AMERICAN UNIVERSITY OF BEIRUT

LEAST GRADIENT PROBLEM

by

JOLIE SIMON LAHOUD

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Approved by:

Dr. Ahmad Sabra, Assistant Professor — Advisor
Mahematics

Dr. Bassam Shayya, Professor — Member of Committee
Mathematics

Dr. Faruk Abi-Khuzam, Professor — Member of Committee
Mathematics

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Abstract
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For a given continuous function g defined on the boundary of Ω where Ω is a bounded lipschitz domain in $\mathbb{R}^n$ satisfying some conditions, we consider proving the existence of a function u in the space of $BV(\Omega)$ that is equal to g on the boundary in the trace sense, and the total variation of its distributional derivative evaluated over Ω is minimal among all such functions,in addition to proving uniqueness when u belongs to $BV(\Omega) \cap C(\overline{\Omega})$.The exposition go deeply in the study of BV theory and sets of finite perimeter.

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Notations

$C_c(X)$	Space of continuous functions with compact support in $X \subset \mathbb{R}^n$
$C(X)$	Space of continuous functions in $X \subset \mathbb{R}^n$
$C_0(X)$	Space of continuous functions f such that $\lim\limits_{\vert x\vert \to \infty} f(x) = 0$

Introduction

The least gradient problem is a problem of minimalization:

$$\inf\{|Du|(\Omega) : u \in BV(\Omega) \cap C(\overline{\Omega}), u = g \text{ on } \partial\Omega\}$$

where Ω is a bounded lipschitz domain in $\mathbb{R}^n$, $g : \partial\Omega \to \mathbb{R}$ continuous. We aim to establish both the existence and uniqueness of a solution to this problem. Additionally, we consider the least gradient problem with a slightly relaxed condition:

$$\inf\{|Du|(\Omega) : u \in BV(\Omega), u = g \text{ on } \partial\Omega\}$$

We will also demonstrate the existence of a solution for this problem. Here the boundary condition is understood in the sense of trace theory in the space of functions of bounded variation BV.

In this thesis, we prove the existence of a solution to this problem under certain conditions on $\partial\Omega$. Specifically, we require that $\partial\Omega$ has non-negative mean curvature (in a weak sense) and is not locally area-minimizing. In two dimensions, these conditions can be replaced with a requirement that the set is strictly convex. Our approach in this thesis is inspired by the work of Sternberg, Williams, and Ziemer in their paper [1] building upon the findings of Bombieri, De Giorgi, and Giusti in [2], which demonstrated, among other things, that the superlevel sets of a function of least gradient are area-minimizing. This result provides the major motivation for the techniques employed. Indeed, this fact, along with the co-area formula, suggests that the existence of a function of least gradient can be established by actually constructing each of its superlevel sets in such a way that they reflect the appropriate boundary condition and that they are area-minimizing.

The thesis is organized as follows. In chapter 1, we start by revisiting some basic ideas in measure theory and introducing new ones.We define the space of bounded variation functions BV, sets of finite perimeters and discuss the coarea formula, that will help us in achieving the existence of a solution. Additionally, we introduce the trace concept. These are essential concepts that provide the groundwork for our study.

In chapter 2, we define the reduced boundary of a set of locally finite perimeter, the set of points where the measure theoretical normal exists.

In Chapter 3, we outline fundamental properties of minimal surfaces, which will serve as valuable tools in our work.

In Chapter 4, we introduce the Least Gradient Problem (LGP) on $BV(\Omega) \cap C(\overline{\Omega})$ and lay out some preliminary ideas.

Finally, chapter 5 is devoted to the explicit construction of the solution to (4.22), as well as the solution to (4.1) and its uniqueness.

Chapter 1

Functions of bounded variation and sets of finite perimeter

1.1 Measure theory

Definition 1.1.1. *Let $(X, \mathcal{A})$ be a measurable space,and let $n \in \mathbb{N}, n \geq 1$,we say that $\mu : \mathcal{A} \to \mathbb{R}^n$ is a measure if:*

1. *$\mu(\emptyset) = 0$,*
2. *(Countable Additivity) For a countable collection $\{A_n\}$ of pairwise disjoint sets in $\mathcal{A}$, we have*
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

If n=1, we say that μ is a real measure, if $n > 1$ we say that μ is a vector measure.

Definition 1.1.2.
1. *A measure μ on X is regular if for each set $A \subset X$ there exists a μ-measurable set B such that $A \subset B$ and $\mu(A) = \mu(B)$.*
2. *A measure μ on $\mathbb{R}^n$ is called Borel if every Borel set is μ-measurable.*
3. *A measure μ on $\mathbb{R}^n$ is Borel regular if μ is Borel and for each set $A \subset \mathbb{R}^n$ there exists a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$.*
4. *A measure μ on $\mathbb{R}^n$ is a Radon measure if μ is Borel regular and $\mu(K) < \infty$ for each compact set $K \subset \mathbb{R}^n$.*
5. *A subset $A \subset X$ is σ-finite with respect to μ if we can write $A = \bigcup_{k\geq 1} B_K$, where B_k is μ-measurable and $\mu(B_k) < \infty$ for $k = 1, 2\cdots$. If X itself is σ-finite we also say that μ is σ-finite.*

Definition 1.1.3 (Total variation measure)**.** *If $\mu : \mathcal{A} \mapsto \mathbb{R}^n$ is a measure, we define its total variation $|\mu|$ for every measurable set E as follows:*

$$|\mu|(E) := \sup\left\{\sum_{h=1}^{\infty} |\mu(E_h)| : E_h \in \mathcal{A}\,\textit{pairwise disjoint}, E = \cup_{h=0}^{\infty} E_h\right\}.$$

Theorem 1.1.1. *[3] Let μ be a measure on $(X, \mathcal{A})$, then $|\mu|$ is a positive finite measure.*

Definition 1.1.4. *(a) Let μ be a positive measure and ν be a real or vector measure on the measure space $(X, \mathcal{A})$. We say that ν is absolutely continuous with respect to μ, and write $\nu << \mu$, if for every $B \in \mathcal{A}$ the following implication holds :*

$$\mu(B) = 0 \implies |\nu|(B) = 0$$

(b) If μ, ν are positive measures, we say that they are mutually singular, and write $\nu \perp \mu$, if there exists $E \in \mathcal{A}$ such that $\mu(E) = 0$ and $\nu(X - E) = 0$. In the case where μ or ν are real or vector-valued, we say that they are mutually singular if $|\mu|$ and $|\nu|$ are so.

Theorem 1.1.2 (Radon-Nikodym)**.** *Let μ be a positive measure and ν be a real or vector measure on the space $(X, \mathcal{A})$ and assume that μ is σ-finite. Then there is a unique pair of $\mathbb{R}^m$-valued measure ν^a, ν^s such that $\nu^a << \mu, \nu^s \perp \mu$ and $\nu = \nu^a + \nu^s$. Moreover, there is a unique function $f \in (L^1(X, \mu))^n$ such that $\nu^a = f\mu$.*
The function f is called the density of ν with respect to μ and is denoted by $\frac{\nu}{\mu}$.

Since trivially each real or vector measure μ is absolutely continuous with respect to $|\mu|$, then we have the following decomposition that we won't prove.

Corollary 1.1.1. *[3][Polar decomposition] Let μ be a $\mathbb{R}^n$-valued measure on the measure space $(X, \mathcal{A})$, then there exists a unique S^{m-1}-valued function $f \in (L^1(X, |\mu|))^n$ such that $\mu = f|\mu|$.*

Proposition 1.1.1. *Let X be a locally compact separable metric space and μ a finite $\mathbb{R}^n$-valued Radon measure on it. Then for every open set $A \in X$ the following equality holds:*

$$|\mu|(A) = \sup\left\{ \sum_{i=1}^{n} \int_X u_i d\mu_i : u \in C_c(A, \mathbb{R}^n), \|u\|_\infty \leq 1 \right\}$$

Proof. By Corollary 1.1.1 there exist a unique S^{m-1}-valued function $f \in (L^1(X, |\mu|))^n$ such that $\mu = f|\mu|$ i.e $\mu = (\mu_1, \mu_2, \cdots, \mu_n) = (f_1|\mu|, \cdots, f_n|\mu|)$. Now fix $A \subset X$ open,and let $u = (u_1, \cdots, u_n) \in C_c(A, \mathbb{R}^n)$ such that $\|u\|_\infty \leq 1$

$$\begin{aligned}
\sum_{i=1}^{n} \int_A u_i \, d\mu_i = \sum_{i=1}^{n} \int_A u_i f_i \, d|\mu| &= \int_A \langle u, f \rangle \, d|\mu| \\
&\leq \int_A |u||f| \, d|\mu| \leq \|u\|_\infty \int_A |f| \, d|\mu| \\
&\leq \|u\|_\infty \int_A d|\mu| \leq \|u\|_\infty |\mu|(A)
\end{aligned}$$

Therefore $\sup\left\{\sum_{i=1}^{n} \int_X u_i \, d\mu_i : u \in C_c(A, \mathbb{R}^n), \|u\|_\infty \leq 1\right\} \leq |\mu|(A)$. Now for the other inequality, using the density of $C_c(A, \mathbb{R}^n)$ in $(L^1(A, |\mu|))^n$ there exist a sequence $(u_h)_h \subset C_c(A, \mathbb{R}^n)$ that converges to f in $(L^1(A, |\mu|))^n$. Moreover, by a truncation argument, we can assume that $\|u_h\|_\infty \leq 1$. Since $u_h = (u_{h,1}, \cdots, u_{h,n})$ converges to $f.\mathbb{1}_A$ in $(L^1(X, |\mu|))^n$ we obtain

$$\lim_{h\to\infty} \sum_{i=1}^{n} \int_X u_{h,i} d\mu_i = \lim_{h\to\infty} \int_A < u_h, f > d|\mu| = \int_A |f|^2 d|\mu| = |\mu|(A).$$

Hence equality holds. □

Theorem 1.1.3 (Riesz). *Let $L : C_c(X, \mathbb{R}^n) \to \mathbb{R}$ be a linear functional which satisfies*

$$\sup\{L(f) : f \in C_c(X, \mathbb{R}^n), |f| \leq 1\} < \infty$$

Then there is a unique $\mathbb{R}^n$-valued Radon measure $\mu = (\mu_1, \cdots, \mu_n)$ on $\mathbb{R}^n$ such that

$$L(f) = \sum_{i=1}^{n} \int_{\mathbb{R}^n} f^i d\mu_i \quad \forall f \in C_c(X, \mathbb{R}^n)$$

And,

$$\|L\| = |\mu|(X)$$

Definition 1.1.5 (Weak* convergence of measures). *Let μ, $(\mu_h)_h$ be R^n-valued Radon measures on X. we say that (μ_h) locally weakly* converges to μ if*

$$\lim_{h\to\infty} \int_X f d\mu_h = \int_X f d\mu$$

for every $f \in C_c(X)$.

If μ and μ_h are finite, we say that (μ_h) weakly converges to μ if*

$$\lim_{h\to\infty} \int_X f d\mu_h = \int_X f d\mu$$

for every $f \in C_0(X)$.

Theorem 1.1.4 (Weak* Compactness). *[3] If (μ_h) is a sequence of finite Radon measures on the l.c.s metric space X with $\sup\{|\mu_h|(X) : h \in \mathbb{N}\} < \infty$,then it has a weakly* converging subsequence.*

Proposition 1.1.2. *[3] Let (μ_h) be a sequence of Radon measures on the l.c.s metric space X locally weakly* converging to μ.Then*

(a) If the measure μ_h are positive, then for every lower semi-continuous function $f : X \to [0, \infty]$

$$\liminf_{h\to\infty} \int_X f d\mu_h \geq \int_X f d\mu$$

And for every upper semi-continuous function $g : X \to [0, \infty)$ with compact support

$$\limsup_{h\to\infty} \int_X g d\mu_h \leq \int_X g d\mu$$

(b) *If $|\mu_h|$ locally weakly* converges to λ, then $\lambda \geq |\mu|$. Moreover if E is a relatively compact μ-measurable set such that $\lambda(\partial E) = 0$, then $\mu_h(E) \to \mu(E)$ as $h \to \infty$. More generally*

$$\int_X f d\mu = \lim_{h\to\infty} \int_X f d\mu_h$$

for any bounded Borel function $f : X \to \mathbb{R}$ with compact support such that the set of its discontinuity points is λ-negligible.

Proposition 1.1.3. *[3] Let (μ_h) be a sequence of positive Radon measures on X, and assume the existence of a positive, finite Radon measure μ in X such that*

$$\lim_{h\to\infty} \mu_h(X) = \mu(X) \quad \text{and} \quad \liminf_{h\to\infty} \mu_h(A) \geq \mu(A)$$

for every $A \subset X$ open. Then

$$\lim_{h\to\infty} \int_X f d\mu_h = \int_X f d\mu$$

f or any bounded continuous function $f : X \to \mathbb{R}$. In particular (μ_h) weakly converges to μ in X.*

Notation. *We denote the average of f over the set E with respect to μ by*

$$⨍_E f d\mu = \frac{1}{\mu(E)} \int_E f d\mu,$$

provided $0 < \mu(E) < \infty$ and the integral on the right is defined.

Theorem 1.1.5 (Lebesgue-Besicovitch differentiation theorem)**.** *Let μ be a radon measure on $\mathbb{R}^n$ and $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then*

$$\lim_{r\to 0} ⨍_{B(x,r)} f d\mu = f(x)$$

for μ a.e $x \in \mathbb{R}^n$.

1.2 Integration by parts

Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary, we recall the Gauss-Green Theorem.

Theorem 1.2.1 (Gauss-Green Theorem)**.** *Suppose $f \in C^1(\overline{\Omega})$. Then*

$$\int_\Omega f_{x_i} dx = \int_{\partial\Omega} f \nu^i dS \qquad i = 1, \cdots, n$$

with $\nu = (\nu^1, \nu^2, \cdots, \nu^n)$ outward pointing unit normal vector field.

Theorem 1.2.2 (Integration by parts formula). *If $f, g \in C^1(\overline{\Omega})$,then*

$$\int_\Omega f g_{x_i} dx = -\int_\Omega g f_{x_i} dx + \int_{\partial\Omega} f g \nu^i dS \qquad \text{for} \quad i = 1, \cdots, n$$

In fact from Gauss-Green theorem:

$$\begin{aligned} \int_\Omega f g_{x_i}\, dx + \int_\Omega g f_{x_i}\, dx &= \int_\Omega (gf)_{x_i}\, dx \\ &= \int_{\partial\Omega} f g \nu^i dS \end{aligned}$$

1.3 Sobolev Spaces

1.3.1 *Weak Derivative*

Motivation: Let Ω open subset of $\mathbb{R}^n$. Given a function $f \in C^1(\Omega)$, then for all $\varphi \in C_c^1(\Omega)$,

$$\int_\Omega f \varphi_{x_i} dx = -\int_\Omega f_{x_i} \varphi dx$$

The problem is when f is not in $C^1(\Omega)$, at least the right-hand side integral will have no meaning hence we will define the weak derivative of f as follows

Definition 1.3.1. *Given a real valued function $f \in L^1_{loc}(\Omega)$ we say that $g \in L^1_{loc}(\Omega)$ is the weak i^{th}-derivative of f if*

$$\int_\Omega f \varphi_{x_i} dx = -\int_\Omega g \varphi dx$$

for all $\varphi \in C_c^1(\Omega)$.

Uniqueness. The weak i^{th}-derivative of f, if it exists, is uniquely defined.

Proof. Assume g and $\tilde{g}$ are weak i^{th}-derivatives of f satisfying

$$\int_\Omega f \varphi_{x_i} dx = -\int_\Omega g \varphi dx = -\int_\Omega \tilde{g} \varphi dx$$

for all $\varphi \in C_c^1(\Omega)$, then

$$\int_\Omega (g - \tilde{g}) \varphi dx = 0$$

for all $\varphi \in C_c^1(\Omega)$.Therefore $g - \tilde{g} = 0$ a.e. □

Definition 1.3.2 (Higher order weak derivatives). *Given a real valued function $f \in L^1_{loc}(\Omega)$, and a multi-index α, we say that $g \in L^1_{loc}(\Omega)$ is the α^{th} weak derivative of f if*

$$\int_\Omega f D^\alpha \varphi dx = (-1)^\alpha \int_\Omega g \varphi dx \quad \text{for all} \quad \varphi \in C_c^\infty(\Omega)$$

Write $g = D^\alpha f$.

Notice that the weak derivative coincide with the classical one if $f \in C^1(\Omega)$, however, a.e. existence of derivative does not imply existence of weak derivatives.

Example 1.3.1. *Let* $\Omega = (0,1) \subset \mathbb{R}$ *and*

$$f(x) = \begin{cases} 0 & \text{if } x \in (0, \frac{1}{2}] \\ 1 & \text{if } x \in (\frac{1}{2}, 1) \end{cases}$$

Assume f *has a weak derivative* g*, then*

$$\int_0^1 f\varphi' dx = -\int_0^1 g\varphi dx$$

for all $\varphi \in C_c^1(\Omega)$.
Case 1: φ *is compactly supported in* $(0, \frac{1}{2}]$ *we get*

$$0 = -\int_0^{\frac{1}{2}} g\varphi dx$$

Hence $g = 0$ *a.e on* $(0, \frac{1}{2}]$.
Case 2: φ *is compactly supported in* $(\frac{1}{2}, 1)$ *we get*

$$\varphi(1^-) - \varphi(\frac{1}{2}) = \int_{\frac{1}{2}}^1 \varphi' dx = -\int_{\frac{1}{2}}^1 g\varphi dx$$

$$0 = -\int_{\frac{1}{2}}^1 g\varphi dx$$

Hence $g = 0$ *a.e on* $(\frac{1}{2}, 1)$.
Therefore $g = 0$ *a.e on* $(0,1)$ *then* $\int_0^1 f\varphi' = 0$ *for all* $\varphi \in C_c^1(\Omega)$ *thus* $f = 0$ *a.e, contradiction.Thus* f *has no weak derivative.*

Example 1.3.2. *Let* $\Omega = B(0,1) \subset \mathbb{R}^n$ *and* $f(x) = \frac{1}{|x|^\alpha}$ *with* $f_{x_i} = -\alpha \frac{x_i}{|x|^{\alpha+2}}$. f *and* f_{x_i} *are in* $L^1_{loc}(\Omega)$ *when* $\alpha < n-1$ *(in* $L^1(\Omega)$*).In fact,*

$$\begin{aligned} \int_{B(0,1)} |f| dx &= \int_0^1 \int_{\partial B(0,r)} |f| dS dr \\ &= \int_0^1 \int_{\partial B(0,r)} \frac{1}{r^\alpha} dS dr \\ &= \int_0^1 \frac{1}{r^\alpha} c_n r^{n-1} dr \\ &= c_n \int_0^1 \frac{1}{r^{\alpha+1-n}} dr < \infty \end{aligned}$$

and

$$
\begin{aligned}
\int_{B(0,1)} |f_{x_i}(x)|dx &= \int_0^1 \int_{\partial B(0,r)} |f_{x_i}|dSdr \\
&\leq \alpha \int_0^1 \int_{\partial B(0,r)} \frac{1}{r^{\alpha+1}} dSdr \\
&\leq \alpha \int_0^1 \frac{1}{r^{\alpha+1}} c_n r^{n-1} dr \\
&= C_n \int_0^1 \frac{1}{r^{\alpha+2-n}} dr < \infty
\end{aligned}
$$

Let $\varphi \in C_c^1(\Omega)$ *and fix* $\epsilon > 0$*, using integration by parts*

$$
\int_{\Omega \setminus B(0,\epsilon)} f\varphi_{x_i} dx = -\int_{\Omega \setminus B(0,\epsilon)} f_{x_i}\varphi dx + \int_{\partial B(0,\epsilon)} f\varphi\nu^i dS
$$

But

$$
\begin{aligned}
|\int_{\partial B(0,\epsilon)} f\varphi\nu^i dS| &\leq \|\varphi\|_\infty \int_{\partial B(0,\epsilon)} |f|\,|\nu^i| dS \\
&\leq \|\varphi\|_\infty \int_{\partial B(0,\epsilon)} \epsilon^{-\alpha} dS \\
&\leq \|\varphi\|_\infty \epsilon^{-\alpha} \int_{\partial B(0,\epsilon)} dS \\
&= c_n \epsilon^{-\alpha+n-1} \to 0 \ as\ \epsilon \to 0
\end{aligned}
$$

Therefore $\int_{\Omega \setminus B(0,\epsilon)} f\varphi_{x_i} dx = -\int_{\Omega \setminus B(0\epsilon)} f_{x_i}\varphi dx$

$$
\begin{aligned}
\int_\Omega f\varphi_{x_i} dx &= \int_{\Omega \setminus B(0,\epsilon)} f\varphi_{x_i} dx + \int_{B(0,\epsilon)} f\varphi_{x_i} dx \\
&= -\int_{\Omega \setminus B(0,\epsilon)} f_{x_i}\varphi dx + \int_{B(0,\epsilon)} f\varphi_{x_i} dx
\end{aligned}
$$

But around 0,

$$
\begin{aligned}
|\int_{B(0,\epsilon)} f\varphi_{x_i} dx| &\leq \int_{B(0,\epsilon)} |f|\,|\varphi_{x_i}|\, dx \\
&\leq \|\varphi_{x_i}\|_\infty \int_{B(0,\epsilon)} |f|\, dx \\
&= \|\varphi_{x_i}\|_\infty \int_0^\epsilon \int_{\partial B(0,r)} \frac{1}{r^\alpha} dSdr \\
&\leq M_i c_n \epsilon^{-\alpha+n} \to 0 \ as\ \epsilon \to 0
\end{aligned}
$$

So $\int_\Omega f\varphi_{x_i} dx = -\int_{\Omega \setminus B(0,\epsilon)} f_{x_i}\varphi dx$*. Now, let*

$$
g_i(x) = \begin{cases} f_{x_i} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}
$$

Hence $\int_\Omega f\varphi_{x_i} dx = -\int_\Omega g_i\varphi dx$*.Therefore* f *has weak derivative* g*.*

1.3.2 *Weak Divergence and Weak Curl*

Definition 1.3.3. *Given $f : \Omega \mapsto \mathbb{R}^n$ in $L^1_{loc}(\Omega, \mathbb{R}^n)$, we say that $g : \Omega \mapsto \mathbb{R} \in L^1_{loc}(\Omega)$ is the weak divergence of f if*

$$\int_\Omega f \cdot \nabla\varphi dx = -\int_\Omega g\varphi dx$$

for all $\varphi \in C^1_c(\Omega)$.Denote $g = div f$

Definition 1.3.4. *Given $f : \Omega \mapsto \mathbb{R}^n$ in $L^1_{loc}(\Omega, \mathbb{R}^n)$,we say that $g : \Omega \mapsto \mathbb{R}^n$ in $L^1_{loc}(\Omega, \mathbb{R}^n)$ is the weak curl of f if*

$$\int_\Omega f \cdot Curl\varphi dx = -\int_\Omega g\varphi dx \qquad \text{for all } \varphi \in C^1_c(\Omega).$$

Denote $g = curl f$

Notice that if $f \in C^1(\Omega)$ above equalities hold for the standard curl and divergence by the Stokes' theorem.

1.3.3 *Sobolev Spaces*

Definition 1.3.5. *For $1 \leq p \leq \infty$, we define $W^{1,p}(\Omega)$ as the set of functions $f \in L^p(\Omega)$ that has weak ith derivative in $L^p(\Omega)$ for every $i = 1, \cdots, n$.*

Notation. *For $f \in W^{1,p}(\Omega)$,let $Df = (D_1 f, D_2 f, \cdots, D_n f)$ with $D_i f$ the weak $i-th$ derivative of f.*

We then define the $W^{1,p}(\Omega)$ norm:

$$\|f\|_{W^{1,p}(\Omega)} = \left(\|f\|_p{}^p + \sum_{i=1}^n \|D_i f\|_p{}^p\right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty$$

$$\|f\|_{W^{1,\infty}(\Omega)} = \|f\|_\infty + \sum_{i=1}^n \|D_i f\|_\infty \text{ for } p = \infty$$

Proposition 1.3.1. *[4]*

1. *($W^{1,p}(\Omega), \|\cdot\|_{W^{1,p}}$) is a Banach space $\forall 1 \leq p \leq \infty$.*
2. *$W^{1,p}(\Omega)$ is separable $\forall 1 \leq p < \infty$.*
3. *$C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for any $1 \leq p < \infty$, with respect to the norm of $W^{1,p}(\Omega)$.*

Example 1.3.3. *Let $\Omega = B(0, 1)$.We verify that if $n > 1$, the unbounded function $f = \log\log(1 + \frac{1}{|x|})$ belongs to $W^{1,n}(\Omega)$. Prove*

(i) $f \in L^n(\Omega)$

(ii) Df *exists*

(iii) $Df \in L^n(\Omega)$

For (i)

$$\begin{aligned}
\int_{B(0,1)} |f|^n dx &\leq \int_{B(0,1)} \log\left(1+\frac{1}{|x|}\right)^n dx \\
&\leq \int_0^1 \int_{\partial B(0,1)} \log\left(\frac{2}{r}\right)^n dS dr \\
&= \int_0^1 \log\left(\frac{2}{r}\right)^n c_n r^{n-1} dr \\
&= c_n \int_0^1 (\log(2)-\log(r))^n r^{n-1} dr \\
&= c_n \left(\frac{1}{n}\log(2)^n - \int_0^1 (\log(2)-\log(r))^{n-1} r^{n-1} dr\right) \quad \textit{(using integration by parts)} \\
&= c_n \left(\frac{1}{n}\log(2) - \frac{\log(2)^{n-1}}{n} - (\frac{n-1}{n}) \int_0^1 (\log(2)-\log(r))^{n-2} r^{n-1} dr\right)
\end{aligned}$$

Proceeding in this fashion and after sufficiently many integration by parts we get that this integral is equal to a constant + $\int_0^1 r^{n-1} dr$ *that is finite since* $n > 1$.

(ii)

$$f_{x_i} = \frac{-x_i}{(|x|^3+|x|^2)log(1+\frac{1}{|x|})} \quad (x \neq 0)$$

Fix $\epsilon > 0$,

$$\int_\Omega f\varphi_{x_i} dx = \int_{\Omega\setminus B(0,\epsilon)} f\varphi_{x_i} dx + \int_{B(0,\epsilon)} f\varphi_{x_i} dx \qquad \forall \varphi \in C_c^1(\Omega) \qquad (1)$$

By integration by parts we have ,

$$\int_{\Omega\setminus B(0,\epsilon)} f\varphi_{x_i} dx = -\int_{\Omega\setminus B(0,\epsilon)} f_{x_i}\varphi dx + \int_{\partial B(0,\epsilon)} f\varphi\nu^i dS \qquad (2)$$

But

$$\begin{aligned}
\int_{\partial B(0,\epsilon)} |f\varphi\nu^i| dS &\leq \|\varphi\|_\infty \int_{\partial B(0,\epsilon)} \left|\log\left(\log\left(1+\frac{1}{|x|}\right)\right)\right| .|\nu^i| dS \\
&\leq \|\varphi\|_\infty \int_{\partial B(0,\epsilon)} \log\left(\log\left(\frac{2}{\epsilon}\right)\right) dS \\
&\leq \|\varphi\|_\infty log(\frac{2}{\epsilon}) c_n \epsilon^{n-1} \to 0 \quad \textit{as } \epsilon \to 0
\end{aligned}$$

Now,

$$\left|\int_{B(0,\epsilon)} f\varphi_{x_i} dx\right| \leq \|\varphi_{x_i}\|_\infty \int_{B(0,\epsilon)} |f| dx \to 0 \quad \textit{as } \epsilon \to 0$$

Letting

$$V_i(x) := \begin{cases} f_{x_i} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

we get $\int_\Omega f\varphi_{x_i}dx = -\int_\Omega V_i\varphi dx$. *Hence* Df *exists and equal to* V.

(iii) We have

$$Df = \frac{1}{|x|(1+|x|)log(1+\frac{1}{|x|})}$$

with $x \neq 0$ *and since* $x \in B(0,1)$ *we get*

$$|Df| \leq \frac{1}{|x|log(1+\frac{1}{|x|})}$$

and notice that

$$\frac{1}{\log(1+\frac{1}{|x|})} \leq \frac{1}{log(\frac{1}{|x|})} \quad \text{when } x \in B(0,1)$$

Now as (i)

$$\int_{B(0,1)} |Df|^n dx \leq \int_{B(0,1)} \frac{1}{(|x|log(\frac{1}{|x|}))^n} \leq \int_0^1 \int_{\partial B(0,1)} \frac{1}{(r\log(\frac{1}{r}))^n} dSdr \leq c_n \int_0^1 \frac{1}{r(log(\frac{1}{r}))^n} dr$$

that is finite since $n > 1$ *by a simple change of variable.*

1.4 Approximation by smooth functions, Mollification

Definition 1.4.1 (Convolution). *The convolution of two functions* f, g *defined in* $\mathbb{R}^n$ *is given by the expression:*

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy$$

whenever this makes sense.It is a commutative and associative operation.

Definition 1.4.2. *Let* μ *be an* $\mathbb{R}^n$*-valued Radon measure in an open set* $\Omega \subset \mathbb{R}^n$, *if* f *is a continuous function, we call the function*

$$\mu * f(x) := \int_\Omega f(x-y)d\mu(y)$$

the convolution between f *and* μ *whenever this makes sense.*

We will introduce functions that will build smooth approximations to given functions in $W^{1,p}(\Omega)$.

Notation. *Let* $\Omega \subset \mathbb{R}^n$ *open subset,* $\epsilon > 0$. *Write* $\Omega_\epsilon = \{x \in \Omega \mid dist(x, \partial\Omega) > \epsilon\}$.

Definition 1.4.3. *(i) Define $\eta \in C^\infty(\mathbb{R}^n)$ by*

$$\eta(x) := \begin{cases} c\ \exp(\frac{1}{|x|^2-1}) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

with $c > 0$ such that $\int_{\mathbb{R}^n} \eta dx = 1$. η is called the standard mollifier.

(ii) For each $\epsilon > 0$, set

$$\eta_\epsilon(x) := \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon}).$$

The functions η_ϵ are C^∞ and satisfy

$$\int_{\mathbb{R}^n} \eta_\epsilon dx = 1, Supp(\eta_\epsilon) \subset B(0,\epsilon)$$

Definition 1.4.4. *If $f : \Omega \mapsto \mathbb{R}$ in $L^1_{loc}(\Omega)$, Define its mollification*

$$f^\epsilon(x) = \eta_\epsilon * f(x) = \int_\Omega \eta_\epsilon(x-y)f(y)dy = \int_{B(0,\epsilon)} \eta_\epsilon(y)f(x-y)dy \qquad \text{for } x \in \Omega_\epsilon$$

Proposition 1.4.1 (Properties of mollifiers)**.** *[4]*

(i) $f^\epsilon \in C^\infty(\Omega_\epsilon)$ for each $\epsilon > 0$.

(ii) $f^\epsilon \to f$ a.e as $\epsilon \to 0$.

(iii) If $f \in C(\Omega)$, then $f^\epsilon \to f$ uniformly on compact subsets of Ω.

(iv) If $1 \leq p < \infty$, and $f \in L^p_{loc}(\Omega)$, then $f^\epsilon \to f$ in $L^p_{loc}(\Omega)$.

Proof. (i) Fix $x \in \Omega_\epsilon, i = 1, \cdots, n$ and h so small that $x + he_i \in \Omega_\epsilon$.Then

$$\begin{aligned}\frac{f^\epsilon(x+he_i) - f^\epsilon(x)}{h} &= \frac{1}{\epsilon^n}\int_\Omega \frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right) f(y)\,dy \\ &= \frac{1}{\epsilon^n}\int_V \frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right) f(y)\,dy\end{aligned}$$

for some open set $V \subset\subset \Omega$. As

$$\frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right) \to \frac{1}{\epsilon}\frac{\partial\eta}{\partial x_i}(\frac{x-y}{\epsilon})$$

uniformly on V, $\frac{\partial f^\epsilon}{\partial x_i}(x)$ exists and equals

$$\int_\Omega \frac{\partial\eta}{\partial x_i}(\frac{x-y}{\epsilon})f(y)dy.$$

Hence $Df^\epsilon(x)$ exists and $Df^\epsilon = D\eta_\epsilon * f$.

(ii) By the Lebesgue differentiation theorem we have

$$\lim_{r\to 0}\int_{B(x,r)} |f(y)-f(x)|dy = 0$$

for a.e $x \in \Omega$. Fix such x, then

$$\begin{aligned}|f^\epsilon(x)-f(x)| &= \left|\int_{B(x,\epsilon)} \eta^\epsilon(x-y)[f(y)-f(x)]dy\right| \\ &\leq \frac{1}{\epsilon^n}\int_{B(x,\epsilon)} \eta\left(\frac{x-y}{\epsilon}\right)|f(y)-f(x)|dy \\ &\leq c\int_{B(x,\epsilon)} |f(y)-f(x)|dy \to 0 \quad \text{as} \quad \epsilon \to 0\end{aligned}$$

(iii) Assume now $f \in C(\Omega)$,and let $V \subset\subset \Omega$, we choose $V \subset\subset W \subset\subset \Omega$, note that f is uniformly continuous on W.Therefore as in the proof of (ii) with uniform convergence we get (iii).

(iv) Let $1 \leq p < \infty$ and $f \in L^p_{loc}(\Omega)$. Choose an open set $V \subset\subset \Omega$ and as above, an open set W so that $V \subset\subset W \subset\subset \Omega$.

Claim: For sufficiently small $\epsilon > 0$, $\|f^\epsilon\|_{L^p(V)} \leq \|f\|_{L^p(W)}$.

Proof of the claim: Let $x \in V$,

$$\begin{aligned}|f^\epsilon(x)| &= \left|\int_{B(x,\epsilon)} \eta_\epsilon(x-y)f(y)dy\right| \\ &\leq \int_{B(x,\epsilon)} \eta_\epsilon^{1-\frac{1}{p}}(x-y)\eta_\epsilon^{\frac{1}{p}}(x-y)|f(y)|dy \\ &\leq \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)dy\right)^{1-\frac{1}{p}}\left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|f(y)|^p dy\right)^{\frac{1}{p}}\end{aligned}$$

Since $\int_{B(x,\epsilon)} \eta_\epsilon(x-y)dy = 1$, this inequality implies

$$\begin{aligned}\int_V |f^\epsilon(x)|^p dx &\leq \int_V \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|f(y)|^p dy\right) dx \\ &\leq \int_W |f(y)|^p \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)dx\right) dy \\ &= \int_W |f(y)|^p dy\end{aligned}$$

□

Now fix $\delta > 0$,and choose $g \in C(W)$ so that

$$\|f-g\|_{L^p(W)} < \delta$$

Then

$$\begin{aligned}\|f^\epsilon - f\|_{L^p(V)} &\le \|f^\epsilon - g^\epsilon\|_{L^p(V)} + \|g^\epsilon - g\|_{L^p(V)} + \|g - f\|_{L^p(V)} \\ &\le 2\|f - g\|_{L^p(W)} + \|g^\epsilon - g\|_{L^p(V)} \quad \text{(By the claim)} \\ &\le 2\delta + \|g^\epsilon - g\|_{L^p(V)}\end{aligned}$$

Now from (iii) $g^\epsilon \to g$ uniformly on V,and hence (iv) is proved.

□

Theorem 1.4.1 (Local approximation by smooth functions)**.** *Assume $f \in W^{1,p}(\Omega)$ for some 1$\le p < \infty$, and set $f^\epsilon = \eta_\epsilon * f$ inΩ_ϵ.Then*

(i) $f^\epsilon \in C^\infty(\Omega_\epsilon)$ *for each* $\epsilon > 0$.

(ii) $Df^\epsilon = \eta_\epsilon * Df = D\eta_\epsilon * f$ *in* Ω_ϵ.

(iii) $f^\epsilon \to f$ *in* $W^{1,p}_{Loc}(\Omega)$ *as* $\epsilon \to 0$.

Proof. As proved previously we have $Df^\epsilon = D\eta_\epsilon * f$. So,

$$Df^\epsilon(x) = D\int_\Omega \eta_\epsilon(x-y)f(y)\,dy = \int_\Omega D_x\eta_\epsilon(x-y)f(y)\,dy = -\int_\Omega D_y\eta_\epsilon(x-y)f(y)\,dy$$

For fixed $x \in \Omega_\epsilon$,the function $\eta(x-\cdot) : y \mapsto \eta_\epsilon(x-y)$ belongs to $C_c^\infty(\Omega)$.Consequently by the definition of the weak derivative

$$\int_\Omega D_y\eta_\epsilon(x-y)f(y)dy = -\int_\Omega \eta_\epsilon(x-y)Df(y)dy$$

Thus $Df^\epsilon(x) = \eta_\epsilon * Df(x)$. □

Proposition 1.4.2. *[4] Suppose Ω is connected and $f \in W^{1,p}(\Omega)$ satisfies $Df = 0$ a.e in Ω. Prove f is constant a.e in Ω.*

Proof. We know that this result is true for C^∞ functions. Let $\epsilon > 0$, Ω_ϵ as defined above, and $f^\epsilon = \eta_\epsilon * f \in C^\infty(\Omega_\epsilon)$. Since $Df = 0$ a.e then $Df^\epsilon = \eta_\epsilon * Df = 0$ a.e, hence $f^\epsilon = a$ a.e on each connected component of Ω_ϵ .Now let $x, y \in \Omega$, since Ω open connected in $\mathbb{R}^n$ there exist $\gamma : [0,1] \mapsto \Omega$ a path connecting x to y. For $\delta = \min_{z\in\gamma} dist(z, \partial\Omega)$ and $\epsilon < \delta$ the whole path lies in Ω_ϵ, hence x and y are in the same connected component of Ω_ϵ, so $f^\epsilon(x) = f^\epsilon(y)$. Since $f^\epsilon \to f$ in $W^{1,p}_{Loc}(\Omega)$, then f is constant a.e in Ω. □

Theorem 1.4.2 (Global Approximation by smooth functions)**.** *[5] Assume Ω is bounded, $\partial\Omega$ is lipschitz.Then if $f \in W^{1,p}(\Omega)$ for some $1 \le p < \infty$, there exists a sequence $(f_k)_{k\ge1} \subset W^{1,p}(\Omega) \cap C^\infty(\overline{\Omega})$ such that $f_k \to f$ in $W^{1,p}(\Omega)$.*

Theorem 1.4.3. *Assume Ω is bounded, $\partial\Omega$ is lipschitz, $1 \le p < n$. Suppose $(f_k)_{k\ge1}$ sequence in $W^{1,p}(\Omega)$ such that*

$$\sup_k \|f_k\|_{W^{1,p}(\Omega)} < \infty$$

Then there exist a subsequence $(f_{k_j})_j$ and $f \in W^{1,p}(\Omega)$ such that $f_{k_j} \to f$ in $L^q(\Omega)$ for each $1 \le q < p^* = \frac{np}{n-p}$.

Proof. Let's try proving it only for $p = q = 1$,since we will later use the compactness in $W^{1,1}(\Omega)$ to prove compactness in BV space.

1. Fix a bounded open set V such that $\Omega \subset\subset V$ and extend each f_k to $g_k \in W^{1,1}(\mathbb{R}^n)$, such that

$$(*)\begin{cases} Suppg_k \subset V \\ \sup_k \|g_k\|_{W^{1,1}(\mathbb{R}^n)} \leq c \sup_k \|f_k\|_{W^{1,1}(\Omega)} < \infty. \end{cases}$$

2. Let $g_k^\epsilon = \eta_\epsilon * g_k$ (the usual mollification ,WLOG supp$(g_k^\epsilon) \subset V$)
3. Claim 1: $\|g_k^\epsilon - g_k\|_{L^1(\mathbb{R}^n)} \leq C_\epsilon$ uniformly in k.(C_ϵ independent of k)

   Proof of Claim 1. Assume g_k is smooth,

$$\begin{aligned} |g_k^\epsilon(x) - g_k(x)| &= \left| \int_\Omega \eta_\epsilon(x-y) g_k(y) dy - g_k(x) \right| \\ &= \left| \int_{B(0,\epsilon)} \frac{1}{\epsilon^n} \eta(\frac{y}{\epsilon}) g_k(x-y) dy - g_k(x) \right| \\ &= \left| \int_{B(0,1)} \eta(y) g_k(x - \epsilon y) dy - \int_{B(0,1)} \eta(y) g_k(x) dy \right| \\ &\leq \int_{B(0,1)} \eta(y) \left| g_k(x-\epsilon y) - g_k(x) \right| dy \\ &= \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \left| Dg_k(x - \epsilon t y) \right| dt dy \end{aligned}$$

   Thus,

$$\begin{aligned} \|g_k^\epsilon - g_k\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |g_k^\epsilon - g_k| dx \\ &\leq \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \left(\int_{\mathbb{R}^n} |Dg_k(x - \epsilon t y)| dx \right) dt dy \\ &\leq \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \|Dg_k\|_{W^{1,1}(\mathbb{R}^n)} dt dy \\ &\leq \epsilon \|Dg_k\|_{W^{1,1}(\mathbb{R}^n)} \\ &\leq C\epsilon \quad \text{by (*)} \end{aligned}$$

   The general case follows by approximation. □

4. Claim 2: For each $\epsilon > 0$, the sequence $(g_k^\epsilon)_{k \geq 1}$ is bounded and equicontinuous on $\mathbb{R}^n$.

   Proof of Claim 2.

$$|g_k^\epsilon(x)| \leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y) |g_k(y)| dy \leq \|\eta_\epsilon\|_\infty \int_{B(x,\epsilon)} |g_k(y)| dy \leq \frac{c_1}{\epsilon^n} \left(\|g_k\|_{L^1(\mathbb{R}^n)} \right) \leq \frac{c}{\epsilon^n},$$

and

$$|Dg_k^\epsilon(x)| \leq \int_{B(x,\epsilon)} |D\eta_\epsilon(x-y)||g_k(y)|dy \leq \frac{c}{\epsilon^{n+1}}$$

□

5. Claim 3: For each $\delta > 0$ there exist a subsequence $(f_{k_j})_{j\geq 1} \subset (f_k)_k$ such that

$$\limsup_{i,j\to\infty} \|f_{k_i} - f_{k_j}\|_{L^1(\Omega)} \leq \delta.$$

Proof of claim 3. Recalling claim 1, we choose $\epsilon > 0$, so small that

$$\sup_k \|g_k^\epsilon - g_k\|_{L^1(\mathbb{R}^n)} \leq \frac{\delta}{3}$$

then by claim 2 and Arzela-Ascoli theorem on $\overline{V}$, we find a subsequence $(g_{k_j}^\epsilon)_{j\geq 1}$ which converges uniformly on $\mathbb{R}^n$. Then

$$\begin{aligned} \|f_{k_i} - f_{k_j}\|_{L^1(\Omega)} &\leq \|g_{k_i} - g_{k_j}\|_{L^1(\mathbb{R}^n)} \\ &\leq \|g_{k_j} - g_{k_j}^\epsilon\|_{L^1(\mathbb{R}^n)} + \|g_{k_j}^\epsilon - g_{k_i}^\epsilon\|_{L^1(\mathbb{R}^n)} + \|g_{k_i}^\epsilon - g_{k_i}\|_{L^1(\mathbb{R}^n)} \\ &\leq \frac{2\delta}{3} + \|g_{k_j}^\epsilon - g_{k_i}^\epsilon\|_{L^1(\mathbb{R}^n)} \\ &\leq \delta \end{aligned}$$

for i, j large enough. □

6. We conclude that there exist a Cauchy subsequence in $L^1(\Omega)$ and hence a convergence subsequence in $L^1(\Omega)$.

□

1.5 Functions of bounded variation

Definition 1.5.1. *Let $f \in L^1(\Omega)$, we say f is a function of bounded variation if the distributional derivative of f is representable by a finite Radon measure in Ω,i.e if there exists a Radon measure,whose total variation is finite on Ω, denoted by $Df = (D_1 f, D_2 f, \cdots, D_n f)$ such that*

$$\int_\Omega f\varphi_{x_i} dx = -\int_\Omega \varphi dD_i f \qquad \forall \varphi \in C_c^1(\Omega) \qquad i = 1, \cdots, n \tag{1.1}$$

The vector space of all functions of bounded variations is denoted by $BV(\Omega)$. We also denote $BV_{loc}(\Omega)$ the space of function $f \in L^1_{loc}(\Omega)$ that has locally bounded variation.

Note that we can also write

$$\int_\Omega f div\varphi \, dx = -\sum_{i=1}^n \int_\Omega \varphi^i dD_i f \qquad \forall \varphi \in C_c^1(\Omega, \mathbb{R}^n). \tag{1.2}$$

The Sobolev space $W^{1,1}(\Omega)$ is contained in $BV(\Omega)$.Indeed for any $f \in W^{1,1}(\Omega)$ the distributional derivative is given by $Df dx$, where Df is the weak derivative of f.But the inclusion is strict, that is, there exist functions f in $BV(\Omega)$ such that their weak derivative does not exist. For instance, the Heaviside function $\mathbb{1}_{[0,\infty)}$ whose distributional derivative is the Dirac measure δ_0 has no weak derivative. In fact we have

$$\int_\mathbb{R} \mathbb{1}_{[0,\infty)} \varphi'(x) dx = \int_0^\infty \varphi'(x) dx = -\varphi(0)$$

and $\int_\mathbb{R} \varphi d\delta_0 = \varphi(0)$ for all $\varphi \in C_c^1(\mathbb{R})$. Therefore δ_0 is the distributional derivative of $\mathbb{1}_{[0,\infty)}$.

Now assume $\mathbb{1}_{[0,\infty)}$ has a weak derivative $g \in L^1(\mathbb{R})$.Let $\varphi \in C_c^1(\mathbb{R})$ such that $\|\varphi\|_\infty \leq 1$, supported in $(-1,1)$ and $\varphi(0) = 1$.

For $n \in \mathbb{N}*$ and $x \in \mathbb{R}$, we set $\varphi_n(x) = \varphi(nx)$ then $\varphi_n \in C_c(\mathbb{R})$ for all n.On the other hand

$$\begin{aligned} |\int_\mathbb{R} g(x)\varphi_n(x) dx| &= |\int_{-\frac{1}{n}}^{\frac{1}{n}} g(x)\varphi_n(x) dx| \\ &\leq \int_{-\frac{1}{n}}^{\frac{1}{n}} |g(x)||\varphi_n(x)| dx \\ &\leq \int_{-\frac{1}{n}}^{\frac{1}{n}} |g(x)| dx \end{aligned}$$

But $|g\mathbb{1}_{(-\frac{1}{n},\frac{1}{n})}| \leq |g|$ and $g\mathbb{1}_{(-\frac{1}{n},\frac{1}{n})}$ converges pointwise to 0. Therefore by dominated convergence theorem we get a contradiction.

Definition 1.5.2. *Let $f \in L^1_{Loc}(\Omega)$. Define $V(f,\Omega)$, the variation of f in Ω, by*

$$V(f,\Omega) = \sup \left\{ \int_\Omega f \operatorname{div} \varphi \, dx \,\middle|\, \varphi \in C_c^1(\Omega, \mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\}.$$

Proposition 1.5.1. *Let $f \in L^1(\Omega)$, then $f \in BV(\Omega)$ if and only if $V(f,\Omega) < \infty$. In addition, $V(f,\Omega)$ coincides with $|Df|(\Omega)$ for any $f \in BV(\Omega)$.*

Proof. First assume $f \in BV(\Omega)$ hence there exists a finite radon measure $Df = (D_1 f, \cdots, D_n f)$ such that

$$\int_\Omega f div(\varphi) dx = -\sum_{i=1}^n \int_\Omega \varphi^i dD_i f \quad \forall \, \varphi \in C_c(\Omega, \mathbb{R}^n).$$

By proposition, we have 1.1.1

$$|Df|(\Omega) = \sup \left\{ \sum_{i=1}^n \int_X \varphi^i dD_i f : \varphi \in C_c(\Omega, \mathbb{R}^n), \|\varphi\|_\infty \leq 1 \right\}$$

Therefore $V(f,\Omega) \leq |Df|(\Omega) < \infty$.Conversely assume $V(f,\Omega) < \infty$.Define $L : C_c^1(\Omega,\mathbb{R}^n) \to \mathbb{R}$ by

$$L(\varphi) = \int_\Omega f div(\varphi) dx$$

a continuous linear functional on $C_c^1(\Omega,\mathbb{R}^n)$.We have

$$V(f,\Omega) = \|L\| = \sup\left\{\frac{|L(\varphi)|}{\|\varphi\|_\infty}, \|\varphi\|_\infty \neq 0\right\} \geq \frac{|L(\varphi)|}{\|\varphi\|_\infty}$$

therefore

$$|L(\varphi)| \leq V(f,\Omega)\|\varphi\|_\infty < \infty \quad (*).$$

For each $\varphi \in C_c(\Omega)$ choose $\varphi_k \in C_c^1(\Omega,\mathbb{R}^n)$ such that $\varphi_k \to \varphi$ uniformly on compact subsets of Ω. Define $\widetilde{L} : C_c(\Omega,\mathbb{R}^n) \to \mathbb{R}$ such that

$$\widetilde{L}(\varphi) = \lim_{k\to\infty} L(\varphi_k)$$

According to $(*)$ $\widetilde{L}(\varphi) < \infty$ (the limit exist) and it is independent of the choice of the sequence $(\varphi_k)_k$. Thus L uniquely extends to a linear functional $\widetilde{L}$ that coincides with L on $C_c^1(\Omega,\mathbb{R}^n)$ (This could be replaced by applying Hahn Banach theorem).Now applying Riesz theorem 1.1.3 on $\widetilde{L}$ we can find an $\mathbb{R}^n$-valued finite radon measure μ on Ω such that

$$\int_\Omega f div(\varphi) dx = \widetilde{L}(\varphi) = \sum_{i=1}^n \int_\Omega \varphi^i d\mu_i \quad \text{for all}\, \varphi \in C_c(\Omega,\mathbb{R}^n).$$

Also $\|L\| = |\mu|(\Omega)$,hence $Df = -\mu$ and thus $f \in BV(\Omega)$. In addition,we've already proven that $V(f,\Omega) \leq |Df|(\Omega)$ and now we have $|Df|(\Omega) = |\mu|(\Omega) = \|L\| \leq V(f,\Omega)$. Therefore $|Df|(\Omega) = V(f,\Omega)$. □

Proposition 1.5.2. *Let* $f \in BV_{loc}(\Omega)$

1. *If* η *is the standard mollifier and* $\Omega_\epsilon = \{x \in \Omega, dist(x,\partial\Omega) > \epsilon\}$, *then* $D(f * \eta_\epsilon) = Df * \eta_\epsilon$ *on* Ω_ϵ.
2. *If* $Df = 0$ *then* f *is constant in any connected component of* Ω.

Proof. 1. I need to prove that

$$\int_\Omega (f * \eta_\epsilon)\varphi_{x_i} dx = -\int_\Omega \varphi(D_i f * \eta_\epsilon) dx \qquad \forall \varphi \in C_c^1(\Omega_\epsilon) \qquad \forall i = 1,\cdots,n$$

In fact,

$$\begin{aligned}
\int_\Omega (f * \eta_\epsilon)\varphi_{x_i} dx &= \int_\Omega \left(\int_\Omega f(y)\eta_\epsilon(x-y)dy \right) \varphi_{x_i} dx \\
&= \int_\Omega \left(\int_\Omega \varphi_{x_i}(x)\eta_\epsilon(x-y)dx \right) f(y)dy \qquad \text{(By Fubini)} \\
&= \int_\Omega (\varphi_{x_i} * \eta_\epsilon)(y) f(y)dy \\
&= \int_\Omega (\varphi * \eta_\epsilon)_{x_i}(y) f(y)dy \\
&= -\int_\Omega (\varphi * \eta_\epsilon) dD_i f \\
&= -\int_\Omega (D_i f * \eta_\epsilon)\varphi dx
\end{aligned}$$

The last equality is true by Fubini theorem and by the symmetry of η_ϵ:

$$\begin{aligned}
\int_\Omega (D_i f * \eta_\epsilon)\varphi dx &= \int_\Omega \left(\int_\Omega \eta_\epsilon(x-y) dD_i f(y) \right) \varphi(x) dx \\
&= \int_\Omega \left(\int_\Omega \eta_\epsilon(x-y)\varphi(x)dx \right) dD_i f(y) \\
&= \int_\Omega \varphi * \eta_\epsilon dD_i f.
\end{aligned}$$

2. For $\epsilon > 0$, $f^\epsilon = \eta_\epsilon * f \in C^\infty(\Omega_\epsilon) \cap L^1_{loc}(\Omega)$. By (1), $Df^\epsilon = Df * \eta_\epsilon$ hence $Df^\epsilon = 0$ on Ω_ϵ so by proposition 1.4.2 $f^\epsilon = c$ a.e on every connected component of Ω_ϵ. But $f^\epsilon \to f$ as $\epsilon \to 0$ a.e in Ω. Therefore $f = c$ in every connected component of Ω.

□

1.5.1 *Approximation by smooth functions*

Theorem 1.5.1 (Lower semi-continuity of variation measure). *Suppose $f_k \in BV(\Omega)$ $(k = 1, \cdots)$ and $f_k \to f$ in $L^1_{loc}(\Omega)$. Then*

$$|Df|(\Omega) \leq \liminf_{k\to\infty} |Df_k|(\Omega)$$

Proof. Since $f_k \in BV(\Omega)$ then $V(f_k, \Omega) = |Df_k|(\Omega)$. Let $\varphi \in C^1_c(\Omega, \mathbb{R}^n)$ such that $\|\varphi\|_\infty \leq 1$, then

$$\int_\Omega f div(\varphi)dx = \lim_{k\to\infty} \int_\Omega f_k div(\varphi)dx = \liminf_{k\to\infty} \int_\Omega f_k div(\varphi)dx \leq \liminf_{k\to\infty} V(f_k, \Omega) = \liminf_{k\to\infty} |Df_k|(\Omega)$$

Note that the first equality in this proof is true since

$$\|(f_k - f)div\varphi\|_1 \leq \|(f_k - f)\|_1 \|div\varphi\|_\infty \to 0 \text{ as } k \to \infty$$

□

Theorem 1.5.2 (Approximation by smooth functions). *Assume $f \in BV(\Omega)$, then there exist functions $(f_k)_{k\geq 1} \subset BV(\Omega) \cap C^\infty(\Omega)$ such that*

1. $\lim\limits_{k\to\infty} f_k = f$ *in* $L^1(\Omega)$
2. $\lim\limits_{k\to\infty} |Df_k|(\Omega) = |Df|(\Omega)$

The converse is also true, i.e if there exist functions $(f_k)_{k\geq 1} \subset C^\infty(\Omega)$ such that

1. $\lim\limits_{k\to\infty} f_k = f$ *in* $L^1(\Omega)$
2. $L = \lim\limits_{k\to\infty} \int_\Omega |Df_k| dx < \infty$

Then $f \in BV(\Omega)$.

Proof. Fix $\epsilon > 0$. Given $m \in \mathbb{N}$, define the open sets

$$\Omega_k = \left\{ x \in \Omega, dist(x, \partial\Omega) > \frac{1}{m+k} \right\} \cap B(0, k+m)$$

note that Ω_k is an increasing sequence.Choose m large enough that $|Df|(\Omega \setminus \Omega_1) < \epsilon\,(*)$ and set $\Omega_0 = \emptyset$. Now define $V_k = \Omega_{k+1} \setminus \overline{\Omega_{k-1}}$. Notice that V_k is an open cover of Ω, hence there exist $(\xi_k)_k$ a partition of unity such that

$$\begin{cases} 0 \leq \xi_k \leq 1 \\ \xi_k \in C_c^\infty(V_k) \\ \sum_{k\geq 1} \xi_k = 1 \qquad \text{on } \Omega. \end{cases}$$

Fix the mollifier η, then for each k, select $\epsilon_k > 0$ so small that

$$(**) \begin{cases} Supp(\eta_{\epsilon_k} * f\xi_k) \subset V_k \\ \int_\Omega |\eta_{\epsilon_k} * f\xi_k - f\xi_k| dx < \frac{\epsilon}{2^k} \\ \int_\Omega |\eta_{\epsilon_k} * fD\xi_k - fD\xi_k| dx < \frac{\epsilon}{2^k} \end{cases}$$

Now define $f_\epsilon = \sum_{k\geq 1} \eta_{\epsilon_k} * f\xi_k$. We have $f_\epsilon \in C^\infty(\Omega)$ because it is a locally finite sum, i.e in some neighborhood of each point $x \in \Omega$ there are only finitely non zero terms in the sum.

Also $\sum_{k\geq 1} \xi_k = 1$ so $f \sum_{k\geq 1} \xi_k = f$ and therefore $\sum_{k\geq 1} f\xi_k = f$.Hence $(**)$ implies:

$$\begin{aligned} \|f_\epsilon - f\|_1 &= \int_\Omega \left| \sum_{k\geq 1} (\eta_{\epsilon_k} * f\xi_k) - \sum_{i\geq 1} f\xi_k \right| dx \\ &\leq \int_\Omega \sum_{k\geq 1} |\eta_{\epsilon_k} * f\xi_k - f\xi_k| dx \\ &= \sum_{k\geq 1} \int_\Omega |(\eta_{\epsilon_k} * f\xi_k) - f\xi_k| dx \\ &\leq \epsilon \end{aligned}$$

Therefore $f_\epsilon \to f$ in $L^1(\Omega)$ as $\epsilon \to 0$.

It remains to show that $|Df_\epsilon|(\Omega) \to |Df|(\Omega)$ as $\epsilon \to 0$.From theorem 1.5.1 we have $|Df|(\Omega) \le \liminf_{\epsilon\to 0} |Df_\epsilon|(\Omega)$.For the reverse inequality , let $\varphi \in C_c^1(\Omega, \mathbb{R}^n)$, $\|\varphi\|_\infty \le 1$,then

$$\begin{aligned}
\int_\Omega f_\epsilon div\varphi\, dx &= \int_\Omega \sum_{k\ge1} (\eta_{\epsilon_k} * f\xi_k) div\varphi\, dx = \sum_{k\ge1}\int_\Omega \sum_{i=1}^n (\eta_{\epsilon_k} * f\xi_k)\varphi_{x_i} dx \\
&= \sum_{k\ge1}\int_\Omega \sum_{i=1}^n (\varphi * \eta_{\epsilon_k})_{x_i} f\xi_k dx = \sum_{k\ge1}\int_\Omega div(\varphi * \eta_{\epsilon_k}) f\xi_k dx \\
&= \sum_{k\ge1}\int_\Omega f div(\xi_k(\varphi * \eta_{\epsilon_k}))dx - \sum_{k\ge1}\int_\Omega fD\xi_k(\varphi * \eta_{\epsilon_k})dx \\
&= \sum_{k\ge1}\int_\Omega f div(\xi_k(\varphi * \eta_{\epsilon_k}))dx - \sum_{k\ge1}\int_\Omega \varphi((\eta_{\epsilon_k} * (fD\xi_k)) - fD\xi_k)dx \\
&:= I_1^\epsilon + I_2^\epsilon
\end{aligned}$$

Having that,by Fubini

$$\int_\Omega fD\xi_k(\eta_{\epsilon_k} * \varphi)dx = \int_\Omega (fD\xi_k * \eta_{\epsilon_k})\varphi dx$$

and $\sum_{k\ge1} D\xi_k = 0$ in Ω, so $\sum_{k\ge1}\int_\Omega fD\xi_k = 0$.

Now note that we have $|\xi_k| \le 1$ by definition, and $|\eta_{\epsilon_k} * \varphi| \le \|\eta_{\epsilon_k}\|_1 \|\varphi\|_\infty \le 1$ therefore $|\xi_k(\eta_{\epsilon_k} * \varphi)| \le 1$. By construction of the sets $V_k's$, each point of Ω belongs to at most three of the sets V_k. So

$$\begin{aligned}
|I_1^\epsilon| &= \left| \int_\Omega f div(\xi_1(\eta_{\epsilon_1} * \varphi))dx + \sum_{k\ge2}\int_\Omega f div(\xi_k(\eta_{\epsilon_k} * \varphi))dx \right| \\
&\le |Df|(V_k) + \sum_{k\ge2} |Df|(V_k) \\
&\le |Df|(\Omega) + \sum_{k\ge2} |Df|(V_k) \\
&\le |Df|(\Omega) + |Df|(V_{k_{\alpha_1}}) + |Df|(V_{k_{\alpha_2}}) + |Df|(V_{k_{\alpha_3}}) \\
&\le |Df|(\Omega) + 3|Df|(\Omega - \Omega_1) \\
&\le |Df|(\Omega) + 3\epsilon \text{ by } (*)
\end{aligned}$$

The above is true since each x belongs to at most 3 V_k's and

$$V_{k_{\alpha_i}} \subset \Omega - \Omega_1 \qquad i = 1, 2, 3.$$

On the other hand, $(**)$ implies that $|I_2^\epsilon| < \epsilon$ Therefore

$$\int_\Omega f_\epsilon div\varphi dx \le |Df|(\Omega) + 4\epsilon$$

Hence $|Df_\epsilon|(\Omega) \leq |Df|(\Omega) + 4\epsilon$. We then get $\lim_{\epsilon\to 0} |Df_\epsilon|(\Omega) = |Df|(\Omega)$.

For the converse and since $f_k \in C^\infty(\Omega)$,notice that by theorem 1.1.4 the finite radon measure $Df_k d\lambda$, where Df_k is the gradient of f_k, has a subsequence that converges weakly* to some $\mathbb{R}^n$-valued measure μ in Ω such that $|\mu|(\Omega) \leq L$, i.e

$$\lim_{k\to\infty} \int_\Omega g D_i f_k dx = \int_\Omega g d\mu \qquad \forall g \in C_0(\Omega)$$

Now by integration by parts for all $\varphi \in C_c^1(\Omega)$,

$$\int_\Omega f_k \varphi_{x_i} dx = -\int_\Omega D_i f_k \varphi dx$$

Let $k \to \infty$ we get

$$\int_\Omega f \varphi_{x_i} dx = -\int_\Omega \varphi d\mu$$

Therefore $f \in BV(\Omega)$ and $Df = \mu$. □

Theorem 1.5.3. *$BV(\Omega)$, endowed with the norm*

$$\|f\|_{BV(\Omega)} := \|f\|_{L^1(\Omega)} + |Df|(\Omega)$$

is a Banach space.

Proof. Clearly $\|-\|_{BV(\Omega)}$ is a norm.We only need to prove that this space is complete.Suppose $(f_j)_j \subset BV(\Omega)$ is a Cauchy sequence, then by definition of the BV norm,$(f_j)_j$ is Cauchy in $L^1(\Omega)$.By completeness of $L^1(\Omega)$,there exist $f \in L^1(\Omega)$ such that $f_j \to f$ in $L^1(\Omega)$. Since $(f_j)_j$ is Cauchy in $BV(\Omega)$, then $\|f_j\|_{BV}$ is bounded. Thus $|Df_j|(\Omega)$ is bounded as $j \to \infty$ and so by semi-continuity theorem 1.5.1

$$|Df|(\Omega) \leq \liminf_{j\to\infty} |Df_j|(\Omega) < \infty$$

hence $f \in BV(\Omega)$.It remains only to show that $f_j \to f$ in $BV(\Omega)$,or since we already have convergence in $L^1(\Omega)$, that $|D(f_j - f)|(\Omega) \to_{j\to\infty} 0$. Suppose $\epsilon > 0, \exists N > 0$, such that

$$\|f_j - f_k\|_{BV} < \epsilon \quad \forall j,k \geq N.$$

This implies that $|D(f_j - f_k)|(\Omega) < \epsilon$, $\forall j,k \geq N$.Now $f_k \to f$ in $L^1(\Omega)$ and so $f_j - f_k \to f_j - f$ in $L^1(\Omega)$. Thus again by semi-continuity

$$|D(f_j - f)|(\Omega) \leq \liminf_{k\to\infty} |D(f_j - f_k)|(\Omega) < \epsilon$$

for arbitrary $\epsilon > 0$, therefore $f_j \to f$ in $BV(\Omega)$. □

1.5.2 *Compactness*

Theorem 1.5.4. *Let $\Omega \subset \mathbb{R}^n$, be open bounded with $\partial\Omega$ lipschitz. Assume $(f_k)_k$ is a sequence in $BV(\Omega)$ satisfying*

$$\sup_k \|f_k\|_{BV(\Omega)} < \infty.$$

Then there exists a subsequence $(f_{k_j})_j$ and a function $f \in BV(\Omega)$ such that

$$f_{k_j} \to f \; in \, L^1(\Omega) \qquad as \; j \to \infty.$$

Proof. For $k = 1, 2, \cdots$ there exist by the approximation theorem $(g_l)_l \subset C^\infty(\Omega) \cap BV(\Omega)$ such that $g_l \to_{l\to\infty} f_k$ in $L^1(\Omega)$ and $\lim\limits_{l\to\infty} |Dg_l|(\Omega) = |Df_k|(\Omega)$

$$\text{i.e}\,(*) \begin{cases} \int_\Omega |g_k - f_k| dx < \frac{1}{k} \\ \sup_k \int_\Omega |Dg_k| dx < \infty \end{cases}.$$

Now since $(g_k) \subset W^{1,1}(\Omega)$ and using compactness of $W^{1,1}(\Omega)$ 1.4.3, there exist $f \in L^1(\Omega)$ and a subsequence $(g_{k_j})_{j\geq 1} \in W^{1,1}(\Omega)$ such that $g_{k_j} \to f$ in $L^1(\Omega)$. But then $(*)$ implies also that $f_{k_j} \to f$ in $L^1(\Omega)$ because $\int_\Omega |f_{k_j} - f| \leq \int_\Omega |f_{k_j} - g_{k_j}| + \int_\Omega |g_{k_j} - f| \to 0$ and $f \in BV(\Omega)$ using lower semicontinuity theorem 1.5.1. □

Definition 1.5.3 (Weak* convergence)**.** *Let $f, f_h \in BV(\Omega)$.we say that (f_h) weakly* converges in $BV(\Omega)$ to f, if (f_h) converges to f in $L^1(\Omega)$ and (Df_h) weakly* converges to Df in Ω, i.e*

$$\lim_{h\to\infty} \int_\Omega \varphi dDf_h = \int_\Omega \varphi dDf \qquad \forall \varphi \in C_0(\Omega).$$

Proposition 1.5.3. *Let $(f_h)_h \subset BV(\Omega)$. (f_h) weakly* converges to f in $BV(\Omega)$ if and only if (f_h) is bounded in $BV(\Omega)$ and converges to f in $L^1(\Omega)$*

Proof. Assume f_h converges weakly* to f then we have converges in $L^1(\Omega)$ and we only need to prove boundedness in $BV(\Omega)$ i.e $\sup_{h\in\mathbb{N}} \|f_h\|_{BV(\Omega)} < \infty$.$(Df_h)_h$ is a sequence of finite radon measures, as a result of Riesz theorem we have there exist a bounded linear functional on $C_0(\Omega)$, L_{Df_h} such that

$$L_{Df_h}(\varphi) = \int_\Omega \varphi dDf_h \qquad \forall \varphi \in C_0(\Omega).$$

Since $\int_\Omega \varphi dDf_h \to \int_\Omega \varphi dDf$ then $|L_{Df_h}(\varphi)| < \infty$, by Banach-Steinhaus theorem (uniform boundedness principle) we get $\|L_{Df_h}\| < \infty$. Therefore $\|L_{Df_h}\| = |Df_h|(\Omega) < \infty$ and thus we conclude boundedness in $BV(\Omega)$.

Conversely assuming $(f_h)_h$ bounded in $BV(\Omega)$ and converges to f in $L^1(\Omega)$, to prove weak* convergence in $BV(\Omega)$ we only need to prove that Df_h weakly* converge to Df in Ω.

By weak* compactness 1.1.4, for any sequence (Df_h) we have a further subsequence that converge weakly* to μ, $Df_{h_k} \to^{w*} \mu$ with μ radon measure. We need to show that $\mu = Df$ and therefore we get $Df_h \to^{w*} Df$. Indeed since $f_h \in BV(\Omega)$ then

$$\int_\Omega f_h \varphi_{x_i} dx = -\int_\Omega \varphi dD_i f_h \qquad \forall \varphi \in C_c^1(\Omega)\, i = 1, \cdots, n.$$

In particular,

$$\int_\Omega f_{h_k} \varphi_{x_i} dx = -\int_\Omega \varphi dD_i f_{h_k}.$$

Letting $k \to \infty$, we get $\int_\Omega f\varphi_{x_i} dx = -\int_\Omega \varphi d\mu_i$ for all $\varphi \in C_c^1(\Omega)$. Hence $\mu = Df$. □

Definition 1.5.4 (Strict convergence). *Let $f, (f_h)_h \in BV(\Omega)$. We say that (f_h) strictly converges in $BV(\Omega)$ to f if $(f_h)_h$ converges to f in $L^1(\Omega)$ and the variations $|Df_h|(\Omega)$ converge to $|Df|(\Omega)$ as $h \to \infty$.*

For $f, g \in BV(\Omega)$ define the distance

$$d(f, g) = \int_\Omega |f - g| dx + ||Df|(\Omega) - |Dg|(\Omega)|.$$

It can be easily checked that d is a distance in $BV(\Omega)$ and it induces strict convergence.

Remark 1.5.1. *Strict convergence implies weak* convergence but the opposite implication is not true in general.*

Take for example $f_h(x) = \frac{\sin hx}{h} \in BV(0, 2\pi)$.$f_h$ weakly converges to 0 in $BV(0, 2\pi)$. In fact,*

1.
$$\int_0^{2\pi} |f_h|\, dx = \int_0^{2\pi} \frac{|\sin(hx)|}{h}\, dx \le \int_0^{2\pi} \frac{1}{h}\, dx = \frac{2\pi}{h} \to 0 \quad \text{as } h \to \infty.$$
Hence f_h converges to 0 in $L^1(0, 2\pi)$.

2.
$$\|f_h\|_{BV} = \int_0^{2\pi} |f_h|\, dx + |Df_h|(0, \pi) \le \frac{2\pi}{h} + 4 \le 2\pi + 4 \quad \forall h \ge 1.$$
And therefore boundedness. With
$$|Df_h|(0, 2\pi) = \int_0^{2\pi} |Df_h| dx = \int_0^{2\pi} |\cos hx| dx = \int_0^{2\pi h} \frac{|\cos x|}{h} dx = 4.$$
But we do not have strict convergence to 0 because $|Df|(0, 2\pi) = 0 \neq 4$.

Proposition 1.5.4. *[3] If $(f_h)_h \subset BV(\Omega)$ strictly converges to f, and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous and positively 1-homogeneous function,we have*

$$\lim_{h\to\infty} \int_\Omega \varphi f\left(\frac{Df_h}{|Df_h|}\right) d|Df_h| = \int_\Omega \varphi f\left(\frac{Df}{|Df|}\right) d|Df|$$

for any bounded continuous function $\varphi : \Omega \to \mathbb{R}$. As consequence, the measures $f\left(\frac{Df_h}{|Df_h|}\right)$ weakly converge in Ω to $f\left(\frac{Df}{|Df|}\right)$, in particular $|Df_h| \to |Df|$ weakly * in Ω.*

The proposition is a particular case of the following theorem.

Theorem 1.5.5 ((Reshetnyak continuity)). *[3] Let Ω be an open subset of $\mathbb{R}^n$ and μ, μ_h be $\mathbb{R}^n$-valued finite Radon measures in Ω if $|\mu_h|(\Omega) \to |\mu|(\Omega)$ then*

$$\lim_{h\to\infty} \int_\Omega f\left(x, \frac{\mu_h}{|\mu_h|}(x)\right) d|\mu_h|(x) = \int_\Omega f\left(x, \frac{\mu}{|\mu|}(x)\right) d|\mu|(x)$$

for every continuous and bounded function $f : \Omega \times S^{n-1} \to \mathbb{R}$.

1.6 Sets of finite perimeter

Definition 1.6.1. *Let E a λ-measurable subset of $\mathbb{R}^n$. For any open set $\Omega \subset \mathbb{R}^n$ the perimeter of E in Ω denoted by $P(E, \Omega)$, is the variation of $\mathbb{1}_E$ in Ω.*

$$P(E, \Omega) := V(\mathbb{1}_E, \Omega) = \sup\left\{ \int_E \operatorname{div}\varphi\, dx \,\middle|\, \varphi \in C_c^1(\Omega, \mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\}.$$

We say E is of finite perimeter in Ω if $P(E, \Omega) < \infty$.

The class of sets of finite perimeter in Ω includes all the sets E, with C^1 boundary inside Ω such that $\mathcal{H}^{n-1}(\Omega \cap \partial E) < \infty$. In fact, by the Gauss-Green theorem, for these sets E we have

$$\int_E div\varphi\, dx = -\int_{\Omega\cap\partial E} \nu_E \cdot \varphi d\mathcal{H}^{n-1} \qquad \forall \varphi \in C_c^1(\Omega, \mathbb{R}^n)$$

where ν_E is the inner unit normal to E.Hence it turns out that

$$P(E, \Omega) = \mathcal{H}^{n-1}(\Omega \cap \partial E).$$

Notice that if $|E \cap \Omega|$ is finite, then $\mathbb{1}_E \in L^1(\Omega)$ and we conclude that E has finite perimeter in Ω if and only if $\mathbb{1}_E \in BV(\Omega)$ and that $P(E, \Omega) = |D\mathbb{1}_E|(\Omega)$ the total variation in Ω of the distributional derivative of $\mathbb{1}_E$. In general we can always say that $\mathbb{1}_E \in BV_{loc}(\Omega)$ whenever E is a set of finite perimeter. Conversely if $\mathbb{1}_E \in BV_{loc}(\Omega)$, then E has finite perimeter in any open set $\Omega' \subset\subset \Omega$, in this case we say that E is a set of locally finite perimeter in Ω.

Theorem 1.6.1. *For any set E of finite perimeter in Ω, the distributional derivative $D\mathbb{1}_E$ is an $\mathbb{R}^n$-valued finite radon measure in Ω.Moreover $P(E, \Omega) = |D\mathbb{1}_E|(\Omega)$, and a generalised Gauss-Green formula holds:*

$$\int_E div\varphi dx = -\int_\Omega \varphi \cdot \nu_E d|D\mathbb{1}_E| \qquad \forall \varphi \in C_c^1(\Omega, \mathbb{R}^n)$$

where $D\mathbb{1}_E = \nu_E|D\mathbb{1}_E|$ the polar decomposition of $D\mathbb{1}_E$.

Proof. We have E is of finite perimeter in Ω, so $\mathbb{1}_E \in BV_{loc}(\Omega)$ and hence the distributional derivative of $\mathbb{1}_E$ is a radon measure in Ω.To show it is finite, it is enough to prove $\sup\{|D\mathbb{1}_E|(K), K \subset \Omega \, \text{compact}\} < \infty$.Since

$$|D\mathbb{1}_E|(K) = P(E,K) \leq P(E,\Omega) \qquad \forall K \subset\subset \Omega \text{ Open}$$

Thus $|D\mathbb{1}_E|(\Omega) < \infty$, and we conclude that $D\mathbb{1}_E$ is a finite Radon measure in Ω. □

Definition 1.6.2 (Convergence in measure). *1. Let f_h, f μ-measurable functions, we say that f_h converge to f in measure if*

$$\lim_{h\to\infty} \mu(\{x \in X, |f_h(x) - f(x)| > \epsilon\}) = 0 \quad \forall \epsilon > 0$$

2. Let E_h, E measurable sets, we say that E_h converges to E in measure in Ω if

$$\mu(\Omega \cap (E_h \triangle E)) \underset{h\to\infty}{\longrightarrow} 0$$

3. Local convergence in measure is the convergence in measure in any open set $A \subset\subset \Omega$.

Remark 1.6.1. *Clearly, the convergence in measures of (E_h) (respectively local convergence in measure) corresponds to convergence in $L^1(\Omega)$ ($L^1_{loc}(\Omega)$) of the characteristic functions $\mathbb{1}_{E_h}$ to $\mathbb{1}_E$.*

Proposition 1.6.1 (Properties of perimeter). *(a) The map $E \mapsto P(E,\Omega)$ is lower semi-continuous with respect to local convergence in measure in Ω.*

(b) $E \to P(E,\Omega)$ is local, $P(E,\Omega) = P(F,\Omega)$ whenever $|\Omega \cap (E \triangle F)| = 0$.

(c) $P(E,\Omega) = P(\mathbb{R}^n \setminus E, \Omega)$, and

$$P(E \cup F, \Omega) + P(E \cap F, \Omega) \leq P(E,\Omega) + P(F,\Omega) \tag{1.3}$$

Proof. We only need to prove (c) since all the above derive directly from the general theory of BV functions.

For $\varphi \in C^1_c(\Omega, \mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} div\varphi \, dx = \int_{\mathbb{R}^n} \varphi_{x_1} + \cdots + \varphi_{x_n} \, dx = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \varphi_{x_1} + \cdots + \varphi_{x_n} dx = 0.$$

So $\int_E div\varphi dx + \int_{\mathbb{R}^n \setminus E} div\varphi dx = 0$ and hence $\int_E div\varphi dx = -\int_{\mathbb{R}^n \setminus E} div\varphi dx$.Therefore $P(E,\Omega) = P(\mathbb{R}^n \setminus E, \Omega)$.

Now for the equality (1.3), we have E, F sets of finite perimeters so $\mathbb{1}_E, \mathbb{1}_F \in BV_{loc}(\Omega)$ hence by theorem 1.5.2 there exist $(u_h)_h, (v_h)_h \in C^\infty(\Omega)$ such that $u_h \to \mathbb{1}_E$, $v_h \to \mathbb{1}_F$ as $h \to \infty$ by a truncation argument $0 \leq u_h \leq 1$, $0 \leq v_h \leq 1$, and

$$\lim_{h\to\infty} \int_\Omega |Du_h| dx = P(E,\Omega),$$

$$\lim_{h\to\infty}\int_\Omega |Dv_h|dx = P(F,\Omega).$$

Then $u_h v_h \to \mathbb{1}_{E\cap F}$ and $u_h + v_h - u_h v_h \to \mathbb{1}_{E\cup F}$ $in\, L^1_{loc}(\Omega)$.By lower semi-continuity,

$$P(E\cap F,\Omega)+P(E\cup F,\Omega) \leq \liminf\int_\Omega |D(u_h v_h)|dx+\liminf\int_\Omega |D(v_h+u_h-u_h v_h)|dx.$$

But notice that

$$\begin{aligned}\int_\Omega |D(u_h v_h)|dx &= \int_\Omega |v_h Du_h + u_h Dv_h|dx \\ &\leq \int_\Omega |Du_h||v_h| + |Dv_h||u_h|dx\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |D(v_h + u_h - u_h v_h)|dx &= \int_\Omega |Dv_h + Du_h - Du_h.v_h - Dv_h.u_h)|dx \\ &\leq \int_\Omega |Du_h(1-v_h)| + |Dv_h(1-u_h)|dx \\ &\leq \int_\Omega |Du_h|(1-v_h) + |Dv_h|(1-u_h)dx.\end{aligned}$$

Adding the two inequalities, we get

$$P(E\cap F,\Omega)+P(E\cup F,\Omega) \leq \liminf\int_\Omega |Du_h|dx+\liminf\int_\Omega |Dv_h|dx = P(E,\Omega)+P(F,\Omega)$$

□

1.6.1 *Coarea formula in* BV *space*

The coarea formula relates the variation of f and the perimeter of its superlevel set. For $f:\Omega\to\mathbb{R}$ and $t\in R$, define the superlevel set $E_t = \{x\in\Omega, f(x) > t\}$.

Lemma 1.6.1. *If $f\in BV(\Omega)$, the map $t\mapsto |D\mathbb{1}_{E_t}|(\Omega) = P(E_t,\Omega)$ is λ-measurable.*

Proof. The mapping $(x,t)\to\mathbb{1}_{E_t}(x)$ is $\lambda\times\lambda$-measurable,because $E_t = f^{-1}((t,\infty))$ measurable set since $f\in L^1(\Omega)$.Thus for each $\varphi\in C^1_c(\Omega,\mathbb{R}^n)$ the function $t\to \int_\Omega \mathbb{1}_{E_t} div\varphi dx = \int_{E_t} div\varphi dx$ is λ-measurable, hence the sup function is λ-measurable and therefore $t\to P(E_t,\Omega)$ is λ-measurable □

Theorem 1.6.2 (Coarea formula)**.** *Let $f\in BV(\Omega)$, then*

(i) E_t has finite perimeter for λ-a.e $t\in\mathbb{R}$.

(ii) $V(f,\Omega) = \int_{-\infty}^{\infty} P(E_t,\Omega)dt = \int_{-\infty}^{\infty} P(\{x\in\Omega, f(x)>t\},\Omega)dt.$

(iii) Conversely,if $f\in L^1(\Omega)$, and $\int_{-\infty}^{\infty} P(E_t,\Omega)dt < \infty$ then $f\in BV(\Omega)$

Proof. First I'll prove (ii) then, (i) and (iii).

(ii) We present the plan of the proof: First I'll show that for all $\varphi \in C_c^1(\Omega, \mathbb{R}^n)$ with $\|\varphi\|_\infty \leq 1$,

$$\int_\Omega f div\varphi \, dx = \int_{-\infty}^{\infty} P(E_t, \Omega) \, dt$$

for $f \geq 0$, then $f \leq 0$ and finally the general case $f = f^+ - f^-$. Then we'll get that $V(f, \Omega) \leq \int_{-\infty}^{\infty} P(E_t, \Omega) dt$ .Next, we prove that we have equality for $f \in BV(\Omega) \cap C^\infty(\Omega)$ and finally for $f \in BV(\Omega)$.

Take $\varphi \in C_c^1(\Omega, \mathbb{R}^n)$, $\|\varphi\|_\infty \leq 1$.

Claim 1: $\int_\Omega f div\varphi \, dx = \int_{-\infty}^{\infty} \int_{E_t} div\varphi \; dxdt$

Proof of the claim. First suppose $f \geq 0$, so

$$f(x) = \int_{\mathbb{R}} \mathbb{1}_{(0, f(x))}(t) \, dt = \int_{\mathbb{R}} \mathbb{1}_{(0,\infty)}(t) \mathbb{1}_{E_t}(x) dt = \int_0^\infty \mathbb{1}_{E_t}(x) dt \qquad a.e \, x \in \Omega$$

Thus

$$\begin{aligned} \int_\Omega f div\varphi dx &= \int_\Omega \big(\int_0^\infty \mathbb{1}_{E_t}(x) dt \big) div\varphi(x) dx \\ &= \int_0^\infty \big(\int_\Omega \mathbb{1}_{E_t}(x) div\varphi(x) dx \big) dt \qquad \textit{By fubini} \\ &= \int_0^\infty \big(\int_{E_t} div\varphi dx \big) dt \end{aligned}$$

Similarly, if $f \leq 0$,

$$f(x) = \int_{\mathbb{R}} \mathbb{1}_{(f(x),0)}(t) dt = \int_{\mathbb{R}} \mathbb{1}_{(-\infty,0)}(t)(\mathbb{1}_{E_t} - 1)(x) dt = \int_{-\infty}^0 (\mathbb{1}_{E_t}(x) - 1) dt$$

Hence

$$\begin{aligned} \int_\Omega f div\varphi dx = \int_\Omega \left(\int_{-\infty}^0 \mathbb{1}_{E_t}(x) - 1 dt \right) div\varphi(x) dx &= \int_{-\infty}^0 \left(\int_\Omega (\mathbb{1}_{E_t}(x) - 1) div\varphi(x) dx \right) dt \\ &= \int_{-\infty}^0 \left(\int_{E_t} div\varphi dx \right) dt \end{aligned}$$

For the general case write, $f = f^+ + (-f^-)$.

$$\begin{aligned} \int_\Omega f div\varphi dx &= \int_\Omega (f^+ + (-f)^-) div\varphi dx \\ &= \int_0^\infty \left(\int_{E_t} div\varphi dx + \int_{-\infty}^0 \int_{E_t} div\varphi dx \right) dt \\ &= \int_{-\infty}^\infty \int_{E_t} div\varphi dx dt \end{aligned}$$

□

Thus for f measurable we always have this inequality

$$\begin{aligned}\int_\Omega f div\varphi dx &= \int_{-\infty}^{\infty}\int_{E_t} div\varphi dx dt \\ &\le \int_{-\infty}^{\infty} \sup\left\{\int_{E_t} \operatorname{div}\varphi\, dx \,\middle|\, \varphi \in C_c^1(\Omega,\mathbb{R}^n),\ \|\varphi\|_\infty \le 1\right\} dt \\ &\le \int_{-\infty}^{\infty} P(E_t,\Omega)dt\end{aligned}$$

Therefore

$$V(f,\Omega) \le \int_{-\infty}^{\infty} P(E_t,\Omega)dt \quad (*)$$

Claim 2: (ii) holds for $f \in BV(\Omega)\cap C^\infty(\Omega)$.

Proof of the claim. Define $m(t) = \int_{\Omega\setminus E_t} |Df|dx = \int_{\{f\le t\}} |Df|dx$. m is non decreasing, so m' exists $\lambda - ae$ with

$$\int_{-\infty}^{\infty} m'(t)dt \le \int_\Omega |Df|dx \qquad (**).$$

Now if I show that $m'(t) \ge P(E_t,\Omega)$ integrating over $\mathbb{R}$ we get the other inequality. To do so fix any $-\infty < t < \infty, r > 0$ and define $\eta : \mathbb{R} \to \mathbb{R}$ by

$$\eta(s) = \begin{cases} 0 & s \le t \\ \frac{s-t}{r} & t \le s \le t+r \\ 1 & s \ge t+r \end{cases}$$

Then

$$\eta'(s) = \begin{cases} \frac{1}{r} & t < s < t+r \\ 0 & s < t \, or\, s > t+r \end{cases}$$

Hence for all $\varphi \in C_c^1(\Omega,\mathbb{R}^n)$,$(\eta o f)\varphi$ is compactly supported in Ω and thus $\int_\Omega div((\eta o f)\varphi)dx = 0$.This implies that

$$-\int_\Omega \eta(f(x))div(\varphi)dx = \int_\Omega \eta'(f(x))Df\cdot\varphi dx = \frac{1}{r}\int_{E_t\setminus E_{t+r}} Df\cdot\varphi dx \qquad (***)$$

Now

$$\begin{aligned}\frac{m(t+r)-m(t)}{r} &= \frac{1}{r}\left(\int_{\Omega\setminus E_{t+r}} |Df|dx - \int_{\Omega\setminus E_t} |Df|dx\right) \\ &= \frac{1}{r}\int_{E_t\setminus E_{t+r}} |Df|dx \\ &\ge \frac{1}{r}\int_{E_t\setminus E_{t+r}} Df.\varphi dx \\ &= -\int_\Omega \eta(f(x))div\varphi dx \qquad \text{by } (***)\end{aligned}$$

For those t such that $m'(t)$ exists i.e $\lambda-ae\, t$, let $r\to 0$,hence $m'(t)\geq -\int_{E_t} div\varphi dx$.Therefore $m'(t)\geq P(E_t,\Omega)$.Thus

$$\int_{-\infty}^{\infty} m'(t)dt \geq \int_{-\infty}^{\infty} P(E_t,\Omega)dt$$

Now using $(**)$ we get

$$V(f,\Omega)=|Df|(\Omega)=\int_{\Omega}|Df|dx\geq\int_{-\infty}^{\infty} m'(t)dt\geq\int_{-\infty}^{\infty} P(E_t,\Omega)dt$$

This estimate and $(*)$ gives that for $f\in BV(\Omega)\cap C^{\infty}(\Omega)$,

$$V(f,\Omega)=\int_{-\infty}^{\infty} P(E_t,\Omega)dt$$

□

Claim3: (ii) holds for $f\in BV(\Omega)$

Proof of the claim. By the approximation theorem, there exists $\{f_k\}_k\subset C^{\infty}(\Omega)$ such that $f_k \underset{k\to\infty}{\to} f$ in $L^1(\Omega)$ and $|Df|(\Omega)=\lim\limits_{k\to\infty}\int_{\Omega}|Df_k|dx=\lim\limits_{k\to\infty}V(f_k,\Omega)<\infty$.

Define $E_t^k=\{x\in\Omega\ f_k(x)>t\}$, notice

$$\begin{aligned}\int_{-\infty}^{+\infty}|\mathbb{1}_{E_t^k}(x)-\mathbb{1}_{E_t}(x)|dt &= \int_{-\infty}^{+\infty}\left|\mathbb{1}_{(\min\{f(x),f_k(x)\},\max\{f(x),f_k(x)\})}(t)\right|dt\\ &= \int_{\min\{f(x),f_k(x)\}}^{\max\{f(x),f_k(x)\}} dt=|f_k(x)-f(x)|\end{aligned}$$

Consequently,

$$\int_{\Omega}|f_k(x)-f(x)|dx=\int_{\Omega}\int_{-\infty}^{+\infty}|\mathbb{1}_{E_t^k}(x)-\mathbb{1}_{E_t}(x)|dxdt=\int_{-\infty}^{+\infty}\int_{\Omega}|\mathbb{1}_{E_t^k}(x)-\mathbb{1}_{E_t}(x)|dtdx$$

Since $f_k\to f$ in $L^1(\Omega)$ then $\int_{-\infty}^{+\infty}\int_{\Omega}|\mathbb{1}_{E_t^k}(x)-\mathbb{1}_{E_t}(x)|dtdx<\infty$. Hence there exists a subsequence which upon re-indexing by k, satisfies $\mathbb{1}_{E_t^k}\to\mathbb{1}_{E_t}$ in $L^1(\Omega)$ for λ almost every t. Then by lower semi-continuity theorem

$$P(E_t,\Omega)\leq\liminf_{k\to\infty}P(E_t^k,\Omega)$$

Thus Fatou's lemma implies

$$\begin{aligned}\int_{-\infty}^{+\infty}P(E_t,\Omega)dt &\leq \int_{-\infty}^{+\infty}\liminf_{k\to\infty}P(E_t^k,\Omega)dt\\ &\leq \liminf_{k\to\infty}\int_{-\infty}^{+\infty}P(E_t^k,\Omega)dt\\ &= \liminf_{k\to\infty}V(f_k,\Omega) \qquad \text{(By Claim 2)}\\ &= \lim_{k\to\infty}V(f_k,\Omega)\\ &= |Df|(\Omega)\\ &= V(f,\Omega)\end{aligned}$$

□

For (i) since $\int_{\mathbb{R}} P(E_t, \Omega)dt < \infty$ hence $P(E_t, \Omega) < \infty$ a.e. And for (iii) if $f \in L^1(\Omega)$ by $(*) V(f, \Omega) \leq \int_{-\infty}^{+\infty} P(E_t, \Omega)dt < \infty$, then $f \in BV(\Omega)$. □

1.6.2 *Trace for BV functions*

Theorem 1.6.3. *[5] Let Ω open bounded with $\partial\Omega$ lipschitz. There exists a bounded linear mapping $T : BV(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$ such that*

$$\int_\Omega f div\varphi dx = -\int_\Omega \varphi \cdot dDf + \int_{\partial\Omega} (\varphi \cdot \nu) Tf d\mathcal{H}^{n-1} \qquad (*)$$

for all $f \in BV(\Omega)$ and $\varphi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$.

Proof. Given $x \in \mathbb{R}^n$, write $x = (x_1, \cdots, x_n) = (x', x_n)$ same for $y = (y', y_n)$.

First assume $f \in BV(\Omega) \cap C^\infty(\Omega)$. Pick $x \in \partial\Omega$,choose $r, h > 0$ and a lipschitz function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$. Define

$$C = C(x, r, h) = \{y \in \mathbb{R}^n \mid |y' - x'| < r,\ |y_n - x_n| < h\}$$

and $\Omega \cap C = \{y \mid |y' - x'| < r, \gamma(y') < y_n < x_n + h\}$. If $0 < \epsilon < \frac{h}{2}$, set

$$C_{\delta,\epsilon} = \{y \in C \mid \gamma(y') + \delta < y_n < \gamma(y') + \epsilon\} \text{ for } 0 \leq \delta < \epsilon < \frac{h}{2}$$

and define $C_\epsilon = C_{0,\epsilon}$, write $C^\epsilon = C \cap \Omega \setminus C_\epsilon$.

For $y \in \partial\Omega \cap C$, we define $f_\epsilon(y) = f(y', \gamma(y') + \epsilon)$, then

$$\begin{aligned}
|f_\delta(y) - f_\epsilon(y)| &= |f(y', \gamma(y') + \delta) - f(y', \gamma(y') + \epsilon)| \\
&= \left| \int_\delta^\epsilon \frac{\partial f}{\partial x_n}(y', \gamma(y') + t) dt \right| \\
&\leq \int_\delta^\epsilon \left| \frac{\partial f}{\partial x_n}(y', \gamma(y') + t) \right| dt \\
&\leq \int_\delta^\epsilon |Df(y', \gamma(y') + t)| dt.
\end{aligned}$$

Consequently,

$$\int_{\partial\Omega\cap C} |f_\delta(y) - f_\epsilon(y)| d\mathcal{H}^{n-1} \leq \int_{\partial\Omega\cap C} \int_\delta^\epsilon |Df(y', \gamma(y') + t)| dt d\mathcal{H}^{n-1} \leq c|Df|(C_{\delta,\epsilon}).$$

Therefore $(f_\epsilon)_{\epsilon>0}$ is Cauchy in $L^1(\partial\Omega \cap C, \mathcal{H}^{n-1})$ thus $Tf = \lim_{\epsilon\to 0} f_\epsilon$ exists in this space. Furthermore, our passing to limits as $\delta \to 0$ we get

$$\int_{\partial\Omega\cap C} |Tf - f_\epsilon| d\mathcal{H}^{n-1} \leq c|Df|(C_\epsilon \cap \Omega) \leq c|Df|(\overline{C_\epsilon} \cap \Omega) \qquad (**)$$

Next fix $\varphi \in C_c^1(C, \mathbb{R}^n)$, then by integration by parts

$$\begin{aligned}
\int_{C^\epsilon} f div\varphi\, dy &= -\int_{C^\epsilon} \varphi \cdot Df dy + \int_{\partial C^\epsilon} f\varphi \cdot \nu dH^{n-1} \\
&= -\int_{C^\epsilon} \varphi \cdot Df dy + \int_{\partial\Omega\cap C} f_\epsilon \varphi_\epsilon \cdot \nu d\mathcal{H}^{n-1} \qquad \text{(By change of variable)}
\end{aligned}$$

Let $\epsilon \to 0$, to find

$$\int_{\Omega\cap C} f div\varphi\, dy = -\int_{\Omega\cap C} \varphi \cdot Df dy + \int_{\partial\Omega\cap C} Tf\varphi \cdot \nu\, d\mathcal{H}^{n-1} (***)$$

Since $\partial\Omega$ is compact, we can cover $\partial\Omega$ with finitely many cylinders $C_i = C(x_i, r_i, h_i)$, $i = 1, \cdots, n$ for which assertions analogous to $(**)$ and $(***)$ hold.Hence there exist $(\xi_k)_k$ a partition of unity such that

$$\begin{cases} 0 \leq \xi_k \leq 1 \\ \xi_k \in C_c^\infty(C_k) \\ \sum_{k\geq 1} \xi_k = 1 \quad \text{on } C \end{cases}$$

$\varphi\xi_k \in C_c^\infty(C_k, \mathbb{R}^n)$ with $\varphi \in C_c^1(C, \mathbb{R}^n)$, then apply above on $\varphi\xi_k$ we get

$$\int_{\Omega\cap C_k} f div(\varphi\xi_k) dy = -\int_{\Omega\cap C_k} \varphi\xi_k \cdot Df dy + \int_{\partial\Omega\cap C_k} Tf\varphi\xi_k \cdot \nu d\mathcal{H}^{n-1}$$

$$\int_{\Omega\cap C_k} f\varphi\cdot\nabla(\xi_k)dy + \int_{\Omega\cap C_k} f\xi_k div(\varphi)dy = -\int_{\Omega\cap C_k} \varphi\xi_k\cdot Df dy + \int_{\partial\Omega\cap C_k} Tf\varphi\xi_k\cdot\nu d\mathcal{H}^{n-1}$$

Summing over k and having that $\sum \xi_k = 1$ then $\sum \nabla\xi_k = \nabla \sum \xi_k = 0$, we get

$$0 + \int_\Omega f div(\varphi)dy = -\int_\Omega \varphi \cdot Df dy + \int_{\partial\Omega} Tf\varphi \cdot \nu d\mathcal{H}^{n-1}$$

hence formula $(*)$ is established.

Now assume that $f \in BV(\Omega)$, choose $f_k \in BV(\Omega) \cap C^\infty(\Omega)$ $(k = 1, \cdots)$ such that

$$f_k \to f \text{ in } L^1(\Omega) \text{ and } |Df_k|(\Omega) \to |Df|(\Omega).$$

In addition, form [5], we have μ_k converges to μ weakly, where μ_k and μ are defined as follows: for $B \subset \mathbb{R}$,

$$\mu_k(B) = \int_{B\cap\Omega} Df_k dx \text{ and } \mu(B) = \int_{B\cap\Omega} dDf$$

Claim: $(Tf_k)_{k\geq 1}$ is a Cauchy sequence in $L^1(\partial\Omega, \mathcal{H}^{n-1})$.

Proof of Claim. As previously, choose a cylinder C, fix $\epsilon > 0, y \in \partial\Omega \cap C$, and define

$$f_k^\epsilon(y) = \frac{1}{\epsilon}\int_0^\epsilon f_k(y', \gamma(y') + t)dt = \frac{1}{\epsilon}\int_0^\epsilon (f_k)_t(y)dt.$$

Then (**) implies

$$\begin{aligned} \int_{\partial\Omega\cap C} |Tf_k - f_k^\epsilon| d\mathcal{H}^{n-1} &= \int_{\partial\Omega\cap C} \left| \frac{1}{\epsilon}\int_0^\epsilon Tf_k dt - \frac{1}{\epsilon}\int_0^\epsilon (f_k)_t dt \right| d\mathcal{H}^{n-1} \\ &\leq \frac{1}{\epsilon}\int_0^\epsilon \int_{\partial\Omega\cap C} |Tf_k - (f_k)_t|\, d\mathcal{H}^{n-1} dt \\ &\leq \frac{1}{\epsilon}\int_0^\epsilon c|Df_k|(\overline{C_\epsilon} \cap \Omega)dt \\ &= c|Df_k|(\overline{C_\epsilon} \cap \Omega). \end{aligned}$$

We have

$$\int_{\partial\Omega\cap C}|Tf_k-Tf_l|d\mathcal{H}^{n-1}\leq\int_{\partial\Omega\cap C}|Tf_k-f_k^\epsilon|d\mathcal{H}^{n-1}+\int_{\partial\Omega\cap C}|Tf_l-f_l^\epsilon|d\mathcal{H}^{n-1}+\int_{\partial\Omega\cap C}|f_k^\epsilon-f_l^\epsilon|d\mathcal{H}^{n-1}.$$

However,

$$\int_{\partial\Omega\cap C}|f_k^\epsilon-f_l^\epsilon|d\mathcal{H}^{n-1}\leq\int_{\partial\Omega\cap C}\frac{1}{\epsilon}\int_0^\epsilon|(f_k)_t-(f_l^\epsilon)_t|d\mathcal{H}^{n-1}dt=\frac{1}{\epsilon}\int_{C_\epsilon}|f_k-f_l|dy$$

hence,

$$\int_{\partial\Omega\cap C}|Tf_k-Tf_l|d\mathcal{H}^{n-1}\leq c(|Df_k|+|Df_l|)(\overline{C_\epsilon}\cap\Omega)+\frac{1}{\epsilon}\int_{C_\epsilon}|f_k-f_l|dy.$$

Thus

$$\limsup_{k,l\to\infty}\int_{\partial\Omega\cap C}|Tf_k-Tf_l|d\mathcal{H}^{n-1}\leq 2c|Df|(\overline{C_\epsilon}\cap\Omega)$$

this is true because $f_k\to f$ in $L^1(\Omega)$,$|Df_k|(\overline{C_\epsilon}\cap\Omega)\to|Df|(\overline{C_\epsilon}\cap\Omega)$. And since the quantity on the right-hand side goes to zero as $\epsilon\to 0$, the claim is proved. □

In view of the claim we may define $Tf=\lim_{k\to\infty}Tf_k$. Finally, $(*)$ holds for each f_k and thus holds in the limit for f.

Note that the definition of Tf does not depend on the choice of the sequence. In fact letting (f_k) and (g_k) two approximating sequences i.e $f_k\to f$ in $L^1(\Omega)$, $|Df_k|(\Omega)\to|Df|(\Omega)$ and $g_k\to g$ in $L^1(\Omega)$, $|Dg_k|(\Omega)\to|Dg|(\Omega)$,

$$\begin{aligned}\|Tf_k-Tg_k\|_1&=\|T(f_k-g_k)\|_1 &&\text{(By linearity of } T)\\ &\leq c\|f_k-g_k\|_{BV} &&\text{(As T is bounded)}\\ &=c(|D(f_k-g_k)|(\Omega)+\|f_k-g_k\|_1)\\ &=c(|Df_k-Dg_k|(\Omega)+\|f_k-g_k\|_1)\\ &\leq c(|Df_k-Df|(\Omega)+|Dg_k-Df|(\Omega)+\|f_k-g_k\|_1+\|g_k-f\|_1)\to_{k\to\infty}0\end{aligned}$$

□

Remark 1.6.2. *The trace function is not injective.Let $f\in BV(\Omega)\cap C_c^\infty(\Omega)$, $f|_{\partial\Omega}=0$ but $f|_{\partial\Omega}=Tf=0$ hence $\ker T\neq 0$ therefore by linearity of T it is not injective.*

Theorem 1.6.4. *[5] Assume Ω is bounded open, $\partial\Omega$ lipschitz. Suppose also $f\in BV(\Omega)$, then for $\mathcal{H}^{n-1}$-a.e $x\in\partial\Omega$,*

$$\lim_{r\to 0}\fint_{B(x,r)\cap\Omega}|f-Tf(x)|dy=0$$

and so

$$Tf(x)=\lim_{r\to 0}\fint_{B(x,r)\cap\Omega}f\,dy$$

Remark 1.6.3. *In particular,if $f \in BV(\Omega) \cap C(\overline{\Omega})$ then $Tf = f|_{\partial\Omega} \quad \mathcal{H}^{n-1} - a.e.$*

Proof. 1. Claim:For $\mathcal{H}^{n-1} - ae\, x \in \partial\Omega, \lim_{r\to 0} \frac{|Df|(B(x,r)\cap\Omega)}{r^{n-1}} = 0$.

Proof of Claim. Fix $\gamma, \delta > \epsilon \geq 0$, and let

$$A_\gamma = \left\{ x \in \partial\Omega \,\middle|\, \limsup_{r\to 0} \frac{|Df|(B(x,r)\cap\Omega)}{r^{n-1}} > \gamma \right\}$$

then for each $x \in A_\gamma, \exists\, 0 < r < \epsilon$ such that

$$\frac{|Df|(B(x,r)\cap\Omega)}{r^{n-1}} \geq \gamma \qquad (*)$$

Using Vitali's covering theorem (conditions are satisfied since $|Df|$ is a finite radon measure and $r < \epsilon$), there exists a countable collection of disjoint balls $\{B(x_i, r_i)\}_{i\in\mathbb{N}}$ satisfying (*) such that $A_\gamma \subset \bigcup_{i=1}^{\infty} B(x_i, 5r_i)$, then

$$\begin{aligned}
\mathcal{H}^{n-1}_{10\delta}(A_\gamma) \leq \sum_{i\geq 1} \mathcal{H}^{n-1}_{10\delta}(B(x_i,5r_i)) &\leq \sum_{i\geq 1} \mathcal{H}^{n-1}(B(x_i,5r_i)) \\
&\leq \sum_{i\geq 1} \alpha_{(n-1)}(5r_i)^{n-1} \\
&\leq \sum_{i\geq 1} \alpha_{(n-1)} 5^{n-1} \frac{|Df|(B(x_i,r_i)\cap\Omega)}{\gamma} \qquad \text{by } (*) \\
&= \frac{c}{\gamma} \sum_{i\geq 1} |Df|(B(x_i,r_i)\cap\Omega) \\
&= \frac{c}{\gamma} |Df| \left(\bigcup_{i\geq 1} B(x_i,r_i) \cap \Omega \right) \\
&\leq \frac{c}{\gamma} |Df|(\Omega^\epsilon)
\end{aligned}$$

where $\Omega^\epsilon = \{x \in \Omega \mid dist(x,\partial\Omega) < \epsilon\}$, take $\epsilon \to 0$ to find $\mathcal{H}^{n-1}_{10\delta}(A_\gamma) = 0$ for all $\delta > 0$. Therefore $\mathcal{H}^{n-1}(A_\gamma) = 0$ and the claim is proved. □

2. Now to prove our theorem, fix a point $x \in \partial\Omega$ such that

$$\lim_{r\to 0} \frac{|Df|(B(x,r)\cap\Omega)}{r^{n-1}} = 0 (\text{ By the claim })$$

and

$$\lim_{r\to 0} \int_{B(x,r)\cap\partial\Omega} |Tf(z) - Tf(x)| d\mathcal{H}^{n-1}(z) = 0.$$

The above is true by the Lebesgue Besicovitch differentiation theorem 1.1.5, since $Tf \in L^1(\partial\Omega, \mathcal{H}^{n-1})$ from the definition of trace.

Let $h = h(r) = 2r \max(1, 4Lip(\gamma))$ and consider the cylinders $C(r) = C(x, r, h)$, observe that for sufficiently small r and as in theorem 1.6.5, (the cylinders $C(r)$ work in place of the cylinders C in the previous proof). Thus estimates similar to those developed in that proof show that

$$\int_{\partial\Omega \cap C(r)} |Tf - f_\epsilon| d\mathcal{H}^{n-1} \leq c|Df|(C(r) \cap \Omega),$$

where

$$f_\epsilon(y) = f(y', \gamma(y') + \epsilon) \qquad y \in C(r) \cap \partial\Omega, 0 < \epsilon < \frac{h(r)}{2}.$$

Consequently,we estimate

$$\int_{B(x,r)\cap\Omega} |Tf(y', \gamma(y')) - f(y)| dy \leq Cr|Df|(C(r) \cap \Omega)$$

Hence we compute,

$$\begin{aligned}
⨍_{B(x,r)\cap\Omega} |f(y) - Tf(x)| dy &\leq ⨍_{B(x,r)\cap\Omega} |Tf(y) - Tf(x)| dy + ⨍_{B(x,r)\cap\Omega} |Tf(y) - f(y)| dy \\
&\leq \frac{c}{r^n} \int_{B(x,r)\cap\Omega} |Tf(y) - Tf(x)| dy + \frac{c}{r^n} \int_{B(x,r)\cap\Omega} |Tf(y) - f(y)| dy \\
&= \frac{c}{r^n} \int_0^r \int_{B(x,r)\cap\partial\Omega} |Tf(y) - Tf(x)| d\mathcal{H}^{n-1} dt \\
&\qquad + \frac{c}{r^n} \int_{B(x,r)\cap\Omega} |Tf(y', \gamma(y')) - f(y)| dy \\
&\leq \frac{c}{r^{n-1}} \int_{C(r)\cap\partial\Omega} |Tf(y) - Tf(x)| d\mathcal{H}^{n-1} + \frac{c}{r^{n-1}} |Df|(B(x,r) \cap \Omega) \\
&\leq o(1) + \frac{c}{r^{n-1}} |Df|(B(x,r) \cap \Omega) \\
&\leq o(1) \qquad \text{as } r \to 0
\end{aligned}$$

Thus,

$$\begin{aligned}
\left| Tf(x) - ⨍_{B(x,r)\cap\Omega} f dy \right| &= \left| ⨍_{B(x,r)\cap\Omega} f(y) - f(y) + Tf(x) dy - ⨍_{B(x,r)\cap\Omega} f dy \right| \\
&\leq \left| ⨍_{B(x,r)\cap\Omega} |f - Tf(x)| dy + ⨍_{B(x,r)\cap\Omega} f(y) dy - ⨍_{B(x,r)\cap\Omega} f(y) dy \right| \to 0
\end{aligned}$$

Therefore, $Tf(x) = \lim_{r\to 0} ⨍_{B(x,r)\cap\Omega} f dy$.

□

1.6.3 *Extension*

Theorem 1.6.5. *Ω open bounded with $\partial\Omega$ lipschitz. Let $f_1 \in BV(\Omega), f_2 \in BV(\mathbb{R}^n \setminus \overline{\Omega})$. Define*

$$\overline{f} = \begin{cases} f_1(x) & x \in \Omega \\ f_2(x) & x \in \mathbb{R}^n \setminus \overline{\Omega} \end{cases}$$

then

$$\overline{f} \in BV(\mathbb{R}^n)$$

and

$$|D\overline{f}|(\mathbb{R}^n) = |Df_1|(\Omega) + |Df_2|(\mathbb{R}^n \setminus \overline{\Omega}) + \int_{\partial\Omega} |Tf_1 - Tf_2| d\mathcal{H}^{n-1}$$

Proof. First we prove that $\overline{f} \in BV(\mathbb{R}^n)$ i.e $V(\overline{f}, \mathbb{R}^n) < \infty$ so it is enough to show $\int_{\mathbb{R}^n} \overline{f} div\varphi dx < \infty$ forall $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $\|\varphi\|_\infty \leq 1$.

1. Let $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ with $\|\varphi\|_\infty \leq 1$, then

$$\begin{aligned} \int_{\mathbb{R}^n} \overline{f} div\varphi dx &= \int_\Omega f_1 div\varphi dx + \int_{\mathbb{R}^n \setminus \overline{\Omega}} f_2 div\varphi dx \\ &= -\int_\Omega \varphi \cdot dDf_1 + \int_{\partial\Omega} \varphi \cdot \nu Tf_1 d\mathcal{H}^{n-1} - \int_{\mathbb{R}^n \setminus \overline{\Omega}} \varphi \cdot dDf_2 + \int_{\partial(\mathbb{R}^n \setminus \overline{\Omega})} \varphi \cdot (-\nu) Tf_2 d\mathcal{H}^{n-1} \\ &\leq \left| -\int_\Omega \varphi \cdot dDf_1 - \int_{\mathbb{R}^n \setminus \overline{\Omega}} \varphi \cdot dDf_2 + \int_{\partial\Omega} (\varphi \cdot \nu)(Tf_1 - Tf_2) d\mathcal{H}^{n-1} \right| \\ &\leq \int_\Omega |\varphi| d|Df_1| + \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\varphi| d|Df_2| + \int_{\partial\Omega} |\varphi \cdot \nu||Tf_1 - Tf_2| d\mathcal{H}^{n-1} \\ &\leq |Df_1|(\Omega) + |Df_2|(\mathbb{R}^n \setminus \overline{\partial\Omega}) + \int_{\partial\Omega} |Tf_1 - Tf_2| dH^{n-1} \\ &< \infty \quad \text{(Since } f_1, f_2 \in BV \text{ and by boundedness of } T) \end{aligned}$$

   Thus $\overline{f} \in BV(\mathbb{R}^n)$, and

$$|D\overline{f}|(\mathbb{R}^n) \leq |Df_1|(\Omega) + |Df_2|(\mathbb{R}^n \setminus \overline{\partial\Omega}) + \int_{\partial\Omega} |Tf_1 - Tf_2| d\mathcal{H}^{n-1}$$

2. We next show equality.
   For all $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} div(\overline{f}\varphi) dx = \int_{\mathbb{R}^n} \overline{f} div(\varphi) dx + \int_{\mathbb{R}^n} \varphi \cdot D\overline{f} dx$$

   since φ is compactly supported, we get

$$0 = \int_{\mathbb{R}^n} \overline{f} div(\varphi) dx + \int_{\mathbb{R}^n} \varphi \cdot dD\overline{f}.$$

   Hence as in 1

$$\begin{aligned} -\int_{\mathbb{R}^n} \varphi \cdot dD\overline{f} &= \int_{\mathbb{R}^n} \overline{f} div(\varphi) dx \\ &= -\int_\Omega \varphi \cdot dDf_1 - \int_{\mathbb{R}^n \setminus \overline{\Omega}} \varphi \cdot dDf_2 + \int_{\partial\Omega} (\varphi \cdot \nu)(Tf_1 - Tf_2) d\mathcal{H}^{n-1}. \end{aligned}$$

   But

$$\begin{aligned} -\int_{\mathbb{R}^n} \varphi \cdot dD\overline{f} &= -\int_\Omega \varphi \cdot dD\overline{f} - \int_{\partial\Omega} \varphi \cdot dD\overline{f} - \int_{\mathbb{R}^n \setminus \overline{\Omega}} \varphi \cdot dD\overline{f} \qquad (*) \\ &= -\int_\Omega \varphi \cdot dDf_1 - \int_{\partial\Omega} \varphi \cdot dD\overline{f} - \int_{\mathbb{R}^n \setminus \overline{\Omega}} \varphi \cdot dDf_2 \end{aligned}$$

With $D\overline{f} = \begin{cases} Df_1 & \text{on } \Omega \\ Df_2 & \text{on } \mathbb{R}^n \setminus \overline{\Omega} \end{cases}$
Consequently, (*) implies

$$-\int_{\partial\Omega} \varphi \cdot dD\overline{f} = \int_{\partial\Omega} (\varphi \cdot \nu)(Tf_1 - Tf_2) d\mathcal{H}^{n-1}$$

Since this is true for any φ, it follows immediately that $dD\overline{f} = (Tf_1 - Tf_2)\nu d\mathcal{H}^{n-1}$ and hence $|D\overline{f}|(\partial\Omega) = \int_{\partial\Omega}(Tf_1 - Tf_2)d\mathcal{H}^{n-1}$

□

1.6.4 *Isoperimteric inequalities, Sobolev's and Poincaré's inequalities for BV*

We now develop some inequalities relating the Lebesgue measure of a set to its perimeter that will be useful in the definition of the measure-theoretic boundary and its properties. We will need the following Sobolev inequalities from [5]

Theorem 1.6.6 (Sobolev's and Poincaré's inequalities for BV)**.** *(i) There exists a constant C_1 such that*

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq C_1 |Df|(\mathbb{R}^n)$$

for all $f \in BV(\mathbb{R}^n)$.

(ii) There exists a constant C_2 such that

$$\|f - (f)_{x,r}\|_{L^{\frac{n}{n-1}}(B(x,r))} \leq C_1 |Df|(B(x,r))$$

for all $B(x,r) \subset \mathbb{R}^n, f \in BV_{loc}(\mathbb{R}^n)$, where $(f)_{x,r} = ⨍_{B(x,r)} f dy$.

(iii) For each $0 < \alpha \leq 1$, there exists a constant $C_3(\alpha)$ such that

$$\|f\|_{L^{\frac{n}{n-1}}(B(x,r))} \leq C_3(\alpha) |Df|(B(x,r))$$

for all $B(x,r) \subset \mathbb{R}^n$ and all $f \in BV_{loc}(\mathbb{R}^n)$ satisfying

$$\frac{|B(x,r) \cap \{f = 0\}|}{|B(x,r)|} \geq \alpha.$$

Theorem 1.6.7 (Isoperimteric inequalities)**.** *Let E be a bounded set of finite perimeter in $\mathbb{R}^n$. Then*

(i) $|E|^{1-\frac{1}{n}} \leq C_1 |D\mathbb{1}_E|(\mathbb{R}^n)$.

(ii) For each ball $B(x,r) \subset \mathbb{R}^n$,

$$\min\{|B(x,r) \cap E|, |B(x,r) \setminus E|\}^{1-\frac{1}{n}} \leq 2C_2 |D\mathbb{1}_E|(B(x,r)).$$

Proof. (i) Let $f = \mathbb{1}_E \in BV(\mathbb{R}^n)$ in assertion (i) of theorem 1.6.6 ,then

$$\|\mathbb{1}_E\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C_1 |D\mathbb{1}_E|(\mathbb{R}^n).$$

(ii) Let $f = \mathbb{1}_{B(x,r)\cap E}$ so $f_{x,r} = \frac{|B(x,r)\cap E|}{|B(x,r)|}$.Thus

$$\begin{aligned}
\int_{B(x,r)} |f - f_{x,r}|^{\frac{n}{n-1}} dy &= \int_{B(x,r)} \left| \frac{\mathbb{1}_{B(x,r)\cap E}|B(x,r)| - |B(x,r)\cap E|}{|B(x,r)|} \right|^{\frac{n}{n-1}} dy \\
&= \int_{B(x,r)\cap E} \left| \frac{|B(x,r)| - |B(x,r)\cap E|}{|B(x,r)|} \right|^{\frac{n}{n-1}} dy + \int_{B(x,r)\cap E^c} \left| \frac{|B(x,r)\cap E|}{|B(x,r)|} \right|^{\frac{n}{n-1}} dy \\
&= \frac{|B(x,r)\cap E^c|}{|B(x,r)|}^{\frac{n}{n-1}} |B(x,r)\cap E| + \frac{|B(x,r)\cap E|}{|B(x,r)|}^{\frac{n}{n-1}} |B(x,r)\cap E^c|.
\end{aligned}$$

If $|B(x,r)\cap E| \le |B(x,r)\cap E^c|$, then

$$\begin{aligned}
\left(\int_{B(x,r)} |f - f_{x,r}|^{\frac{n}{n-1}} dy \right)^{1-\frac{1}{n}} &\ge \left| \frac{|B(x,r)\cap E^c|}{|B(x,r)|} \right| |B(x,r)\cap E|^{1-\frac{1}{n}} \\
&\ge \frac{1}{2} \min\{|B(x,r)\cap E|, |B(x,r) - E|\}^{1-\frac{1}{n}}.
\end{aligned}$$

The other case is similar.

□

Chapter 2

Reduced boundary

2.1 Reduced Boundary

Let E be a set of finite perimeter, denote ν_E, the measurable function such that $D\mathbb{1}_E = -\nu_E|D\mathbb{1}_E|$.

Definition 2.1.1. *Let $x \in \mathbb{R}^n$. We say that $x \in \partial^* E$, the reduced boundary of E, if*

(i) $|D\mathbb{1}_E|(B(x,r)) = P(E, B(x,r)) > 0 \quad \forall r > 0$

(ii) $\lim\limits_{r \to 0} ⨍_{B(x,r)} \nu_E d|D\mathbb{1}_E| = \nu_E(x)$

(iii) $|\nu_E(x)| = 1$

Remark 2.1.1. *According to Lebesgue-Besicovitch differentiation theorem1.1.5,$|D\mathbb{1}_E|(\mathbb{R}^n - \partial^* E) = 0$.*

Lemma 2.1.1. *Let $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$, then for each $x \in \mathbb{R}^n$,*

$$\int_{E \cap B(x,r)} div\varphi dy = \int_{B(x,r)} \varphi \cdot \nu_E d|D\mathbb{1}_E| + \int_{E \cap \partial B(x,r)} \varphi \cdot \nu d\mathcal{H}^{n-1} \qquad \text{for } \lambda - ae\, r > 0$$

with ν the outward unit normal to $\partial B(x,r)$.

Proof. Assume $h : \mathbb{R}^n \to \mathbb{R}$ smooth, then

$$\int_E div(\varphi h) dy = \int_E h div\varphi dy + \int_E Dh \cdot \varphi dy$$

$$\begin{aligned}
\int_E div(\varphi h)\, dy &= \int_{\mathbb{R}^n} \mathbb{1}_E div(\varphi h)\, dy \\
&= -\int_{\mathbb{R}^n} \varphi h \cdot dD\mathbb{1}_E \\
&= \int_{\mathbb{R}^n} (\varphi h) \cdot \nu_E\, d|D\mathbb{1}_E|
\end{aligned}$$

We get

$$\int_{\mathbb{R}^n} h\varphi \cdot \nu_E \, d|D\mathbb{1}_E| = \int_E h div\varphi \, dy + \int_E Dh \cdot \varphi \, dy \quad (*)$$

Let

$$g_\epsilon(s) := \begin{cases} 1 \quad 0 \le s \le r \\ \frac{r-s+\epsilon}{\epsilon} \quad r \le s \le r+\epsilon \\ 0 \quad s \ge r+\epsilon \end{cases}$$

and notice

$$g'_\epsilon(s) = \begin{cases} 0 \quad 0 \le s < r \text{ or } s > r+\epsilon \\ \frac{-1}{\epsilon} \quad r < s < r+\epsilon \end{cases}.$$

Now let $h_\epsilon(y) := g_\epsilon(|y-x|)$,

$$Dh_\epsilon(y) = \begin{cases} 0 \quad |y-x| < r \text{ or } |y-x| > r+\epsilon \\ \frac{-1}{\epsilon}\frac{y-x}{|y-x|} \quad r < |y-x| < r+\epsilon \end{cases}$$

By approximation, (*) holds for h_ϵ (with a partition of unity for the smoothness)

$$\begin{aligned} \int_{\mathbb{R}^n} h_\epsilon\varphi \cdot \nu_E d|D\mathbb{1}_E| &= \int_E h_\epsilon div\varphi dy + \int_E Dh_\epsilon \cdot \varphi dy \\ &= \int_E h_\epsilon div\varphi dy - \frac{1}{\epsilon}\int_{E\cap\{y: r<|y-x|<r+\epsilon\}} \varphi \cdot \frac{y-x}{|y-x|} dy \end{aligned}$$

Let $\epsilon \to 0$,

$$\int_{B(x,r)} 1\varphi \cdot \nu_E d|D\mathbb{1}_E| = \int_{E\cap B(x,r)} div\varphi dy - \int_{E\cap\partial B(x,r)} \varphi \cdot \nu d\mathcal{H}^{n-1} \quad \lambda - ae\, r > 0.$$

□

Lemma 2.1.2. *There exist positive constants $A_1, A_2, \cdots, A_5$ depending only on n, such that for each $x \in \partial^* E$,*

1. $\liminf_{r\to 0} \frac{|B(x,r)\cap E|}{r^n} > A_1 > 0$
2. $\liminf_{r\to 0} \frac{|B(x,r)\cap E^c|}{r^n} > A_2 > 0$
3. $\liminf_{r\to 0} \frac{|D\mathbb{1}_E|(B(x,r))}{r^{n-1}} > A_3 > 0$
4. $\limsup_{r\to 0} \frac{|D\mathbb{1}_E|(B(x,r))}{r^{n-1}} \le A_4$
5. $\limsup_{r\to 0} \frac{|D\mathbb{1}_{E\cap B(x,r))}|(\mathbb{R}^n)}{r^{n-1}} \le A_5$

Proof.

Step 1: Fix $x \in \partial^* E$. According to lemma 2.1.1 , for $\lambda - a.e\, r > 0$, and for every $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $\|\varphi\|_\infty \leq 1$,

$$\begin{aligned}\left|\int_{B(x,r)\cap E} div\varphi\, dy\right| &\leq \left|\int_{B(x,r)} \varphi \cdot \nu_E d|D\mathbb{1}_E|\right| + \left|\int_{E\cap\partial B(x,r)} \varphi \cdot \nu\, d\mathcal{H}^{n-1}\right| \\ &\leq \int_{B(x,r)} |\varphi| d|D\mathbb{1}_E| + \mathcal{H}^{n-1}(E \cap \partial B(x,r)) \\ &\leq \int_{B(x,r)} d|D\mathbb{1}_E| + \mathcal{H}^{n-1}(E \cap \partial B(x,r)) \\ &= |D\mathbb{1}_E|(B(x,r)) + \mathcal{H}^{n-1}(E \cap \partial B(x,r))\end{aligned}$$

Hence,

$$|D\mathbb{1}_{E\cap B(x,r)}|(\mathbb{R}^n) \leq |D\mathbb{1}_E|(B(x,r)) + \mathcal{H}^{n-1}(E \cap \partial B(x,r)) \quad (*)$$

On the other hand, choose $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $\varphi = \nu_E(x)$ on $B(x,r)$. From the proof of lemma 2.1.1 we get

$$\int_{B(x,r)} \nu_E(x) \cdot \nu_E d|D\mathbb{1}_E| = \int_{E\cap B(x,r)} div(\nu_E(x))\, dy - \int_{E\cap\partial B(x,r)} \nu_E(x) \cdot \nu d\mathcal{H}^{n-1} \qquad (**)$$

Note that $\mathrm{div}(\nu_E(x))dy = 0$.

Since $x \in \partial^* E$,

$$\lim_{r\to 0} \nu_E(x) \cdot \fint_{B(x,r)} \nu_E d|D\mathbb{1}_E| = |\nu_E(x)|^2 = 1$$

Thus for $\lambda - a.e$ and sufficiently small $r > 0$, say $0 < r < r_0 = r_0(x)$, we have

$$\frac{\nu_E(x) \cdot \int_{B(x,r)} \nu_E d|D\mathbb{1}_E|}{|D\mathbb{1}_E|(B(x,r))} \geq \frac{1}{2}.$$

Hence $(**)$ implies

$$\frac{1}{2}|D\mathbb{1}_E|(B(x,r)) \leq \mathcal{H}^{n-1}(E \cap \partial B(x,r)) \qquad (***)$$

This and $(*)$ give

$$|D\mathbb{1}_{E\cap B(x,r)}|(\mathbb{R}^n) \leq 3\mathcal{H}^{n-1}(E \cap \partial B(x,r)) \quad (****)$$

for a.e $0 < r < r_0$.

Step 2: Write $g(r) = |B(x,r) \cap E|$ then $g(r) = \int_0^r \mathcal{H}^{n-1}(\partial B(x,s) \cap E)ds$. It is absolutely continuous, and $g'(r) = \mathcal{H}^{n-1}(\partial B(x,r) \cap E)$ for a.e $r > 0$. Using the isoperimetric inequality 1.6.7 and $(****)$, we compute

$$g(r)^{1-\frac{1}{n}} = |B(x,r) \cap E|^{1-\frac{1}{n}} \leq c_1|D\mathbb{1}_{(B(x,r)\cap E)}|(\mathbb{R}^n) \leq C_1 g'(r)$$

for $a.e \quad r \in (0, r_0)$ Thus

$$\frac{1}{C_1} \le g(r)^{\frac{1}{n}-1} g'(r) = n(g^{\frac{1}{n}}(r))'$$

implying that

$$(g^{\frac{1}{n}}(r))' \ge \frac{1}{C_1 n}$$

Hence $g^{\frac{1}{n}}(r) \ge \frac{r}{C_1 n}$ and $g(r) \ge \frac{r^n}{(C_1 n)^n}$ for $0 < r < r_0$ .Therefore

$$\frac{g(r)}{r^n} = \frac{|B(x,r) \cap E|}{r^n} \ge \frac{1}{(C_1 n)^n}$$

for r sufficiently small. This proves (1).

Step 3: Since for all $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$

$$\int_E div\varphi \, dx + \int_{\mathbb{R}^n \setminus E} div\varphi \, dx = \int_{\mathbb{R}^n} div\varphi \, dx = 0$$

it follows that $|D\mathbb{1}_E| = |D\mathbb{1}_{\mathbb{R}^n \setminus E}|$ with $\nu_E = -\nu_{\mathbb{R}^n \setminus E}$ then statement (2) follows from (1), by taking $g(r) = |B(x,r) \cap E^c|$.

Step 4: According to the relative isoperimetric inequality 1.6.7,

$$c \min \left\{ \frac{|B(x,r) \cap E|}{r^n}, \frac{|B(x,r) \cap E^c|}{r^n} \right\}^{1-\frac{1}{n}} \le \frac{|D\mathbb{1}_E|(B(x,r))}{r^{n-1}}.$$

Hence (3) follows from (1) and (2).

Step 5: By $(***)$,

$$|D\mathbb{1}_E|(B(x,r)) \le 2\mathcal{H}^{n-1}(E \cap \partial B(x,r)) \le Cr^{n-1} \quad (0 < r < r_0)$$

this is (4).

Step 6: Statement (5) is a consequence of $(*)$ and (4).

□

2.1.1 *Blow up*

Definition 2.1.2. *For each $x \in \partial^* E$, define the hyperplane*

$$H(x) = \{y \in \mathbb{R}^n \mid \nu_E(x) \cdot (y - x) = 0\}$$

and the half spaces

$$H^+ = \{y \in \mathbb{R}^n \mid \nu_E(x) \cdot (y - x) \ge 0\}.$$

$$H^-(x) = \{y \in \mathbb{R}^n \mid \nu_E(x) \cdot (y - x) \le 0\}$$

Notation. *Fix $x \in \partial^* E, r > 0$, and set*

$$E_r = \{y \in \mathbb{R}^n \mid r(y-x) + x \in E\}.$$

Remark 2.1.2. *Observe $y \in E \cap B(x,r)$ if and only if $g_r(y) \in E_r \cap B(x,1)$ where $g_r(y) = (\frac{y-x}{r}) + x$.*

Theorem 2.1.1 (Blow up of reduced boundary)**.** *Assume $x \in \partial^* E$. Then*

$$\mathbb{1}_{E_r} \to \mathbb{1}_{H^-(x)} \quad in\, L^1_{loc}(\mathbb{R}^n) \qquad as \quad r \to 0$$

Thus for small enough $r > 0, E \cap B(x,r)$ approximately equals the half ball $H^-(x) \cap B(x,r)$.

Proof. 1. First of all we may assume

$$\begin{cases} x = 0,\, \nu(0) = e_n = (0, \cdots, 0, 1) \\ H(0) = \{y \in \mathbb{R}^n, y_n = 0\} \\ H^+(0) = \{y \in \mathbb{R}^n, y_n \geq 0\} \\ H^-(0) = \{y \in \mathbb{R}^n, y_n \leq 0\} \end{cases}$$

2. Choose any sequence $r_k \to 0$. It will be enough to show that there exists a subsequence $(s_j)_j \subset (r_k)_k$ for which $\mathbb{1}_{E_{s_j}} \to \mathbb{1}_{H^-(0)}$ in $L^1_{Loc}(\mathbb{R}^n)$.
3. Fix $L > 0$, and let $D_r = E_r \cap B(0,L), g_r(y) = \frac{y}{r}$. Then for any $\varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), \|\varphi\|_\infty \leq 1$. We have

$$\begin{aligned} \int_{D_r} div\varphi, dz &= \frac{1}{r^{n-1}} \int_{E \cap B(0,rL)} div(\varphi \circ g_r) dy \\ &= \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} (\varphi \circ g_r) \cdot \nu_{E \cap B(0,rL)} d|D\mathbb{1}_{E \cap B(0,rL)}| \\ &\leq \left| \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} (\varphi \circ g_r) \cdot \nu_{E \cap B(0,rL)} d|D\mathbb{1}_{E \cap B(0,rL)}| \right| \\ &\leq \frac{|D\mathbb{1}_{E \cap B(0,rL)}|(\mathbb{R}^n)}{r^{n-1}} \\ &\leq c < \infty \end{aligned}$$

for all $r \in (0,1]$, according to lemma 2.1.2 (5).

Consequently,

$$|D\mathbb{1}_{D_r}|(\mathbb{R}^n) \leq c < \infty \quad (0 < r \leq 1)$$

and furthermore,

$$\|\mathbb{1}_{D_r}\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \mathbb{1}_{D_r} dx = |D_r| \leq |B(0,L)| < \infty$$

Hence $\|\mathbb{1}_{D_r}\|_{BV(\mathbb{R}^n)} = \|\mathbb{1}_{D_r}\|_{L^1(\mathbb{R}^n)} + |D\mathbb{1}_{D_r}|(\mathbb{R}^n) < \infty$, for all $0 < r \leq 1$. So by the compactness theorem 1.5.4 there exists a subsequence $(s_j)_{j\geq 1} \subset (r_k)_{k\geq 1}$ and a function $f \in BV_{loc}(\mathbb{R}^n)$ such that, writing $E_j = E_{s_j}$, we have

$$\mathbb{1}_{E_j} \to f \text{ in } L^1_{loc}(\mathbb{R}^n)$$

we may also assume $\mathbb{1}_{E_j} \to f$ λ-a.e. Hence $f(x) \in \{0, 1\}$ for $\lambda-$ae x and so

$$f = \mathbb{1}_F \quad \lambda - a.e$$

where $F \subset \mathbb{R}^n$ has locally finite perimeter. Hence if $\varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int_F div\varphi \, dy = \int_{\mathbb{R}^n} \varphi \cdot \nu_F d|D\mathbb{1}_F| \qquad (*)$$

for some $|D\mathbb{1}_F|$-measurable function ν_F, with $|\nu_F| = 1 \, |D\mathbb{1}_F|-$ae.

We must prove $F = H^-(0)$.

4. Claim 1: $\nu_F = e_n = \nu_E(0) \quad |D\mathbb{1}_F|-$ae.

   Proof of claim 1: Let us write $\nu_j = \nu_{E_j}$ then if $\varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \varphi \cdot \nu_j d|D\mathbb{1}_{E_j}| = \int_{E_j} div\varphi \, dy \quad (j = 1, 2 \cdots).$$

   Since $\mathbb{1}_{E_j} \to \mathbb{1}_F$ in L^1_{loc}, we see from the above and $(*)$ that

$$\int_{\mathbb{R}^n} \varphi \cdot \nu_j d|D\mathbb{1}_{E_j}| \underset{j\to\infty}{\longrightarrow} \int_{\mathbb{R}^n} \varphi \cdot \nu_F d|D\mathbb{1}_F|$$

   Thus $\nu_j|D\mathbb{1}_{E_j}| \to \nu_F|D\mathbb{1}_F|$ weakly in the sense of Radon measures.Consequently,by 1.1.2 for each $L > 0$, for which $|D\mathbb{1}_F|(\partial B(0, L)) = 0$, hence for all but at most countably many $L > 0$ we have

$$\int_{B(0,L)} \nu_j d|D\mathbb{1}_{E_j}| \underset{j\to\infty}{\longrightarrow} \int_{B(0,L)} \nu_F d|D\mathbb{1}_F| \quad (**)$$

   On the other hand, for all φ as above

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi \cdot \nu_j d|D\mathbb{1}_{E_j}| &= \int_{\mathbb{R}^n} \mathbb{1}_{E_j} div\varphi \, dx \\ &= \int_{E_j} div\varphi \, dx \\ &= \frac{1}{s_j^{n-1}} \int_E div(\varphi \circ g_{s_j}) dy \\ &= \frac{1}{s_j^{n-1}} \int_{\mathbb{R}^n} (\varphi \circ g_{s_j}) \cdot \nu_E d|D\mathbb{1}_E| \end{aligned}$$

$$\begin{aligned}
\int_{B(0,L)} \varphi \cdot \nu_j d|D\mathbb{1}_{E_j}| &= \int_{B(0,L)\cap E_j} div\varphi \, dx \\
&= \frac{1}{s_j^{n-1}} \int_{B(0,s_jL)\cap E} div(\varphi \circ g_{s_j}) dy \\
&= \frac{1}{s_j^{n-1}} \int_{B(0,s_jL)} (\varphi \circ g_{s_j}) \cdot \nu_E d|D\mathbb{1}_E|
\end{aligned}$$

Whence

$$(***) \begin{cases} |D\mathbb{1}_{E_j}|(B(0,L)) = \frac{1}{s_j^{n-1}}|D\mathbb{1}_E|(B(0,s_jL)) \\ \int_{B(0,L)} \nu_j d|D\mathbb{1}_{E_j}| = \frac{1}{s_j^{n-1}} \int_{B(0,s_jL)} \nu_E d|D\mathbb{1}_E| \end{cases}$$

Therefore

$$\begin{aligned}
\lim_{j\to\infty} ⨍_{B(0,L)} \nu_j d|D\mathbb{1}_{E_j}| &= \lim_{j\to\infty} ⨍_{B(0,s_jL)} \nu_E d|D\mathbb{1}_E| \text{ (by (***))} \\
&= \nu_E(0) = e_n \text{ since } 0 \in \partial^* E.
\end{aligned}$$

If $|D\mathbb{1}_E|(\partial B(0,L)) = 0$, by lower semi-continuity theorem 1.5.1

$$\begin{aligned}
|D\mathbb{1}_F|(B(0,L)) &\leq \liminf_{j\to\infty} |D\mathbb{1}_{E_j}|(B(0,L)) \\
&= \liminf_{j\to\infty} \int_{B(0,L)} \nu_j d|D\mathbb{1}_{E_j}| \\
&= \lim_{j\to\infty} \int_{B(0,L)} e_n \cdot \nu_j d|D\mathbb{1}_{E_j}| \\
&= \int_{B(0,L)} e_n \cdot \nu_F d|D\mathbb{1}_F| \quad by\,(**)
\end{aligned}$$

Since $|\nu_F| = 1$, $|D\mathbb{1}_F|-$a.e, the above inequality forces $\nu_F = e_n$, $|D\mathbb{1}_F| - a.e.$ In fact, having $\int_{B(0,L)} d|D\mathbb{1}_F| \leq \int_{B(0,L)} e_n \cdot \nu_F d|D\mathbb{1}_F|$, we get

$$\int_{B(0,L)} (1 - e_n \cdot \nu_F) d|D\mathbb{1}_F| \leq 0$$

and $e_n \cdot \nu_F \leq |e_n| \cdot |\nu_F| = 1$ so $(1 - e_n \cdot \nu_F) \geq 0$, hence $\int_{B(0,L)}(1 - e_n \cdot \nu_F) d|D\mathbb{1}_F| = 0$, and therefore $1 - e_n \cdot \nu_F = 0\, a.e$ thus $e_n = \nu_F\, a.e.$

It also follows from the above inequality that

$$|D\mathbb{1}_F|(B(0,L)) = \lim_{j\to\infty} |D\mathbb{1}_{E_j}|(B(0,L))$$

whenever $|D\mathbb{1}_F|(\partial B(0,L)) = 0$. □

5. Claim 2: F is a half space.

Proof of claim 2: By claim 1, for all $\varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, $\int_F div\varphi\, dz = \int_{\mathbb{R}^n} \varphi \cdot e_n d|D\mathbb{1}_F|$. Fix $\epsilon > 0$, and let $f^\epsilon = \eta_\epsilon * \mathbb{1}_F \in C^\infty(\mathbb{R}^n)$, where η_ϵ is the usual mollifier. So

$$\begin{aligned}\int_{\mathbb{R}^n} f^\epsilon div\varphi\, dz &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \eta_\epsilon(x-z)\mathbb{1}_F(x) div\varphi(z)\, dxdz \\ &= \int_F \int_{\mathbb{R}^n} \eta_\epsilon(x-z) div\varphi(z)\, dzdx \\ &= \int_F \eta_\epsilon * div\varphi dx \\ &= \int_F div(\eta_\epsilon * \varphi) dz \\ &= \int_{\mathbb{R}^n} (\eta_\epsilon * \varphi) \cdot e_n d|D\mathbb{1}_F|.\end{aligned}$$

But $Df^\epsilon = \eta_\epsilon * D\mathbb{1}_F$, and $f^\epsilon \in C^\infty(\mathbb{R}^n)$ hence

$$\int_{\mathbb{R}^n} f^\epsilon div\varphi\, dz = -\int_{\mathbb{R}^n} \varphi d D f^\epsilon = -\int_{\mathbb{R}^n} \varphi \cdot \nabla f^\epsilon dz.$$

Thus

$$\frac{\partial f^\epsilon}{\partial z_i} = 0\, (i = 1, \cdots, n-1), \frac{\partial f^\epsilon}{\partial z_n} \leq 0.$$

As $f_\epsilon \to \mathbb{1}_F\, \lambda-$a.e when $\epsilon \to 0$, we conclude that up to set of measure zero $F = \{y \in \mathbb{R}^n \mid y_n \leq \gamma\}$ for some $\gamma \in \mathbb{R}$. □

6. Claim 3: $F = H^-(0)$

Proof of Claim 3: We must show $\gamma = 0$. Assume $\gamma > 0$. Since $\mathbb{1}_{E_j} \to \mathbb{1}_F$ in $L^1_{loc}(\mathbb{R}^n)$, we have $B(0,\gamma) = B(0,\gamma) \cap F$.

$$\alpha(n)\gamma^n = |B(0,\gamma)| = |B(0,\gamma) \cap F| = \lim_{j\to\infty} |B(0,\gamma) \cap E_j| = \lim_{j\to\infty} \frac{|B(0,\gamma s_j) \cap E|}{s_j^n}$$

Thus

$$\lim_{j\to\infty} \frac{|B(0,\gamma s_j) \cap E|}{s_j^n \alpha(n)\gamma^n} = 1.$$

But we know that

$$\frac{|B(0,\gamma s_j) \cap E|}{s_j^n \alpha(n)\gamma^n} + \frac{|B(0,\gamma s_j) \cap E^c|}{s_j^n \alpha(n)\gamma^n} = 1$$

Contradiction to lemma 2.1.2 (2). Similarly , the case $\gamma < 0$, leads to contradiction to lemma 2.1.2(1). □

□

The following result describes the local behaviour of E around a point in ∂E.

Corollary 2.1.1. *[4] Assume $x \in \partial^* E$. Then*
(i)

$$\lim_{r\to 0} \frac{|B(x,r) \cap E \cap H^+(x)|}{r^n} = 0$$
$$\lim_{r\to 0} \frac{|(B(x,r) - E) \cap H^-(x)|}{r^n} = 0$$

(ii)

$$\lim_{r\to 0} \frac{|D\mathbb{1}_E|(B(x,r))}{\alpha(n-1)r^{n-1}} = 1$$

Definition 2.1.3. *A unit vector $\nu_E(x)$ for which (i) holds is called the measure theoretic unit outer normal to E at x.*

2.1.2 *Structure theorem for sets of finite perimeter*

Lemma 2.1.3. *[4] There exists a constant C,depending only on n, such that*

$$\mathcal{H}^{n-1}(B) \le C|D\mathbb{1}_E|(B)$$

for all $B \subset \partial^* E$.

Lemma 2.1.4 (Whitney's extension theorem). *[4] Let C be a closed set and assume $f : C \to \mathbb{R}$, $d : C \to \mathbb{R}$ continuous functions,and for each compact set $K \subset C$,*

$$\rho_k(\delta) = \sup\left\{ \frac{f(y) - f(x) - d(x)\cdot(y-x)}{|y-x|} \,\middle|\, 0 < |x-y| \le \delta,\ x, y \in K \right\} \to 0$$

as $\delta \to 0$.

Then there exists a function $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ such that

1. $\overline{f} \in C^1$.
2. $\overline{f} = f, D\overline{f} = d$ *on C.*

Theorem 2.1.2 (Structure theorem for sets of finite perimeter). *Assume E has locally finite perimeter in $\mathbb{R}^n$. Then*
(i)

$$\partial^* E = \bigcup_{k=1}^{\infty} K_k \cup N$$

where $|D\mathbb{1}_E|(N) = 0$ and K_k are compact subsets of C^1-hypersurfaces S_k .

(ii) $\nu_E|_{S_k}$ *is normal to* $S_k\,(k = 1, \cdots .)$.

(iii) $|D\mathbb{1}_E| = \mathcal{H}^{n-1}|_{\partial^* E}$

Proof. For each $x \in \partial^* E$, we have according to corollary 2.1.1,

$$\begin{cases} \lim_{r\to 0} \frac{|B(x,r)\cap E\cap H^+(x)|}{r^n} = 0 \\ \lim_{r\to 0} \frac{|(B(x,r)-E)\cap H^-(x)|}{r^n} = 0 \end{cases} \qquad (*)$$

Using Egoroff's Theorem, we see that there exist $|D\mathbb{1}_E|-$measurable sets F_n such that

$$|D\mathbb{1}_E|(\partial^* E \setminus F_n) < \frac{1}{n}.$$

Hence $|D\mathbb{1}_E|\big(\partial^* E \setminus \bigcup_{i=1}^{\infty} F_i\big) = 0$. Therefore we can find a sequence of disjoint $|D\mathbb{1}_E|-$measurable sets $\{F_i\}_{i=1}^{\infty} \subset \partial^* E$ such that

$$\begin{cases} |D\mathbb{1}_E|\big(\partial^* E \setminus \bigcup_{i=1}^{\infty} F_i\big) = 0, |D\mathbb{1}_E|(F_i) < \infty, \text{ and} \\ \text{The convergence in (2.1.2) is uniform for } x \in F_i,\ (i = 1, \cdots) \end{cases}$$

Then by Lusin's Theorem, for each i there exist disjoint compact sets $\{E_i^j\}_{j=1}^{\infty} \subset F_i$ such that

$$\begin{cases} |D\mathbb{1}_E|\big(F_i \setminus \bigcup_{i=1}^{\infty} E_i^j\big) = 0, \\ \nu_E|_{E_i^j} \text{ is continuous.} \end{cases}$$

Re-index the sets $\{E_i^j\}_{i,j}^{\infty}$ and call them $\{K_k\}_{k=1}^{\infty}$.Then letting

$$N = \partial^* E \setminus \bigcup_{k\geq 1} K_k,$$

we have

$$\begin{aligned} |D\mathbb{1}_E|\left(\partial^* E \setminus \bigcup_{k=1}^{\infty} K_k\right) &= |D\mathbb{1}_E|\left(\left(\partial^* E \setminus \bigcup_{i=1}^{\infty} F_i\right) \cup \left(\bigcup_{i=1}^{\infty} F_i \setminus \bigcup_{k=1}^{\infty} K_k\right)\right) \\ &\leq |D\mathbb{1}_E|\left(\partial^* E \setminus \bigcup_{i=1}^{\infty} F_i\right) + \sum_{i=1}^{\infty} |D\mathbb{1}_E|\left(F_i \setminus \bigcup_{k=1}^{\infty} K_k\right) \\ &= 0. \end{aligned}$$

Then,

$$\begin{cases} \partial^* E = \bigcup_{k=1}^{\infty} K_k \cup N,\ |D\mathbb{1}_E|(N) = 0, \\ \text{the convergence in (2.1.2) is uniform on } K_k, (\text{since } K_k \subset F_i), \text{ and} \\ \nu_E|_{K_k} \text{is continuous } (k = 1, 2\cdots) \end{cases} \qquad (**)$$

Now define for $\delta > 0$,

$$\rho_k(\delta) = \sup\left\{\frac{|\nu_E(x)\cdot(y-x)|}{|y-x|} : 0 < |x-y| \leq \delta,\ x, y \in K_k\right\}$$

Claim 3: For each $k = 1, 2\cdots$, $\rho_k(\delta) \to 0$ as $\delta \to 0$.

Proof of claim 3: We may assume k=1. Fix $0 < \epsilon < 1$. By $(*)$ and $(**)$ there exist $0 < \delta < 1$ such that if $z \in K_1$ and $r < 2\delta$, then

$$\begin{cases} |B(x,r) \cap E \cap H^+(z)| < \frac{\epsilon^n}{2^{n+2}} \alpha(n) r^n \\ |B(x,r) \cap E \cap H^-(z)| > \alpha(n)\big(\frac{1}{2} - \frac{\epsilon^n}{2^{n+2}}\big) r^n \end{cases} \qquad (***)$$

To prove Claim 3, we shall prove that: for every $x, y \in K_1$, such that $|x-y| < \delta$, we have $|\frac{\nu_E.(y-x)}{|y-x|}| < \epsilon$, hence we get $\sup |\frac{\nu_E.(y-x)}{|y-x|}| < \epsilon$. Since this is true for arbitrary epsilon we get that $\rho_k(\delta) \to 0$ as $\delta \to 0$.

Assume now $x, y \in K_1, 0 < |x-y| \le \delta$.

Case 1. $\nu_E(x) \cdot (y-x) > \epsilon|y-x|$. Since $\epsilon < 1$,

$$B(y, \epsilon|x-y|) \subset H^+(x) \cap B(x, 2|x-y|) \qquad (****)$$

To see this observe that if $z \in B(y, \epsilon|x-y|)$, then $z = y + w$, where $|w| \le |x-y|$, whence

$$\nu_E(x) \cdot (z-x) = \nu_E(x) \cdot (y-x) + \nu_E(x) \cdot w > \epsilon|x-y| - |w| \ge 0$$

with $-|w| \le \nu_E \cdot w \le |w|$. Therefore $z \in H^+(x)$ and $|z-x| \le |y-x| + |w| \le |y-x| + \epsilon|x-y| < 2|x-y|$. On the other hand, $(***)$ with $z = x$ and $r = 2|x-y|$ implies,

$$\begin{aligned} |E \cap B(x, 2|x-y|) \cap H^+(x)| &< \frac{\epsilon^n}{2^{n+2}} \alpha(n)(2|x-y|)^n \\ &= \frac{\epsilon^n \alpha(n)}{4} |x-y|^n \end{aligned}$$

and $(***)$ with $z = y$ implies,

$$\begin{aligned} |E \cap B(y, \epsilon|x-y|)| &\ge |E \cap B(y, \epsilon|x-y|) \cap H^-(y)) \\ &\ge \frac{\epsilon^n |x-y|^n \alpha(n)}{2} \big(1 - \frac{\epsilon^n}{2^{n+1}}\big) \\ &> \frac{\epsilon^n \alpha(n)}{4} |x-y|^n \end{aligned}$$

However, by applying $\lambda|_E$ to both sides of $(****)$ we get a contradiction.

Case 2. $\nu_E(x) \cdot (y-x) \le -\epsilon|y-x|$.

This similarly leads to a contradiction. □

Now we apply Whitney's extension theorem 2.1.4 with $f = 0$, $d = \nu_E$ on K_k, to get $\overline{f_k} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{cases} \overline{f_k} = 0 \ \text{ on } K_k \\ D\overline{f_k} = d = \nu_E \ \text{ on } K_k \end{cases}$$

Let $S_k = \{x \in \mathbb{R}^n \mid \overline{f_k} = 0 \ \ \& \ \ |D\overline{f_k}| > \frac{1}{2}\}$ $k = 1, 2, \cdots$, that is the pre-image of 0 where 0 is a regular point. So by the implicit function theorem S_k is a C^1

$(n-1)$-dimensional sub-manifold of $\mathbb{R}^n$. Hence $\nu_E|_{S_k} = D\overline{f_k}|_{S_k}$ exists and is normal since $K_k \subset S_k$.This proves (i) and (ii).

To prove (iii) choose a Borel set $B \subset \partial^* E$ and prove $|D\mathbb{1}_E|(B) = \mathcal{H}^{n-1}(B)$. According to the previous lemma 2.1.3,

$$\mathcal{H}^{n-1}(B \cap N) \leq C|D\mathbb{1}_E|(B \cap N) = 0.$$

Thus we may as well assume $B \subset \cup_{k=1}^{\infty} K_k$, and in fact $B \subset K_1$ (since K_i's are disjoint). We have just proved that there exist a C^1-hypersurface $K_1 \subset S_1$. Let

$$\nu = \mathcal{H}^{n-1}|_{S_1}.$$

Since S_1 is C^1, by corollary 2.1.1

$$\lim_{r\to 0} \frac{\nu(B(x,r))}{\alpha(n-1)r^{n-1}} = \lim_{r\to 0} \frac{\mathcal{H}^{n-1}(B(x,r) \cap S_1)}{\mathcal{H}^{n-1}(B(x,r)} = \lim_{r\to 0} \frac{\mathcal{H}^{n-1}(B(x,r) \cap S_1)}{\alpha(n-1)r^{n-1}} = 1 \quad (x \in B).$$

Thus corollary 2.1.1 (ii) implies that

$$\lim_{r\to 0} \frac{\nu(B(x,r))}{\alpha(n-1)r^{n-1}} = \lim_{r\to 0} \frac{|D\mathbb{1}_E|(B(x,r))}{\alpha(n-1)r^{n-1}}$$

then

$$\lim_{r\to 0} \frac{\nu(B(x,r))}{|D\mathbb{1}_E|(B(x,r))} = 1 \qquad (x \in B).$$

Also, we have that $\nu << |D\mathbb{1}_E|$. In fact let $A \subset \partial^* E$, $|D\mathbb{1}_E|(A) = 0$, from Lemma 2.1.3

$$\mathcal{H}^{n-1}(A) \leq c|D\mathbb{1}_E|(A) = 0$$

but $\nu(A) = \mathcal{H}^{n-1}(A \cap S_1) \leq \mathcal{H}^{n-1}(A)$ therefore $\nu(A) = 0$. Hence since ν and $|D\mathbb{1}_E|$ are Radon measures, with $\nu << |D\mathbb{1}_E|$ [4, Theorem 2, section 1.6.2] implies

$$\begin{aligned}\nu(B) = \mathcal{H}^{n-1}(B \cap S_1) = \int_B \nu_E dD\mathbb{1}_E = \int_E d|D\mathbb{1}_E| \\ = H^{n-1}(B) = \int_B d|D\mathbb{1}_E| = |D\mathbb{1}_E|(B).\end{aligned}$$

Remark 2.1.3. *If ∂E is C^1 then $\partial^* E = \partial E$ and $\nu(x)$ the unit normal vector to ∂E at x coincides with the measure theoretic normal $\nu_E(x)$.*

□

2.2 The measure theoretic boundary

Definition 2.2.1. *We define the measure theoretic boundary of E, $\partial_M E$, as*

$$\partial_M E = \left\{ x : 0 < \limsup_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} \right\} \cap \left\{ x : \liminf_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} < 1 \right\}$$

Remark 2.2.1. *$\partial_M E = \mathbb{R}^n - (E^0 \cup E^1)$ with E^1 and E^0 are respectively the measure theoretic interior and exterior. $\partial_M E$ is the set of points where the density is neither 0 nor 1.*

Lemma 2.2.1. *(i) $\partial^* E \subset \partial_M E$.*

(ii) $\mathcal{H}^{n-1}(\partial_M E \setminus \partial^* E) = 0$.

Proof. (i) Follows from lemma 2.1.2. Let $x \in \partial^* E$,

$$0 < A_1 < \liminf_{r \to 0} \frac{|E \cap B(x,r)|}{r^n} \leq \limsup_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|}$$

and

$$\liminf_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = 1 - \liminf_{r \to 0} \frac{|E^c \cap B(x,r)|}{|B(x,r)|}$$

But $\liminf_{r \to 0} \frac{|E^c \cap B(x,r)|}{|B(x,r)|} > 0$, then $\liminf_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} < 1$. Therefore $x \in \partial_M E$.

(ii) Since the mapping

$$r \mapsto \frac{|B(x,r) \cap E|}{r^n}$$

is continuous, if $x \in \partial_M E$, there exists $0 < \alpha < 1$ and $r_j \to 0$ such that

$$\lim_{j \to \infty} \frac{|B(x,r_j) \cap E|}{\alpha(n) r_j^n} = \alpha,$$

but we have

$$|B(x,r_j)| = |B(x,r_j) \cap E^c| + |B(x,r_j) \cap E| = \alpha(n) r_j^n$$

then we get

$$\frac{|B(x,r_j) \cap E^c|}{\alpha(n) r_j^n} = \frac{\alpha(n) r_j^n - |B(x,r_j) \cap E|}{\alpha(n) r_j^n} = 1 - \alpha_j$$

Therefore,

$$\min\{|B(x,r_j) \cap E|, |B(x,r_j) \cap E^c|\} = \min\{\alpha_j, 1 - \alpha_j\} \alpha(n) r_j^n$$

By the relative Isoperimetric inequality1.6.7,

$$\begin{aligned} \min\{|B(x,r_j) \cap E|, |B(x,r_j) \cap E^c|\}^{\frac{n-1}{n}} &= (\min\{\alpha_j, 1-\alpha_j\}\alpha(n))^{\frac{n-1}{n}} r_j^{n-1} \\ &\leq 2c_2 |D\mathbb{1}_E|(U(x,r_j)) \end{aligned}$$

Therefore,

$$\frac{(\min\{\alpha_j, 1-\alpha_j\}\alpha(n))^{\frac{n-1}{n}}}{2c_2} \leq \frac{|D\mathbb{1}_E|(U(x,r_j))}{r_j^{n-1}}$$

Hence

$$\limsup_{r\to 0} \frac{(\min\{\alpha_j, 1-\alpha_j\}\alpha(n))^{\frac{n-1}{n}}}{2c_2} \le \limsup_{r\to 0} \frac{|D\mathbb{1}_E|(U(x,r_j))}{r_j^{n-1}}.$$

By definition of measure theoretic boundary we get

$$\limsup_{r\to 0} \frac{(\min\{\alpha_j, 1-\alpha_j\}\alpha(n))^{\frac{n-1}{n}}}{2c_2} > 0$$

and therefore

$$\limsup_{r\to 0} \frac{|D\mathbb{1}_E|(B(x,r)}{r^{n-1}} > 0.$$

Now to prove $\mathcal{H}^{n-1}(\partial_M E \backslash \partial^* E) = 0$, we have $x \in \partial_M E$ and $\limsup_{r\to 0} \frac{|D\mathbb{1}_E|(B(x,r)}{r^{n-1}} > 0$. Fix $\delta > 0$, let

$$F = \left\{ B(x,r) \middle| x \in \partial_M E \setminus \partial^* E, B(x,r) \subset \mathbb{R}^n \setminus \partial^* E, r < \frac{\delta}{10} \;\&\; |D\mathbb{1}_E|(B(x,r)) > c\, r^{n-1} \right\}.$$

By Vitali's covering theorem, there exist a countable disjoint family of balls in F such that $\partial_M E \setminus \partial^* E \subset \bigcup_{i\ge 1} B(x_i, 5r_i)$. Therefore,

$$\begin{aligned}
\mathcal{H}^{n-1}_{10\delta}(\partial_M E \setminus \partial^* E) &\le c \sum_{i\ge 1} \alpha(n-1)(5r_i)^{n-1} \\
&\le C \sum_{i\ge 1} (r_i)^{n-1} \\
&\le c' \sum_{i\ge 1} |D\mathbb{1}_E|(B(x_i, r_i)) \\
&= c' |D\mathbb{1}_E|(\bigcup_{i\ge 1} B(x_i, r_i)) \\
&\le c' |D\mathbb{1}_E|(\mathbb{R}^n \setminus \partial^* E) \\
&= 0
\end{aligned}$$

let $\delta \to 0$, we get $\mathcal{H}^{n-1}(\partial_M E \setminus \partial^* E) = 0$.

□

Proposition 2.2.1. *Let E be a set of finite perimeter,*

$$|D\mathbb{1}_E|(\Omega) = P(E,\Omega) = \mathcal{H}^{n-1}(\Omega \cap \partial_M E) = \mathcal{H}^{n-1}(\Omega \cap \partial^* E)$$

Proof. Since $\partial^* E \subset \partial_M E$ then $\mathcal{H}^{n-1}(\Omega \cap \partial^* E) \le \mathcal{H}^{n-1}(\Omega \cap \partial_M E)$. I need to prove that $\mathcal{H}^{n-1}(\Omega \cap \partial_M E) \le \mathcal{H}^{n-1}(\Omega \cap \partial^* E)$. By Lemma 2.2.1 $\mathcal{H}^{n-1}(\partial_M E \cap (\partial^* E)^c) = \mathcal{H}^{n-1}(\partial_M E \setminus \partial^* E) = 0$. Hence

$$\begin{aligned}
\mathcal{H}^{n-1}(\Omega \cap \partial_M E) &\le \mathcal{H}^{n-1}(\Omega \cap \partial_M E \cap \partial^* E) + \mathcal{H}^{n-1}(\Omega \cap \partial_M E \cap (\partial^* E)^c) \\
&\le \mathcal{H}^{n-1}(\Omega \cap \partial^* E) + \mathcal{H}^{n-1}(\partial_M E \cap (\partial^* E)^c) \\
&\le \mathcal{H}^{n-1}(\Omega \cap \partial^* E)
\end{aligned}$$

Thus $\mathcal{H}^{n-1}(\Omega \cap \partial^* E) = \mathcal{H}^{n-1}(\Omega \cap \partial_M E)$. By the structure theorem 2.1.2, $P(E,\Omega) = \mathcal{H}^{n-1}(\Omega \cap \partial^* E)$. Therefore we get the equality. □

Theorem 2.2.1 (Generalized Gauss-Green Theorem). *Let $E \subset \mathbb{R}^n$ have locally finite perimeter. Then*

(i) $\mathcal{H}^{n-1}(\partial_M E \cap K) < \infty$ *for each compact set* $K \subset R^n$.

(ii) For $\mathcal{H}^{n-1}$*-a.e* $x \in \partial_M E$*,there is a unique measure theoretic unit outer normal* $\nu_E(x)$*, such that*

$$\int_E div\varphi\, dx = \int_{\partial_M E} \varphi \cdot \nu_E d\mathcal{H}^{n-1} \qquad (*)$$

for all $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$.

Proof. We know that

$$\begin{aligned}\int_E div\varphi\, dx &= \int_{\mathbb{R}^n} \varphi \cdot \nu_E d|D\mathbb{1}_E| \\ &= \int_{\partial^* E} \varphi \cdot \nu_E d|D\mathbb{1}_E| + \int_{\mathbb{R}^n \setminus \partial^* E} \varphi \cdot \nu_E d|D\mathbb{1}_E|\end{aligned}$$

But $|D\mathbb{1}_E|(\mathbb{R}^n \setminus \partial^* E) = 0$, and $|D\mathbb{1}_E| = \mathcal{H}^{n-1}|_{\partial_M E} = \mathcal{H}^{n-1}|_{\partial^* E}$. Hence

$$\int_E div\varphi dx = \int_{\partial^* E} \varphi \cdot \nu_E d|D\mathbb{1}_E| = \int_{\partial^* E} \varphi \cdot \nu_E d\mathcal{H}^{n-1} = \int_{\partial_M E} \varphi \cdot \nu_E d\mathcal{H}^{n-1}$$

□

Proposition 2.2.2. *E is of finite perimeter if and only if* $\mathcal{H}^{n-1}(\partial_M E) < \infty$.

Proof. If E is of finite perimeter then $P(E, \mathbb{R}^n) = \mathcal{H}^{n-1}(\partial_M E)) < \infty$. Conversely, assume $\mathcal{H}^{n-1}(\partial_M E) < \infty$. Let $\varphi \in C_c^1(\mathbb{R}^n)$ such that $\|\varphi\|_\infty < 1$, by the generalised Gauss-Green theorem

$$\begin{aligned}\left|\int_E div\varphi dx\right| &= \left|\int_{\partial_M E} \varphi \cdot \nu_E d\mathcal{H}^{n-1}\right| \\ &\leq \int_{\partial_M E} |\varphi \cdot \nu_E| d\mathcal{H}^{n-1} \\ &\leq \mathcal{H}^{n-1}(\partial_M E) < \infty.\end{aligned}$$

Therefore E is of finite perimeter. □

Remark 2.2.2. *The equality* $P(E, \Omega) = \mathcal{H}^{n-1}(\Omega \cap \partial_M E) = \mathcal{H}^{n-1}(\Omega \cap \partial^* E)$ *whenever* $P(E, \Omega) < \infty$*, implies that sets of finite perimeter are defined only up to sets of measure zero. In other words, each set determines an equivalence class of sets of finite perimeter. In order to avoid this ambiguity, whenever a set of finite perimeter E is considered, we shall always employ the measure theoretic closure as the set to represent E. Thus with this convention, we have*

$$x \in E \textit{ if and only if } \limsup_{r \to 0} \frac{|E \cap B(x, r)|}{|B(x, r)|} > 0 \tag{2.1}$$

Proposition 2.2.3.

$$\overline{\partial^* E} = \partial E$$

with the convention: $x \in E$ *if and only if* $\limsup_{r\to 0} \frac{|E\cap B(x,r)|}{|B(x,r)|} > 0$.

Proof. To prove that $\partial^* E$ dense in ∂E, I need to show that $\forall x \in \partial E, \forall r > 0, B(x,r) \cap \partial^* E \neq \phi$.

Claim : For any ball B, if $B \cap \partial^* E = \emptyset$, then $B \cap \partial E = \emptyset$.

Hence by this claim, for $x \in \partial E, \forall r > 0, B(x,r) \cap \partial E \neq \emptyset$,we get $B(x,r) \cap \partial^* E \neq \emptyset$. Therefore $\overline{\partial^* E} = \partial E$.

Proof of claim : Let B be an open a ball such that $B \cap \partial^* E = \emptyset \quad (\partial^* E \subset B^c)$. We have

$$P(E,B) = |D\mathbb{1}_E|(B) = \mathcal{H}^{n-1}(B \cap \partial^* E) = \mathcal{H}^{n-1}(\emptyset) = 0.$$

Therefore $|D\mathbb{1}_E|(B) = \int_B d|D\mathbb{1}_E| = 0$. Hence

$$|D\mathbb{1}_E| = 0 \text{ a.e on } B$$

and $D\mathbb{1}_E = \nu_E |D\mathbb{1}_E| = 0$ a.e on B. Since B is connected we get $\mathbb{1}_E = cst$ a.e on B (0 or 1),hence $E \subset B$ or $E \subset B^c$.Using the convention we get

$$\partial_M E \subset E$$

so $\partial^* E \subset \partial_M E \subset E$. But $\partial^* E \subset B^c$, then E cannot be included in B for otherwise $\partial^* E \subset B$. Hence $E \subset B^c$ and $E \cap B = \phi$, therefore $\overline{E} \subset \overline{B^c} = B^c$. But $\partial E \subset \overline{E} \subset B^c$, thus $\partial E \cap B = \emptyset$ and claim is proved. □

□

Chapter 3

Minimal Surfaces

3.1 Minimal surfaces

Minimal surfaces are surfaces in space which locally minimize the area, in the sense that any small enough piece of the surface has the smallest area among all surfaces with the same boundary.In this chapter, we will outline certain characteristics of minimal surfaces, crucial for proving the existence of a solution to the Least Gradient Problem in chapter 4.These characteristics will be listed without proof, serving as foundational knowledge for our later analyses.

Definition 3.1.1. *A surface M is said to be a minimal surface, if at each point,its mean curvature H is zero.*

If we have a surface that is obtained as the graph of a function $z = f(x, y)$, take a parametrization $x(u, v) = (u, v, f(u, v))$ we get

$$H = \frac{(1 + f_v^2)f_{uu} + (1 + f_u^2)f_{vv} - 2f_u f_v f_{uv}}{2(1 + f_u^2 + f_v^2)^{\frac{3}{2}}}$$

By this formula we get the following proposition.

Proposition 3.1.1. *[6] $M \subset \mathbb{R}^3$ is minimal if and only if*

$$(1 + f_v^2)f_{uu} + (1 + f_u^2)f_{vv} - 2f_u f_v f_{uv} = 0$$

This is called the minimal surface equation.

Its divergence form is given by

$$div(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}) = 0$$

Example 3.1.1 (Examples of minimal surfaces in $\mathbb{R}^3$)**.** *[6]*

1. *(Catenoid) A catenoid is a surface of revolution generated by a catenary $y(x) = cosh(x)$ and parameterized by $x(u, v) = (u, cosh(u)cos(v), cosh(u)sin(v))$. It has a mean curvature $H = 0$.*

2. *(Helicoid) The mean curvature for the helicoid parameterized by* $x(u,v) = (v\cos(u), v\sin(u), u)$ *is also 0*

Proposition 3.1.2. *Minimal surfaces in* $\mathbb{R}^2$ *are straight lines.*

Proof. Clearly from definition 3.1.1 straight lines are minimal surfaces in $\mathbb{R}^2$.

Conversely, let C be a minimal surface in $\mathbb{R}^2$ i.e a curve with endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$, we want to prove that C is a straight line. The curve C can be parameterized by $\gamma(t) = (x(t), y(t))$ with $x(0) = x_1, y(0) = y_1, x(1) = x_2, y(1) = y_2$.

$$\begin{aligned}
L(C) &= \int_0^1 \|\gamma'\|\, dt \\
&= \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\, dt \\
&= \int_0^1 |x'(t) + iy'(t)|\, dt \\
&\geq \left| \int_0^1 x'(t) + iy'(t)\, dt \right| \\
&= |x(1) - x(0) + i(y(1) - y(0))| \\
&= |x_2 - x_1 + i(y_2 - y_1)| \\
&= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\end{aligned}$$

But the length of a straight line with endpoints A and B is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, therefore C is a straight line. □

Definition 3.1.2. *We say that* $u \in C^1(\Omega)$ *is a weak supersolution (subsolution) of the minimal surface equation in* Ω *if*

$$\int_\Omega \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1 + |\nabla u|^2}} dx \geq 0 \quad (\leq) \quad \textit{whenever} \quad \varphi \in C_c^1(\Omega), \varphi \geq 0.$$

The strong form is called the minimal surfaces equation

$$-div \left(\frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) = 0 \qquad \forall x \in \Omega$$

Lemma 3.1.1. *[7] Suppose* W *is an open subset of* $\mathbb{R}^{n-1}$*. If* $v_1, v_2 \in C_c^1(W)$ *are respectively weak super and subsolution of the minimal surface equation in* W *and if* $v_1(x_0') = v_2(x_0')$ *for some* $x_0' \in W$ *while* $v_1(x') \geq v_2(x')$ *for all* $x' \in W$*, then*

$$v_1(x') = v_2(x')$$

for all x' in some closed ball contained in W *centered at* x_0'.

Definition 3.1.3. *Let* E *be a set of locally finite perimeter,* U *bounded, open set. Let*

$$\begin{aligned}
\psi(E, U) &= |D\mathbb{1}_E|(U) - \inf\{|D\mathbb{1}_F|(U), E\Delta F \subset\subset U\} \\
&= P(E, U) - \inf\{P(F, U), E\Delta F \subset\subset U\}
\end{aligned}$$

where $E\Delta F$ *denotes the symmetric difference of* E *and* F.

Definition 3.1.4. *We say that ∂E is area-minimizing in U if $\psi(E, U) = 0$ and locally area-minimizing if $\psi(E, U) = 0$ whenever U is bounded.*

Theorem 3.1.1. *[6] If M is area-minimizing, then M is a minimal surface.*

Definition 3.1.5. *Let $U \subset \mathbb{R}^n$, we say that a function $u \in BV_{loc}(\Omega)$ has least gradient with respect to U if for every $v \in BV_{loc}(U)$ with compact support $K \subset U$*

$$\int_K |Du| \leq \int_K |D(u + v)|$$

If $U = \mathbb{R}^n$ we say that u is of least gradient.

Equivalently, we say that u is of least gradient if u is a solution of

$$inf\{|Du|(\Omega) : u \in BV(\Omega) \cap C(\overline{\Omega}), u = g \ \text{ on } \ \partial\Omega\}$$

The following theorem, established by Bombieri, De Giorgi, and Giusti in [2],lays the foundation for proving the existence and uniqueness of the solution to the least gradient problem.

Theorem 3.1.2. *[2] If u is of least gradient then $\partial\{u \geq t\}$ is area-minimizing for each t.*

3.2 Regularity of Minimal surfaces and tangent cones

Theorem 3.2.1. *[8] If $n \geq 2$, Ω is an open set in $\mathbb{R}^n$, and ∂E is area-minimizing in Ω, then $\Omega \cap \partial^* E$ is an analytic hypersurface , while the singular set of E in Ω, $\sigma(E; \Omega) = \Omega \cap (\partial E - \partial^* E)$,satisfies the following properties:*

1. *if $2 \leq n \leq 7$, then $\sigma(E; \Omega)$ is empty;*
2. *if $n = 8$, then $\sigma(E; \Omega)$ has no accumulation points in Ω;*
3. *if $n \geq 9$, then $\mathcal{H}^s(\sigma(E; \Omega)) = 0$ for every $s > n - 8$.*

We've proved in theorem 2.1.1 that for $0 \in \partial^* E \subset \partial E$ and for each sequence $(r_i)_{i \geq 1}$ with $r_i \to 0$, there exist a sequence such that $\mathbb{1}_{E_{r_i}} \to \mathbb{1}_C$ in $L^1_{loc}(\mathbb{R}^n)$ where C is a set of locally finite perimeter. Now assuming that ∂E is area-minimizing further properties are added to C.

Theorem 3.2.2. *[9] Suppose E is a minimal set such that $0 \in \partial E$.For $t > 0$, let $E_t = \{x \in \mathbb{R}^n : tx \in E\}$. Then for every sequence $(t_j)_j$ tending to zero there exists a subsequence $(s_j)_j$ such that E_{s_j} converges locally in $\mathbb{R}^n$ to a set C. Moreover C is a minimal cone. The cone C is called a tangent cone to E at 0.*

Proposition 3.2.1. *[9] If E is regular at 0, then C must be a half space. In fact the converse is also true: If C is a half space then ∂E is regular in a neighbourhood of 0. That is there exists $r > 0$ such that $B(0, r) \cap \partial E$ is a real analytic hypersurface.*

Thus the set E can only have singularities if there exist minimal cones in $\mathbb{R}^n$ which have singularities.

Proposition 3.2.2. *[9] Minimal cones with singularities at 0 cannot exist in $\mathbb{R}^n$ with $n \leq 7$ hence the regularity of minimal surfaces in $\mathbb{R}^n$, $n \leq 7$.*

This result is the best possible since the cone

$$S = \{x \in \mathbb{R}^8; x_1^2 + x_2^2 + x_3^2 + x_4^2 < x_5^2 + x_6^2 + x_7^2 + x_8^2\}$$

is a singular minimal cone in $\mathbb{R}^8$ as proved in [2].

Theorem 3.2.3. *[1] Let $E_1 \subset E_2$ and suppose ∂E_1 and ∂E_2 are area-minimizing in an open set $U \subset R^n$. Further, suppose $x \in (\partial E_1) \cap (\partial E_2) \cap U$ then ∂E_1 and ∂E_2 agree in some neighborhood of x.*

This Theorem is trivial in 2 dimensions, where minimal surfaces are straight lines.

Chapter 4

Least Gradient Problem

4.1 Introduction

Let Ω be a bounded lipschitz domain in $\mathbb{R}^n$ and $g : \partial\Omega \to \mathbb{R}$ continuous function. We consider the following variational problem

$$\inf\{|Du|(\Omega) : u \in BV(\Omega) \cap C(\overline{\Omega}), u = g \text{ on } \partial\Omega\} \tag{4.1}$$

We will prove that a solution to this problem exists provided that $\partial\Omega$ satisfies the following conditions:

1. For every $x \in \partial\Omega$, there exists $\epsilon_0 > 0$ such that for every set of finite perimeter $A \subset\subset B(x, \epsilon_0)$
$$P(\Omega, \mathbb{R}^n) \leq P(\Omega \cup A, \mathbb{R}^n) \tag{4.2}$$

2. For every $x \in \partial\Omega$, and every $\epsilon > 0$, there exists a set of finite perimeter $A \subset\subset B(x, \epsilon)$ such that
$$P(\Omega, B(x, \epsilon)) > P(\Omega \setminus A, B(x, \epsilon)) \tag{4.3}$$

The first condition states that $\partial\Omega$ has non-negative mean curvature in the weak sense, while the second states that $\partial\Omega$ is not locally area-minimizing with respect to interior variations.Also if $\partial\Omega$ is smooth, then both conditions together are equivalent to the condition that the mean curvature of $\partial\Omega$ is positive on a dense set.

4.2 Preliminaries

Let
$$[a, b] = \{\cap I; I \text{ an interval containing } g(\partial\Omega)\}$$

Proposition 4.2.1. *[9] The boundary data g, admits a continuous extension*

$$G \in BV(\mathbb{R}^n \setminus \overline{\Omega}) \cap C(\mathbb{R}^n \setminus \Omega)$$

and we can require that $Supp\, G \subset B(0, R)$ *where* R *is chosen such that* $\Omega \subset\subset B(0, R)$.

Proof. Let $R>0$ such that $\Omega \subset\subset B(0,R)$ and denote $\widetilde{\Omega} = B(0,R)\setminus\overline{\Omega}$.There exist $u \in C^\infty(\widetilde{\Omega})$ such that

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\widetilde{\Omega}} = \begin{cases} g & \partial\Omega \\ 0 & \partial B(0,R) \end{cases} \end{cases}.$$

let

$$G = \begin{cases} u & \widetilde{\Omega} = B(0,R)\setminus\overline{\Omega} \\ 0 & \text{Otherwise} \end{cases}$$

Hence $G \in C(\mathbb{R}^n \setminus \Omega)$ and from gradient estimate on harmonic functions [10, Inequality 2.31]

$$|DG| \leq \frac{n}{dist(y,\partial\widetilde{\Omega})}\sup_{\widetilde{\Omega}}|G| < \infty$$

Therefore

$$\int_{\mathbb{R}^n\setminus\overline{\Omega}} |DG| = \int_{\widetilde{\Omega}} |DG| < \infty$$

□

We have

$$G \in BV(\mathbb{R}^n \setminus \overline{\Omega}) \cap C(\mathbb{R}^n\setminus\Omega) \quad \text{with} \quad G = g \quad \text{on} \quad \partial\Omega \tag{4.4}$$

For each $t \in [a,b]$, let

$$\mathcal{L}_t = (\mathbb{R}^n \setminus \Omega) \cap \{x : G(x) \geq t\} \tag{4.5}$$

Note that by the Coarea formula

$$P(\mathcal{L}_t, \mathbb{R}^n\setminus\overline{\Omega}) < \infty.$$

Let

$$T = [a,b] \cap \{t : P(\mathcal{L}_t, \mathbb{R}^n \setminus \overline{\Omega}) < \infty\}. \tag{4.6}$$

Proposition 4.2.2. $\mathcal{H}^{n-1}(\partial_M \mathcal{L}_t) = P(\mathcal{L}_t, \mathbb{R}^n\setminus\overline{\Omega}) + \mathcal{H}^{n-1}(\partial_M\mathcal{L}_t \cap \partial\Omega) < \infty$

Proof. Using Proposition 2.2.1 and the fact that $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ with Ω being bounded,

$$\begin{aligned}\mathcal{H}^{n-1}(\partial_M\mathcal{L}_t) &= \mathcal{H}^{n-1}(\partial_M\mathcal{L}_t\cap\partial\Omega) + \mathcal{H}^{n-1}(\partial_M\mathcal{L}_t\cap\Omega) + \mathcal{H}^{n-1}(\partial_M\mathcal{L}_t\cap(\mathbb{R}^n\setminus\overline{\Omega}))\\ &= \mathcal{H}^{n-1}(\partial_M\mathcal{L}_t\cap\partial\Omega) + \mathcal{H}^{n-1}(\partial_M\mathcal{L}_t\cap\Omega) + P(\mathcal{L}_t,\mathbb{R}^n\setminus\overline{\Omega}).\end{aligned}$$

But $\mathcal{H}^{n-1}(\partial_M\mathcal{L}_t\cap\Omega) = 0$ given that $\mathcal{L}_t \subset \mathbb{R}^n\setminus\Omega$. Therefore

$$\mathcal{H}^{n-1}(\partial_M\mathcal{L}_t) = \mathcal{H}^{n-1}(\partial_M\mathcal{L}_t\cap\partial\Omega) + P(\mathcal{L}_t,\mathbb{R}^n\setminus\overline{\Omega}) < \infty.$$

□

Theorem 4.2.1. *For each $t \in T$, there exists a solution for the following problems:*

$$\min\{P(E,\mathbb{R}^n) : E \setminus \overline{\Omega} = \mathcal{L}_t \setminus \overline{\Omega}\} \tag{4.7}$$

$$\max\{|E|,\ E \text{ is a solution of } (4.7)\} \tag{4.8}$$

Proof. Denote $m = \inf\{P(E,\mathbb{R}^n) : E \setminus \overline{\Omega} = \mathcal{L}_t \setminus \overline{\Omega}\}$. By definition of the inf there exists a sequence $(E_k)_k$, such that $E_k \setminus \overline{\Omega} = \mathcal{L}_t \setminus \overline{\Omega}$ for all k, and

$$\lim_{k\to\infty} P(E_k,\mathbb{R}^n) = m.$$

Recall that $\Omega \subset\subset B(0,R)$, and $G = 0$ in $\mathbb{R}^n \setminus B(0,R)$. We have

$$\|\mathbb{1}_{E_k}\|_{BV(B(0,R))} = \|\mathbb{1}_{E_k}\|_{L^1(B(0,R))} + |D\mathbb{1}_{E_k}|(B(0,R))$$

but $\|\mathbb{1}_{E_k}\|_{L^1(B(0,R))}$ is finite and $|D\mathbb{1}_{E_k}|(B(0,R))$ is bounded since $|D\mathbb{1}_{E_k}|(B(0,R)) \leq P(E_k,\mathbb{R}^n)$ and $P(E_k,\mathbb{R}^n)$ is bounded from the fact that it is convergent,we get $\sup\|\mathbb{1}_{E_k}\|_{BV(B(0,R))} < \infty$.By the compactness theorem 1.5.4 there exist a subsequence $(\mathbb{1}_{E_{k_j}}) \subset BV(B(0,R))$ such that

$$\mathbb{1}_{E_{k_j}} \underset{j\to\infty}{\to} \mathbb{1}_E \quad \text{in } L^1(B(0,R))$$

with $E \subset \mathbb{R}^n$ such that $E \setminus \overline{\Omega} = \mathcal{L}_t \setminus \overline{\Omega}$. By the lower semi-continuity theorem 1.5.1

$$|D\mathbb{1}_E|(B(0,R)) \leq \liminf_{j\to\infty} |D\mathbb{1}_{E_{k_j}}|(B(0,R)).$$

We get $P(E,B(0,R)) \leq \liminf_{j\to\infty} P(E_{k_j},B(0,R)) \leq \liminf_{j\to\infty} P(E_{k_j},\mathbb{R}^n)$, but

$$\begin{aligned} P(E,\mathbb{R}^n) &= |D\mathbb{1}_E|(\mathbb{R}^n) \\ &= |D\mathbb{1}_E|(B(0,R)) + |D\mathbb{1}_E|(\mathbb{R}^n \setminus B(0,R)) \\ &= \int_{B(0,R)} d|D\mathbb{1}_E| + \int_{\mathbb{R}^n\setminus B(0,R)} \nu_E dD\mathbb{1}_E. \end{aligned}$$

$\mathbb{1}_E$ is constant outside the ball since $G = 0$ and $E \setminus \overline{\Omega} = \mathcal{L}_t \setminus \overline{\Omega}$,therefore

$$P(E,B(0,R)) = P(E,\mathbb{R}^n).$$

Thus $P(E,\mathbb{R}^n) \leq \liminf P(E_{k_j},\mathbb{R}^n) = m$, hence $P(E,\mathbb{R}^n) = m$.

To prove (4.8), denote $M = \sup\{|E| : E \text{ solution of } (4.7)\}$, by definition of the sup there exists a sequence $(E_k)_k$ such that

$$|E_k| \to M.$$

Again there exist a subsequence $(\mathbb{1}_{E_{k_j}}) \subset BV(B(0,R))$ such that

$$\mathbb{1}_{E_{k_j}} \to \mathbb{1}_E \quad \text{in } L^1(B(0,R)).$$

with E solution to (4.7). Notice that

$$\|\mathbb{1}_{E_{k_j}} - \mathbb{1}_E\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\mathbb{1}_{E_{k_j}} - \mathbb{1}_E| d\lambda = \int_{(E_{k_j}\setminus E)\cup(E\setminus E_{k_j})} 1 d\lambda = |E_{k_j}\Delta E|,$$

and

$$|E_{k_j}\Delta E| = |(E_{k_j}\setminus E)\cup(E\setminus E_{k_j})| \geq |E_{k_j}\setminus E| = |E_{k_j}| - |E_{k_j}\cap E| \geq |E_{k_j}| - |E|.$$

We then get

$$M \geq |E| \geq |E_{k_j}| - \|\mathbb{1}_{E_{k_j}} - \mathbb{1}_E\|_{L^1(\mathbb{R}^n)}.$$

With the right-hand-side term converging to M as j tends to infinity,we get $|E| = M$ and the max exists.

This maximum is unique. In fact, assume E_1, E_2 solution maximizers of (4.8). We notice that

$$(E_1 \cup E_2) \setminus \overline{\Omega} = (E_1 \setminus \overline{\Omega}) \cup (E_2 \setminus \overline{\Omega}) = \mathcal{L}_t \setminus \overline{\Omega}$$

and

$$(E_1 \cap E_2) \setminus \overline{\Omega} = (E_1 \setminus \overline{\Omega}) \cap (E_2 \setminus \overline{\Omega}) = \mathcal{L}_t \setminus \overline{\Omega}.$$

Hence $E_1 \cup E_2$, and $E_1 \cap E_2$ are competitors of (4.7), thus

$$P(E_1, \mathbb{R}^n) \leq P(E_1 \cup E_2, \mathbb{R}^n) \text{ and } P(E_1, \mathbb{R}^n) \leq P(E_1 \cap E_2, \mathbb{R}^n)$$

$$P(E_2, \mathbb{R}^n) \leq P(E_1 \cup E_2, \mathbb{R}^n) \text{ and } P(E_2, \mathbb{R}^n) \leq P(E_1 \cap E_2, \mathbb{R}^n).$$

However by (1.3),

$$P(E_1 \cup E_2, \mathbb{R}^n) + P(E_1 \cap E_2, \mathbb{R}^n) \leq P(E_1, \mathbb{R}^n) + P(E_2, \mathbb{R}^n)$$

then $E_1 \cup E_2$ and $E_1 \cap E_2$ are minimizes of (4.7). Therefore

$$|E_1| \geq |E_1 \cup E_2| = |E_1| + |E_2 \setminus E_1|$$

and

$$|E_2| \geq |E_1 \cup E_2| = |E_2| + |E_1 \setminus E_2|$$

which implies that $|E_1 \Delta E_2| = 0$, completing the proof of uniqueness. □

Notation. For every $t \in T$, we denote by E_t the unique solution to (4.8).

4.3 Properties of the set E_t

We have Ω a Lipschitz domain, then for each $x_0 \in \partial\Omega, \partial\Omega$ can be represented locally as the graph of a non-negative lipschitz function h, defined on some ball $B'(x_0, r) \subset \mathbb{R}^{n-1}$, where $x'_0 \in \mathbb{R}^{n-1}$, that is

$$\{(x', h(x')) : x' \in B'(x'_0, r)\} \subset \partial\Omega.$$

Notation. *$B'(x_0', r)$ and x' denote elements in $\mathbb{R}^{n-1}$ and thus they will be distinguished from their n-dimensional counterparts $B(x_0, r)$ and x. We assume our configuration is oriented in such a way that*

$$\{(x', x''): 0 < x'' < h(x')\} \subset \Omega$$

Proposition 4.3.1. *[3] If Ω is a lipschitz domain, then Ω is a set of finite perimeter and*

$$P(\Omega, U) = \mathcal{H}^{n-1}(\partial^*\Omega \cap U) = \mathcal{H}^{n-1}(\partial\Omega),$$

whenever $U \subset \mathbb{R}^n$ is an open set.

Lemma 4.3.1. *If Ω is a lipschitz domain, with non-negative mean curvature in the sense of (4.2), then the function h, whose graph represents $\partial\Omega$ locally, is a weak supersolution of the minimal surface equation. That is , for r sufficiently small*

$$\int_{B'(x_0',r)} \frac{\nabla h \cdot \nabla\varphi}{\sqrt{1+|\nabla h|^2}}\, dx' \geq 0 \quad \textit{whenever } \varphi \in C_c^1(B'(x_0', r)), \varphi \geq 0.$$

Proof. For $t \geq 0$, and $\varphi \in C_c^1(B'(x_0', r)), \varphi \geq 0$, let

$$\begin{aligned} f(t) &= \int_{B'(x_0',r)} \sqrt{1+|\nabla h|^2 + 2t\nabla h\cdot\nabla\varphi + t^2|\nabla\varphi|^2} dx' \\ &= \int_{B'(x_0',r)} \sqrt{1+|\nabla h + t\nabla\varphi|^2} dx', \end{aligned}$$

and

$$A = \{(x', x''), h(x') \leq x'' \leq h(x') + t\varphi(x'), x' \in B'(x_0', r)\}.$$

Assuming that r has been chosen sufficiently small so that A is of finite perimeter and $A \subset\subset B(x_0, r)$ with $x_0 = (x_0', x_0'')$, condition (4.2) can be invoked, we have $P(\Omega) \leq P(A \cup \Omega)$, and hence

$$0 \leq P(A\cup\Omega) - P(\Omega) = \mathcal{H}^{n-1}(\partial(A\cup\Omega)) - \mathcal{H}^{n-1}(\partial\Omega) = f(t) - f(0)$$

Therefore f has a minimum at $t = 0$, then $f'(0) \geq 0$, but

$$\begin{aligned} f'(t) &= \int_{B'(x_0',r)} \left(\sqrt{1+|\nabla h|^2 + 2t\nabla h\cdot\nabla\varphi + t^2|\nabla\varphi|^2}\right)' dx' \\ &= \int_{B'(x_0',r)} \frac{\nabla h\cdot\nabla\varphi + t|\nabla\varphi|^2}{\sqrt{1+|\nabla h|^2 + 2t\nabla h\cdot\nabla\varphi + t^2|\nabla\varphi|^2}} dx' \end{aligned}$$

and

$$f'(0) = \int_{B'(x_0',r)} \frac{\nabla h\cdot\nabla\varphi}{\sqrt{1+|\nabla h|^2}} \geq 0$$

□

Lemma 4.3.2. *For almost all $t \in [a, b], \partial E_t \cap \partial\Omega \subset g^{-1}(t)$.*

Proof. We Show first that ∂E_t is locally area-minimizing in a neighborhood of each point $x_0 \in (\partial E_t \cap \partial\Omega) \setminus g^{-1}(t)$. Since $x_0 \notin g^{-1}(t)$ so either $g(x_0) < t$ or $g(x_0) > t$.

First assume $g(x_0) < t$. Since $x_0 \in \partial\Omega$, and we have $G = g$ on $\partial\Omega$, then $G(x_0) = g(x_0) < t$ with G continuous on $\mathbb{R}^n \setminus \Omega$, thus there exists $\epsilon > 0$, such that $B(x_0, \epsilon) \cap \mathcal{L}_t = \emptyset$. We will assume that $\epsilon < \epsilon_0$, where ϵ_0 appears in condition (4.2), we proceed by taking a variation F satisfying $F \Delta E_t \subset\subset B(x_0, \epsilon)$. We have for every $A \subset\subset B(x_0, \epsilon_0)$,

$$\begin{aligned} P(A \cap \Omega, \mathbb{R}^n) + P(A \cup \Omega, \mathbb{R}^n) &\leq P(A, \mathbb{R}^n) + P(\Omega, \mathbb{R}^n) \quad \text{by (1.3)} \\ &\leq P(A, \mathbb{R}^n) + P(A \cup \Omega, \mathbb{R}^n) \quad \text{by (4.2)} \end{aligned}$$

Hence

$$P(A \cap \Omega, \mathbb{R}^n) \leq P(A, \mathbb{R}^n). \tag{4.9}$$

Define $F' = (F \setminus B(x_0, \epsilon)) \cup (F \cap \overline{\Omega})$. Since $E_t \Delta F \subset\subset B(x_0, \epsilon)$ then $E_t \setminus \overline{\Omega}$ and $F \setminus \overline{\Omega}$ coincides outside the ball and so

$$\begin{aligned} F' \setminus \overline{\Omega} &= (F \setminus B(x_0, \epsilon)) \setminus \overline{\Omega} \\ &= (F \setminus \overline{\Omega}) \setminus B(x_0, \epsilon) \\ &= (E_t \setminus \overline{\Omega}) \setminus B(x_0, \epsilon) \\ &= (\mathcal{L}_t \setminus \overline{\Omega}) \setminus B(x_0, \epsilon) \ \ (E_t \text{ solution to (4.8)}) \\ &= \mathcal{L}_t \setminus \overline{\Omega} \quad (\mathcal{L}_t \cap B(x_0, \epsilon) = \emptyset) \end{aligned}$$

Thus F' is admissible in (4.7) and therefore

$$P(E_t, \mathbb{R}^n) \leq P(F', \mathbb{R}^n)$$

It remains to show $P(F', \mathbb{R}^n) \leq P(F, \mathbb{R}^n)$. First observe that $F' \cap B(x_0, \epsilon) = F \cap B(x_0, \epsilon) \cap \overline{\Omega}$. In fact,

$$\begin{aligned} F' \cap B(x_0, \epsilon) &= (F \cap B^c(x_0, \epsilon)) \cup (F \cap \overline{\Omega})) \cap B(x_0, \epsilon) \\ &= F \cap \overline{\Omega} \cap B(x_0, \epsilon). \end{aligned}$$

Moreover since $F' \subset F$, then

$$\begin{aligned} F' \Delta F = F \setminus F' &= F \cap (F \setminus B(x_0, \epsilon))^c \cap (F \cap \overline{\Omega})^c \\ &= F \cap B(x_0, \epsilon) \cap \overline{\Omega}^c \\ &= (F \setminus \overline{\Omega}) \cap B(x_0, \epsilon) \\ &= (((F \setminus E_t) \cup E_t) \setminus \overline{\Omega}) \cap B(x_0, \epsilon) \\ &= ((F \setminus E_t) \setminus \overline{\Omega}) \cup (E_t \cap \overline{\Omega}^c \cap B(x_0, \epsilon) \\ &\subset F \Delta E_t \subset\subset B(x_0, \epsilon) \end{aligned}$$

We used above the facts that $F \Delta E_t \subset\subset B(x_0, \epsilon)$ and $(E_t \setminus \overline{\Omega}) \cap B(x_0, \epsilon) = \emptyset$. Hence,

$$\begin{aligned} P(F, \mathbb{R}^n) - P(F', \mathbb{R}^n) &= P(F, B(x_0, \epsilon)) - P(F', B(x_0, \epsilon)) \\ &= P(F \cap B(x_0, \epsilon), B(x_0, \epsilon)) - P(F \cap B(x_0, \epsilon) \cap \overline{\Omega}, B(x_0, \epsilon)) \\ &= P(F \cap B(x_0, \epsilon), B(x_0, \epsilon)) - P(F \cap B(x_0, \epsilon) \cap \Omega, B(x_0, \epsilon)) \\ &= P(F \cap B(x_0, \epsilon), \mathbb{R}^n) - P(F \cap B(x_0, \epsilon) \cap \Omega, \mathbb{R}^n) \end{aligned} \tag{4.10}$$

The above inequality with $A = F \cap B(x_0, \epsilon)$ in (4.9) we get $P(F, \mathbb{R}^n) - P(F', \mathbb{R}^n) \geq 0$. This implies that $P(E_t, \mathbb{R}^n) \leq P(F, \mathbb{R}^n)$ or equivalently $P(E_t, B(x_0, \epsilon)) \leq P(F, B(x_0, \epsilon))$ when $g(x_0) < t$.

Now for the case $g(x_0) > t$. Since $G(x_0) = g(x_0) > t$, the continuity of G in Ω^c implies that as above there exists $\epsilon > 0$ such that $B(x_0, \epsilon) \subset \{x : G(x) \geq t\}$, i.e $B(x_0, \epsilon) \cap (\mathbb{R}^n \setminus \Omega) \subset \mathcal{L}_t$, implying that $B(x_0, \epsilon) \setminus \Omega \subset \mathcal{L}_t$ provided ϵ is sufficiently small and $\epsilon < \epsilon_0$. Let F be a variation such that $F \Delta E_t \subset\subset B(x_0, \epsilon)$, and define $F' = F \cup (B(x_0, \epsilon) \setminus \overline{\Omega})$, then

$$\begin{aligned}
F' \setminus \overline{\Omega} &= (F \setminus \overline{\Omega}) \cup (B(x_0, \epsilon) \setminus \overline{\Omega}) \\
&= ((F \setminus B(x_0, \epsilon)) \setminus \overline{\Omega}) \cup (B(x_0, \epsilon) \setminus \overline{\Omega}) \\
&= ((E_t \setminus B(x_0, \epsilon)) \setminus \overline{\Omega}) \cup (B(x_0, \epsilon) \setminus \overline{\Omega}) \\
&= (\mathcal{L}_t \setminus B(x_0, \epsilon) \setminus \overline{\Omega}) \cup (B(x_0, \epsilon) \setminus \overline{\Omega}) \qquad \text{but } B(x_0, \epsilon) \setminus \overline{\Omega} \subset \mathcal{L}_t \\
&= (\mathcal{L}_t \setminus B(x_0, \epsilon) \setminus \overline{\Omega}) \cup (\mathcal{L}_t \setminus \overline{\Omega} \cap B(x_0, \epsilon)) \\
&= \mathcal{L}_t \setminus \overline{\Omega}
\end{aligned}$$

Thus, since F' is a competitor for (4.7), it follows that $P(E_t, \mathbb{R}^n) \leq P(F', \mathbb{R}^n)$. Then it remains to show

$$P(F', \mathbb{R}^n) \leq P(F, \mathbb{R}^n). \tag{4.11}$$

For this, note that $E_t \Delta F \subset\subset B(x_0, \epsilon)$ and $B(x_0, \epsilon) \setminus \Omega = B(x_0, \epsilon) \cap \mathcal{L}_t \subset E_t$ implies $(F')^c \cap B(x_0, \epsilon) = F^c \cap B(x_0, \epsilon) \cap \overline{\Omega}$ and $(F')^c \Delta F^c \subset\subset B(x_0, \epsilon)$. In fact,

$$(F)^c \Delta (F')^c = (F)^c \setminus (F')^c = F^c \cap F' = F^c \cap (B(x_0, \epsilon) \setminus \overline{\Omega})$$

But we have $B(x_0, \epsilon) \setminus \overline{\Omega} \subset E_t$, therefore

$$(F)^c \Delta (F')^c \subset F^c \cap E_t \subset F \Delta E_t \subset\subset B(x_0, \epsilon).$$

Since

$$P(F, \mathbb{R}^n) - P(F', \mathbb{R}^n) = P(F^c, \mathbb{R}^n) - P((F')^c, \mathbb{R}^n)$$

then (4.11) follows from (4.10) with F and F' replaced by F^c and $(F')^c$.

We have thus demonstrated that ∂E_t is area-minimizing in $B(x_0, \epsilon)$. We will show that this leads to a contradiction.

Assume first $g(x_0) < t$, so that $G < t$ on $(\mathbb{R}^n \setminus \Omega) \cap B(x_0, \epsilon)$ provided ϵ has been chose sufficiently small. Consequently, since for $g(x_0) < t$ we have $B(x_0, \epsilon) \cap \mathcal{L}_t = \emptyset$, so that $(E_t \setminus \overline{\Omega}) \cap B(x_0, \epsilon) = (\mathcal{L}_t \setminus \overline{\Omega}) \cap B(x_0, \epsilon) = \emptyset$.With

$$E_t \cap B(x_0, \epsilon) = (E_t \cap \overline{\Omega} \cap B(x_0, \epsilon)) \cup ((E_t \setminus \overline{\Omega}) \cap B(x_0, \epsilon)$$

we get

$$E_t \cap B(x_0, \epsilon) \subset \overline{\Omega} \cap B(x_0, \epsilon) \tag{4.12}$$

Recall that Ω is a lipschitz domain, we can represent its boundary locally by a lipschitz function h.Thus with $x_0 \in \partial E_t \cap \partial\Omega \setminus g^{-1}(t)$, we express $\partial\Omega$ locally about

x_0 as $\{(x', h(x')); x' \in B'(x'_0, \epsilon')\}$ where $x_0 = (x'_0, x''_0)$ and $x''_0 = h(x'_0) > 0$. For simplicity of notation take $x'_0 = 0$. The number ϵ' is chosen so that $\epsilon' < \epsilon$ and that

$$\{(x', h(x')) : |x'| \leq \epsilon'\} \subset B(x_0, \epsilon) \tag{4.13}$$

We define the half-infinite cylinder above $B'(0, \epsilon')$ as $C = B'(0, \epsilon') \times [0, \infty)$, we may assume

$$\Omega \cap C = \{(x', x''); |x'| < \epsilon, 0 \leq x'' < h(x')\}$$

Now consider the solution of the minimal surface equation on $B'(0, \epsilon')$ relative to the boundary data $f = h|_{\partial B'(0,\epsilon')}$. Thus we let v be the unique solution of

$$div\left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}}\right) = 0 \ \text{ on } B'(0, \epsilon')$$

$$v = f \quad \text{on} \quad \partial B'(0, \epsilon')$$

By lemma 4.3.1, we have that h is a weak super-solution of the minimal surface equation, and by [10, Theorem 10.7] we have that $h \geq v$ on $\overline{B'(0, \epsilon')}$. In fact, $h > v$ on $B'(0, \epsilon')$, because the set $\{h = v\}$ is obviously closed in $B'(0, \epsilon')$ and it is also open in $B'(0, \epsilon')$, because of lemma 3.1.1. Hence if $\{h = v\}$ is not empty, then $h = v$ in $B'(0, \epsilon')$, but this would contradict (4.3) because with v being a solution to the minimal surface equation we get

$$\{(x', h(x'), x' \in B'(0, \epsilon')\} = \{(x', v(x')), x' \in B'(0, \epsilon')\}$$

hence for any $A \subset\subset B(0, \epsilon), P(\Omega, B(x, \epsilon)) \leq P(\Omega \setminus A, B(x, \epsilon))$. Therefore $\{h = v\} = \emptyset$ in $B'(0, \epsilon')$. Consequently with $\delta = h(0) - v(0)$, we have $\delta > 0$.

Now consider a 1-parameter family of graphs, $v_\tau(x') = v(x') + \tau$ and let

$$\tau^* = \max\{\tau : \text{there exists } x' \in \overline{B'(0, \epsilon')} \text{ such that } (x', v_\tau(x')) \in \partial E_t \cap \overline{\Omega}\}.$$

Note that,$\tau^* \geq \delta$ since $x_0 = (0, h(0)) \in \partial E_t \cap \overline{\Omega}$ and $h(0) = v(0) + \delta$. Let

$$V_{\tau^*} = \{(x', x'') : |x'| < \epsilon', x'' \leq v(x') + \tau^*\}.$$

In view of our choice of ϵ', observe that

$$E_t \cap \{x : |x'| < \epsilon'\} \subset V_{\tau^*}.$$

Observe also that if a point $(x', v_{\tau^*}(x'))$ is an element of $\partial E_t \cap \overline{\Omega}$, then $|x'| < \epsilon'$, for otherwise i.e if $|x'| = \epsilon'$, we would have that $h(x') = v(x')$ but $v_{\tau^*}(x') = v(x') + \tau^* \leq h(x') = v(x')$ which would imply that $\tau^* \leq 0$ contradicting that $\tau^* \geq \delta > 0$. Thus the set $\partial[E_t \cap \{x : |x'| < \epsilon'\}] \cap \{(x', v_{\tau^*}(x')) : |x'| < \epsilon'\}$ is non-empty and according to Theorem 3.2.3, with v solution to the minimal surface equation, it is open as well as closed in the connected set $\{(x', v_{\tau^*}(x')) : |x'| < \epsilon'\}$. This implies that

$$\{(x', v_{\tau^*}(x') : |x'| < \epsilon'\} \subset \partial[E_t \cap \{x : |x'| < \epsilon'\}]. \tag{4.14}$$

Since $\tau^* \geq \delta > 0$, it follows that $v_{\tau^*}(x') = v(x') + \tau^* > v(x') = h(x')$ whenever $|x'| = \epsilon'$ having that $v = h$ on $\partial B'(0, \epsilon')$. Consequently, using the continuity of v_{τ^*}, the graph $\{(x', v_{\tau^*}(x')) : |x'| < \epsilon'\}$ contains points in $\mathbb{R}^n \setminus \overline{\Omega}$, say $(y', v_{\tau^*}(y')), |y'| < \epsilon'$, as well as points in $\overline{\Omega} \cap B(x_0, \epsilon)$ say $(z', v_{\tau^*}(z')), |z'| < \epsilon'$. The point $(y', v_{\tau^*}(y')), |y'| < \epsilon'$ could possibly be an element of $\mathbb{R}^n \setminus B(x_0, \epsilon)$. Consider the line segment L, in $B'(x_0, \epsilon')$ that joins y' and z'. Let a' be that point on L closest to y' with the property that $(a', v_{\tau^*}(a')) \in \partial\Omega$. Then all points a on L that are closer to y' than a' and that are sufficiently near a' have the property that $(a, v_{\tau^*}(a)) \in (\mathbb{R}^n \setminus \overline{\Omega}) \cap B(x_0, \epsilon)$. Here we have used (4.13) and continuity of v_{τ^*}. Then $(a, v_{\tau^*}(a)) \in \{(x', v_{\tau^*}(x')) : |x'| < \epsilon'\} \subset \partial[E_t \cap \{x : |x'| < \epsilon'\}]$. Therefore

$$E_t \cap B(x_0, \epsilon) \cap \mathbb{R}^n \setminus \overline{\Omega} \neq \emptyset$$

contradicting (4.12). This contradiction was due to the assumption that $g(x_0) < t$ and the fact that ∂E_t is area-minimizing in $B(x_0, \epsilon)$. Similarly for the case $g(x_0) > t$. Finally we get $\partial E_t \cap \partial\Omega \subset g^{-1}(t)$. □

We've shown that for almost all $t \in [a, b], \partial E_t \cap \partial\Omega \subset g^{-1}(t)$. We want to ultimately identify $E_t \cap \overline{\Omega}$ as the set $\{u \geq t\}$ (up to a set of measure zero) for almost all t. We will need the following Lemma.

Lemma 4.3.3. *If $s, t \in T = [a, b] \cap \{t : P(\mathcal{L}_t, \mathbb{R}^n \setminus \overline{\Omega}) < \infty\}$ with $s < t$, then $E_t \subset\subset E_s$.*

Proof. We first show $E_t \subset E_s$.

Note that, with $s < t$ we have $\mathcal{L}_t \subset \mathcal{L}_s$.Then

$$(E_s \cap E_t) \setminus \overline{\Omega} = (E_s \setminus \overline{\Omega}) \cap (E_t \setminus \overline{\Omega}) = (\mathcal{L}_s \setminus \overline{\Omega}) \cap (\mathcal{L}_t \setminus \overline{\Omega}) = (\mathcal{L}_t \setminus \overline{\Omega})$$

thus $E_s \cap E_t$ is a competitor with E_t in (4.7). Also,

$$(E_s \cup E_t) \setminus \overline{\Omega} = (E_s \setminus \overline{\Omega}) \cup (E_t \setminus \overline{\Omega}) = (\mathcal{L}_s \setminus \overline{\Omega}) \cup (\mathcal{L}_t \setminus \overline{\Omega}) = (\mathcal{L}_s \setminus \overline{\Omega})$$

$E_s \cup E_t$ is a competitor with E_s in (4.7). Thus

$$P(E_s \cap E_t, \mathbb{R}^n) \geq P(E_t, \mathbb{R}^n) \quad \text{and} \quad P(E_s \cup E_t, \mathbb{R}^n) \geq P(E_s, \mathbb{R}^n).$$

As

$$P(E_s \cap E_t, \mathbb{R}^n) + P(E_s \cup E_t, \mathbb{R}^n) \leq P(E_s, \mathbb{R}^n) + P(E_t, \mathbb{R}^n)$$

we get $P(E_s \cup E_t, \mathbb{R}^n) \leq P(E_s, \mathbb{R}^n)$, and hence $P(E_s \cup E_t, \mathbb{R}^n) = P(E_s, \mathbb{R}^n)$. Similarly $P(E_s \cap E_t, \mathbb{R}^n) = P(E_t, \mathbb{R}^n)$. But we have $E_s \subset E_s \cup E_t$, then $|E_s| \leq |E_s \cup E_t|$. E_s solution to (4.8) thus $|E_s| \geq |E_s \cup E_t|$ and therefore $|E_s \cup E_t| = |E_s|$. Now,

$$|E_s \cup E_t| = |E_s \cup (E_t \setminus E_s)| = |E_s| + |E_t \setminus E_s| = |E_s \cup E_t| + |E_t \setminus E_s|$$

Hence $|E_t \setminus E_s| = 0$. Using the convention (2.1) , we can now prove that $E_t \subset E_s$. Let $x \in E_t$, then

$$\limsup_{r \to 0} \frac{|E_t \cap B(x, r)|}{|B(x, r)|} > 0.$$

On my way to prove that $\limsup_{r\to 0} \frac{|E_s\cap B(x,r)|}{|B(x,r)|} > 0$,knowing that

$$\limsup_{r\to 0} \frac{|E_s \cap B(x,r)|}{|B(x,r)|} \geq \limsup_{r\to 0} \frac{|E_s \cap E_t \cap B(x,r)|}{|B(x,r)|}.$$

$E_t = (E_t \setminus E_s) \cup (E_t \cap E_s)$, then

$$E_t \cap B(x,r) = ((E_t \setminus E_s) \cap B(x,r)) \cup (E_t \cap E_s \cap B(x,r)),$$

and we have $|E_t \setminus E_s| = 0$, hence $|(E_t \setminus E_s) \cap B(x,r)| = 0$. Therefore

$$\limsup_{r\to 0} \frac{|E_s \cap B(x,r)|}{|B(x,r)|} \geq \limsup_{r\to 0} \frac{|E_s \cap E_t \cap B(x,r)|}{|B(x,r)|} = \limsup_{r\to 0} \frac{|E_t \cap B(x,r)|}{|B(x,r)|} > 0.$$

Thus $x \in E_s$ and $E_t \subset E_s$. It remains to show that this containment is in fact compact, i.e that $\overline{E_t} \subset int(E_s)$. $E_t \subset E_s$, then $int(E_t) \subset int(E_s)$, hence to prove that $\overline{E_t} \subset int(E_s)$, we only need to prove $\partial E_t \subset int(E_s)$ i.e prove $\partial E_t \cap \partial E_s = \emptyset$. First outside $\overline{\Omega}$, we have

$$E_t \setminus \overline{\Omega} = \mathcal{L}_t \setminus \overline{\Omega} \subset\subset \mathcal{L}_s \setminus \overline{\Omega}$$

This is due to the fact that $\mathcal{L}_t \subset\subset \mathcal{L}_s$ since $s < t$. In $\overline{\Omega}$, we show that $\partial E_t \cap \partial E_s \cap \overline{\Omega} = \emptyset$. We have $\partial E_t \cap \partial E_s \cap \partial\Omega = \emptyset$, in fact by Lemma 4.3.2, we have

$$\partial E_t \cap \partial\Omega \subset g^{-1}(t) = \{x : g(x) = t\}$$

and

$$\partial E_s \cap \partial\Omega \subset g^{-1}(s) = \{x : g(x) = s\}.$$

Since $s < t, g^{-1}(t) \cap g^{-1}(s) = \emptyset$, therefore $\partial E_t \cap \partial E_s \cap \partial\Omega = \emptyset$. Finally,assume by contradiction that $S = \partial E_t \cap \partial E_s \cap \Omega \neq \emptyset$. Since $E_t \subset E_s$, and $\partial E_s, \partial E_t$ are area minimizing in Ω then by 3.2.3 we get that S is open relative to ∂E_s and clearly S is closed relative to ∂E_s thus S is both open and closed relative to ∂E_s and therefore S is equal to a connected component of ∂E_s that do not intersect $\partial\Omega$, now by [1] and [7, theorem 4.4 part 2, 3, 4] we get a contradiction.Therefore $\partial E_t \cap \partial E_s \cap \Omega = \emptyset$ and we conclude that $E_t \subset\subset E_s$.

□

4.4 Construction of the solution

In this Chapter, we will construct the solution u of the Least gradient problem (4.1). For this purpose define the set

$$A_t = \overline{E_t \cap \Omega}$$

Proposition 4.4.1. *For $t \in T$,*

$$\{g > t\} \subset (E_t)^i \cap \partial\Omega \subset A_t \cap \partial\Omega \tag{4.15}$$

$$\overline{\{g > t\}} \subset A_t \cap \partial\Omega \subset \overline{E_t} \cap \partial\Omega \subset \{g \geq t\} \tag{4.16}$$

Proof. 1. For the first inclusion. Let $x_0 \in \{g > t\}$ i.e $g(x_0) > t$. Similarly to the proof of lemma 4.3.2, there exist $\epsilon > 0$ such that $G(x) > t$, $\forall x \in B(x_0, \epsilon)$ and we get

$$B(x_0, \epsilon) \setminus \Omega \subset E_t \subset \overline{E_t}.$$

Hence $x_0 \in \overline{E_t}$.But since $\partial E_t \cap \partial\Omega \subset g^{-1}(t)$ then $x_0 \notin \partial E_t$ and thus $x_0 \in (E_t)^i \cap \partial\Omega$.

For the second inclusion, let $x \in (E_t)^i \cap \partial\Omega$, then $\forall r > 0, B(x, r) \cap E_t \neq \emptyset$. $x \in \partial\Omega$ then $B(x, r) \cap \Omega \neq \emptyset$, therefore $B(x, r) \cap E_t \cap \Omega \neq \emptyset$. This implies that $x \in \overline{E_t \cap \Omega}$ and hence

$$x \in \overline{E_t \cap \Omega} \cap \partial\Omega = A_t \cap \partial\Omega.$$

2. From (4.15) we get

$$\overline{\{g > t\}} \subset \overline{A_t \cap \partial\Omega} = A_t \cap \partial\Omega.$$

Also we know $A_t = \overline{E_t \cap \Omega} \subset \overline{E_t} \cap \overline{\Omega}$, then

$$A_t \cap \partial\Omega \subset \overline{E_t} \cap \overline{\Omega} \cap \partial\Omega = \overline{E_t} \cap \partial\Omega.$$

Now for the last inclusion of (4.16) write $\overline{E_t} \cap \partial\Omega$ as $((E_t)^i \cup \partial E_t) \cap \partial\Omega$ but from lemma 4.3.2 $\partial E_t \cap \partial\Omega \subset \{g = t\} \subset \{g \geq t\}$, then it is enough to prove $(E_t)^i \cap \partial\Omega \subset \{g \geq t\}$. Assume $x \in (E_t)^i \cap \partial\Omega$, then $\exists r > 0$ such that $B(x, r) \subset E_t$. In addition we have

$$G(y) \geq t \quad \forall y \in \mathcal{L}_t \setminus \Omega$$

and $\mathcal{L}_t \setminus \overline{\Omega} = E_t \setminus \overline{\Omega}$. Hence $G(y) \geq t, \forall y \in (E_t \setminus \overline{\Omega}) \cup (\mathcal{L}_t \cap \partial\Omega)$. Finally by continuity of G, with $B(x, r) \subset E_t$ we get that $G(x) = g(x) \geq t$, and the inclusion is proven.

□

Note that by lemma 4.3.3 and (4.16)

$$A_t \subset\subset A_s$$

relative to the topology on $\overline{\Omega}$ whenever $s, t \in T$ with $s < t$.

Now we define our candidate solution

$$u(x) = \sup\{t \in T \mid x \in A_t\}. \tag{4.17}$$

Theorem 4.4.1. *The function u defined by (4.17) satisfies the following:*

1. *$u = g$ on $\partial\Omega$*
2. *u is continuous on $\overline{\Omega}$*
3. *$A_t \subset \{u \geq t\}$ for all $t \in T$ and $|\{u \geq t\} \setminus A_t| = 0$ for almost all $t \in T$*

Proof. 1. Let $x_0 \in \partial\Omega$ and suppose $g(x_0) = t$.

If $s < t$, from (4.15) we have $\{g > s\} \subset (E_s)^i \cap \partial\Omega$ hence with $g(x_0) > s$ we get $x_0 \in (E_s)^i \cap \partial\Omega$, consequently $x_0 \in A_s \cap \partial\Omega$ and thus $x_0 \in A_s$ for all $s \in T$ such that $s < t$. By definition of $u(x)$, we have $u(x_0) \geq s$ for all $s \in T, s < t$. By letting $s \to t$ we get $u(x_0) \geq t$.

To show that $u(x_0) = t$, suppose by contradiction that $u(x_0) = \tau > t$, hence $\sup\{t \in T, x_0 \in A_t\} = \tau$ and this implies that for all $\epsilon > 0$ there exists $r \in \{t \in T, x_0 \in A_t\}$ such that $r > \tau - \epsilon$.For a convenient ϵ, $r_\epsilon \in (t, \tau) \cap T$ and $x_0 \in A_r$. But by (4.16) $A_r \cap \partial\Omega \subset \{g \geq r\}$ which is a contradiction since $g(x_0) = t < r$. Therefore $u = g$ on $\partial\Omega$.

2. First we will prove

$$(i)\{u \geq t\} = \bigcap_{s \in T, s<t} A_s \quad (ii)\{u > t\} = \bigcup_{s \in T, s>t} A_s$$

(i)

$$\begin{aligned} x \in \bigcap_{s \in T, s<t} A_s &\implies x \in A_s, s \in T, s < t \\ &\implies \sup\{t \in T, x \in A_t\} \ \geq t \end{aligned}$$

Hence $\bigcap_{s \in T, s<t} A_s \subset \{u \geq t\}$.
Conversely, for $x \in \{u \geq t\}$

$$\begin{aligned} u(x) \geq t &\implies u(x) > s \, \forall s < t \\ &\implies \exists s < s' \leq t \text{ such that } x \in A_{s'} \\ &\implies A_{s'} \subset A_s \\ &\implies x \in A_s \, \forall s < t \\ &\implies x \in \bigcap_{s \in T, s<t} A_s \end{aligned}$$

(ii)

$$\begin{aligned} u(x) > t &\implies \sup\{t \in T, x \in A_t\} > t \\ &\implies \exists t_0 \in T, x \in A_{t_0} \text{such that } t_0 > t \\ &\implies x \in \bigcup_{s \in T, s>t} A_s \end{aligned}$$

Conversely,if $x \in A_{s_0}$ for some $s_0 > t$ then $u(x) \geq s_0 > t$ and hence $x \in \{u > t\}$.

The set $\{u \geq t\}$ is closed since it is an arbitrary union of closed sets, and $\{u > t\}$ is open relative to $\overline{\Omega}$. To prove this let $x_0 \in \bigcup_{s \in T, s>t} A_s$, then there exists $s_0 > t$ such that $x_0 \in A_{s_0}$, take α such that $t < \alpha < s_0$ (such α exists since $dist(\partial A_{s_0}, \partial A_t) > 0$), then $A_{s_0} \subset\subset A_\alpha$ hence there exist an open set U such that $A_{s_0} \subset U \subset A_\alpha$. Therefore $U \subset \bigcup_{s\in T, s>t} A_s$ and $x_0 \in A_{s_0} \subset U$. Thus $\{u > t\} = \bigcup_{s\in T, s>t} A_s$ is open relative to $\overline{\Omega}$. With $\{u \geq t\}$ closed and $\{u > t\}$ open we get u continuous on $\overline{\Omega}$.

3. Clearly by definition of u being the supremum we get

$$A_t \subset \{u \geq t\} \text{ for all } t \in T.$$

But $\{u \geq t\} \setminus A_t \subset u^{-1}(t)$. In fact, let x such that $u(x) \geq t$ and $x \notin A_t$, assume that $u(x) > t$ then there exist $s_0 \in T$ such that $x \in A_{s_0}$ and $s_0 > t$, but this implies that $A_{s_0} \subset A_t$, hence $x \in A_t$ which is a contradiction, therefore $u(x) = t$.Now since $|u^{-1}(\{t\})| = 0$ we get $|\{u \geq t\} \setminus A_t| = 0$ for almost all $t \in T$.

□

Theorem 4.4.2. *If Ω is a bounded lipschitz domain that satisfies* (4.2) *and* (4.3)*, then the function u defined by* (4.17) *is a solution to* (4.1)*.*

Proof. Let $v \in BV(\Omega) \cap C(\overline{\Omega}), v = g$ on $\partial\Omega$, be a competitor in (4.1).Recall the extension $G \in BV(\mathbb{R}^n \setminus \overline{\Omega})$ of g. Now define an extension $\overline{v} \in BV(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ of v by $\overline{v} = G$ in $\mathbb{R}^n \setminus \overline{\Omega}$.

Let $F_t = \{\overline{v} \geq t\}$. It is sufficient to show that

$$P(E_t, \Omega) \leq P(F_t, \Omega) \tag{4.18}$$

for almost every $t \in T$, because then $v \in BV(\Omega)$ and the coarea formula 1.6.2 would imply that

$$\int_{-\infty}^{+\infty} P(E_t, \Omega)dt \leq \int_{-\infty}^{+\infty} P(F_t, \Omega)dt = |Dv|(\Omega) < \infty.$$

Hence $u \in BV(\Omega)$, furthermore $|Du|(\Omega) \leq |Dv|(\Omega)$.

We know that E_t is a solution of (4.7) while $F_t \setminus \overline{\Omega} = \mathcal{L}_t \setminus \overline{\Omega}$ almost everywhere, hence

$$P(E_t, \mathbb{R}^n) \leq P(F_t, \mathbb{R}^n). \tag{4.19}$$

Next note that

$$\begin{aligned} P(E_t, \mathbb{R}^n) &= \mathcal{H}^{n-1}(\mathbb{R}^n \cap \partial^* E_t) \\ &= \mathcal{H}^{n-1}(\partial^* E_t \cap \partial\Omega) + \mathcal{H}^{n-1}(\partial^* E_t \cap \Omega) + \mathcal{H}^{n-1}(\partial^* E_t \cap \overline{\Omega}^c) \\ &= \mathcal{H}^{n-1}(\partial^* E_t \cap \partial\Omega) + P(E_t, \Omega) + \mathcal{H}^{n-1}(\partial^* \mathcal{L}_t \setminus \overline{\Omega}) \\ &\geq \mathcal{H}^{n-1}(\partial^* \mathcal{L}_t \setminus \overline{\Omega}) + P(E_t, \Omega). \end{aligned} \tag{4.20}$$

Observe also that

$$P(F_t, \mathbb{R}^n) = \mathcal{H}^{n-1}(\partial^* F_t \setminus \overline{\Omega}) + \mathcal{H}^{n-1}(\partial^* F_t \cap \Omega) + \mathcal{H}^{n-1}(\partial^* F_t \cap \partial\Omega).$$

We claim that $\mathcal{H}^{n-1}(\partial^* F_t \cap \partial\Omega) = 0$ for almost all t.By this claim

$$\begin{aligned} P(F_t, \mathbb{R}^n) &= \mathcal{H}^{n-1}(\partial^* \mathcal{L}_t \setminus \overline{\Omega}) + \mathcal{H}^{n-1}(\partial^* F_t \cap \Omega) \\ &= \mathcal{H}^{n-1}(\partial^* \mathcal{L}_t \setminus \overline{\Omega}) + P(F_t, \Omega). \end{aligned} \tag{4.21}$$

By (4.19) and (4.20) we get

$$\mathcal{H}^{n-1}(\partial^* \mathcal{L}_t \setminus \overline{\Omega}) + P(E_t, \Omega) \leq \mathcal{H}^{n-1}(\partial^* \mathcal{L}_t \setminus \overline{\Omega}) + P(F_t, \Omega).$$

Therefore

$$P(E_t, \Omega) \leq P(F_t, \Omega)$$

and (4.18) established.

Proof of the claim: $\mathcal{H}^{n-1}(\partial^* F_t \cap \partial\Omega) = 0$ for almost all t.

Since $\overline{v} \in C(\mathbb{R}^n)$, we have

$$\partial^* F_t \subset \partial F_t \subset \overline{v}^{-1}(t)$$

but $\mathcal{H}^{n-1}(\overline{v}^{-1}(t) \cap \partial\Omega) = 0$ for all but countably many t, since $\mathcal{H}^{n-1}(\partial\Omega) < \infty$.Thus $\mathcal{H}^{n-1}(\partial^* F_t \cap \partial\Omega) = 0$ for all but countably many t. □

□

Theorem 4.4.3. *If Ω is bounded lipschitz domain that satisfies (4.2) and (4.3),then the function u defined by (4.17) is a solution to*

$$\inf\{|Dv|(\Omega) : v \in BV(\Omega), v = g \text{ on } \partial\Omega\}, \tag{4.22}$$

where $g : \partial\Omega \to \mathbb{R}$ is continuous. Here $v = g$ on $\partial\Omega$ is understood in the sense of trace theory in BV.

Proof. Since

$$\{|Dv|(\Omega) : v \in BV(\Omega) \cap C(\overline{\Omega}), v = g \text{ on } \partial\Omega\} \subset \{|Dv|(\Omega) : v \in BV(\Omega), v = g \text{ on } \partial\Omega\}$$

the inf in (4.22) is less or equal to the inf in (4.1). We need to prove that they are equal.

We will proceed as in theorem 4.4.2. Let $v \in BV(\Omega)$ and $\overline{v}$ its extension as defined in the previous proof. Note that since g is continuous on $\partial\Omega$ and $G \in C(\mathbb{R}^n \setminus \Omega)$ then $\overline{v} \in BV(\mathbb{R}^n) \cap C(\mathbb{R}^n \setminus \Omega)$. We only need to prove

$$P(E_t, \Omega) \leq P(F_t, \Omega)$$

for almost every $t \in T$ where $F_t = \{\overline{v} \geq t\}$.As in the proof of the previous theorem we have

$$P(E_t, \mathbb{R}^n) \leq P(F_t, \mathbb{R}^n) \text{ and } P(E_t, \mathbb{R}^n) \geq \mathcal{H}^{n-1}(\partial^* \mathcal{L}_t \setminus \overline{\Omega}) + P(E_t, \Omega).$$

We need to show that $P(F_t, \mathbb{R}^n) = \mathcal{H}^{n-1}(\partial^* \mathcal{L}_t \setminus \overline{\Omega}) + P(F_t, \Omega)$ i.e $\mathcal{H}^{n-1}(\partial^* F_t \cap \partial\Omega) = 0$. We have that g is the trace in $\partial\Omega$ of $\overline{v} \in BV(\Omega)$. By theorem 1.6.4

$$\lim_{r\to 0} \fint_{B(x,r)\cap\Omega} |\overline{v}(y) - g(x)|dy = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-almost all } x \in \partial\Omega. \tag{4.23}$$

On my way to prove that $\partial^* F_t \cap \partial\Omega \subset g^{-1}(t)$. For this, consider $x \in \partial^* F_t \cap \partial\Omega$ that satisfies (4.23), for such an x observe that $g(x) = t$. Indeed, if $g(x) < t$, say $g(x) = t - \epsilon$ then

$$\begin{aligned}
0 &= \lim_{r\to 0} \frac{1}{|B(x,r)\cap\Omega|} \int_{B(x,r)\cap\Omega} |\overline{v}(y) - g(x)|dy \\
&= \lim_{r\to 0} \frac{1}{|B(x,r)\cap\Omega|} \left(\int_{B(x,r)\cap\Omega\cap\{\overline{v}<t\}} |\overline{v}(y) - g(x)|dy + \int_{B(x,r)\cap\Omega\cap\{\overline{v}\geq t\}} |\overline{v}(y) - g(x)|dy \right) \\
&\geq \limsup_{r\to 0} \frac{1}{|B(x,r)\cap\Omega|} \left(\int_{B(x,r)\cap\Omega\cap\{\overline{v}\geq t\}} |\overline{v}(y) - g(x)|dy \right) \\
&\geq \limsup_{r\to 0} \frac{\epsilon |B(x,r)\cap\Omega\cap\{\overline{v}\geq t\}|}{|B(x,r)\cap\Omega|}
\end{aligned}$$

Therefore

$$\limsup_{r\to 0} \frac{|B(x,r)\cap\Omega\cap\{\overline{v}\geq t\}|}{|B(x,r)\cap\Omega|} = 0$$

Using also the fact that g is the trace of $\overline{v} \in BV(\mathbb{R}^n \setminus \Omega)$ we employ a similar argument and get

$$\limsup_{r\to 0} \frac{|B(x,r)\cap(\mathbb{R}^n\setminus\Omega)\cap\{\overline{v}\geq t\}|}{|B(x,r)\cap(\mathbb{R}^n\setminus\Omega)|} = 0.$$

Now,

$$\begin{aligned}
\limsup_{r\to 0} \frac{|B(x,r)\cap\{\overline{v}\geq t\}|}{|B(x,r)|} &= \limsup_{r\to 0} \frac{|B(x,r)\cap\Omega\cap\{\overline{v}\geq t\}| + |B(x,r)\cap(\mathbb{R}^n\setminus\Omega)\cap\{\overline{v}\geq t\}|}{|B(x,r)\cap\Omega| + |B(x,r)\cap(\mathbb{R}^n\setminus\Omega)|} \\
&= \limsup_{r\to 0} \Big(\frac{|B(x,r)\cap(\mathbb{R}^n\setminus\Omega)\cap\{\overline{v}\geq t\}|}{|B(x,r)\cap\Omega| + |B(x,r)\cap(\mathbb{R}^n\setminus\Omega)|} \\
&\qquad + \frac{|B(x,r)\cap\Omega\cap\{\overline{v}\geq t\}|}{|B(x,r)\cap\Omega| + |B(x,r)\cap(\mathbb{R}^n\setminus\Omega)|} \Big) \\
&\leq \limsup_{r\to 0} \Big(\frac{|B(x,r)\cap(\mathbb{R}^n\setminus\Omega)\cap\{\overline{v}\geq t\}|}{|B(x,r)\cap(\mathbb{R}^n\setminus\Omega)|} \\
&\qquad + \frac{|B(x,r)\cap\Omega\cap\{\overline{v}\geq t\}|}{|B(x,r)\cap\Omega|} \Big) \\
&= 0.
\end{aligned}$$

Hence we conclude that

$$\limsup_{r\to 0} \frac{|B(x,r)\cap\{\overline{v}\geq t\}|}{|B(x,r)|} = 0.$$

This implies that $x \notin \partial_M F_t$ and since $\partial^* F_t \subset \partial_M F_t$ we get $x \notin \partial^* F_t$ which is a contradiction. We do the same if $g(x) > t$ (take $g(x) = \epsilon + t$). Hence $g(x) = t$ thus $\partial^* F_t \cap \partial\Omega \subset g^{-1}(t)$ and therefore $\mathcal{H}^{n-1}(\partial^* F_t \cap \partial\Omega) \leq \mathcal{H}^{n-1}(g^{-1}(t)) = 0$ for all but countably many t. Thus

$$P(F_t, \mathbb{R}^n) = \mathcal{H}^{n-1}(\partial^* \mathcal{L}_t \setminus \overline{\Omega}) + P(F_t, \Omega)$$

And as in the previous theorem we get that u is a solution. □

4.5 Uniqueness

Theorem 4.5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded lipschitz domain satisfying (4.2) and (4.3). Suppose $u_1, u_2 \in C(\overline{\Omega}) \cap BV(\Omega)$ are minimizers of (4.1) with boundary data g_1 and g_2, respectively. If $g_1 \geq g_2$ on $\partial\Omega$, then $u_1 \geq u_2$ in $\overline{\Omega}$.*

Proof. Suppose there exist $x_0 \in \Omega$ such that $u_1(x_0) < u_2(x_0)$. Choose real numbers s and t such that $u_1(x_0) < s < t < u_2(x_0)$.

Let $A = \overline{\Omega} \cap \{u_1 \geq s\}$ and $B = \overline{\Omega} \cap \{u_2 \geq t\}$. In view of theorem 4.4.3, we have

$$|Du_i|(\Omega) = \inf\{|Dv|(\Omega) : v \in BV(\Omega), v = g_i \text{ on } \partial\Omega\}.$$

Consequently, by theorem 3.1.2 we conclude that ∂A and ∂B are area-minimizing relative to Ω.
We will now proceed to show that $\partial A = \partial(A \cup B)$ by establishing that $\partial A \cap \partial(A \cup B)$ is both open and closed relative to both ∂A and $\partial(A \cup B)$. Then this will lead to a contradiction.

First note that

$$B \setminus A \subset\subset \Omega. \tag{4.24}$$

To prove this we will use the continuity of $u_2 - u_1$ and the compactness of $\partial\Omega$: The continuity of $u_2 - u_1$ implies that $\{x : (u_2 - u_2)(x) \geq t - s\} = (u_2 - u_1)^{-1}([t - s, \infty))$ is closed. Now clearly, $B \setminus A \subset \{x : (u_2 - u_2)(x) \geq t - s\}$.Therefore

$$\overline{B \setminus A} \subset \overline{\{x : (u_2 - u_2)(x) \geq t - s\}} = \{x : (u_2 - u_2)(x) \geq t - s\}.$$

We still need to prove that $\{x : (u_2 - u_2)(x) \geq t - s\} \cap \partial\Omega = \emptyset$. Assume not,then there exist $x_0 \in \partial\Omega \cap \{x : (u_2 - u_2)(x) \geq t - s\}$, but on $\partial\Omega$ we have $u_2 = g_2$ and $u_1 = g_1$ with $g_1 \geq g_2$ hence $(u_2 - u_1)(x_0) \leq 0$ and $0 < t - s \leq (u_2 - u_1)(x_0) \leq 0$,which is a contradiction. Thus

$$\overline{B \setminus A} \subset \{x : (u_2 - u_2)(x) \geq t - s\} \subset\subset \Omega$$

and $B \setminus A \subset\subset \Omega$. Form this inclusion it follows that

$$(1) \quad P(A, \Omega) \leq P(A \cup B, \Omega) \text{ and } (2) \quad P(B, \Omega) \leq P(A \cap B, \Omega),$$

In fact to prove (1), notice that ∂A is area-minimizing then by definition $P(A, \Omega) \leq P(F, \Omega)$ for all F where $A \Delta F \subset\subset \Omega$. Take $F = A \cup B$,

$$A \Delta F = (A \cup (A \cup B)) \setminus (A \cap (A \cup B)) = (A \cup B) \setminus A = B \setminus A \subset\subset \Omega.$$

Therefore $P(A,\Omega) \leq P(A \cup B,\Omega)$. Same for (2), with ∂B area minimizing and $F = A \cap B$. Now invoking (1.3), we get

$$P(A,\Omega) = P(A \cup B,\Omega) \text{ and } P(B,\Omega) = P(A \cap B,\Omega).$$

Therefore $A \cup B$ and $A \cap B$ are area-minimizing relative to Ω. Clearly $\partial A \cap \partial(A \cup B)$ is closed relative to ∂A and $\partial(A \cup B)$. From theorem 3.2.3 and (4.24) , with $\partial A, \partial(A \cup B)$ both area-minimizing and $A \subset A \cup B$ we get that $\partial A \cap \partial(A \cup B)$ is open relative to ∂A and $\partial(A \cup B)$.

Before proceeding, we employ the following elementary topological observation: If X and Y are sets such that $X \cap Y$ is open and closed relative to both X and Y then any component of either X or Y that intersects $X \cap Y$ is necessarily contained in $X \cap Y$. Consequently, each component of X is either contained in Y or disjoint from Y. Similarly, each component of Y is either contained in X or is disjoint form X.

Thus with $X = \partial A$ and $Y = \partial(A \cup B)$ we may conclude that $\partial A \subset \partial(A \cup B)$ because any component of ∂A disjoint from $\partial(A \cup B)$ would be contained in Ω, which is an impossibility.In fact, assume that there exist $C \subset \partial A$ such that $C \cap \partial(A \cup B) = \emptyset$ and $C \subset \Omega$.

Claim:$\partial A \cap \partial\Omega \cap (\partial(A \cup B))^c = \emptyset$.

Proof of the claim: Let $x \in \partial A \cap \partial\Omega$, then $u_1(x) = g_1(x) = s \geq u_2(x) = g_2(x)$.Let $\epsilon > 0$ and take $N_\epsilon(x)$, since $x \in \partial A$ then $N_\epsilon(x) \cap A \neq \emptyset$ therefore $N_\epsilon(x) \cap (A \cup B) \neq \emptyset$ and also $N_\epsilon(x) \cap A^c \neq \emptyset$. We can also see that $\partial A \cap \partial\Omega \subset B^c$ implying that $N_\epsilon(x) \cap B^c \neq \emptyset$, hence $N_\epsilon(x) \cap (A \cup B)^c \neq \emptyset$, thus $x \in \partial(A \cup B)$ therefore $\partial A \cap \partial\Omega \subset \partial(A \cup B)$ and thus $\partial A \cap \partial\Omega \cap (\partial(A \cup B))^c = \emptyset$. □

Now by this claim and as in the proof of Lemma 4.3.3,we get that $\partial A \cap (\partial(A \cup B))^c \cap \Omega = \emptyset$ which is a contradiction with our assumption.Therefore we get that $\partial A \subset \partial(A \cup B)$.

This same argument can be applied with $X = \partial(A \cup B)$ and $Y = \partial A$ and conclude that $\partial(A \cup B) \subset \partial A$ hence $\partial A = \partial(A \cup B)$.

We will finally show that this leads to a contradiction. Let $S = B^i \setminus A$ then $B^i \setminus A \subset B \setminus A \subset\subset \Omega$ so $S = B^i \setminus A \subset\subset \Omega$ and $\partial S \subset \partial A \cup \partial B$. However, it is not possible that $\partial S \subset \partial A$.Assume by contradiction that $\partial S \subset \partial A$,then ∂S is area-minimizing with $S \subset\subset \Omega$ but this is impossible because we get $\partial S \cap \partial A \cap \partial\Omega = \emptyset$ since $S \subset\subset \Omega$. Hence as above we get $\partial S \cap \partial A \cap \Omega = \emptyset$ but $\partial S \subset \partial A$, hence $\partial S \cap \partial A \cap \Omega = \partial S \cap \Omega = \emptyset$ but this is not true since $S \subset\subset \Omega$.

Thus there is $x^* \in \partial S \cap (\partial B \setminus \partial A)$ and an open set U containing x^* such that $U \cap A = \emptyset$ this implies that $(A \cup B) \cap U = B \cap U$ and therefore $\partial(A \cup B) \cap U = \partial B \cap U$ and since $x^* \in \partial B \cap U$, we get $x^* \in \partial(A \cup B) \cap U$ and therefore $x^* \in \partial(A \cup B)$ but $x^* \notin \partial A$ which contradicts that $\partial(A \cup B) = \partial A$.Finally, we conclude that $u_1 \geq u_2$ in $\overline{\Omega}$. □

Corollary 4.5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded lipschitz domain satisfying (4.2) and (4.3). If $u_1, u_2 \in C(\overline{\Omega}) \cap BV(\Omega)$ are solutions to (4.1) relative to their own boundary* *data, then*

$$\sup_{\Omega} |u_1 - u_2| = \sup_{\partial\Omega} |u_1 - u_2|$$

In particular, the solution to (4.1) *is unique.*

Proof. Let $u^* = u_1 + \sup_{\partial\Omega} |u_1 - u_2|$. u^* has least gradient and $u^* \geq u_2$ on $\partial\Omega$ hence by theorem 4.5.1 we get $u^* \geq u_2$ on $\overline{\Omega}$, that is $u_1 \geq u_2 - \sup_{\partial\Omega} |u_1 - u_2|$.

Reversing the roles u_1 and u_2 we get $u_2 \geq u_1 - \sup_{\partial\Omega} |u_1 - u_2|$, then $|u_1 - u_2| \leq \sup_{\partial\Omega} |u_1 - u_2|$ on $\overline{\Omega}$. Thus $\sup_{\overline{\Omega}} |u_1 - u_2| \leq \sup_{\partial\Omega} |u_1 - u_2|$ but $\sup_{\partial\Omega} |u_1 - u_2| \leq \sup_{\overline{\Omega}} |u_1 - u_2|$ and therefore

$$\sup_{\Omega} |u_1 - u_2| = \sup_{\partial\Omega} |u_1 - u_2|.$$

In particular,with same boundary value we get that the solution to (4.1) is unique,

$$\sup_{\Omega} |u_1 - u_2| = 0.$$

Therefore $u_1 = u_2$ on Ω.

□

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