AMERICAN UNIVERSITY OF BEIRUT

L-SERIES OF HARMONIC MAASS FORMS

by
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Abstract
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We introduce Harmonic Maass forms and present some of their analytic properties. We continue to define their related L-series by using Laplace transform, and prove their functional equations. Our primary objective is to develop a converse theorem for these L-series in both integral and non-integral weights. This became achievable through the definition of the mentioned L-series on a broader class of test functions. To illustrate the idea, we first present an outline of the special case of weakly holomorphic modular forms on $SL_2(\mathbb{Z})$ and then extend it to Harmonic Maass forms. Subsequently, we consider an example of using the converse theorem.

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Chapter 1

Introduction

The analysis of holomorphic modular forms and Maass forms is often guided by the study of their L-series. Specifically, a connection has been established between L-series and a particular category of harmonic Maass forms known as weakly holomorphic modular forms, leading to interesting findings. Despite these achievements, the investigation of this L-series has not been as thorough as that of the modular objects. Additionally, the definition of L-series has not been extended to all harmonic Maass forms.

In this thesis, we define the L-series of harmonic Maass forms using the Laplace transform and prove their functional equations. With this definition, we succeed in establishing a converse theorem. To illustrate the idea, we first outline the special case of weakly holomorphic modular forms on $SL_2(\mathbb{Z})$.

The first chapter introduces some basic number theoretic functions and their properties. These functions will be needed in later chapters. We define modular forms on the full modular group and give their related properties. We also introduce Hecke operators and show that the L-series associated with Hecke eigenforms have a product expansion. The second chapter serves as a comprehensive introduction to harmonic Maass forms in both integer and half-integer weights. Their transformation properties characterize them under the modular group and specific harmonic conditions. Additionally, in the same chapter, we define the raising and lowering operators that play a crucial role in changing the weight of these forms. In the context of half-integer weights, harmonic Maass forms are characterized by their action under $\Gamma_0(N)$ and Dirichlet characters. Also, it provides insights into their Fourier expansions, ensuring absolute convergence. In chapter 3 the main goal is to prove the functional equation of the $L-$series $L_f(\phi)$ when f is a harmonic Maass form. Finally, chapter 4 discusses a converse theorem for $L-$series of general harmonic Maass forms.

1.1 Definitions and Basic Properties

In this section, we introduce some elementary number theoretic functions needed for our work in later chapters.

1.1.1 *Whittaker Functions*

Whittaker functions are special functions that represent solutions to a particular second-order ordinary differential equation.

Whittaker's normalized differential equation is the second-order ordinary differential equation

$$\frac{\partial^2}{\partial y^2}G(y) + \left(\frac{-1}{4} + \frac{k}{y} + \frac{\frac{1}{4} - \nu^2}{y^2}\right)G(y) = 0$$

for smooth functions $G : (0, \infty) \to \mathbb{C}$ and $\nu \notin \frac{-1}{2}\mathbb{N}$. We have two solutions $M_{k,\nu}(y)$ and $W_{k,\nu}(y)$ with different behavior as $y \to \infty$:

$$M_{k,\nu}(y) \sim \frac{\Gamma(1 + 2\nu)}{\Gamma(\frac{1}{2} + \nu + k)} e^{\frac{1}{2}y} y^{-k}$$

and

$$W_{k,\nu}(y) \sim e^{-\frac{1}{2}y} y^{-k}.$$

1.1.2 *Definition and Properties of Gamma Function*

Definition 1.1.1. *The Gamma function is defined by the integral formula*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \tag{1.1}$$

1. *The Gamma function is analytic in the Region $Re(z) > 0$.*
2. *The Gamma function satisfies $\Gamma(s+1) = s\Gamma(z)$, so it can be extended for all s except the negative integers and zero.*

Note that these are some of the many properties of the Gamma function.

Proof. Regarding (1), first, we split the integral, so:

$$\begin{aligned} \Gamma(z) &= \int_0^\infty e^{-t} t^{z-1} dt \\ &= \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt \end{aligned}$$

Now,for $0 \le t \le 1$ knowing that $e^{-t} \le 1$,

$$\begin{aligned} \left| \int_{\frac{1}{n}}^{\frac{1}{m}} e^{-t} t^{z-1} dt \right| &\le \int_{\frac{1}{n}}^{\frac{1}{m}} t^{Re(z)-1} dt \\ &\le \infty \end{aligned}$$

And

$$\left| \int_n^\infty e^{-t} t^{z-1} dt \right| \leq \int_n^\infty e^{-t} t^{Re(z)-1} dt$$
$$\leq \int_n^\infty e^{-\frac{1}{2}t} dt$$
$$\leq 2e^{-\frac{1}{2}n}$$

this tends to zero as $n \to \infty$, giving uniform convergence. Hence we get that the Gamma function is analytic for $Re(z) > 0$.

The equality in (2) is easily done by integration by parts:

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt$$
$$= [-t^z e^{-t}]_0^\infty + \int_0^\infty z t^{z-1} e^{-t} dt$$
$$= \int_0^\infty z t^{z-1} e^{-t} dt$$
$$= z\Gamma(z).$$

□

The incomplete Gamma function is defined as:

$$\Gamma(r, z) := \int_z^\infty e^{-t} t^{r-1} dt. \tag{1.2}$$

When $z \neq 0$, $\Gamma(r, z)$ is an entire function of r. We note the asymptotic relation for $x \in \mathbb{R}$

$$\Gamma(s, x) \sim x^{s-1} e^{-x} \tag{1.3}$$

as $|x| \to \infty$. In this chapter, we will also be using the following formula for the incomplete Gamma function

$$\Gamma(n+1,z) = n!e^{-z}e_n(z). \tag{1.4}$$

where

$$e_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}. \tag{1.5}$$

1.1.3 *The Legendre-Jacobi-Kronecker Symbol*

Let p be an odd prime, then for any $a \in \mathbb{Z}$ the congruence

$$x^2 \equiv a (modp)$$

can have no solution (a is a quadratic non-residue mod p), one solution if $a \equiv 0(modp)$, or two solutions(a is a quadratic residue mod p).
Define the Legendre symbol $\left(\frac{a}{b}\right)$ to be

$$\left(\frac{a}{b}\right) = \begin{cases} -1 \text{ if a is quadratic non-residue,} \\ 0 \text{ if } a = 0, \\ 1 \text{ if a is a quadratic residue.} \end{cases} \tag{1.6}$$

Proposition 1.1.1. *1. The Legendre Symbol is multiplicative,*

$$\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right).$$

In particular, the product of two quadratic non-residues is a quadratic residue.

2. We have the congruence

$$a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right)(modp).$$

3. There are as many quadratic residues as non-residues modp, i.e. $\frac{p-1}{2}$.

Theorem 1.1.1. *Let p be an odd prime. Then:*

1.
$$\left(\frac{-1}{p}\right) \equiv (-1)^{\frac{p-1}{2}},$$
and
$$\left(\frac{2}{p}\right) \equiv (-1)^{\frac{p^2-1}{8}}.$$

2. *If q is an odd prime different from p, then we have the reciprocity law:*
$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}.$$

Now, we will extend the definition of the Legendre symbol.

Definition 1.1.2. *We define the Kronecker (or Kronecker-Jacobi) symbol $\left(\frac{a}{b}\right)$ for any a and b in $\mathbb{Z}$ in the following way.*

1. *If $b = 0$ then* $\left(\frac{a}{0}\right) = \begin{cases} 1 \text{ if } a = \pm 1, \\ 0 \text{ otherwise.} \end{cases}$

2. *For $b \neq 0$ write $b = \prod p$, where p are not necessarily distinct primes (including $p = 2$), or $p = -1$ to take care of the sign. Then we set*
$$\left(\frac{a}{b}\right) = \prod \left(\frac{a}{p}\right),$$
where $\left(\frac{a}{p}\right)$ is the Legendre symbol defined previously for $p > 2$, and where we define
$$\left(\frac{a}{1}\right) = \begin{cases} 0 & \text{if } a \text{ is even,} \\ (-1)^{\frac{a^2-1}{8}} & \text{if } a \text{ is odd.} \end{cases}$$

and also

$$\left(\frac{a}{-1}\right) = \begin{cases} 1 \text{ if } a \geq 0, \\ -1 \text{ if } a < 0. \end{cases}$$

From the properties of the Legendre symbol, one has the following properties of the Kronecker symbol.

Theorem 1.1.2. *1.* $\left(\frac{a}{b}\right) = 0$ *if and only if* $(a, b) \neq 1$.

2. *For all* a, b, c *we have*

$$\left(\frac{ab}{c}\right) = \left(\frac{a}{c}\right)\left(\frac{b}{c}\right),$$

and

$$\left(\frac{a}{bc}\right) = \left(\frac{a}{b}\right)\left(\frac{a}{c}\right), bc \neq 0.$$

3. $b > 0$ *being fixed, the symbol* $\left(\frac{a}{b}\right)$ *is periodic in* a *of period* $b \neq 2(mod 4)$, *otherwise it is periodic of period* $4b$.

4. $a \neq 0$ *being fixed, the symbol* $\left(\frac{a}{b}\right)$ *is periodic in* b *of period* $|a|$ *if* $a \equiv 0/1(mod 4)$, *otherwise it is periodic of period* $4|a|$.

5. *Let* p *and* q *be positive odd integers not necessarily prime, we have:*

   - $$\left(\frac{-1}{p}\right) \equiv (-1)^{\frac{p-1}{2}},$$

     and

     $$\left(\frac{2}{p}\right) \equiv (-1)^{\frac{p^2-1}{8}}.$$

   - *If* q *is an odd prime different from* p*, then we have the reciprocity law:*

     $$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}.$$

1.1.4 *Dirichlet Character*

Definition 1.1.3. *Let G be a group. A character of G is a group homomorphism $\chi : G \to \mathbb{C}^*$, where $\mathbb{C}^*$ is the multiplicative group of non-zero complex numbers. The set of characters of G is written $\mathcal{G}$.*

Lemma 1.1.1. *If G is finite, then $\mathcal{G}$ is also a finite group.*

We define the Dirichlet character of modulus m where m is a positive integer by the function $\chi : \mathbb{Z} \to \mathbb{C}$ such that for all $a, b \in \mathbb{Z}$ we have:

1. $\chi(ab) = \chi(a)\chi(b)$, χ is multiplicative.
2. $\chi(a) \begin{cases} = 0 \text{ if } (a,m) > 1 \\ \neq 0 \text{ if } (a,m) = 1 \end{cases}$
3. $\chi(a+m) = \chi(a)$, χ is periodic with period m.

We denote the trivial/principal character by χ_0, and it is equal to

$$\chi_0(a) = \begin{cases} 0 \text{ if } (a,m) > 1 \\ 1 \text{ if } (a,m) = 1. \end{cases}$$

One can easily see that property (1) implies that

$$\chi(a^n) = \chi(a)^n.$$

From property (2) we can get that

$$\chi(1) = 1$$

and from property (3)

$$\text{if } a \equiv b (mod m), \text{ then } \chi(a) = \chi(b).$$

In addition, if χ and χ' are two characters for the same modulus, so is their product $\chi\chi'$:

$$\chi\chi'(a) = \chi(a)\chi'(a).$$

The principal character is an identity

$$\chi\chi_0(a) = \chi(a)\chi_0(a) = \begin{cases} 0 \times 0 = \chi(a) \text{ if } (a, m) > 1 \\ \chi(a) \times 1 = \chi(a) \text{ if } (a, m) = 1. \end{cases}$$

Primitive Dirichlet Characters

Given a Dirichlet character χ_1 of modulus m_1 dividing m_2, we can always create a Dirichlet character χ_2 of modulus m_2 by defining

$$\chi_2(n) \begin{cases} := \chi_1(n) \text{ for } n \in (\mathbb{Z}/m_2\mathbb{Z})^\times \\ := 0 \text{ otherwise} \end{cases}$$

Provided m_2 is divisible by a prime p that does not divide m_1, the Dirichlet characters χ_1 and χ_2 will not be the same. In the same process, we can create infinitely many characters that differ from χ_1 in the same way. We will distinguish between the characters that do and do not arise this way.

Definition 1.1.4. *Let χ_1 and χ_2 be Dirichlet characters modulus m_1 and m_2 respectively, with $m_1|m_2$. If $\chi_1(n) = \chi_2(n)$ for $n \in (\mathbb{Z}/m_2\mathbb{Z})^\times$, then χ_2 is induced by χ_1.*

Definition 1.1.5. *A Dirichlet character is primitive if it is not induced by any Dirichlet character other than itself. Dirichlet character is called principal when it is induced by 1(and is primitive $\iff \chi = 1$)*

It is important to note that every Dirichlet character χ is induced by a primitive

Dirichlet Character $\tilde{\chi}$ that is uniquely determined by χ.

Definition 1.1.6. *Let χ be a character mod q. We say d is a quasiperiod of χ if $\chi(m) = \chi(n)$ whenever $m \equiv n(modd)$ and $(mn, q) = 1$. The least quasiperiod of χ is called the conductor of χ.*

Lemma 1.1.2. *Let χ be a Dirichlet character mod q. The conductor of χ is a divisor of q.*

Proof. Let d be a quasiperiod of χ, put $g = (d, q)$. We must show that g is also a quasiperiod of χ.
Suppose

$$m \equiv n(\text{mod } d) \tag{1.7}$$

and (mn,q)=1. Using (1.7) and $g = (d, q)$, by Euclid's algorithm, there is $x, y \in \mathbb{Z}$ such that $m - n = dx + qy$. Thus

$$\chi(m) = \chi(m - qy) = \chi(dx + n) = \chi(n).$$

Thus g is a quasiperiod of χ. □

Definition 1.1.7. *A Dirichlet character χ mod q is called primitive when is has conductor q.*

Orthogonality of Dirichlet Characters

Proposition 1.1.2. *Let $a(modq)$ be a reduced residue class. Then*

$$\sum_{\chi on \mathcal{G}} \chi(n)\bar{\chi}(a) = \begin{cases} \phi(q) \ \textit{if } n \equiv a(modq) \\ 0 \ \textit{otherwise.} \end{cases}$$

Where $\mathcal{G}$ is a group formed by the set of Dirichlet characters mod q.

First note that

$$\bar{\chi}(a) = \frac{1}{\chi(a)} = \chi(a^{-1})$$

Let ψ be a character in $\mathcal{G}$, and note that χ varies over all the characters in $\mathcal{G}$, so does $\psi\chi$. Therefore

$$\psi(na^{-1}) \sum_{\chi \in \mathcal{G}} \chi(na^{-1}) = \sum_{\chi \in \mathcal{G}} \psi(na^{-1})\chi(na^{-1}) = \sum_{\chi \in \mathcal{G}} \psi\chi(na^{-1}) = \sum_{\chi \in \mathcal{G}} \chi(na^{-1})$$

Hence either $\sum_{\chi \in \mathcal{G}} \chi(na^{-1}) = 0$ or $\psi(na^{-1}) = 1$.
If $\sum_{\chi \in \mathcal{G}} \chi(na^{-1}) \neq 0$ then $\psi(na^{-1}) = 1$ which implies that $na^{-1} \equiv 1(modq)$ so, $n \equiv a(modq)$. Which clearly implies that our sum is $\phi(q)$.

Proposition 1.1.3. *Let $q \in \mathbb{N}$. Then*

$$\sum_{\chi mod\ q} \chi(n) = \begin{cases} \phi(q) \ \text{if}\ n \equiv 1 \ mod\ q \\ 0 \ \text{otherwise}\ . \end{cases}$$

And

$$\sum_{n=1,(n,q)=1}^{q} \chi(n) = \begin{cases} \phi(q) \ \text{if}\ \chi = \chi_0 \\ 0 \ \text{if}\ \chi \neq \chi_0, \end{cases}$$

Proposition 1.1.4. *Let q be any natural number and let $\chi(modq)$ be a Dirichlet character. If χ and ψ are two characters $(modq)$ then*

$$\sum_{n(modq)} \chi(n)\bar{\psi}(n) = \begin{cases} \phi(q) \ \text{if}\ \chi = \psi \\ 0 \ \text{if not.} \end{cases}$$

Using (proposition 1.1.3) for the character $\chi\psi$, we immediately get our desired equal-

ity.

we say that $\chi(a)$ is real if all its values are real (they must be 0 or ± 1). χ is real if and only if $\chi^2 = \chi_0$. Real characters are Kronecker symbols.

1.2 Modular Forms.

Denote $\mathbb{H}$ to be the upper half plane, i.e $\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$.

Definition 1.2.1. *The full modular group is*

$$\Gamma(1) = SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ where } a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

It is generated by $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Which means that every matrix $A \in \Gamma(1)$ can be expressed in the form $A = T^{n_1} S T^{n_2} S ... S T^{n_k}$.

Note that the full modular group $\Gamma(1) = SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ by:

$$\text{For } z \in \mathbb{C}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1), \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}$$

Definition 1.2.2. *The principal congruence subgroup of level N of $SL_2(\mathbb{Z})$ is given by*

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (modN) \right\}$$

Definition 1.2.3. *Let $\Gamma \subset SL_2(\mathbb{Z})$ be a congruence subgroup of $SL_2(\mathbb{Z})$.*

A fundamental domain $\mathfrak{F}_\mathfrak{d}$ of Γ is an open subset of the upper half plane such that :

- *No two distinct points of $\mathfrak{F}_\mathfrak{d}$ are equivalent under Γ.*

- *Every point in the upper half plane is equivalent to some point in the closure of $\mathfrak{F}_{\mathfrak{d}}$ under Γ.*

Definition 1.2.4. *For each $A \in \Gamma(1)$, we define the Eichler length $l(M)$ of the matrix A by:*

$$l(A) := min\{\frac{\epsilon_1}{2} + |n_1 + ... + |n_k| + k : M = T^{\epsilon_1} S^{n_1} T S^{n_2} T ... T S^{n_k}\}.$$

And $l(I) := 0$.

Theorem 1.2.1. *The standard fundamental domain of the full modular group $\Gamma(1)$ is given by the open set:*

$$\mathfrak{F}_{\mathfrak{d}(1)} = \{z \in \mathbb{H} : |z| > 1, |Re(z)| < \frac{1}{2}\}$$

Definition 1.2.5. *An automorphic factor j is a map*

$$j : Mat_2(\mathbb{R})) \times \mathbb{C} \to \mathbb{C}$$

$$(M, z) \to j(M, z) = cz + d$$

Where $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $a, b, c, d \in \mathbb{R}$

Definition 1.2.6. *Let Γ be a subgroup of $\Gamma(1)$ with a finite index.*
The multiplier is a function $v : \Gamma \to \mathbb{C}$ such that $|v(M)| = 1, \forall M \in \Gamma$.

We can easily verify that $v(-I) = e^{\pi i k}$, $v(I) = 1$, $v(T) = e^{\frac{\pi i k}{2}}$, and $v(S) = e^{\frac{\pi i k}{6}}$.

A modular form of weight k is a function F satisfying the transformation law:

$$F(Mz) = v(M) j(M, z)^k F(z)$$

Definition 1.2.7. *Let $k \in 2\mathbb{Z}$. A modular form of weight k is a function $f : \mathbb{H} \to \mathbb{C}$ such that:*

1. *f is holomorphic on $\mathbb{H}$.*

2. *$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), z \in \mathbb{H}$.*

3. *f has Fourier expansion of the form: $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$.*

4. *$f(z)$ is bounded as $Im(z) \to \infty$.*

Note that modular forms are of even weights since for

$$I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, z \in \mathbb{C}, f(z) = f(-Iz) = (-1)^k f(z)$$

.

If $c(0) = 0$, the function f is called a cusp form.

Definition 1.2.8. *Let $k > 2$, we define the Eisenstein series of weight k,*
$G_k : \mathbb{H} \to \mathbb{C}$ such that $G_k(z) = \sum_{m\in\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$,
Its Fourier expansion is given by:

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n+1}^{\infty} \sigma_{k-1}(n)e^{2\pi inz}$$

Note that Eisenstein series are entire modular forms of weight k, for $k > 2$.

Theorem 1.2.2. *If $f \in S_{2k}$ with $f(z) = \sum_{n=0}^{\infty} c(n)e^{2\pi in}$, then the Fourier coefficient of f satisfies $c(n) = \mathcal{O}(n^k)$.*

Proof. The series $f(z) = \sum_{n=0}^{\infty} c(n)e^{2\pi in}$ converges absolutely if $|x| < 1$.

Since $c(0) = 0$ we can remove a factor x and write:

$$\begin{aligned} |f(z)| &= |x|\left|\sum_{n=1}^{\infty} c(n)x^{n-1}\right| \\ &\leq |x|\sum_{n=1}^{\infty} |c(n)||x|^{n-1}. \end{aligned} \tag{1.8}$$

if $z \in \mathfrak{F}_{\Gamma(1)}$ then $z = u + iv$ where $v \geq \frac{\sqrt{3}}{3} > \frac{1}{2}$, thus

$$\begin{aligned} |x| &= |e^{2\pi iz}| \\ &= |e^{2\pi i(u+iv)}| \\ &= |e^{2\pi iu-2\pi v}| \\ &= |e^{-2\pi v}|, \text{ since } v > \frac{1}{2} \text{ we obtain} \\ &< |e^{-\pi}| \end{aligned}$$

combining (1) and (2) we obtain:

$$\begin{aligned} |f(z)| &\leq |x|\sum_{n=1}^{\infty} |c(n)||x|^{n-1} \\ &\leq |x|\sum_{n=1}^{\infty} |c(n)|e^{-\pi(n-1)}, \text{ since } |x| < |e^{-\pi}| \\ &\leq A|x|, \text{ where } A = \sum_{n=1}^{\infty} |c(n)|e^{-\pi(n-1)} \\ &= Ae^{-2\pi v} \end{aligned}$$

Hence,

$$|f(z)|v^k \leq Av^k e^{-2\pi v}$$

. Now, define $g(z) = \frac{1}{2}|z - \bar{z}| = v,$ where $z \in \mathbb{H}$. Then for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1),$

$$g(Az) = \frac{1}{2}|Az - A\bar{z}| = |cz + d|^{-2}g(z),$$

so

$$g^k(Az) = |cz + d|^{-2k}g^k(z),$$

and the product

$$\phi(z) = |f(z)|g^k(z)$$

is invariant under transformations if $\Gamma(1)$, since

$$\begin{aligned} \phi(Az) &= |f(z)|g^k(Az) \\ &= (cz + d)^2kf(z)|cz + d|^{-2k}g(z), \\ &= f(z)g(z) \\ &= \phi(z) \end{aligned}$$

Furthermore, $\phi(z)$ is continuous in $\mathfrak{F}_{\Gamma(1)}$, and tends to zero as v tends to infinity. Thus, $\phi(z)$ is bounded in $\mathfrak{F}_{\Gamma(1)}$. Being invariant under$\Gamma(1)$, we conclude that it is bounded in (H), so let's say

$$\phi(z) \leq M, \forall z \in \mathbb{H}.$$

$$\rightarrow |f(z)| \leq Mv^{-k}.$$

By Cauchy's residue theorem, we have

$$c(n) = \int_0^1 f(u + iv)x^{-n}\, du$$

so,

$$|c(n)| \leq \int_0^1 |f(u+iv)x^{-n}|\, du$$

$$|c(n)| \leq Mv^{-k}|x|^{-n} = Mv^{-k}e^{2\pi i v}, \forall v > 0$$

When $v = \frac{1}{n}$, we get $|c(n)| \leq Mn^k e^{\pi} = \mathcal{O}(n^k)$. □

It is important to mention that:

- The only entire modular forms of weight zero are the constant functions.
- Every nonconstant entire modular form has weight $k \geq 4$, where k is even.
- If k is odd, if $k < 0$,or $k = 2$, the only entire modular form of weight k is the zero function.
- The only entire cusp form of weight < 12 is the zero function.

Theorem 1.2.3. *Let $k \in 2\mathbb{Z}$, and let $F : \mathbb{H} \to \mathbb{C}$ be a function satisfying the two relations:*

$$F(z+1) = F(z),$$

$$F\left(\frac{-1}{z}\right) = z^k F(z)$$

for all $z \in \mathbb{H}$. Then, F satisfies

$$F(Mz) = j(M,z)^k F(z)$$

for all $z \in \mathbb{H}$ and all $M \in \Gamma(1)$.

Proof. Each element $M \in \Gamma(1)$ can be expressed in the generators S and T of $\Gamma(1)$, and has an Eichler length. We will make an induction argument in the Eichler length to prove the theorem. Consider first the case where $l(M) = 0$. This implies that

$M = I$ and $F(Mz) = j(M,z)^k F(z)$ simplifies to

$$F(z) = F(Iz) = 1^k F(z).$$

Suppose that $F(Mz) = j(M,z)^k F(z)$ holds for all the elements $M' \in \Gamma(1)$ with Eichler length $l(M') < m$ for some $m \in \mathbb{N}$. Now we consider an element $M \in \Gamma(1)$ with Eichler length $l(M) = m$. Since $\Gamma(1)$ is generated by S and T, we know that M can be written as at least one of the following three cases:

$$M = M'S, M = M'S^{-1}, M = M'T,$$

where M' has a smaller Eichler length. Assume $M = M'S$ for some $M' \in \Gamma(1)$ with $l(M') \leq m-1$. Hence

$$\begin{aligned} F(Mz) = F(M'Sz) \qquad\qquad & = j(M', Sz)^k F(Sz) \\ = j(M', Sz)^k j(S,z)^k F(z) & \\ = j(M'S, z)^k F(z) & \\ = j(M,z)^k F(z). & \end{aligned}$$

The other two cases $M = M'S^{-1}$ and $M = M'T$ follow by the same argument. □

1.2.1 *The Mellin Transform*

Definition 1.2.9. *Let $f : (0,\infty) \to \mathbb{C}$ be a Lebesgue-integrable function. Its Mellin transform is defined as*

$$\mathcal{M}(f)(s) := \int_0^\infty f(t) t^{s-1} dt$$

for all $s \in \mathbb{C}$.

Now, let $k \in 2\mathbb{N},$ and $F \in S_k(\Gamma(1))$. The cusp forms F has a Fourier expansion of the form:

$$F(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}, \forall z \in \mathbb{H}. \tag{1.9}$$

The Fourier coefficients $a(n)$ satisfy the growth estimate:

$$a(n) = \mathcal{O}(n^{\frac{k}{2}}) \text{ as } n \to \infty. \tag{1.10}$$

We associate with F the L-series of F given by:

$$L(s) = \sum_{n \in \mathbb{N}} a(n) n^{-s}, Re(s) > \frac{k}{2} + 1 \tag{1.11}$$

Definition 1.2.10. *Let $k \in 2\mathbb{N}$, and let $f \in S_k$. Define the completed L-function $\Lambda(f, s)$ of f by taking the Mellin transform of f along the upper imaginary axis*

$$\Lambda(f, s) = \int_0^{\infty} f(iy) y^{s-1} dy, \forall s \in \mathbb{C}.$$

Lemma 1.2.1. *$\Lambda(f, s)$ is well defined $\forall s \in \mathbb{C}$.*

Proof. Consider the following integral:

$$\int_0^{\infty} y^{s-1} F(iy) dy.$$

Splitting the path of integration by 1, and then substituting y by $\frac{1}{y}$, we get:

$$\begin{aligned}
\int_0^{\infty} y^{s-1} F(iy) dy &= \int_0^{1} y^{s-1} F(iy) dy + \int_1^{\infty} y^{s-1} F(iy) dy \\
&= \int_1^{\infty} y^{-s-1} F(iy) dy + \int_1^{\infty} y^{s-1} F(iy) dy \\
&= \int_1^{\infty} y^{k-s-1} F(iy) dy + \int_1^{\infty} y^{s-1} F(iy) dy \\
&= \int_1^{\infty} (y^{k-s-1} + y^{s-1}) F(iy) dy
\end{aligned}$$

This integral exists since $F(iy) = \mathcal{O}(e^{-y\delta})$ as $y \to \infty$ for some $\delta > 0$. Thus

$$\Lambda(f, s) = \int_0^\infty y^{s-1} F(iy) dy$$

exists $\forall s$. □

Theorem 1.2.4. *We have*

$$\Lambda(f, s) = \frac{\Gamma(s)}{(2\pi)^s} L(f, s) \tag{1.12}$$

for $R(s) > 1 + \frac{k}{2}$, *where*

$$\Gamma(s) = \int_0^\infty e^{-y} y^{s-1} dy \tag{1.13}$$

is the Gamma function.

Proof. Let $s \in \mathbb{H}$ with $Re(s) > 1 + \frac{k}{2}$. Since, $f \in S_k$, we have $a(n) = \mathcal{O}(n^{\frac{k}{2}})$, thus we can apply the dominated convergence theorem.

$$\begin{aligned}
\Lambda(f, s) &= \int_0^\infty f(iy) y^{s-1} dy \\
&= \int_0^\infty \sum_{n=1}^\infty a(n) e^{-2\pi i n y} y^{s-1} dy \\
&= \int_0^\infty \sum_{n=1}^\infty (2\pi n y)^{-s} e^{-\tau} \tau^{s-1} dy \\
&= (2\pi)^{-s} \sum_{n=1}^\infty \frac{a(n)}{n^s} \int_0^\infty e^{-\tau} \tau^{s-1} dy \\
&= \frac{\Gamma(s)}{(2\pi)^s} L(f, s).
\end{aligned}$$

□

Lemma 1.2.2. *The function* $\Lambda(f, s)$ *of the cusp form* F *extends holomorphically to the complex plane and satisfies the functional relation*

$$\Lambda(f, k - s) = (-1)^{\frac{k}{2}} \Lambda(f, s), \forall s \in \mathbb{C} \tag{1.14}$$

Proof. First recall that

$$F\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z\right) = F\left(\frac{-1}{z}\right) = z^k F(z), \forall z \in \mathbb{H}.$$

And that F is invariant under the map $z \to \frac{-1}{z}$, and acts on the upper imaginary axis as $y \to \frac{1}{y}$. So applying this substitution on $\Lambda(f, s)$, we get:

$$\begin{aligned}
\Lambda(f, s) &= \int_0^\infty F(iy) y^{s-1} dy \\
&= -\int_\infty^0 F(\frac{-1}{iy}) y^{1-s} y^{-2} dy \\
&= \int_0^\infty (iy)^k F(iy) y^{-1-s} dy \\
&= i^k \int_0^\infty F(iy) y^{k-1-s} dy \\
&= i^k \Lambda(f, k-s).
\end{aligned}$$

Now, since $i^k = (-i)^k = (-1)^{\frac{k}{2}}$, we get our desired equality:

$$\Lambda(f, k-s) = (-1)^{\frac{k}{2}} \Lambda(f, s).$$

□

Proposition 1.2.1. *let $F \in S_k(\Gamma(1))$ be a cusp form of weight $k \in 2\mathbb{N}$. Then the L-series defined by $L(s) := \sum_{n \in \mathbb{N}} a(n) n^{-s}$ converges absolutely for $Re(s) > \frac{k}{2}$. And its L-function $\Lambda(f, s) = \int_0^\infty F(iy) y^{s-1} dy$ extends to a holomorphic function on $\mathbb{C}$ and satisfies the functional equation $\Lambda(f, k-s) = (-1)^{\frac{k}{2}} \Lambda(f, s)$.*

Definition 1.2.11. *Let $a, b \in \mathbb{R}$ such that $a < b$, and let $\phi : \{s \in \mathbb{C} : a < Re(s) < b \to \mathbb{C}\}$ be a holomorphic function satisfying the asymptotic estimate*

$$\phi(s) \to 0 \textit{ uniformly as } |Im(s)| \to \infty$$

We also, assume that the path integral $\int_{c-i\infty}^{c+i\infty} |\phi(s)|ds$ *exits for every* c *with* $a + \delta < c < b - \delta$*.(arbitrary but fixed* δ*)*

The inverse Mellin transform is defined by

$$\mathcal{M}^{-1}(\phi)(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \phi(s)x^{-s}ds, x > 0.$$

Lemma 1.2.3. *Let* $L(s) = \sum_{n\in\mathbb{N}} a(n)n^{-s}$ *be an L-series that is absolutely convergent on the half-plane*$Re(s) > \frac{k}{2} + 1$*. Assume that its associated L-function* $\Lambda(f, s) = (2\pi)^{-s}\Gamma(s)L(s)$ *is a holomorphic function on the same half-plane.*

Then, the function $f : i(0, \infty) \to \mathbb{C}$ *on the upper imaginary axis defined by* $f := \mathcal{M}^{-1}(\Lambda(f, s))$ *satisfies:*

$$f(iy) = \sum_{n=1}^{\infty} a(n)e^{-2\pi ny}, y > 0.$$

Proof. Using Stirling's estimate of the $\Gamma-$function:

$$|\Gamma(s)| \leq C_2 e^{\frac{-\pi|t|}{2}}|t|^{\sigma-\frac{1}{2}}, C_2 > 0.$$

We find an upper bound of $\Lambda(f, s)$,

$$\begin{aligned}|\Lambda(f, s)| &= |(2\pi)\Gamma(s)L(s)| \\ &= (2\pi)^{-\sigma}|L(s)||\Gamma(s)| \\ &\leq C_2(2\pi)^{-\sigma}\left(\sum_{n=1}^{\infty}|a(n)|n^{-\sigma}\right)e^{\frac{-\pi|t|}{2}}|t|^{\sigma-\frac{1}{2}}.\end{aligned}$$

for $s = \sigma + it$ with $\sigma > \frac{k}{2} + 1$ bounded, $|t| > 2$, and $C_2 > 0$.Now, since $L(s)$ is convergent absolutely for $Re(s) > \frac{k}{2} + 1$, we have $a(n) = \mathcal{O}(n^{\frac{k}{2}})$. So, we can apply the inverse Mellin transform to $\Lambda(f,s)$. Taking $c > 1 + \frac{k}{2}$, we get:

$$\begin{aligned}
\mathcal{M}^{-1}(\Lambda(f,s))(y) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Lambda(f,s)y^{-s}ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (2\pi)^{-s}\Gamma(s)L(s)y^{-s}ds, \forall y > 0 \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (2\pi)^{-s}\Gamma(s)\sum_{n\in\mathbb{N}} a(n)n^{-s}y^{-s}ds \\
&= \sum_{n\in\mathbb{N}} a(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (2\pi n y)^{-s}\Gamma(s)ds \\
&= \sum_{n\in\mathbb{N}} a(n)e^{-2\pi n y}.
\end{aligned}$$

we get the last equality since we know that

$$\Gamma(s) := \int_0^\infty e^{-y}y^{s-1} = \mathcal{M}(e^{-y})(s), Re(s) > 0.$$

□

Lemma 1.2.4. *For $k \in 2\mathbb{N}$, let $L(s)$ be an L-series that is absolutely convergent on the half-plane $Re(s) > \frac{k}{2} + 1$. Assume that its associated L-function $\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(s)$ extends to a holomorphic function on $\mathbb{C}$ and satisfies the functional equation $\Lambda(f,s) = (-1)^{\frac{k}{2}}\Lambda(f,k-s)$. Then the function $f : i(0,\infty) \to \mathbb{C}$ on the upper imaginary axis defined by $f := \mathcal{M}^{-1}(\Lambda(f,s))$ satisfies:*

$$(iy)^k f(iy) = f\left(\frac{-1}{iy}\right), y > 0.$$

Proof. We already have:

$$f(iy) = \mathcal{M}^{-1}(\Lambda(f,s)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Lambda(f,s)y^{-s}ds,$$

for $y > 0$. Now, applying the functional equation $\Lambda(f,s) = (-1)^{\frac{k}{2}}\Lambda(f,k-s)$, we get:

$$\begin{aligned}
f(iy) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Lambda(f,s)y^{-s}ds \\
&= \frac{(-1)^{\frac{k}{2}}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Lambda(f,k-s)y^{-s}ds \\
&= \frac{-(-1)^{\frac{k}{2}}}{2\pi i}\int_{k-c+i\infty}^{k-c-i\infty} \Lambda(f,s)y^{s-k}ds \\
&= y^{-k}\frac{(-1)^{\frac{k}{2}}}{2\pi i}\int_{k-c-i\infty}^{k-c+i\infty} \Lambda(f,s)\left(\frac{1}{y}\right)^{-s}ds
\end{aligned}$$

for all $y > 0$.

The integral on the right-hand side is again interpreted as the inverse Mellin transform, integrating along the path $(k-c) = i\mathbb{R}$ instead of $c + i\mathbb{R}$. Hence, we have:

$$\begin{aligned}
f(iy) &= y^{-k}\frac{(-1)^{\frac{k}{2}}}{2\pi i}\int_{k-c-i\infty}^{k-c+i\infty} \Lambda(f,s)\left(\frac{1}{y}\right)^{-s}ds \\
&= (-1)^{\frac{k}{2}}y^{-k}\mathcal{M}(\Lambda)\left(\frac{1}{y}\right) \\
&= (-1)^{\frac{k}{2}}y^{-k}f\left(i\frac{1}{y}\right)
\end{aligned}$$

for all $y > 0$. And, since k is even, we have $-1 = i^{-2}$, so we get:

$$f(iy) = (iy)^k f\left(\frac{-1}{iy}\right).$$

□

Lemma 1.2.5. *Let $f : i(0,\infty) \to \mathbb{C}$ be a function given by the Fourier expansion*

$$f(iy) = \sum_{n=1}^{\infty} a(n)e^{-2\pi ny}, y > 0$$

such that the series $\sum_{n=1}^{\infty} a(n)e^{-2\pi ny}$ converges absolutely for all $y > 0$. Then f

extends holomorphically to a function $\tilde{f} : \mathbb{H} \to \mathbb{C}$ given by

$$\tilde{f} = \sum_{n=1}^{\infty} a(n)e^{2\pi nz}, z \in \mathbb{H}.$$

The function $\tilde{f}$ satisfies $\tilde{f}(z+1) = \tilde{f}(z), \forall z \in \mathbb{H}$.

Proof. Using the triangular inequality, we get the following inequality:

$$\left| \sum_{n=1}^{\infty} a(n)e^{2\pi nz} \right| \leq \sum_{n=1}^{\infty} |a(n)|e^{-2\pi ny},$$

where $z = x + iy \in \mathbb{H}$ with real x and $y > 0$. Since $\sum_{n=1}^{\infty}$ converges absolutely for every $y > 0$, we see that the left-hand side of the above inequality converges. Therefore:

$$\tilde{f} : \mathbb{H} \to \mathbb{C}$$

$$z \to \tilde{f}(z) := \sum_{n=1}^{\infty} a(n)e^{2\pi nz}$$

is well defines and forms a holomorphic function on $\mathbb{H}$. The periodicity of $\tilde{f}$ is clearly got from the Fourier expansion using $e^{2\pi in} = 1$. □

Collecting the results of the previous three Lemmas we get the following converse statement.

Proposition 1.2.2. *For $k \in 2\mathbb{N}$, let $L(s) = \sum_{n \in \mathbb{N}} a(n)n^{-s}$ be an L-series that is absolutely convergent on the half-plane $Re(s) > \frac{k}{2} + 1$. Assume that its associated L- function $\Lambda(f, s) := (2\pi)^{-s}\Gamma(s)L(s)$ extends to a holomorphic function on $\mathbb{C}$ and satisfies the functional equation $\Lambda(f, s) = (-1)^{\frac{k}{2}}\Lambda(f, k-s)$. Then the function $f : \mathbb{H} \to \mathbb{C}$ defined by:*

$$f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi nz}$$

is a cusp forms of weight k with trivial multiplier on the full modular group.

Proof. According to Lemma 1.2.3 $f = \mathcal{M}^{-1}(\Lambda)$ is well defined on the upper imaginary axis and has the required Fourier expansion. And Lemma 1.2.5 shows that f extends holomorphically to $\mathbb{H}$ and satisfies $f(z) = f(z+1)$ for all $z \in mathbbH$. Also, Lemma 1.2.4 shows the relation $j(T,z)^k f(z) = f(Tz)$ for all $z \in i(0,\infty)$ on the upper imaginary axis. This relation holds for all $z \in \mathbb{H}$ since f is holomorphic. Now using theorem 1.2.3, we have that f satisfies the required transformation property :$f(Mz) = j(M,z)^k f(z)$ for all $z \in \mathbb{H}$ and all $M \in \Gamma(1)$. Hence, $f \in S_k(\Gamma(1))$. □

1.3 Hecke Operators

One should note that Fourier coefficients provide information about the behavior of a modular form under the action of Hecke operators, and can be used to study the L-functions. However, in the next chapter, while studying Harmonic Maass forms this won't be the case. First, let's define the slash operator,

$$(f|_k M)(z) = j(M,z)^{-k} f(Mz)$$

where f is a meromorphic function on $\mathbb{H}$, and k a real number.

Lemma 1.3.1. *Let $k \in 2\mathbb{Z}$, and $f : \mathbb{H} \to \mathbb{C}$. We have,*

$$(f|_k M)|_k V = f|_k MV$$

$\forall M, V \in GL_2(\mathbb{R})$.

Proof. We have

$$\begin{aligned}\big((f|_k M)|_k V\big)(z) &= j(V,z)^{-k}\big(f|_k M\big)(Vz)\\ &= j(V,z)^{-k} j(M,Vz)^{-k} f(M(Vz))\\ &= j(MV,z)^{-k} f(MVz)\\ &= (f|_k MV)(z), \forall z \in \mathbb{H}.\end{aligned}$$

Let $n \in \mathbb{N}$. We define the set R_* of finite linear combinations of elements $Mat_2(\mathbb{Z})$ with integer coefficients. □

Definition 1.3.1. *Let f be a meromorphic function on $\mathbb{H}$ and k be a real number. We define the slash operator of weight k on R_* by*

$$f|_k \sum_i a_i M_i := \sum_i a_i\big(j(M,z)^{-k} f(M_i z)\big).$$

Definition 1.3.2. *Let k be an integer and let $f : \mathbb{H} \to \mathbb{C}$. For $n \in \mathbb{N}$, we define the Hecke operator of index n by*

$$\begin{aligned}(T_n f)(z) &:= n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right)\\ &= n^{k-1} f|_k \left(\sum_{d|n} \sum_{b=0}^{d-1} \begin{pmatrix} \frac{n}{d} & b \\ 0 & d \end{pmatrix}\right)(z)\end{aligned} \tag{1.15}$$

If $n = p$ prime:

$$(T_p F)(z) = p^{k-1} f(pz) + \frac{1}{p} \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right)$$

Theorem 1.3.1. *If $f \in M_k(\Gamma(1))$ and has the Fourier expansion*

$$f(z) = \sum_{m=0}^{\infty} c(m) e^{2\pi i m z},$$

then $T_n f$ has the Fourier expansion

$$(T_n f)(z) = \sum_{m=0}^{\infty} \gamma(m) e^{2\pi i m z},$$

where

$$\gamma(m) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{nm}{d^2}\right).$$

Proof.

$$(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right),$$

And

$$f(z) = \sum_{m=0}^{\infty} c(m) e^{2\pi i m z},$$

Thus

$$f\left(\frac{nz+bd}{d^2}\right) = \sum_{m=0}^{\infty} c(m) e^{2\pi i m \frac{nz+bd}{d^2}}$$

Therefore

$$\begin{aligned}
(T_n f)(z) &= n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} c(m) e^{2\pi i m \frac{nz+bd}{d^2}} \\
&= \sum_{m=0}^{\infty} \sum_{d|n} \left(\frac{n}{d}\right)^{k-1} c(m) e^{\frac{2\pi i m n z}{d^2}} \frac{1}{d} \sum_{b=0}^{d-1} (e^{\frac{2\pi i m}{d}})^b \\
&= \sum_{m=0}^{\infty} \sum_{d|n, d|m} \left(\frac{n}{d}\right)^{k-1} c(m) e^{\frac{2\pi i m n z}{d^2}} \\
&= \sum_{q=0}^{\infty} \sum_{d|n} \left(\frac{n}{d}\right)^{k-1} c(qd) e^{\frac{2\pi i q n z}{d}}, \text{ (d divides m so m=qd)} \\
&= \sum_{q=0}^{\infty} \sum_{d|n} d^{k-1} c\left(\frac{qn}{d}\right) e^{2\pi i q d z}, \text{ (we replace d by } \frac{n}{d}) \\
&= \sum_{q=0}^{\infty} \sum_{d|n} d^{k-1} c\left(\frac{mn}{d^2}\right) e^{2\pi i m z}.
\end{aligned}$$

□

Notice that letting $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with determinant $n = ad$, (2) can be written as

$$(T_n f)(z) = n^{k-1} \sum_{a \geq 1, ad=n, 0 \leq b < d} d^{-k} f(Az) = \frac{n}{d} \sum_{a \geq 1, ad=n, 0 \leq b < d} a^k f(Az).$$

Which can also be defined as

$$(T_n f)(z) = \frac{1}{n} \sum_A a^k f(Az). \tag{1.16}$$

Theorem 1.3.2. *In every equivalence class of* $\Gamma(n)$ *there is a representative triangular form* $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, *where* $d > 0$

Theorem 1.3.3. *if* $A_1 \in \Gamma(1)$ *and* $V_1 \in \Gamma$, *then there exist matrices* $A_2 \in \Gamma(n)$, *and* $V_2 \in \Gamma$ *such that*

$$A_1 V_1 = V_2 A_2.$$

Also, if $A_i = \begin{pmatrix} \alpha_i & \beta_i \\ 0 & \delta_i \end{pmatrix}$, *and,* $\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$ *for* $i = 1, 2$ *then we have*

$$a_1(\gamma_2 A_2 z + \delta_2) = a_2(\gamma_1 z + \delta_1).$$

Theorem 1.3.4. *If* $f \in M_k$ *and* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, *then*

$$(Tnf)(Vz) = (\gamma z + \delta)^k (Tnf)(z)$$

Proof. We use the representation in (3) to write:

$$(T_n f)(z) = \frac{1}{n} \sum_{A_1} a_1^k f(A_1 z).$$

where $A_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}$. Replacing z by Vz, we obtain:

$$(T_n f)(Vz) = \frac{1}{n} \sum_{A_1} a_1^k f(A_1 V z). \tag{1.17}$$

By The previous theorem, there exists $\begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} \in \Gamma(n)$ and $\begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \in \Gamma$ such that

$$A_1 V = V_2 A_2$$

and

$$a_1(\gamma_2 A_2 z + \delta_2) = a_2(\gamma z + \delta).$$

Thus,

$$\begin{aligned} a_1^k f(A_1 V z) &= a_1^k f(V_2 A_2 z) \\ &= a_1^k (\gamma_2 A_2 z + \delta_2) f(A_2 z) \text{ (since f is an entire modular form)} \\ &= a_2^k (\gamma z + \delta)^k f(A_2 z). \end{aligned}$$

Hence, (4) becomes:

$$\begin{aligned} (T_n f)(Vz) &= \frac{1}{n} \sum_{A_1} a_2^k (\gamma z + \delta)^k f(A_2 z) \\ &= \frac{1}{n} (\gamma z + \delta)^k \sum_{A_1} a_2^k f(A_2 z) \\ &= (\gamma z + \delta)^k (Tnf)(z). \end{aligned}$$

□

Theorem 1.3.5. *If $f \in M_k$ then, $Tnf \in M_k$. Furthermore, if $f \in S_k$ then, $Tnf \in S_k$.*

Definition 1.3.3. *A non-zero function satisfying a relation of the form*

$$T_n f = \lambda(n) f$$

for some complex scalar $\lambda(n)$ *is called an eigenvalue of* T_n*. If* f *is an eigenform so is* cf *for every* $c \neq 0$*. Furthermore, if* f *is an eigenform for every Hecke operator* $T_n, n \geq 1$ *then* f *is called a simultaneous eigenform. A simultaneous eigenform is said to be normalized if* $c(1) = 1$*, where* $f(z) = \sum_{m=0}^{\infty} e^{2piimz}$.

Definition 1.3.4. *Let* $k \in \mathbb{R}$ *be a constant and* $\{a(n)\}_{n \in \mathbb{N}}$ *be a sequence of complex numbers satisfying the growth estimate*

$$c(n) = \mathcal{O}(n^k), \text{ for } n \in \mathbb{Z},$$

the function

$$L(s) := \sum_{n=1}^{\infty} c(n) n^{-s}$$

is called the L-series of $(a(n))_n$.

Note that, Hecke found a connection between each modular form with Fourier series $f(z) = c(0) + \sum_{n=1}^{\infty} c(n) e^{2\pi i n z}$, and the L-series $L(s) = \sum_{n=1}^{\infty} c(n) n^{-s}$ formed with the same coefficient except for $c(0)$. If $f \in S_{2K},$ then $c(n) = \mathcal{O}(n^k),$ and, if $f \in M_{2K},$ then $c(n) = \mathcal{O}(n^{2k-1})$. Thus, the L-series is convergent absolutely for $Re(s) > 2k$ when $f \in M_{2K}$ since

$$\begin{aligned}
|L(f,s)| &= \left| \sum_{n=1}^{\infty} \frac{c(n)}{n^s} \right| \\
&\leq \sum \left| \frac{c(n)}{n^s} \right| \\
&\leq \sum \frac{An^{2k-1}}{n^{Re(s)}}, \text{where A is a constant.} \\
&= \sum \frac{A}{n^{Re(s)-2k+1}} \text{ which is convergent for } Re(s) > 2k.
\end{aligned}$$

Similarly, the L-series is convergent absolutely for $Re(s) > k+1$ when $f \in S_{2K}$.

Theorem 1.3.6. *If the coefficient $c(n)$ satisfies the multiplicative property*

$$c(m)c(n) = \sum_{d|(m,n)} d^{2k-1} c\left(\frac{mn}{d^2}\right) \tag{1.18}$$

the Dirichlet series will have an Euler product representation of the form

$$L(s) = \prod_p \frac{1}{1 - c(p)p^{-s} + p^{2k-1}p^{-2s}}, \tag{1.19}$$

absolutely convergent with the Dirichlet series.

Proof. Since the coefficients are multiplicative, according to ([1], Theorem11.7), when the Dirichlet series converges absolutely, we have

$$L(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \prod_p \left[1 + \sum_{n=1}^{\infty} c(p^n)p^{-ns}\right]. \tag{1.20}$$

Now, (1.18) with ([2], Theorem6.13) implies

$$c(p)c(p^n) = c(p^{n+1}) + p^{2k-1}c(p^{n-1}) \tag{1.21}$$

for each prime p. Using this we get

$$
\begin{aligned}
&(1 - c(p)x + p^{2k-1}x^2)\left(1 + \sum_{n=1}^{\infty} c(p^n)x^n\right) \\
&= 1 - c(p)x + p^{2k-1}x^2 + \sum_{n=1}^{\infty} c(p^n)x^n - c(p)x\sum_{n=1}^{\infty} c(p^n)x^n + p^{2k-1}x^2\sum_{n=1}^{\infty} c(p^n)x^n \\
&= 1 - c(p)x + p^{2k-1}x^2 + \sum_{n=1}^{\infty} c(p^n)x^n + p^{2k-1}x^2\sum_{n=1}^{\infty} c(p^n)x^n - \sum_{n=1}^{\infty}[c(p^{n+1}) + p^{2k-1}c(p^{n-1})]x^{n+1} \\
&= 1 - c(p)x + p^{2k-1}x^2 + c(p)x + \sum_{n=2}^{\infty} c(p^n)x^n + p^{2k-1}x^2\sum_{n=1}^{\infty} c(p^n)x^n - \sum_{n=1}^{\infty} c(p^{n+1})x^{n+1} \\
&\qquad - p^{2k-1}x^2 - p^{2k-1}\sum_{n=2}^{\infty} c(p^{n-1})x^{n+1} \\
&= 1
\end{aligned}
\tag{1.22}
$$

for all $|x| < 1$. Now taking $x = p^{-s}$, we see that (1.20) reduces to (1.19).

□

Chapter 2

Harmonic Maass Forms

Bruinier and Funke [3] wrote an important paper on the theory of geometric theta lifts. In their paper, they defined the notion of a harmonic Maass form. The nonholomorphic modular forms constructed by Zwegers turned out to be weight 1/2 harmonic Maass forms. This coincidental development ignited research on harmonic Maass forms.

Harmonic Maass forms are functions that satisfy certain transformation properties under the modular group, combined with certain harmonic conditions. They are generalizations of classical modular forms and have recently been related to many different topics in number theory: Ramanujan's mock theta functions, Dyson's rank-generating functions, Borcherds products, and central values and derivatives of quadratic twists of modular L-functions.

In this chapter, we recall the definition and basic properties of harmonic Maass forms, while also defining they Fourier expansions ensuring absolute convergence.

2.1 Properties of Harmonic Maass Forms for Even Weights

Definition 2.1.1. *Let $k \in 2\mathbb{Z}$. A twice continuously differentiable function $f; \mathbb{H} \to \mathbb{C}$ is called a weak Harmonic Maass form of weight k and trivial multiplier on the full modular group $\Gamma(1)$, if it satisfies the following three conditions:*

1. $f(Mz) = j(M, z)^k f(z) \forall z \in \mathbb{H}$ *and* $M \in \Gamma(1)$.
2. $f(z) = \mathcal{O}(e^{Cy})$ *as* $Im(z) = y \to \infty$.
3. $\Delta_k f = 0$.

Where

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}$$

is the hyperbolic Laplace operator of weight k.

Note that $\Delta_k f = 0$ explains the word Harmonic in the name weakly Harmonic Maass forms since the functions vanishing under the Laplace operator are called harmonic functions.

Lemma 2.1.1. *Let f be a weak Harmonic Maass form of weight $k \in 2\mathbb{Z}$. Then f has a unique decomposition into two functions f_+ and f_-, such that:*

$$f(z) = f_+(z) + f_-(z).$$

$\forall z \in \mathbb{H}$ *where there is suitable $n_-, n_+ \in \mathbb{Z}$ such that:*

$$f_+(z) = \sum_{n\in\mathbb{Z}, n\geq n_+} a_+(n)e^{2\pi inz},$$

and

$$f_-(z) = a_-(0)Im(z)^{1-k} + \sum_{n\in\mathbb{Z}, n\leq n_-, n\neq 0} a_-(n)\Gamma(1-k, -4\pi n Im(z))e^{2\pi i n z}.$$

Definition 2.1.2. *Let $k \in 2\mathbb{Z}$. We define the subset $H_k^+ = H_k^+(\Gamma(1))$ of $H_k(\Gamma(1))$ by $f \in H_k^+$ if and only if $f \in H_k$ and f satisfies the condition:*
There exists a nonpositive number $n \in \mathbb{Z}$ and a polynomial:

$$P_f = \sum_{n_+\leq n\leq 0} c_f(n)q^n \in \mathbb{C}[q^{-1}],$$

where $q = e^{2\pi i z}$, such that:

$$f(z) - P_f(e^{2\pi i z}) = \mathcal{O}(e^{-cIm(z)})$$

as $Im(z) \to \infty$ for some $c > 0$.
Such harmonic Maass form f has a Fourier expansion of the form:

$$\begin{aligned} f(z) &= f_+(z) + f_-(z) \\ &= \sum_{n\in\mathbb{Z}, n\geq n_+} a_+(n)e^{2\pi i n z} + \sum_{n\in\mathbb{Z}, n<0} a_-(n)\Gamma(1-k, 4\pi|n|Im(z))e^{2\pi i n z}. \end{aligned}$$

$\forall z \in \mathbb{H}$. *Note that the expansion of f_- only contains terms with negative index n.*

We denote the space of real analytic functions f on $\mathbb{H}$ satisfying the following transformation property

$$f(Mz) = v(M)j(M,z)^k f(z),$$

by $A_k = A_k(\Gamma(1))$.

Definition 2.1.3. *We define the raising and lowering operators of weight k by:*

$$R_k = 2i\frac{\partial}{\partial z} + \frac{k}{y},$$

and

$$L_k = -2iy^2\frac{\partial}{\partial \bar{z}},$$

with $z = x + iy \in \mathbb{H}$.

The raising and lowering operators R_k and L_k raise and lower the weight k of real analytic functions f on $\mathbb{H}$ that satisfy $f(Mz) = v(M)j(M,z)^k f(z)$ to $k+2$ and $k-2$ respectively.

Lemma 2.1.2. *For $k \in 2\mathbb{Z}$, we have:*

$$L_k : A_k(\Gamma(1)) \to A_{k-2}(\Gamma(1))$$

and

$$R_k : A_k(\Gamma(1)) \to A_{k+2}(\Gamma(1)).$$

Proposition 2.1.1. *Let $k \in 2\mathbb{Z}$. The differential operator ξ_k given by:*

$$\begin{aligned}(\xi_k f)(z) :&= Im(z)^{k-2}\overline{L_k f(z)}\\ &= R_{-k} Im(z)^k \overline{f(z)}.\end{aligned}$$

defines an anti-linear mapping

$$\xi_k : H_k(\Gamma(1)) \to M^!_{2-k}(\Gamma(1))$$

Proof. We have

$$R_k = 2i\frac{\partial}{\partial z} + \frac{k}{y}$$

so,

$$\begin{aligned}
R_{-k}(y^k\overline{f(z)}) &= 2i\frac{\partial}{\partial z}(y^k\overline{f(z)}) + \frac{-k}{y}\overline{f(z)}\\
&= 2i\frac{\partial}{\partial z}\left(\left(\frac{z-\bar{z}}{2i}\right)^k\overline{f(z)}\right) - ky^{k-1}\overline{f(z)}\\
&= 2i\left(\left(\frac{z-\bar{z}}{2i}\right)^k\frac{\partial}{\partial z}\overline{f(z)}\right) + k\left(\frac{z-\bar{z}}{2i}\right)^{k-1}\overline{f(z)} - ky^{k-1}\overline{f(z)}\\
&= y^{k-2}\left(2iy^2\frac{\partial}{\partial z}\overline{f(z)}\right)\\
&= y^{k-2}\overline{\left(-2iy^2\frac{\partial}{\partial \bar{z}}f(z)\right)}\\
&= Im(z)^{k-2}\overline{L_k f(z)},\ \text{since } L_k = -2iy^2\frac{\partial}{\partial \bar{z}}.
\end{aligned}$$

□

Note that, restricting ξ_k to the space $H_k^+(\Gamma(1))$, we have the map:

$$\xi_k : H_k^+(\Gamma(1)) \to S_{2-k,1}(\Gamma(1)).$$

2.2 Properties of Harmonic Maass Forms for Half Integer Weights

Let $k \in \frac{1}{2}\mathbb{Z}$, let Δ_k denote the weight k hyperbolic Laplacian on $\mathbb{H}$ given by

$$\Delta_k := -4y^2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} + 2iky\frac{\partial}{\partial \bar{z}} \tag{2.1}$$

where $z = x + iy$ with $x, y \in \mathbb{R}$. For $k \in \frac{1}{2} + \mathbb{Z}$. We define the action $|_k$ of $\Gamma_0(N)$, for $4|N$, on smooth function $f : \mathbb{H} \to C$ as follows:

$$(f|_k\gamma)(z) := \left(\frac{c}{d}\right)\epsilon_d^{2k}(cz+d)^{-k}f(\gamma z),\ \text{for all } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(N). \tag{2.2}$$

Where $\left(\frac{c}{d}\right)$ is the Kronecker symbol, and for an odd integer d, we set

$$\epsilon_d = \begin{cases} 1 \text{ if } d \equiv 1 mod 4, \\ i \text{ if } d \equiv 3 mod 4, \end{cases} \tag{2.3}$$

so that $\epsilon_d^2 = \left(\frac{-1}{d}\right)$.

Let $W_M = \begin{pmatrix} 0 & -\sqrt{M}^{-1} \\ \sqrt{M} & 0 \end{pmatrix}$ for $M \in \mathbb{N}$. We have

$$(f|_k W_M)(z) = (f|_k W_M^{-1})(z) = f(W_M z)(-i\sqrt{M}z)^{-k}. \tag{2.4}$$

For $a \in \mathbb{R}_+$, we have

$$\left(f|_k \begin{pmatrix} \frac{1}{a} & b \\ 0 & a \end{pmatrix}\right)(z) = a^{-k} f\left(\frac{z + ba}{a^2}\right). \tag{2.5}$$

Now, we state the definition of Harmonic Maass forms.

Definition 2.2.1. *Let $N \in \mathbb{N}$ when $k \in \frac{1}{2}+\mathbb{Z}$. Let ψ be a Dirichlet character modulo N. A harmonic Maass form of weight k and Dirichlet character ψ for $\Gamma_0(N)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ such that:*

1. *For all $\gamma = \begin{pmatrix} * & * \\ & d \end{pmatrix} \in \Gamma_0(N)$, we have $f|_k\gamma = \psi(d)f$.*
2. *$\Delta_k(f) = 0$, which means the f is an eigenfunction of the hyperbolic Laplacian with eigenvalue zero.*
3. *For each $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, there is a polynomial $P(z) \in \mathbb{C}[e^{-2\pi z}]$, such that*

$$f(\gamma z)(cz + d)^{-k} - P(z) = \mathcal{O}(e^{-\epsilon y}) \text{ as } y \to \infty, \text{ for some } \epsilon > 0.$$

Let $H_k(N,\psi)$ denote the space of Harmonic Maass forms with weight k and character ψ for $\Gamma_0(N)$. To describe the Fourier expansions of the elements of $H_k(N,\psi)$, recall the incomplete gamma function (1.2) and its asymptotic behavior (1.1.2). Thus, now we can state the following theorem,

Theorem 2.2.1. *Let $k \in \mathbb{Z}$. Each $f \in H_k(N,\psi)$ has the absolute convergent Fourier expansion*

$$f(z) = \sum_{n \geq -n_0} a(n) e^{2\pi n z} + \sum_{n<0} b(n)\Gamma(1-k, -4\pi n y) e^{2\pi i n z} \tag{2.6}$$

for some $a(n), b(n) \in \mathbb{C}$, and $n_0 \in \mathbb{N}$.

Chapter 3

L-series

In this chapter, we define the L-series for general harmonic Maass forms. Subsequently, we introduce the renormalized partial derivative and its corresponding L-series, which will be utilized later in proving the converse theorem. Following this, we present the integral form of the L-series and define the twisted function, enabling us to derive the functional equations L_{f_χ} and $L_{\delta_k f_\chi}$. With this, we succeed in establishing a converse theorem that does not appear to have been formulated and proved before.

let L be the Laplace transform mapping each smooth function $\phi : \mathbb{R}_+ \to \mathbb{C}$ to

$$(L\phi)(s) = \int_0^\infty e^{-st}\phi(t)dt \tag{3.1}$$

for each $s \in \mathbb{C}$ for which the integral converges absolutely. And let f a weakly holomorphic cusp form of weight $k \in 2\mathbb{Z}$ for $SL_2(\mathbb{Z})$ with expansion

$$f(z) = \sum_{\substack{n=-n_0,\\ n\neq 0}}^{\infty} a(n)e^{2\pi i n z}. \tag{3.2}$$

Define $\mathcal{F}_f$ to be the space of test functions $\phi : \mathbb{R}_+ \to \mathbb{C}$ such that

$$\sum_{\substack{n=-n_0,\\ n\neq 0}}^{\infty} |a(n)|(L|\phi|)(2\pi n)$$

converges. Then, define the L-series map $L_f : \mathcal{F}_f \to \mathbb{C}$ by

$$L_f(\phi) = \sum_{\substack{n=-n_0,\\ n\neq 0}}^{\infty} a(n)(L\phi)(2\pi n). \tag{3.3}$$

We will state the converse theorem in the special case of weakly holomorphic modular cusp forms for $SL_2(\mathbb{Z})$. Thus, the statement and proof for all harmonic Maass forms will be given in the next sections.

Theorem 3.0.1. *Let $(a(n))_{n\geq -n_0}$ be a sequence of complex numbers such that $a(n) = \mathcal{O}(e^{c\sqrt{n}})$ as $n \to \infty$, for some $c > 0$. For each $z \in \mathbb{H}$, let*

$$f(z) = \sum_{\substack{n=-n_0,\\ n\neq 0}}^{\infty} a(n)e^{2\pi i n z}.$$

Suppose that the function $L_f(\phi)$ defined for each compactly supported smooth $\phi : \mathbb{R}_+ \to \mathbb{C}$ by (3.3), satisfies:

$$L_f(\phi) = i^k L_f(\check{\phi}). \tag{3.4}$$

where $\check{\phi}$ is given by

$$\check{\phi}(x) = x^{k-2}\phi\left(\frac{1}{x}\right).$$

Then f is a weakly holomorphic cusp form of weight $k \in \mathbb{Z}$ for $SL_2(\mathbb{Z})$.

The next theorem is an example of the way the functional equation and the converse theorem can be used.

The main application of our construction is a summation formula for harmonic lifts via the operator ξ_{2-k}. This operator maps a $2-k$ harmonic Maass form f to a weight k holomorphic cusp form

$$\xi_{2-k} f := 2iy^{2-k} \frac{\bar{\partial} f}{\partial \bar{z}}.$$

where $z = x + iy$.

The operator ξ_{2-k} is surjective [4], and finding a preimage for a given cusp form is a fundamental problem in the theory of harmonic Maass forms. Despite its significance, it remains an open question concerning the explicit computation of the holomorphic part of a harmonic Maass form g when a holomorphic cusp form is already known.

In the following theorem, we state in the special case of level one and even weight, and then in the following sections, we will state it and prove it in general.

Theorem 3.0.2. *Let f be a weight $k \in \mathbb{N}$ holomorphic cusp form with Fourier expansion*

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}.$$

Suppose that g is a weight $2-k$ harmonic Maass form such that $\xi_{2-k} g = f$ with Fourier expansion

$$g(z) = \sum_{n \geq -n_0} c^{+}(n) e^{2\pi i n z} + \sum_{n<0} c^{-}(n) \Gamma(k-1, -4\pi n y) e^{2\pi i n z},$$

Then, for every smooth, compactly supported $\phi : \mathbb{R}_{+} \to \mathbb{R}$, we have

$$\sum_{n \geq -n_0} c^{+}(n) \int_0^{\infty} \phi(y) \left(e^{-2\pi n y} - (-iy)^k 2 e^{\frac{-2\pi n y}{y}} \right) dy$$
$$= \sum_{l=0}^{k2} \sum_{n>0} \overline{a(n)} \Bigg(\frac{(k-2)!}{l!} (4\pi n)^{1-k+l} \int_0^{\infty} e^{-2\pi n y} y^l \phi(y) dy$$
$$+ \frac{2^{l+1}}{k-1} (8\pi n)^{\frac{-(k+1)}{2}} \int_0^{\infty} e^{-\pi n y} y^{\frac{k}{2}-1} \phi(y) M_{1-\frac{k}{2}+l, \frac{k-1}{2}}(2\pi n y) dy,$$

where $M_{k,\mu}(z)$ is the Whittaker hypergeometric function.

This is one of the first times where the summation formulas have appeared in the study of harmonic Maass forms.

Now, we will establish the relationship between our L-series and the classical L-series of holomorphic cusp forms. For $s \in \mathbb{C}$, let

$$I_s(x) := (2\pi)^s x^{s-1} \frac{1}{\Gamma(s)}. \tag{3.5}$$

Then, for $u > 0$ and $Re(s) > 0$,

$$\begin{aligned}
(LI_s)(u) &= \int_0^\infty e^{-ut} I_s(t) dt \\
&= \int_0^\infty e^{-ut} (2\pi)^s t^{s-1} \frac{1}{\Gamma(s)} dt \\
&= \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty e^{-ut} t^{s-1} dt \\
&= \frac{(2\pi)^s}{\Gamma(s)} \frac{\Gamma(s)}{u^s} \\
&= \left(\frac{2\pi}{u}\right)^s.
\end{aligned} \tag{3.6}$$

Let f be a holomorphic cusp form for $\Gamma_0(N)$ of weight $k \in \mathbb{Z}$ with Fourier expansion $f(z) = \sum_{n=1}^\infty a(n) e^{2\pi i n z}$. Since $a(n) = \mathcal{O}(n^{\frac{k-1}{2}+\epsilon})$, for any $s \in \mathbb{C}$ with $Re(s) > \frac{k-1}{2}$ [2], we have $I_s \in \mathcal{F}_f$ and

$$L_f(I_s) = \sum_{n=1}^\infty \frac{a(n)}{n^s} \tag{3.7}$$

is the usual L-series of f, i.e, for the holomorphic cusp forms, the L-series is the usual/classical L-series.

However, in the case of weakly holomorphic cusp forms, f can be expressed as $f(z) = \sum_{n=-n_0, n\neq 0}^\infty a(n) e^{2\pi i n z}$ where n_0 is the largest integer such that $a(n_0) \neq 0$.

Their L-series is defined by: for any fixed $t_0 > 0$

$$L(s, f) := \sum_{n \geq n_0, n \neq 0} \frac{a(n)\Gamma(s, 2\pi n t_0)}{(2\pi n)^s} + i^k \sum_{n \geq n_0, n \neq 0} \frac{a(n)\Gamma(k-s, \frac{2\pi n}{t_0})}{(2\pi n)^{k-s}} \tag{3.8}$$

for all $s \in \mathbb{C}$, where $\Gamma(s, x)$ is the incomplete gamma function (1.2) which continues entirely as a function in $s \in \mathbb{C}$ for $x \neq 0$.

For fixed $T > 0$, we define the characteristic function

$$\mathbb{1}_T(x) := \begin{cases} 1 \text{ when } x > T, \\ 0 \text{ otherwise} \end{cases}$$

Then, with I_s defined as in (3.5), we have, for $t_0 > 0$ and $u > 0$.

$$\begin{aligned} L(I_s \mathbb{1}_{t_0})(u) &= \int_0^\infty e^{-ut} I_s(t) \mathbb{1}_{t_0}(t) dt \\ &= \frac{(2\pi)^s}{\Gamma(s)} \int_{t_0}^\infty e^{-ut} t^{s-1} dt, \text{ letting } x = ut, dx = udt \\ &= \frac{(2\pi)^s}{\Gamma(s)} \int_{ut_0}^\infty e^{-x} \left(\frac{x}{u}\right)^{s-1} \frac{1}{u} dx \\ &= \frac{(2\pi)^s}{\Gamma(s)} \frac{1}{u^s} \int_{ut_0}^\infty e^{-x} x^{s-1} dx \\ &= \frac{(2\pi)^s}{u^s} \frac{\Gamma(s, ut_0)}{\Gamma(s)}. \end{aligned} \tag{3.9}$$

The incomplete Gamma function has an analytic continuation giving an entire function of s when $u \neq 0$. Thus, we interpret $L(I_s \mathbb{1}_{t_0})(u)$ as the analytic continuation of $\Gamma(s, ut_0)$.

Now, by the growth estimate of the Fourier coefficient $a(n)$ and the asymptotic behavior of the incomplete Gamma function, we deduce that, for any $t_0 > 0$,

$$L_f(I_s \mathbb{1}_{t0}) = \sum_{n \geq -n_0, n \neq 0} \frac{a(n)}{n^s} \frac{\Gamma(s, ut_0)}{\Gamma(s)}$$

converges absolutely.

The construction of the L-series on a broader class of test functions than on $\{I_s \mathbb{1}_{t0} : s \in \mathbb{C}\}$ makes a converse theorem possible. The dependence on the test function goes through the Laplace transform. As a result, both ϕ and $\check{\phi}$ belong to the domain of absolute convergence of the L-series L_f. This cannot happen in the case of standard L-series of holomorphic cusp forms because there is no value of s for which both I_s and $\check{I}_s$ belong to the same domain of absolute convergence.

3.1 L-series Associated with Harmonic Maass Forms

Let $\mathcal{C}(\mathbb{R}, \mathbb{C})$ be the space of piecewise smooth complex-valued function on $\mathbb{R}$. Recall (3.1) the notation for the Laplace transform of the function ϕ in $\mathbb{R}_+$, where the integral is absolutely convergent.

Now, for s in $\mathbb{C}$, we define

$$\phi_s(x) := \phi(x)x^{s-1},$$

and note that for $s = 1$, $\phi_1 = \phi$.

Let M be a positive integer and $k \in \frac{1}{2}\mathbb{Z}$. For each f on $\mathbb{H}$,

$$f(z) = \sum_{n \geq -n_0} a(n) e^{\frac{2\pi i n z}{M}} + \sum_{n<0} b(n) \Gamma\left(1-k, \frac{-4\pi n y}{M}\right) e^{\frac{2\pi i n z}{M}} \tag{3.10}$$

is the absolute convergence series, where the convergence is obtained by the growth estimates of the Fourier coefficients $a(n)$ and $b(n)$ and by the asymptotic behavior of the incomplete Gamma function.

Let $\mathcal{F}_f$ be the space of functions $\phi \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ such that the integral defining $(L\phi)(s)$ and $(L\phi_{2-k})(s)$ respectively converge absolutely for all s with $Re(s) > -2\pi n_0$ and

$Re(s) > 0$. The following series converges:

$$\sum_{n>\geq -n_0} |a(n)||(L|\phi|)\left(\frac{2\pi n}{M}\right) + \sum_{n<0} |b(n)|\left(\frac{-4\pi n}{M}\right)^{1-k} \int_0^\infty \frac{(L|\phi_{2-k}|)\left(\frac{-2\pi n(2t+1)}{D}\right)}{(1+t)^k} dt. \tag{3.11}$$

Definition 3.1.1. *Let M be a positive integer and $k \in \frac{1}{2}\mathbb{Z}$. Let f be a function on $\mathbb{H}$ given by the Fourier expansion (3.10). The L-series of f is defined to be the map $L_f : \mathcal{F}_f \to \mathbb{C}$ such that, for $\phi \in \mathcal{F}_f$,*

$$\begin{aligned} L_f(\phi) = &\sum_{n>\geq -n_0} a(n)(L\phi)\left(\frac{2\pi n}{M}\right) \\ &+ \sum_{n<0} b(n)\left(\frac{-4\pi n}{M}\right)^{1-k} \int_0^\infty \frac{(L\phi_{2-k})\left(\frac{-2\pi n(2t+1)}{D}\right)}{(1+t)^k} dt. \end{aligned} \tag{3.12}$$

To prove the converse theorem for non-holomorphic elements of $H_k(N,\psi)$, we will need the renormalized version of the partial derivative in terms of x:

$$(\delta_k f)(z) := z\frac{\partial f}{\partial x}(z) + \frac{k}{2}f(z), \tag{3.13}$$

where $z = x+iy$. This operator will be used to ensure the vanishing of the eigenfunction F of the Laplacian, since it is not enough to show vanishing on the imaginary axis, it is also required to have $\frac{\partial F}{\partial x} = 0$.

So, now recalling the Fourier expansion (3.10), we have:

$$\begin{aligned} (\delta_k f)(z) = &\frac{k}{2}f(z) + \sum_{n\geq -n_0} a(n)\left(\frac{2\pi i n z}{M}\right)e^{\frac{2\pi i n z}{M}} \\ &+ \sum_{n<0} b(n)\left(\frac{2\pi i n z}{M}\right)\Gamma\left(1-k, \frac{-4\pi n y}{M}\right)e^{\frac{2\pi i n z}{M}}. \end{aligned} \tag{3.14}$$

Now, we will assign a class of function $\mathcal{F}_{\delta_k f}$ and an L-series map $\mathcal{F}_{\delta_k f} \to \mathbb{C}$ to it.

Let $\mathcal{F}_{\delta_k f}$ be the set of functions $\phi \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ such that the following series converges:

$$\begin{aligned} &2\pi \sum_{n > \geq -n_0} |na(n)| (L\phi_2) \left(\frac{2\pi n}{M} \right) \\ &+ 2\pi \sum_{n<0} |nb(n)| \left(\frac{-4\pi n}{M} \right)^{1-k} \int_0^\infty \frac{(L\phi_{2-k})\left(\frac{-2\pi n(2t+1)}{M}\right)}{(1+t)^k} dt. \end{aligned} \tag{3.15}$$

We let $L_{\delta_k f}$ be such that for $\phi \in \mathcal{F}_{\delta_k f}$,

$$\begin{aligned} L_{\delta_k f} &:= \frac{k}{2} L_f(\phi) - \frac{2\pi}{M} \sum_{n > \geq -n_0} na(n)(L\phi_2)\left(\frac{2\pi n}{M}\right) \\ &+ \frac{2\pi}{M} \sum_{n<0} nb(n) \left(\frac{-4\pi n}{M}\right)^{1-k} \int_0^\infty \frac{(L\phi_{2-k})\left(\frac{-2\pi n(2t+1)}{M}\right)}{(1+t)^k} dt, \end{aligned} \tag{3.16}$$

converges absolutely.

Lemma 3.1.1. *Let f be a function on $\mathbb{H}$ as a series in (3.10). For $\phi \in \mathcal{F}_f$, the L-series $L_f(\phi)$ can be given by*

$$L_f(\phi) = \int_0^\infty f(iy)\phi(y)dy. \tag{3.17}$$

Similarly, for $\phi \in \mathcal{F}_{\delta_k f}$,

$$L_{\delta_k f}(\phi) = \int_0^\infty (\delta_k f) f(iy)\phi(y)dy. \tag{3.18}$$

where $\delta_k f$ is defined in (3.13) and $L_{\delta_k f}$ in (3.16).

Proof. From (3.10), we get

$$f(iy) = \sum_{n \geq -n_0} a(n) e^{\frac{-2\pi n y}{M}} + \sum_{n<0} b(n)\Gamma\left(1-k, \frac{-4\pi n y}{M}\right) e^{\frac{-2\pi n y}{M}}.$$

We multiply both sides by $\phi(y)$ and then integrate over $(0, \infty)$, we get:

$$\int_0^\infty f(iy)\phi(y) = \int_0^\infty \sum_{n \geq -n_0} a(n) e^{\frac{-2\pi n y}{M}} \phi(y) dy + \int_0^\infty \sum_{n<0} b(n) \Gamma\left(k-1, \frac{-4\pi n y}{M}\right) e^{\frac{-2\pi n y}{M}} \phi(y) dy.$$

Now, since $\phi \in \mathcal{F}_f$, we can interchange the order of summation and integration, we get:

$$\int_0^\infty f(iy)\phi(y) = \sum_{n \geq -n_0} \int_0^\infty a(n) e^{\frac{-2\pi n y}{M}} \phi(y) dy + \sum_{n<0} \int_0^\infty b(n) \Gamma\left(1-k, \frac{-4\pi n y}{M}\right) e^{\frac{-2\pi n y}{M}} \phi(y) dy.$$

For the holomorphic part of the series $L_f(\phi)$,

$$\sum_{n \geq -n_0} \int_0^\infty a(n) e^{\frac{-2\pi n y}{M}} \phi(y) dy = \sum_{n \geq -n_0} a(n) \int_0^\infty a(n) e^{\frac{-2\pi n y}{M}} \phi(y) dy = \sum_{n \geq -n_0} a(n) (L\phi)\left(\frac{2\pi n}{M}\right).$$

For the non-holomorphic part, using

$$\Gamma(a, z) = z^a e^{-z} \int_0^\infty \frac{e^{-zt}}{(1+t)^{1-a}} dt$$

for $Re(z) > 0$, we get

$$\begin{aligned}
&\sum_{n<0}\int_0^\infty b(n)\Gamma\Big(1-k,\frac{-4\pi ny}{M}\Big)e^{\frac{-2\pi ny}{M}}\phi(y)dy \\
&=\sum_{n<0}b(n)\int_0^\infty \Gamma\Big(1-k,\frac{-4\pi ny}{M}\Big)e^{\frac{-2\pi ny}{M}}\phi(y)dy \\
&=\sum_{n<0}b(n)\int_0^\infty \bigg(\Big(\frac{-4\pi ny}{M}\Big)^{1-k}e^{\frac{4\pi ny}{M}}\int_0^\infty \frac{e^{\frac{4\pi nyt}{M}}}{(1+t)^k}dt\bigg)e^{\frac{-2\pi ny}{M}}\phi(y)dy \\
&=\sum_{n<0}b(n)\int_0^\infty \Big(\frac{-4\pi n}{M}\Big)^{1-k}y^{1-k}e^{\frac{2\pi ny}{M}}\phi(y)\int_0^\infty \frac{e^{\frac{4\pi nyt}{M}}}{(1+t)^k}dtdy \\
&=\int_0^\infty\int_0^\infty \Big(\frac{-4\pi n}{M}\Big)^{1-k}\phi_{2-k}(y)e^{\frac{2\pi ny(1+2t)}{M}}\times\frac{1}{(1+t)^k}dydt \\
&=\Big(\frac{-4\pi n}{M}\Big)^{1-k}\int_0^\infty \frac{L_{\phi_{2-k}}(\frac{-2\pi n(2t+1)}{M})}{(1+t)^k}dt.
\end{aligned}$$

Hence

$$\begin{aligned}
L_f(\phi) &= \int_0^\infty f(iy)\phi(y)dy \\
&= \sum_{n\geq -n_0}a(n)(L\phi)\Big(\frac{2\pi n}{M}\Big)+\sum_{n<0}b(n)\Big(\frac{-4\pi n}{M}\Big)^{1-k}\int_0^\infty \frac{L_{\phi_{2-k}}(\frac{-2\pi n(2t+1)}{M})}{(1+t)^k}dt.
\end{aligned}$$

We proceed similarly for $L_{\delta_k f}(\phi)$, having (3.14), we get:

$$(\delta_k f)(iy) = \frac{k}{2}f(iy)+\sum_{n\geq -n_0}na(n)\Big(\frac{-2\pi ny}{M}\Big)e^{\frac{-2\pi ny}{M}}+\sum_{n<0}nb(n)\Big(\frac{-2\pi ny}{M}\Big)\Gamma\Big(1-k,\frac{-4\pi ny}{M}\Big)e^{\frac{-2\pi ny}{M}}$$

We multiply both sides by $\phi(y)$ and then integrate over $(0, \infty)$, we get:

$$\int_0^\infty (\delta_k f)(iy)\phi(y) = \int_0^\infty \frac{k}{2} f(iy)\phi(y)dy \tag{3.19}$$

$$+ \int_0^\infty \sum_{n \geq -n_0} na(n)\left(\frac{-2\pi ny}{M}\right)e^{\frac{-2\pi ny}{M}}\phi(y)dy \tag{3.20}$$

$$+ \int_0^\infty \sum_{n<0} nb(n)\left(\frac{-2\pi ny}{M}\right)\Gamma\left(1-k, \frac{-4\pi ny}{M}\right)e^{\frac{-2\pi ny}{M}}\phi(y)dy \tag{3.21}$$

We just proved above that

$$\int_0^\infty \frac{k}{2} f(iy)\phi(y)dy = \frac{k}{2} L_f(\phi)$$

As before, we can interchange summation and integral,

$$\begin{aligned}
\int_0^\infty \sum_{n \geq -n_0} a(n)\left(\frac{-2\pi ny}{M}\right)e^{\frac{-2\pi ny}{M}}\phi(y)dy &= \frac{-2\pi}{M} \sum_{n \geq -n_0} na(n) \int_0^\infty y\phi(y)e^{\frac{-2\pi ny}{M}}dy \\
&= \frac{-2\pi}{M} \sum_{n \geq -n_0} na(n) \int_0^\infty \phi_2(y)e^{\frac{-2\pi ny}{M}}dy \\
&= \frac{-2\pi}{M} \sum_{n \geq -n_0} na(n)(L\phi_2)\left(\frac{2\pi n}{M}\right).
\end{aligned}$$

Finally, regarding (3.18),

$$\begin{aligned}
&\int_0^\infty \sum_{n<0} nb(n)\left(\frac{-2\pi ny}{M}\right)\Gamma\left(1-k, \frac{-4\pi ny}{M}\right)e^{\frac{-2\pi ny}{M}}\phi(y)dy \\
&\qquad = \frac{-2\pi}{M} \sum_{n<0} nb(n) \int_0^\infty \Gamma\left(1-k1, \frac{-4\pi ny}{M}\right)e^{\frac{-2\pi ny}{M}}\phi(y)dy \\
&\qquad + \frac{-2\pi}{M} \sum_{n<0} nb(n)\left(\frac{-4\pi n}{M}\right)^{1-k} \int_0^\infty \frac{L_{\phi_{2-k}}(\frac{-2\pi n(2t+1)}{M})}{(1+t^k)}dt.
\end{aligned}$$

Hence

$$\begin{aligned}
L_{\delta_k f}(\phi) &= \int_0^\infty (\delta_k f) f(z)\phi(y)dy \\
&= \frac{k}{2}L_f(\phi) + \frac{-2\pi}{M}\sum_{n\geq -n_0} na(n)(L\phi_2)\left(\frac{2\pi n}{M}\right) \\
&+ \frac{-2\pi}{M}\sum_{n<0} nb(n)\left(\frac{-4\pi n}{M}\right)^{1-k}\int_0^\infty \frac{L_{\phi_{2-k}}(\frac{-2\pi n(2t+1)}{M})}{(1+t^k)}dt.
\end{aligned}$$

This is our desired result. □

Our goal in the next theorem is to state and prove the functional equation of the L-series $L_f(\phi)$ when $f \in H_k(N,\psi)$.

Let f be a function on $\mathbb{H}$ with Fourier expansion (3.10) where $M = 1$

$$f(z) = \sum_{n\geq -n_0} a(n)e^{2\pi inz} + \sum_{n<0} b(n)\Gamma\left(1-k, -4\pi ny\right)e^{2\pi inz} \tag{3.22}$$

Define the generalized Gauss sum for a Dirichlet character χ modulo D and $n \in \mathbb{Z}$

$$\tau_\chi(n) := \sum_{u mod D} \chi(u)e^{2\pi in\frac{u}{D}}. \tag{3.23}$$

Then we define the twisted function f_χ as

$$
\begin{aligned}
f_\chi(z) :&= D^{\frac{k}{2}} \sum_{u \bmod D} \overline{\chi(u)} \left(f|_k \begin{pmatrix} \frac{1}{\sqrt{D}} & \frac{u}{\sqrt{D}} \\ 0 & \sqrt{D} \end{pmatrix} \right)(z) \\
&= D^{\frac{k}{2}} \sum_{u \bmod D} \overline{\chi(u)} \frac{1}{(\sqrt{D})^k} f\left(\frac{z+u}{D}\right) \\
&= \sum_{u \bmod D} \overline{\chi(u)} f\left(\frac{z+u}{D}\right) \\
&= \sum_{u \bmod D} \overline{\chi(u)} \Big[\sum_{n \geq n_0} a(n) e^{2\pi i n \frac{z+u}{D}} + \sum_{n<0} b(n)\Gamma(1-k, -4\pi n y) e^{2\pi i n \frac{z+u}{D}} \Big] \\
&= \sum_{u \bmod D} \sum_{n \geq n_0} \overline{\chi(u)} e^{2\pi i n \frac{u}{D}} a(n) e^{2\pi i n \frac{z}{D}} + \sum_{u \bmod D} \sum_{n<0} \overline{\chi(u)} e^{2\pi i n \frac{u}{D}} b(n)\Gamma(1-k, -4\pi n y) e^{2\pi i n \frac{z}{D}} \\
&= \sum_{n \geq n_0} a(n) \tau_{\bar{\chi}}(n) e^{2\pi i n \frac{z}{D}} + \sum_{n<0} \tau_{\bar{\chi}}(n) b(n) \Gamma(1-k, -4\pi n y) e^{2\pi i n \frac{z}{D}}.
\end{aligned}
\tag{3.24}
$$

Then the L-series for f_χ is

$$
\begin{aligned}
L_{f_\chi}(\phi) &= \sum_{n \geq -n_0} \tau_{\bar{\chi}}(n) a(n) (L\phi)\left(\frac{2\pi n}{D}\right) \\
&\quad + \sum_{n<0} \tau_{\bar{\chi}}(n) b(n) \left(\frac{-4\pi n}{D}\right)^{1-k} \int_0^\infty \frac{L_{\phi_{2-k}}(\frac{-2\pi n(2t+1)}{M})}{(1+t)^k} dt
\end{aligned}
\tag{3.25}
$$

and for $\delta_k f_\chi$

$$
\begin{aligned}
L_{\delta_k f_\chi}(\phi) &= \frac{k}{2} L_{f_\chi}(\phi) + \frac{-2\pi}{M} \sum_{n \geq -n_0} n \tau_{\bar{\chi}}(n) a(n) (L\phi_2)\left(\frac{2\pi n}{D}\right) \\
&\quad + \frac{-2\pi}{M} \sum_{n<0} n \tau_{\bar{\chi}}(n) b(n) \left(\frac{-4\pi n}{D}\right)^{1-k} \int_0^\infty \frac{L_{\phi_{2-k}}(\frac{-2\pi n(2t+1)}{M})}{(1+t^k)} dt.
\end{aligned}
\tag{3.26}
$$

for $\phi \in \mathcal{F}_{f_\chi} \cap \mathcal{F}_{\delta k(f_\chi)}$. For $a \in \frac{1}{2}\mathbb{Z}, M \in \mathbb{N}$ and $\phi : \mathbb{R}_+ \to \mathbb{C}$, We define:

$$
(\phi|_a W_M)(x) := (Mx)^{-a} \phi\left(\frac{1}{Mx}\right), \forall x > 0,
\tag{3.27}
$$

where $W_M = \begin{pmatrix} 0 & -\sqrt{M}^{-1} \\ \sqrt{M} & 0 \end{pmatrix}$.

Also, define the set $S_c(\mathbb{R}_+)$ of complex-valued, compactly supported, and piecewise smooth functions on $\mathbb{R}_+$, which satisfy: for any $y \in \mathbb{R}_+$, there exists $\phi \in S_c(\mathbb{R}_+)$ such that $\phi(y) \neq 0$.

Now, we are ready to prove the functional equation of $L_f(\phi)$ and its twists.

Theorem 3.1.1. *Fix $k \in \frac{1}{2}\mathbb{Z}$. Let $N \in \mathbb{N}$, and let ψ be a Dirichlet character modulo N. when $k \in \frac{1}{2} + \mathbb{Z}$, assume $4|N$. Suppose that f is an element of $H_k(N, \psi)$ with expansion (3.10) and that χ is a character modulo D with $(D, N) = 1$. Consider the maps $L_{f\chi}, L_{\delta k f\chi} : \mathcal{F}_{f\chi} \cap \mathcal{F}_{\delta k f\chi} \to \mathbb{C}$ given (3.25) and (3.26). Set*

$$g := f|_k W_N$$

and $\mathcal{F}_{f,g} := \{\phi \in \mathcal{F}_f \cap \mathcal{F}_{\delta k f} : \phi|_{2-k} W_N \in \mathcal{F}_g \cap \mathcal{F}_{\delta_{kg}}\}$. Then $\mathcal{F}_{f,g} \neq 0$ and we have the following functional equations. For each $\phi \in \mathcal{F}_{f,g}$. if $k \in \mathbb{Z}$,

$$L_{f\chi}(\phi) = i^k \frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}} L_{g\bar{\chi}}(\phi|_{2-k} W_N), \tag{3.28}$$

$$L_{\delta_k f\chi}(\phi) = i^k \frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}} L_{\delta_k g\bar{\chi}}(\phi|_{2-k} W_N). \tag{3.29}$$

For each $\phi \in \mathcal{F}_{f,g}$, if $k \in \frac{1}{2} + \mathbb{Z}$,

$$L_{f\chi}(\phi) = \psi_D(-1)^{k-\frac{1}{2}} \psi_D(N) \frac{\chi(-N)\psi(D)}{\epsilon_D N^{-1+\frac{k}{2}}} L_{g\bar{\chi}\psi_D}(\phi|_{2-k} W_N), \tag{3.30}$$

$$L_{\delta_k f\chi}(\phi) = \psi_D(-1)^{k-\frac{1}{2}} \psi_D(N) \frac{\chi(-N)\psi(D)}{\epsilon_D N^{-1+\frac{k}{2}}} L_{\delta k g\bar{\chi}\psi_D}(\phi|_{2-k} W_N), \tag{3.31}$$

Where $\psi_D(N) = \left(\frac{u}{D}\right)$ is the real Dirichlet character modulo D.

Note that in this theorem, we denote by $\mathcal{F}_f$ the space of compactly supported functions.

Proof. First, by the definition of $S_c(\mathbb{R}_+)$ we see that it is contained in $\mathcal{F}_f$ and $\mathcal{F}_{\delta kf}$, so $\mathcal{F}_{f,g}$ non-zero. Also, for $\phi \in S_c(\mathbb{R}_+)$, with $Supp(\phi) \subset (c_1, c_2)$ where $c_1, c_2 > 0$, then for all $x > 0$

$$\begin{aligned} L(|\phi|)(x) &= \int_{c_1}^{c_2} |\phi(y)| e^{-xy} dy \\ &\ll_{c_1,c_2,\phi} e^{-xc_1} \end{aligned}$$

Using the growth estimates of the Fourier coefficients of $a(n), b(n)$ we deduce that (3.11) converge.

Since the Dirichlet character is bounded, we have that $\tau_{\bar{\chi}}(n)$ is bounded, using it with $L_{f\chi}(\phi)$, we get that for $\phi \in \mathcal{F}_f$, then $\phi \in \mathcal{F}_{f\chi}$ for all χ.

In order to prove the functional equations for $L_{f\chi}(\phi)$ and $L_{\delta_k f\chi}(\phi)$, we will take two cases.

Case1 : $k \in \mathbb{Z}$.

Using the identity

$$\begin{aligned} W_n \begin{pmatrix} \frac{1}{\sqrt{D}} & \frac{u}{\sqrt{D}} \\ 0 & \sqrt{D} \end{pmatrix} W_N^{-1} &= W_N^{-1} \begin{pmatrix} \frac{1}{\sqrt{D}} & \frac{u}{\sqrt{D}} \\ 0 & \sqrt{D} \end{pmatrix} W_N^{-1} \\ &= \begin{pmatrix} D & -v \\ -Nu & \frac{1+Nuv}{D} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{D}} & \frac{v}{\sqrt{D}} \\ 0 & \sqrt{D} \end{pmatrix} \end{aligned} \tag{3.32}$$

valid for $u, v \in \mathbb{Z}$ with $(u, D) = 1$ and $Nuv \equiv -1(modD)$, with the definition of $g = f|_k W_N$, we get that

$$f_\chi|_k W_N = \chi(-N)\psi(D) g_{\bar{\chi}}. \tag{3.33}$$

we have

$$
\begin{aligned}
L_{f\chi}(\phi) &= \int_0^\infty f_\chi(iy)\phi(y)dy \\
&= \int_0^\infty f_\chi\left(\frac{i}{Ny}\right)\phi\left(\frac{1}{Ny}\right)N^{-1}y{-}1y{-}1dy \text{ by change of variable}
\end{aligned}
$$

Note that

$$
\begin{aligned}
(f_\chi|_k W_N)(iy) &= f_\chi(W_N(iy))(\sqrt{N}iy)^{-k} \\
&= f_\chi\left(\frac{1/\sqrt{N}}{\sqrt{N}iy}\right)(\sqrt{N})^{-k}(iy)^{-k} \\
&= f_\chi\left(\frac{i}{Ny}\right)N^{\frac{-k}{2}}i^{-k}y^{-k}.
\end{aligned}
$$

Thus

$$
f_\chi\left(\frac{i}{Ny}\right) = (f_\chi|_k W_N)(iy)N^{\frac{k}{2}}i^k y^k. \tag{3.34}
$$

so,

$$
\begin{aligned}
L_{f\chi}(\phi) &= \int_0^\infty (f_\chi|_k W_N)(iy)N^{\frac{k}{2}}i^k y^k y^{-2}N^{-1}\phi\left(\frac{1}{Ny}\right)dy \\
&= i^k\chi(-N)\psi(D)N^{\frac{k}{2}-1}\int_0^\infty g_{\bar{\chi}}(iy)y^{k-2}\phi\left(\frac{1}{Ny}\right)dy \\
&= \frac{i^k\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}}\int_0^\infty g_{\bar{\chi}}(iy)(Ny)^{k-2}\phi\left(\frac{1}{Ny}\right)dy \text{ by multiplying by } \frac{N^{k-2}}{N^{k-2}} \\
&= \frac{i^k\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}}L_{g\bar{\chi}}(\phi|_{2-k}W_N).
\end{aligned}
$$

Which gives us the first functional equation (3.28).

Regarding the equality (3.29) we apply the operator δ_k to (3.33), we get:

$$
\begin{aligned}
(\delta(f_\chi|_k W_N))(z) &= \frac{k}{2}(f_{\bar{\chi}}|_k W_N)(z) + z\frac{\partial}{\partial x}(f_\chi|_k W_N)(z) \\
&= \chi(-N)\psi(D)(\delta_k g_{\bar{\chi}})(z)
\end{aligned}
\tag{3.35}
$$

where

$$(f_\chi|_k W_N)(z) = (\sqrt{N}z)^{-k} f_\chi(W_N z)$$
$$= (\sqrt{N}z)^{-k} f_\chi\biggl(\frac{-1}{Nz}\biggr).$$

so

$$\begin{aligned}(\delta(f_\chi|_k W_N))(z) &= \frac{k}{2}(f_\chi|_k W_N)(z) + z\frac{\partial}{\partial x}\biggl((\sqrt{N}z)^{-k} f_\chi\biggl(\frac{-1}{Nz}\biggr)\biggr) \\ &= ((\delta_k f_\chi|_k W_N)(z).\end{aligned} \tag{3.36}$$

We get

$$((\delta_k f_\chi)|_k W_N)(z) = \chi(-N)\psi(D)(\delta_k g_{\bar{\chi}})(z). \tag{3.37}$$

We deduce that the differential operator δ_k and the action of W_N via $|_k$ almost commute with each other.

Proceeding as before:

$$\begin{aligned}L_{\delta_k f\chi}(\phi) &= \int_0^\infty (\delta_k f\chi)(iy)\phi(y)dy \\ &= \int_0^\infty (\delta_k f\chi)\biggl(\frac{i}{Ny}\biggr)\phi\biggl(\frac{1}{Ny}\biggr)N^{-1}y^{-2}dy\end{aligned} \tag{3.38}$$

We know that

$$(f_\chi|_k W_N)(iy) = f_\chi\biggl(\frac{i}{Ny}\biggr)(\sqrt{N}(iy))^{-k}$$
$$= f_\chi\biggl(\frac{i}{Ny}\biggr)N^{\frac{-k}{2}} i^{-k} y^{-k}$$

So

$$f_\chi\biggl(\frac{i}{Ny}\biggr) = (f_\chi)|_k W_N)(iy) N^{\frac{k}{2}} i^k y^k$$

Applying the operator δ_k to both sides we get

$$\begin{aligned}\delta_k\left(f_\chi\left(\frac{i}{Ny}\right)\right) &= \delta_k((f_\chi|_k W_N)(iy))N^{\frac{k}{2}} i^k y^k \\ &= ((\delta_k f_\chi)|_k W_N)(iy)N^{\frac{k}{2}} i^k y^k \text{ by (3.37)}\end{aligned}$$

Therefore

$$\begin{aligned}L_{\delta_k f\chi}(\phi) &= \int_0^\infty ((\delta_k f_\chi)|_k W_N)(iy)N^{\frac{k}{2}} i^k y^k \phi\left(\frac{1}{Ny}\right)N^{-1}y^{-2}dy \\ &= \int_0^\infty \chi(-N)\psi(D)(\delta_k g_{\bar\chi})(iy)N^{\frac{k}{2}-1} i^k y^{k-2}\phi\left(\frac{1}{Ny}\right)dy \\ &= \chi(-N)\psi(D)i^k \times \frac{1}{N^{k-1}}\int_0^\infty (\delta_k g_{\bar\chi})(iy)N^{\frac{k}{2}-1}y^{k-2}\phi\left(\frac{1}{Ny}\right) \times N^{k-2}dy \\ &= \chi(-N)\psi(D)i^k N^{1-\frac{k}{2}}\int_0^\infty (\delta_k g_{\bar\chi})(iy)(\phi|_{2-k}W_N)(y)dy \\ &= 0\chi(-N)\psi(D)i^k N^{1-\frac{k}{2}} L_{\delta k f\chi}(\phi|_{2-k}W_N).\end{aligned} \tag{3.39}$$

Which is our desired result.

Moving on to case two where $k \in \frac{1}{2}+\mathbb{Z}$, according to ([5], Lemma 3.4) we have

$$\begin{aligned}g(\gamma z)(cz+d)^{-k} &= \psi(a)\epsilon_a^{-2k}\left(\frac{-bN}{a}\right)(f|_k W_N)(z) \\ &= \overline{\psi(d)}\epsilon_d^{-2k}\left(\frac{c}{d}\right)\left(\frac{N}{d}\right)g(z).\end{aligned} \tag{3.40}$$

since $a \equiv d \pmod 4, ad \equiv 1 \pmod{(-bN)}$ and $-bc \equiv 1 \pmod d$.

Again according to ([5], Proposition 5.1) , we have

$$f_\chi\left(\frac{-1}{Nz}\right)(-i\sqrt{N}z)^{-k} = \psi_D(-1)^{k-\frac{1}{2}}\psi_d(-1)^{k-\frac{1}{2}}\psi_D(N)\frac{\chi(-N)\psi(D)}{\epsilon_d}g_{\bar\chi\psi(D)}(z). \tag{3.41}$$

Now, we proceed as in the first case to get our desired functional equations. □

Recall that $\phi_s(x) = \phi(x)x^{s-1}$. We will show that for $y > 0$ and $s \in \mathbb{C}$ with $Re(s) > \frac{1}{2}$, we get

$$(L|\phi_s|)(y) \leq \Big(L(|\phi|^2)(y)\Big) y^{-Re(s)+\frac{1}{2}} (\Gamma(2Re(s)-1))^{\frac{1}{2}}. \tag{3.42}$$

Proof.

$$\begin{aligned}
(L|\phi_s|)(y) &= \int_0^\infty e^{-yt}|\phi_s(t)|dt \\
&= \int_0^\infty e^{-yt}|\phi(t)||t^{s-1}|dt \\
&= \int_0^\infty e^{\frac{yt}{2}}|\phi(t)|e^{\frac{yt}{2}}||t^{s-1}|dt \\
&\leq \left(\int_0^\infty (e^{\frac{yt}{2}}|\phi(t)|)^2 dt\right)^{\frac{1}{2}} \left(\int_0^\infty (e^{\frac{yt}{2}}|t^{s-1}|)^2 dt\right)^{\frac{1}{2}} \text{ By C.S inequality}
\end{aligned}$$

we know that

$$(L|\phi|^2)(y) = \int_0^\infty (e^{\frac{-yt}{2}}|\phi(t)|)^2 dt$$

And

$$\begin{aligned}
\int_0^\infty (e^{\frac{yt}{2}}|t^{s-1}|)^2 dt &= \int_0^\infty e^{-yt} t^{2Re(s)-2} dt \\
&= \int_0^\infty e^{-x}\left(\frac{x}{y}\right)^{2Re(s)-2} \times \frac{1}{y}\, dx \text{ letting } x = yt \\
&= \int_0^\infty e^{-x} x^{2Re(s)-2} \frac{1}{y^{2Re(s)-2+1}}\, dx \\
&= y^{-2Re(s)+1} \int_0^\infty e^{-x} x^{2Re(s)-2}\, dx \\
&= y^{-2Re(s)+1} \Gamma(2Re(s)-1)
\end{aligned}$$

Therefore we get our desired inequality (3.42). □

Now, for a given function f on $\mathbb{H}$ with series expansion (3.10) with $M = 1$, consider $\phi \in \mathcal{F}_f$. We have that

$$\sum_{n>\geq -n_0} |a(n)| \left((L|\phi|^2) \left(\frac{2\pi n}{1} \right) \right)^{\frac{1}{2}} + \sum_{n<0} |b(n)| \left(\frac{-4\pi n}{1} \right)^{1-k} \int_0^\infty \frac{((L|\phi_{2-k}|^2)\left(\frac{-2\pi n(2t+1)}{1}\right))^{\frac{1}{2}}}{(1+t)^k} dt. \tag{3.43}$$

converges. Then with (3.42) we have $\phi_s \in \mathcal{F}_f$ for Re(s)$> \frac{1}{2}$.

Theorem 3.1.2. *Let $k \in \mathbb{Z}$ and $f \in H_k(N, \psi)$. Set $g = f|_k W_N$ and $n_0 \in \mathbb{N}$ be such that $f(z)$ and $g(z)$ are $\mathcal{O}(e^{2\pi n_0 y})$ as $y = Im(z)$ tends to ∞. Suppose $\phi \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ is a non-zero function such that, for some $\epsilon > 0, \phi(x)$ and $\phi(x^{-1})$ are $o(e^{2\pi(n_0+\epsilon)x})$ as $x \to \infty$. We further assume that the series (3.43) converges. Then the L-series*

$$L(s, f, \phi) := L_f(\phi_s) \tag{3.44}$$

converges absolutely for $Re(s) > \frac{1}{2}$, has analytic continuation to all $s \in \mathbb{C}$ and satisfies the functional equation

$$L(s, f, \phi) = N^{-s-\frac{k}{2}+1} i^k L(1-s, g, \phi|_{1-k} W_N). \tag{3.45}$$

Note that the test functions for which the series $L_f(\phi)$ and the integral $\int_0^\infty f(iy)\phi(y)dy$ converge are different. And having that the integral converges for any $s \in \mathbb{C}$ gives analytic continuation for $L_f(\phi)$ for any $s \in \mathbb{C}$.

Proof. Due to the growth of $\phi(y)$, we deduce tht $L(|\phi|^2)(y)$ converges absolutely for $y \geq -2\pi n_0$. This combined with the note before the proof we get that $\phi_s \in \mathcal{F}_f$ for

Re(s)> $\frac{1}{2}$.

$$\begin{aligned}
L(s,f,\phi) &= L_f(\phi_s)\\
&= \int_0^\infty f(iy)\phi_s(y)dy\\
&= \int_0^\infty f(iy)y^{s-1}\phi(y)dy\\
&= \int_0^{\frac{1}{\sqrt{N}}} f(iy)y^{s-1}\phi(y)dy + \int_{\frac{1}{\sqrt{N}}}^\infty f(iy)y^{s-1}\phi(y)dy
\end{aligned}$$

In the first integral we let $y = \frac{1}{Nx}$, and in the second, we let $y = x$, we get

$$L(s,f,\phi) = \int_{\frac{1}{\sqrt{N}}}^\infty f\left(\frac{i}{Nx}\right)\phi\left(\frac{1}{Nx}\right)\frac{1}{N^s}\times\frac{1}{x^{s+1}}dx + \int_{\frac{1}{\sqrt{N}}}^\infty f(ix)x^{s-1}\phi(x)dx \tag{3.46}$$

using

$$f\left(\frac{i}{Nx}\right)(f|_k W_N)(ix)(\sqrt{N}ix)^K = g(ix)i^k N^{\frac{k}{2}}x^k,$$

and (3.27), with $a = 1-k$, we get for $Re(s) > \frac{1}{2}$

$$\begin{aligned}
L(s,f,\phi) &= \int_{\frac{1}{\sqrt{N}}}^\infty g(ix)i^k N^{\frac{k}{2}}x^k(\phi|_{1-k}W_N)(x)(Nx)^{1-k}\times\frac{1}{N^s}\times\frac{1}{x^{s+1}}dx + \int_{\frac{1}{\sqrt{N}}}^\infty f(ix)x^{s-1}\phi(x)dx\\
&= i^k N^{\frac{-k}{2}+1-s}\int_{\frac{1}{\sqrt{N}}}^\infty g(ix)(\phi|_{1-k}W_N)(x)x^{-s}dx + \int_{\frac{1}{\sqrt{N}}}^\infty f(ix)x^{s-1}\phi(x)dx.
\end{aligned} \tag{3.47}$$

Now,

$$\begin{aligned}
L(1-s,g,\phi|_{1-k}W_N) &= L_g((\phi|_{1-k}W_N)_{1-s}) \\
&= \int_0^\infty g(iy)(\phi|_{1-k}W_N)_{1-s}(y)ds \\
&= \int_0^\infty g(iy)y^{-s}(\phi|_{1-k}W_N)(y)ds \\
&= \int_0^{\frac{1}{\sqrt{N}}} g(iy)y^{-s}(\phi|_{1-k}W_N)(y)ds + \int_{\frac{1}{\sqrt{N}}}^\infty g(iy)y^{-s}(\phi|_{1-k}W_N)(y)ds
\end{aligned} \tag{3.48}$$

where :

$$\begin{aligned}
\int_0^{\frac{1}{\sqrt{N}}} g(iy)y^{-s}(\phi|_{1-k}W_N)(y)ds &= \int_0^{\frac{1}{\sqrt{N}}} f\left(\frac{i}{Ny}\right)i^{-k}N^{\frac{-k}{2}}y^{-k}\phi\left(\frac{1}{Ny}\right)(Ny)^{-1+k}y^{-s}dy \\
&= i^{-k}N^{\frac{k}{2}-1}\int_0^{\frac{1}{\sqrt{N}}} f\left(\frac{i}{Ny}\right)\phi\left(\frac{1}{Ny}\right)y^{-s-1}dy \\
&= i^{-k}N^{\frac{k}{2}-1+s}\int_{\frac{1}{\sqrt{N}}}^\infty f(ix)\phi(x)x^{s-1}dx
\end{aligned}$$

And letting $y=x$ in the second integral of (3.48), we get

$$\begin{aligned}
L(1-s,g,\phi|_{1-k}W_N) &= i^{-k}N^{\frac{k}{2}-1+s}\int_{\frac{1}{\sqrt{N}}}^\infty f(ix)\phi(x)x^{s-1}dx + \int_{\frac{1}{\sqrt{N}}}^\infty g(ix)x^{-s}(\phi|_{1-k}W_N)(x)dx \\
&= i^{-k}N^{\frac{-k}{2}+1-s}\int_{\frac{1}{\sqrt{N}}}^\infty g(ix)x^{-s}(\phi|_{1-k}W_N)(x)x^{-s}dx + \int_{\frac{1}{\sqrt{N}}}^\infty f(ix)\phi(x)x^{s-1}dx.
\end{aligned} \tag{3.49}$$

Hence, we got our desired equality (3.48). □

Chapter 4

The converse theorem

We will first state and prove the converse theorem. Then, we will demonstrate that it is possible, in the case of integer weights, to formulate the converse theorem using primitive characters. As an application, we will state and prove a summation formula for the holomorphic part of a harmonic lift of a given cusp form.

Theorem 4.0.1. *Let N be a positive integer and ψ a Dirichlet character modulo N. For $j \in \{1,2\}$, let $(a_j(n))_{n\geq -n_0}$ for some integer n_0 and $(b_j(n))_{n<0}$ be sequence of complex numbers such that $a_j(n), b_j(n) = \mathcal{O}(e^{C\sqrt{N}})$ as $|n| \to \infty$ for some constant $C > 0$. We define smooth functions $f_j : \mathbb{H} \to \mathbb{C}$ given the following Fourier expansions associated to given sequences:*

$$f_j(z) = \sum_{n\geq -n_0} a_j(n)e^{2\pi inz} + \sum_{n<0} b_j(n)\Gamma(1-k, -4\pi ny)e^{2\pi inz} \tag{4.1}$$

for all $D \in \{1, 2, ..., N^2 - 1\}$, $gcd(D,N) = 1$, let χ be a Dirichlet character modulo D. For any $\phi \in S_c(\mathbb{R}_+)$, for any D and χ, we assume that, if $k \in \mathbb{Z}$

$$L_{f1\chi}(\phi) = i^k \frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}} L_{f2\bar{\chi}}(\phi|_{2-k}W_N), \tag{4.2}$$

and

$$L_{\delta k f 1\chi}(\phi) = i^k \frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}} L_{\delta k f 2\bar{\chi}}(\phi|_{2-k}W_N). \quad (4.3)$$

If $k \in \frac{1}{2} + \mathbb{Z}$

$$L_{f1\chi}(\phi) = \psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\frac{\chi(-N)\psi(D)}{\epsilon_D N^{-1+\frac{k}{2}}} L_{f2\bar{\chi}\psi_D}(\phi|_{2-k}W_N), \quad (4.4)$$

$$L_{\delta k(f1\chi)}(\phi) = \psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\frac{\chi(-N)\psi(D)}{\epsilon_D N^{-1+\frac{k}{2}}} L_{\delta k(f2\bar{\chi}\psi_D)}(\phi|_{2-k}W_N). \quad (4.5)$$

Here $\psi_D(u) = \left(\frac{u}{D}\right)$ *belongs to* $H_k(\Gamma_0(N), \psi)$.

Then the function f_1 *is a harmonic Maass form with weight* k *and Nebentypus character* ψ *for* $\Gamma_0(N)$ *and* $f_2 = f_1|_k W_N$.

There is a freedom of choice of the test functions in this theorem. We reduce the size of the set of test functions, so we assume that our functional equations hold only for the family of test functions $\phi_s(x) = x^{s-1}\phi(x)(s \in \mathbb{C})$ for a single $\phi \in S_c(\mathbb{R}_+)$. In this setting, we will be proving the converse theorem.

Proof. Using the growth conditions of $a_j(n), b_j(n)$, and the asymptotic behaviour of $\Gamma(s,x)$, we can see that the smooth function $f_j(z)$ given by (4.1) converges absolutely to a smooth function of $\mathbb{H}$ for $j \in \{1,2\}$. Since the form of the Fourier expansion of f_1 and f_2 is similar to one of harmonic Maass forms, we can directly say that f_1 and f_2 satisfy conditions (2) and(3) of definition (2.2.1). Similarly, for any Dirichlet character χ modulo D, the twisted functions

$$f_{j\chi}(z) = \sum_{n \geq -n_0} \tau_{\overline{\chi}} a_j(n) e^{2\pi i n z/D} + \sum_{n<0} \tau_{\overline{\chi}} b_j(n)\Gamma(1-k, -4\pi n y/D) e^{2\pi i n z/D}. \quad (4.6)$$

converges absolutely for $j \in \{1,2\}$, since χ is also bounded. So is the operator

$$\begin{aligned}\delta_k(f_j\chi)(z) &= z\frac{\partial f_{j\chi}}{\partial x}(z) + \frac{k}{2}f_{j\chi}(z).\\ &= \frac{k}{2}f_j(z) + \sum_{n\geq -n_0} a_j(n)e^{2\pi inz/D}\frac{2\pi inz}{D} + \sum_{n<0} b_j(n)\frac{2\pi inz}{D}\Gamma(1-k,-4\pi ny/D)e^{2\pi inz/D}.\end{aligned}$$

for $j \in \{1,2\}$.

Note that for any $s \in \mathbb{C}$, and $\phi \in S_c(\mathbb{R}_+), \phi_s(y) = y^{s-1}\phi(y) \in S_c(\mathbb{R}_+)$. In fact, since $\phi \in S_c(\mathbb{R}_+)$, there is $0 < c_1 < c_2$, and $C > 0$ such that $Supp(\phi) \subset [c_1, c_2]$, and $|\phi(y)| \leq C$ for any $y > 0$. Thus for $j \in \{1,2\}$ and $n > 0$, we have

$$\begin{aligned}|a_j(n)|(L|\phi_s|)\left(\frac{2\pi n}{D}\right) &= |a_j(n)|\int_{c_1}^{c_2} e^{\frac{-2\pi ny}{D}}|\phi_s(y)|dy\\ &= |a_j(n)|\int_{c_1}^{c_2} e^{\frac{-2\pi ny}{D}}|y^{s-1}||\phi(y)|dy\\ &\leq |a_j(n)|\int_{c_1}^{c_2} Ce^{\frac{-2\pi ny}{D}}y^{Re(s)-1}dy\\ &\leq C|a_j(n)|e^{\frac{-2\pi nc_1}{D}}max\{c_1^{Re(s)-1}, c_2^{Re(s)-1}\}(c_2-c_1).\end{aligned} \tag{4.7}$$

Using (4.7) , the boundedness of $\tau_{\overline{\chi}}(n)$ and $a_j(n)$, and the convergence of $(L|\phi_s|)(\frac{2\pi n}{D})$, we get :

$$\begin{aligned}&\sum_{n\geq -n_0} |\tau_{\overline{\chi}}(n)||a_j(n)|(L|\phi_s|)\left(\frac{2\pi n}{D}\right)\\ &= \sum_{n\geq -n_0}^{0} |\tau_{\overline{\chi}}(n)||a_j(n)|(L|\phi_s|)\left(\frac{2\pi n}{D}\right) + \sum_{n\geq 1} |\tau_{\overline{\chi}}(n)||a_j(n)|(L|\phi_s|)\left(\frac{2\pi n}{D}\right)\\ &\leq \infty.\end{aligned}$$

for any $s \in \mathbb{C}$ and for any Dirichlet character χ modulo D.

Likewise, for $n < 0,$ and $t > 0:$

$$\begin{aligned}(L\phi_{s+1-k})\left(\frac{-2\pi n(2t+1)}{D}\right) &= \int_{c_1}^{c_2} e^{\frac{2\pi n(2t+1)y}{D}} |\phi_{s+1-k}(y)| dy \\ &= \int_{c_1}^{c_2} e^{\frac{2\pi n(2t+1)y}{D}} |y^{s-k}||\phi(y)| dy \\ &\le C \int_{c_1}^{c_2} e^{\frac{2\pi n(2t+1)y}{D}} y^{Re(s)-k} dy \\ &\ll e^{\frac{2\pi n(2t+1)c_1}{D}} max\{c_1^{Re(s)-k}, c_2^{Re(s)-k}\}.\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{n<0} |\tau_{\overline{\chi}}(n)||b_j(n)| \left|\frac{4\pi n}{D}\right|^{1-k} \int_0^\infty \frac{(L|\phi_{s+1-k}|)\left(\frac{-2\pi n(2t+1)}{D}\right)}{(1+t)^k} dt \\ &\ll \sum_{n<0} |\tau_{\overline{\chi}}(n)||b_j(n)| \left|\frac{4\pi n}{D}\right|^{1-k} \int_0^\infty \frac{1}{(1+t)^k} e^{\frac{2\pi n(2t+1)c_1}{D}} max\{c_1^{Re(s)-k}, c_2^{Re(s)-k}\} \\ &= \sum_{n<0} |\tau_{\overline{\chi}}(n)||b_j(n)| \left|\frac{4\pi n}{D}\right|^{1-k} \int_0^\infty \frac{1}{(1+t)^k} e^{\frac{4\pi n c_1 t}{D} + \frac{2\pi n c_1}{D}} max\{c_1^{Re(s)-k}, c_2^{Re(s)-k}\} \\ &= max\{c_1^{Re(s)-k}, c_2^{Re(s)-k}\} \left(\int_0^\infty \frac{1}{(1+t)^k} e^{\frac{4\pi n c_1 t}{D}} dt\right) \sum_{n<0} |\tau_{\overline{\chi}}(n)||b_j(n)| \left|\frac{4\pi n}{D}\right|^{1-k} e^{\frac{2\pi n c_1}{D}}.\end{aligned}$$

converges for any $s \in \mathbb{C}$, and for any Dirichlet character χ modulo D. So, ϕ_s satisfies (3.11) for $f_{j\chi}$, hence it belongs to $\mathcal{F}_{f1\chi} \cap \mathcal{F}_{f2\overline{\chi}}$. Hence by Weierstrass theorem , we conclude that $L_{fj\chi}(\phi_s)$ is an analytic function on $s \in \mathbb{C}$.

Now, using the Mellin inversion formula, we have

$$\mathcal{M}^{-1}(\mathcal{M}(f_{j\chi}(iy)\phi(y))) = f_{j\chi}(iy)\phi(y)$$

Where

$$\mathcal{M}(f_{j\chi}(iy)\phi(y))(s) = \int_0^\infty f_{j\chi}(iy)\phi(y)y^{s-1}dy = L_{fj\chi}(\phi_s),$$

And

$$\mathcal{M}^{-1}(L_{fj\chi}(\phi_s))(y) = \frac{1}{2\pi i}\int_\sigma L_{fj\chi}(\phi_s)y^{-s}ds.$$

Thus

$$f_{j\chi}(iy)\phi(y) = \frac{1}{2\pi i}\int_{\sigma} L_{fj\chi}(\phi_s)y^{-s}ds. \tag{4.8}$$

for all $\sigma \in \mathbb{R}$. Now, similar to the work done above, we can conclude that $L_{\delta k(fj\chi)}(\phi_s)$ is analytic for any $s \in \mathbb{C}$ and for any Dirichlet character χ modulo D. Again, using the Mellin inversion formula, we deduce the following

$$\delta_{k(fj\chi)}(iy)\phi(y) = \frac{1}{2\pi i}\int_{\sigma} L_{\delta k(fj\chi)}(\phi_s)y^{-s}ds. \tag{4.9}$$

Using integration by parts, we get

$$\begin{aligned} L_{f_{1\chi}}(\phi_s) &= \int_0^{\infty} f_{1\chi}(iy)\phi_s(y)dy \\ &= \int_0^{\infty} f_{1\chi}(iy)\phi(y)y^{s-1}dy \\ &= \left[\frac{y^s}{s}f_{1\chi}(iy)\phi(y)\right]_0^{\infty} - \frac{1}{s}\int_0^{\infty}\frac{\partial f_{1\chi}(iy)\phi(y)}{\partial y}y^s dy \end{aligned} \tag{4.10}$$

since ϕ is compactly supported, it vanishes at 0 and ∞. Then

$$\begin{aligned} |L_{f_{1\chi}}(\phi_s)| &= \left| -\frac{1}{s}\int_0^{\infty}\frac{\partial f_{1\chi}(iy)\phi(y)}{\partial y}y^s dy\right| \\ &\leq \frac{1}{|s|}\int_0^{\infty}\left|\frac{\partial f_{1\chi}(iy)\phi(y)}{\partial y}\right|y^{Re(s)}dy \\ &\to 0 \text{ ,as } |Im(s)| \to 0. \end{aligned} \tag{4.11}$$

Similarly,

$$\begin{aligned} L_{\delta_{k(f1\chi)}}(\phi_S) &\leq \frac{1}{|s|}\int_0^{\infty} y^{Re(s)}\left|\frac{\partial(\delta_{k(f1\chi)}\phi(y))}{\partial y}\right|dy \\ &\to 0 \text{ as } Im(s) \to 0. \end{aligned} \tag{4.12}$$

We can see that $L_{(f1\chi)}(\phi_s)$ converges uniformly to 0 as $|Im(s)| \to 0$ for $Re(s)$ in any compact set in $\mathbb{C}$. Hence, we change the line of integration from $Re(s) = \sigma$ to

$Re(s) = k - \sigma$, so

$$f_{1\chi}(iy)\phi(y) = \frac{1}{2\pi i}\int_{k-\sigma} L_{f1\chi}(\phi_s)y^{-s}ds,$$

then change the variable s to $k - s$, we get

$$f_{1\chi}(iy)\phi(y) = \frac{1}{2\pi i}\int_{\sigma} L_{f1\chi}(\phi_{k-s})y^{-k+s}ds. \tag{4.13}$$

Similarly, we get:

$$\delta_k(f_{1\chi})(iy)(\phi)(y) = \frac{1}{2\pi i}\int_{\sigma} L_{\delta k(f1\chi)}(\phi_{k-s})y^{s-k}ds. \tag{4.14}$$

To proceed we have two separate 2 cases:

1. $k \in \mathbb{Z}$,
2. $k \in \frac{1}{2} + \mathbb{Z}$.

Starting with case (1), using (4.2) with (4.13),

$$\begin{aligned} f_{1\chi}(iy)\phi(y) &= \frac{1}{2\pi i}\int_{\sigma} L_{f1\chi}(\phi_{k-s})y^{-k+s}ds \\ &= i^k\frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}}\frac{1}{2\pi i}\int_{\sigma} L_{f2\bar{\chi}}(\phi_{k-s}|_{2-k}W_N)y^{-k+s}ds \end{aligned} \tag{4.15}$$

We have that $\phi_{k-s}|_{2-k}W_N \in \mathcal{F}_{f1\chi} \cap \mathcal{F}_{f2\bar{\chi}}$, and for each $y > 0$:

$$\begin{aligned} (\phi_{k-s}|_{2-k}W_N)(y) &= (Ny)^{k-2}\phi_{k-s}\left(\frac{1}{Ny}\right) \\ &= (Ny)^{k-2}\phi\left(\frac{1}{Ny}\right)\left(\frac{1}{Ny}\right)^{k-s-1} \\ &= (Ny)^{s-1}\phi\left(\frac{1}{Ny}\right) \end{aligned} \tag{4.16}$$

So, we get

$$\begin{aligned}L_{f2\bar{\chi}}(\phi_{k-s}|_{2-k}W_N) &= \int_0^\infty f_{2\bar{\chi}}(iy)(\phi_{k-s}|_{2-k}W_N)(y)dy \\ &= \int_0^\infty f_{2\bar{\chi}}(iy)(Ny)^{s-1}\phi\left(\frac{1}{Ny}\right)dy.\end{aligned} \tag{4.17}$$

Again, using the Mellin Inversion formula,

$$\frac{1}{N}f_{2\bar{\chi}}\left(\frac{-1}{iNy}\right)\phi(y) = \frac{1}{2\pi i}\int_\sigma L_{f2\bar{\chi}}(\phi_{k-s}|_{2-k}W_N)y^s ds. \tag{4.18}$$

Therefore, using (4.15) and (4.18), we get

$$\begin{aligned}f_{1\chi}(iy)\phi(y) &= i^k\frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}}\frac{1}{2\pi i}\int_\sigma L_{f2\bar{\chi}}(\phi_{k-s}|_{2-k}W_N)y^{-k+s}ds \\ &= i^k\frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}}y^{-k}\frac{1}{N}f_{2\bar{\chi}}\left(\frac{-1}{iNy}\right)\phi(y) \\ &= i^k\frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}}}f_{2\bar{\chi}}\left(\frac{-1}{iNy}\right)\phi(y).\end{aligned} \tag{4.19}$$

Similarly,

$$\begin{aligned}L_{\delta k}(f_{2\bar{\chi}})(\phi_{k-s}|_{2-k}W_N) &= \int_0^\infty (\delta_k(f_{2\bar{\chi}}))(iy)(\phi_{k-s}|_{2-k}W_N)(y)dy \\ &= \int_0^\infty (\delta_k(f_{2\bar{\chi}}))(iy)(Ny)^{s-1}\phi\left(\frac{1}{Ny}\right)dy.\end{aligned} \tag{4.20}$$

Using (4.1) and (4.14),

$$\delta_k(f_{1\chi})(iy)\phi(y) = -i^k\frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}}\frac{1}{2\pi i}\int_\sigma L_{\delta kf2\bar{\chi}}(\phi_{k-s}|_{2-k}W_N)y^{-k+s}ds. \tag{4.21}$$

By the Mellin inversion,

$$N^{-1}\delta_k(f_{2\bar{\chi}})\left(\frac{-1}{iNy}\right)(\phi)(y) = \left(\frac{1}{2\pi i}\right)\int_\sigma L_{\delta kf2\bar{\chi}}(\phi_{k-s}|_{2-k}W_N)y^s ds. \tag{4.22}$$

Therefore,

$$\delta_k(f_{1\chi})(iy)\phi(y) = i^{k+2}\frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}}}y^{-k}\delta_k(f_{2\bar{\chi}})\left(\frac{-1}{iNy}\right)\phi(y). \tag{4.23}$$

Since $\phi \in S_c(\mathbb{R}_+)$, so for $y \in \mathbb{R}_+$, such that $\phi(y) \neq 0$, thus the following relations are true for all $y > 0$

$$f_{1\chi}(iy) = i^{k}\frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}}}f_{2\bar{\chi}}\left(\frac{-1}{iNy}\right). \tag{4.24}$$

$$\delta_k(f_{1\chi})(iy) = i^{k+2}\frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}}}y^{-k}\delta_k(f_{2\bar{\chi}})\left(\frac{-1}{iNy}\right). \tag{4.25}$$

Now, define

$$F_\chi(z) := f_{1\chi}(z) - \chi(-N)\psi(D)(f_{2\bar{\chi}}|_k W_N^{-1})(z). \tag{4.26}$$

Notice that

$$\begin{aligned} F_\chi(iy) &= f_{1\chi}(iy) - \chi(-N)\psi(D)(f_{2\bar{\chi}}|_k W_N^{-1})(iy) \\ &= i^{k}\frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}}}y^{-k}f_{2\bar{\chi}}\left(\frac{-1}{iNy}\right) - \chi(-N)\psi(D)(-\sqrt{N}iy)^{-k}f_{2\bar{\chi}}\left(\frac{-1}{Nyi}\right) \\ &= 0 \end{aligned} \tag{4.27}$$

Where $(-i)^{-k} = (-1)^{-k}i^{-k} = (1/i^2)^{-k}i^{-k} = i^k$, and using the renormalized version of partial derivative we get $\frac{\partial}{\partial x}F_\chi(iy) = 0$.

Since $f_{1\chi}$ and $f_{2\bar{\chi}}$ are defined as the Fourier series of $e^{2\pi inz}$ and $\Gamma(1-k, -4\pi ny)e^{2\pi inz}$., they are eigenfunctions of the hyperbolic Laplacian. Therefore F_χ is an eigenfunction of the Laplace operator. The vanishing of F_χ and $\frac{\partial}{\partial x}F_\chi$ on the imaginary axis implies that $F_\chi = 0$ ([6], Lemma1.9.2), thus

$$f_{1\chi}(z) = \chi(-N)\psi(D)(f_{2\bar{\chi}}|_k W_N^{-1})(z). \tag{4.28}$$

Note that for $D = 1$,

$$\tau_\chi(n) = \sum_{u=1} \chi(u) e^{2\pi i n \frac{u}{1}} = 1 \tag{4.29}$$

so, $f_{1\chi}(z) = f_1(z)$, $f_{2\bar{\chi}}(z) = f_2(z)$ and $\psi(D) = 1$ so we get

$$f_1 = f_2|_k W_N^{-1}$$

Applying the slash operator to (3.24), then using (3.32),

$$\begin{aligned}
f_{2\bar{\chi}}|_k W_N^{-1}(z) &= \sum_{v \bmod D} \chi(v) \left(f_2|_k \begin{pmatrix} \frac{1}{\sqrt{D}} & \frac{v}{\sqrt{D}} \\ 0 & \sqrt{D} \end{pmatrix} |_k W_N^{-1} \right)(z) \\
&= \sum_{\substack{v \bmod D \\ (v,D)=1 \\ -Nuv \equiv 1 \pmod{D}}} \chi(v) \left(f_2|_k W_N^{-1}|_K \begin{pmatrix} D & -u \\ -Nv & \frac{1+Nuv}{D} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{D}} & \frac{u}{\sqrt{D}} \\ 0 & \sqrt{D} \end{pmatrix} \right)(z) \\
&= \sum_{\substack{v \bmod D \\ (v,D)=1 \\ -Nuv \equiv 1 \pmod{D}}} \chi(v) \left(f_1|_k \begin{pmatrix} D & -u \\ -Nv & \frac{1+Nuv}{D} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{D}} & \frac{u}{\sqrt{D}} \\ 0 & \sqrt{D} \end{pmatrix} \right)(z)
\end{aligned} \tag{4.30}$$

Now using (4.28),

$$\begin{aligned}
f_{1\chi}(z) &= \chi(-N)\psi(D)(f_{2\bar{\chi}}|_k W_N^{-1})(z) \\
&= \chi(-N)\psi(D) \sum_{\substack{v \bmod D \\ (v,D)=1 \\ -Nuv \equiv 1 \pmod{D}}} \chi(v) \left(f_1|_k \begin{pmatrix} D & -u \\ -Nv & \frac{1+Nuv}{D} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{D}} & \frac{u}{\sqrt{D}} \\ 0 & \sqrt{D} \end{pmatrix} \right)(z) \\
&= \psi(D) \sum_{\substack{v \bmod D \\ (v,D)=1 \\ -Nuv \equiv 1 \pmod{D}}} \overline{\chi(u)} \left(f_1|_k \begin{pmatrix} D & -u \\ -Nv & \frac{1+Nuv}{D} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{D}} & \frac{u}{\sqrt{D}} \\ 0 & \sqrt{D} \end{pmatrix} \right)(z)
\end{aligned} \tag{4.31}$$

Since $\chi(-N)\chi(v) = \chi(-Nv) = \chi(u^{-1}) = \overline{\chi(u)}$.

We know that

$$f_{1\chi}(z) = \sum_{u \bmod D} \overline{\chi(u)} f_1\left(\frac{z+u}{D}\right)$$

using (4.31), ([7])

$$\sum_{u \bmod D} \overline{\chi(u)} f_1\left(\frac{z+u}{D}\right) = \sum_{u \bmod D} \chi(u)\psi(D) \times \sum_{\substack{v \bmod D \\ (v,D)=1 \\ -Nuv\equiv 1 \bmod D}} \overline{\chi(u)} f_1|_k \begin{pmatrix} D & -u \\ -Nv & \frac{1+Nuv}{D} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{D}} & \frac{u}{\sqrt{D}} \\ 0 & \sqrt{D} \end{pmatrix} (z)$$

which is equivalent to

$$f_1(z) = \sum_{u \bmod D} \sum_{\substack{v \bmod D \\ (v,D)=1 \\ -Nuv\equiv 1 \bmod D}} \psi(D) f_1|_k \begin{pmatrix} D & -u \\ -Nv & \frac{1+Nuv}{D} \end{pmatrix}$$

Taking the sum over all characters modulo D, we see that for each u, v such that $-Nuv \equiv 1 \bmod D$, we have

$$f_1(z) = \psi(D) f_1|_k \begin{pmatrix} D & -u \\ -Nv & \frac{1+Nuv}{D} \end{pmatrix} \tag{4.32}$$

Which is equivalent to

$$f_1|_k \begin{pmatrix} \frac{1+Nuv}{D} & u \\ Nv & D \end{pmatrix} = \psi(D) f_1. \tag{4.33}$$

So we have proved that f_1 satisfies condition (1) of Definition (2.2.1).

Moving on to case (2), using (4.4) with (4.8),

$$\begin{aligned} f_{1\chi}(iy)\phi(y) &= \frac{1}{2\pi i}\int_\sigma L_{f1\chi}(\phi_s)y^{-s}ds \\ &= \psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\frac{\chi(-N)\psi(D)}{\epsilon_D N^{-1+\frac{k}{2}}}\frac{1}{2\pi i}\int_\sigma L_{f2\bar{\chi}\psi_D}(\phi_s|_{2-k}W_N)y^{-s}ds \end{aligned} \tag{4.34}$$

As we proceeded before, having $L_{f2\bar{\chi}\psi_D}(\phi_s)$ converges uniformly for Re(s) we change the line of integration from Re(s)$=\sigma$ to Re(s)$=k-\sigma$. then we change the variable s to $k-s$, we get,

$$f_{1\chi}(iy)\phi(y) = \psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\frac{\chi(-N)\psi(D)}{\epsilon_D N^{-1+\frac{k}{2}}}\frac{1}{2\pi i}\int_\sigma L_{f2\bar{\chi}\psi_D}(\phi_{k-s}|_{2-k}W_N)y^{-k+s}ds. \tag{4.35}$$

Using (4.16), we get

$$\begin{aligned} L_{f2\bar{\chi}\psi_D}(\phi_{k-s}|_{2-k}W_N) &= \int_0^\infty f_{2\bar{\chi}\psi_D}(iy)(\phi_{k-s}|_{2-k}W_N)(y)dy \\ &= \int_0^\infty f_{2\bar{\chi}\psi_D}(iy)(Ny)^{s-1}\phi\left(\frac{1}{Ny}\right)dy. \end{aligned} \tag{4.36}$$

And the Mellin inversion formula gives us

$$\frac{1}{N}f_{2\bar{\chi}\psi_D}\left(\frac{-1}{iNy}\right)\phi(y) = \frac{1}{2\pi i}\int_\sigma L_{f2\bar{\chi}\psi_D}(\phi_{k-s}|_{2-k}W_N)y^s ds. \tag{4.37}$$

Therefore,

$$\begin{aligned} f_{1\chi}(iy)\phi(y) &= \psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\frac{\chi(-N)\psi(D)}{\epsilon_D N^{-1+\frac{k}{2}}}\frac{1}{N}y^{-k}f_{2\bar{\chi}\psi_D}\left(\frac{-1}{iNy}\right)\phi(y) \\ &= \psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\frac{\chi(-N)\psi(D)}{\epsilon_D N^{\frac{k}{2}}}y^{-k}f_{2\bar{\chi}\psi_D}\left(\frac{-1}{iNy}\right)\phi(y). \end{aligned} \tag{4.38}$$

Since $\phi \in S_c(\mathbb{R}_+)$, so for any $y \in \mathbb{R}_+$, $\phi(y) \neq 0$, thus the following relation is true for all $y > 0$,

$$\begin{aligned} f_{1\chi}(iy) &= \psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\frac{\chi(-N)\psi(D)}{\epsilon_D N^{\frac{k}{2}}}y^{-k}f_{2\bar{\chi}\psi_D}\left(\frac{-1}{iNy}\right) \\ &= \psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\frac{\chi(-N)\psi(D)}{\epsilon_D}N^{\frac{-k}{2}}y^{-k}f_{2\bar{\chi}\psi_D}\left(\frac{-1}{iNy}\right) \\ &= \psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\frac{\chi(-N)\psi(D)}{\epsilon_D}(f_{1\bar{\chi}\psi D}|_k W_N^{-1})(iy). \end{aligned} \tag{4.39}$$

Using (4.5) with (4.14).

$$\delta_k(f_{1\chi})(iy)\phi(y) = \psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\frac{\chi(-N)\psi(D)}{\epsilon_D N^{-1+\frac{k}{2}}}\frac{1}{2\pi i}\int_\sigma L_{\delta k(f2\bar{\chi}\psi_D)}(\phi_{k-s}|_{2-k}W_N)y^{s-k}ds$$

Where the Mellin inversion formula gives us

$$-N^{-1}\delta_k(f_{2\bar{\chi}\psi_D})\left(\frac{-1}{iNy}\right)\phi(y) = \frac{1}{2\pi i}\int_\sigma L_{\delta k f 2\bar{\chi}\psi_D}(\phi_{k-s}|_{2-k}W_N))y^s ds. \tag{4.40}$$

Therefore,

$$\delta_k(f_{1\chi})(iy)\phi(y) = -\psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\frac{\chi(-N)\psi(D)}{\epsilon_D N^{\frac{k}{2}}}\delta_k(f_{2\bar{\chi}\psi_D})\left(\frac{-1}{iNy}\right)\phi(y) \tag{4.41}$$

Since $\phi \in S_c(\mathbb{R}_+)$, so for $y \in \mathbb{R}_+$, such that $\phi(y) \neq 0$, thus the following relation are true for all $y > 0$,

$$\delta_k(f_{1\chi})(iy) = -\psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\frac{\chi(-N)\psi(D)}{\epsilon_D N^{\frac{k}{2}}}\delta_k(f_{2\bar{\chi}\psi_D})\left(\frac{-1}{iNy}\right). \tag{4.42}$$

Now, we define

$$F_\chi(z) = f_{1\chi}(z) - \psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\chi(-N)\psi(D)\epsilon_D^{-1}(f_{2\bar{\chi}\psi D}|_k W_N^{-1})(z). \tag{4.43}$$

As before, $F_\chi(iy) = 0$ and $\frac{\partial}{\partial x}F_\chi(iy) = 0$. We also know that F_χ is an eigenfunction of the Laplace operator, so we have shown the vanishing of the general eigenfunction of the Laplacian, $F_\chi = 0$, thus

$$f_{1\chi}(z) = \psi_D(-1)^{k-\frac{1}{2}}\psi_D(N)\chi(-N)\psi(D)\epsilon_D^{-1}(f_{2\bar{\chi}\psi D}|_k W_N^{-1})(z). \tag{4.44}$$

Similarly as in the previous case, we have

$$f_{1\chi}(z) = \psi(D) \sum_{\substack{v \bmod D \\ (v,D)=1 \\ -Nuv\equiv 1 \pmod D}} \overline{\chi(u)} f_1|_k \begin{pmatrix} D & -u \\ -Nv & \frac{1+Nuv}{D} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{D}} & \frac{u}{\sqrt{D}} \\ 0 & \sqrt{D} \end{pmatrix} (z) \tag{4.45}$$

Taking the sum over all characters modulo D, we deduce for each u, v such that $-Nuv \equiv 1 \bmod D$, we have

$$f_1(z) = \psi(D) f_1|_k \begin{pmatrix} D & -u \\ -Nv & \frac{1+Nuv}{D} \end{pmatrix} \tag{4.46}$$

Which is equivalent to

$$f_1|_k \begin{pmatrix} \frac{1+Nuv}{D} & u \\ Nv & D \end{pmatrix} = \psi(D) f_1. \tag{4.47}$$

So we have proved that f_1 satisfies condition (1) of Definition (2.2.1).

□

Corollary 4.0.1. *With the notation of the previous theorem, let* $(a_j(n))_{n\geq -n_0}$ *for* $j = 1,2$ *be sequences of complex numbers such that* $j = 1,2$ *be sequences of complex numbers such that* $a_j(n) = \mathcal{O}(e^{c\sqrt{n}})$ *as* $n \to \infty$, *for some* $C > 0$. *Define the holomorphic functions* $f_j : \mathbb{H} \to \mathbb{C}$ *by the following Fourier expansions:*

$$f_j(z) = \sum_{n\geq -n_0} a_j(n) e^{e\pi i n z}. \tag{4.48}$$

For all $D \in 1,2,...,N^2-1$, $gcd(D,N) = 1$, *let* χ *be a Dirichlet character modulo*

D. For each D, χ and any $\phi \in S_c(\mathbb{R}_+)$, we assume that,

$$L_{f1\chi} = i^k \frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}} L_{f2\bar{\chi}}(\phi|_2 - kW_N) \tag{4.49}$$

if $k \in \mathbb{Z}$. Then, the function f_1 is a weakly holomorphic form with weight k and character ψ for $\Gamma_0(N)$, and $f_2 = f_1|_k W_N$.

Proof. The proof is the same as the proof of the previous theorem except that we do not need the equations of $L_{\delta_k(f_{k\chi})}(\phi)$ due to the holomorphicity of f_1 and f_2. $\square$

4.1 Alternative Converse Theorem for Integral Weight

We will state and prove the converse theorem so that only primitive characters are required in the statement. Recall the following notation of the Gauss sum of a character χ modulo D :

$$\tau(\chi) := \sum_{m \bmod D} \chi(m) e^{2\pi i \frac{m}{D}}. \tag{4.50}$$

When χ is primitive, we have $\tau_{\bar{\chi}}(n) = \chi(n)\tau(\bar{\chi})$.

Theorem 4.1.1. *Let $k \in \mathbb{Z}, N \in \mathbb{N}$ and ψ a Dirichlet character modulo N. For $j \in \{1, 2\}$, let $(a_j(n))_{n \geq -n_0}$ for some integers n_0 and $(b_j(n))_{n<0}$ be sequences of complex numbers such that $a_j(n), b_j(n) = \mathcal{O}(e^{C\sqrt{|n|}})$ as $n \to \infty$ for some constant $C > 0$. We define smooth functions $f_j : \mathbb{H} \to \mathbb{C}$ given by the following Fourier expansions associated with the given sequences:*

$$f_j(z) = \sum_{n \geq -n_0} a(n) e^{2\pi n z} + \sum_{n<0} b(n) \Gamma(1-k, -4\pi n y) e^{2\pi i n z}. \tag{4.51}$$

For all $D \in \mathbb{N}, (D, N) = 1$, all primitive Dirichlet characters χ modulo D and all

$\phi \in S_c(\mathbb{R}_+)$, *we assume that,*

$$L_{f1\chi}(\phi) = i^k \frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}} L_{f2\bar{\chi}}(\phi|_{2-k}W_N), \tag{4.52}$$

and

$$L_{\delta k f1\chi}(\phi) = -i^k \frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}} L_{\delta k f2\bar{\chi}}(\phi|_{2-k}W_N). \tag{4.53}$$

Then, the function f_1 *belongs to* $H_k(\Gamma_0(N), \psi)$.

Proof. In the previous theorem, we got

$$f_{1\chi}(z) = \chi(-N)\psi(D)(f_{2\bar{\chi}}|_k W_N^{-1})(z) \tag{4.54}$$

without depending on the character χ whether it is primitive or not, and since the other assumptions of the theorem are the same, we deduce:

$$(f_{1\chi}|_k W_N)(z) = \chi(-N)\psi(D) f_{2\bar{\chi}}(z). \tag{4.55}$$

Now, let $\tilde{f}_{j\chi}(z) := \frac{\chi(-1)\tau(\chi)}{D} f_{j\chi}(Dz)$ for $j = 1, 2$,

$$\begin{aligned}
\tilde{f}_{1\chi}|_k W_{ND^2}(z) &= (D\sqrt{N}z)^{-k} \tilde{f}_{1\chi}(W_{ND^2}z) \\
&= (D\sqrt{N}z)^{-k} \tilde{f}_{1\chi}\left(\frac{-1}{D^2\sqrt{N}z}\right) \\
&= (D\sqrt{N}z)^{-k} f_{1\chi}\left(\frac{-1}{D\sqrt{N}z}\right)\frac{\chi(-1)\tau(\chi)}{D} \\
&= (f_{1\chi}|_k W_n)(Dz)\frac{\chi(-1)\tau(\chi)}{D} \\
&= \chi(-N)\psi(D) f_{2\bar{\chi}}(Dz)\frac{\chi(-1)\tau(\chi)}{D} \\
&= \chi(-N)\psi(D)\tilde{f}_{2\bar{\chi}}(z).
\end{aligned} \tag{4.56}$$

This coincides with ((5.13)-[8]) where we can say if $c(r)$ is any function of the non-zero residue classes modulo D such that $\sum c(r) = 0$, we have

$$\sum_{\substack{r \bmod D,\\ (r,D)=1}} c(r) g|_k \begin{pmatrix} D & -r \\ -Nm & s \end{pmatrix} \begin{pmatrix} 1 & \frac{r}{D} \\ 0 & 1 \end{pmatrix} \equiv \sum_{\substack{r \bmod D,\\ (r,D)=1}} c(r)\psi(D) g|_k \begin{pmatrix} 1 & \frac{r}{D} \\ 0 & 1 \end{pmatrix} \tag{4.57}$$

Then

$$\sum_{\substack{r \bmod D,\\ (r,D)=1}} c(r) \begin{pmatrix} D & -r \\ -Nm & s \end{pmatrix} \begin{pmatrix} 1 & \frac{r}{D} \\ 0 & 1 \end{pmatrix} \equiv \sum_{\substack{r \bmod D,\\ (r,D)=1}} c(r)\psi(D) \begin{pmatrix} 1 & \frac{r}{D} \\ 0 & 1 \end{pmatrix} \tag{4.58}$$

modΩ , where Ω is the annihilator of g. If D is prime, we take c to be 1 on the residue class of r, -1 on the class of $-r$, and 0 elsewhere. Then by matrix operation, we get

$$\left[\begin{pmatrix} D & -r \\ -Nm & s \end{pmatrix} - \psi(D)\right] \begin{pmatrix} 1 & \frac{2r}{D} \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} D & r \\ Nm & s \end{pmatrix} - \psi(D), \tag{4.59}$$

and

$$\begin{aligned} &\left[\begin{pmatrix} D & r \\ Nm & s \end{pmatrix} - \psi(D)\right] \begin{pmatrix} s & -r \\ -Nm & D \end{pmatrix} \begin{pmatrix} 1 & \frac{2r}{D} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D & -r \\ -Nm & s \end{pmatrix} \begin{pmatrix} 1 & \frac{2r}{D} \\ 0 & 1 \end{pmatrix} \\ &\equiv \left[\begin{pmatrix} D & -r \\ -Nm & s \end{pmatrix} \psi(D)\right] \begin{pmatrix} 1 & \frac{2r}{D} \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{4.60}$$

Let

$$g := f_2|_k\gamma - \psi(D) f_2, \tag{4.61}$$

where $\gamma = \begin{pmatrix} D & r \\ Nm & t \end{pmatrix}$. And g satisfies $g = g|_k M$ which is clearly seen by (4.59) and (4.60), where $M = \begin{pmatrix} 1 & \frac{2r}{D} \\ \frac{-2Nm}{t} & -3 + \frac{4}{Dt} \end{pmatrix}$ is an elliptic element of infinite order with fixed point

$$\frac{1}{2\times\frac{-2Nm}{t}}\left[4-\frac{4}{Dt}\pm\sqrt{\left(-2+\frac{4}{Dt}\right)^2-4\left(\frac{-3Dt+4+4Nmr}{Dt}\right)}\right]$$
$$=\frac{1}{DmN}\big(1-Dt\pm\sqrt{1-D(2+Nmr)t+D^2t^2}\big), \tag{4.62}$$

Now, define $g_1 := f_2|_k\begin{pmatrix} D' & r' \\ Nm & t\end{pmatrix} - \psi(D)f_2$, where

$$\begin{pmatrix} D' & r' \\ Nm & t\end{pmatrix}=\begin{pmatrix} 1 & -u' \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & b \\ Nm & d\end{pmatrix}\begin{pmatrix} 1 & -v \\ 0 & 1\end{pmatrix}$$

with $D' = a - u'Nm$, $r' = -av + u'vNm + b - u'd$ and $a = d - Nc$. Then as we proceeded before, we get $g_1|_kM_1 = g_1$ where $\begin{pmatrix} 1 & \frac{2r'}{D'} \\ \frac{-2Nm}{t} & -3+\frac{4}{D't}\end{pmatrix}$ is an infinite order elliptic matrix with fixed point

$$\frac{1}{DmN}\big(1-Dt\pm\sqrt{1-D(2+Nmr)(mN+t)+D^2t(mN+t)}\big). \tag{4.63}$$

Also,

$$f_2|_k\begin{pmatrix} D' & r' \\ Nm & t\end{pmatrix}=f_2|_k\begin{pmatrix} 1 & -u' \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & b \\ Nm & d\end{pmatrix}\begin{pmatrix} 1 & -v \\ 0 & 1\end{pmatrix}$$
$$=f_2|_k\begin{pmatrix} a & b \\ Nm & d\end{pmatrix}\begin{pmatrix} 1 & -v \\ 0 & 1\end{pmatrix} \tag{4.64}$$
$$=f_2|_k\begin{pmatrix} D & r \\ Nm & t\end{pmatrix}$$

Since

$$\begin{pmatrix} a & b \\ Nm & d\end{pmatrix}=\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}\begin{pmatrix} D & r \\ Nm & t\end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & 1\end{pmatrix}, \tag{4.65}$$

where $D = a - uNm, s = d - vNm$, and $r = -av + uvNm + b - ud$. and

$$\begin{aligned} f_2|_k \begin{pmatrix} a & b \\ Nm & d \end{pmatrix} &= f_2|_k \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D & r \\ Nm & t \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \\ &= f_2|_k \begin{pmatrix} D & r \\ Nm & t \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{4.66}$$

which is equivalent to

$$f_2|_k \begin{pmatrix} a & b \\ Nm & d \end{pmatrix} \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} = f_2|_k \begin{pmatrix} D & r \\ Nm & t \end{pmatrix}$$

Thus, we get

$$g_1 = f_2|_k \begin{pmatrix} D & r \\ Nm & t \end{pmatrix} - \psi(D) f_2$$

$$g_1 = g.$$

Therefore, g is invariant under two infinite order elliptic matrices that do not have fixed points in common, therefore g is constant (Theorem 3.11-[9]). And Hence $g = 0$. Thus, we have

$$f_2|_k \begin{pmatrix} D & r \\ Nm & s \end{pmatrix} = \psi(D) f_2$$

Hence

$$
\begin{aligned}
f_2|_k \begin{pmatrix} a & b \\ Nm & D \end{pmatrix} &= f_2|_k \begin{pmatrix} D & r \\ Nm & s \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \\
&= \psi(D) f_2|_k \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \\
&= \psi(D) f_2 \\
&= \psi(a) f_2, \text{ since } D \equiv a \pmod{N}
\end{aligned} \tag{4.67}
$$

So f_2 satisfied the third condition of a Harmonic Maass form, and as before from the form of its fourier expansion we have the other two conditions satisfied. Hence $f_2 \in H_k(\gamma_0(N), \psi)$, then so is f_1. □

Example of using the converse theorem

Using the previous two theorems, we will give an alternative proof of the following statement: if $k \in \mathbb{N}$ and f is a weight $2-k$ holomorphic cusp form, then the $(k-1)$ derivative of f is a weakly holomorphic cusp form of weight k.

Proposition 4.1.1. *Let $k \in 2\mathbb{N}$, and let $f \in S^!_{2-k}$ for $SL_2(\mathbb{Z})$ with Fourier expansion (3.2). Then the function f_1 given by*

$$f_1(z) = \sum_{\substack{n=-n_0 \\ n \neq 0}} a(n)(2\pi n)^{k-1} e^{2\pi i n z} \tag{4.68}$$

is an element of $S^!_k$.

Proof. Since $f \in S^!_{2-k}$, $n^{k-1}a(n) = \mathcal{O}(e^{c\sqrt{n}})$ as $n \to \infty$ for some $c > 0$. For $\phi \in S_c(\mathbb{R}_+)$, we have

$$L_{f1}(\phi) = \sum_{\substack{n=-n_0, \\ n \neq 0}}^{\infty} (2\pi n)^{k-1} a(n)(L\phi)(2\pi n), \tag{4.69}$$

And

$$L_f(\alpha(\phi)) = \sum_{\substack{n=-n_0,\\ n\neq 0}}^{\infty} a(n)L(\alpha(\phi))(2\pi n)$$
$$= \sum_{\substack{n=-n_0,\\ n\neq 0}}^{\infty} (2\pi n)^{k-1}a(n)(L\phi)(2\pi n) \tag{4.70}$$
$$= L_{f1}(\phi).$$

Where

$$\alpha(\phi)(x) := L^{-1}(u^{k-1}(L\phi)(u))(x). \tag{4.71}$$

Now using (4.1.8 [10]) and ϕ being supported in $(c_1, c_2) \subset \mathbb{R}_{>0}$, we have : $(L\phi^{(k-1)})(u) = u^{k-1}(L\phi)(u) - u^{k-2}\phi(0) - u^{k-3}\phi'(0) - \ldots = u^{k-1}(L\phi)(u)$. Then

$$\alpha(\phi) = L^{-1}(u^{k-1}(L\phi)(u)) = L^{-1}(L\phi^{(k-1)}) = \phi^{(k-1)}, \tag{4.72}$$

thus, $\alpha(\phi) \in \mathcal{F}_f$. Using theorem (3.1.1), we get

$$L_{f1}(\phi) = L_f(\alpha(\phi)) = i^{2-k}L_f(\alpha(\phi)|_k W_1). \tag{4.73}$$

And,

$$L_{f1}(\phi|_{2-k}W_1) = L_f(\alpha(\phi|_{2-k}W_1)). \tag{4.74}$$

We claim that

$$\alpha(\phi)|_k W_1 = -\alpha(\phi|_{2-k}W_1), \tag{4.75}$$

which is equivalent to

$$-u^{k-1}(L(\phi|_{2-k}W_1))(u) = L(\alpha(\phi|_{2-k}W_1))$$
$$= L(-\alpha(\phi)|_k W_1) \tag{4.76}$$
$$= -L(\alpha(\phi)|_k W_1)(u).$$

So it is enough to prove (4.76) for $u > 0$ since both sides are holomorphic.

Let $p_l(x) = x^l$, for $l \in \mathbb{Z}, l \geq 1$. By ([10]-4.2.3) for $u > 0$, we have

$$\frac{1}{l!}(Lp_l)(u) = p_{-l-1}(u) = u^{-l-1}. \tag{4.77}$$

Using (4.71). we get

$$\phi(x) = L^{-1}(u^{-k+1}(L\alpha(\phi)))(x) \tag{4.78}$$

$\Longleftrightarrow$

$$\begin{aligned}
\phi(x) &= L^{-1}(p_{1-k}(L\alpha(\phi)))(x) \\
&= L^{-1}\left[\frac{1}{(k-2)!}Lp_{k-2}L\alpha(\phi)\right](x) \\
&= \frac{1}{(k-2)!}\int_0^x p_{k-2}(x-t)\alpha(\phi)(t)dt \\
&= \frac{1}{(k-2)!}\int_0^x (x-t)^{k-2}\alpha(\phi)(t)dt.
\end{aligned} \tag{4.79}$$

And

$$\begin{aligned}
L(\phi|_{2-k}W_1)(u) &= \int_0^\infty (\phi|_{2-k}W_1)(x)e^{-ux}dx \\
&= \int_0^\infty x^{k-2}\phi\left(\frac{1}{x}\right)e^{-ux}dx \\
&= \int_0^\infty x^{k-2}\frac{1}{(k-2)!}\int_0^{\frac{1}{x}}\left(\frac{1}{x}-t\right)^{k-2}\alpha(\phi)(t)dte^{-ux}dx \\
&= \frac{1}{(k-2)!}\int_0^\infty x^{k-2}\int_0^{\frac{1}{x}}\frac{(1-tx)^{k-2}}{x^{k-2}}\alpha(\phi)(t)dte^{-ux}dx \\
&= \frac{1}{(k-2)!}\int_0^\infty\int_0^{\frac{1}{x}}(1-tx)^{k-2}\alpha(\phi)(t)dte^{-ux}dx \\
&= \frac{1}{(k-2)!}\int_0^\infty\int_0^1(1-t)^{k-2}\alpha(\phi)\left(\frac{t}{x}\right)\frac{1}{x}dte^{-ux}dx \\
&= \frac{1}{(k-2)!}\int_0^\infty\int_0^1(1-t)^{k-2}\alpha(\phi)\left(\frac{1}{x}\right)\frac{1}{xt}e^{-uxt}tdtdx \\
&= \frac{1}{(k-2)!}\int_0^\infty\frac{1}{x}\alpha(\phi)\left(\frac{1}{x}\right)\left(\int_0^1(1-t)^{k-2}e^{-uxt}dt\right)dx
\end{aligned}$$

Using the fact that

$$\gamma^*(a,z) = \frac{1}{\Gamma(a)} \int_0^1 t^{a-1} e^{-zt} dt, \mathrm{Re}(a) > 0, \tag{4.80}$$

we get

$$\begin{aligned} \gamma^*(k-1,-ux) &= \frac{1}{\Gamma(k-1)} \int_0^1 t^{k-2} e^{uxt} dt \\ &= \frac{1}{\Gamma(k-1)} \int_0^1 (1-t)^{k-2} e^{ux} e^{-uxt} dt. \end{aligned} \tag{4.81}$$

Then,

$$L(\phi|_{2-k} W_1) = \frac{1}{(k-2)!} \int_0^\infty \frac{1}{x} \alpha(\phi) \left(\frac{1}{x}\right) \Gamma(k-1) e^{-ux} \gamma^*(k-1,-ux) dx$$

now using

$$\gamma^*(a,z) = \frac{z^{-a}}{\Gamma(a)} \gamma(a,z), \tag{4.82}$$

we get:

$$\begin{aligned} L(\phi|_{2-k} W_1) &= \frac{1}{(k-2)!} \int_0^\infty \frac{1}{x} \alpha(\phi) \left(\frac{1}{x}\right) \Gamma(k-1) e^{-ux} \times \frac{(-ux)^{1-k}}{\Gamma(k-1)} \gamma(k-1,-ux) dx \\ &= \frac{1}{(k-2)!} \int_0^\infty \frac{1}{x} \alpha(\phi) \left(\frac{1}{x}\right) (-ux)^{1-k} e^{-ux} \gamma(k-1,-ux) dx \end{aligned}$$

And with

$$\gamma(n+1,z) = n! \left(1 - e^{-z} \sum_{k=0}^{n} \frac{z^k}{k!}\right) \tag{4.83}$$

we obtain:

$$\begin{aligned} L(\phi|_{2-k}W_1) &= \frac{1}{(k-2)!}\int_0^\infty \frac{1}{x}\alpha(\phi)\left(\frac{1}{x}\right)(-ux)^{1-k}(k-2)!e^{-ux}\left(1-e^{ux}\sum_{j=0}^{k-2}\frac{(-ux)^j}{j!}\right)dx \\ &= \int_0^\infty \frac{1}{x}\alpha(\phi)\left(\frac{1}{x}\right)(-ux)^{1-k}e^{-ux}\left(1-e^{ux}\sum_{j=0}^{k-2}\frac{(-ux)^j}{j!}\right)dx \\ &= \int_0^\infty \frac{1}{x}\alpha(\phi)\left(\frac{1}{x}\right)(-ux)^{1-k}e^{-ux}dx + \int_0^\infty \sum_{j=0}^{k-2}\frac{(-ux)^j}{j!}\frac{1}{x}\alpha(\phi)\left(\frac{1}{x}\right)(-ux)^{1-k}dx \end{aligned} \tag{4.84}$$

where

$$\begin{aligned} \int_0^\infty \frac{1}{x}(\alpha(\phi)\left(\frac{1}{x}\right)(-ux)^{1-k}e^{-ux}dx &= \int_0^\infty (\alpha\phi)|_k W_1)(x)e^{-ux}(-u)^{1-k}dx \\ &= (-u)^{1-k}\int_0^\infty (\alpha\phi)|_k W_1)(x)e^{-ux}dx \\ &= (-u)^{1-k}L(\alpha(\phi)|_k W_1)(u). \end{aligned} \tag{4.85}$$

And

$$\begin{aligned} \int_0^\infty \sum_{j=0}^{k-2}\frac{(-ux)^j}{j!}\frac{1}{x}\alpha(\phi)\left(\frac{1}{x}\right)(-ux)^{1-k}dx &= \sum_{j=0}^{k-2}\frac{(-u)^{j+1-k}}{j!}\int_0^\infty x^{j-k}\alpha(\phi)\left(\frac{1}{x}\right)dx \\ &= \sum_{j=0}^{k-2}\frac{(-u)^{j+1-k}}{j!}\int_0^\infty x^{k-2-j}\alpha(\phi)(x)dx. \end{aligned} \tag{4.86}$$

Thus,

$$L(\phi|_{2-k}W_1) = (-u)^{1-k}L(\alpha(\phi)|_k W_1)(x) + \sum_{j=0}^{k-2}\frac{(-u)^{j+1-k}}{j!}\int_0^\infty x^{k-2-j}\alpha(\phi)(x)dx. \tag{4.87}$$

For $j \in [0, k-2]$, by using integration by parts and the fact that ϕ is compactly supported, we get

$$\int_0^\infty \alpha(\phi)(x)x^j dx = \int_0^\infty \phi^{(k-1)}(x)x^j dx = 0 \tag{4.88}$$

Thus,

$$L(\phi|_{2-k}W_1) = (-u)^{1-k}L(\alpha(\phi)|_k W_1)(x) \tag{4.89}$$

since k is even, it is equivalent to

$$-u^{k-1}L(\phi|_{2-k}W_1) = L(\alpha(\phi)|_k W_1)(x) \tag{4.90}$$

Therefore, we have proved (4.75). Combining this with (4.73) and (4.74), we have

$$\begin{aligned} L_{f1}(\phi) &= i^{2-k}L_f(\alpha(\phi)|_k W_1) \\ &= -i^{2-k}L_f(\alpha(\phi|_{2-k}W_1)) \\ &= -i^{2-k}L_{f1}(\phi|_{2-k}W_1) \\ &= i^k L_{f1}(\phi|_{2-k}W_1). \end{aligned} \tag{4.91}$$

Which implies by the previous corollary that f_1 is a weakly holomorphic form with weight k for $SL_2(\mathbb{Z})$. □

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