) \\ g'(1) \end{pmatrix} = f'(1)$$

and the latter we already have in terms of p. Similarly for c_3 and c_4. □

Proposition 4.1.3. *With the same setting as in Corollary 4.1.2, we have that if $c_4 \neq 0$, then $2c_2 + c_4 \neq 0$. If this is the case, or if $c_4 = 0$ and $\alpha = 0$, then we have f as displayed in Proposition 4.1.1, with:*

$$\alpha = \frac{c_4}{2c_2 + c_4}$$

and:

$$v = -\frac{2c_2^2}{2c_2 + c_4}$$

and then, since $|\alpha| < 1$, we have $|\alpha|^2 - 1 \neq 0$, so $|c_4|^2 - |2c_2 + c_4|^2 \neq 0$, so that $|c_2|^2 + \Re(\overline{c_2}c_4) \neq 0$, and:

$$c_1 = -\frac{2|c_2|^4}{|c_2|^2 + \Re(\overline{c_2}c_4)}$$

and:

$$c_3 = -\frac{2|c_2|^2\overline{c_2}c_4}{|c_2|^2 + \Re(\overline{c_2}c_4)}$$

Else, $c_4 = 0$ and $\alpha \neq 0$, and then $v = 0$, thus $f \equiv 0$, so that $c_i = 0$ for $i = 1, 2, 3, 4$. Therefore, in particular, f is completely determined by $g'(1)$ and $g''(1)$.
As a useful remark, c_1 is given in terms of α and v by:

$$c_1 = \frac{2|v|^2}{|\alpha|^2 - 1}$$

and c_3 is given in terms of α and c_1 or α and v by:

$$c_3 = \frac{2\alpha}{1-\alpha}c_1 = \frac{4\alpha|v|^2}{(|\alpha|^2 - 1)(1-\alpha)}$$

Proof. Suppose first that $c_4 \neq 0$. As $g'(1) = c_2$, we have $v = (\alpha - 1)c_2$, and as $g''(1) = c_4$, we have that $\dfrac{-2\alpha}{(1-\alpha)^2}v = c_4$, so that $\dfrac{-2\alpha}{(1-\alpha)^2}(\alpha - 1)c_2 = c_4$, and hence $2\alpha c_2 = (1-\alpha)c_4$, and this gives $(2c_2 + c_4)\alpha = c_4$, which implies that $2c_2 + c_4 \neq 0$ (else, $c_4 = 0$, which is impossible), thus $\alpha = \dfrac{c_4}{2c_2 + c_4}$. Now we just substitute back to get v:

$$v = (\alpha - 1)c_2 = \left(\frac{c_4}{2c_2 + c_4} - 1\right)c_2 = -\frac{2c_2^2}{2c_2 + c_4}$$

Now:

$$c_1 = f_0'(1) = \frac{2|v|^2}{|\alpha|^2 - 1} = 2\frac{4|c_2|^4}{|2c_2 + c_4|^2}\frac{1}{\dfrac{|c_4|^2}{|2c_2 + c_4|^2} - 1}$$

$$= \frac{8|c_2|^4}{|c_4|^2 - |2c_2 + c_4|^2}$$
$$= \frac{8|c_2|^4}{|c_4|^2 - (4|c_2|^2 + |c_4|^2 + 4\Re(\overline{c_2}c_4)}$$
$$= -\frac{2|c_2|^4}{|c_2|^2 + \Re(\overline{c_2}c_4)}$$

and:

$$c_3 = f_0''(1) = \frac{4\alpha|v|^2}{(|\alpha|^2 - 1)(1-\alpha)}$$
$$= 4\frac{c_4}{2c_2 + c_4}\frac{4|c_2|^4}{|2c_2 + c_4|^2}\frac{1}{\left(\frac{|c_4|^2}{|2c_2+c_4|^2} - 1\right)\left(1 - \frac{c_4}{2c_2 + c_4}\right)}$$
$$= \frac{16c_4|c_2|^4}{(2c_2 + c_4 - c_4)(|c_4|^2 - |2c_2 + c_4|^2)}$$
$$= \frac{16c_4|c_2|^4}{2c_2(-4|c_2|^2 - 4\Re(\overline{c_2}c_4))}$$
$$= -\frac{2|c_2|^2\overline{c_2}c_4}{|c_2|^2 + \Re(\overline{c_2}c_4)}$$

Moreover, if one compares c_1 and c_3 in terms of α and v, one readily sees that:

$$c_3 = \frac{2\alpha}{1-\alpha}c_1$$

Finally, if $c_4 = 0$, then if $\alpha = 0$, $g'(1) = c_2$ implies $v = -c_2$. Else, if $\alpha \neq 0$, then $g''(1) = c_4$ implies $v = 0$, and this means that $f \equiv 0$. □

Proposition 4.1.4. *Fix $n = 2$, so that $Q = \{(w, z) \in \mathbb{C}^2, \Re w = |z|^2\}$. Let f be a stationary disk attached to Q, with $f(1) = 0$. Write $f'(1) = (c_1, c_2)^t$ and $f''(1) = (c_3, c_4)^t$. Let $q = (q_0, q') \in \mathbb{C}^2$ be a given point in the region bounded by Q. Suppose that $f(0) = q$. Then, we either have $c_1 = 0$, and then $f \equiv 0$ and $q = 0$, or $c_1 \in \mathbb{R} \setminus \{0\}$, in which case we necessarily have that $c_1\overline{c_3} + 4|c_2|^2\overline{c_1} \neq 0$, and q_0 and q' are given in terms of the c_i by:*

$$q_0 = \frac{2c_1^2}{\overline{c_3} + 4|c_2|^2}$$

and:

$$q' = \frac{2c_1c_2}{\overline{c_3} + 4|c_2|^2}$$

Proof. The expressions for $f'(1)$ and $f''(1)$ that we obtained in Corollary 4.1.1 give us this set of equations:

$$\frac{|q_0|^2}{2(|q'|^2 - \Re q_0)} = c_1 \tag{1}$$

$$\frac{\overline{q_0}q'}{2(|q'|^2 - \Re q_0)} = c_2 \tag{2}$$
$$\frac{(q_0 - 2|q'|^2)|q_0|^2}{2(|q'|^2 - \Re q_0)^2} = c_3 \tag{3}$$
$$\frac{(q_0 - 2|q'|^2)\overline{q_0}q'}{2(|q'|^2 - \Re q_0)^2} = c_4 \tag{4}$$

Notice how from (1) we have $c_1 \in \mathbb{R}$. If $c_1 = 0$, then $q_0 = 0$ as well, and then $c_2 = c_3 = c_4 = 0$, so by Proposition 4.1.3, $\alpha = v = 0$, and so we must have $f \equiv 0$, and then $q = 0$.
So, assume that $c_1 \neq 0$. Then $q_0 \neq 0$ as well. From (1), we have:

$$|q'|^2 - \Re q_0 = \frac{|q_0|^2}{2c_1}$$

and if we substitute this in (2), we get:

$$\frac{\overline{q_0}q'}{2\left(\frac{|q_0|^2}{2c_1}\right)} = c_2$$

so that $\frac{\overline{q_0}q'c_1}{q_0\overline{q_0}} = c_2$, and so $q' = \frac{c_2}{c_1}q_0$.
Now, substituting these information in (3) gives us:

$$\frac{\left(q_0 - 2\left|\frac{c_2}{c_1}q_0\right|^2\right)|q_0|^2}{2\left(\frac{|q_0|^4}{4c_1^2}\right)} = c_3$$

i.e.

$$\frac{2\left(q_0 - \frac{2|c_2|^2}{c_1^2}|q_0|^2\right)c_1^2}{|q_0|^2} = c_3$$

hence,

$$\frac{2(c_1^2 - 2|c_2|^2\overline{q_0})}{\overline{q_0}} = c_3$$

and this reduces to:

$$2(c_1^2 - 2|c_2|^2\overline{q_0}) = c_3\overline{q_0}$$

rearranging, we get:

$$(c_3 + 4|c_2|^2)\overline{q_0} = 2c_1^2$$

We already know that $c_1 \neq 0$, so that necessarily $c_3 + 4|c_2|^2 \neq 0$, and one gets after cojugating:

$$(\overline{c_3} + 4|c_2|^2)q_0 = 2c_1^2$$

and therefore,

$$q_0 = \frac{2c_1^2}{\overline{c_3} + 4|c_2|^2}$$

and:

$$q' = \frac{c_2}{c_1} q_0 = \frac{2c_1 c_2}{\overline{c_3} + 4|c_2|^2}$$

and this completes the proof. □

4.2 In the Context of the Automorphisms of Q

In this section, we mention without proof the automorphisms of Q, and we discuss how they relate to the stationary disks attached to Q.

4.2.1 *The Automorphisms we are After*

The main way automorphisms of Q interact with stationary disks is straightforward: the automorphisms map stationary disks attached to Q to stationary disks attached to Q in a non-trivial manner. It will turn out that this is enough to understand the automorphism group of Q. We start by displaying a tweaked version of the list of automorphisms mentioned in [9].

Theorem 4.2.1. *The group of automorphisms of Q is generated by the families $H_{-2}, H_{-1}, H_0^1, H_0^2, H_1$ and H_2, which are given by the following formulas:*

$$H_{-2}(w, z) = (w - ir, z)$$

$$H_{-1}(w, z) = \big(w + ||a||^2 + 2\langle \overline{a}, z\rangle, z + a\big)$$

$$H_0^1(w, z) = (\lambda^2 w, \lambda z)$$

$$H_0^2(w, z) = (w, Uz)$$

$$H_1(w, z) = \frac{1}{1 - 2i\langle \overline{b}, z\rangle + ||b||^2 w}(w, z + iwb)$$

$$H_2(w, z) = \frac{1}{1 + isw}(w, z)$$

where $r, s, \lambda \in \mathbb{R}$ with $\lambda > 0$, $a, b \in \mathbb{C}^n$, and $U \in M_{n\times n}(\mathbb{C})$ a unitary matrix with respect to $(z, z') \mapsto \langle \overline{z}, z'\rangle$, are parameters of these families.

Proposition 4.2.1. *The generators of the group of automorphisms of Q, listed above, map stationary disks of Q to stationary disks of Q.*

Proof. See Proposition 3.2.2 for the more general proof that stationary disks are biholomorphic invariants. □

What follows is **almost** a proof and is not the most natural way to approach this fact. There are a few **non-trivial** gaps that one should fill, but most of the calculations and required conditions are displayed here. This is what it looks like if one would want to verify this by "brute force". However, as we have already mentioned, this fact is straightforward if one looks at Proposition 3.2.2.

Proof. Recall that the stationary disks attached to Q are given by the family of maps $f : \overline{\Delta} \to \mathbb{C}^{n+1}$ parameterized by $v, u \in \mathbb{C}^n$, $y_0 \in \mathbb{R}$ and $\alpha \in \Delta$, with $f = (f_0, f_1, \dots, f_n)$, where:

$$f_0(\zeta) = \overline{v}^t v + 2\overline{v}^t u \frac{\zeta}{1-\alpha\zeta} + \frac{\overline{u}^t u}{1-|\alpha|^2} \frac{1+\alpha\zeta}{1-\alpha\zeta} + iy_0$$

and $\forall j \geq 1$,

$$f_j(\zeta) = v_j + u_j \frac{\zeta}{1-\alpha\zeta}$$

Let $\Phi : \mathbb{C}^n \times \mathbb{C}^n \times \Delta \times \mathbb{R} \to \left\{ g : \overline{\Delta} \to \mathbb{C}^{n+1}, g \text{ is a stationary disk attached to } Q \right\}$ be the map given by: $\Phi(v, u, \alpha, y_0) = f$ as displayed above.
Let's fix such a stationary disk $f = \Phi(v, u, \alpha, y_0)$ and feed it into each of the H's, trying to compute the corresponding parameters for the outputs $H \circ f$. For convenience, we will use the notation f' for $(f_1, \dots, f_n)$. So that one has:

$$f'(\zeta) = v + u \frac{\zeta}{1-\alpha\zeta}$$

It's clear that $H_{-2} \circ f = \Phi(v, u, \alpha, y_0 - r)$ is a stationary disk attached to Q.
For H_{-1}, let's look at the second component of $H_{-1}(f(\zeta))$:

$$f'(\zeta) + a = v + u \frac{\zeta}{1-\alpha\zeta} + a = (v+a) + u \frac{\zeta}{1-\alpha\zeta}$$

hence, we will tailor the first component of $H_{-1}(f(\zeta))$ in such a way as to make the $v + a$ appear and we compute the rest of the new parameters. This first component is just:

$$f_0(\zeta) + \|a\|^2 + 2\langle a, f'(\zeta) \rangle$$

which is:

$$\|v\|^2 + 2\langle \overline{v}, u \rangle \frac{\zeta}{1-\alpha\zeta} + \frac{\|u\|^2}{1-|\alpha|^2} \frac{1+\alpha\zeta}{1-\alpha\zeta} + iy_0 + \|a\|^2 + 2 \left\langle \overline{a}, v + u \frac{\zeta}{1-\alpha\zeta} \right\rangle$$

equivalently:

$$\|v\|^2 + \langle \overline{a}, v \rangle + \langle \overline{v}, a \rangle + \|a\|^2 + 2\langle \overline{v+a}, u \rangle \frac{\zeta}{1-\alpha\zeta} + \frac{\|u\|^2}{1-|\alpha|^2} \frac{1+\alpha\zeta}{1-\alpha\zeta} + iy_0 + \langle \overline{a}, v \rangle - \langle \overline{v}, a \rangle$$

i.e.

$$\|v+a\| + 2\langle \overline{v+a}, u \rangle \frac{\zeta}{1-\alpha\zeta} + \frac{\|u\|^2}{1-|\alpha|^2} \frac{1+\alpha\zeta}{1-\alpha\zeta} + i\left(y_0 + 2\Im\langle \overline{a}, v \rangle\right)$$

and we conclude that $H_{-2} \circ f = \Phi\left(v + a, u, \alpha, y_0 + 2\Im\langle \overline{a}, v\rangle\right)$ is again a stationary disk attached to Q.
For H_0^1 and H_0^2, we clearly have $H_0^1 \circ f = \Phi(\lambda v, \lambda u, \alpha, y_0)$ and $H_0^2 \circ f = \Phi(Uv, Uu, \alpha, y_0)$, stationary disks attached to Q.
For H_1, we are going to write $H_1 \circ f = \Phi(v', u', \alpha', y_0')$ and solve for v', u', α' and y_0'. First, let's consider the case $u = 0$. In this case, $f_0(\zeta) = \|v\|^2 + iy_0$ and $f'(\zeta) = v$, so that f is a constant function. Since f is attached to Q and H_1 is an automorphism of Q, we know that $H_1(f(\zeta)) = (A, B) \in Q$, so that $\Re A = \|B\|^2$. Let $v' = B$ and $y_0' = \Im A$. Then, one has that $H_1(f(\zeta)) = (\|v'\|^2 + iy_0', v')$, and so $H_1 \circ f = \Phi(v', 0, 0, y_0')$ is indeed a stationary disk attached to Q (we did not need to take $\alpha' = 0$, surely any $\alpha' \in \Delta$ works here).
Now, we consider the case $u \neq 0$. First, let's look at the expression:

$$\left.\frac{1}{1 - 2i\langle \overline{b}, z\rangle + \|b\|^2 w}\right|_{(z,w)=f(\zeta)}$$

and let's play around with it. This is just:

$$\frac{1}{1 - 2i\langle \overline{b}, f'(\zeta)\rangle + \|b\|^2 f_0(\zeta)} = \frac{1 - \alpha\zeta}{\eta + \mu\zeta}$$

where:

$$\eta := 1 - 2i\langle \overline{b}, v\rangle + \|b\|^2\|v\|^2 + \|b\|^2\frac{\|u\|^2}{1 - |\alpha|^2} + iy_0\|b\|^2$$

$$\mu := -\alpha + 2i\alpha\langle \overline{b}, v\rangle - 2i\langle \overline{b}, u\rangle - \|b\|^2\|v\|^2\alpha + 2\|b\|^2\langle \overline{v}, u\rangle + \|b\|^2\frac{\|u\|^2}{1 - |\alpha|^2}\alpha - i\alpha y_0\|b\|^2$$

$$\alpha' := -\frac{\mu}{\eta}$$

We need to verify many things. First, that $\eta \neq 0$. Second, that this α' satisfies $|\alpha'| < 1$. And third, that this α' is indeed the α' we are looking for.
Let's write:

$$\eta = \left(1 + 2\Im\langle \overline{b}, v\rangle + \|b\|^2\|v\|^2 + \|b\|^2\frac{\|u\|^2}{1 - |\alpha|^2}\right) + i\left(y_0\|b\|^2 - 2\Re\langle \overline{b}, v\rangle\right)$$

If $\Im\eta \neq 0$, there's nothing to show. Suppose that $\Im\eta = 0$, so that:

$$\eta = 1 + 2\Im\langle \overline{b}, v\rangle + \|b\|^2\|v\|^2 + \|b\|^2\frac{\|u\|^2}{1 - |\alpha|^2}$$

By the Cauchy-Schwarz inequality, one has:

$$\left|\Im\langle \overline{b}, v\rangle\right| \leq |\langle \overline{b}, v\rangle| \leq \|b\|\|v\|$$

hence,

$$1 + 2\Im\langle \overline{b}, v\rangle + \|b\|^2\|v\|^2 \geq 1 - 2\|b\|\|v\| + \|b\|^2\|v\|^2 = (1 - \|b\|\|v\|)^2$$

so that:

$$\eta \geq (1 - \|b\|\|v\|)^2 + \|b\|^2 \frac{\|u\|^2}{1-|\alpha|^2}$$

hence, $\eta \geq 0$. If in fact $\eta = 0$, then we must have:

$$1 - \|b\|\|v\| = 0$$

and:

$$\|b\|^2 \frac{\|u\|^2}{1-|\alpha|^2} = 0$$

hence b cannot possibly be 0, so that we must have $u = 0$, but this is also not possible. Therefore, $\eta \neq 0$.
Now, one needs to check that $|\alpha'| < 1$, and we will leave this for the interested reader.
We will now show that this α' works for our purpose.
First, we look at the second component of $H_1(f(\zeta))$. This is:

$$\frac{1}{\eta} \frac{(1-\alpha\zeta)\left(v + u\frac{\zeta}{1-\alpha\zeta} + if_0(\zeta)b\right)}{1-\alpha'\zeta}$$

which we can re-write as:

$$\frac{1}{\eta}\frac{v+(u-\alpha v)\zeta}{1-\alpha'\zeta} + \frac{1}{\eta}\frac{\left(\|v\|^2 + \frac{\|u\|^2}{1-|\alpha|^2} + iy_0\right)b + \left(-\alpha\|v\|^2 + 2\langle\overline{v}, u\rangle + \frac{\|u\|^2}{1-|\alpha|^2}\alpha - iy_0\alpha\right)b\zeta}{1-\alpha'\zeta}$$

This must equal:

$$v' + u'\frac{\zeta}{1-\alpha'\zeta}$$

Upon multiplying both sides by $\eta(1-\alpha'\zeta)$ and regrouping, one gets on the LHS:

$$\left[v + \left(\|v\|^2 + \frac{\|u\|^2}{1-|\alpha|^2} + iy_0\right)b\right] + \left[u - \alpha v + \left(-\alpha\|v\|^2 + 2\langle\overline{v}, u\rangle + \frac{\|u\|^2}{1-|\alpha|^2}\alpha - iy_0\alpha\right)b\right]\zeta$$

and on the RHS:

$$\eta v' + \eta(u' - \alpha' v')\zeta$$

so that:

$$v' = \frac{1}{\eta}v + \frac{1}{\eta}\left(\|v\|^2 + \frac{\|u\|^2}{1-|\alpha|^2} + iy_0\right)b$$

and:

$$u' = \frac{1}{\eta}u + \alpha' v' - \frac{\alpha}{\eta}v + \frac{1}{\eta}\left(-\alpha\|v\|^2 + 2\langle\overline{v}, u\rangle + \frac{\|u\|^2}{1-|\alpha|^2}\alpha - iy_0\alpha\right)b$$

Let's now write out the first component of $H_1(f(\zeta))$. This is just:

$$\frac{1}{\eta}\frac{(1-\alpha\zeta)f_0(\zeta)}{1-\alpha'\zeta}$$

which, as mentioned above, we can rearrange as:

$$\frac{1}{\eta}\frac{\left(\|v\|^2+\frac{\|u\|^2}{1-|\alpha|^2}+iy_0\right)+\left(-\alpha\|v\|^2+2\langle\overline{v},u\rangle+\frac{\|u\|^2}{1-|\alpha|^2}\alpha-iy_0\alpha\right)\zeta}{1-\alpha'\zeta}$$

this should equal:

$$\|v'\|^2+2\langle\overline{v'},u'\rangle\frac{\zeta}{1-\alpha'\zeta}+\frac{\|u'\|^2}{1-|\alpha'|^2}\frac{1+\alpha'\zeta}{1-\alpha'\zeta}+iy_0'$$

Upon equating these two and multiplying through by $\eta(1-\alpha'\zeta)$, one gets on the LHS:

$$\left(\|v\|^2+\frac{\|u\|^2}{1-|\alpha|^2}+iy_0\right)+\left(-\alpha\|v\|^2+2\langle\overline{v},u\rangle+\frac{\|u\|^2}{1-|\alpha|^2}\alpha-iy_0\alpha\right)\zeta$$

and on the RHS:

$$\eta\left(\|v'\|^2+\frac{\|u'\|^2}{1-|\alpha'|^2}+iy_0'\right)+\left(\mu\|v'\|^2+2\eta\langle\overline{v'},u'\rangle-\frac{\|u'\|^2}{1-|\alpha'|^2}\mu+iy_0'\mu\right)\zeta$$

so that one gets:

$$\begin{cases}\eta\|v'\|^2+\frac{\|u'\|^2}{1-|\alpha'|^2}\eta+iy_0'\eta=\|v\|^2+\frac{\|u\|^2}{1-|\alpha|^2}+iy_0\\ \mu\|v'\|^2+2\eta\langle\overline{v'},u'\rangle-\frac{\|u'\|^2}{1-|\alpha'|^2}\mu+iy_0'\mu=-\alpha\|v\|^2+2\langle\overline{v},u\rangle+\frac{\|u\|^2}{1-|\alpha|^2}\alpha-iy_0\alpha\end{cases}$$

and this gives us two ways of writing y_0', let's go with the first one:

$$y_0'=\frac{1}{\eta}y_0-\frac{i}{\eta}\|v\|^2-\frac{i}{\eta}\frac{\|u\|^2}{1-|\alpha|^2}+i\frac{\|u'\|^2}{1-|\alpha'|^2}+i\|v'\|^2$$

For completeness, one must check that $y_0'\in\mathbb{R}$, and must verify that y_0' satisfies the equation we did not use.
Finally, we show this for H_2. As with H_1, we write $H_2\circ f=\Phi(v',u',\alpha',y_0')$ and solve for these parameters.
If $v=u=0$, then $f(\zeta)=(iy_0,0)$, and:

$$H_1(f(\zeta))=\frac{1}{1-sy_0}(iy_0,0)$$

so that $H_1\circ f=\Phi\left(0,0,0,\frac{y_0}{1-sy_0}\right)$ is once again a stationary disk attached to Q.
Else, let's proceed with the fraction:

$$\frac{1}{1+isf_0(\zeta)}=\frac{1}{1+is\left(\|v\|^2-2\langle\overline{v},u\rangle\frac{\zeta}{1-\alpha\zeta}+\frac{\|u\|^2}{1-|\alpha|^2}\frac{1+\alpha\zeta}{1-\alpha\zeta}+iy_0\right)}$$

This is just:

$$\frac{1-\alpha\zeta}{1-\alpha\zeta+is\left(\|v\|^2-\alpha\|v\|^2\zeta+2\langle\overline{v},u\rangle\zeta+\frac{\|u\|^2}{1-|\alpha|^2}(1+\alpha\zeta)+iy_0-i\alpha y_0\zeta\right)}$$

which we will rewrite as:

$$\frac{1-\alpha\zeta}{\eta+\mu\zeta}=\frac{1}{\eta}\frac{1-\alpha\zeta}{1-\alpha'\zeta}$$

where:

$$\eta := 1+is\|v\|^2+is\frac{\|u\|^2}{1-|\alpha|^2}-sy_0$$

$$\mu := -\alpha-i\alpha s\|v\|^2+2is\langle\overline{v},u\rangle+is\alpha\frac{\|u\|^2}{1-|\alpha|^2}+\alpha sy_0$$

$$\alpha' := -\frac{\mu}{\eta}$$

Notice that $\eta=0 \iff sy_0=1$ and $s\left(\|v\|^2+\frac{\|u\|^2}{1-|\alpha|^2}\right)=0 \implies u=v=0$, a contradiction. Hence, $\eta\neq 0$.
As above, one needs to verify that $|\alpha'|<1$.
Let's show now that this α' works for our purpose. The second component of $H_2(f(\zeta))$ is just:

$$\frac{1}{\eta}\frac{(1-\alpha\zeta)\left(v+u\frac{\zeta}{1-\alpha\zeta}\right)}{1-\alpha'\zeta}=\frac{1}{\eta}\frac{v+(u-\alpha v)\zeta}{1-\alpha'\zeta}$$

and this should equal:

$$v'+u'\frac{\zeta}{1-\alpha'\zeta}$$

Equating these two and multiplying through by $\eta(1-\alpha'\zeta)$, one gets:

$$v+(u-\alpha v)\zeta=\eta v'+(\eta u'-\eta\alpha' v')\zeta$$

so that:

$$v'=\frac{1}{\eta}v$$

and

$$u'=\alpha' v'+\frac{1}{\eta}(u-\alpha v)$$

Looking now at the first component of $H_1(f(\zeta))$:

$$\frac{1}{\eta}\frac{(1-\alpha\zeta)\left(\|v\|^2+2\langle\overline{v},u\rangle\frac{\zeta}{1-\alpha\zeta}+\frac{\|u\|^2}{1-|\alpha|^2}\frac{1+\alpha\zeta}{1-\alpha\zeta}+iy_0\right)}{1-\alpha'\zeta}$$

this is:

$$\frac{1}{\eta}\frac{\left(\|v\|^2+\frac{\|u\|^2}{1-|\alpha|^2}+iy_0\right)+\left(-\|v\|^2\alpha+2\langle\overline{v},u\rangle+\frac{\|u\|^2}{1-|\alpha|^2}\alpha-i\alpha y_0\right)\zeta}{1-\alpha'\zeta}$$

and this must equal:

$$\|v'\|^2+2\langle\overline{v'},u'\rangle\frac{\zeta}{1-\alpha'\zeta}+\frac{\|u'\|^2}{1-|\alpha'|^2}\frac{1+\alpha'\zeta}{1-\alpha'\zeta}+iy_0'$$

Multiplying through by $\eta(1-\alpha'\zeta)$, one gets on the LHS:

$$\left(\|v\|^2+\frac{\|u\|^2}{1-|\alpha|^2}+iy_0\right)+\left(-\|v\|^2\alpha+2\langle\overline{v},u\rangle+\frac{\|u\|^2}{1-|\alpha|^2}\alpha-i\alpha y_0\right)\zeta$$

and on the RHS:

$$\eta\left(\|v'\|^2+\frac{\|u'\|^2}{1-|\alpha'|^2}+iy_0'\right)+\eta\left(-\alpha'\|v'\|^2+2\langle\overline{v'},u'\rangle+\alpha'\frac{\|u'\|^2}{1-|\alpha'|^2}=iy_0'\alpha'\right)\zeta$$

comparing the constant parts, we get:

$$y_0'=\frac{-i}{\eta}\|v\|^2+i\|v'\|^2-\frac{i}{\eta}\frac{\|u\|^2}{1-|\alpha|^2}+i\frac{\|u'\|^2}{1-|\alpha'|^2}+\frac{1}{\eta}y_0$$

And finally, for completion, one must check that y_0' is indeed real and satisfies the other equation. $\square$

4.2.2 *Connecting to Stationary Disks*

We go back to our stationary disks and lay out the notations and alogrithm that we will be carrying out in Chapter 5 to nail down our desired automorphisms.

Definition 4.2.1. *Given a hypersurface $M\subset\mathbb{C}^m$ with $0\in M$, we define $\mathrm{Aut}(M,0)$ to be the group of automorphisms of M which map 0 to 0, which is known as the **isotropy group** of M. What we will care about in the subsequent pages are these isotropic automorphisms.*

Remark 4.2.1. *Given $F:U\subset\mathbb{C}^2\to\mathbb{C}^2$, and supposing that F is twice complex-differentiable on some neighborhood $V\subset U$ of $a\in U$, we know that DF lands in $\mathcal{L}(\mathbb{C}^2,\mathbb{C}^2)$, and then $DF(a)$, looked at as a matrix, is a 2×2 matrix with complex entries, so it's determined by 4 numbers in $\mathbb{C}$. Moreover, we have that D^2F lands in $\mathcal{L}(\mathbb{C}^2,\mathcal{L}(\mathbb{C}^2,\mathbb{C}^2))\equiv\mathcal{L}(\mathbb{C}^2,\mathbb{C}^2;\mathbb{C}^2)$, so that $D^2F(a)$ is determined by 8 complex numbers, and because of symmetry in the mixed second order partial derivatives, we only need 6 numbers in total to determine it.*

Proposition 4.2.2. *Let $U\subset\mathbb{C}^2$ be open and $H:U\to\mathbb{C}^2$ be of class C^2. Let $V\subset\mathbb{C}$ be open and $f:V\to\mathbb{C}^2$ be of class C^2 with $f(V)\subset U$. Write $H=(H_1,H_2)$ and $f=(f_0,g)$, and write the coordinates in $\mathbb{C}^2$ as (w,z). Let $h=(h_0,k)$ be the* *map $h = H \circ f$. Then, with the understanding that what is being fed into the partial derivatives is $f(\zeta)$, we have:*

$$h_0'(\zeta) = \frac{\partial H_1}{\partial w} f_0'(\zeta) + \frac{\partial H_1}{\partial z} g'(\zeta)$$

$$k'(\zeta) = \frac{\partial H_2}{\partial w} f_0'(\zeta) + \frac{\partial H_2}{\partial z} g'(\zeta)$$

$$h_0''(\zeta) = \frac{\partial^2 H_1}{\partial w^2}(f_0'(\zeta))^2 + 2\frac{\partial^2 H_1}{\partial w \partial z} g'(\zeta) f_0'(\zeta) + \frac{\partial^2 H_1}{\partial z^2}(g'(\zeta))^2 + \frac{\partial H_1}{\partial w} f_0''(\zeta) + \frac{\partial H_1}{\partial z} g''(\zeta)$$

$$k''(\zeta) = \frac{\partial^2 H_2}{\partial w^2}(f_0'(\zeta))^2 + 2\frac{\partial^2 H_2}{\partial w \partial z} g'(\zeta) f_0'(\zeta) + \frac{\partial^2 H_2}{\partial z^2}(g'(\zeta))^2 + \frac{\partial H_2}{\partial w} f_0''(\zeta) + \frac{\partial H_2}{\partial z} g''(\zeta)$$

Remark 4.2.2. *This was a direct consequence of the Chain Rule (Theorem 2.1.5). We can also formulate things in terms of linear/multilinear maps, but we prefer this display because we are going to employ precisely these equations in the next few lines.*

From here on, we will fix $n = 2$.

Notation 4.2.1. *Given $H = (H_1, H_2) \in \mathrm{Aut}(Q, 0)$, with abuse of notation, will denote by Λ the 2-jet of H at 0: $\Lambda = (\Lambda_0, \Lambda_1, \Lambda_2)$, where $\Lambda_0 = 0$, and:*

$$\Lambda_1 = \begin{pmatrix} \Lambda_w^{(1)} & \Lambda_z^{(1)} \\ \Lambda_w^{(2)} & \Lambda_z^{(2)} \end{pmatrix}$$

and:

$$\Lambda_2 = \begin{pmatrix} \Lambda_{ww}^{(1)} & \Lambda_{wz}^{(1)} & \Lambda_{zz}^{(1)} \\ \Lambda_{ww}^{(2)} & \Lambda_{wz}^{(2)} & \Lambda_{zz}^{(2)} \end{pmatrix}$$

where the first and second rows of Λ_1 are ∂H_1 and ∂H_2 respectively, and the first and second rows of Λ_2 are the vectors of second partial derivatives of the H_i in the obvious manner.

Proposition 4.2.3. *Let $f = (f_0, g)$ be a stationary disk attached to Q with $f(1) = 0$. Let $H \in \mathrm{Aut}(Q, 0)$, and let Λ be the 2-jet of H at 0. Let $h = (h_0, k)$ be the stationary disk attached to Q given by $h = H \circ f$. We then have $h(1) = 0$, and with the c_i as in Proposition 4.1.3 and $c_2 = k'(1)$ and $c_4 = k''(1)$, one has:*

$$c_2' = \Lambda_w^{(2)} c_1 + \Lambda_z^{(2)} c_2$$

and

$$c_4' = \Lambda_{ww}^{(2)} c_1^2 + 2\Lambda_{wz}^{(2)} c_1 c_2 + \Lambda_{zz}^{(2)} c_2^2 + \Lambda_w^{(2)} c_3 + \Lambda_z^{(2)} c_4$$

Proof. We have $h(1) = H(f(1)) = H(0) = 0$. Now, by the previous proposition, one has:

$$c_2' = \frac{\partial H_2}{\partial w}(0,0) f_0'(1) + \frac{\partial H_2}{\partial z}(0,0) g'(1) = \Lambda_w^{(2)} c_1 + \Lambda_z^{(2)} c_2$$

We compute c_4' in the same manner. □

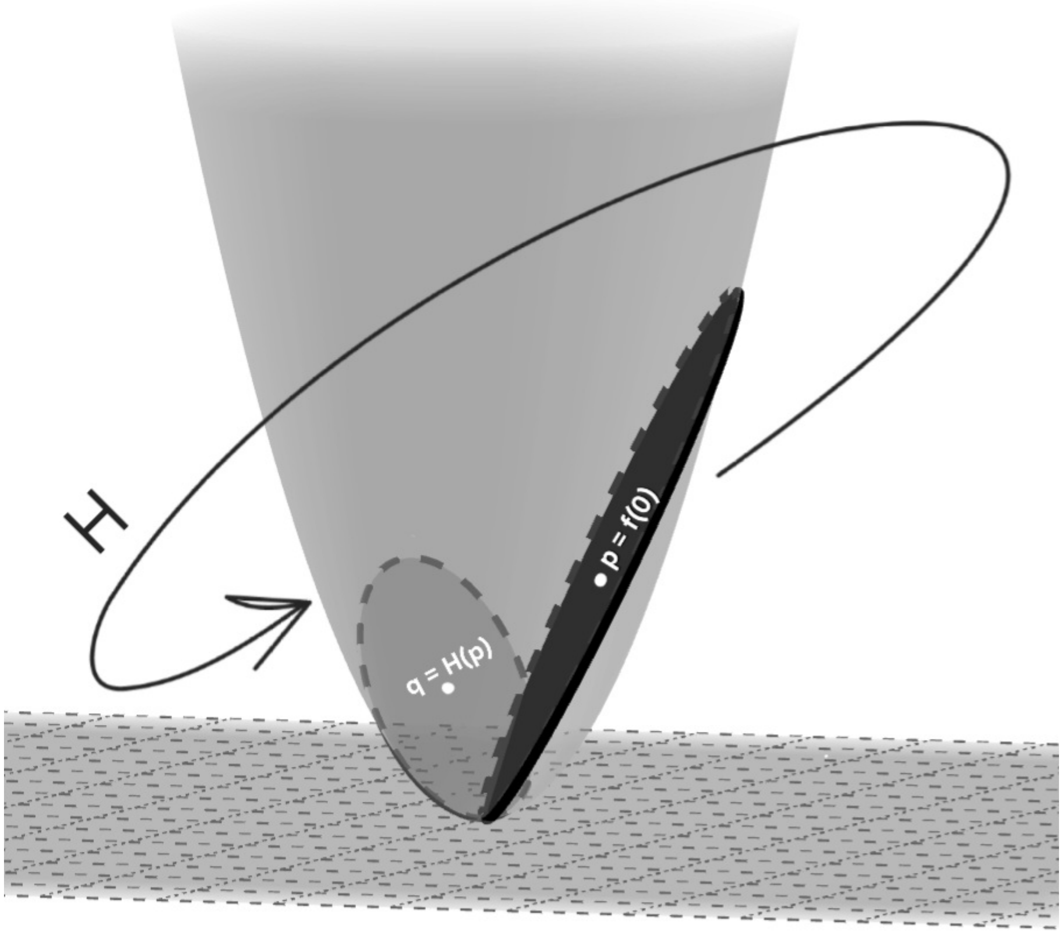

Figure 4.3: In this loose pciture, the black disk is f and the dark grey disk is h. We are trying to understand H by observing how its jets act on the disks.

Remark 4.2.3. *By Proposition 4.1.3, this completely determines h, albeit with a few conditions that one must be careful about. In fact, if α' and v' denote the parameters of h, then one has:*

$$\alpha' = \frac{c_4'}{2c_2' + c_4'}$$

and

$$v' = -\frac{2c_2'^2}{2c_2' + c_4'}$$

and h is given by:

$$h_0(\zeta) = \frac{2(1-\alpha')|v'|^2}{1-|\alpha'|^2}\frac{1-\zeta}{1-\alpha'\zeta}$$

and

$$k(\zeta) = \frac{1-\zeta}{1-\alpha'\zeta}v'$$

and one has the following conditions: $|\alpha'| < 1$, $c_3' = \dfrac{2\alpha'}{1-\alpha'}c_1'$, *and:*

$$c_1' = -\frac{2|c_2'|^4}{|c_2'|^2 + \Re(\overline{c_2'}c_4')}$$

Now the picture is as follows. Having $f(0) = 1$ and knowing the c_2 and c_4 for f completely determines f, and given only the 2-jet of an $H \in \mathrm{Aut}(Q, 0)$, by above we can determine the c_2' and c_4' for h, and these, along with the "inversion" work in Proposition 4.1.4, allow us to determine h.
By definition, we have $h(0) = H(f(0)) = H(p)$. Thus, we are able to find $H(p)$ for every p that is the center of an attached stationary disk f with $f(1) = 0$. But we already know from Proposition 4.1.2 that such an f can be found for any p that is not the origin (which is all we need, we know that H maps the origin to the origin). Hence, we are able to find H itself.
A subtlety here is that for this procedure to really give us an automorphism, we should impose holomorphy. As we will see in Chapter 5, this process doesn't a priori give us an automorphism because of holomorphy issues.

Chapter 5

Parameterizing the Automorphisms

We remind the reader that the parameters of our computations will be Λ, the second jet at 0 of our supposed automorphism H, and $p = (p_0, p')$. We let f be the stationary disk whose center is p and which satisfies $f(1) = 0$. We also put $h = H \circ f$. Recall that if we write $c_1 = f_0'(1)$, $c_2 = g'(1)$, $c_3 = f_0''(1)$ and $c_4 = g''(1)$, and if the c_j' are to denote the analogous parameters for h, then one has:

$$c_2' = \Lambda_w^{(2)} c_1 + \Lambda_z^{(2)} c_2$$

and:

$$c_4' = \Lambda_{ww}^{(2)} c_1^2 + 2\Lambda_{wz}^{(2)} c_1 c_2 + \Lambda_{zz}^{(2)} c_2^2 + \Lambda_w^{(2)} c_3 + \Lambda_z^{(2)} c_4$$

where:

$$c_1 = \frac{|p_0|^2}{2(|p'|^2 - \Re p_0)}$$

$$c_2 = \frac{\overline{p_0} p'}{2(|p'|^2 - \Re p_0)}$$

$$c_3 = \frac{(p_0 - 2|p'|^2)|p_0|^2}{2(|p'|^2 - \Re p_0)^2}$$

and:

$$c_4 = \frac{(p_0 - 2|p'|^2)\overline{p_0} p'}{2(|p'|^2 - \Re p_0)^2}$$

And once we have the c_2' and c_4', we may reconstruct h as follows:

$$\alpha' = \frac{c_4'}{2c_2' + c_4'}$$

and

$$v' = -\frac{2c_2'^2}{2c_2' + c_4'}$$

and h is given by:

$$h_0(\zeta) = \frac{2(1-\alpha')|v'|^2}{1 - |\alpha'|^2} \frac{1-\zeta}{1-\alpha'\zeta}$$

and

$$k(\zeta) = \frac{1-\zeta}{1-\alpha'\zeta}\upsilon'$$

On the other hand, we may also determine the c_1' and c_3' from the c_2' and c_4', as Proposition 4.1.3 shows that:

$$c_1' = -\frac{2|c_2'|^4}{|c_2'|^2 + \Re(\overline{c_2'}c_4')}$$

and:

$$c_3' = -\frac{2|c_2'|^2\overline{c_2'}c_4'}{|c_2'|^2 + \Re(\overline{c_2'}c_4')}$$

and these allow us, for instance, to determine $q = h(0) = (q_0, q')$ as displayed in Proposition 4.1.4:

$$q_0 = \frac{2c_1'^2}{\overline{c_3'} + 4|c_2'|^2}$$

and:

$$q' = \frac{2c_1'c_2'}{\overline{c_3'} + 4|c_2'|^2}$$

5.1 Our Direct Approach

In this section, we nail down the isotropic automorphisms of Q using the most direct approach and the information we have ourselves about the stationary disks attached to the origin.

5.1.1 *Computing with the* q_0 *and* q'

We first turn our attention to the q_0 and q' mentioned above. We write them explicitly in terms of Λ and p, and then we study their holomorphicity.

Proposition 5.1.1. *If p is not the origin, then we have:*

$$q_0 = \frac{4|\varphi|^2|p_0|^2p_0}{(2|p'|^2 - |p_0|^2\Re\omega)(2-p_0\omega)}$$

and:

$$q' = \frac{2p_0\varphi}{2-p_0\omega}$$

or, most explicitly,

$$q' = \frac{2\left(\Lambda_w^{(2)}p_0 + \Lambda_z^{(2)}p'\right)^2}{2\Lambda_w^{(2)}p_0 + 2\Lambda_z^{(2)}p' - \Lambda_{ww}^{(2)}p_0^2 - 2\Lambda_{wz}^{(2)}p'p_0 - \Lambda_{zz}^{(2)}p'^2}$$

where:

$$\varphi = \varphi(p_0, p') = \Lambda_w^{(2)} + \frac{p'}{p_0}\Lambda_z^{(2)}$$

and:

$$\psi = \psi(p_0, p') = \Lambda^{(2)}_{ww} + \frac{2p'}{p_0}\Lambda^{(2)}_{wz} + \frac{p'^2}{p_0^2}\Lambda^{(2)}_{zz}$$

and:

$$\omega = \omega(p_0, p') = \frac{\psi(p_0, p')}{\varphi(p_0, p')} = \frac{\psi}{\varphi}$$

Proof. Note that we may write $c_2 = \dfrac{p'}{p_0}c_1$ and $c_4 = \dfrac{p'}{p_0}c_3$, and recall that $c_1 \in \mathbb{R}$. We have:

$$\begin{aligned}
c_2' &= \Lambda^{(2)}_w c_1 + \Lambda^{(2)}_z c_2 \\
&= \Lambda^{(2)}_w c_1 + \frac{p'}{p_0}\Lambda^{(2)}_z c_1 \\
&= \left(\Lambda^{(2)}_w + \frac{p'}{p_0}\Lambda^{(2)}_z\right) c_1 \\
&= \varphi c_1
\end{aligned}$$

and:

$$\begin{aligned}
c_4' &= \Lambda^{(2)}_{ww} c_1^2 + 2\Lambda^{(2)}_{wz} c_1 c_2 + \Lambda^{(2)}_{zz} c_2^2 + \Lambda^{(2)}_w c_3 + \Lambda^{(2)}_z c_4 \\
&= \Lambda^{(2)}_{ww} c_1^2 + 2\frac{p'}{p_0}\Lambda^{(2)}_{wz} c_1^2 + \frac{p'^2}{p_0^2}\Lambda^{(2)}_{zz} c_1^2 + \Lambda^{(2)}_w c_3 + \frac{p'}{p_0}\Lambda^{(2)}_z c_3 \\
&= \left(\Lambda^{(2)}_{ww} + 2\frac{p'}{p_0}\Lambda^{(2)}_{wz} + \frac{p'^2}{p_0^2}\Lambda^{(2)}_{zz}\right) c_1^2 + \left(\Lambda^{(2)}_w + \frac{p'}{p_0}\Lambda^{(2)}_z\right) c_3 \\
&= \psi c_1^2 + \varphi c_3
\end{aligned}$$

Note that:

$$c_3 = \frac{(p_0 - 2|p'|^2)c_1}{|p'|^2 - \Re p_0}$$

and then,

$$\begin{aligned}
c_1' &= \frac{-2|c_2'|^4}{|c_2'|^2 + \Re(\overline{c_2'}c_4')} \\
&= \frac{-2|\varphi|^4 c_1^4}{|\varphi|^2 c_1^2 + \Re(\overline{\varphi} c_1 c_4')} \\
&= \frac{-2|\varphi|^2 c_1^3}{c_1 + \Re(c_4'/\varphi)} \\
&= \frac{-2|\varphi|^2 c_1^3}{c_1 + \Re\{c_1^2\omega + c_3\}} \\
&= \frac{-2|\varphi|^2 c_1^3}{c_1 + c_1^2\Re\omega + \dfrac{c_1}{|p'|^2 - \Re p_0}(\Re p_0 - 2|p'|^2)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-2|\varphi|^2 c_1^2}{1 + c_1 \Re\omega + \dfrac{\Re p_0 - 2|p'|^2}{|p'|^2 - \Re p_0}} \\
&= \frac{-2|\varphi|^2 c_1^2 (|p'|^2 - \Re p_0)}{|p'|^2 - \Re p_0 + \dfrac{|p_0|^2}{2}\Re\omega + \Re p_0 - 2|p'|^2} \\
&= \frac{4|\varphi|^2 c_1^2 (|p'|^2 - \Re p_0)}{2|p'|^2 - |p_0|^2 \Re\omega}
\end{aligned}
$$

Next,

$$
\begin{aligned}
c_3' &= \frac{c_1' c_4'}{c_2'} \\
&= \frac{c_1'}{c_1} \frac{c_4'}{\varphi} \\
&= \frac{4|\varphi|^2 c_1 (|p'|^2 - \Re p_0)}{2|p'|^2 - |p_0|^2 \Re\omega} (c_1^2 \omega + c_3) \\
&= \frac{4|\varphi|^2 c_1 (|p'|^2 - \Re p_0)}{2|p'|^2 - |p_0|^2 \Re\omega} \left(c_1^2 \omega + \frac{p_0 - 2|p'|^2}{|p'|^2 - \Re p_0} c_1 \right) \\
&= \frac{4|\varphi|^2 c_1^2}{2|p'|^2 - |p_0|^2 \Re\omega} \left(c_1 (|p'|^2 - \Re p_0)\omega + p_0 - 2|p'|^2 \right) \\
&= \frac{4|\varphi|^2 c_1^2}{2|p'|^2 - |p_0|^2 \Re\omega} \left(\frac{|p_0|^2}{2}\omega + p_0 - 2|p'|^2 \right) \\
&= \frac{2|\varphi|^2 c_1^2 (|p_0|^2 \omega + 2p_0 - 4|p'|^2)}{2|p'|^2 - |p_0|^2 \Re\omega}
\end{aligned}
$$

Now we may compute q_0:

$$
\begin{aligned}
q_0 &= \frac{2c_1'^2}{\overline{c_3'} + 4|c_2'|^2} \\
&= \frac{2\left(\dfrac{16|\varphi|^4 c_1^4 (|p'|^2 - \Re p_0)^2}{(2|p'|^2 - |p_0|^2 \Re\omega)^2} \right)}{\dfrac{2|\varphi|^2 c_1^2 (|p_0|^2 \overline{\omega} + 2\overline{p_0} - 4|p'|^2)}{2|p'|^2 - |p_0|^2 \Re\omega} + 4|\varphi|^2 c_1^2} \\
&= \frac{16|\varphi|^2 c_1^2 (|p'|^2 - \Re p_0)^2}{(2|p'|^2 - |p_0|^2 \Re\omega)(|p_0|^2 \overline{\omega} + 2\overline{p_0} - 4|p'|^2 + 2(2|p'|^2 - |p_0|^2 \Re\omega))} \\
&= \frac{4|\varphi|^2 |p_0|^4}{(2|p'|^2 - |p_0|^2 \Re\omega)(2\overline{p_0} - |p_0|^2 \omega)} \\
&= \frac{4|\varphi|^2 |p_0|^2 p_0}{(2|p'|^2 - |p_0|^2 \Re\omega)(2 - p_0 \omega)}
\end{aligned}
$$

Finally, we compute q':

$$
q' = \frac{c_2'}{c_1'} q_0
$$

$$= \frac{\frac{\varphi c_1}{4|\varphi|^2 c_1^2(|p'|^2 - \Re p_0)}}{2|p'|^2 - |p_0|^2\Re\omega} q_0$$
$$= \frac{2|p'|^2 - |p_0|^2\Re\omega}{4\overline{\varphi}c_1(|p'|^2 - \Re p_0)} q_0$$
$$= \frac{2|p'|^2 - |p_0|^2\Re\omega}{2\overline{\varphi}|p_0|^2} \frac{4|\varphi|^2|p_0|^4}{(2|p'|^2 - |p_0|^2\Re\omega)(2\overline{p_0} - |p_0|^2\omega)}$$
$$= \frac{2\varphi|p_0|^2}{2\overline{p_0} - |p_0|^2\omega}$$
$$= \frac{2p_0\varphi}{2 - p_0\omega}$$

which matches the claim.
Furthermore, if we now write φ, ψ and ω in terms of Λ and p, we have:

$$p_0\varphi = \Lambda_w^{(2)} p_0 + \Lambda_z^{(2)} p'$$

and:

$$p_0\omega = p_0\frac{\psi}{\varphi} = \frac{\Lambda_{ww}^{(2)}p_0^2 + 2\Lambda_{wz}^{(2)}p'p_0 + \Lambda_{zz}^{(2)}p'^2}{\Lambda_w^{(2)}p_0 + \Lambda_z^{(2)}p'}$$

and if one substitutes into the expression of q', one gets:

$$q' = \frac{2\left(\Lambda_w^{(2)}p_0 + \Lambda_z^{(2)}p'\right)}{2 - \frac{\Lambda_{ww}^{(2)}p_0^2 + 2\Lambda_{wz}^{(2)}p'p_0 + \Lambda_{zz}^{(2)}p'^2}{\Lambda_w^{(2)}p_0 + \Lambda_z^{(2)}p'}}$$
$$= \frac{2\left(\Lambda_w^{(2)}p_0 + \Lambda_z^{(2)}p'\right)^2}{2\Lambda_w^{(2)}p_0 + 2\Lambda_z^{(2)}p' - \Lambda_{ww}^{(2)}p_0^2 - 2\Lambda_{wz}^{(2)}p'p_0 - \Lambda_{zz}^{(2)}p'^2}$$

and this finishes the proof. □

Remark 5.1.1. *Notice that q' is a holomorphic function of p_0 and p', regardless of whether or not the Λ's are coming from an automorphism. Hence, we only need to focus our attention on q_0.*

Remark 5.1.2. *The map $\overline{M} \ni (p_0, p') \mapsto (q_0, q')$ maps Q to Q in a bijective manner. Indeed, we have:*

$$q_0 = \frac{4|p_0\varphi|^2}{(2|p'|^2 - |p_0|^2\Re\omega)} \frac{p_0}{(2 - p_0\omega)}$$

so that:

$$\Re q_0 = \frac{4|p_0\varphi|^2}{(2|p'|^2 - |p_0|^2\Re\omega)} \cdot \frac{1}{2}\left(\frac{p_0}{2 - p_0\omega} + \frac{\overline{p_0}}{2 - \overline{p_0\omega}}\right)$$

i.e.

$$\Re q_0 = \frac{4|p_0\varphi|^2}{(2|p'|^2 - |p_0|^2\Re\omega)} \frac{2\Re p_0 - |p_0|^2\Re\omega}{|2 - p_0\omega|^2}$$

and:

$$|q'|^2 = \frac{4|p_0\varphi|^2}{|2-p_0\omega|^2}$$

Hence,

$$\Re q_0 = |q'|^2$$
$$\Longleftrightarrow$$
$$\frac{4|p_0\varphi|^2}{(2|p'|^2 - |p_0|^2\Re\omega)}\frac{2\Re p_0 - |p_0|^2\Re\omega}{|2-p_0\omega|^2} = \frac{4|p_0\varphi|^2}{|2-p_0\omega|^2}$$
$$\Longleftrightarrow$$
$$2\Re p_0 - |p_0|^2\Re\omega = 2|p'|^2 - |p_0|^2\Re\omega$$
$$\Longleftrightarrow$$
$$\Re p_0 = |p'|^2$$

Proposition 5.1.2. *The map $(p_0, p') \mapsto q_0$ corresponds to an automorphism of M if and only if $\Lambda_z^{(2)} \neq 0$ and either (i) or (ii), where:*

(i) $\Lambda_w^{(2)} = 0$, $\Lambda_{ww}^{(2)} = 0$, $\Lambda_{zz}^{(2)} = 0$, *and* $\Re\left(\Lambda_z^{(2)}\overline{\Lambda_{wz}^{(2)}}\right) = 0$

(ii) $\Lambda_w^{(2)} \neq 0$, *and the following non-linear system of* 5 *equations in* $\Lambda_w^{(2)}$, $\Lambda_z^{(2)}$, $\Lambda_{ww}^{(2)}$, $\Lambda_{wz}^{(2)}$, *and* $\Lambda_{zz}^{(2)}$ *has a solution:*

$$2\left|\Lambda_w^{(2)}\right|^2\overline{\Lambda_w^{(2)}} - \Lambda_z^{(2)}\overline{\Lambda_w^{(2)}\Lambda_{wz}^{(2)}} + \Lambda_{wz}^{(2)}\overline{\Lambda_w^{(2)}\Lambda_z^{(2)}} + \left|\Lambda_z^{(2)}\right|^2\Lambda_{ww}^{(2)} = 0 \tag{5.1}$$
$$4\Lambda_z^{(2)}\overline{\Lambda_w^{(2)}} + \Lambda_{zz}^{(2)}\overline{\Lambda_z^{(2)}} = 0 \tag{5.2}$$
$$-2\left|\Lambda_w^{(2)}\right|^2\overline{\Lambda_{wz}^{(2)}} + \Lambda_{ww}^{(2)}\overline{\Lambda_w^{(2)}\Lambda_z^{(2)}} + 2\Lambda_w^{(2)}\overline{\Lambda_{ww}^{(2)}\Lambda_z^{(2)}} = 0 \tag{5.3}$$
$$2\left|\Lambda_w^{(2)}\right|^2\overline{\Lambda_z^{(2)}} + \left|\Lambda_z^{(2)}\right|^2\overline{\Lambda_{wz}^{(2)}} + \Lambda_{wz}^{(2)}\overline{\Lambda_z^{(2)}}^2 - \Lambda_z^{(2)}\overline{\Lambda_w^{(2)}\Lambda_{zz}^{(2)}} = 0 \tag{5.4}$$
$$2\Lambda_w^{(2)}\overline{\Lambda_{wz}^{(2)}\Lambda_z^{(2)}} + \Lambda_{ww}^{(2)}\overline{\Lambda_z^{(2)}}^2 - 2\left|\Lambda_w^{(2)}\right|^2\overline{\Lambda_{zz}^{(2)}} = 0 \tag{5.5}$$

Proof. By the Cauchy-Riemann equations, we know that q_0 is holomorphic if and only if $\dfrac{\partial q_0}{\partial \overline{p_0}} = 0$ and $\dfrac{\partial q_0}{\partial \overline{p'}} = 0$. Hence, we compute these derivatives and solve the equations. In fact, these equations hold if and only if:

$$\left(\frac{\partial\,\text{num}}{\partial\overline{p_0}}\right)\text{denom} - \left(\frac{\partial\,\text{denom}}{\partial\overline{p_0}}\right)\text{num} = 0$$

and:

$$\left(\frac{\partial\,\text{num}}{\partial\overline{p'}}\right)\text{denom} - \left(\frac{\partial\,\text{denom}}{\partial\overline{p'}}\right)\text{num} = 0$$

where $\text{num} = \overline{p_0\varphi}$ and $\text{denom} = 4|p'|^2 - 2|p_0|^2\Re\omega$ are the "potentially non-holomorphic" parts of q_0. Indeed, by Proposition 5.1.1, we have:

$$q_0 = \frac{\text{num}}{\text{denom}} \cdot \frac{8p_0^2\varphi}{2-p_0\omega}$$

and we know that, then, the function q_0 is holomorphic if and only if $\dfrac{\text{num}}{\text{denom}}$ satisfies the Cauchy-Riemann equations.
We have $\dfrac{\partial\,\text{num}}{\partial\overline{p_0}} = \overline{\Lambda_w^{(2)}}$, and:

$$
\begin{aligned}
2|p_0|^2\Re\omega &= 2\Re\big\{|p_0|^2\omega\big\} \\
&= 2\Re\left\{ \overline{p_0}p_0 \frac{\Lambda_{ww}^{(2)} + \dfrac{2p'}{p_0}\Lambda_{wz}^{(2)} + \dfrac{p'^2}{p_0^2}\Lambda_{zz}^{(2)}}{\Lambda_w^{(2)} + \dfrac{p'}{p_0}\Lambda_z^{(2)}} \right\} \\
&= 2\Re\left\{ \frac{\Lambda_{ww}^{(2)}p_0^2\overline{p_0} + 2\Lambda_{wz}^{(2)}p'p_0\overline{p_0} + \Lambda_{zz}^{(2)}p'^2\overline{p_0}}{\Lambda_w^{(2)}p_0 + \Lambda_z^{(2)}p'} \right\} \\
&= \frac{\Lambda_{ww}^{(2)}p_0^2\overline{p_0} + 2\Lambda_{wz}^{(2)}p'p_0\overline{p_0} + \Lambda_{zz}^{(2)}p'^2\overline{p_0}}{\Lambda_w^{(2)}p_0 + \Lambda_z^{(2)}p'} + \frac{\overline{\Lambda_{ww}^{(2)}}\,\overline{p_0}^2p_0 + 2\overline{\Lambda_{wz}^{(2)}}\,\overline{p'}\,\overline{p_0}p_0 + \overline{\Lambda_{zz}^{(2)}}\,\overline{p'}^2p_0}{\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}}
\end{aligned}
$$

so that denom is given by:

$$
4p'\overline{p'} - \frac{\Lambda_{ww}^{(2)}p_0^2\overline{p_0} + 2\Lambda_{wz}^{(2)}p'p_0\overline{p_0} + \Lambda_{zz}^{(2)}p'^2\overline{p_0}}{\Lambda_w^{(2)}p_0 + \Lambda_z^{(2)}p'} - \underbrace{\frac{\overline{\Lambda_{ww}^{(2)}}\,\overline{p_0}^2p_0 + 2\overline{\Lambda_{wz}^{(2)}}\,\overline{p'}\,\overline{p_0}p_0 + \overline{\Lambda_{zz}^{(2)}}\,\overline{p'}^2p_0}{\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}}}_{\Omega}
$$

Now,

$$
\frac{\partial}{\partial\overline{p_0}}\left(\frac{\Lambda_{ww}^{(2)}p_0^2\overline{p_0} + 2\Lambda_{wz}^{(2)}p'p_0\overline{p_0} + \Lambda_{zz}^{(2)}p'^2\overline{p_0}}{\Lambda_w^{(2)}p_0 + \Lambda_z^{(2)}p'}\right) = \frac{\Lambda_{ww}^{(2)}p_0^2 + 2\Lambda_{wz}^{(2)}p'p_0 + \Lambda_{zz}^{(2)}p'^2}{\Lambda_w^{(2)}p_0 + \Lambda_z^{(2)}p'}
$$

and $\dfrac{\partial\Omega}{\partial\overline{p_0}}$ is given by:

$$
\frac{\left(2\overline{\Lambda_{ww}^{(2)}}\,\overline{p_0}p_0 + 2\overline{\Lambda_{wz}^{(2)}}\,\overline{p'}p_0\right)\left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}\right) - \overline{\Lambda_w^{(2)}}\left(\overline{\Lambda_{ww}^{(2)}}\,\overline{p_0}^2p_0 + 2\overline{\Lambda_{wz}^{(2)}}\,\overline{p'}\,\overline{p_0}p_0 + \overline{\Lambda_{zz}^{(2)}}\,\overline{p'}^2p_0\right)}{\left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}\right)^2}
$$

which reduces, upon expanding the numerator, to:

$$
\frac{\partial\Omega}{\partial\overline{p_0}} = \frac{\overline{\Lambda_{ww}^{(2)}\Lambda_w^{(2)}}\,\overline{p_0}^2p_0 + 2\overline{\Lambda_{ww}\Lambda_z^{(2)}}\,\overline{p_0}\,\overline{p'}p_0 + 2\overline{\Lambda_{wz}\Lambda_z^{(2)}}\,\overline{p'}^2p_0 - \overline{\Lambda_{zz}^{(2)}\Lambda_w^{(2)}}\,\overline{p'}^2p_0}{\left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}\right)^2}
$$

and this way we have ourselves $\dfrac{\partial\,\text{denom}}{\partial\overline{p_0}}$. We then come back to the equation:

$$
\left(\frac{\partial\,\text{num}}{\partial\overline{p_0}}\right)\text{denom} - \left(\frac{\partial\,\text{denom}}{\partial\overline{p_0}}\right)\text{num} = 0
$$

and we write out each term explicitly.
Upon multiplying both sides by $\left(\Lambda_w^{(2)} p_0 + \Lambda_z^{(2)} p'\right)\left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}\right)$, we get:

$$\text{term}_1 - \text{term}_2 - \text{term}_3 + \text{term}_4 + \text{term}_5 = 0$$

where:

$$\begin{aligned}
\text{term}_1 &= 4\overline{\Lambda_w^{(2)}} p'\overline{p'}\left(\Lambda_w^{(2)} p_0 + \Lambda_z^{(2)} p'\right)\left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}\right) \\
\text{term}_2 &= \overline{\Lambda_w^{(2)}}\left(\Lambda_{ww}^{(2)} p_0^2 \overline{p_0} + 2\Lambda_{wz}^{(2)} p' p_0 \overline{p_0} + \Lambda_{zz}^{(2)} p'^2 \overline{p_0}\right)\left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}\right) \\
\text{term}_3 &= \overline{\Lambda_w^{(2)}}\left(\overline{\Lambda_{ww}^{(2)}}\,\overline{p_0}^2 p_0 + 2\overline{\Lambda_{wz}^{(2)}}\,\overline{p'}\,\overline{p_0} p_0 + \overline{\Lambda_{zz}^{(2)}}\,\overline{p'}^2 p_0\right)\left(\Lambda_w^{(2)} p_0 + \Lambda_z^{(2)} p'\right) \\
\text{term}_4 &= \left(\Lambda_{ww}^{(2)} p_0^2 + 2\Lambda_{wz}^{(2)} p' p_0 + \Lambda_{zz}^{(2)} p'^2\right)\left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}\right)^2
\end{aligned}$$

and term_5 is given by:

$$\left(\overline{\Lambda_{ww}^{(2)}\Lambda_w^{(2)}}\,\overline{p_0}^2 p_0 + 2\overline{\Lambda_{ww}\Lambda_z^{(2)}}\,\overline{p_0}\,\overline{p'} p_0 + 2\overline{\Lambda_{wz}\Lambda_z^{(2)}}\,\overline{p'}^2 p_0 - \overline{\Lambda_{zz}^{(2)}\Lambda_w^{(2)}}\,\overline{p'}^2 p_0\right)\left(\Lambda_w^{(2)} p_0 + \Lambda_z^{(2)} p'\right)$$

Upon full expansion, one gets the equation:

$$\sum_{j=1}^{6} \text{term}'_j = 0$$

where:

$$\begin{aligned}
\text{term}'_1 &= 4\left|\Lambda_w^{(2)}\right|^2 \overline{\Lambda_w^{(2)}} |p_0|^2 |p'|^2 + 4\Lambda_z^{(2)} \overline{\Lambda_w^{(2)}}^2 \overline{p_0} |p'|^2 p' - 2\left|\Lambda_w^{(2)}\right|^2 \Lambda_{wz}^{(2)} |p_0|^2 p_0 p' \\
\text{term}'_2 &= -2\Lambda_z^{(2)} \overline{\Lambda_w^{(2)}\Lambda_{wz}^{(2)}} |p_0|^2 |p'|^2 + \Lambda_{ww}^{(2)} \overline{\Lambda_w^{(2)}\Lambda_z^{(2)}} |p_0|^2 p_0 \overline{p'} + 2\Lambda_{wz}^{(2)} \overline{\Lambda_w^{(2)}\Lambda_z^{(2)}} |p_0|^2 |p'|^2 \\
\text{term}'_3 &= \Lambda_{zz}^{(2)} \overline{\Lambda_w^{(2)}\Lambda_z^{(2)}}\,\overline{p_0} |p'|^2 p' + 4\left|\Lambda_w^{(2)}\right|^2 \overline{\Lambda_z^{(2)}} p_0 |p'|^2 \overline{p'} + 4\left|\Lambda_z^{(2)}\right|^2 \overline{\Lambda_w^{(2)}} |p'|^4 \\
\text{term}'_4 &= 2\Lambda_w^{(2)} \overline{\Lambda_{ww}^{(2)}\Lambda_z^{(2)}} |p_0|^2 p_0 \overline{p'} + 2\left|\Lambda_z^{(2)}\right|^2 \overline{\Lambda_{ww}^{(2)}} |p_0|^2 |p'|^2 + 2\Lambda_w^{(2)} \overline{\Lambda_{wz}^{(2)}\Lambda_z^{(2)}} p_0^2 \overline{p'}^2 \\
\text{term}'_5 &= 2\left|\Lambda_z^{(2)}\right|^2 \overline{\Lambda_{wz}^{(2)}} p_0 |p'|^2 \overline{p'} + \Lambda_{ww}^{(2)} \overline{\Lambda_z^{(2)}}^2 p_0^2 \overline{p'}^2 + 2\Lambda_{wz}^{(2)} \overline{\Lambda_z^{(2)}}^2 p_0 |p'|^2 \overline{p'} \\
\text{term}'_6 &= \Lambda_{zz}^{(2)} \overline{\Lambda_z^{(2)}}^2 |p'|^4 - 2\left|\Lambda_w^{(2)}\right|^2 \overline{\Lambda_{zz}^{(2)}} p_0^2 \overline{p'}^2 - 2\Lambda_z^{(2)} \overline{\Lambda_w^{(2)}\Lambda_{zz}^{(2)}} p_0 |p'|^2 \overline{p'}
\end{aligned}$$

Next, we group these terms together according to the p's, and we get the form of the equation we are seeking, which is a polynomial equation in p_0, $\overline{p_0}$, p', and $\overline{p'}$:

$$\eta_1 |p_0|^2 |p'|^2 + \eta_2 \overline{p_0} |p'|^2 p' + \eta_3 |p_0|^2 p_0 \overline{p'} + \eta_4 p_0 |p'|^2 \overline{p'} + \eta_5 |p'|^4 + \eta_6 p_0^2 \overline{p'}^2 = 0$$

where:

$$\begin{aligned}
\eta_1 &= 4\left|\Lambda_w^{(2)}\right|^2 \overline{\Lambda_w^{(2)}} - 2\Lambda_z^{(2)} \overline{\Lambda_w^{(2)}\Lambda_{wz}^{(2)}} + 2\Lambda_{wz}^{(2)} \overline{\Lambda_w^{(2)}\Lambda_z^{(2)}} + 2\left|\Lambda_z^{(2)}\right|^2 \Lambda_{ww}^{(2)} \\
\eta_2 &= 4\Lambda_z^{(2)} \overline{\Lambda_w^{(2)}}^2 + \Lambda_{zz}^{(2)} \overline{\Lambda_w^{(2)}\Lambda_z^{(2)}}
\end{aligned}$$

$$
\begin{aligned}
\eta_3 &= -2\left|\Lambda_w^{(2)}\right|^2 \overline{\Lambda_{wz}^{(2)}} + \Lambda_{ww}^{(2)}\overline{\Lambda_w^{(2)}\Lambda_z^{(2)}} + 2\Lambda_w^{(2)}\overline{\Lambda_{ww}^{(2)}\Lambda_z^{(2)}} \\
\eta_4 &= 4\left|\Lambda_w^{(2)}\right|^2 \overline{\Lambda_z^{(2)}} + 2\left|\Lambda_z^{(2)}\right|^2 \overline{\Lambda_{wz}^{(2)}} + 2\Lambda_{wz}^{(2)}\overline{\Lambda_z^{(2)}}^2 - 2\Lambda_z^{(2)}\overline{\Lambda_w^{(2)}\Lambda_{zz}^{(2)}} \\
\eta_5 &= 4\left|\Lambda_z^{(2)}\right|^2 \overline{\Lambda_w^{(2)}} + \Lambda_{zz}^{(2)}\overline{\Lambda_z^{(2)}}^2 \\
\eta_6 &= 2\Lambda_w^{(2)}\overline{\Lambda_{wz}^{(2)}\Lambda_z^{(2)}} + \Lambda_{ww}^{(2)}\overline{\Lambda_z^{(2)}}^2 - 2\left|\Lambda_w^{(2)}\right|^2 \overline{\Lambda_{zz}^{(2)}}
\end{aligned}
$$

This cannot happen unless $\eta_j = 0$ for each $j \in \{1,2,3,4,5,6\}$. As a result, one gets a system of 6 equations given by $\eta_j = 0$, $1 \leq j \leq 6$.
We will now show that the second equation given by the $\overline{p'}$ derivative gives the same set of equations.
We have $\dfrac{\partial\,\mathrm{num}}{\partial \overline{p'}} = \overline{\Lambda_z^{(2)}}$, and:

$$
\frac{\partial}{\partial \overline{p'}}\left(4p'\overline{p'} - \frac{\Lambda_{ww}^{(2)}p_0^2\overline{p_0} + 2\Lambda_{wz}^{(2)}p'p_0\overline{p_0} + \Lambda_{zz}^{(2)}p'^2\overline{p_0}}{\Lambda_w^{(2)}p_0 + \Lambda_z^{(2)}p'}\right) = 4p'
$$

also, $\dfrac{\partial\Omega}{\partial \overline{p'}}$ is given by:

$$
\frac{\left(2\overline{\Lambda_{wz}^{(2)}}|p_0|^2 + 2\overline{\Lambda_{zz}^{(2)}}\,\overline{p'}p_0\right)\left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}\right) - \overline{\Lambda_z^{(2)}}\left(\overline{\Lambda_{ww}^{(2)}}\,\overline{p_0}^2p_0 + 2\overline{\Lambda_{wz}^{(2)}}\,\overline{p'}\,\overline{p_0}p_0 + \overline{\Lambda_{zz}^{(2)}}\,\overline{p'}^2p_0\right)}{\left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}\right)^2}
$$

which simplifies to:

$$
\frac{\partial\Omega}{\partial \overline{p'}} = \frac{2\overline{\Lambda_{wz}^{(2)}\Lambda_w^{(2)}}|p_0|^2\overline{p_0} + 2\overline{\Lambda_{zz}^{(2)}\Lambda_w^{(2)}}\,\overline{p'}|p_0|^2 + \overline{\Lambda_{zz}^{(2)}\Lambda_z^{(2)}}p_0\overline{p'}^2 - \overline{\Lambda_z^{(2)}\Lambda_{ww}^{(2)}}\,\overline{p_0}^2p_0}{\left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}\right)^2}
$$

which thus gives us $\dfrac{\partial\,\mathrm{denom}}{\partial \overline{p'}}$.
As before now, we write out fully and explicitly the equation:

$$
\left(\frac{\partial\,\mathrm{num}}{\partial \overline{p'}}\right)\mathrm{denom} - \left(\frac{\partial\,\mathrm{denom}}{\partial \overline{p'}}\right)\mathrm{num} = 0
$$

We multiply both sides of the equation by $\left(\Lambda_w^{(2)}p_0 + \Lambda_z^{(2)}p'\right)\left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}\right)$, and we get ourselves:

$$
\mathrm{term}_1'' - \mathrm{term}_2'' - \mathrm{term}_3'' - \mathrm{term}_4'' + \mathrm{term}_5'' = 0
$$

where:

$$
\begin{aligned}
\mathrm{term}_1'' &= 4\overline{\Lambda_z^{(2)}}p'\overline{p'}\left(\Lambda_w^{(2)}p_0 + \Lambda_z^{(2)}p'\right)\left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}\right) \\
\mathrm{term}_2'' &= \overline{\Lambda_z^{(2)}}\left(\Lambda_{ww}^{(2)}p_0^2\overline{p_0} + 2\Lambda_{wz}^{(2)}p'p_0\overline{p_0} + \Lambda_{zz}^{(2)}p'^2\overline{p_0}\right)\left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{term}_3'' &= \overline{\Lambda_z^{(2)}} \left(\overline{\Lambda_{ww}^{(2)}}\,\overline{p_0}^2 p_0 + 2\overline{\Lambda_{wz}^{(2)}}\,\overline{p'}\,\overline{p_0} p_0 + \overline{\Lambda_{zz}^{(2)}}\,\overline{p'}^2 p_0 \right) \left(\Lambda_w^{(2)} p_0 + \Lambda_z^{(2)} p' \right) \\
\mathrm{term}_4'' &= 4p' \left(\Lambda_w^{(2)} p_0 + \Lambda_z^{(2)} p' \right) \left(\overline{\Lambda_w^{(2)}}\,\overline{p_0} + \overline{\Lambda_z^{(2)}}\,\overline{p'} \right)^2
\end{aligned}
$$

and term_5'' is given by:

$$
\left(2\overline{\Lambda_{wz}^{(2)} \Lambda_w^{(2)}} |p_0|^2 \overline{p_0} + 2\overline{\Lambda_{zz}^{(2)} \Lambda_w^{(2)}}\,\overline{p'} |p_0|^2 + \overline{\Lambda_{zz}^{(2)} \Lambda_z^{(2)}} p_0 \overline{p'}^2 - \overline{\Lambda_z^{(2)} \Lambda_{ww}^{(2)}}\,\overline{p_0}^2 p_0 \right) \left(\Lambda_w^{(2)} p_0 + \Lambda_z^{(2)} p' \right)
$$

Fully expanded, this equation is:

$$
\sum_{j=1}^{6} \mathrm{term}_j''' = 0
$$

where:

$$
\begin{aligned}
\mathrm{term}_1''' &= -4 \left| \Lambda_w^{(2)} \right|^2 \overline{\Lambda_w^{(2)}} |p_0|^2 \overline{p_0} p' - 4\Lambda_z^{(2)} \overline{\Lambda_w^{(2)}}^2 \overline{p_0}^2 p'^2 + 2 \left| \Lambda_w^{(2)} \right|^2 \overline{\Lambda_{wz}^{(2)}} |p_0|^4 \\
\mathrm{term}_2''' &= 2\Lambda_z^{(2)} \overline{\Lambda_w^{(2)} \Lambda_{wz}^{(2)}} |p_0|^2 \overline{p_0} p' - \Lambda_{ww}^{(2)} \overline{\Lambda_w^{(2)} \Lambda_z^{(2)}} |p_0|^4 - 2\Lambda_{wz}^{(2)} \overline{\Lambda_w^{(2)} \Lambda_z^{(2)}} |p_0|^2 \overline{p_0} p' \\
\mathrm{term}_3''' &= -\Lambda_{zz}^{(2)} \overline{\Lambda_w^{(2)} \Lambda_z^{(2)}}\,\overline{p_0}^2 p'^2 - 4 \left| \Lambda_w^{(2)} \right|^2 \overline{\Lambda_z^{(2)}} |p_0|^2 |p'|^2 - 4 \left| \Lambda_z^{(2)} \right|^2 \overline{\Lambda_w^{(2)}}\,\overline{p_0} |p'|^2 p' \\
\mathrm{term}_4''' &= -2\Lambda_w^{(2)} \overline{\Lambda_{ww}^{(2)} \Lambda_z^{(2)}} |p_0|^4 - 2 \left| \Lambda_z^{(2)} \right|^2 \overline{\Lambda_{ww}^{(2)}} |p_0|^2 \overline{p_0} p' - 2\Lambda_w^{(2)} \overline{\Lambda_z^{(2)} \Lambda_{wz}^{(2)}} |p_0|^2 p_0 \overline{p'} \\
\mathrm{term}_5''' &= -2 \left| \Lambda_z^{(2)} \right|^2 \overline{\Lambda_{wz}^{(2)}} |p_0|^2 |p'|^2 - \Lambda_{ww}^{(2)} \overline{\Lambda_z^{(2)}}^2 |p_0|^2 p_0 \overline{p'} - 2\Lambda_{wz}^{(2)} \overline{\Lambda_z^{(2)}}^2 |p_0|^2 |p'|^2 \\
\mathrm{term}_6''' &= -\Lambda_{zz}^{(2)} \overline{\Lambda_z^{(2)}}^2 \overline{p_0} |p'|^2 p' + 2 \left| \Lambda_w^{(2)} \right|^2 \overline{\Lambda_{zz}^{(2)}} |p_0|^2 p_0 \overline{p'} + 2\Lambda_z^{(2)} \overline{\Lambda_w^{(2)} \Lambda_{zz}^{(2)}} |p_0|^2 |p'|^2
\end{aligned}
$$

Upon grouping these terms with respect to the p's, one gets precisely:

$$
-\eta_1 |p_0|^2 \overline{p_0} p' - \eta_2 \overline{p_0}^2 p'^2 - \eta_3 |p_0|^4 - \eta_4 |p_0|^2 |p'|^2 - \eta_5 \overline{p_0} |p'|^2 p' - \eta_6 |p_0|^2 p_0 \overline{p'} = 0
$$

so that:

$$
\eta_1 |p_0|^2 \overline{p_0} p' + \eta_2 \overline{p_0}^2 p'^2 + \eta_3 |p_0|^4 + \eta_4 |p_0|^2 |p'|^2 + \eta_5 \overline{p_0} |p'|^2 p' + \eta_6 |p_0|^2 p_0 \overline{p'} = 0
$$

and, although this is a different equation than the one we got before in terms of the monomials in the p's, its result is exactly the same: it holds iff each $\eta_j = 0$, $1 \leq j \leq 6$, and we get our original system again. We now proceed to discuss this system.

First, note if $\Lambda_z^{(2)} = 0$, then $\eta_1 = 0$ gives $\Lambda_w^{(2)} = 0$ as well, and this implies that the Jacobian of the automorphism which q_0 corresponds to is non-invertible at the origin. However, we know that diffeomorphisms have invertible Jacobians, so we know that this case is invalid. Hence, we have $\Lambda_z^{(2)} \neq 0$.

If $\Lambda_w^{(2)} = 0$, then $\eta_1 = 0$ gives us $\Lambda_{ww}^{(2)} = 0$, and $\eta_5 = 0$ gives us $\Lambda_{zz}^{(2)} = 0$. This makes all the equations trivial except for $\eta_4 = 0$, and $\eta_4 = 0$, upon substituting $\Lambda_w^{(2)} = 0$ and dividing both sides by $\overline{\Lambda_z^{(2)}}$, precisely says that:

$$
\Lambda_z^{(2)} \overline{\Lambda_{wz}^{(2)}} + \Lambda_{wz}^{(2)} \overline{\Lambda_z^{(2)}} = 0
$$

i.e.

$$\Re\left(\Lambda_z^{(2)}\overline{\Lambda_{wz}^{(2)}}\right) = 0$$

and this gives us the case (i).
On the other hand, if $\Lambda_w^{(2)} \neq 0$, then we may divide both sides of $\eta_2 = 0$ by $\overline{\Lambda_w^{(2)}}$ to get (5.2). If one divides both sides of the equation $\eta_5 = 0$ by $\overline{\Lambda_z^{(2)}}$, one gets precisely (5.2) again. The equations $\eta_1 = 0$ and $\eta_4 = 0$ can be divided both sides by 2 and from this one gets (5.1) and (5.4) respectively. The equation $\eta_3 = 0$ is (5.3), and the equation $\eta_6 = 0$ is (5.5).
This completes the proof. □

5.1.2 *Solving for the Λ's and Concluding*

We now have ourselves the conclusions that allow us to get the desired isotropic automorphisms, and in this subsection, this is what we do. We will also show how the family of automorphisms we get is generated by the automorphisms as listed in Theorem 4.2.1.

Proposition 5.1.3. *The automorphisms corresponding to the case (i) of Proposition 5.1.2 are the ones of the form:*

$$H(w, z) = \frac{1}{\lambda_1 - \lambda_2 w}\left(|\lambda_1|^2\lambda_1 w, \lambda_1^2 z\right)$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1 \neq 0$ and $\Re\left(\lambda_1\overline{\lambda_2}\right) = 0$.

Proof. We have $\Lambda_w^{(2)} = \Lambda_{ww}^{(2)} = \Lambda_{zz}^{(2)} = 0$, and writing $\lambda_1 = \Lambda_z^{(2)}$ and $\lambda_2 = \Lambda_{wz}^{(2)}$, we have $\Re\left(\lambda_1\overline{\lambda_2}\right) = 0$. Also, note that $\lambda_1 \neq 0$ by Proposition 5.1.2.
Using the notations of Proposition 5.1.1, we have $\varphi = \lambda_1\dfrac{p'}{p_0}$, $\psi = 2\lambda_2\dfrac{p'}{p_0}$, and $\omega = \dfrac{\psi}{\varphi} = 2\dfrac{\lambda_2}{\lambda_1}$. Hence,

$$\Re\omega = \frac{\lambda_2}{\lambda_1} + \frac{\overline{\lambda_2}}{\overline{\lambda_1}} = \frac{\lambda_1\overline{\lambda_2} + \overline{\lambda_2}\lambda_1}{|\lambda_1|^2} = \frac{2\Re\left(\lambda_1\overline{\lambda_2}\right)}{|\lambda_1|^2} = 0$$

so that:

$$q_0 = \frac{4|\lambda_1 p'|^2 p_0}{2|p'|^2\left(2 - 2\dfrac{\lambda_2}{\lambda_1}p_0\right)} = \frac{|\lambda_1|^2 p_0}{1 - \dfrac{\lambda_2}{\lambda_1}p_0} = \frac{|\lambda_1|^2\lambda_1 p_0}{\lambda_1 - \lambda_2 p_0}$$

On the other hand,

$$q' = \frac{2p_0\varphi}{2 - p_0\omega} = \frac{2\lambda_1 p'}{2 - 2\dfrac{\lambda_2}{\lambda_1}p_0} = \frac{\lambda_1^2 p'}{\lambda_1 - \lambda_2 p_0}$$

In our notation, p_0 (the first component) corresponds to w, and p' (the second component) corresponds to z. Hence, q_0 and q' being, respectively, the first and second components of the map H, one gets:

$$H(w,z) = \left(\frac{|\lambda_1|^2\lambda_1 w}{\lambda_1 - \lambda_2 w}, \frac{\lambda_1^2 z}{\lambda_1 - \lambda_2 w}\right)$$

i.e.

$$H(w,z) = \frac{1}{\lambda_1 - \lambda_2 w}\left(|\lambda_1|^2\lambda_1 w, \lambda_1^2 z\right)$$

as claimed. □

Proposition 5.1.4. *The automorphisms corresponding to the case (ii) of the Proposition 5.1.2 are the ones of the form:*

$$H(w,z) = \frac{1}{2\lambda_1\overline{\lambda_2} + 4|\lambda_1|^2 z - \overline{\lambda_2}\lambda_3 w}\left(2|\lambda_2|^2\lambda_1\overline{\lambda_2}w, 2\lambda_1\overline{\lambda_2}\left(\lambda_2 z + \lambda_1 w\right)\right)$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}\setminus\{0\}$ *are such that* $\Re\left(\lambda_1\overline{\lambda_3}\right) = -\dfrac{2|\lambda_1|^4}{|\lambda_2|^2}$.

Proof. In the case (ii), we have $\Lambda_w^{(2)} \neq 0$. Also, note that we already have $\Lambda_z^{(2)} \neq 0$. Let us write $\Lambda_w^{(2)} = \lambda_1, \Lambda_z^{(2)} = \lambda_2$ and $\Lambda_{ww}^{(2)} = \lambda_3$. From the equation (5.2), one has:

$$\Lambda_{zz}^{(2)} = -\frac{4\lambda_2\overline{\lambda_1}}{\overline{\lambda_2}}$$

If we substitute this into (5.4), we get:

$$2|\lambda_1|^2\overline{\lambda_2} + |\lambda_2|^2\overline{\Lambda_{wz}^{(2)}} + \overline{\lambda_2}^2\Lambda_{wz}^{(2)} - \lambda_2\overline{\lambda_1}\left(-\frac{4\lambda_2\overline{\lambda_1}}{\overline{\lambda_2}}\right) = 0$$

i.e.

$$\overline{\lambda_2}^2\Lambda_{wz}^{(2)} + |\lambda_2|^2\overline{\Lambda_{wz}^{(2)}} = -6|\lambda_1|^2\overline{\lambda_2}$$

hence,

$$\overline{\lambda_2}\Lambda_{wz}^{(2)} + \lambda_2\overline{\Lambda_{wz}^{(2)}} = -6|\lambda_1|^2$$

so that:

$$\Re\left(\lambda_2\overline{\Lambda_{wz}^{(2)}}\right) = -3|\lambda_1|^2$$

One may write (5.1) as:

$$2|\lambda_1|^2\overline{\lambda_1} + \overline{\lambda_1}\left(\overline{\lambda_2}\Lambda_{wz}^{(2)} - \lambda_2\overline{\Lambda_{wz}^{(2)}}\right) + |\lambda_2|^2\overline{\lambda_3} = 0$$

i.e.

$$\overline{\lambda_1}\left(2i\Im\left(\overline{\lambda_2}\Lambda_{wz}^{(2)}\right)\right) = -2|\lambda_1|^2\overline{\lambda_1} - |\lambda_2|^2\overline{\lambda_3}$$

hence,

$$i\Im\big(\overline{\lambda_2}\Lambda^{(2)}_{wz}\big) = -|\lambda_1|^2 - \frac{|\lambda_2|^2\overline{\lambda_3}}{2\overline{\lambda_1}}$$

so that:

$$\underbrace{i\Im\big(\overline{\lambda_2}\Lambda^{(2)}_{wz}\big)}_{\in i\mathbb{R}} = \underbrace{-|\lambda_1|^2 - \Re\left\{\frac{|\lambda_2|^2\overline{\lambda_3}}{2\overline{\lambda_1}}\right\}}_{\in\mathbb{R}} + \underbrace{\left(\Re\left\{\frac{|\lambda_2|^2\overline{\lambda_3}}{2\overline{\lambda_1}}\right\} - \frac{|\lambda_2|^2\overline{\lambda_3}}{2\overline{\lambda_1}}\right)}_{\in i\mathbb{R}} \tag{5.6}$$

and this warrants two conclusions.
First:

$$-|\lambda_1|^2 - \Re\left\{\frac{|\lambda_2|^2\overline{\lambda_3}}{2\overline{\lambda_1}}\right\} = 0$$

so:

$$-|\lambda_2|^2\Re\left(\frac{\overline{\lambda_3}}{\overline{\lambda_1}}\right) = 2|\lambda_1|^2$$

i.e.

$$\Re\left(\frac{\lambda_3}{\lambda_1}\right) = -\frac{2|\lambda_1|^2}{|\lambda_2|^2}$$

Note also that:

$$|\lambda_1|^2\Re\left(\frac{\lambda_3}{\lambda_1}\right) = \Re\left\{|\lambda_1|^2\frac{\lambda_3}{\lambda_1}\right\} = \Re\left\{\overline{\lambda_1}\lambda_1\frac{\lambda_3}{\lambda_1}\right\} = \Re\big(\overline{\lambda_1}\lambda_3\big) = \Re\big(\lambda_1\overline{\lambda_3}\big)$$

so:

$$\Re\big(\lambda_1\overline{\lambda_3}\big) = |\lambda_1|^2\Re\left(\frac{\lambda_3}{\lambda_1}\right)$$

thus, we may also write:

$$\Re\big(\lambda_1\overline{\lambda_3}\big) = -\frac{2|\lambda_1|^4}{|\lambda_2|^2}$$

which is part of our claim.
Notice that if $\lambda_3 = 0$, then this very last equality implies that $\lambda_1 = 0$, and this is impossible. So that, also as claimed, one has $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \setminus \{0\}$.
The second conclusion from (5.6) is that:

$$i\Im\big(\overline{\lambda_2}\Lambda^{(2)}_{wz}\big) = \Re\left\{\frac{|\lambda_2|^2\overline{\lambda_3}}{2\overline{\lambda_1}}\right\} - \frac{|\lambda_2|^2\overline{\lambda_3}}{2\overline{\lambda_1}}$$

and so,

$$\begin{aligned}
\overline{\lambda_2}\Lambda^{(2)}_{wz} &= \Re\big(\overline{\lambda_2}\Lambda^{(2)}_{wz}\big) + i\Im\big(\overline{\lambda_2}\Lambda^{(2)}_{wz}\big) \\
&= -3|\lambda_1|^2 + \Re\left\{\frac{|\lambda_2|^2\overline{\lambda_3}}{2\overline{\lambda_1}}\right\} - \frac{|\lambda_2|^2\overline{\lambda_3}}{2\overline{\lambda_1}} \\
&\overset{(\star)}{=} -3|\lambda_1|^2 + \frac{|\lambda_2|^2\lambda_3}{2\lambda_1} - \Re\left(\frac{|\lambda_2|^2\lambda_3}{2\lambda_1}\right)
\end{aligned}$$

$$= -3|\lambda_1|^2 + \frac{|\lambda_2|^2\lambda_3}{2\lambda_1} - \frac{|\lambda_2|^2}{2}\Re\left(\frac{\lambda_3}{\lambda_1}\right)$$
$$= -3|\lambda_1|^2 + \frac{|\lambda_2|^2\lambda_3}{2\lambda_1} - \frac{|\lambda_2|^2}{2}\left(-\frac{2|\lambda_1|^2}{|\lambda_2|^2}\right)$$
$$= -2|\lambda_1|^2 + \frac{|\lambda_2|^2\lambda_3}{2\lambda_1}$$

where, to get $(\star)$, we used the fact that for all $z \in \mathbb{C}$, one has $\Re\overline{z} - \overline{z} = z - \Re z$, which can be easily checked.
Thus, dividing through by $\overline{\lambda_2}$ one gets:

$$\Lambda^{(2)}_{wz} = -\frac{2|\lambda_1|^2}{\overline{\lambda_2}} + \frac{\lambda_2\lambda_3}{2\lambda_1}$$

With the information we have now, let us note that:

$$(5.3)$$
$$\Longleftrightarrow$$
$$-2|\lambda_1|^2\left(-\frac{2|\lambda_1|^2}{\lambda_2} + \frac{\overline{\lambda_2\lambda_3}}{2\overline{\lambda_1}}\right) + \lambda_3\overline{\lambda_1\lambda_2} + 2\lambda_1\overline{\lambda_3\lambda_2} = 0$$
$$\Longleftrightarrow$$
$$\frac{4|\lambda_1|^4}{\lambda_2} - \lambda_1\overline{\lambda_2\lambda_3} + \lambda_3\overline{\lambda_1\lambda_2} + 2\lambda_1\overline{\lambda_3\lambda_2} = 0$$
$$\Longleftrightarrow$$
$$\frac{4|\lambda_1|^4}{\lambda_2} + \overline{\lambda_2}\left(\lambda_1\overline{\lambda_3} + \overline{\lambda_1}\lambda_3\right) = 0$$
$$\Longleftrightarrow$$
$$2\overline{\lambda_2}\Re\left(\lambda_1\overline{\lambda_3}\right) = -\frac{4|\lambda_1|^4}{\lambda_2}$$
$$\Longleftrightarrow$$
$$\Re\left(\lambda_1\overline{\lambda_3}\right) = -\frac{2|\lambda_1|^4}{|\lambda_2|^2}$$

Hence, our solution is consistent with (5.3), and (5.3) does not present to us any new information.
We also have the same situation with (5.5).
Indeed,

$$(5.5)$$
$$\Longleftrightarrow$$
$$2\lambda_1\overline{\lambda_2}\left(\frac{-2|\lambda_1|^2}{\lambda_2} + \frac{\overline{\lambda_2\lambda_3}}{2\overline{\lambda_1}}\right) + \lambda_3\overline{\lambda_2}^2 - 2|\lambda_1|^2\left(\frac{-4\overline{\lambda_2}\lambda_1}{\lambda_2}\right) = 0$$
$$\Longleftrightarrow$$

$$-\frac{4|\lambda_1|^2\lambda_1\overline{\lambda_2}}{\lambda_2}+\frac{\lambda_1\overline{\lambda_2}^2\overline{\lambda_3}}{\overline{\lambda_1}}+\lambda_3\overline{\lambda_2}^2+\frac{8|\lambda_1|^2\lambda_1\overline{\lambda_2}}{\lambda_2}=0$$

$$\Longleftrightarrow$$

$$\frac{4|\lambda_1|^2\lambda_1}{\lambda_2}+\frac{\lambda_1\overline{\lambda_2\lambda_3}}{\overline{\lambda_1}}+\lambda_3\overline{\lambda_2}=0$$

$$\Longleftrightarrow$$

$$\frac{\overline{\lambda_2}\left(\lambda_1\overline{\lambda_3}+\overline{\lambda_1}\lambda_3\right)}{\overline{\lambda_1}}=-\frac{4|\lambda_1|^2\lambda_1}{\lambda_2}$$

$$\Longleftrightarrow$$

$$\lambda_1\overline{\lambda_3}+\overline{\lambda_1}\lambda_3=-\frac{4|\lambda_1|^4}{|\lambda_2|^2}$$

$$\Longleftrightarrow$$

$$\Re\big(\lambda_1\overline{\lambda_3}\big)=-\frac{2|\lambda_1|^4}{|\lambda_2|^2}$$

and so, our solution is also consistent with (5.5), and (5.5) does not provide us with any new information.
This completes the solution to the system in (ii), and now we may substitute the expressions we got for $\Lambda^{(2)}_{wz}$ and $\Lambda^{(2)}_{zz}$ in terms of the λ_1, λ_2 and λ_3, as well as use the conditions on λ_1, λ_2 and λ_3 to figure out q_0 and q'.
With the notations of Proposition 5.1.1, one has:

$$p_0\varphi=\lambda_1 p_0+\lambda_2 p'$$

and:

$$\begin{aligned}p_0^2\psi&=\lambda_3 p_0^2+2\Lambda^{(2)}_{wz}p'p_0+\Lambda^{(2)}_{zz}p'^2\\&=\lambda_3 p_0^2+\left(-\frac{4|\lambda_1|^2}{\overline{\lambda_2}}+\frac{\lambda_2\lambda_3}{\lambda_1}\right)p'p_0-\frac{4\lambda_2\overline{\lambda_1}}{\overline{\lambda_2}}p'^2\end{aligned}$$

so,

$$p_0\omega=p_0\frac{\psi}{\varphi}=\frac{p_0^2\psi}{p_0\varphi}=\frac{\lambda_3 p_0^2+\left(-\frac{4|\lambda_1|^2}{\overline{\lambda_2}}+\frac{\lambda_2\lambda_3}{\lambda_1}\right)p'p_0-\frac{4\lambda_2\overline{\lambda_1}}{\overline{\lambda_2}}p'^2}{\lambda_1 p_0+\lambda_2 p'}$$

We recall that:

$$q_0=\frac{4|p_0\varphi|^2p_0}{(2|p'|^2-\Re(\overline{p_0}p_0\omega))(2-p_0\omega)}$$

and that:

$$q'=\frac{2p_0\varphi}{2-p_0\omega}$$

The reader may verify by direct manipulation and comparison, and without needing any special assumptions such as the relation we obtained above between the λ_i's, that the following symbolic equality follows from the very latter expressions of $p_0\varphi, p_0\omega$ and q_0:

$$q_0 = \frac{4\lambda_1\overline{\lambda_2}\left|\lambda_1 p_0 + \lambda_2 p'\right|^2 p_0}{\left(2|p'|^2 - \Re\left\{\frac{\lambda_3}{\lambda_1}|p_0|^2 - 4\frac{\overline{\lambda_1}}{\overline{\lambda_2}}p'\overline{p_0}\right\}\right)\left(2\lambda_1\overline{\lambda_2} + 4|\lambda_1|^2 p' - \overline{\lambda_2}\lambda_3 p_0\right)}$$

Now, observe that:

$$\begin{aligned}
2|p'|^2 - \Re\left\{\frac{\lambda_3}{\lambda_1}|p_0|^2 - 4\frac{\overline{\lambda_1}}{\overline{\lambda_2}}p'\overline{p_0}\right\} &= 2|p'|^2 - |p_0|^2\Re\left\{\frac{\lambda_3}{\lambda_1}\right\} + 4\Re\left\{\frac{\overline{\lambda_1}}{\overline{\lambda_2}}p'\overline{p_0}\right\} \\
&= 2|p'|^2 - |p_0|^2\left(-\frac{2|\lambda_1|^2}{|\lambda_2|^2}\right) + 4\Re\left\{\frac{\overline{\lambda_1}}{\overline{\lambda_2}}p'\overline{p_0}\right\} \\
&= \frac{2}{|\lambda_2|^2}\left(|\lambda_2 p'|^2 + |\lambda_1 p_0|^2 + 2\Re\left\{|\lambda_2|^2\frac{\overline{\lambda_1}}{\overline{\lambda_2}}p'\overline{p_0}\right\}\right) \\
&= \frac{2}{|\lambda_2|^2}\left(|\lambda_2 p'|^2 + |\lambda_1 p_0|^2 + 2\Re\left(\overline{\lambda_1 p_0}\lambda_2 p'\right)\right) \\
&= \frac{2}{|\lambda_2|^2}\left(\overline{\lambda_1 p_0}\lambda_1 p_0 + \overline{\lambda_1 p_0}\lambda_2 p' + \overline{\lambda_2 p'}\lambda_1 p_0 + \overline{\lambda_2 p'}\lambda_2 p'\right) \\
&= \frac{2}{|\lambda_2|^2}\left(\overline{\lambda_1 p_0} + \overline{\lambda_2 p'}\right)\left(\lambda_1 p_0 + \lambda_2 p'\right) \\
&= \frac{2}{|\lambda_2|^2}|\lambda_1 p_0 + \lambda_2 p'|^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
q_0 &= \frac{4\lambda_1\overline{\lambda_2}\left|\lambda_1 p_0 + \lambda_2 p'\right|^2 p_0}{\left(\frac{2}{|\lambda_2|^2}|\lambda_1 p_0 + \lambda_2 p'|^2\right)\left(2\lambda_1\overline{\lambda_2} + 4|\lambda_1|^2 p' - \overline{\lambda_2}\lambda_3 p_0\right)} \\
&= \frac{2|\lambda_2|^2\lambda_1\overline{\lambda_2}p_0}{2\lambda_1\overline{\lambda_2} + 4|\lambda_1|^2 p' - \overline{\lambda_2}\lambda_3 p_0}
\end{aligned}$$

On the other hand, one can also verify as above that the following symbolic equality follows merely from the above-displayed expressions of $p_0\varphi$, $p_0\omega$ and q':

$$q' = \frac{2\lambda_1\overline{\lambda_2}\left(\lambda_2 p' + \lambda_1 p_0\right)}{2\lambda_1\overline{\lambda_2} + 4|\lambda_1|^2 p' - \overline{\lambda_2}\lambda_3 p_0}$$

Finally, as our p_0 corresponds to w and our p' corresponds to z, one gets the automorphism:

$$H(w,z) = \left(\frac{2|\lambda_2|^2\lambda_1\overline{\lambda_2}w}{2\lambda_1\overline{\lambda_2} + 4|\lambda_1|^2 z - \overline{\lambda_2}\lambda_3 w}, \frac{2\lambda_1\overline{\lambda_2}\left(\lambda_1 w + \lambda_2 z\right)}{2\lambda_1\overline{\lambda_2} + 4|\lambda_1|^2 z - \overline{\lambda_2}\lambda_3 w}\right)$$

or, in other words,

$$H(w,z) = \frac{1}{2\lambda_1\overline{\lambda_2} + 4|\lambda_1|^2 z - \overline{\lambda_2}\lambda_3 w}\left(2|\lambda_2|^2\lambda_1\overline{\lambda_2}w, 2\lambda_1\overline{\lambda_2}\left(\lambda_1 w + \lambda_2 z\right)\right)$$

This completes the proof. □

Corollary 5.1.1. *The group of isotropic automorphisms of Q is generated by the families of functions $H_{\tau,\kappa}$ and $H_{\alpha,\beta,\gamma}$ given by:*

$$H_{\tau,\kappa}(w,z) = \frac{1}{\tau - \kappa w}\left(|\tau|^2\tau w, \tau^2 z\right)$$

and:

$$H_{\alpha,\beta,\gamma}(w,z) = \frac{1}{2\alpha\overline{\beta} + 4|\alpha|^2 z - \overline{\beta}\gamma w}\left(2|\beta|^2\alpha\overline{\beta}w, 2\alpha\overline{\beta}\left(\beta z + \alpha w\right)\right)$$

where $\kappa \in \mathbb{C}$, $\tau, \alpha, \beta, \gamma \in \mathbb{C} \setminus \{0\}$, $\Re(\tau\overline{\kappa}) = 0$, and $\Re(\alpha\overline{\gamma}) = -\dfrac{2|\alpha|^4}{|\beta|^2}$

Proof. This is merely a restatement of the propositions 5.1.3 and 5.1.4. □

Corollary 5.1.2. *Every function H displayed in Theorem 4.2.1 and satisfying $H(0,0) = (0,0)$ is an isotropic automorphism of Q.*

Proof. These (families of) functions are H_0^1, H_0^2, H_1 and H_2. Note that, since $n = 1$ in our work, we should write:

$$H_0^2(w,z) = (w, \eta z)$$

and:

$$H_1(w,z) = \frac{1}{1 - 2i\overline{b}z + |b|^2 w}(w, z + ibw)$$

for some $\eta, b \in \mathbb{C}$, with $|\eta| = 1$.
To get H_0^1, one can just take $\lambda_1 = \lambda > 0$ and $\lambda_2 = 0$ in Proposition 5.1.3: we have $\lambda_1 \neq 0$ and $\Re\big(\lambda_1\overline{\lambda_2}\big) = 0$ by default, and with these values of λ_1 and λ_2, we get precisely $H(w,z) = (\lambda^2 w, \lambda z)$.
To get H_0^2, it is enough to take $\lambda_1 = \eta$ and $\lambda_2 = 0$ in Proposition 5.1.3. As $|\eta| = 1$, we have $\eta \neq 0$, and surely we have $\Re\big(\lambda_1\overline{\lambda_2}\big) = 0$. One gets $H(w,z) = (|\eta|^2 w, \eta z)$, i.e. $H(w,z) = (w, \eta z)$, as desired.
For H_2, we also use Proposition 5.1.3, this time with $\lambda_1 = 1$ and $\lambda_2 = -is$. We have $\Re\big(\lambda_1\overline{\lambda_2}\big) = \Re(is) = 0$ since $s \in \mathbb{R}$, and one gets exactly:

$$H(w,z) = \frac{1}{1 + isw}(w,z)$$

Finally, for H_1, if $b = 0$, then H_1 is just the identity map, and this is already obtainable from Proposition 5.1.3 by taking $\lambda_1 = 1$ and $\lambda_2 = 0$. If $b \neq 0$, we take $\lambda_1 = ib$, $\lambda_2 = 1$, and $\lambda_3 = -2i|b|^2 b$ in Proposition 5.1.4. We indeed have $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \setminus \{0\}$. Moreover,

$$\Re\big(\lambda_1\overline{\lambda_3}\big) = \Re\big(ib(2i|b|^2\overline{b})\big) = -2|b|^4$$

and:

$$-\frac{2|\lambda_1|^4}{|\lambda_2|^2} = -2|ib|^4 = -2|b|^4$$

so that $\Re\big(\lambda_1\overline{\lambda_3}\big) = -\dfrac{2|\lambda_1|^4}{|\lambda_2|^2}$.
With these values of the λ_i, we get:

$$H(w,z) = \frac{1}{2ib + 4|b|^2 z - (-2i|b|^2 b)w}(2ibw, 2ib(z+ibw))$$

i.e.

$$H(w,z) = \frac{2ib}{2ib\,(1 - 2ibz + |b|^2 w)}(w, z+ibw)$$

hence,

$$H(w,z) = \frac{1}{1 - 2ibz + |b|^2 w}(w, z+ibw)$$

which is precisely H_1. □

Corollary 5.1.3. *The group of isotropic automorphisms of Q is generated by the isotropic automorphisms displayed in Theorem 4.2.1 .*

Proof. Consider an automorphism H of the type displayed in Proposition 5.1.3. In other words, let $\lambda_1, \lambda_2 \in \mathbb{C}$ satisfy $\lambda_1 \neq 0$ and $\Re\big(\lambda_1\overline{\lambda_2}\big) = 0$, and write:

$$H(w,z) = \frac{1}{\lambda_1 - \lambda_2 w}\left(|\lambda_1|^2\lambda_1 w, \lambda_1^2 z\right)$$

Upon dividing through by λ_1, we can write:

$$H(w,z) = \frac{1}{1 - \dfrac{\lambda_2}{\lambda_1}w}\left(|\lambda_1|^2 w, \lambda_1 z\right)$$

Put $\lambda = |\lambda_1|$. As $\lambda_1 \neq 0$, we have $\lambda > 0$. Also put $\eta = \dfrac{\lambda_1}{|\lambda_1|}$. Clearly, $|\eta| = 1$. Finally, put $s = -\dfrac{1}{|\lambda_1|^2}\Im\dfrac{\lambda_2}{\lambda_1}$. We have:

$$\Re\left(\frac{\lambda_2}{\lambda_1}\right) = \frac{1}{2}\left(\frac{\lambda_2}{\lambda_1} + \frac{\overline{\lambda_2}}{\overline{\lambda_1}}\right) = \frac{\Re\big(\lambda_1\overline{\lambda_2}\big)}{|\lambda_1|^2} = 0$$

so:

$$\frac{\lambda_2}{\lambda_1} = i\Im\left(\frac{\lambda_2}{\lambda_1}\right)$$

hence:

$$i|\lambda_1|^2 s = -\frac{\lambda_2}{\lambda_1}$$

and thus, with λ being the parameter of H_0^1, η being that of H_0^2, and s being that of H_2, we have:

$$H_2 \circ H_0^1 \circ H_0^2(w,z) = H_2\left(H_0^1\left(H_0^2(w,z)\right)\right)$$

$$
\begin{aligned}
&= H_2\left(H_0^1\left((w, \eta z)\right)\right) \\
&= H_2\left(\lambda^2 w, \lambda \eta z\right) \\
&= H_2\left(|\lambda_1|^2 w, \lambda_1 z\right) \\
&= \frac{1}{1 + is\left(|\lambda_1|^2 w\right)}\left(|\lambda_1|^2 w, \lambda_1 z\right) \\
&= \frac{1}{1 - \dfrac{\lambda_2}{\lambda_1} w}\left(|\lambda_1|^2 w, \lambda_1 z\right)
\end{aligned}
$$

and this shows that $H = H_2 \circ H_0^1 \circ H_0^2$.
Now consider an automorphism H of the type displayed in Proposition 5.1.4, and divide through by $2\lambda_1\overline{\lambda_2}$ to get:

$$
H(w, z) = \frac{1}{1 + \dfrac{2\overline{\lambda_1}}{\overline{\lambda_2}} z - \dfrac{\lambda_3}{2\lambda_1} w}\left(|\lambda_2|^2 w, \lambda_2 z + \lambda_1 w\right)
$$

Let $\lambda = \left|\dfrac{1}{\lambda_2}\right| = \dfrac{1}{|\lambda_2|}$, and let $\eta = \dfrac{|\lambda_2|}{\lambda_2}$. Then, $\lambda > 0$ and $|\eta| = 1$, and we will take these to be the parameters of H_0^1 and H_0^2 respectively. Take $b = \dfrac{-i\lambda_1}{|\lambda_2|^2}$ to be the parameter of H_1, and, finally, take $s = \dfrac{1}{2|\lambda_2|^2}\Im\left(\dfrac{\lambda_3}{\lambda_1}\right)$ to be the parameter of H_2. Observe, then, that:

$$
\frac{\lambda_3}{\lambda_1} = \Re\left(\frac{\lambda_3}{\lambda_1}\right) + i\Im\left(\frac{\lambda_3}{\lambda_1}\right) = -\frac{2|\lambda_1|^2}{|\lambda_2|^2} + 2i|\lambda_2|^2 s
$$

hence,

$$
is - \frac{\lambda_3}{2|\lambda_2|^2\lambda_1} = is - \frac{1}{2|\lambda_2|^2}\left(-\frac{2|\lambda_1|^2}{|\lambda_2|^2} + 2i|\lambda_2|^2 s\right) = \frac{|\lambda_1|^2}{|\lambda_2|^4} = |b|^2
$$

Now:

$$
\begin{aligned}
H_2 \circ H \circ H_0^1 \circ H_0^2(w, z) &= H_2\left(H_1\left(H_0^1\left(w, \frac{|\lambda_2|}{\lambda_2} z\right)\right)\right) \\
&= H_2\left(H\left(\frac{1}{|\lambda_2|^2} w, \frac{1}{\lambda_2} z\right)\right) \\
&= H_2\left(\frac{1}{1 + \dfrac{2\overline{\lambda_1}}{|\lambda_2|^2} z - \dfrac{\lambda_3}{2|\lambda_2|^2\lambda_1} w}\left(w, z + \frac{\lambda_1}{|\lambda_2|^2} w\right)\right) \\
&= H_2\left(\frac{1}{1 - 2i\bar{b}z - \dfrac{\lambda_3}{2|\lambda_2|^2\lambda_1} w}\left(w, z + ibw\right)\right)
\end{aligned}
$$

$$= H_2 \left(\frac{w}{1 - 2i\bar{b}z - \dfrac{\lambda_3}{2|\lambda_2|^2\lambda_1} w}, \frac{z + ibw}{1 - 2i\bar{b}z - \dfrac{\lambda_3}{2|\lambda_2|^2\lambda_1} w} \right)$$

Thus, $H_2 \circ H \circ H_0^1 \circ H_0^2(w, z)$ is:

$$\frac{1}{1 + is \left(\dfrac{w}{1 - 2i\bar{b}z - \dfrac{\lambda_3}{2|\lambda_2|^2\lambda_1} w} \right)} \left(\frac{w}{1 - 2i\bar{b}z - \dfrac{\lambda_3}{2|\lambda_2|^2\lambda_1} w}, \frac{z + ibw}{1 - 2i\bar{b}z - \dfrac{\lambda_3}{2|\lambda_2|^2\lambda_1} w} \right)$$

which is just:

$$\frac{1}{1 - 2i\bar{b}z - \dfrac{\lambda_3}{2|\lambda_2|^2\lambda_1} w + isw} (w, z + ibw)$$

therefore,

$$\begin{aligned} H_2 \circ H \circ H_0^1 \circ H_0^2(w, z) &= \frac{1}{1 - 2i\bar{b}z + \left(is - \dfrac{\lambda_3}{2|\lambda_2|^2\lambda_1} \right) w} (w, z + ibw) \\ &= \frac{1}{1 - 2i\bar{b}z + |b|^2 w} (w, z + ibw) \\ &= H_1(w, z) \end{aligned}$$

Hence, $H_2 \circ H \circ H_0^1 \circ H_0^2 = H_1$, so that: $H = H_2^{-1} \circ H_1 \circ (H_0^2)^{-1} \circ (H_0^1)^{-1}$.
This completes the proof. □

5.2 Suggestions for Future Work

In this section, we give very brief outlines and discussions of suggested methods for analogous future work regarding our general question.

5.2.1 *The Method of Horizontal Disks*

As we hinted previously, the reason our computations were straightforward was that we had the simplest model of a strongly pseudoconvex hypersurface. In general, the situation isn't as workable, and since strong pseudoconvexity is only a local property, we can only hope to have something that looks like our paraboloid in a neighborhood of the considered point. It turns out that the local automorphisms of the paraboloid are also the global ones, but this generally doesn't apply. This means that we will need to restrict our attention to the disks living inside the given arbitrary neighborhood.

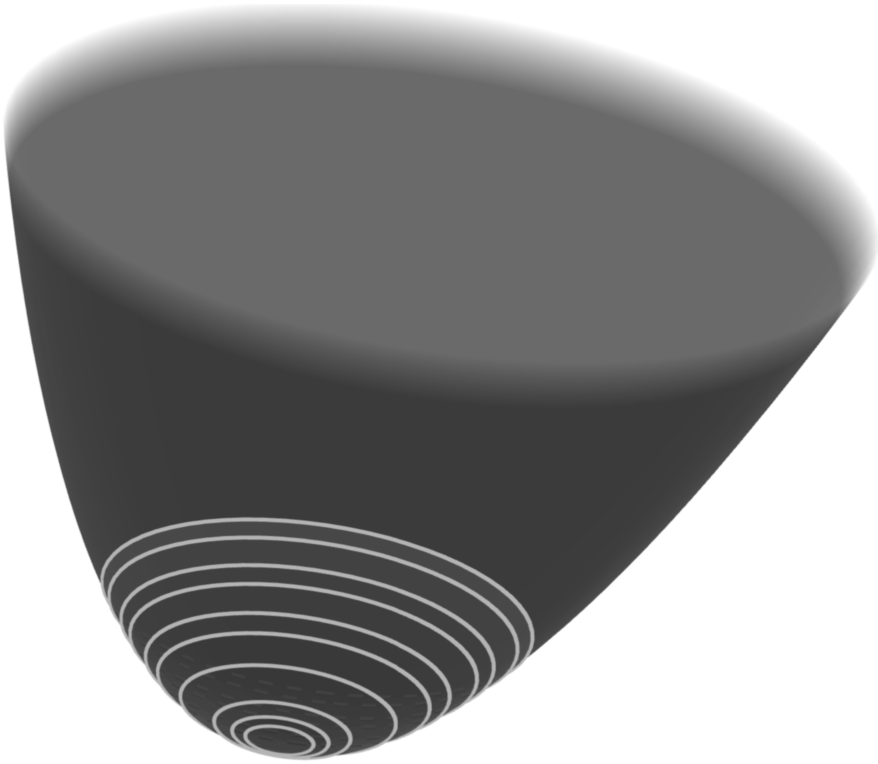

Figure 5.1: The horizontal disks (white) extend arbitrarily close to the origin. We can always work with them regardless of which neighborhood we are "stuck" in. It's not per se the fact that they are horizontal that is most relevant, but the fact that they are stationary disks living arbitrarily close to the origin.

Thus, we suggest a general approach to finding the automorphisms based on this neighborhood restriction. For simplicity, we pretend that - locally - we are living in this simple paraboloid which we studied in Chapters 4 and 5. This will help us give concrete pointers, and one can glean from these pointers our intuition regarding this local issue.

First, we assume we are living in a neighborhood U of the origin and we consider as starting parameters the 2-jets at the origin of an automorphism H. We fix a **slanted disk** f living inside U, which we may pretend is given by something like $f(\zeta) = (2(1-\zeta), 1-\zeta)$ (its precise expression doesn't matter for our purposes here, what matters is that it is sufficiently small and that it is of the type we discussed in our work). We have $f(1) = 0$, and it can be checked that f is stationary and attached to our paraboloid, so that f is of the type we discussed previously. By continuity and compactness, it can be argued that f has a "slanted neighborhood" $V \subset U$, which allows the doing of our previous work to determine H inside V in terms of its 2-jets at the origin. Our goal is then to further determine this H on the entire neighborhood U.

Let us define a **horizontal disk** attached to Q to be any stationary disk $\Gamma : \overline{\Delta} \to \mathbb{C}^2$ attached to Q and given by $\Gamma(\zeta) = (|c|^2 + iy_0, c\zeta)$ for some $c \in \mathbb{C}$ and $y_0 \in \mathbb{R}$. This

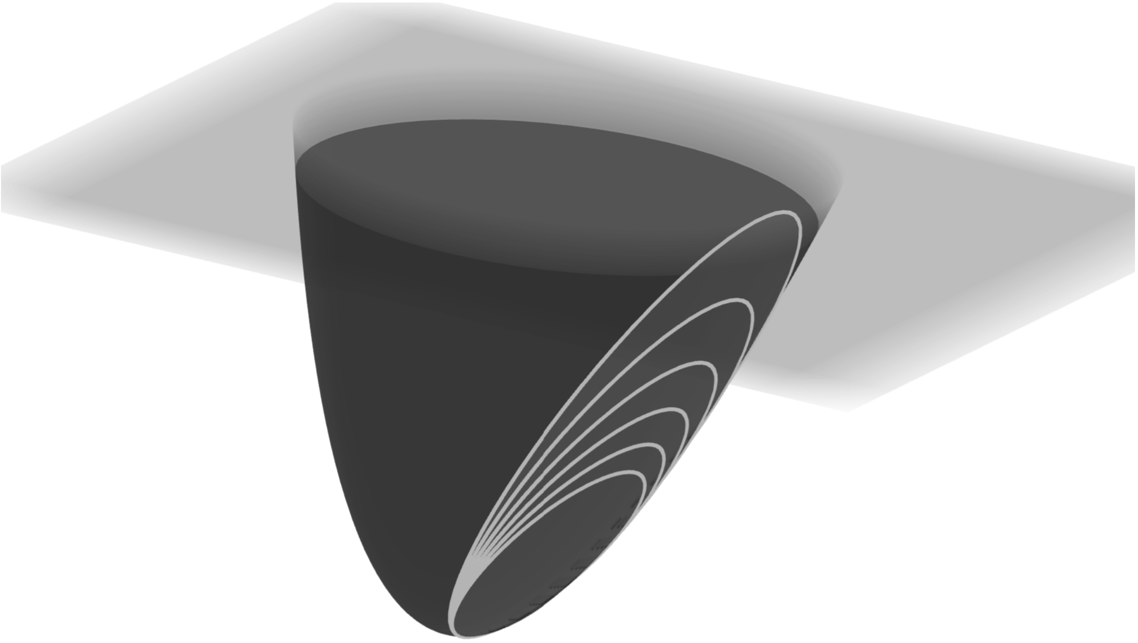

Figure 5.2: Here, we are "cornered" in the displayed neighborhood, bounded by this plane. A "slanted" neighborhood of the kind we are discussing is represented by this aggregation of (white) disks.

is naturally what we would understand by the term "horizontal": it has a constant first component. Note that - generally speaking - it is a rarity for such disks i.e. the horizontal ones to be stationary, and we shouldn't expect this. However, one can try to capture stationary "perturbations" of such horizontal disks. We will discuss this in the next subsection. For now, let us assume that we have this simple enough situation where the Γ's as defined are all stationary, which indeed they are.
The reason these Γ's are relevant is threefold. First and most importantly, they are stationary. Second, they have very simple expressions. Third, they inhabit any given neighborhood of the origin, allowing one to choose from them in a very useful manner however close one is from the origin.
We start with a point $p = (p_0, p') \in U$. We pick out an attached horizontal disk Γ which passes through p. To do this, we just pick $y_0 = \Im p_0$ and choose c in such a manner that $|c| = \sqrt{\Re p_0}$. Then, the corresponding Γ will pass through p at $\zeta_0 = p'/c$. Since $p \in M$, $\Re p_0 > |p'|^2$, so $|c| > |p'|$. Hence, $|\zeta_0| < 1$, i.e. $\zeta_0 \in \Delta$, so the disk indeed passes through p.
We find out the point of intersection r of our horizontal disk Γ with the slanted disk f we mentioned (the white point in Figure 5.3). To find r, we solve $|c|^2 + iy_0 = 2c\zeta$, i.e. $p_0 = 2c\zeta$. So, f indeed passes through r at $\zeta_1 = p_0/2c$.
We want now this ζ_1 to play the role played previously by $\zeta = 1$, and so we want r to play the role previously played by the origin. What follows then is once again doing the procedure of Chapter 4 and Chapter 5 to determine $H(p)$. Indeed, we have H inside V, so we have the 2-jets of H at r. The chain rule, once applied to $h := H \circ \Gamma$ at $\zeta = \zeta_1$, produces the 2-jets of h at $\zeta = \zeta_1$. Assuming we have

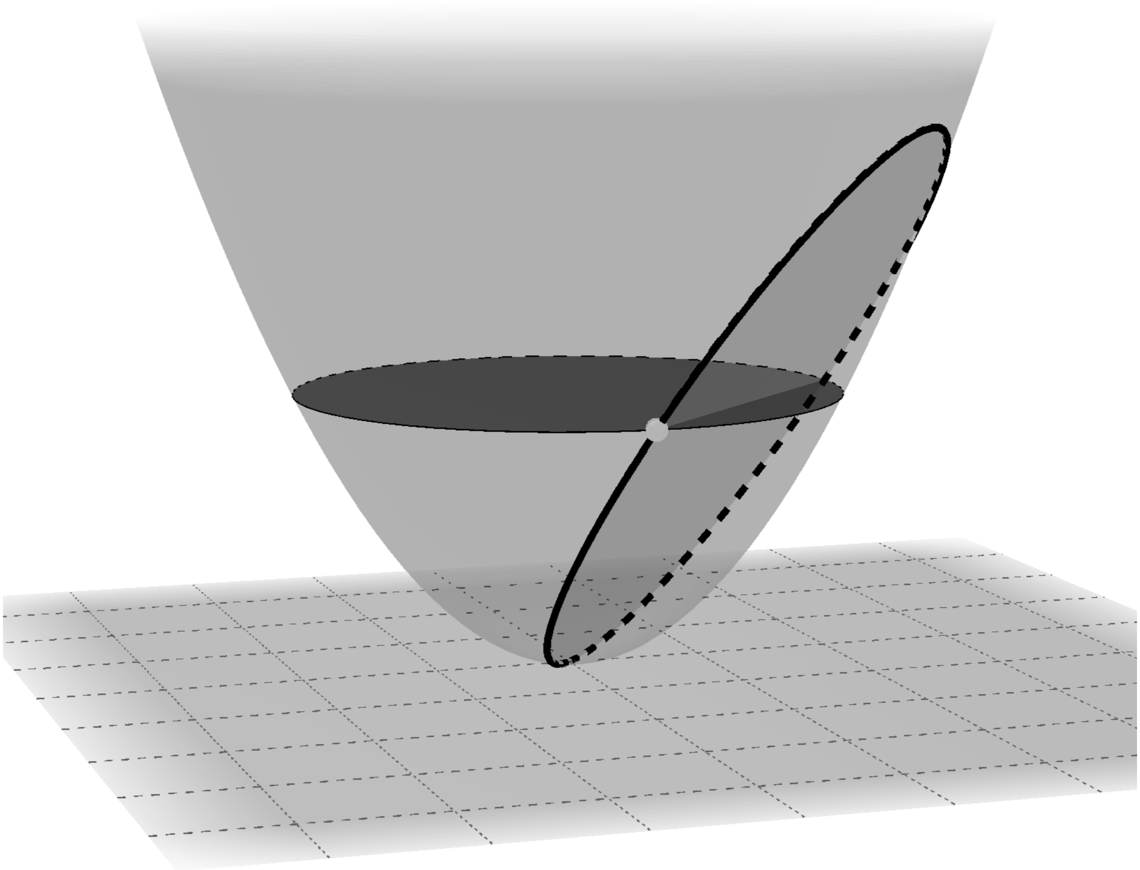

Figure 5.3: The intuitive picture we have in our mind with horizontal disks. Shown as the white point in the figure, this point of intersection of our horizontal disk with an appropriately chosen slanted disk we suggest will play the role previously played by the origin. Note that, mostly, these disks won't intersect in this manner, i.e. won't intersect on the boundary.

the procedure to nail down h starting from its 2-jets at $\zeta = \zeta_1$ (which should be doable but is a bit more messy because $h(\zeta_1)$ is not the origin), we then evaluate: $H(p) = H(\Gamma(\zeta_0)) = h(\zeta_0)$. Finally, by imposing holomorphy like before, one obtains H on all of U.

5.2.2 *Perturbation Problems*

The previous subsection concerned situations that, although are more complicated than our model of work, are still nice enough to allow similar procedures with the only significant constraint being the restriction to an arbitrary neighborhood of the point considered. However, we might generally encounter hypersurfaces of strongly pseudoconvex type which are very perturbed and might not be as workable.

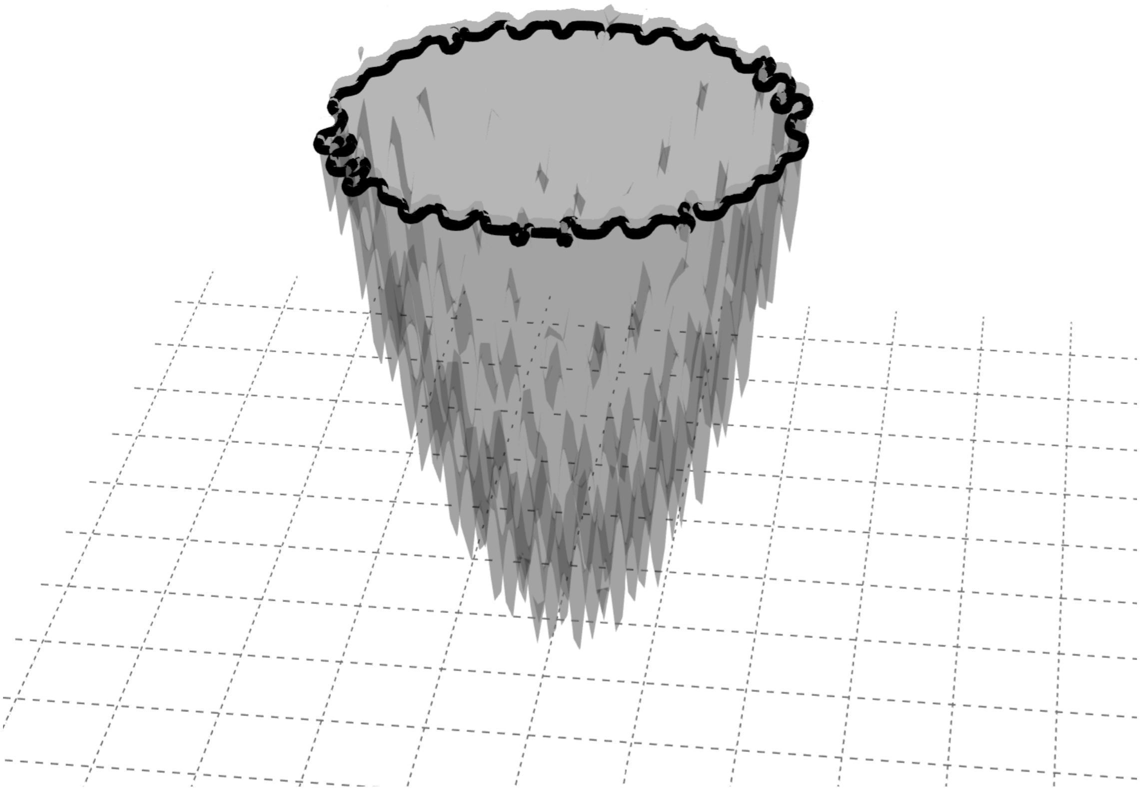

Figure 5.4: A "wiggly" hypersurface like this one might prove challenging to work with even using the horizontal disk approach. The black curve at the top is the boundary of a cross-section, to help the reader better visualize such a hypersurface.

First, for such hypersurfaces, we might not necessarily have explicit formulas for the attached stationary disks. Second, even if such explicit formulas are known, we cannot hope to have stationary "horizontal sections", and the slanted disks we talked about might behave differently.
For this type of situation, we propose the idea of a holistic perturbation, where one adds to the entire admixture a parameter - say similar in spirit to the ones found in the classical Chern-Moser normal form - which serves to measure in a sense the "deviation" of the hypersurface from being a genuine sphere locally. This parameter is 0 if the hypersurface is a locally a sphere, and "how much" it is different from 0 serves to indicate how heavily perturbed it is.
In fact, this parameter is really the defining function itself, so it is not a number or a parameter in the usual sense. This will mean that instead of working in a finite dimensional space with numerical parameters, one will need to work with infinite dimensional (Banach) spaces allowing for appropriate incorporation of the Implicit Function theorem (Theorem 2.1.7). See [8] for related discussions. It can be argued that the whole procedure as applied to such a perturbed hypersurface will be a perturbation of the procedure outlined in Chapter 4 and Chapter 5.

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