AMERICAN UNIVERSITY OF BEIRUT

MONGE-AMPÈRE EQUATION

by

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Abstract of the Thesis of

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In this work, we focus on studying the Aleksandrov solution of the Monge-Ampère equation. Initially, we develop the notion of a normal mapping and discuss its properties through proving concepts from convex analysis. Moreover, we define the Monge-Ampère measure over a Borel sigma algebra as well as proving the maximum and comparison principles of this equation. We conclude our study with solving the homogeneous and non-homogeneous Dirichlet problems for the Monge-Ampère operator.

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Introduction

The Monge-Ampère equation has received a great deal of attention in recent years due to its significant implications in various fields. It is classified as a fully non-linear degenerate elliptic partial differential equation. Its classical form is given by

$$\det D^2u = f(x, u, \nabla u) \ \ in \ \ \Omega \tag{1}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open convex set, $u : \Omega \to \mathbb{R}$ is a convex function, and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ is given.

This equation was initially introduced by the French mathematician Gaspard Monge in 1784 and later by André-Marie Ampère in 1820. Although it has been studied for a long time ago, it is still an active area of research which arises in many important problems in analysis, geometry, and physics. One of its interesting applications in differential geometry is the "prescribed Gaussian curvature equation" where the Gauss curvature $K(x)$ of the graph of a function u on $\mathbb{R}^n$ at $(x, u(x))$ is given by (1) with $f = K(x)(1 + |\nabla u|^2)^{(n+2)/2}$ [1]. The Monge-Ampère equation also arises strongly in Optimal Transport. Given μ and μ^* measures with compact support, consider the optimal map minimizing $\int_{\mathbb{R}^n} |T(x)-x|^2 d\mu(x)$ over all measure-preserving maps T from μ to μ^*. It turns out that T exists and is given by the subdifferential of a convex function u in $\mathbb{R}^n$ such that u satifies (1) with $f = g(x)/h(\nabla u(x))$ where $d\mu = g(x)dx$ and $d\mu^* = h(y)dy$ [2]. Another field where the equation appears is atmospheric sciences, in particular, meteorology. In fact, the semigeostrophic equations can be reformulated as a coupled Monge-Ampère/transport (MA/TR) problem after suitable changes of variables [3].

Moreover, as we hope for a smooth function u to solve (1), the Russian mathematician A. D. Aleksandrov introduced a notion of weak solutions to the Monge-Ampère equation called Aleksandrov solutions (or generalized solutions). As a consequence, the study of smoothness of such solutions has become a center of interest to many researchers. This notion is defined as follows: to a convex function $u : \Omega \to \mathbb{R}$, one associates a measure Mu in Ω that will be defined later in Chapter 3, and u is called an Aleksandrov solution to (1) if Mu has density f. In practice, it is useful to consider a Borel measure as the right hand side of (1) in order to prove the existance and uniqueness of solutions. So given a nonnegative Borel measure μ inside Ω, we call u an Aleksandrov solution of $\det D^2u = \mu$ if $Mu = \mu$ in Ω.

In this thesis, we are interested in the study of convex Aleksandrov solution u

to the Dirichlet problem

$$\begin{cases} \det D^2u = \mu & in \ \ \Omega \\ u = g & on \ \ \partial\Omega \end{cases} \tag{2}$$

where $g : \partial\Omega \to \mathbb{R}$ is a continuous function. We begin our work in Chapter 1 with some preliminaries concerning covex sets and convex functions. Chapter 2 then presents the notion of a normal mapping $\partial u : \Omega \to \mathcal{P}(\mathbb{R}^n)$ of a function $u : \Omega \to \mathbb{R}$ and discuss its related properties in convex analysis.

In Chapter 3, we define a Borel $\sigma - algebra$ on Ω to a continuous function u that contains subsets E such that $\partial u(E)$ is Lebesgue measurable. We then introduce the Monge-Ampère measure Mu associated to u in Ω where $Mu(E) = |\partial u(E)|$. If $u \in C^2(\Omega)$, we proved that Mu is a measure with density $\det D^2u$ which asserts the notion of weak Aleksandrov solutions. After that, in Section 3.3, we discuss an interesting property of the Monge-Ampère measure that is the stability under uniform convergence which will imply the closedness of Aleksandrov solutions under uniform limits. In Sections 3.4 and 3.5, we consider the maximum and comparison principles that are fundamental in the study of Monge-Ampère operator.

Finally, in Chapter 4, we solve the homogeneous Dirichlet problem and then prove the existance and uniqueness of a convex Aleksandrov solution u to (2).

Our research is based on the classical book *The Monge-Ampère Equation* by *Cristian E. Gutiérrez* [4]. We also rely on the book *Convex Analysis* by *R. T. Rockafellar* [5] as well as on the book *Functions of Bounded Variation and Free Discontinuity Problems* by *L. Ambrosio, N. Fusco, and D. Pallara* [6] for the measure theoretic results.

Chapter 1

Preliminaries From Convex Analysis

1.1 Convex Sets and Supporting Hyperplanes

Definition 1.1.1. *A set $\Omega \subseteq \mathbb{R}^n$ is convex if and only if for every x,y $\in \Omega$, the straight line $(1-t)x + ty \in \Omega$ whenever $t \in [0,1]$.*

Definition 1.1.2. *A set $\Omega \subseteq \mathbb{R}^n$ is strictly convex if and only if for every $x, y \in \bar{\Omega}$, the straight line $(1-t)x + ty \in \Omega^{\text{o}}$ whenever $t \in (0,1)$.*

Proposition 1.1.3. *Intersection of convex sets is convex.*

Proof. Let Ω_1 and Ω_2 be two convex sets. Let $x, y \in \Omega_1 \cap \Omega_2$ and $t \in (0,1)$. We have $x, y \in \Omega_1$ with $(1-t)x + ty \in \Omega_1$ since Ω_1 is convex, and $x, y \in \Omega_2$ with $(1-t)x + ty \in \Omega_2$ since Ω_2 is convex. Thus $(1-t)x + ty \in \Omega_1 \cap \Omega_2$. $\square$

Proposition 1.1.4. *Interior of a convex set is convex.*

Proof. Let Ω be a convex set. Let $x, y \in \Omega^{\text{o}}$ and $t \in (0,1)$. We claim that $z = (1-t)x + ty \in \Omega^{\text{o}}$. As $y \in \Omega^{\text{o}}$, $\exists r > 0$ such that $B(y,r) \subseteq \Omega$. We show that $B(z, rt) \subseteq \Omega$ and thus $z \in \Omega^{\text{o}}$. Let $u \in B(z, rt)$ and write $v = y + \frac{1}{t}(u - z)$. Hence $v \in B(y,r) \subseteq \Omega$ and $u = (1-t)x + tv$. Since Ω is convex with $x, v \in \Omega$, then $u \in \Omega$. $\square$

Proposition 1.1.5. *Closure of a convex set is convex.*

Proof. Let Ω be a convex set. Let $x, y \in \bar{\Omega}$ and $t \in (0,1)$. There exists $(x_n)_n \subseteq \Omega$ and $(y_n)_n \subseteq \Omega$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. As Ω is convex, $(1-t)x_n + ty_n \in \Omega$ $\forall n \in \mathbb{N}$ with $\lim_{n\to\infty} ((1-t)x_n + ty_n) = (1-t)x + ty$. Therefore $(1-t)x + ty \in \bar{\Omega}$. $\square$

Proposition 1.1.6. *If Ω is a convex set with nonempty interior, then for every x in the interior of Ω and y in the closure of Ω, the line $(1-t)x + ty \in \Omega^\circ$ whenever $0 \le t < 1$.*

Proof. Let $x \in \Omega^{\mathrm{o}}$, $y \in \bar{\Omega}$, and $t \in (0,1)$. We claim that $z = (1-t)x + ty \in \Omega^{\mathrm{o}}$. As $x \in \Omega^{\mathrm{o}}$, $\exists \epsilon > 0$ such that $B(x,\epsilon) \subseteq \Omega$. We first show that $\forall y' \in \Omega$ and $z' = (1-t)x+ty'$, $B(z',(1-t)\epsilon) \subseteq \Omega$. Let $u \in B(z',(1-t)\epsilon)$ and write $v = x + \frac{1}{1-t}(u-z')$. Hence $v \in B(x,\epsilon) \subseteq \Omega$ and $u = (1-t)v + ty'$. Since Ω is convex with $v, y' \in \Omega$, then $u \in \Omega$. Now, consider the ball $B(y, \frac{1-t}{t}\epsilon)$. As $y \in \bar{\Omega}$, $\exists y' \in \Omega \cap B(y, \frac{1-t}{t}\epsilon)$, and we write $z = (1-t)x + ty' + t(y-y')$. Then $z \in B(z',(1-t)\epsilon) \subseteq \Omega$ as we proved before, and thus $z \in \Omega^{\mathrm{o}}$. ☐

Proposition 1.1.7. *If Ω is a convex set with nonempty interior, then $\Omega^{\mathrm{o}} = (\bar{\Omega})^{\mathrm{o}}$.*

Proof. We have
$$\Omega \subseteq \bar{\Omega} \implies \Omega^{\mathrm{o}} \subseteq (\bar{\Omega})^{\mathrm{o}}$$
Suppose now $y \in (\bar{\Omega})^{\mathrm{o}}$. $\exists \epsilon > 0$ such that $B(y,\epsilon) \subseteq \bar{\Omega}$. Let $x \in \Omega^{\mathrm{o}}$ with $x \neq y$ and $0 < \delta < \frac{\epsilon}{|y-x|}$. We obtain $z = x + (1+\delta)(y-x) = y + \delta(y-x) \in B(y,\epsilon) \subseteq \bar{\Omega}$. We can write $y = (1-t)x + tz$ with $0 < t = \frac{1}{1+\delta} < 1$. Hence, by Proposition 1.1.6, $y \in \Omega^{\mathrm{o}}$.

☐

Proposition 1.1.8. *If Ω is a convex set with nonempty interior, then $\partial\Omega = \partial\bar{\Omega}$.*

Proof. This is a direct result from definition of boundary and Proposition 1.1.7. ☐

Definition 1.1.9. *A hyperplane Π in $\mathbb{R}^n$ is given in cartesian coordinates as follows:*
$$\{x \in \mathbb{R}^n : p \cdot x = b\}$$
with p a non-zero vector in $\mathbb{R}^n$ and b an arbitrary real constant.

A hyperplane Π divides $\mathbb{R}^n$ into two closed half spaces:
$$\Pi^+ = \{x \in \mathbb{R}^n : p \cdot x \geq b\} \qquad \text{and} \qquad \Pi^- = \{x \in \mathbb{R}^n : p \cdot x \leq b\}.$$

Definition 1.1.10. *Given a set $\Omega \subseteq \mathbb{R}^n$, and $x_0 \in \partial\Omega$. We say that the hyperplane Π supports Ω at x_0 if and only if $x_0 \in \Pi$ and Ω is contained in one of the two closed subspaces $\Pi^{\pm}$.*

To elaborate, here Π passes through x_0, so there exists p (normal) such that Π is given by:
$$p \cdot (x - x_0) = 0.$$
If Π supports Ω at x_0, then
$$\Omega \subseteq \{x : p \cdot (x - x_0) \geq 0\} \qquad \text{or} \qquad \Omega \subseteq \{x : p \cdot (x - x_0) \leq 0\}.$$
We can replace p by $-p$ and get the following definition.

Definition 1.1.11. *Given $\Omega \subseteq \mathbb{R}^n$, and $x_0 \in \partial\Omega$. Ω has a supporting hyperplane at x_0 if and only if there exists $p \in \mathbb{R}^n \setminus \{0\}$ such that for every $x \in \Omega$, we have $p \cdot (x - x_0) \geq 0$.*

Theorem 1.1.12. *Given $\Omega \subseteq \mathbb{R}^n$ a closed convex set. For each $x_0 \in \partial\Omega$, there exists a hyperplane Π supporting Ω at x_0.*

Proof. Take $x_0 \in \partial\Omega$.
Case 1: Ω is bounded

Step 1. As $x_0 \in \partial\Omega$, $\forall \epsilon > 0$, $B(x_0, \epsilon) \cap \Omega^{\complement} \neq \emptyset$. Hence $\forall k \in \mathbb{N}$, $\exists y_k \in B(x_0, 1/k) \cap \Omega^{\complement}$. We claim that for each $k \in \mathbb{N}$, there exists a unique $x_k \in \partial\Omega$ such that $d(y_k, \Omega) = |y_k - x_k|$.

Starting with the definition of the distance, we have $d(y_k, \Omega) = \inf_{x\in\Omega} |y_k - x|$, and $0 \leq d(y_k, \Omega) \leq |y_k - x| < \infty$ for some $x \in \Omega$. By definition of infimum, $\forall n \in \mathbb{N}$, $\exists x_n \in \Omega$ such that

$$d(y_k, \Omega) \leq |y_k - x_n| < d(y_k, \Omega) + \frac{1}{n}.$$

which implies that $\lim\limits_{n\to\infty} |y_k - x_n| = d(y_k, \Omega)$. Moreover, $|y_k - x_n| \leq d(y_k, \Omega) + 1$ $\forall n \in \mathbb{N}$. Then $x_n \in B(y_k, d(y_k, \Omega) + 2)$ $\forall n \in \mathbb{N}$ obtaining $(x_n)_n$ is a bounded sequence. By Bolzano-Weierstrass Theorem, $(x_n)_n$ has a convergent subsequence, say without relabeling that

$$x_n \to x_k$$

with $x_k \in \Omega$ as Ω is closed. Therefore $\lim\limits_{n\to\infty} |y_k - x_n| = |y_k - x_k|$, and by uniqueness of limit, we get $d(y_k, \Omega) = |y_k - x_k|$.

Now, we show that x_k is unique. Suppose there exists $x'_k \in \Omega$ such that $x'_k \neq x_k$ and $d(y_k, \Omega) = |y_k - x'_k|$. As Ω is convex, we have $\frac{x_k + x'_k}{2} \in \Omega$, and thus

$$d(y_k, \Omega) \leq \left| y_k - \frac{x_k + x'_k}{2} \right| \leq \left| \frac{y_k - x_k}{2} \right| + \left| \frac{y_k - x'_k}{2} \right| = \frac{d(y_k, \Omega)}{2} + \frac{d(y_k, \Omega)}{2} = d(y_k, \Omega).$$

This implies that

$$|y_k - x_k + y_k - x'_k| = |y_k - x_k| + |y_k - x_k|.$$

Squaring the equality, we get

$$\langle y_k - x_k, y_k - x'_k \rangle = |y_k - x_k||y_k - x'_k|$$

which is equivalent to $(y_k - x_k)$ and $(y_k - x'_k)$ being linearly dependent. We obtain $(y_k - x_k) = \lambda(y_k - x'_k)$ for some $\lambda \in \mathbb{R}$. However, we know that $|y_k - x_k| = |y_k - x'_k|$, so $\lambda = 1$ or $\lambda = -1$. If $\lambda = -1$, then $\left| y_k - \frac{x_k + x'_k}{2} \right| = 0$ which is a contradiction since $y_k \notin \Omega$ and $\frac{x_k + x'_k}{2} \in \Omega$. Therefore $\lambda = 1$ and $x_k = x'_k$.

To end this step, it remains to show that $x_k \in \partial\Omega$. Since $y_k \notin \Omega$, then $d(y_k, \partial\Omega) \leq d(y_k, \Omega)$. Similar to what we proved before, for $\partial\Omega$ is a closed set, $\exists x'_k \in \partial\Omega$ such that

$$d(y_k, \partial\Omega) = |y_k - x'_k|.$$

Suppose now $x_k \notin \partial\Omega$, so $|y_k - x'_k| < |y_k - x_k|$ which is a contradiction since $|y_k - x_k| = \inf_{x\in\Omega} |y_k - x|$ and $x'_k \in \Omega$.

Step 2. We show that the hyperplane passing through x_k with normal $(x_k - y_k)$ supports Ω at x_k.

Consider the plane

$$\Pi : (x_k - y_k) \cdot (x - x_k) = 0.$$

It is sufficient to show that $(x_k - y_k) \cdot (x - x_k) \geq 0 \;\; \forall x \in \Omega$ (see Definition 1.1.11). Suppose $\exists\, x' \in \Omega$ such that $(x_k - y_k) \cdot (x' - x_k) < 0$ and take the line

$$\ell : \ell(t) = (1-t)x_k + tx', \quad t \in \mathbb{R}.$$

We know that $\ell(t) \in \Omega \;\forall\; t \in [0,1]$ since Ω is convex. Also,

$$|y_k - \ell(t)|^2 = |y_k - x_k - t(x' - x_k)|^2 = |y_k - x_k| - 2t(y_k - x_k) \cdot (x' - x_k) + t^2|x' - x_k|^2.$$

We notice that it is an equation of parabola with minimum $(\hat{t}, \ell(\hat{t}))$ where

$$\hat{t} = \frac{(y_k - x_k) \cdot (x' - x_k)}{|x' - x_k|^2} > 0.$$

Then $\exists\; 0 < t_1 < 1$ such that $t_1 < \hat{t}$ and $|y_k - \ell(t_1)| < |y_k - \ell(0)| = |y_k - x_k|$. This is a contradiction as $\ell(t_1) \in \Omega$ and $|y_k - x_k| = d(y_k, \Omega)$.

Step 3. We claim that $\exists z \in \mathbb{R}^n$ such that the hyperplane passing through x_0 with normal $(x_0 - z)$ supports Ω at x_0.

Since Ω is bounded in this case, then there exists a ball B containing Ω. Let z_k be the intersection of the ray from x_k to y_k and the boundary of B. We first want to show that

$$d(z_k, \Omega) = |z_k - x_k|.$$

For the same reasoning as in Step 1, there exists a unique $x_k' \in \partial\Omega$ such that $d(z_k, \Omega) = |z_k - x_k'|$. Suppose $x_k' \neq x_k$. From Step 2, we know that the hyperplane passing through x_k' with normal $(x_k' - z_k)$ supports Ω at x_k'. Hence $(x_k' - z_k) \cdot (x_k - x_k') \geq 0$ as $x_k \in \Omega$, and by Step 2, $(x_k - y_k) \cdot (x_k' - x_k) \geq 0$ as $x_k' \in \Omega$. This gives that $(x_k - z_k) \cdot (x_k' - x_k) \geq 0$ since $(x_k - z_k)$ is collinear with $(x_k - y_k)$ with same direction. However, we obtain

$$(x_k' - z_k) \cdot (x_k - x_k') = -(x_k - x_k')^2 - (x_k - z_k) \cdot (x_k' - x_k) < 0$$

which is a contradiciton.

Consider now the sequence $(y_k)_k \subseteq B(x_0, 1/k) \cap \Omega^{\complement}$. Since $|y_k - x_0| < \frac{1}{k} \;\; \forall k \in \mathbb{N}$, so $\lim\limits_{k\to\infty} y_k = x_0$. Moreover, we have the sequence $(x_k)_k \subseteq \partial\Omega$ with Ω closed and bounded, so $(x_k)_k$ is a bounded sequence. By Bolzano-Weierstrass Theorem, $(x_k)_k$ has a convergent subsequence, say without relabeling that $x_k \to x$. But

$$\lim_{k\to\infty} |y_k - x_k| = \lim_{k\to\infty} d(y_k, \Omega) = d(x_0, \Omega) = 0,$$

then $\lim\limits_{k\to\infty} x_k = x_0$. Also, the sequence $(z_k)_k \subseteq \bar{B}$ is bounded, hence again by Bolzano-Weierstrass Theorem, $(z_k)_k$ has a convergent subsequence, say without relabeling that $z_k \to z$. As we have $d(z_k, \Omega) = |z_k - x_k| \ \forall k \in \mathbb{N}$, we apply limit both sides and obtain $d(z, \Omega) = |z - x_0|$. Therefore, as we proved in Step 2, the hyperplane passing through x_0 with normal $(x_0 - z)$ supports Ω at x_0. This ends the proof for this case.

Case 2: Ω is unbounded

Let B be a closed ball of center x_0 and radius r. Then $B \cap \Omega$ is closed bounded convex set by Proposition 1.1.3. From Case 1, there exists a hyperplane Π supporting $B \cap \Omega$ at x_0 which means there exists $p \in \mathbb{R}^n \setminus \{0\}$ such that for every $x \in B \cap \Omega$, we have $p \cdot (x - x_0) \geq 0$. We want to show that Π supports Ω at x_0.

Suppose $\exists\, x_1 \in B^{\complement} \cap \Omega$ such that $p \cdot (x_1 - x_0) < 0$. We have $(1-t)x_0 + tx_1 \in \Omega \ \forall t \in (0,1)$ since Ω is convex, and $\exists\, t' \in (0,1)$ such that $x' = (1-t')x_0 + t'x_1 \in B$. Therefore $x' \in B \cap \Omega$ and hence $p \cdot (x' - x_0) \geq 0$. However,

$$p \cdot (x' - x_0) = p \cdot ((1-t')x_0 + t'x_1 - x_0) = t'p \cdot (x_1 - x_0) < 0$$

which is a contradiction.

$\square$

Corollary 1.1.13. *Given $\Omega \subseteq \mathbb{R}^n$ an open convex set. For each $x_0 \in \partial\Omega$, there exists a hyperplane Π supporting Ω at x_0.*

Proof. Let $x_0 \in \partial\Omega$. We have $x_0 \in \bar{\Omega}$ with $\bar{\Omega}$ is a closed convex set by Proposition 1.1.5. Consequently, by Proposition 1.1.8 and Theorem 1.1.12, there exists a hyperplane Π supporting $\bar{\Omega}$ at x_0. Hence it supports Ω at x_0.

$\square$

Corollary 1.1.14. *If Ω is an open convex set, then for every x, y in the boundary of Ω, the line $(1-t)x + ty \in \Omega$ whenever $0 < t < 1$ or $(1-t)x + ty \in \partial\Omega$ whenever $0 \leq t \leq 1$.*

Proof. Let $x, y \in \partial\Omega$. We have $(1-t)x + ty \in \bar{\Omega} \ \forall t \in (0,1)$ as $\bar{\Omega}$ is convex by Proposition 1.1.5. Hence for each $t \in (0,1)$, $(1-t)x + ty \in \Omega$ or $(1-t)x + ty \in \partial\Omega$. Suppose $\exists\, t_1, t_2 \in (0,1)$ such that $z_1 = (1-t_1)x + t_1 y \in \Omega$ and $z_2 = (1-t_2) + t_2 y \in \partial\Omega$. Then, by Corollary 1.1.13, there exists a supporting hyperplane Π to Ω at z_2 i.e. $\Omega \subseteq \Pi^+$ and thus $\Omega^{\mathrm{o}} = \Omega \subseteq (\Pi^+)^{\mathrm{o}}$. This implies that $p \cdot (z_1 - z_2) > 0$ as $z_1 \in \Omega$. Now, we write

$$p \cdot (z_1 - z_2) = p \cdot (x - z_2) + t_1 p \cdot (y - x) = (1-t_1)p \cdot (x - z_2) + t_1 p \cdot (y - z_2).$$

Moreover, $z_2 \in \Pi$ and z_2 belong to the line (xy), then either $(xy) \subset \Pi$ or (xy) intersects Π at z_2. If $(xy) \subset \Pi$, then $p \cdot (x - z_2) = 0$ and $p \cdot (y - z_2) = 0$. We obtain $p.(z_1 - z_2) = 0$ which is a contradiction. If $(xy) \not\subseteq \Pi$, then we will have $x \in \Pi^+$ and $y \in \Pi^-$ that is also a contradiction. $\square$

Corollary 1.1.15. *If Ω is an open convex set with $\Omega \subsetneq \mathbb{R}^n$, then $\bar{\Omega} \subsetneq \mathbb{R}^n$.*

Proof. Suppose $\bar{\Omega} = \mathbb{R}^n$. Since $\Omega \subsetneq \mathbb{R}^n$, then there exists $z \notin \Omega$ with $z \in \mathbb{R}^n = \bar{\Omega}$ obtaining $z \in \partial\Omega$. By Corollary 1.1.13, there exists a supporting hyperplane Π to Ω at z. This implies $\Omega \subseteq \Pi^+$ and $\bar{\Omega} \subseteq \Pi^+$. Hence, $\mathbb{R}^n \subseteq \Pi^+$, that is $\Pi^+ = \mathbb{R}^n$ which is a contradiction. □

Corollary 1.1.16. *If Ω is an open convex, then $\bar{\Omega}$ is equal to the intersections of all the upper closed half-spaces formed by the supporting hyperplanes to it at boundary points, that is*

$$\bar{\Omega} = \bigcap_{x_0 \in \partial\Omega} \bigcap_{p \in N(x_0)} \Pi^+_{x_0,p}$$

with

$$N(x_0) = \{p : |p| = 1, \Pi_{x_0,p} \text{ supports } \Omega \text{ at } x_0\}.$$

Proof. We know that $\Omega \subseteq \Pi^+_{x_0,p}$ $\forall p \in N(x_0), \forall x_0 \in \partial\Omega$. This implies that $\bar{\Omega} \subseteq \Pi^+_{x_0,p}$ $\forall p \in N(x_0), \forall x_0 \in \partial\Omega$ since $\Pi^+_{x_0,p}$ is a closed set (see Definition 1.1.10). Thus

$$\bar{\Omega} \subseteq \bigcap_{x_0 \in \partial\Omega} \bigcap_{p \in N(x_0)} \Pi^+_{x_0,p}.$$

Suppose now $y \notin \bar{\Omega}$. By Corollary 1.1.13, and as we proceed in the proof of Theorem 1.1.12, $\exists$ $x_0 \in \partial\Omega$ and unit normal vector $p = \frac{x_0 - y}{|x_0 - y|}$ such that the hyperplane $\Pi_{x_0,p} : p \cdot (x - x_0) = 0$ supports Ω at x_0. Hence $\Omega \subseteq \Pi^+_{x_0,p}$ with $y \notin \Pi^+_{x_0,p}$. Therefore

$$y \notin \bigcap_{x_0 \in \partial\Omega} \bigcap_{p \in N(x_0)} \Pi^+_{x_0,p}.$$

□

1.2 Convex Functions in One Dimension

Definition 1.2.1. *Let $f : (a,b) \to \mathbb{R}$ a function with $a, b \in \mathbb{R}$. f is said to be convex if and only if*

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) \quad \forall t \in [0,1], \forall x, y \in (a,b).$$

Theorem 1.2.2. *Let $f : (a,b) \to \mathbb{R}$. f is convex if and only if for all $s, t, u \in (a,b)$ with $s < t < u$, we have*

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}. \tag{1.1}$$

Proof. Suppose f is convex, and let $s, t, u \in (a,b)$ such that $s < t < u$. Then $\exists r \in (0,1)$ such that $t = (1-r)s + ru$ with

$$\frac{f(t) - f(s)}{t - s} \leq \frac{(1-r)f(s) + rf(u) - f(s)}{(1-r)s + ru - s} = \frac{r(f(u) - f(s))}{r(u - s)} = \frac{f(u) - f(s)}{u - s}$$

and

$$\frac{f(u)-f(t)}{u-t} \geq \frac{f(u)-(1-r)f(s)-rf(u)}{u-(1-r)s+ru} = \frac{(1-r)(f(u)-f(s))}{(1-r)(u-s)} = \frac{f(u)-f(s)}{u-s}.$$

Conversely, suppose f satisfy (1.1). Let $x, y \in (a,b)$, and consider without loss of generality that $x < y$. Let $r \in (0,1)$ with $t = (1-r)x + ry$, then $x < t < y$. Hence we get

$$\frac{f(t)-f(x)}{t-x} \leq \frac{f(y)-f(t)}{y-t}.$$

This gives that

$$\frac{f((1-r)x+ry)-f(x)}{r(y-x)} \leq \frac{f(y)-f((1-r)x+ry)}{(1-r)(y-x)}$$

which implies $(1-r)f((1-r)x+ry)-(1-r)f(x) \leq rf(y)-rf((1-r)x+ry)$. Therefore $f((1-r)x+ry) \leq (1-r)f(x)+rf(y)$ and then f is convex. □

Proposition 1.2.3. *Let $f : (a,b) \to \mathbb{R}$ a convex function, then f is bounded in any closed subinterval.*

Proof. Let $[c,d] \subseteq (a,b)$. For every $x \in [c,d]$, $\exists\, t_x \in [0,1]$ such that $x = (1-t_x)c + t_x d$. By convexity of f, we obtain

$$f(x) \leq (1-t_x)f(c)+t_x f(d) \leq (1-t_x)\max(f(c),f(d))+t_x\max(f(c),f(d)) = \max(f(c),f(d)).$$

□

Theorem 1.2.4. *Let $f : (a,b) \to \mathbb{R}$ be a convex function then f is continuous.*

Proof. Let $x_0 \in (a,b)$. Then $x_0 \in [x_0-\delta, x_0+\delta] \subseteq (a,b)$ for some $\delta > 0$. Take $0 < h < 1$ such that $\sqrt{h} < \delta$. We have $x_0 < x_0 + h < x_0 + \sqrt{h}$. Thus Theorem 1.2.2 implies the inequality

$$\frac{f(x_0+h)-f(x_0)}{h} \leq \frac{f(x_0+\sqrt{h})-f(x_0)}{\sqrt{h}},$$

that is $f(x_0+h)-f(x_0) \leq \sqrt{h}(f(x_0+\sqrt{h})-f(x_0))$. However, by Proposition 1.2.3, f is bounded on $[x_0-\delta, x_0+\delta]$ and thus $f(x_0+h)-f(x_0) \leq M\sqrt{h}$ for some $M > 0$. Now apply limit superior both sides, we obtain

$$\limsup_{h\to 0}(f(x_0+h)-f(x_0)) \leq 0.$$

Similarly, with $x_0 - \sqrt{h} < x_0 < x_0 + h$ and Theorem 1.2.2, we get

$$\frac{f(x_0+h)-f(x_0)}{h} \geq \frac{f(x_0)-f(x_0-\sqrt{h})}{\sqrt{h}}.$$

This implies that $f(x_0+h)-f(x_0) \geq \sqrt{h}(f(x_0)-f(x_0-\sqrt{h}) \geq -M\sqrt{h}$. Applying limit inferior both sides, we get

$$\liminf_{h\to 0}(f(x_0+h)-f(x_0)) \geq 0.$$

Hence we obtain $\lim\limits_{h\to 0}(f(x_0+h)-f(x_0)) = 0$ and thus f is continuous at x_0 with x_0 is arbitrary in (a,b). Therefore f is continuous on (a,b). □

Proposition 1.2.5. *Let $f:(a,b)\to\mathbb{R}$ be a real valued differentiable function. f is convex if and only if f' is increasing.*

Proof. Suppose f is convex, and let $s,t,u \in (a,b)$ such that $s<t<u$. By Theorem 1.2.2, we have

$$\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}.$$

Let $t\to s^+$ in the left inequality and $t\to u^-$ in the right inequality implies respectively that

$$f'(s) \leq \frac{f(u)-f(s)}{u-s} \qquad \textit{and} \qquad \frac{f(u)-f(s)}{u-s} \leq f'(u).$$

Therefore we get that $f'(s)\leq f'(u)$ and f' is then increasing.

Conversely, suppose f' is increaing. Let $s,t,u\in(a,b)$ such that $s<t<u$. f is differentiable, then by Mean Value Theorem, there exists $x\in(s,t)$ and $y\in(t,u)$ such that

$$\frac{f(t)-f(s)}{t-s} = f'(x) \leq f'(y) = \frac{f(u)-f(t)}{u-t}$$

as f' is increasing. Therefore, by Theorem 1.2.2, we obtain that f is convex. □

Proposition 1.2.6. *Let $f:(a,b)\to\mathbb{R}$ be a twice differentiable function. f is convex if and only if $f''\geq 0$.*

Proof. Direct result from Proposition 1.2.5. □

Theorem 1.2.7. *Let $f:(a,b)\to\mathbb{R}$ be a differentiable function. If f is convex, then $f(y)\geq f(x)+f'(x)(y-x)$ for all $x,y\in(a,b)$.*

Proof. Let $x,y\in(a,b)$.

If $x<y$: let $c\in(a,b)$ such that $x<c<y$. Then, by Theorem 1.2.2, we have

$$\frac{f(y)-f(x)}{y-x} \geq \frac{f(x)-f(c)}{x-c}.$$

Letting $c\to x^+$ both sides gives

$$\frac{f(y)-f(x)}{y-x} \geq f'(x).$$

This implies that $f(y)\geq f(x)+f'(x)(y-x)$.

If $x > y$: let $c \in (a.b)$ such that $y < c < x$. Again, by Theorem 1.2.2, we get

$$\frac{f(x) - f(c)}{x - c} \geq \frac{f(x) - f(y)}{x - y}.$$

Letting $c \to x^-$ both sides implies that

$$f'(x) \geq \frac{f(x) - f(y)}{x - y}.$$

Therefore $f(y) \geq f(x) + f'(x)(y - x)$.

If $x = y$: the inequality holds, obviously. $\square$

1.3 Convex Functions in Higher Dimensions

Definition 1.3.1. *Let Ω be an open convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. u is said to be convex if and only if*

$$u((1-t)x + ty) \leq (1-t)u(x) + tu(y) \quad \forall t \in [0,1], \forall x, y \in \Omega.$$

Definition 1.3.2. *Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. We define the graph of u to be the set*

$$G(u) = \{(x, u(x)) \in \Omega \times \mathbb{R} : x \in \Omega\}.$$

We define the epigraph of u to be the set

$$epi(u) = \{(x, y) \in \Omega \times \mathbb{R} : y \geq u(x)\}.$$

Proposition 1.3.3. *Let Ω be an open convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. u is a convex function if and only if $epi(u)$ is a convex subset of $\mathbb{R}^{n+1}$.*

Proof. Suppose u is convex. Let (x_1, y_1), $(x_2, y_2) \in epi(u)$ and $t \in (0,1)$. We claim that $(1-t)(x_1, y_1) + t(x_2, y_2) \in epi(u)$. We have $u((1-t)x_1 + tx_2) \leq (1-t)u(x_1) + tu(x_2)$ by convexity of u. Also, $u(x_1) \leq y_1$ and $u(x_2) \leq y_2$ since (x_1, y_1), $(x_2, y_2) \in epi(u)$. This implies $u((1-t)x_1 + tx_2) \leq (1-t)y_1 + ty_2$. Hence $((1-t)x_1 + tx_2, (1-t)y_1 + ty_2) \in epi(u)$.

Conversely, suppose $\mathrm{epi}(u)$ is convex. Let $x_1, x_2 \in \Omega$ and $t \in [0,1]$. We have $(x_1, u(x_1)), (x_2, u(x_2)) \in epi(u)$ and thus $(1-t)(x_1, u(x_1)) + t(x_2, u(x_2)) \in epi(u)$ since $\mathrm{epi}(u)$ is convex. We get $u((1-t)x_1 + tx_2) \leq (1-t)u(x_1) + tu(x_2)$. $\square$

Proposition 1.3.4. *Let Ω be an open convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ convex, then u is bounded in any compact subset of Ω.*

Proof. We show first by induction that if u is convex on the hybercube $[-1,1]^n$, then

$$|u(x)| \leq \max(|u(\pm e_1)|, \cdots, |u(\pm e_n)|).$$

If $n = 1$, then we proceed as in the proof of Proposition 1.2.3 to get that $|u(x)| \leq \max(|u(-1)|, |u(1)|)$. Assume the result is true in dimension $n-1$ with $u : [-1,1]^n \mapsto$

$\mathbb{R}$ convex. Notice that the restriction of u on each face of the cube is convex and by the induction hypothesis bounded. In fact

$$\sup_{x\in\partial[-1,1]^n} |u(x)| \leq \max(|u(\pm e_1)|, \cdots, |u(\pm e_n)|).$$

Take $x \in (-1,1)^n$, and consider the line L_x from a vertex, say $e_1 = (1, 0, \cdots, 0)$, to x. It will intersect the boundary of the cube at a point P_x. We have that u restricted to L_x is a convex one variable function, then Proposition 1.2.3 implies that

$$|u(x)| \leq \max(|u(e_1)|, |u(P_x)|) \leq \max(|u(\pm e_1)|, \cdots, |u(\pm e_n))|.$$

Let $\Omega' \subseteq \Omega$ open, and u convex on Ω'. Let $x_0 \in \Omega'$ then there exists a closed hyper-cube $Q_\delta = \{x : \|x - x_0\|_\infty \leq \delta\} \subseteq \Omega'$. Rescaling and translating above argument, we get that u is bounded on Q_δ.

More generally, for $K \subseteq \Omega$ compact, we take Ω' open such that $K \subseteq \Omega' \subseteq \overline{\Omega'} \subset \Omega$. By compactness of K and since Ω' is open, then K can be covered by finitely many closed hypercubes $Q_1, Q_2, \cdots, Q_{n_K}$ contained in Ω'. Hence by above part for every $x \in K$,

$$|u(x)| \leq \sup(|u(Q_1)|, \cdots, |u(Q_{n_K})|) < \infty.$$

$\square$

Theorem 1.3.5. *Let Ω be an open convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. If u is convex then u is continuous.*

Proof. Let $x_0 \in \Omega$. There exists $\varepsilon > 0$ such that $\overline{B(x_0, \varepsilon)} \subseteq \Omega$. From Proposition 1.3.4, u is bounded on $\overline{B(x_0, \varepsilon)}$, say $|u(x)| \leq M$. For $x \in B(x_0, \varepsilon)$, let z_x be the intersection of the line from x_0 to x with $\partial B(x_0, \varepsilon)$. $u|_{[x_0, z_x]}$ is a one variable convex function, then from Theorem 1.2.2

$$u(x) - u(x_0) \leq \frac{u(z_x) - u(x_0)}{|z_x - x_0|}|x - x_0| \leq \frac{2M}{\varepsilon}|x - x_0|.$$

Switching the roles of x and x_0, we conclude that for every $x \in B(x_0, \varepsilon)$

$$|u(x) - u(x_0)| \leq \frac{2M}{\varepsilon}|x - x_0|.$$

Hence u is continuous at x_0. $\square$

Theorem 1.3.6. *Let Ω be an open convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a differentiable function. If u is convex, then $u(y) \geq u(x) + \nabla u(x)^T(y - x)$ for all $x, y \in \Omega$.*

Proof. Suppose u is convex. Let $x, y \in \Omega$, and define on $[0, 1]$

$$f(t) = u(x + t(y - x)).$$

Clearly, f is differentiable as it is a composition of two differentiable functions. We claim that f is convex. Let $r \in [0,1]$ and $t_0, t_1 \in [0,1]$. Replacing x with $(1-r)x+rx$ and applying convexity of u, we obtain

$$\begin{aligned} f((1-r)t_0 + rt_1) &= u((1-r)(x + t_0(y-x)) + r(x + t_1(y-x))) \\ &\leq (1-r)u(x + t_0(y-x)) + ru(x + t_1(y-x)) \\ &= (1-r)f(t_0) + rf(t_1) \end{aligned}$$

and hence f is convex. By Theorem 1.2.7, we have $f(1) \geq f(0) + f'(0)(1-0)$. Substituting $f(1) = u(y)$, $f(0) = u(x)$, and $f'(t) = \nabla u(x + t(y-x))^T(y-x)$ in the inequality, we conclude that $u(y) \geq u(x) + \nabla u(x)^T(y-x)$. $\square$

Theorem 1.3.7. *Let Ω be an open convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a twice differentiable function. If u is convex, then its Hessian matrix is positive semi-definite, that is $D^2u(x) \geq 0$ for all $x \in \Omega$.*

Proof. Suppose u is convex. Let $x, y \in \Omega$, and define on $[0,1]$

$$f(t) = u((x + t(y-x)).$$

f is twice differentiable as it is the composition of two twice differentiable functions. Similarly, as we proved in Theorem 1.3.6, f is also convex. Therefore, by Proposition 1.2.6, we get that $f''(t) \geq 0$ for all $t \in [0,1]$. This implies that

$$(y-x)^T D^2u(x + t(y-x))(y-x) \geq 0 \quad \forall t \in [0,1].$$

In particular, for $t = 0$,

$$(y-x)^T D^2u(x)(y-x) \geq 0 \tag{1.2}$$

and this is true for any $x, y \in \Omega$. Now, let λ be an eigenvalue of the Hessian matrix $D^2u(x)$, then there exists a unit eigenvector $w \neq 0$ corresponding to λ, i.e $(D^2u(x))w = \lambda w$. Since Ω is open, then there exists $r > 0$ such that $B(x,r) \subseteq \Omega$. Letting $y = x + rw$, we substitute in (1.2) and obtain that $\lambda|w|^2 \geq 0$ which implies $\lambda \geq 0$. Therefore, $D^2u(x)$ is positive semidefinite. $\square$

Chapter 2

Normal Mappings

2.1 Definitions

Definition 2.1.1. *Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. A vector $p \in \mathbb{R}^n$ is a subgradient of u at $x_0 \in \Omega$ if and only if $u(x) \geq u(x_0) + p \cdot (x - x_0)\ \forall x \in \Omega$.*

Remark 2.1.2. *Suppose $\exists\ p \in \mathbb{R}^n$ a subgradient of u at x_0. Then we get that $y \geq u(x) \geq u(x_0) + p \cdot (x - x_0)\ \forall (x, y) \in epi(u)$. This implies that*

$$(p, -1) \cdot ((x, y) - (x_0, u(x_0)) \leq 0 \quad \forall (x, y) \in epi(u).$$

Thus the plane

$$H : (p, -1) \cdot ((x, y) - (x_0, u(x_0)) = 0$$

is a supporting hyperplane to epi(u) at $(x_0, u(x_0))$ (See Definition 1.1.10).

Remark 2.1.3. *Suppose $\exists\ p \in \mathbb{R}^n$ a subgradient of u at x_0. Then we say that the affine function $L(x) = u(x_0) + p \cdot (x - x_0)$ is a supporting hyperplane to u at $(x_0, u(x_0))$.*

Definition 2.1.4. *Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. The normal mapping of u is a set valued function given by*

$$\partial u : \Omega \to \mathcal{P}(\mathbb{R}^n)$$

such that for each $x_0 \in \Omega$, $\partial u(x_0)$ is the set of all subgradients of u at x_0, i.e

$$\partial u(x_0) = \{p \in \mathbb{R}^n : u(x) \geq u(x_0) + p \cdot (x - x_0)\ \forall x \in \Omega\}.$$

$\partial u(x_0)$ is called the subdifferential of u at x_0

For a set $E \subseteq \Omega$, we define

$$\partial u(E) = \bigcup_{x \in E} \partial u(x).$$

2.2 Properties of subdifferential

Proposition 2.2.1. *Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. The subdifferential of u at $x_0 \in \Omega$ is a convex subset of $\mathbb{R}^n$.*

Proof. Let $p_1, p_2 \in \partial u(x_0)$ and $t \in [0, 1]$. We need to prove that $(1-t)p_1 + tp_2 \in \partial u(x_0)$. In fact, for $x \in \Omega$,

$$\begin{aligned} u(x_0) + ((1-t)p_1 + tp_2) \cdot (x - x_0) &= (1-t)(u(x_0) + p_1 \cdot (x - x_0)) + t(u(x_0) + p_2 \cdot (x - x_0)) \\ &\leq (1-t)u(x) + tu(x) \\ &= u(x). \end{aligned}$$

□

Theorem 2.2.2. *Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a continuous function. If $K \subset \Omega$ is a compact set, then $\partial u(K)$ is a compact subset of $\mathbb{R}^n$.*

Proof. Let $K \subset \Omega$ compact. We will show that $\partial u(K)$ is compact using sequential compactness definition.

Take a sequence $(p_n)_n \subset \partial u(K)$. First, let's show that $(p_n)_n$ is a bounded sequence. We have $p_n \in \partial u(K) = \bigcup_{x \in K} \partial u(x)$. Therefore, for each $n \in \mathbb{N}$, there exists $x_n \in K$ such that $p_n \in \partial u(x_n)$. We get

$$u(x) \geq u(x_n) + p_n \cdot (x - x_n) \quad \forall x \in \Omega. \tag{2.1}$$

Now, for $0 < \delta < 1$, we define $K_\delta = \{x \in \Omega : dist(x, K) \leq \delta\}$. We have
i) $K_\delta \subseteq \Omega \subseteq \mathbb{R}^n$.
ii) K_δ is closed: Letting $g(x) = dist(x, K)$, then $K_\delta = g^{-1}([0, \delta])$. So it is an inverse image of a closed set under a continuous function.
iii) K_δ is bounded: As K is compact, then there exists $R > 0$ such that $K \subseteq B(0, R)$. Thus, by definition of K_δ, $K_\delta \subseteq B(0, R + 4\delta)$. Hence, K_δ is a compact subset of $\mathbb{R}^n$.

Now, let $y \in \mathbb{R}^n$ such that $|y| = 1$. Then $x_n + \delta y \in K_\delta \ \forall n \in \mathbb{N}$ since $dist(x_n + \delta y, K) \leq |x_n + \delta y - x_n| = \delta$. Moreover, by substituting in (2.1),

$$u(x_n + \delta y) \geq u(x_n) + \delta p_n \cdot y \quad \forall n \in \mathbb{N}.$$

If $p_n \neq 0$, take $y = \frac{p_n}{|p_n|}$. We get

$$\max_{x \in K_\delta} u(x) \geq u(x_n + \delta y) \geq u(x_n) + \delta |p_n| \geq \min_{x \in K} u(x) + \delta |p_n|.$$

As u is continuous with K and K_δ are compact, $\max_{x \in K_\delta} u(x)$ and $\min_{x \in K} u(x)$ are finite, obtaining that

$$|p_n| \leq \frac{\max_{x \in K_\delta} u(x) - \min_{x \in K} u(x)}{\delta} \quad \forall n \in \mathbb{N}.$$

This implies that $(p_n)_n \subseteq \partial u(K) \subseteq \mathbb{R}^n$ is a bounded sequence. By Bolzano-Weierstrass Theorem, $(p_n)_n$ has a convergent subsequence, say without relabeling that $p_n \xrightarrow[n \to \infty]{} p_0$.

It remains to show that $p_0 \in \partial u(K)$. Also $(x_n) \subset K$, then by sequential compactness property of K, (x_n) has a convergent subsequence, say without relabeling that $x_n \xrightarrow[n\to\infty]{} x_0$ with $x_0 \in K$. As u is continuous, applying limit as $n \to \infty$ in (2.1) implies that

$$u(x) \geq u(x_0) + p_0 \cdot (x - x_0) \quad \forall x \in \Omega$$

concluding $p_0 \in \partial u(x_0) \subseteq \partial u(K)$. □

Theorem 2.2.3. *Let Ω be an open convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. u is convex if and only if $\partial u(x_0) \neq \emptyset$ for all $x_0 \in \Omega$.*

Proof. Suppose u is convex in Ω, and let $x_0 \in \Omega$. Then $(x_0, u(x_0)) \in G(u) \subseteq \partial epi(u)$ (See Definition 1.3.2). Proposition 1.3.3 implies that epi(u) is a convex subset of $\mathbb{R}^{n+1}$ since u is convex. Hence, by Theorem 1.1.13, there exists a supporting hyperplane to epi(u) at $(x_0, u(x_0))$, i.e there exists $\hat{p} \in \mathbb{R}^{n+1} \setminus \{0\}$ such that

$$\hat{p} \cdot ((x, y) - (x_0, u(x_0)) \geq 0 \quad \forall (x, y) \in epi(u)$$

(see Definition 1.1.11). Let $\hat{p} = (p, p_{n+1})$ with $p \in \mathbb{R}^n$ and $p_{n+1} \in \mathbb{R}$. Also, let $\epsilon > 0$ and take $x = x_0$ with $y = u(x_0) + \epsilon$. Hence $(x, y) \in epi(u)$ and

$$0 \leq (p, p_{n+1}) \cdot (x - x_0, y - u(x_0)) = (p, p_{n+1}) \cdot (0, \epsilon).$$

This implies that $\epsilon p_{n+1} \geq 0$, i.e $p_{n+1} \geq 0$. Suppose now that $p_{n+1} = 0$. We then obtain that $p \cdot (x - x_0) \geq 0$ for all $x \in \Omega$. However, Ω is open with $x_0 \in \Omega$, thus for $\delta > 0$ small enough we have $x_1 = x_0 + \delta e_i \in \Omega$, and $x_2 = x_0 - \delta e_i \in \Omega$. Hence, we get $\delta p \cdot e_i \geq 0$ and $-\delta p \cdot e_i \geq 0$, and so $p \cdot e_i = 0$ for every i concluding that $p = 0$ that is a contradiction. Therefore, we get $p_{n+1} > 0$ and we can write

$$(p, 1) \cdot ((x, y) - (x_0, u(x_0))) \geq 0 \quad \forall (x, y) \in epi(u).$$

Thus $p \cdot (x - x_0) + y - u(x_0) \geq 0$ $\forall (x, y) \in epi(u)$ which is equivalent to $y \geq u(x_0) + p \cdot (x - x_0)$ $\forall (x, y) \in epi(u)$. In particular, for $y = u(x)$, we reach that

$$u(x) \geq u(x_0) + p \cdot (x - x_0) \quad \forall x \in \Omega.$$

Hence $p \in \partial u(x_0)$ and $\partial u(x_0) \neq \emptyset$.

Conversely, we suppose that $\partial u(x_0) \neq \emptyset$ for all $x_0 \in \Omega$. Let $x_1, x_2 \in \Omega$ with $x_0 = (1 - t)x_1 + tx_2$. Then $x_0 \in \Omega$ as Ω is convex, and $\partial u(x_0) \neq \emptyset$. So there exist $p \in \mathbb{R}^n$ such that $u(x) \geq u(x_0) + p \cdot (x - x_0)$ $\forall x \in \Omega$. In particular, $u(x_1) \geq u(x_0) + p \cdot (x_1 - x_0)$ and $u(x_2) \geq u(x_0) + p \cdot (x_2 - x_0)$. We obtain

$$\begin{aligned}(1 - t)u(x_1) + tu(x_2) &\geq (1 - t)(u(x_0) + p \cdot (x_1 - x_0)) + t(u(x_0) + p \cdot (x_2 - x_0)) \\ &= (1 - t)u(x_0) + tp \cdot (x_0 - x_2) + tu(x_0) + tp \cdot (x_2 - x_0) \\ &= u(x_0) = u((1 - t)x_1 + tx_2).\end{aligned}$$

Therefore u is convex. □

Theorem 2.2.4. *Let Ω be an open convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a convex function. If $K \subset \Omega$ is compact, then u is Lipschitz continuous in K with Lipschitz constant $C(K, u) = \sup\{|p|, p \in \partial u(K)\}$.*

Proof. Let $x, y \in \Omega$. By Theorem 2.2.3, we have $\partial u(x) \neq \emptyset$ and $\partial u(y) \neq \emptyset$. Let $p_1 \in \partial u(x)$ and $p_2 \in \partial u(y)$. Then $u(z) \geq u(x)+p_1\cdot(z-x)$ and $u(z) \geq u(y)+p_2\cdot(z-y)$ for all $z \in \Omega$. In particular, $u(y) \geq u(x) + p_1 \cdot (y - x)$ and $u(x) \geq u(y) + p_2 \cdot (x - y)$. This implies

$$u(y) - u(x) \geq p_1 \cdot (y - x) \geq -|p_1||y - x|$$

and

$$u(x) - u(y) \geq p_2 \cdot (x - y) \geq -|p_2||x - y|.$$

Now, let $C = \sup\{|p|, p \in \partial u(K)\}$. C is finite as $\partial u(K)$ is compact from Theorem 2.2.2. We obtain

$$-C|x - y| \leq u(x) - u(y) \leq C|x - y|$$

which is equivalent to

$$|u(x) - u(y)| \leq C|x - y|.$$

Therefore, u is Lipschitz continuous in K. $\square$

Theorem 2.2.5. *Let Ω be an open, bounded, and convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a convex function with $u \leq 0$ on $\partial\Omega$. If $p \in \partial u(x_0)$ for some $x_0 \in \Omega$, then*

$$|p| \leq \frac{-u(x_0)}{dist(x_0, \partial\Omega)}.$$

Proof. Let $p \in \partial u(x_0)$ for some $x_0 \in \Omega$.

First, let's show that $u(x) \leq 0$ for all $x \in \Omega$. Let $x \in \Omega$. From properties of Ω, we know that there exists $x_1, x_2 \in \partial\Omega$ and $t \in (0, 1)$ such that $x = (1-t)x_1 + tx_2$. But u is convex with $u(x_1), u(x_2) \leq 0$. Therefore, we get $u(x) \leq (1 - t)u(x_1) + tu(x_2) \leq 0$.

Now, if $p = 0$, then

$$0 \leq \frac{-u(x_0)}{dist(x_0, \partial\Omega)}$$

and we are done. If $p \neq 0$, we know that $u(x) \geq u(x_0) + p \cdot (x - x_0)$ $\forall x \in \Omega$ as $p \in \partial u(x_0)$. Let $0 < r < dist(x_0, \partial\Omega)$ and take $x = x_0 + r\frac{p}{|p|}$. We get $|x - x_0| = r < dist(x_0, \partial\Omega)$ obtaining $x \in \Omega$ and $0 \geq u(x) \geq u(x_0) + r|p|$. Letting $r \to dist(x_0, \partial\Omega)$ implies that

$$|p| \leq \frac{-u(x_0)}{dist(x_0, \partial\Omega)}.$$

$\square$

Theorem 2.2.6. *Let Ω be an open convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. If u is convex in Ω and differentiable at $x_0 \in \Omega$, then $\partial u(x_0) = \{\nabla u(x_0)\}$.*

Proof. Let $x_0 \in \Omega$.

First, as u is differentiable at x_0, Theorem 1.3.6 implies that $\nabla u(x_0) \in \partial u(x_0)$ Now, let $p \in \partial u(x_0)$. Then

$$u(x) \geq u(x_0) + p \cdot (x - x_0) \quad \forall x \in \Omega. \tag{2.2}$$

As Ω is open, $x_0 \in \Omega$, and for $h > 0$ arbitrary small , we have by (2.2), $u(x_0+he_i) \geq u(x_0) + hp \cdot e_i$ and $u(x_0 - he_i) \geq u(x_0) - hp \cdot e_i$ for all $1 \leq i \leq n$. We write $p = (p_1, p_2, ..., p_n)$. Therefore $hp_i \leq u(x_0+he_1)-u(x_0)$ and $-hp_i \leq u(x_0-he_1)-u(x_0)$ for all $1 \leq i \leq n$. This means

$$p_i \leq \frac{u(x_0 + he_i) - u(x_0)}{h}$$

and

$$p_i \geq \frac{u(x_0 - he_i) - u(x_0)}{-h}$$

for all $1 \leq i \leq n$. Letting $h \to 0$, we get $p_i \leq \frac{\partial u}{\partial x_i}(x_0)$ and $p_i \geq \frac{\partial u}{\partial x_i}(x_0)$ for all $1 \leq i \leq n$. Therefore $p_i = \frac{\partial u}{\partial x_i}(x_0)$ $\forall 1 \leq i \leq n$ and thus $p = \nabla u(x_0)$. $\square$

Theorem 2.2.7. *Let Ω be an open convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. If u is convex in Ω and $\partial u(x_0) = \{p\}$ for some $x_0 \in \Omega$, then u is differentiable at x_0 with $p = \nabla u(x_0)$.*

Proof. Let $x_0 \in \Omega$ with $\partial u(x_0) = \{p\}$.

Step 1. Suppose $h : (a, b) \to \mathbb{R}$ with $a, b \in \mathbb{R}$ a convex function such that $h(t) \geq 0$ $\forall t \in (a, b)$ with $h(0) = 0$ and $\partial h(0) = \{c\}$. We claim that h is differentiable at $t = 0$ and $h'(0) = c$.

Let

$$s(t) = \frac{h(t)}{t}.$$

Let $0 < t_1 < t_2$. As h is convex, Theorem 1.2.2 implies that

$$\frac{h(t_1) - h(0)}{t_1 - 0} \leq \frac{h(t_2) - h(0)}{t_2 - 0},$$

that is $s(t_1) \leq s(t_2)$. Therefore s is increasing in $(0, \infty)$ with $s(t) \geq 0$ for $t > 0$, hence $m = inf\{s(t) : t > 0\} = \lim_{t \to 0^+} s(t)$ exists with $m \geq 0$. Now, let $t_2 < t_1 < 0$. Again, as h is convex, Theorem 1.2.2 gives that

$$\frac{h(0) - h(t_2)}{0 - t_2} \leq \frac{h(0) - h(t_1)}{0 - t_1}$$

which means $s(t_1) \leq s(t_2)$. Then s is increasing in $(-\infty, 0)$ with $s(t) \leq 0$ for $t < 0$, thus $k = \sup\{s(t) : t < 0\} = \lim_{t \to 0^-} s(t)$ exists and $k \leq 0$.

We get $s(t) \geq m$ $\forall t > 0$ which gives $h(t) \geq mt$ $\forall t > 0$. Moreover, h is always positive, then $h(t) \geq mt$ $\forall t \leq 0$. Therefore, we have $h(t) \geq mt$ $\forall t \in (a, b)$ and $h(t) \geq h(0) + m(t - 0)$ $\forall t \in (a, b)$ obtaining that $m \in \partial h(0)$. In addition, $s(t) \leq$

$k\ \ \forall t < 0$ i.e $h(t) \geq kt\ \ \forall t < 0$. Again, as h is always positive, we get $h(t) \geq kt\ \ \forall t \geq 0$ and so $h(t) \geq h(0) + k(t-0)\ \ \forall t \in (a,b)$ obtaining $k \in \partial h(0)$. This implies that $m = n = c$ which means

$$\lim_{t\to 0^+} \frac{h(t)}{t} = \lim_{t\to 0^-} \frac{h(t)}{t} = c.$$

Therefore, h is differentiable at $t = 0$ with $h'(0) = c$ which completes this step.

Step 2. Suppose $g : \Omega \to \mathbb{R}$ is a convex function such that $0 \in \Omega$, $g(x) \geq 0\ \ \forall x \in \Omega$, $g(0) = 0$, and $\partial g(0) = \{0\}$. We claim that g is differentiable at 0 with $\nabla g(0) = 0$. Fix $x \neq 0$ in Ω and let

$$h(t) = g(tx).$$

First, as Ω is convex, $(1-t)0 + tx \in \Omega\ \ \forall t \in [0,1]$. Hence h is well defined on $[0,1]$. Next, we show that h is convex. Let $t_1, t_2, \lambda \in [0,1]$. Since g is convex, we have

$$h((1-\lambda)t_1 + \lambda t_2) = g(((1-\lambda)t_1 + \lambda t_2)x) \leq (1-\lambda)g(t_1x) + \lambda g(t_2x) = (1-\lambda)h(t_1) + \lambda h(t_2).$$

Also, we have that $h(0) = g(0) = 0$, and $h(t) \geq 0\ \ \forall t \in [0,1]$.

Now, to use Step 1, it remains to show that $\partial h(0) = \{0\}$. Clearly, $0 \in \partial h(0)$. Let $c \in \partial h(0)$, then $h(t) \geq ct\ \ \forall t \in [0,1]$, i.e $g(tx) \geq ct\ \ \forall t \in [0,1]$. Take the line

$$\ell : \{(tx, ct),\ \ t \in \mathbb{R}\} \subseteq \mathbb{R}^{n+1}$$

so ℓ supports h at $t = 0$. By construction of h and convexity of g, there exists a plane Π containing ℓ supporting g at 0. Since Π is passing through the origin, we write $\Pi : y = q \cdot x$ for some $q \in \mathbb{R}^n$. Also, as Π contains ℓ, we obtain $ct = q \cdot tx$ which implies that $c = q \cdot x$. However $q \in \partial g(0) = \{0\}$, therefore $c = 0$ obtaining $\partial h(0) = 0$. By Step 1, h is differentiable at $t = 0$ with $h'(0) = 0$. We get

$$\lim_{t\to 0} \frac{h(t)}{t} = \lim_{t\to 0} \frac{g(tx)}{t} = 0$$

and this is true for each $x \neq 0$ in Ω.

Proceeding in the proof of differentiability of g at 0, we let $\delta > 0$ such that $[-\delta,\delta]^n \subseteq \Omega$. Let v_i be vertices of $[-\delta,\delta]^n$. The convex hull of the vertices equals $[-\delta,\delta]$ and is identical to the set of all their convex combinations. Let x be such that $|x| = \delta$. Thus, we can write $x = \sum_{i=1}^k \lambda_i v_i$ with $\sum_{i=1}^k \lambda_i = 1$ and $0 \leq \lambda_i \leq 1$. By convexity of g, we get

$$\frac{g(tx)}{t} = \frac{g(\sum_{i=1}^k \lambda_i(tv_i))}{t} \leq \frac{\sum_{i=1}^k \lambda_i g(tv_i)}{t} \leq \frac{\sum_{i=1}^k g(tv_i)}{t}$$

and this is true for all x such that $|x| = \delta$. Set now

$$h_i(t) = g(tv_i).$$

Hence

$$\sup_{|x|=\delta} \frac{g(tx)}{t} \leq \sum_{i=1}^k \frac{h_i(t)}{t}.$$

But we proved above that h_i is differentiable at $t = 0$ with $h_i'(0) = 0 \ \forall i$. Then

$$\lim_{t \to 0} \frac{h_i(t)}{t} = 0 \quad \forall i$$

which implies

$$\lim_{t \to 0} \sup_{|x| = \delta} \frac{g(tx)}{t} = 0.$$

Finally, let y be such that $|y| < \delta$. We write

$$\frac{g(y)}{|y|} = \delta \, \frac{g\left(\frac{|y|}{\delta} \frac{\delta y}{|y|}\right)}{\frac{|y|}{\delta}}$$

with $\left|\frac{\delta y}{|y|}\right| = \delta$. Therefore

$$\frac{g(y)}{|y|} \leq \delta \sup_{|x|=\delta} \frac{g\left(\frac{|y|}{\delta} x\right)}{\frac{|y|}{\delta}}.$$

Using what we proved above, we get

$$\lim_{y \to 0} \frac{g(y)}{|y|} = 0$$

and thus g is differentiable at 0 with $\nabla g(0) = 0$. This proves our claim.

Step 3: Back to our main claim, we need to show that u is differentiable at x_0 with $\nabla u(x_0) = p$ given that $\partial u(x_0) = \{p\}$. There exists $r > 0$ such that $B(x_0, r) \subseteq \Omega$. Define a real valued function g on $B(0, r)$ such that

$$g(x) = u(x_0 + x) - u(x_0) - p \cdot x.$$

First, we show that $g(x) \geq 0 \ \forall x \in dom(g)$. In fact, as $p \in \partial u(x_0)$, we have $u(y) \geq u(x_0) + p \cdot (y - x_0) \ \forall y \in \Omega$. In particular, $u(x + x_0) - u(x_0) - p \cdot x \geq 0 \ \forall x \in dom(g)$ which gives that $g(x) \geq 0 \ \forall x \in dom(g)$. Also, by convexity of u, g is convex with $g(0) = 0$. In addition, we can see that $\partial g(0) = \{0\}$. Obviously, $0 \in \partial g(0)$. Let $q \in \partial g(0)$. We have $g(x) \geq q \cdot x \ \forall x \in dom(g)$ obtaining $u(x_0 + x) - u(x_0) - p \cdot x \geq q \cdot x \ \forall x \in dom(g)$. In particular, $u(y) \geq u(x_0) + (p + q) \cdot (y - x_0) \ \forall y \in \Omega$. Hence $p + q \in \partial u(0)$ then $p + q = p$ which gives that $q = 0$.

Now, by Step 2, g is differentiable at $x = 0$ with

$$\lim_{x \to 0} \frac{g(x)}{|x|} = 0.$$

This implies that

$$\lim_{x \to 0} \frac{u(x_0 + x) - u(x_0) - p \cdot x}{|x|} = 0$$

i.e.

$$\lim_{z \to x_0} \frac{u(z) - u(x_0) - p \cdot (z - x_0)}{|z - x_0|} = 0.$$

This proves that u is differentiable at x_0 and $\nabla u(x_0) = p$. □

2.3 Examples

Example 2.3.1. *Consider $u : \mathbb{R} \to \mathbb{R}$ defined by $u(x) = |x|$.*

Obviously, u is convex in $\mathbb{R}$. Thus Theorem 2.2.3 implies that $\partial u(x) \neq \emptyset \ \forall x \in \mathbb{R}$. When $x > 0$ or $x < 0$, we know that u is differentiable at x. Hence, by Theorem 2.2.6, $\partial u(x) = \{\nabla u(x)\} = \{1\}$ when $x > 0$ and $\partial u(x) = \{\nabla u(x)\} = \{-1\}$ when $x < 0$.

Now, for $x = 0$, we let $p \in \partial u(0)$. We have $|y| \geq py \ \forall y \in \mathbb{R}$. Taking $y > 0$, we obtain $p \leq 1$. Taking $y < 0$, we obtain $p \geq -1$. Hence $\partial u(0) \subseteq [-1, 1]$. Conversely, if we let $p \in [-1, 1]$, we get $px \leq |x| \ \forall x \in \mathbb{R}$ which implies that $p \in \partial u(0)$. Therefore, $\partial u(0) = [-1, 1]$.

Example 2.3.2. *Let $\Omega = B(x_0, r) \subseteq \mathbb{R}^n$, the ball of center $x_0 \in \mathbb{R}^n$ and radius $r > 0$. Let u be the function defined on Ω whose graph in $\mathbb{R}^{n+1}$ is the upside-down right cone with vertex at $(x_0, 0)$ and base on the plane $x_{n+1} = h$ for $h > 0$. We write*

$$u(x) = \frac{h}{r}|x - x_0|.$$

First, u is convex in Ω. Indeed, let $x, y \in \Omega$ and $t \in (0, 1)$. We have

$$u((1-t)x+ty) = \frac{h}{r}|(1-t)(x-x_0)+t(y-x_0)| \leq (1-t)\frac{h}{r}|x-x_0|+t\frac{h}{r}|y-x_0| = (1-t)u(x)+tu(y).$$

Hence, by Theorem 2.2.3, we get that $\partial u(x) \neq \emptyset \ \forall x \in \Omega$.

If $x \neq x_0$, we have $0 < |x - x_0| < r$ with u is differentiable at x. By Theorem 2.2.6, we obtain $\partial u(x) = \{\nabla u(x)\}$. Now, let's calculate the gradient of u at x. Writing

$$u(x) = \frac{h}{r}\sqrt{\sum_{i=1}^{n}(x_i - x_{0,i})^2}$$

then $\frac{\partial u}{\partial x_i}(x) = \frac{h}{r}\frac{x_i - x_{0,i}}{|x - x_0|}$ obtaining that $\nabla u(x) = \frac{h}{r}\frac{x - x_0}{|x - x_0|}$.

Now, if $x = x_0$, we let $p \in \partial u(x_0)$ such that $p \neq 0$. So $u(x) \geq u(x_0) + p \cdot (x - x_0) \ \forall x \in \Omega$, that is

$$\frac{h}{r}|x - x_0| \geq p \cdot (x - x_0) \quad \forall x \in \Omega.$$

Take $x = x_0 + k\frac{p}{|p|}$ with $0 < k < r$. As $|x - x_0| = k < r$, then $x \in \Omega$ and $|p| \leq \frac{h}{r}$. Therefore $\partial u(x_0) \subseteq \overline{B(0, h/r)}$. Conversely, if $p \in \overline{B(0, h/r)}$ and $x \in \Omega$, we have

$$u(x) - u(x_0) = \frac{h}{r}|x - x_0| \geq |p||x - x_0| \geq p \cdot (x - x_0)$$

thus $p \in \partial u(x_0)$. We conclude that $\partial u(x_0) = \overline{B(0, h/r)}$.

Therefore, we reach to

$$\partial u(x) = \begin{cases} \{\frac{h}{r}\frac{x - x_0}{|x - x_0|}\} & if x \neq x_0 \\ \\ \overline{B(0, h/r)} & if x = x_0 \end{cases}.$$

Chapter 3

Monge-Ampère Measure

3.1 Legendre Transform

Definition 3.1.1. *Let Ω an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. The Legendre transform of u is a function $u^* : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ given by*

$$u^*(p) = \sup_{x \in \Omega}(x \cdot p - u(x)).$$

Proposition 3.1.2. *Let Ω be an open bounded subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a bounded function. Then u^* is finite and convex in $\mathbb{R}^n$.*

Proof. Let $p_1, p_2 \in \mathbb{R}^n$ and $t \in [0, 1]$. Then

$$\begin{aligned} u^*((1-t)p_1 + tp_2) &= \sup_{x \in \Omega}(x \cdot ((1-t)p_1 + tp_2) - u(x)) \\ &\le (1-t)\sup_{x \in \Omega}(x \cdot p_1 - u(x)) + t\sup_{x \in \Omega}(x \cdot p_2 - u(x)) \\ &= (1-t)u^*(p_1) + tu^*(p_2). \end{aligned}$$

□

Proposition 3.1.3. *Let Ω be an open bounded subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a bounded function. Let $x_0 \in \Omega$ and $p_0 \in \mathbb{R}^n$. $p_0 \in \partial u(x_0)$ if and only if $u^*(p_0) = x_0 \cdot p_0 - u(x_0)$.*

Proof. Suppose $p_0 \in \partial u(x_0)$. Then $u(x) \ge u(x_0) + p_0 \cdot (x - x_0)$ $\forall x \in \Omega$, that is $x \cdot p_0 - u(x) \le x_0 \cdot p_0 - u(x_0)$ $\forall x \in \Omega$. We get

$$\sup_{x \in \Omega}(x \cdot p_0 - u(x)) \le x_0 \cdot p_0 - u(x_0)$$

thus $u^*(p_0) \le x_0 \cdot p_0 - u(x_0)$. However, by definition of u^*, $x_0 \cdot p_0 - u(x_0) \le u^*(p_0)$. Therefore $u^*(p_0) = x_0 \cdot p_0 - u(x_0)$. Conversely, suppose $u^*(p_0) = x_0 \cdot p_0 - u(x_0)$. This gives that $x \cdot p_0 - u(x) \le x_0 \cdot p_0 - u(x_0)$ $\forall x \in \Omega$ which implies $u(x) \ge u(x_0) + p_0 \cdot (x - x_0)$ $\forall x \in \Omega$. Thus $p_0 \in \partial u(x_0)$. □

Proposition 3.1.4. *Let Ω be an open bounded subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a bounded function. If $N(x) = \{p \in \mathbb{R}^n : u^*(p) = x \cdot p - u(x)\}$ then $Nu(x) = \partial u(x)$.*

Proof. Direct result from Proposition 3.1.3. □

Proposition 3.1.5. *Let Ω be an open, bounded, and convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a bounded function. If u is convex and differentiable in Ω, then $u^*(Du(x)) = x \cdot Du(x) - u(x)$ for each $x \in \Omega$.*

Proof. As u is convex and differentiable, Theorem 2.2.6 implies that $\partial u(x) = \{Du(x)\}$ $\forall x \in \Omega$. By Proposition 3.1.3, we get $u^*(Du(x)) = x \cdot Du(x) - u(x)\ \ \forall x \in \Omega$.

□

Proposition 3.1.6. *Let Ω be an open bounded subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a bounded function. Let $x_0 \in \Omega$ and $p_0 \in \mathbb{R}^n$. If $p_0 \in \partial u(x_0)$ then $x_0 \in \partial u^*(p_0)$.*

Proof. Let $p_0 \in \partial u(x_0)$. We claim that $x_0 \in \partial u^*(p_0)$, that is $u^*(p) \geq u^*(p_0) + x_0 \cdot (p - p_0)\ \forall p \in \mathbb{R}^n$. By Proposition 3.1.3, we proved that $u^*(p_0) = x_0 \cdot p_0 - u(x_0)$. Hence, it is sufficient to show that $u^*(p) \geq x_0 \cdot p - u(x_0)\ \ \forall p \in \mathbb{R}^n$. But by definition of u^*, the inequality is verified. □

Proposition 3.1.7. *Let Ω be an open, bounded, and convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a bounded function. If u, u^* both convex and differentiable, then $Du^*(Du(x)) = x$ for each $x \in \Omega$.*

Proof. Since u, u^* are both convex and differentiable, Theorem 2.2.6 implies that $\partial u(x) = \{Du(x)\}\ \forall x \in \Omega$ and $\partial u^*(p) = \{Du^*(p)\}\ \forall p \in \mathbb{R}^n$. Since $p = Du(x) \in \partial u(x)$, Proposition 3.1.6 gives that $x \in \partial u^*(p)$. Therefore $x = Du^*(p) = Du^*(Du(x))$.

□

Proposition 3.1.8. *Let Ω be an open, bounded, and convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a bounded function. Then $(u^*)^*(x) \leq u(x)$ for each $x \in \Omega$.*

Proof. By definition of u^*, we have $x \cdot p - u(x) \leq u^*(p)\ \forall x \in \Omega, \forall p \in \mathbb{R}^n$, that is $p \cdot x - u^*(p) \leq u(x)\ \forall x \in \Omega, \forall p \in \mathbb{R}^n$. We conclude that

$$(u^*)^*(x) = \sup_{p \in \mathbb{R}^n} (p \cdot x - u^*(p)) \leq u(x) \quad \forall x \in \Omega.$$

□

Proposition 3.1.9. *Let Ω be an open bounded subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a bounded function. Let $x_0 \in \Omega$. If $\partial u(x_0) \neq \emptyset$ then $(u^*)^*(x_0) = u(x_0)$.*

Proof. Assume $\partial u(x_0) \neq \emptyset$. Then there exists $p_0 \in \mathbb{R}^n$ such that $p_0 \in \partial u(x_0)$. By Proposition 3.1.3, we have $u^*(p_0) = x_0 \cdot p_0 - u(x_0)$ which implies

$$u(x_0) = x_0 \cdot p_0 - u^*(p_0) \leq \sup_{p \in \mathbb{R}^n} (x_0 \cdot p - u^*(p)) = (u^*)^*(x_0).$$

Hence, using Proposition 3.1.8, we get $(u^*)^*(x_0) = u(x_0)$. □

Proposition 3.1.10. *Let Ω be an open, bounded, and convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a bounded function. u is convex in Ω if and only if $(u^*)^*(x) = u(x)$ for each $x \in \Omega$.*

Proof. Suppose u is convex. By Theorem 2.2.3, $\partial u(x) \neq \emptyset$ $\forall x \in \Omega$. Hence, by Proposition 3.1.9, we get $(u^*)^*(x) = u(x)$ $\forall x \in \Omega$. Coversely, suppose $(u^*)^*(x) = u(x)$ $\forall x \in \Omega$. $(u^*)^*$ is convex in Ω by Proposition 3.1.2. Therefore u is convex in Ω. □

Proposition 3.1.11. *Let Ω be an open, bounded, and convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a bounded function. Let $x_0 \in \Omega$ and $p_0 \in \mathbb{R}^n$. If u is convex in Ω, then $p_0 \in \partial u(x_0)$ if and only if $x_0 \in \partial u^*(p_0)$.*

Proof. The necessary condition is always true by Proposition 3.1.6. Suppose now $x_0 \in \partial u^*(p_0)$. By Proposition 3.1.3, we get $(u^*)^*(x_0) = p_0 \cdot x_0 - u^*(p_0)$. Also, by Proposition 3.1.10, we obtain $u(x_0) = p_0 \cdot x_0 - u^*(p_0)$, that is $u^*(p_0) = x_0 \cdot p_0 - u(x_0)$. Again, Proposition 3.1.3 implies that $p_0 \in \partial u(x_0)$. □

3.2 Monge-Ampère Measure

Lemma 3.2.1. *Let Ω be an open convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. If u is convex in Ω, then u is differentiable a.e. in Ω.*

Proof. The proof follows from the fact that u is locally Lipschitz, see [5]. □

Notation. $|\cdot|$ denotes the Lebesgue measure.

Theorem 3.2.2. *Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a continuous function. The set*

$$\{p \in \mathbb{R}^n : \exists x, y \in \Omega, x \neq y \text{ and } p \in \partial u(x) \cap \partial u(y)\}$$

has Lebesgue measure zero.

Proof. Case 1: Ω is bounded

Assume u is bounded in Ω. Let $p_0 \in \partial u(x) \cap \partial u(y)$ for $x \neq y$ in Ω. By Proposition 3.1.6, this implies that $x, y \in \partial u^*(p_0)$. But $x \neq y$, thus $\partial u^*(p_0)$ is not a singleton. Hence, by Theorem 2.2.6, u^* is not differentiable at p_0. Therefore, we get $\{p \in \mathbb{R}^n : \exists x, y \in \Omega, x \neq y \text{ and } p \in \partial u(x) \cap \partial u(y)\} \subseteq \{p \in \mathbb{R}^n : u^* \text{ is not differentiable at } p\}$. Now, Proposition 3.1.2 gives that u^* is convex. So using Lemma 3.2.1, u^* is differentiable a.e.. By monotonicity of the Lebesgue measure, we reach that

$$|\{p \in \mathbb{R}^n : \exists x, y \in \Omega, x \neq y \text{ and } p \in \partial u(x) \cap \partial u(y)\}| = 0.$$

Case 2: Ω is unbounded

As Ω is open, we can write $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ with $(\Omega_k)_k$ is an increasing sequence of open sets such that $\overline{\Omega_k}$ are compact. By continuity of u in Ω and Case 1, we obtain that for each $k \in \mathbb{N}$ the set

$$N_k = \{p \in \mathbb{R}^n : \exists x, y \in \Omega_k, x \neq y \textit{ and } p \in \partial(u|_{\Omega_k})(x) \cap \partial(u|_{\Omega_k})(y)\}$$

has lebesgue measure zero. Now, we show that $N = \{p \in \mathbb{R}^n : \exists x, y \in \Omega, x \neq y \textit{ and } p \in \partial u(x) \cap \partial u(y)\} \subseteq \bigcup_{k=1}^{\infty} N_k$ and then by countable subadditivity of Lebesgue measure we conclude that N is Lebesgue null set. Let $p \in N$, then $\exists x, y \in \Omega$, $x \neq y$, and $p \in \partial u(x) \cap u(y)$. Since Ω_k are increasing subsets of Ω, so $\exists k \in \mathbb{N}$ such that $x, y \in \Omega_k$ and $u(z) \geq u(x) + p \cdot (z - x)$ $\forall z \in \Omega_k$ and $u(z) \geq u(y) + p \cdot (z - y)$ $\forall z \in \Omega_k$ obtaining $p \in N_k$. ☐

Theorem 3.2.3. *Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ a continuous function. The class*

$$\mathcal{A} = \{E \subseteq \Omega : \partial u(E) \textit{ is Lebesgue measurable}\}$$

is the Borel $\sigma-$algebra in Ω.

Proof. We prove that $\mathcal{A}$ satisfies the three properties of a $\sigma-$algebra.

First, we let $(E_n)_n \subseteq \mathcal{A}$ and claim that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$. We write

$$\partial u\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} \partial u(E_n)$$

which follows immediately from Definition 2.1.4. Moreover, we have $\bigcup_{n=1}^{\infty} E_n \subseteq \Omega$ and $\partial u(E_n)$ is Lebesgue measurable $\forall n \in \mathbb{N}$. This implies that $\bigcup_{n=1}^{\infty} \partial u(E_n)$ is Lebesgue measurable which proves our claim.

Secondly, we show that $\Omega \in \mathcal{A}$. Given Ω open, it can be written as a countable union of compact subsets (write $\Omega = \bigcup_{n=1}^{\infty}(\{x \in \Omega : dist(x, \partial\Omega) \geq \frac{1}{n}\} \cap \overline{B(0, n)})$). By Theorem 2.2.2, we have $\partial u(K)$ is Lebesgue measurable for K any compact subset of Ω obtaining $K \in \mathcal{A}$. Then Ω is a countable union of elements in $\mathcal{A}$ and then $\Omega \in \mathcal{A}$.

Thirdly, we let $E \in \mathcal{A}$ and claim that $\Omega \setminus E \in \mathcal{A}$. Let's write

$$\partial u(\Omega \setminus E) = (\partial u(\Omega \setminus E) \cap \partial u(E)) \cup (\partial u(\Omega \setminus E) \setminus \partial u(E)).$$

However, $p \in (\partial u(\Omega \setminus E) \setminus \partial u(E))$ if and only if $\exists x \in \Omega \setminus E$ such that $p \in \partial u(x)$ and $p \notin \partial u(E)$ which is also equivalent to $p \in (\partial u(\Omega) \setminus \partial u(E))$. Thus

$$\partial u(\Omega \setminus E) = (\partial u(\Omega \setminus E) \cap \partial u(E)) \cup (\partial u(\Omega) \setminus \partial u(E)).$$

Moreover, Theorem 3.2.2 implies that $|\partial u(\Omega \setminus E) \cap \partial u(E)| = 0$ obtaining that $\partial u(\Omega \setminus E) \cap \partial u(E)$ is Lebesgue null set and hence $\partial u(\Omega \setminus E) \cap \partial u(E)$ is Lebesgue measurable. On the other hand, we have $\Omega, E \in \mathcal{A}$, so $\partial u(\Omega) \setminus \partial u(E)$ is also Lebesgue measurable. Therefore, we get $\partial u(\Omega \setminus E)$ is Lebesgue measurable and thus $\Omega \setminus E \in \mathcal{A}$.

Now, it remains to show that $\mathcal{A}$ is Borel. As $\mathcal{A}$ is a $\sigma - algebra$, it is sufficient to show that it contains all open subsets of Ω. Let O be any open subset of Ω. Similarly, as we proved in the second step, O can be written as a countable union of compact subsets and thus $O \in \mathcal{A}$. ☐

Proposition 3.2.4. *Let Ω be an open, bounded, and convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a bounded function. If u is convex in Ω, then for each Borel set $F \subseteq \mathbb{R}^n$, the set*

$$(\partial u)^{-1}(F) = \{x \in \Omega : \partial u(x) \cap F \neq \emptyset\}$$

is Lebesgue measurable.

Proof. Let F be any Borel subset of $\mathbb{R}^n$.

We first show that

$$(\partial u)^{-1}(F) = \partial u^*(F).$$

Let $x \in (\partial u)^{-1}(F)$. Then $\partial u(x) \cap F \neq \emptyset$ i.e. $\exists p \in \mathbb{R}^n$ such that $p \in \partial u(x) \cap F$. By Proposition 3.1.6, we get $x \in \partial u^*(p)$ with $p \in F$ which implies that $x \in \partial u^*(F)$. Now, let $x \in \partial u^*(F)$. So $\exists p \in F$ such that $x \in \partial u^*(p)$. As u is convex, Proposition 3.1.11 implies that $p \in \partial u(x)$ obtaining $\partial u(x) \cap F \neq \emptyset$ and $x \in (\partial u)^{-1}(F)$.

Now, u^* is convex by Proposition 3.1.2, and thus continuous by Theorem 1.3.5. Then for $u^* \in C(\mathbb{R}^n)$, Theorem 3.2.3 gives that the set

$$\mathcal{A}' = \{F \subseteq \mathbb{R}^n : \partial u^*(F) \text{ is Lebesgue measurable}\}$$

is Borel $\sigma-algebra$ in $\mathbb{R}^n$. But F is a Borel subset of $\mathbb{R}^n$, then $F \in \mathcal{A}'$. Hence $\partial u^*(F)$ is Lebesgue measurable which means $(\partial u)^{-1}(F)$ is Lebesgue measurable. □

Theorem 3.2.5. *Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ a continuous function. Consider the Borel $\sigma-algebra$ $\mathcal{A} = \{E \subseteq \Omega : \partial u(E)$ is Lebesgue measurable$\}$. Then the set function*

$$Mu : \mathcal{A} \to \overline{\mathbb{R}}$$

defined by

$$Mu(E) = |\partial u(E)|$$

is a measure, finite on compacts, that is called the Monge-Ampère measure associated with the function u.

Proof. We show that Mu satisfies the two properties of a measure.
i) $Mu(\emptyset) = |\partial u(\emptyset)| = |\emptyset| = 0$.
ii) Let $(E_n)_n$ be a sequence of disjoint sets in $\mathcal{A}$. We claim that

$$Mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n=1}^{\infty} Mu(E_n)$$

that is

$$\left|\partial u\left(\bigcup_{n\in\mathbb{N}} E_n\right)\right| = \sum_{n=1}^{\infty} |\partial u(E_n)|.$$

From Definition 2.1.4, we have $\partial u\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \bigcup_{n\in\mathbb{N}} \partial u(E_n)$. We then let $F_n = \partial u(E_n)$ for each $n \in \mathbb{N}$ and write

$$\bigcup_{n\in\mathbb{N}} F_n = F_1 \cup (F_2 \setminus F_1) \cup (F_3 \setminus F_2 \cup F_1) \cup (F_4 \setminus F_3 \cup F_2 \cup F_1) \cup .....$$

By the $\sigma-$additivity of Lebesgue measure, we get

$$\left|\bigcup_{n\in\mathbb{N}} F_n\right| = \sum_{n=1}^{\infty} |F_n \setminus F_{n-1} \cup F_{n-2} \cup ... \cup F_1|.$$

On the other hand, we write

$$F_n = (F_n \cap (F_{n-1} \cup F_{n-2} \cup ... \cup F_1)) \cup (F_n \setminus (F_{n-1} \cup F_{n-2} \cup ... \cup F_1))$$

Again, by $\sigma-$additivity of lebesgue measure, we have

$$|F_n| = |F_n \cap (F_{n-1} \cup F_{n-2} \cup ... \cup F_1)| + |F_n \setminus (F_{n-1} \cup F_{n-2} \cup ... \cup F_1)|.$$

Since $E_i \cap E_j = \emptyset$ for $i \neq j$, Theorem 3.2.2 implies that $|F_i \cap F_j| = |\partial u(E_i) \cap \partial u(E_j)| = 0$. Therefore $|F_n \cap (F_{n-1} \cup F_{n-2} \cup ... \cup F_1)| = 0$. Thus, we end with

$$|F_n| = |F_n \setminus (F_{n-1} \cup F_{n-2} \cup ... \cup F_1)|.$$

We conclude that

$$|\bigcup_{n\in\mathbb{N}} \partial u(E_n)| = |\bigcup_{n\in\mathbb{N}} F_n| = \sum_{n=1}^{\infty} |F_n \setminus (F_{n-1} \cup F_{n-2} \cup ... \cup F_1)| = \sum_{n=1}^{\infty} |F_n| = \sum_{n=1}^{\infty} |\partial u(E_n)|$$

which ends our claim. Hence Mu satisfies i) and ii) which implies that Mu is a measure.

Now, it remains to show that Mu is finite on compacts. Let K be any compact subset of Ω. We have by Theorem 2.2.2 that $\partial u(K)$ is compact subset of $\mathbb{R}^n$, then $K \in \mathcal{A}$ and $Mu(K) = |\partial u(K)| < \infty$. □

Theorem 3.2.6. *(Sard's Lemma) [7]*
Let Ω be an open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ a C^1 function in Ω. If $S_0 = \{x \in \Omega : \det f'(x) = 0\}$ then $|f(S_0)| = 0$.

Theorem 3.2.7. *Let Ω be an open convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ a C^2 convex function in Ω. The Monge-Ampère measure associated with u satisfies*

$$Mu(E) = \int_E \det D^2u(x)dx$$

for every Borel set $E \subseteq \Omega$.

Proof. Since $u \in C^2(\Omega)$ convex, then by Theorem 1.3.7, we have $D^2u(x) \geq 0$ for every $x \in \Omega$. Define the set

$$A = \{x \in \Omega : D^2u(x) > 0\}.$$

First, we claim that Du is injective on A. In fact, Let $x_1, x_2 \in A$ such that $Du(x_1) = Du(x_2)$. Since u is convex and differentiable in Ω, Theorem 1.3.6 gives that

$$u(y) \geq u(x) + Du(x) \cdot (y - x) \quad \forall x, y \in \Omega.$$

Hence, applying the inequality for $x = x_1$ and $y = x_2$ then for $x = x_2$ and $y = x_1$, we get $u(x_2) \geq u(x_1) + Du(x_1) \cdot (x_2 - x_1)$ and $u(x_1) \geq u(x_2) + Du(x_2) \cdot (x_1 - x_2)$. But $Du(x_1) = Du(x_2)$, then we obtain that for every $x_1, x_2 \in A$

$$u(x_1) - u(x_2) = Du(x_1) \cdot (x_1 - x_2) = Du(x_2) \cdot (x_1 - x_2). \tag{3.1}$$

Now, define

$$g(\lambda) = u((1-\lambda)x_2 + \lambda x_1), \quad \lambda \in [0, 1].$$

As $u \in C^2(\Omega)$, then $g \in C^2([0, 1])$. Notice that, from the chain rule

$$\begin{aligned} g'(\lambda) &= Du((1-\lambda)x_2 + \lambda x_1) \cdot (x_1 - x_2) \\ g''(\lambda) &= \langle D^2((1-\lambda)x_2 + \lambda x_1)(x_2 - x_1), (x_2 - x_1)\rangle \end{aligned}$$

Recalling Taylor's Formula,

$$g(1) = g(0) + g'(0) + \int_0^1 g''(t)(1-t)dt,$$

hence plugging the values of g, we get

$$\begin{aligned} u(x_1) = u(x_2) &+ Du(x_2) \cdot (x_1 - x_2) \\ &+ \int_0^1 (1-t)\langle D^2u(x_2 + t(x_1 - x_2))(x_2 - x_1), (x_2 - x_1)\rangle dt. \end{aligned}$$

Comparing (3.1) with the above equality, we get

$$\int_0^1 (1-t)\langle D^2u(x_2 + t(x_1 - x_2))(x_2 - x_1), (x_2 - x_1)\rangle dt = 0. \tag{3.2}$$

Moreover, since u is convex, from Theorem 1.3.7

$$\langle D^2u(x_2 + t(x_1 - x_2))(x_2 - x_1), (x_2 - x_1)\rangle \geq 0$$

Also, as $u \in C^2(\Omega)$, we obtain that the integrand is continuous and non-negative in (3.2) . Thus $\langle D^2u(x_2)(x_1 - x_2), (x_1 - x_2)\rangle = 0$. But since $x_2 \in A$ then $D^2u(x_2) > 0$ concluding that $x_1 = x_2$. Therefore Du is injective on A which ends our claim.

We next apply Sard's Lemma, Theorem 3.2.6, with $f = Du$ and

$$S_0 = \{x \in A : \det D^2u(x) = 0\} = \Omega \setminus A,$$

and get $|Du(\Omega \setminus A)| = 0$. Now, let $E \subseteq \Omega$ be any Borel set. We have

$$Du(E) = Du(E \cap A) \cup Du(E \setminus A).$$

As S_0 is closed, $E \cap A$ and $E \setminus A$ are also Boret sets. We know that the Monge-Ampère measure associated to u is defined on Borel sets and it is $\sigma-$additive. Then we obtain

$$Mu(E) = Mu(E \cap A) + Mu(E \setminus A) = |Du(E \cap A)| + |Du(E \setminus A)| = |Du(E \cap A)|$$

Finally, we have Du is a diffeomorphism on the open set A, then by the change of variable formula [8], we get that

$$|Du(E \cap A)| = \int_{E \cap A} |\det D^2 u(x)| dx.$$

Therefore, since $\det D^2 u = 0$ on $\Omega \setminus A$ that has measure zero, we obtain

$$Mu(E) = \int_{E \cap A} \det D^2 u(x) dx = \int_E \det D^2 u(x) dx.$$

This ends our proof. $\square$

Proposition 3.2.8. *Let Ω be an open convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ a continuous function. For $c > 0$, we have $M(cu) = c^n Mu$.*

Proof. We show that $\partial(cu)(E) = c\partial u(E)$ for $E \subseteq \Omega$ any Borel subset.

$$\begin{aligned} p \in \partial(cu)(E) &\iff \exists\, x_0 \in E \ \text{ such that } \ p \in \partial(cu)(x_0) \\ &\iff cu(x) \geq cu(x_0) + p \cdot (x - x_0) \quad \forall x \in \Omega \\ &\iff q = \frac{p}{c} \in \partial u(x_0) \ \text{ with } \ x_0 \in E \\ &\iff p \in c\partial u(E). \end{aligned}$$

Therefore

$$M(cu)(E) = |\partial(cu)(E)| = |c\partial u(E)| = c^n |\partial u(E)| = c^n Mu(E).$$

$\square$

Example 3.2.9. *We complete this section with an example where we recall the following result in linear algebra: If A is an invertible $n \times n$ matrix, and x, y are two n-dimensional column vectors, then*

$$\det(A + xy^T) = (1 + y^T A^{-1} x) \det A.$$

[9]

We have u is the cone of Example 2.3.2. We want to calculate the Monge-Ampère measure associated with u of any Borel set $E \subset \Omega$.

If $x_0 \notin E$, u is twice differentiable on E and thus by Theorem 3.2.7, we have

$$Mu(E) = \int_E \det D^2 u(x) dx.$$

In Example 2.3.2, we calculate $\frac{\partial u}{\partial x_i}(x) = \frac{h}{r} \frac{x_i - x_{0,i}}{|x - x_0|}$. Hence

$$\frac{\partial^2 u}{\partial x_j \partial x_i}(x) = \frac{h}{r} \left(\frac{\delta_{ij}}{|x - x_0|} - \frac{(x_i - x_{0,i})(x_j - x_{0,j})}{|x - x_0|^3} \right) = \frac{h}{r|x - x_0|} \left(\delta_{ij} - \frac{(x_i - x_{0,i})(x_j - x_{0,j})}{|x - x_0|^2} \right)$$

where $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$. *Hence*

$$D^2u(x) = \frac{h}{r|x-x_0|}\left(I_n - \frac{(x-x_0)(x-x_0)^T}{|x-x_0|}\right)$$

where I_n *is the identity matrix. Using that fact recalled at the begining, we get*

$$\det D^2u(x) = \left(\frac{h}{r|x-x_0|}\right)^n \left(1 - \frac{(x-x_0)\cdot(x-x_0)}{|x-x_0|^2}\right) = 0.$$

Therefore, $Mu(E) = 0$.

If $x_0 \in E$, *we have*

$$Mu(E) = Mu(E \cap \{x_0\}) + Mu(E \setminus \{x_0\}) = Mu(\{x_0\}) = |\partial u(x_0)| = |B(0, h/r)|$$

where the latter equality follows from Example 2.3.2.

We conclude that

$$Mu = |B(0, h/r)|\, \delta_{x_0}$$

where δ_{x_0} *is the Dirac measure.*

3.3 Weak Convergence of Monge-Ampère Measure

Lemma 3.3.1. *Let* Ω *be an open convex subset of* $\mathbb{R}^n$. *Let* $(u_k)_k$ *be a sequence of real-valued convex functions in* Ω *such that*

$$u_k \xrightarrow[k\to\infty]{} u \quad \textit{uniformly on compact subsets of } \Omega.$$

(i) If $K \subset \Omega$ *is compact, then*

$$\limsup_{k\to\infty} \partial u_k(K) \subseteq \partial u(K),$$

and by Fatou's Lemma

$$\limsup_{k\to\infty} |\partial u_k(K)| \leq |\partial u(K)|.$$

(ii) If $U \subset \Omega$ *is open such that* $\bar{U} \subset \Omega$, *then*

$$\partial u(U) \subseteq \liminf_{k\to\infty} \partial u_k(U)$$

for almost every point in $\partial u(U)$, *and by Fatou's Lemma*

$$|\partial u(U)| \leq \liminf_{k\to\infty} |\partial u_k(U)|.$$

Proof. (i) Let $K \subset \Omega$ be a compact set. We first claim that

$$\limsup_{k\to\infty} \partial u_k(K) \subseteq \partial u(K). \tag{3.3}$$

Let

$$p \in \limsup_{k\to\infty} \partial u_k(K) = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \partial u_n(K)$$

Then $\forall\, k \in \mathbb{N},\ \exists\, n_k \geq k$, such that $p \in \partial u_{n_k}(K) = \bigcup_{x\in K} \partial u_{n_k}(x)$, that is $\forall\, k \in \mathbb{N},\ \exists\, n_k \geq k$ and $x_{n_k} \in K$ such that $p \in \partial u_{n_k}(x_{n_k})$. We obtain the sequence $(x_{n_k})_k \subseteq K$. Hence, by sequentially compactness of K, $(x_{n_k})_k$ has a convergent subsequence to a point in K, say without relabeling that

$$x_{n_k} \xrightarrow[k\to\infty]{} x_0$$

with $x_0 \in K$ and $p \in \partial u_{n_k}(x_{n_k})\ \ \forall k \in \mathbb{N}$. Thus

$$u_{n_k}(x) \geq u_{n_k}(x_{n_k}) + p \cdot (x - x_{n_k}) \quad \forall x \in \Omega \quad \forall k \in \mathbb{N}.$$

By uniform convergence of u_{n_k}, we get $u(x) \geq u(x_0) + p \cdot (x - x_0)\ \forall x \in \Omega$ obtaining $p \in \partial u(x_0)$ with $x_0 \in K$. Therefore $p \in \partial u(K)$ which proves our claim.

Now, we show that

$$\limsup_{k\to\infty} |\partial u_k(K)| \leq |\partial u(K)|.$$

First, we claim that

$$\partial u_k(K) \subseteq B(0, C(K)) \quad \forall k \in \mathbb{N}$$

where $C(K)$ is a constant depending only on K. Let $k \in \mathbb{N}$ and $p \in \partial u_k(K)$. Then $\exists x_0 \in K$ such that $p \in \partial u_k(x_0)$. Thus

$$u_k(x) \geq u_k(x_0) + p \cdot (x - x_0) \quad \forall x \in \Omega.$$

As K is compact and Ω open, we can always find V open such that $K \subset V \subset \bar{V} \subset \Omega$. Let $0 < r < dist(x_0, \partial V)$ and take in particular $x_p = x_0 + r\frac{p}{|p|}$. So $|x_p - x_0| = r < dist(x_0, \partial V)$ which implies $x_p \in V \subseteq \Omega$ and $u_k(x_p) \geq u_k(x_0) + r|p|$. We obtain

$$|p| \leq \frac{|u_k(x_p)| + |u_k(x_0)|}{r}.$$

Knowing that u_k is uniformly convergent sequence on $\bar{V}$ with u_k bounded in $\bar{V}$ since u_k are continuous by Theorem 1.3.5, then u_k are uniformly bounded in $\bar{V}$. Hence $\exists C_1(K)$ a constant dependeing only on K such that

$$|p| \leq \frac{2C_1(K)}{r}.$$

Letting $r \to dist(x_0, \partial V)$, we get

$$|p| \leq \frac{2C_1(K)}{dist(x_0, \partial V)} = C(K)$$

and then $p \in B(0, C(K))$. We then have

$$\limsup_{k\to\infty} |\partial u_k(K)| = \limsup_{k\to\infty} \int_{\partial u_k(K)} dx = \limsup_{k\to\infty} \int_{B(0,C(K))} \chi_{\partial u_k(K)} dx.$$

However $\partial u_k(K)$ are Lebesgue measurable sets by Theorem 2.2.2 and thus $\chi_{\partial u_k(K)}$ are Lebesgue measurable functions. Besides, $\chi_{\partial u_k(K)} \leq 1$ with $\int_{B(0,C(K))} 1dx = |B(0, C(K))| < \infty$. Therefore, by Fatou's Lemma

$$\limsup_{k\to\infty} |\partial u_k(K)| \leq \int_{B(0,C(K))} \limsup_{k\to\infty} \chi_{\partial u_k(K)} dx \leq \int_{\mathbb{R}^n} \limsup_{k\to\infty} \chi_{\partial u_k(K)} dx.$$

Moreover, using the fact that

$$\limsup_{k\to\infty} \chi_{\partial u_k(K)} = \chi_{\limsup_{k\to\infty} \partial u_k(K)}$$

and (3.3), we conclude

$$\limsup_{k\to\infty} |\partial u_k(K)| \leq \int_{\mathbb{R}^n} \chi_{\partial u(K)} dx = |\partial u(K)|.$$

(ii) Let $U \subset \Omega$ be an open set such that $\bar{U} \subset \Omega$. Let

$$S = \{p \in \mathbb{R}^n : \exists x, y \in \Omega, x \neq y \ \ and \ \ p \in \partial u(x) \cap \partial u(y)\},$$

hence S is a Lebesgue null set by Theorem 3.2.2. We claim that

$$\partial u(U) \setminus S \subseteq \liminf_{k\to\infty} \partial u_k(U).$$

Let $p \in \partial u(U) \setminus S$, then there exists a unique $x_0 \in U$ such that $p \in \partial u(x_0)$ and $p \notin \partial u(x_1)$ $\forall x_1 \in \Omega$, $x_1 \neq x_0$. Let $x_1 \in \Omega$ with $x_1 \neq x_0$. We have $u(x) \geq u(x_0) + p \cdot (x - x_0)$ $\forall x \in \Omega$. In particular, $u(x_1) \geq u(x_0) + p \cdot (x_1 - x_0)$. Suppose $u(x_1) = u(x_0) + p \cdot (x_1 - x_0)$, then $u(x) \geq u(x_1) - p \cdot (x_1 - x_0) + p \cdot (x - x_0)$ $\forall x \in \Omega$ obtaining $u(x) \geq u(x_1) + p \cdot (x - x_1)$ $\forall x \in \Omega$. This implies that $p \in \partial u(x_1)$ which is a contradiction. Therefore

$$u(x_1) > u(x_0) + p \cdot (x_1 - x_0) \quad \forall x_1 \in \Omega, \quad x_1 \neq x_0. \tag{3.4}$$

Case 1: $\bar{U}$ is bounded

Consider the line

$$\ell(x) = u(x_0) + p \cdot (x - x_0)$$

and

$$\delta = \min\{u(x) - \ell(x) : x \in \partial U\}.$$

δ is the minimum of a continuous function over a compact set, thus δ is attained in ∂U. However, as $x_0 \in U$, (3.4) gives that $u(x) - \ell(x) > 0$ $\forall x \in \partial U$ and hence $\delta > 0$.

By uniform convergence of u_k on compact subsets of Ω, $\exists\, k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$, $\forall x \in \bar{U}$, $|u_k(x) - u(x)| < \frac{\delta}{2}$. Now, for $k \geq k_0$, let

$$\delta_k = \max\{\ell(x) - u_k(x) + \frac{\delta}{2} : x \in \bar{U}\}.$$

δ_k is the maximum of a continuous function over a compact set, so δ_k is attained in $\bar{U}$. Thus $\exists\, x_k \in \bar{U}$ such that $\delta_k = \ell(x_k) - u_k(x_k) + \frac{\delta}{2}$. With $x_0 \in U$,

$$\delta_k \geq \ell(x_0) - u_k(x_0) + \frac{\delta}{2} = u(x_0) - u_k(x_0) + \frac{\delta}{2} > -\frac{\delta}{2} + \frac{\delta}{2} = 0.$$

Suppose now $x_k \in \partial U$. We get $\delta \leq u(x_k) - \ell(x_k)$ that gives

$$\delta_k = \ell(x_k) - u(x_k) + u(x_k) - u_k(x_k) + \frac{\delta}{2} < -\delta + \frac{\delta}{2} + \frac{\delta}{2} = 0$$

which is a contradiction. Thus $x_k \in U$. Next, we show that $p \in \partial u_k(x_k)$. We have $\delta_k = u(x_0)+p\cdot(x_k-x_0)-u_k(x_k)+\frac{\delta}{2}$ and by definition of δ_k, $u_k(x) \geq u_k(x_k)+p\cdot(x-x_k)$ $\forall x \in \bar{U}$. Since u_k is convex in Ω and U is open, $u_k(x) \geq u_k(x_k) + p \cdot (x - x_k)$ $\forall x \in \Omega$ obtaining $p \in \partial u_k(x_k)$ with $x_k \in U$ which means $p \in \partial u_k(U)$. But this is true for all $k \geq k_0$, thus $p \in \bigcup_{k_0=1}^{\infty} \bigcap_{k=k_0}^{\infty} \partial u_k(U)$. Therefore

$$p \in \liminf_{k\to\infty} \partial u_k(U)$$

which completes the proof for this case.

<u>Case 2: $\bar{U}$ is unbounded</u>

Since U is open, we write $U = \bigcup_{j=1}^{\infty} U_j$ with U_j open and $\bar{U}_j$ compact. Then, from Case 1,

$$\partial u(U) = \bigcup_{j=1}^{\infty} \partial u(U_j) \subseteq \bigcup_{j=1}^{\infty} \liminf_{k\to\infty} \partial u_k(U_j) \quad a.e.\ in\ \partial u(U).$$

However, $\partial u_k(U_j) \subseteq \partial u_k(U)$ $\forall j \in \mathbb{N}$ which implies

$$\liminf_{k\to\infty} \partial u_k(Uj) \subseteq \liminf_{k\to\infty} \partial u_k(U) \quad \forall j \in \mathbb{N}$$

obtaining

$$\bigcup_{j=1}^{\infty} \liminf_{k\to\infty} \partial u_k(U_j) \subseteq \liminf_{k\to\infty} \partial u_k(U).$$

Therefore, we get

$$\partial u(U) \subseteq \liminf_{k\to\infty} \partial u_k(U) \quad a.e.\ in\ \partial u(U).$$

Now, we continue to show that

$$|\partial u(U)| \leq |\liminf_{k\to\infty} |\partial u_k(U)|.$$

We have

$$|\partial u(U)| = \int_{\partial u(U)} d\lambda = \int_{\mathbb{R}^n} \chi_{\partial u(u)} d\lambda \leq \int_{\mathbb{R}^n} \chi_{\liminf\limits_{k\to\infty} \partial u_k(U)} d\lambda.$$

On the other side, we know that

$$\chi_{\liminf\limits_{k\to\infty} \partial u_k(U)} = \liminf_{k\to\infty} \chi_{\partial u_k(U)}.$$

So we get

$$|\partial u(U)| \leq \int_{\mathbb{R}^n} \liminf_{k\to\infty} \chi_{\partial u_k(U)} d\lambda.$$

However, as U is open, $\partial u_k(U)$ is Lebesgue measurable (see Theorem 3.2.3), and hence $\chi_{\partial u_k(U)}$ are non-negative measurable functions. Therefore, by Fatou's Lemma, we get

$$|\partial u(U)| \leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} \chi_{\partial u_k(U)} d\lambda = \liminf_{k\to\infty} \int_{\partial u_k(U)} d\lambda = \liminf_{k\to\infty} |\partial u_k(U)|$$

which completes our proof. □

In the following lemma, we used some techniques from [6].

Lemma 3.3.2. *Let Ω be an open subset of $\mathbb{R}^n$. Let $(\mu_k)_k$ and μ be Borel measures in Ω that are finite on compact sets. Suppose that*

(a) $\limsup\limits_{k\to\infty} \mu_k(F) \leq \mu(F)$ *for each $F \subset \Omega$ compact, and*

(b) $\liminf\limits_{k\to\infty} \mu_k(G) \geq \mu(G)$ *for each $G \subset \Omega$ open.*

Then

$$\mu_k \to \mu \quad \textit{weakly},$$

that is,

$$\int_\Omega f(x) d\mu_k \to \int_\Omega f(x) d\mu$$

for all f continuous with compact support in Ω (or for all f continuous and bounded in Ω if $\mu_k(\Omega)$ and $\mu(\Omega)$ are finite).

Proof. Initially, we work on case $f \geq 0$ with f continuous on compact support in Ω. We first claim that

$$\liminf_{k\to\infty} \int_\Omega f d\mu_k \geq \int_\Omega f d\mu.$$

Notice that we can write

$$f(x) = \int_0^\infty \chi_{\{f>t\}}(x) dt.$$

In fact, for $x \in \Omega$, $\exists\, r \geq 0$ such that $f(x) = r$. This implies that

$$\chi_{\{f>t\}}(x) = \begin{cases} 1 & 0 \leq t < r \\ 0 & t \geq r \end{cases}$$

and thus

$$\int_0^\infty \chi_{\{f>t\}}(x)dt = \int_0^r dt = r = f(x). \tag{3.5}$$

Moreover, since Ω can be written as a countable union of increasing sequence of compact subsets with μ_k in Ω finite on compact subsets, we obtain that μ_k is $\sigma - finite$ in Ω. We then apply Tonelli's Theorem for non-negative measurable functions in $\Omega \times (0, \infty)$ and get that

$$\int_\Omega f d\mu_k = \int_\Omega \left(\int_0^\infty \chi_{\{f>t\}} dt \right) d\mu_k = \int_0^\infty \left(\int_\Omega \chi_{\{f>t\}} d\mu_k \right) dt = \int_0^\infty \mu_k(\{f > t\})dt.$$

Note that $t \to \mu_k(\{f > t\})$ is decreasing and thus Lebesgue measurable. Similarly, we have

$$\int_\Omega f d\mu = \int_0^\infty \mu(\{f > t\})dt.$$

Now,

$$\liminf_{k\to\infty} \int_\Omega f d\mu_k = \liminf_{k\to\infty} \int_0^\infty \mu_k(\{f > t\})dt \geq \int_0^\infty \liminf_{k\to\infty} \mu_k(\{f > t\})dt$$

using Fatou's Lemma as $\mu_k(\{f > t\})$ are non-negative measurable functions. On the other side, f is continuous, then $\{f > t\}$ is open in Ω, and by the given property (b), we get

$$\liminf_{k\to\infty} \mu_k(\{f > t\}) \geq \mu(\{f > t\})$$

and so

$$\liminf_{k\to\infty} \int_\Omega f d\mu_k \geq \int_0^\infty \mu(\{f > t\})dt = \int_\Omega f d\mu.$$

To end this case, it remains to show that

$$\limsup_{k\to\infty} \int_\Omega f d\mu_k \leq \int_\Omega f d\mu.$$

Since f is non-negative continuous function with compact support, then there exists $B > 0$ such that $0 \leq f \leq B$, and similar to (3.5) we can write

$$f(x) = \int_0^B \chi_{\{f\geq t\}}(x)dt.$$

Again by Tonelli's Theorem, we have for each $k \in \mathbb{N}$

$$\int_\Omega f d\mu_k = \int_{suppf} f d\mu_k = \int_0^B \left(\int_{suppf} \chi_{\{f\geq t\}} d\mu_k \right) dt = \int_0^B \mu_k(suppf \cap \{f \geq t\})dt$$

and similarly

$$\int_\Omega f d\mu = \int_0^B \mu(suppf \cap \{f \geq t\})dt. \tag{3.6}$$

Now,

$$\limsup_{k\to\infty} \int_\Omega f d\mu_k = \limsup_{k\to\infty} \int_{suppf} f d\mu_k = \limsup_{k\to\infty} \int_0^B \mu_k(suppf \cap \{f \geq t\})dt.$$

By the given property (a), we know that

$$\limsup_{k\to\infty} \mu_k(suppf) \leq \mu(suppf) < \infty$$

since *suppf* is compact in Ω and μ is finite on compact subsets of Ω. Hence, the sequence is bounded above, and we have

$$\mu_k(suppf \cap \{f \geq t\}) \leq \mu_k(suppf) \leq M \quad \forall k \in \mathbb{N}$$

for some $M \geq 0$ with $\int_0^B M dt = MB < \infty$. Thus, we can apply Fatou's Lemma to obtain

$$\limsup_{k\to\infty} \int_\Omega f d\mu_k \leq \int_0^B \limsup_{k\to\infty} \mu_k(suppf \cap \{f \geq t\})dt.$$

But again using property (a), as $(suppf \cap \{f \geq t\})$ is compact, we have

$$\limsup_{k\to\infty} \mu_k(suppf \cap \{f \geq t\}) \leq \mu(suppf \cap \{f \geq t\})$$

obtaining

$$\limsup_{k\to\infty} \int_\Omega f d\mu_k \leq \int_0^B \mu(suppf \cap \{f \geq t\})dt = \int_\Omega f d\mu.$$

More generally, for f any continuous function with compact support in Ω. We can write $f = f^+ - f^-$ with $f^+, f^- \geq 0$ and for each $k \in \mathbb{N}$,

$$\int_\Omega f d\mu_k = \int_\Omega f^+ d\mu_k - \int_\Omega f^- d\mu_k$$

and

$$\int_\Omega f d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu.$$

Since f^+ and f^- are bounded functions with compact support, then we can apply Case 1 on f^+ and f^- and get that

$$\lim_{k\to\infty} \int_\Omega f d\mu_k = \lim_{k\to\infty} \int_\Omega f^+ d\mu_k - \lim_{k\to\infty} \int_\Omega f^- d\mu_k = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu = \int_\Omega f d\mu.$$

$\square$

Theorem 3.3.3. *Let Ω be an open convex subset of $\mathbb{R}^n$. Let $(u_k)_k$ be a sequence of real-valued convex functions in Ω such that*

$$u_k \xrightarrow[k\to\infty]{} u \quad \textit{uniformly on compact subsets of } \Omega.$$

Then, the associated Monge-Ampère measure Mu_k converge to Mu weakly, that is,

$$Mu_k \xrightarrow[k\to\infty]{} Mu \quad \textit{weakly}.$$

Proof. This directly follows from Lemmas 3.3.1 and 3.3.2. □

3.4 Aleksandrov's Maximum Principle

Lemma 3.4.1. *Let Ω be an open bounded subset of $\mathbb{R}^n$. Let u and v be real-valued convex functions such that $u, v \in C(\bar{\Omega})$. If $u = v$ on $\partial\Omega$ and $v \geq u$ in Ω then $\partial v(\Omega) \subseteq \partial u(\Omega)$.*

Proof. Let $p \in \partial v(\Omega)$ and $x_0 \in \Omega$ such that $p \in \partial v(x_0)$. Thus

$$v(x) \geq v(x_0) + p \cdot (x - x_0) \quad \forall x \in \Omega. \tag{3.7}$$

Let

$$A = \sup_{x\in\Omega}\{v(x_0) + p \cdot (x - x_0) - u(x)\}.$$

We have $A \geq v(x_0) - u(x_0) \geq 0$ since $v \geq u$ in Ω. Also, A is attained in $\bar{\Omega}$ as it is the supremum of a continuous function over a compact set. Hence

$$A = v(x_0) + p \cdot (x_1 - x_0) - u(x_1) \quad \text{for some } x_1 \in \bar{\Omega}.$$

Now, we show that $v(x_0) + p \cdot (x - x_0) - A$ is a supporting hyperplane to u at some point in Ω. From the definition of A, we have for every $x \in \Omega$

$$u(x) \geq v(x_0) + p \cdot (x - x_0) - A = u(x_1) + p \cdot (x - x_1).$$

If $x_1 \in \Omega$, then $p \in \partial u(x_1) \subseteq \partial u(\Omega)$, and we are done. Otherwise, if $x_1 \in \partial\Omega$, since $u = v$ on $\partial\Omega$ then from (3.7)

$$A = v(x_0) + p \cdot (x_1 - x_0) - u(x_1) = v(x_0) + p \cdot (x_1 - x_0) - v(x_1) \leq 0.$$

Thus $A = 0$ and for every $x \in \Omega$

$$u(x) \geq v(x_0) + p \cdot (x - x_0) \geq u(x_0) + p \cdot (x - x_0)$$

obtaining that $p \in \partial u(x_0) \subseteq \partial u(\Omega)$. □

Theorem 3.4.2. *(Aleksandrov's Maximum Principle)*
Let Ω be an open, bounded, and convex subset of $\mathbb{R}^n$ with $diam(\Omega) = \Delta$. Let u be a real-valued convex function such that $u \in C(\bar{\Omega})$. If $u = 0$ on $\partial\Omega$, then

$$(-u(x))^n \leq C_n \Delta^{n-1} dist(x, \partial\Omega)\, |\partial u(\Omega)| \quad \forall x \in \Omega,$$

with C_n is a constant depending only on the dimension n.

Proof. Let $x_0 \in \Omega$ and let v be the convex function whose graph is the upside-down cone of vertex $(x_0, u(x_0))$ and base Ω with $v = 0$ on $\partial\Omega$.

Step 1. We show that $\partial v(\Omega) \subseteq \partial u(\Omega)$. In fact, we have $u = v = 0$ on $\partial\Omega$, and notice that also from the construction of v and the convexity of u that $u \leq v$ in Ω. Indeed, let $x \in \Omega$ and $(x, v(x)) \in G(v)$ (see Definition 1.3.2) with v is a cone. Then $\exists z \in \partial\Omega$ and $t \in [0, 1]$ such that

$$(x, v(x)) = (1-t)(x_0, v(x_0)) + t(z, v(z)) = (1-t)(x_0, u(x_0)) + t(z, 0).$$

However $(x_0, u(x_0)), (z, 0) \in epi(u)$ (see Definition 1.3.2) with $epi(u)$ is a convex set as u is a convex function by Proposition 1.3.3. Hence, $(x, v(x)) \in epi(u)$ and thus $v(x) \geq u(x)$. Now, applying Lemma 3.4.1, we conclude that $\partial v(\Omega) \subseteq \partial u(\Omega)$.

Step 2. Let's show that $\partial v(\Omega)$ is a convex subset of $\mathbb{R}^n$. In fact, we show that $\partial v(\Omega) = \partial v(x_0)$ which is convex by Proposition 2.2.1.

Let $p \in \partial v(\Omega)$, then $\exists x_1 \in \Omega$ such that $p \in \partial v(x_1)$, that is

$$v(x) \geq v(x_1) + p \cdot (x - x_1) \quad \forall x \in \Omega. \tag{3.8}$$

If $x_1 = x_0$, we are done. If $x_1 \neq x_0$, we claim that $v(x_1) + p \cdot (x - x_1)$ is a supporting hyperplane to v at x_0 since v is a cone. It is sufficient to show that $v(x_1) = v(x_0) + p \cdot (x_1 - x_0)$. We have from (3.8) $v(x_0) \geq v(x_1) + p \cdot (x_0 - x_1)$ and then it remains to prove the reverse inequality.

Since $(x_1, v(x_1)) \in G(v)$ with v is a cone, then $\exists z \in \partial\Omega$ and $\exists t \in [0, 1]$ such that

$$(x_1, v(x_1)) = (1-t)(x_0, v(x_0)) + t(z, v(z)) = (1-t)(x_0, u(x_0)) + t(z, 0). \tag{3.9}$$

From the continuity of v on $\bar{\Omega}$, (3.8) extends to $x \in \bar{\Omega}$ which implies that

$$(p, -1) \cdot ((x, v(x)) - (x_1, v(x_1))) \leq 0 \quad \forall x \in \bar{\Omega}.$$

In particular for $x = z$ and from (3.9):

$$(p, -1) \cdot ((z, 0) - (1-t)(x_0, u(x_0)) - t(z, 0)) \leq 0,$$

that is

$$(1-t)(p, -1) \cdot ((z, 0) - (x_0, u(x_0))) \leq 0,$$

concluding that

$$(p, -1) \cdot ((z, 0) - (x_0, u(x_0))) \leq 0. \tag{3.10}$$

On the other hand,

$$\begin{aligned} v(x_1) - v(x_0) + p \cdot (x_0 - x_1) &= (p, -1) \cdot ((x_0, v(x_0)) - (x_1, v(x_1))) \\ &= (p, -1) \cdot ((x_0, v(x_0)) - (1-t)(x_0, v(x_0)) - t(z, 0)) \\ &= t(p, -1) \cdot ((x_0, v(x_0)) - (z, 0)) \\ &\geq 0 \quad \text{(from (3.10))}. \end{aligned}$$

Therefore, $v(x_1) \geq v(x_0) + p \cdot (x_1 - x_0)$ and thus $v(x_1) = v(x_0) + p \cdot (x_1 - x_0)$. This implies that $v(x) \geq v(x_0) + p \cdot (x - x_0)$ $\forall x \in \Omega$ and $p \in \partial v(x_0)$ which completes this step.

Step 3. We prove that $\exists q_0 \in \partial v(\Omega)$ such that

$$|q_0| = \frac{-u(x_0)}{dist(x_0, \partial\Omega)}.$$

We know that $dist(x_0, \partial\Omega)$ is attained in $\partial\Omega$ as it is the infimum of a continuous function over a compact set, i.e $\exists x_1 \in \partial\Omega$ such that

$$|x_1 - x_0| = dist(x_0, \partial\Omega). \tag{3.11}$$

Corollary 1.1.13 implies the existence of the supporting hyperplane Π_{x_1} to Ω at x_1. We claim that the hyperplane $H \subseteq \mathbb{R}^{n+1}$ containing Π_{x_1} and passing through $(x_0, u(x_0))$ supports v at $(x_0, u(x_0))$. We start by proving that $x_1 - x_0$ is normal to Π_{x_1}. We write the equation of the line passing through x_1 in Π_{x_1} having a slope p:

$$L_p : x(t) = x_1 + tp \quad t \in \mathbb{R}.$$

Let $\gamma(t) = |x(t) - x_0|^2$, $t \in \mathbb{R}$. From (3.11), $\gamma(t)$ has minimum at $t = 0$, and so $\gamma'(0) = 0$. We have

$$\gamma'(t) = 2(x(t) - x_0) \cdot x'(t) = 2(x_1 + tp - x_0) \cdot p$$

then $\gamma'(0) = 2(x_1 - x_0) \cdot p = 0$. Therefore $(x_1 - x_0) \perp L_p$ for every line in Π_{x_1} passing through x_1 which implies that $(x_1 - x_0)$ is normal to Π_{x_1}. Now, let's move and find the normal to H, call it $\hat{q} = (q_1, q_2, ..., q_n, q_{n+1})$. By definition of H, we know that $\hat{q}$ must be orthogonal to $(x_1 - x_0, -u(x_0))$ and to L_p for every line in Π_{x_1} with slope p. Take $\hat{q} = (x_1 - x_0, q_{n+1})$. We have from above $\hat{q} \cdot (p, 0) = (x_1 - x_0) \cdot p = 0$. It remains to find q_{n+1} so that $\hat{q} \cdot (x_1 - x_0, -u(x_0)) = 0$, that is $|x_1 - x_0|^2 - q_{n+1}u(x_0) = 0$ which implies

$$q_{n+1} = \frac{|x_1 - x_0|^2}{u(x_0)}.$$

Therefore,

$$\hat{q} = \left(x_1 - x_0, \frac{|x_1 - x_0|^2}{u(x_0)} \right) \quad \text{normal to } H.$$

We obtain

$$(H) : \hat{q} \cdot (\hat{x} - (x_0, u(x_0)) = 0 \quad with\ \hat{x} \in \mathbb{R}^{n+1}.$$

Bach to our claim, we need to check that H supports v at $(x_0, u(x_0))$. It is sufficient to show that

$$\hat{q} \cdot ((x, v(x)) - (x_0, v(x_0)) \geq 0 \quad \forall x \in \Omega \tag{3.12}$$

(see Definition 1.1.10). However, due to the geometry of v, we know that for each $(x, v(x)) \in G(v)$, $\exists z \in \partial\Omega$ and $t \in [0, 1]$ such that

$$(x, v(x)) = (1 - t)(x_0, v(x_0)) + t(z, v(z)) = (1 - t)(x_0, u(x_0)) + t(z, 0).$$

Hence, it is sufficient to show that

$$\hat{q} \cdot ((z, 0) - (x_0, v(x_0))) \geq 0 \quad \forall z \in \partial\Omega.$$

But

$$\begin{aligned}\hat{q}\cdot((z,0)-(x_0,v(x_0))) &= \left(x_1-x_0,\frac{|x_1-x_0|^2}{u(x_0)}\right)\cdot(z-x_0,-u(x_0))\\ &= (x_1-x_0)\cdot(z-x_0)-|x_1-x_0|^2\\ &= (x_1-x_0)\cdot(z-x_1)\\ &\geq 0 \quad (\text{by definition of } \Pi_{x_1})\end{aligned}$$

Now, it remains to find q_0. Writing,

$$\hat{q}=\left(x_1-x_0,\frac{|x_1-x_0|^2}{u(x_0)}\right)=\frac{|x_1-x_0|^2}{u(x_0)}\left(\frac{x_1-x_0}{|x_1-x_0|^2}u(x_0),1\right),$$

we let

$$q_0=\frac{x_0-x_1}{|x_1-x_0|^2}(u(x_0)).$$

From (3.12), we get $(q_0,-1)\cdot((x,v(x))-(x_0,v(x_0))\leq 0\ \forall x\in\Omega$. This implies $v(x)\geq v(x_0)+q_0\cdot(x-x_0)\ \forall x\in\Omega$ that gives $q_0\in\partial v(x_0)$ with

$$|q_0|=\frac{-u(x_0)}{|x_1-x_0|}.$$

This ends this step.

Step 4. We claim that $B\left(0,\frac{-u(x_0)}{\Delta}\right)\subseteq\partial v(\Omega)$.

Let $p\in B\left(0,\frac{-u(x_0)}{\Delta}\right)$ then

$$-v(x_0)\geq|p|\Delta\geq p\cdot(z-x_0)\quad\forall z\in\partial\Omega.$$

This gives

$$v(z)=0\geq v(x_0)+p\cdot(z-x_0)\quad\forall z\in\partial\Omega. \tag{3.13}$$

Take $x\in\Omega$. As v is a cone, $\exists z\in\partial\Omega$ and $t\in[0,1]$ such that

$$(x,v(x))=(1-t)(x_0,v(x_0))+t(z,v(z)),$$

which implies from (3.13) that

$$v(x)=(1-t)v(x_0)+tv(z)\geq v(x_0)+p\cdot(t(z-x_0))=v(x_0)+p\cdot(x-x_0)$$

obtaining $p\in\partial v(x_0)=\partial v(\Omega)$.

Step 5. From Step 3, $|q_0|\geq\frac{-u(x_0)}{\Delta}$, thus $q_0\notin B\left(0,\frac{-u(x_0)}{\Delta}\right)$. Now, let $\mathcal{H}$ be the convex hull of $B\left(0,\frac{-u(x_0)}{\Delta}\right)$ and q_0, hence

$$|\mathcal{H}|=C_n\left(\frac{-u(x_0)}{\Delta}\right)^{n-1}|q_0|$$

with C_n a constant depending only on the dimension n. However, as $\partial v(\Omega)$ is a convex set containing $B\left(0, \frac{-u(x_0)}{\Delta}\right)$ and q_0 (from Step 2, 3, and 4), then

$$\mathcal{H} \subseteq \partial v(\Omega).$$

Therefore

$$|\mathcal{H}| \leq |\partial v(\Omega)| \leq |\partial u(\Omega)| \quad \text{(from Step 1)}$$

which implies

$$C_n \left(\frac{-u(x_0)}{\Delta}\right)^{n-1} \left(\frac{-u(x_0)}{dist(x_0, \partial\Omega)}\right) \leq |\partial u(\Omega)|.$$

We finally obtain

$$(-u(x_0))^n \leq C_n \, \Delta^{n-1} \, dist(x_0, \partial\Omega) \, |\partial u(\Omega)|$$

with x_0 arbitrary in Ω. □

3.5 Comparison Principle

Lemma 3.5.1. *If A and B are symmetric and positive semi-definite $n \times n$ nmatrices, then*

$$\det(A + B) \geq \det A + \det B.$$

Proof. First, we consider the case when $\det A > 0$. A is diagonalizable by orthogonal matrices, i.e. there exists O orthogonal matrix and D diagonal matrix such that $A = ODO^{-1}$. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the entries of D, and let $C = O\sqrt{D}O^{-1}$ where $\sqrt{D}$ is the diagonal matrix with entries $\sqrt{\lambda_1}, \sqrt{\lambda_2}, ..., \sqrt{\lambda_n}$. C is symmetric since

$$C^T = (O^{-1})^T(\sqrt{D})^T O^T = O\sqrt{D}O^{-1} = C$$

as O is orthogonal ($O^{-1} = O^T$) and $\sqrt{D}$ is diagonal. Also, C is positive definite. Indeed, as A is positive definite, then $\lambda_i > 0 \ \forall 0 \leq i \leq n$ which implies $\sqrt{\lambda_i} > 0 \ \forall 0 \leq i \leq n$. Moreover,

$$C^2 = O\sqrt{D}O^{-1}O\sqrt{D}O^{-1} = ODO^{-1} = A.$$

Now, we write

$$\frac{\det(A + B)}{\det A} = \det C^{-1} \det(A + B) \det C^{-1} = \det(C^{-1}AC^{-1} + C^{-1}BC^{-1}) = \det(I + C^{-1}BC^{-1}).$$

However, $C^{-1}BC^{-1}$ is positive semi-definite. This follows from the fact that B is positive semi-definite and C^{-1} is symmetric. Indeed, for $x \in \mathbb{R}^n$, $x^T(C^{-1}BC^{-1})x = (C^{-1}x)^T B(C^{-1}x) \geq 0$. Letting $\beta_1, \beta_2, ..., \beta_n$ be eigenvalues of $C^{-1}BC^{-1}$, we get $\beta_i \geq 0 \ \forall 0 \leq i \leq n$. Therefore, we can say that

$$\det(I + C^{-1}BC^{-1}) = (1 + \beta_1)(1 + \beta_2)...(1 + \beta_n) \geq 1 + \beta_1\beta_2...\beta_n = 1 + \det(C^{-1}BC^{-1}).$$

We obtain

$$\frac{\det(A+B)}{\det A} \geq 1 + \det(C^{-1}BC^{-1}) = 1 + \frac{\det B}{(\det C)^2} = \frac{\det A + \det B}{\det A}.$$

This implies that $\det(A+B) \geq \det A + \det B$.

Now, if $\det A = 0$ and $\det B = 0$, then $\det(A+B) \geq 0$ since $A+B$ is positive semi-definite.

Finally, if $\det A = 0$ and $\det B > 0$, we then apply the first case on B and get that $\det(A+B) \geq \det A + \det B = \det B$. □

Lemma 3.5.2. *Let Ω be an open, bounded, and convex subset of $\mathbb{R}^n$. Let u and v be a real-valued convex functions such that $u, v \in C(\bar{\Omega})$. We have*

$$M(u+v) \geq Mu + Mv.$$

Proof. We start with case $u, v \in C^2(\Omega)$. Since u and v are convex functions, Theorem 1.3.7 implies that $D^2u(x)$ and $D^2v(x)$ are positive semi-definite matrices $\forall x \in \Omega$. Also, as $u, v \in C^2(\Omega)$, then $D^2u(x)$ and $D^2v(x)$ are symmetric matrices $\forall x \in \Omega$. By Theorem 3.2.7, and Lemma 3.5.1, we conclude that for any Borel set $E \subseteq \Omega$,

$$M(u+v)(E) = \int_E \det(D^2(u+v)(x))\,dx \geq \int_E \det D^2u(x)\,dx + \int_E \det D^2v(x)\,dx = Mu(E) + Mv(E)$$

Now if $u \notin C^2(\Omega)$ or $v \notin C^2(\Omega)$, we can approximate u and v by sequences $u_k, v_k \in C^2(\Omega)$ convex functions respectively converging uniformly on compact subsets of Ω [10], that is,

$$u_k \xrightarrow[k\to\infty]{} u, \quad v_k \xrightarrow[k\to\infty]{} v \quad \text{uniformly on compact subsets of } \Omega.$$

This gives $u_k + v_k \xrightarrow[k\to\infty]{} u + v$ uniformly on compact subsets of Ω. Theorem 3.3.3 implies that Mu_k, Mv_k and $M(u_k + v_k)$ converge weakly to Mu, Mv and $M(u+v)$ respectively. Equivalently,

$$\int_\Omega f dMu_k \xrightarrow[k\to\infty]{} \int_\Omega f dMu, \quad \int_\Omega f dMv_k \xrightarrow[k\to\infty]{} \int_\Omega f dMv, \quad \int_\Omega f dM(u_k+v_k) \xrightarrow[k\to\infty]{} \int_\Omega f dM(u+v)$$

for all f continuous and bounded in Ω.

Let $E \subseteq \Omega$ be open, we show that there exists a sequence of positive bounded continuous functions $\{f_n\}_n$ such that

$$f_n \xrightarrow[n\to\infty]{} \chi_E.$$

Construct the sequence of sets $\{F_n\}_n$ such that $F_n = \{x \in \Omega \mid d(x, E^c) \geq \frac{1}{n}\}$. Notice that $\{F_n\}_n$ is an increasing sequence of closed sets with $E = \cup_{n=1}^{\infty} F_n$. Define the function $f_n : \Omega \to \mathbb{R}$

$$f_n(x) = \frac{d(x, E^c)}{d(x, E^c) + d(x, F_n)}.$$

Clearly , $0 \leq f_n \leq 1 \ \forall n \in \mathbb{N}$. Also, since E^c and F_n are disjoint closed sets in Ω, then f_n is continuous on Ω for all $n \in \mathbb{N}$ [11]. Besides, $\{f_n\}_n$ is increaing sequence of functions. Indeed, let $x \in \Omega$. $F_n \subseteq F_{n+1}$ implies $d(x, F_n) \geq d(x, F_{n+1}) \ \forall n \in \mathbb{N}$ obtaining $f_n(x) \leq f_{n+1}(x) \ \forall n \in \mathbb{N}$. Moreover,

$$\lim_{n\to\infty} f_n(x) = \chi_E(x) \quad \forall x \in \Omega.$$

To see this, let $x \in \Omega$. If $x \in E^c$, hence $f_n(x) = 0 = \chi_E(x) \ \forall n \in \mathbb{N}$. If $x \in E$, then there exists $n_0 \in \mathbb{N}$ such that $x \in F_n \ \forall n \geq n_0$. Thus $d(x, F_n) = 0 \ \forall n \geq n_0$ which gives that

$$\lim_{n\to\infty} f_n(x) = \frac{d(x, E^c)}{d(x, E^c)} = 1 = \chi_E(x).$$

Therefore, we obtain an increasing sequence $\{f_n\}_n$ of positive, bounded, and continuous functions. Applying the first case on u_k and v_k, we get

$$\det(D^2u_k(x) + D^2v_k(x)) \geq \det(D^2u_k(x)) + \det(D^2v_k(x)) \quad \forall x \in \Omega.$$

Besides, Theorem 3.2.7 implies for any $f \geq 0$,

$$\begin{aligned} \int_\Omega f dM(u_k + v_k) &= \int_\Omega f \det(D^2u_k + D^2v_k)dx \\ &\geq \int_\Omega f \det D^2u_k dx + \int_\Omega f \det D^2v_k dx \\ &= \int_\Omega f dMu_k + \int_\Omega f dMv_k. \end{aligned}$$

Hence, we have

$$\int_\Omega f_n dM(u_k + v_k) \geq \int_\Omega f_n dMu_k + \int_\Omega f_n dMv_k \quad \forall n \in \mathbb{N}.$$

By weak convergence, letting $k \to \infty$, we get

$$\int_\Omega f_n dM(u + v) \geq \int_\Omega f_n dMu + \int_\Omega f_n dMv \quad \forall n \in \mathbb{N}.$$

Now, we let $n \to \infty$, and by monotone convergence theorem we obtain

$$\int_\Omega \chi_E dM(u + v) \geq \int_\Omega \chi_E dMu + \int_\Omega \chi_E dMv$$

which implies

$$M(u + v)(E) \geq Mu(E) + Mv(E)$$

with $E \subseteq \Omega$ any open set.

More generally, if $E \subseteq \Omega$ is any borel set. Let $G \subseteq \Omega$ be any open set containing E. We showed that

$$M(u + v)(G) \geq Mu(G) + Mv(G) \geq Mu(E) + Mv(E) \tag{3.14}$$

and by outer regularity of $M(u+v)$, we have

$$M(u+v)(E) = \inf\{M(u+v)(G) \mid G \supseteq E, G \text{ is open in } \Omega\}.$$

Therefore, taking infimum on (3.14), we get

$$\inf_{\substack{G\supseteq E\\ Gopen}} M(u+v)(G) \geq Mu(E) + Mv(E)$$

obtaining

$$M(u+v)(E) \geq Mu(E) + Mv(E).$$

□

Theorem 3.5.3. *Let Ω be an open, bounded, and convex subset of $\mathbb{R}^n$. Let u and v be real-valued convex functions such that $u, v \in C(\bar{\Omega})$.*
If

$$|\partial u(E)| \leq |\partial v(E)| \quad \textit{for every Borel set } E \subseteq \Omega,$$

then

$$\min_{x\in\bar{\Omega}}\{u(x) - v(x)\} = \min_{x\in\partial\Omega}\{u(x) - v(x)\}.$$

Proof. Since $u, v \in C(\bar{\Omega})$ with $\bar{\Omega}$ and $\partial\Omega$ are compact sets, then both minimums are finite numbers. Let

$$a = \min_{x\in\bar{\Omega}}\{u(x) - v(x)\} \qquad \text{and} \qquad b = \min_{x\in\partial\Omega}\{u(x) - v(x)\}.$$

First, notice that $a \leq b$. Suppose $a < b$, thus the minimum over $\bar{\Omega}$ is attained for some $x_0 \in \Omega$, i.e. $a = u(x_0) - v(x_0)$. Take $\delta > 0$ small. Let

$$w(x) = v(x) + \delta|x - x_0|^2 + \frac{b+a}{2}$$

and

$$G = \{x \in \bar{\Omega} : u(x) < w(x)\}.$$

We have $x_0 \in G$. In fact, $x_0 \in \Omega$ and

$$u(x_0) - w(x_0) = u(x_0) - v(x_0) - \delta|x_0 - x_0|^2 - \frac{b+a}{2} = \frac{a-b}{2} < 0.$$

Moreover, if $x \in \partial\Omega$, $b \leq u(x) - v(x)$ and

$$w(x) = v(x) + \delta|x - x_0|^2 + \frac{b+a}{2} \leq u(x) + \delta(diam(\Omega))^2 + \frac{a-b}{2} < u(x)$$

with assumption on δ that $\delta(diam\Omega)^2 < \frac{b-a}{2}$. We obtain that $w(x) < u(x)\ \forall x \in \partial\Omega$. Therefore, $G \subseteq \Omega$ and $\partial G \subseteq \{x \in \Omega : u(x) = w(x)\}$. Hence, G is a bounded open set with $u, w \in C(\bar{G})$, $u < w$ in G, and $u = w$ on ∂G. So Lemma 3.4.1 implies that

$$\partial w(G) \subseteq \partial u(G). \tag{3.15}$$

On the other hand, we have

$$\partial w = \partial \left(v + \delta|x - x_0|^2 + \frac{b+a}{2} \right) = \partial(v + \delta|x - x_0|^2).$$

The latter equality follows from the following. Let $p \in \partial w(x)$ for some $x \in \Omega$, thus $w(y) \geq w(x) + p \cdot (y - x) \;\; \forall y \in \Omega$ which implies

$$v(y) + \delta|y - x_0|^2 + \frac{b+a}{2} \geq v(x) + \delta|x - x_0|^2 + \frac{b+a}{2} + p \cdot (y - x) \quad \forall y \in \Omega.$$

Subtracting $\frac{b+a}{2}$ both sides of the inequality gives that

$$v(y) + \delta|y - x_0|^2 \geq v(x) + \delta|x - x_0|^2 + p \cdot (y - x) \quad \forall y \in \Omega$$

obtaining $p \in \partial(v + \delta|x - x_0|^2)(x)$. The converse follows similarly.

Using Lemma 3.5.2, we get

$$|\partial(v + \delta|x - x_0|^2)(G)| \geq |\partial v(G)| + |\partial(\delta|x - x_0|^2)(G)|$$

with

$$|\partial(\delta|x - x_0|^2)(G)| = \int_G det D^2(\delta|x - x_0|)dx = \int_G (2\delta)^n dx = (2\delta)^n|G|$$

by Theorem 3.2.7. Therefore, we obtain using (3.15)

$$|\partial u(G)| \geq |\partial w(G)| \geq |\partial v(G)| + |\partial(\delta|x - x_0|^2)(G)| = |\partial v(G)| + (2\delta)^n|G| > |\partial v(G)|$$

which contradicts the given. $\square$

Corollary 3.5.4. *Let Ω be an open, bounded, and convex subset of $\mathbb{R}^n$. Let u and v be real-valued convex functions such that $u, v \in C(\bar{\Omega})$.*
If

$$|\partial u(E)| = |\partial v(E)| \quad \textit{for every Borel set } E \subseteq \Omega$$

with $u = v$ on $\partial\Omega$, then $u = v$ in Ω.

Proof. Applying the comparison principle, Theorem 3.5.3, with the fact $|\partial u(E)| \leq |\partial v(E)|$ implies

$$\min_{x \in \bar{\Omega}}\{u(x) - v(x)\} = \min_{x \in \partial\Omega}\{u(x) - v(x)\}.$$

But $u = v$ on $\partial\Omega$, hence we obtain $u(x) \geq v(x) \;\; \forall x \in \Omega$. Similarly, applying comparison principle, Theorem 3.5.3, with the fact $|\partial v(E)| \leq |\partial u(E)|$ implies $v(x) \geq u(x) \;\; \forall x \in \Omega$. Therefore, we get $u(x) = v(x) \;\; \forall x \in \Omega$. $\square$

Chapter 4

Aleksandrov Solution

4.1 The Homogeneous Dirichlet Problem

Definition 4.1.1. *Let Ω be an open convex subset of $\mathbb{R}^n$ and μ a Borel measure in Ω. A convex function $u : \Omega \to \mathbb{R}$ is called an Aleksandrov solution (or generalized solution) to the Monge-Ampère equation*

$$detD^2u = \mu$$

if the Monge-Ampère measure associated with u equals to μ, i.e.

$$Mu = \mu.$$

Theorem 4.1.2. *Let Ω be an open, bounded, and strictly convex subset of $\mathbb{R}^n$ and $g : \partial\Omega \to \mathbb{R}$ a continuous function. There exists a unique convex function $u \in C(\bar{\Omega})$ Aleksandrov solution of the problem*

$$\begin{cases} detD^2u = 0 & in\ \ \Omega \\ u = g & on\ \ \partial\Omega \end{cases}. \tag{4.1}$$

Proof. Let

$$\mathcal{F} = \{a(x) : a \text{ is an affine function and } a \leq g \text{ on } \partial\Omega\}.$$

We have $\mathcal{F} \neq \emptyset$. In fact, as g is continuous on compact a set, g is bounded and $|g(x)| \leq C \ \forall x \in \partial\Omega$. Letting $a = -C - 1$, we have $a \in \mathcal{F}$. Now, define

$$u(x) = \sup\{a(x) : a \in \mathcal{F}\}.$$

Step 1. We show that u is convex in Ω. We have u is a pointwise supremum of affine functions, then u is convex in Ω. To see this, let $x, y \in \Omega$ and $t \in [0, 1]$. Since a is affine, we have for every $a \in \mathcal{F}$

$$a((1-t)x + ty) = (1-t)a(x) + ta(y) \leq (1-t)u(x) + tu(y).$$

Thus taking the sup over all a we get

$$u((1-t)x+ty) \leq (1-t)u(x)+tu(y).$$

Step 2. We prove that $u = g$ on $\partial\Omega$. First, as $a(x) \leq g(x)$ $\forall a \in \mathcal{F}$ $\forall x \in \partial\Omega$, then $u(x) \leq g(x)$ $\forall x \in \partial\Omega$.

It remains to show that $u(\xi) \geq g(\xi)$ $\forall \xi \in \partial\Omega$. Let $\xi \in \partial\Omega$ and let $\epsilon > 0$. By definition of continuity of g on ξ, there exists $\delta > 0$ such that $\forall x \in \partial\Omega$,

$$|x-\xi| < \delta \implies |g(x)-g(\xi)| < \epsilon.$$

Since Ω is convex, Corollary 1.1.13 implies the existence of a supporting hyperplane Π to Ω at ξ. Let $P(x) = 0$ be the equation of Π. Assume $\Omega \subseteq \Pi^+$ i.e. $\Omega \subseteq \{x : P(x) \geq 0\}$. We claim that there exists $\eta > 0$ such that

$$S = \{x \in \bar{\Omega} : P(x) \leq \eta\} \subseteq B(\xi, \delta).$$

Suppose $\forall \eta > 0$,

$$\{x \in \bar{\Omega} : P(x) \leq \eta\} \cap (B(\xi,\delta))^c \neq \emptyset,$$

then for every $n \in \mathbb{N}$, there exists

$$y_n \in \{x \in \bar{\Omega} : P(x) \leq \frac{1}{n}\} \cap (B(\xi,\delta))^c.$$

We obtain a sequence $(y_n)_n \subset \bar{\Omega}$ with $P(y_n) \leq \frac{1}{n}$ and $|y_n - \xi| \geq \delta$. As $\bar{\Omega}$ is bounded, $(y_n)_n$ is a bounded sequence. By Bolzano-Weirstrass, it has a convergent subsequence, say

$$y_{n_k} \xrightarrow[k\to\infty]{} y_0$$

with $y_0 \in \bar{\Omega}$. Besides, $P(y_{n_k}) \leq \frac{1}{k}$, then $P(y_0) \leq 0$. However, $y_0 \in \bar{\Omega} \subseteq \{x : P(x) \geq 0\}$, hence $P(y_0) \geq 0$ which implies that $P(y_0) = 0$. Also, $|y_{n_k} - \xi| \geq \delta$, then $|y_0 - \xi| \geq \delta > 0$ which means $y_0 \neq \xi$. We obtain $y_0, \xi \in \bar{\Omega}$ with $y_0 \neq \xi$, and with Ω open and strictly convex, we get

$$(1-t)y_0 + t\xi \in \Omega \subseteq \{x : P(x) > 0\} \quad \forall t \in (0,1).$$

But, as P plane and $P(y_0) = P(\xi) = 0$, we have for $t \in (0,1)$,

$$P((1-t)y_0 + t\xi) = (1-t)P(y_0) + tP(\xi) = 0$$

which is a contradiction. This ends the proof of our claim.

Now, we let

$$M = \min\{g(x) : x \in \partial\Omega \ \ and \ \ P(x) \geq \eta\}$$

and take

$$a(x) = g(\xi) - \epsilon - AP(x) \tag{4.2}$$

where A is positive constant. We have $a \leq g$ on $\partial\Omega$. In fact, if $x \in \partial\Omega \cap S$, then $x \in \partial\Omega \cap B(\xi,\delta)$ and $|g(x) - g(\xi)| < \epsilon$. Therefore,

$$g(x) > g(\xi) - \epsilon \geq g(\xi) - \epsilon - AP(x) = a(x).$$

If $x \in \partial\Omega \cap S^c$, hence $P(x) > \eta$ and

$$\begin{aligned} g(x) \geq M = a(x) + M - a(x) &= a(x) + M - g(\xi) + \epsilon + AP(x) \\ &> a(x) + M - g(\xi) + \epsilon + A\eta. \end{aligned}$$

Taking

$$A \geq \max\left\{\frac{g(\xi) - \epsilon - M}{\eta}, 0\right\}, \tag{4.3}$$

we get

$$g(x) > a(x)$$

in this case. Therefore, we have a is an affine function with $a < g$ on $\partial\Omega$, thus $a \in \mathcal{F}$.

Since $u(\xi) \geq a(\xi) = g(\xi) - \epsilon$, letting $\epsilon \to 0$, we end with $u(\xi) \geq g(\xi)$ and thus $u(\xi) = g(\xi)$.

Step 3. We claim that $u \in C(\bar{\Omega})$. Actually, as we proved that u is convex in Ω, then u is continuous in Ω by Theorem 1.3.5. It remains to show that u is continuous up to boundary $\partial\Omega$.

Let $\xi \in \partial\Omega$ with $(x_n)_n \subset \bar{\Omega}$ be such that

$$x_n \xrightarrow[n\to\infty]{} \xi.$$

We prove that

$$u(x_n) \xrightarrow[n\to\infty]{} u(\xi) = g(\xi).$$

We first show that

$$\liminf_{n\to\infty} u(x_n) \geq g(\xi).$$

Let $\epsilon > 0$ and consider $a(x)$ as in Step 2 (see (4.2) and (4.3)). We proved that $a \in \mathcal{F}$ obtaining $u(x_n) \geq a(x_n)\ \ \forall n \in \mathbb{N}$, so

$$\liminf_{n\to\infty} u(x_n) \geq \liminf_{n\to\infty} a(x_n) = \liminf_{n\to\infty}(g(\xi) - \epsilon - AP(x_n)) = g(\xi) - \epsilon - AP(\xi) = g(\xi) - \epsilon.$$

Taking $\epsilon \to 0$, we reach our goal.

Second, we show that

$$\limsup_{n\to\infty} u(x_n) \leq g(\xi).$$

As Ω is convex, there exists a harmonic function h in Ω such that $h \in C(\bar{\Omega})$ and $h = g$ on $\partial\Omega$ [12]. Now, we let $a \in \mathcal{F}$, so a is affine and $a \leq g$ on $\partial\Omega$. Thus a is harmonic in Ω and $a \leq h$ on $\partial\Omega$. We obtain $a - h$ is harmonic in Ω since Ω open with $a - h \leq 0$ on $\partial\Omega$. By the maximum principle, $a - h$ attains its maximum on $\partial\Omega$. This implies that $a - h \leq 0$ in $\bar{\Omega}$, that is, $a \leq h$ in $\bar{\Omega}$. But this is true for any $a \in \mathcal{F}$, thus by definition of u, we have $u \leq h$ in $\bar{\Omega}$. Hence, $u(x_n) \leq h(x_n)\ \ \forall n \in \mathbb{N}$. Therefore,

$$\limsup_{n\to\infty} u(x_n) \leq \limsup_{n\to\infty} h(x_n) = h(\xi) = g(\xi)$$

as h is continuous on $\bar{\Omega}$. We reach that

$$\lim_{n\to\infty} u(x_n) = g(\xi) = u(\xi)$$

and so u is continuous at ξ for every $\xi \in \partial\Omega$.

Step 4. We show that the Monge-Ampère measure associated to u is equal to zero in Ω, i.e. $Mu = 0$ in Ω. Actually, if we show that $Mu(\Omega) = 0$, by monotonicity of Mu we obtain that $Mu(E) = 0$ for all Borel sets $E \subseteq \Omega$, and thus $Mu = 0$ in Ω. We know using Theorem 3.2.2 that the set

$$N = \{p \in \mathbb{R}^n : \exists x, y \in \Omega, x \neq y \ \ and \ \ p \in \partial u(x) \cap \partial u(y)\}$$

is Lebesgue null set. We claim that $\partial u(\Omega) \subseteq N$ to conclude our claim.

First, as u is convex, then $\partial u(\Omega) \neq \emptyset$ by Theorem 2.2.3. Let $p \in \partial u(\Omega)$, thus there exists $x_0 \in \Omega$ such that $p \in \partial u(x_0)$, and so by continuity up to $\partial\Omega$ we get that

$$u(x) \geq u(x_0) + p \cdot (x - x_0) \quad \forall x \in \bar{\Omega}. \tag{4.4}$$

Let

$$a(x) = u(x_0) + p \cdot (x - x_0).$$

Since $u = g$ on $\partial\Omega$, we have $g \geq a$ on $\partial\Omega$. Suppose that $g > a$ on $\partial\Omega$, then there exists $n \in \mathbb{N}$ such that $g \geq a + \frac{1}{n}$ on $\partial\Omega$. We obtain $a + \frac{1}{n} \in \mathcal{F}$ and $u(x) \geq a(x) + \frac{1}{n} \ \ \forall x \in \bar{\Omega}$. In particular, $u(x_0) \geq a(x_0) + \frac{1}{n} = u(x_0) + \frac{1}{n}$ which is a contradiction. Therefore, there exists $\xi \in \partial\Omega$ such that

$$g(\xi) = a(\xi).$$

Now, since Ω is strictly convex, then $(1-t)\xi + tx_0 \in \Omega \ \forall t \in (0, 1)$. Let $z = (1-t)\xi + tx_0$ for some $t \in (0, 1)$. By convexity of u, we have

$$u(z) \leq (1-t)u(\xi) + tu(x_0).$$

But $u(\xi) = g(\xi) = a(\xi)$ and $u(x_0) = a(x_0)$, thus

$$u(z) \leq (1-t)a(\xi) + ta(x_0) = a(z).$$

Hence

$$u(z) = a(z) = u(x_0) + p \cdot (z - x_0).$$

From (4.4), we get

$$u(x) \geq u(x_0) + p \cdot (x - z) + p \cdot (z - x_0) = u(z) + p \cdot (x - z) \qquad \forall x \in \Omega$$

obtaing that $p \in \partial u(z)$. Therefore, we have $x_0, z \in \Omega$ with $x_0 \neq z$ and $p \in \partial u(x_0) \cap \partial u(z)$, and thus $p \in N$ which ends the proof of this step.

Step 5. From Steps 1,2,3, and 4 we obtain that the convex function $u \in C(\bar{\Omega})$ is an Aleksandrov solution to the Dirichlet problem (4.1). To end the proof, it remains to show that u is unique. Suppose there exists a convex function $v \in C(\bar{\Omega})$ Aleksandrov solution to (4.1) with $v \neq u$. We get that $Mu = Mv = 0$ and $u = v$ on $\partial\Omega$. By Comparison principle 3.5.3, we obtain that $u = v$ in Ω. Therefore u is unique. □

4.2 The Nonhomogeneous Dirichlet Problem

Definition 4.2.1. *Let Ω be an open, bounded, and convex subset of $\mathbb{R}^n$ and $g : \partial\Omega \to \mathbb{R}$ a continuous function. Let μ a Borel measure in Ω. We define the set*

$$\mathcal{F}(\mu, g) = \{v \in C(\bar{\Omega}) : v \text{ is convex in } \Omega,\ Mv \geq \mu, \text{ and } v = g \text{ on } \partial\Omega\}.$$

Remark 4.2.2. *Assume Ω is strictly convex and $\mathcal{F}(\mu, g) \neq \emptyset$. We have all functions in $\mathcal{F}(\mu, g)$ are uniformly bounded from above and we can define*

$$U(x) = \sup\{v(x) : v \in \mathcal{F}(\mu, g)\}.$$

To see this, we let $v \in \mathcal{F}(\mu, g)$. By Theorem 4.1.2, there exists a unique convex function $w \in C(\bar{\Omega})$ Aleksandrov solution to the problem

$$\begin{cases} detD^2 w = 0 & in\ \ \Omega \\ w = g & on\ \ \partial\Omega \end{cases}. \tag{4.5}$$

Hence, $Mw = 0 \leq \mu \leq Mv$. By Comparison principle 3.5.3, we get that

$$\min_{x \in \bar{\Omega}}\{w(x) - v(x)\} = \min_{x \in \partial\Omega}\{w(x) - v(x)\}.$$

But $w = v = g$ on $\partial\Omega$, then $w(x) - v(x) \geq 0\ \ \forall x \in \Omega$. This implies that

$$v(x) \leq w(x) \leq \sup_{x \in \bar{\Omega}} |w(x)| = M \quad \forall x \in \Omega$$

where M is positive constant since $w \in C(\bar{\Omega})$ with $\bar{\Omega}$ compact. Therefore,

$$\sup_{x \in \Omega} v(x) \leq M.$$

Lemma 4.2.3. *Let Ω be an open, bounded, and strictly convex subset of $\mathbb{R}^n$ and $g : \partial\Omega \to \mathbb{R}$ continuous function. Let μ_j, μ be Borel measures in Ω and $u_j \in C(\bar{\Omega})$ convex real valued functions such that:*

1. *For each j, u_j is Aleksandrov solution to the problem*

$$\begin{cases} detD^2 u_j = \mu_j & in\ \ \Omega \\ u_j = g & on\ \ \partial\Omega \end{cases}. \tag{4.6}$$

2. *$\mu_j \to \mu$ weakly in Ω.*

3. *$\mu_j(\Omega) \leq B\ \ \forall j \in \mathbb{N}$ where B is a constant.*

Then $(u_j)_j$ has a subsequence that converges uniformly on compact subsets of Ω to a convex function $u \in C(\bar{\Omega})$ Aleksandrov solution to the problem

$$\begin{cases} detD^2 u = \mu & in\ \ \Omega \\ u = g & on\ \ \partial\Omega \end{cases}. \tag{4.7}$$

Proof. We show that there exists a subsequence $(u_{j_k})_k$ that converges uniformly on compact subsets of Ω.

Step 1. We show that the sequence $(u_j)_j$ is uniformly bounded in Ω. First, notice that $u_j \in \mathcal{F}(\mu_j, g)$ for each $j \in \mathbb{N}$. From Remark 4.2.2, we have $u_j(x) \leq w(x) \leq M' \ \forall x \in \Omega, \forall j \in \mathbb{N}$. Therefore u_j are uniformly bounded from above.

We claim that u_j are uniformly bounded from below. Let $\xi \in \partial\Omega$ and $\epsilon > 0$. Since Ω is convex, Corollary 1.1.13 implies the existance of a supporting hyperplane Π to Ω at ξ. Let $P(x) = 0$ be the equation of Π. Assume $\Omega \subseteq \Pi^+$ i.e. $\Omega \subseteq \{x : P(x) \geq 0\}$. Take

$$a(x) = g(\xi) - \epsilon - A'P(x)$$

where $A' = \max\{A, B\}$ and A is the constant given in the proof of Theorem 4.1.2 (see (4.3)). We showed in the proof of Theorem 4.1.2 that

$$a(x) \leq g(x) \quad \forall x \in \partial\Omega.$$

Set

$$v_j(x) = u_j(x) - a(x).$$

We have $v_j \in C(\bar{\Omega})$ and v_j are convex in Ω. On $\partial\Omega$, $u_j(x) = g(x)$ by (4.6), and hence $v_j(x) = g(x) - a(x) \geq 0$. We consider now the following two cases.

If $v_j(x) \geq 0$ for all $x \in \Omega$, we get

$$u_j(x) \geq a(x) \geq \inf_{x \in \bar{\Omega}} a(x).$$

Hence u_j are uniformly bounded from below.

Now, if $\exists\, x_0 \in \Omega$ such that $v_j(x_0) < 0$, let $G = \{x \in \Omega : v_j(x) \leq 0\}$. We have $G \neq \emptyset$, compact and convex. Moreover, $\partial G \subseteq \{x \in \Omega : v_j(x) = 0\}$. Letting $\Delta_1 = diam(G)$, since $v_j \in C(G)$ with $v_j = 0$ on ∂G then applying Aleksandrov's maximum principle 3.4.2 on G, we obtain

$$(-v_j(x))^n \leq C_n \, \Delta_1^{n-1} \, dist(x, \partial G) \, Mv_j(G) \quad \forall x \in G,$$

where C_n is a constant depending only on the dimension n. However, $G \subseteq \Omega$, then letting $\Delta = diam(\Omega)$

$$(-v_j(x))^n \leq C_n \, \Delta^{n-1} \, dist(x, \partial\Omega) \, Mv_j(\Omega) \quad \forall x \in G.$$

We have $Mu_j(\Omega) = \mu_j(\Omega) \leq B$ by given properties 1 and 3 with $u_j = v_j + a$. Using Lemma 3.5.2, we get from Theorem 3.2.7

$$B \geq Mu_j(\Omega) \geq Mv_j(\Omega) + Ma(\Omega) = Mv_j.$$

This implies that

$$(-v_j(x))^n \leq C_n \, \Delta^{n-1} \, dist(x, \partial\Omega) \, B \quad \forall x \in G,$$

which is equivalent to

$$-v_j(x) \leq (C_n \, \Delta^{n-1} \, dist(x, \partial\Omega) \, B)^{\frac{1}{n}} \quad \forall x \in G,$$

that is
$$v_j(x) \geq -(C_n \Delta^{n-1} dist(x, \partial\Omega)\, B)^{\frac{1}{n}} \quad \forall x \in G.$$
We obtain
$$u_j(x) - a(x) \geq -(C_n \Delta^{n-1} dist(x, \partial\Omega)\, B)^{\frac{1}{n}} \quad \forall x \in G.$$
Also if $x \in \Omega \setminus G$, we have $v_j(x) > 0$, i.e. $u_j(x) - a(x) > 0$. Thus the inequality holds for any $x \in \Omega$. We get that

$$u_j(x) \geq a(x) - (C_n \Delta^{n-1} dist(x, \partial\Omega)\, B)^{\frac{1}{n}} \quad \forall x \in \Omega \tag{4.8}$$
$$= g(\xi) - \epsilon - A' P(x) - (C_n \Delta^{n-1} dist(x, \partial\Omega)\, B)^{\frac{1}{n}} \quad \forall x \in \Omega. \tag{4.9}$$

With $P(x) \leq \sup_{x \in \bar{\Omega}} P(x) := \tilde{P}$. Finally, since $dist(x, \partial\Omega) \leq diam(\Omega) = \Delta$, we obtain
$$u_j(x) \geq g(\xi) - \epsilon - A'\tilde{P} - \Delta (C_n B)^{\frac{1}{n}} \quad \forall x \in \Omega.$$
Therefore, u_j are uniformly bounded from below in Ω. Thus u_j are uniformly bounded in Ω.

Step 2. We show that u_j is an equicontinuous sequence in compact subsets of Ω. Since u_j are convex in Ω, Theorem 2.2.4 implies that for $K \subset \Omega$ compact, u_j is Lipschitz continuous in K with constant

$$C(K, j) = \sup\{|p| : p \in \partial u_j(K)\}.$$

We claim that $C(K, j)$ is uniformly bounded in j. Let $p \in \partial u_j(K)$. Then there exists $x_1 \in K$ such that $p \in \partial u_j(x_1)$. Thus

$$u_j(x) \geq u_j(x_1) + p \cdot (x - x_1) \quad \forall x \in \Omega.$$

Let $0 < r < dist(x_1, \partial\Omega)$, and take in particular $x_p = x_1 + r\frac{p}{|p|}$. So $|x_p - x_1| = r < dist(x_1, \partial\Omega)$ which implies $x_p \in \Omega$ and $u_j(x_p) \geq u_j(x_1) + r|p|$. We obtain

$$|p| \leq \frac{|u_j(x_p)| + |u_j(x_1)|}{r}.$$

Knowing that u_j is a uniformly bounded sequence in Ω, we get

$$|p| \leq \frac{2C}{r}$$

where C is independent of j and x, and letting $r \to dist(x_1, \partial\Omega)$, we have

$$|p| \leq \frac{2C}{dist(x_1, \partial\Omega)}.$$

Moreover, we have $dist(x_1, \partial\Omega) \geq dist(K, \partial\Omega)$, hence

$$|p| \leq \frac{2C}{dist(K, \partial\Omega)}$$

and this is true for any $p \in \partial u_j(K)$. Therefore,

$$C(K, j) \leq \frac{2C}{dist(K, \partial\Omega)}.$$

We get that u_j is Lipschitz in K with constant

$$C(K) = \frac{2C}{dist(K, \partial\Omega)}.$$

Then $\forall \epsilon > 0$, $\exists\, \delta = \frac{\epsilon}{C(K)}$ such that

$$|x - y| < \delta \implies |u_j(x) - u_j(y)| < \epsilon \quad \forall x, y \in K.$$

Hence, we obtain that u_j is an equicontinuous sequence in K.

Now, by Arzelà-Ascoli, there exists a subsequence of $(u_j)_j$ that converges uniformly on compact subsets of Ω, say without relabeling that

$$u_j \xrightarrow[j\to\infty]{} u.$$

Step 3. Define $u = g$ on $\partial\Omega$. We claim that u is convex in Ω and $u \in C(\bar{\Omega})$. Convexity of u follows directly from the fact that it is the uniform limit of a sequence of convex functions. Also, this implies that $u \in C(\Omega)$. It remains to show that u is continuous at points in $\partial\Omega$.

Let $\xi \in \partial\Omega$. From Remark 4.2.2, we have $u_j(x) \leq w(x)\ \ \forall j \in \mathbb{N}$ where $w \in C(\bar{\Omega})$ is a convex Aleksandrov solution of problem (4.5). Besides, from (4.9) and the fact that $dist(x, \partial\Omega) \leq |x - \xi|$, we obtain $\forall j \in \mathbb{N}\, \forall x \in \Omega$,

$$w(x) \geq u_j(x) \geq g(\xi) - \epsilon - A'P(x) - (C_n\Delta^{n-1}B)^{\frac{1}{n}}|x - \xi|^{\frac{1}{n}}. \tag{4.10}$$

Letting $j \to \infty$, we get

$$w(x) \geq u(x) \geq g(\xi) - \epsilon - A'P(x) - (C_n\Delta^{n-1}B)^{\frac{1}{n}}|x - \xi|^{\frac{1}{n}} \quad \forall x \in \Omega.$$

But

$$\lim_{x\to\xi} w(x) = w(\xi) = g(\xi)$$

as $w \in C(\bar{\Omega})$, and

$$\lim_{x\to\xi}(g(\xi) - \epsilon - A'P(x) - (C_n\Delta^{n-1}B)^{\frac{1}{n}}|x - \xi|^{\frac{1}{n}}) = g(\xi) - \epsilon.$$

Letting $\epsilon \to 0$ and by Squeeze theorem, we obtain

$$\lim_{x\to\xi} u(x) = g(\xi) = u(\xi)$$

and thus u is continuous at ξ.

Step 4. To end the proof, it remains to show that u is Aleksandrov solution to problem (4.7). Now, since

$$u_j \xrightarrow[j\to\infty]{} u \quad \text{uniformly on compact subsets of } \Omega,$$

Lemma 3.3.3 implies that

$$Mu_j \xrightarrow[j\to\infty]{} Mu \quad \textit{weakly}.$$

However, from given properties 1 and 2, we have

$$\mu_j = Mu_j \xrightarrow[j\to\infty]{} \mu \quad \textit{weakly}.$$

Hence

$$Mu = \mu.$$

This ends the proof. □

Lemma 4.2.4. *Let Ω be an open, bounded, and strictly convex subset of $\mathbb{R}^n$ and μ a Borel measure in Ω with $\mu(\Omega) < \infty$. Then there exists a sequence $(\mu_k)_k$ that converges weakly to μ such that $\mu_k(\Omega) \leq B$ $\forall k \in \mathbb{N}$ and for each $k \in \mathbb{N}$,*

$$\mu_k = \sum_{j=1}^{N_k} a_j^k \delta_{x_j^k}$$

with $x_j^k \in \Omega$ and $a_j^k > 0$.

Proof. Since Ω is bounded, there exists $N > 0$ such that $\bar{\Omega} \subseteq [-N, N]^n := Q_N$. Fix $k \in \mathbb{N}$, divide Q_N into cubes with disjoint interiors and with diameter $\frac{1}{k}$. Call them $Q_1^k, Q_2^k, ..., Q_{N_k}^k$ and let $\Omega_j^k = Q_j^k \cap \Omega$ while removing ones with empty interior. Therefore, there exists a disjoint family $\{\Omega_j^k\}_{j=1}^{N_k}$ of Borel subsets of Ω such that $diam(\Omega_j^k) < \frac{1}{k}$ and

$$\Omega = \bigcup_{j=1}^{N_k} \Omega_j^k.$$

Now, take $x_j^k \in \Omega_j^k$ and let

$$\mu_k = \sum_{j=1}^{N_k} \mu(\Omega_j^k) \delta_{x_j^k}.$$

As $\mu(\Omega) < \infty$, we have

$$\mu_k(\Omega) = \sum_{j=1}^{N_k} \mu(\Omega_j^k) = \mu(\Omega) < \infty \quad \forall k \in \mathbb{N}. \tag{4.11}$$

Moreover, let f be a continuous function with compact support in Ω. Let $\epsilon > 0$. From uniform continuity of f, there exists $\delta > 0$ such that $\forall x, y \in \Omega$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{\mu(\Omega) + 1}.$$

Consider $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \delta$. Therefore, $\forall k \geq k_0$, we have

$$\begin{aligned}
\left|\int_\Omega f d\mu_k - \int_\Omega f d\mu\right| &= \sum_{j=1}^{N_k} \left(\int_{\Omega_j^k} f d\mu_k - \int_{\Omega_j^k} f d\mu\right) \\
&= \sum_{j=1}^{N_k} \left(\int_{\Omega_j^k} f(x_j^k) d\mu - \int_{\Omega_j^k} f(x) d\mu\right) \\
&\leq \sum_{j=1}^{N_k} \int_{\Omega_j^k} |f(x_j^k) - f(x)| d\mu \\
&< \frac{\epsilon}{1+\mu(\Omega)} \sum_{j=1}^{N_k} \int_{\Omega_j^k} d\mu \\
&< \varepsilon.
\end{aligned}$$

Hence

$$\int_\Omega f(x) d\mu_k \xrightarrow[k\to\infty]{} \int_\Omega f(x) d\mu$$

which ends the proof. $\square$

Theorem 4.2.5. *Let Ω be an open, bounded, and strictly convex subset of $\mathbb{R}^n$ and $g : \partial\Omega \to \mathbb{R}$ continuous function. Let μ be a Borel measure in Ω with $\mu(\Omega) < \infty$. There exists a unique convex function $u \in C(\bar{\Omega})$ Aleksandrov solution of the problem*

$$\begin{cases} detD^2 u = \mu & in \;\; \Omega \\ u = g & on \;\; \partial\Omega \end{cases}$$

Proof. First, we know from Lemma 4.2.4 that there exists a sequence $(\mu_k)_k$ that converges weakly to μ such that $\mu_k(\Omega) \leq B \;\; \forall k \in \mathbb{N}$ and for each $k \in \mathbb{N}$,

$$\mu_k = \sum_{j=1}^{N_k} a_j^k \delta_{x_j^k}$$

with $x_j^k \in \Omega$ and $a_j^k > 0$. Hence, if for each $k \in \mathbb{N}$, we show that there exists a unique convex function $u_k \in C(\bar{\Omega})$ Aleksandrov solution of the Dirichlet problem

$$\begin{cases} detD^2 u_k = \mu_k & in \;\; \Omega \\ u_k = g & on \;\; \partial\Omega \end{cases} \tag{4.12}$$

then the theorem follows from Lemma 4.2.3.

Therefore, we may assume without loss of generality that

$$\mu = \sum_{i=1}^{N} a_i \delta_{x_i}$$

with $x_i \in \Omega$ and $a_i > 0$. We show that there exists a unique convex function $U \in C(\bar{\Omega})$ Aleksandrov solution of the problem

$$\begin{cases} detD^2u = \mu & in\ \Omega \\ u = g & on\ \partial\Omega. \end{cases}$$

Step 1. We prove that $\mathcal{F}(\mu, g) \neq \emptyset$ where from Definition 4.2.1

$$\mathcal{F}(\mu, g) = \{v \in C(\bar{\Omega}) : v \text{ is convex in } \Omega,\ Mv \geq \mu, \text{ and } v = g \text{ on } \partial\Omega\}.$$

Let

$$f(x) = \frac{1}{w_n^{\frac{1}{n}}} \sum_{i=1}^{N} a_i^{\frac{1}{n}} |x - x_i|$$

where w_n is the volume of the unit ball in $\mathbb{R}^n$. Clearly, f is convex with $f \in C(\mathbb{R}^n)$, and thus $g - f : \partial\Omega \to \mathbb{R}$ is continuous. Consider the following homogeneous Dirichlet problem with data $g - f$:

$$\begin{cases} detD^2u = 0 & in\ \Omega \\ u = g - f & on\ \partial\Omega \end{cases}. \tag{4.13}$$

By Theorem 4.1.2, there exists a unique convex function $u \in C(\bar{\Omega})$ Aleksandrov solution of (4.13). We claim that $v = u + f \in \mathcal{F}(\mu, g)$. Initially, it is obvious that v is convex in Ω with $v \in C(\bar{\Omega})$ as u, and f are convex and in $C(\bar{\Omega})$. Moreover, on $\partial\Omega$, $u = g - f$ and hence $v = g$. It remains to show that $Mv \geq \mu$ in Ω. We have

$$\begin{aligned}
Mv &= M(u{+}f) \\
&\geq Mu + Mf \quad \text{(by Lemma 3.5.2)} \\
&= 0 + Mf \quad \text{(from (4.13))} \\
&= M\left(\frac{1}{w_n^{\frac{1}{n}}} \sum_{i=1}^{N} a_i^{\frac{1}{n}} |x - x_i|\right) \\
&= \frac{1}{w_n} M\left(\sum_{i=1}^{N} a_i^{\frac{1}{n}} |x - x_i|\right) \quad \text{(by Proposition 3.2.8)} \\
&\geq \frac{1}{w_n} \sum_{i=1}^{N} M(a_i^{\frac{1}{n}} |x - x_i|) \quad \text{(again by Lemma 3.5.2).}
\end{aligned}$$

However, from Example 2.3.2 and Example 3.2.9, we have

$$M(a_i^{\frac{1}{n}} |x - x_i|) = |B(0, a_i^{\frac{1}{n}})| \delta_{x_i} = |a_i^{\frac{1}{n}} B(0, 1)| \delta_{x_i} = a_i w_n \delta_{x_i}.$$

We obtain

$$Mv \geq \sum_{i=1}^{N} a_i \delta_{x_i} = \mu.$$

Therefore, $v \in \mathcal{F}(\mu, g)$ and $\mathcal{F}(\mu, g) \neq \emptyset$.

By Remark 4.2.2, we can now define

$$U(x) = \sup\{v(x) : v \in \mathcal{F}(\mu, g)\}.$$

Step 2. Let $u, v \in \mathcal{F}(\mu, g)$ and define ϕ on $\bar{\Omega}$ such that $\phi(x) = \max\{u(x), v(x)\}$. We claim that $\phi \in \mathcal{F}(\mu, g)$. First, ϕ is convex in Ω. To see this, let $x, y \in \Omega$ and $t \in [0, 1]$. Since u and v both are convex in Ω, we have

$$u((1-t)x + ty) \leq (1-t)u(x) + tu(y) \leq (1-t)\phi(x) + t\phi(y)$$

and

$$v((1-t)x + ty) \leq (1-t)v(x) + tv(y) \leq (1-t)\phi(x) + t\phi(y).$$

Hence

$$\phi((1-t)x + ty) = \max\{u((1-t)x + ty), v((1-t)x + ty)\} \leq (1-t)\phi(x) + t\phi(y).$$

Also, $\phi \in C(\bar{\Omega})$. This follows from the fact that ϕ can be written as

$$\phi = \frac{(u+v) + |u-v|}{2}$$

given $u, v \in C(\bar{\Omega})$. Moreover, $\phi = g$ on $\partial\Omega$. Let $x \in \partial\Omega$, as $u(x) = v(x) = g(x)$, then $\phi(x) = \max\{u(x), v(x)\} = g(x)$.

It remains to show that $M\phi \geq \mu$ to prove our claim. Consider the sets

$$\Omega_0 = \{x \in \Omega : u(x) = v(x)\}$$

$$\Omega_1 = \{x \in \Omega : u(x) > v(x)\}$$

$$\Omega_2 = \{x \in \Omega : u(x) < v(x)\}.$$

Let E be a Borel set. If $E \subseteq \Omega_1$, then $\partial\phi(E) \supseteq \partial u(E)$. In fact, let $p \in \partial u(E)$, so $\exists\, x' \in E$ such that $p \in \partial u(x')$ i.e.

$$u(x) \geq u(x') + p \cdot (x - x') \quad \forall x \in \Omega.$$

But on $E \subseteq \Omega_1$, $\phi(x') = u(x')$. Thus, we obtain

$$\phi(x) \geq u(x) \geq \phi(x') + p \cdot (x - x') \quad \forall x \in \Omega$$

which implies $p \in \partial\phi(E)$. Similarly, we show that if $E \subseteq \Omega_2$, then $\partial\phi(E) \supseteq \partial v(E)$. Also, if $E \subseteq \Omega_0$, so $\partial\phi(E) \supseteq \partial u(E)$ and $\partial\phi(E) \supseteq \partial v(E)$.

More generally, let $E \subseteq \Omega$ any Borel subset. We write $E = (E \cap \Omega_0) \cup (E \cap \Omega_1) \cup (E \cap \Omega_2)$ and obtain by countable additivity of $M\phi$ and μ that

$$\begin{aligned} M\phi(E) &= M\phi(E \cap \Omega_0) + M\phi(E \cap \Omega_1) + M\phi(E \cap \Omega_2) \\ &\geq Mu(E \cap \Omega_0) + Mu(E \cap \Omega_1) + Mv(E \cap \Omega_2) \\ &\geq \mu(E \cap \Omega_0) + \mu(E \cap \Omega_1) + \mu(E \cap \Omega_2) \quad (\text{as } u, v \in \mathcal{F}(\mu, g)) \\ &= \mu(E). \end{aligned}$$

Step 3. At the end of Step 1, we defined

$$U(x) = \sup\{v(x) : v \in \mathcal{F}(\mu, g)\}.$$

In this step, we show that for each $y \in \Omega$, there exists a sequence $v_m \in \mathcal{F}(\mu, g)$ converging uniformly on compact subsets of Ω to a function $w \in \mathcal{F}(\mu, g)$ with $w(y) = U(y)$.

From Step 1, $\mathcal{F}(\mu, g) \neq \emptyset$. Let $v_0 \in \mathcal{F}(\mu, g)$ and $y \in \Omega$. From definition of $U(y)$, there exists a sequence $v'_m \in \mathcal{F}(\mu, g)$ such that

$$v'_m(y) \xrightarrow[m\to\infty]{} U(y).$$

Consider now v_m on $\bar{\Omega}$ such that $v_m(x) = \max\{v_0(x), v'_m(x)\}$. From Step 2, $v_m \in \mathcal{F}(\mu, g)$ $\forall m \in \mathbb{N}$. Hence $v'_m(y) \leq v_m(y) \leq U(y)$ $\forall m \in \mathbb{N}$. Letting $m \to \infty$, we get

$$v_m(y) \xrightarrow[m\to\infty]{} U(y). \tag{4.14}$$

From Remark 4.2.2, we have $v_m(x) \leq W(x)$ $\forall j \in \mathbb{N}$ where $W \in C(\bar{\Omega})$ convex Aleksandrov solution of problem (4.5), that is v_m is uniformly bounded above. Also, $v_m(x) \geq v_0(x) \geq \inf_{\bar{\Omega}} v_0(x) = M$ $\forall x \in \Omega\, \forall m \in \mathbb{N}$ as $v_0 \in C(\bar{\Omega})$. This implies that v_m is uniformly bounded in Ω. Hence, for the same reasoning we have in Lemma 4.2.3, Step 1, we obtain that v_m is an equicontinuous sequence in compact subsets of Ω. By Arzelà-Ascoli, there exists a subsequence of $(v_m)_m$ that converges uniformly on compact subsets of Ω, say without relabeling that

$$v_m \xrightarrow[m\to\infty]{} w.$$

(4.14) implies that $w(y) = U(y)$. Define $w = g$ on $\partial\Omega$. We claim that $w \in \mathcal{F}(\mu, g)$. First, the convexity of w is a direct result as it is the uniform limit of convex functions. Also, as $v_m \in C(\bar{\Omega})$ and w is the uniform limit on compact subsets of Ω, thus $w \in C(\Omega)$. To show that w is also continuous on $\partial\Omega$, we let $\xi \in \partial\Omega$. We know that

$$v_0(x) \leq v_m(x) \leq W(x) \quad \forall x \in \Omega.$$

Letting $m \to \infty$, we get

$$v_0(x) \leq w(x) \leq W(x) \quad \forall x \in \Omega.$$

However,

$$\lim_{x\to\xi} v_0(x) = v_0(\xi) = g(\xi) \qquad \text{and} \qquad \lim_{x\to\xi} W(x) = W(\xi) = g(\xi)$$

since $v_0, W \in C(\bar{\Omega})$. By Squeeze theorem, we conclude that

$$\lim_{x\to\xi} w(x) = g(\xi) = w(\xi)$$

which implies that w is continuous at ξ with ξ arbitrary in $\partial\Omega$. Therefore $w \in C(\bar{\Omega})$.

It remains to show that $Mw \geq \mu$. Consider $K \subseteq \Omega$ compact. Lemma 3.3.1 implies that

$$Mw(K) \geq \limsup_{m\to\infty} Mv_m(K).$$

But $Mv_m(K) \geq \mu(K)$ $\forall m \in \mathbb{N}$. Thus $Mw(K) \geq \mu(K)$ for any $K \subset \Omega$ compact. More generally, let $E \subseteq \Omega$ any Borel set. We have $Mw(E) \geq Mw(K) \geq \mu(K)$ for any $K \subseteq E$ compact. This implies that

$$Mw(E) \geq \sup_{\substack{K\subseteq E \\ K compact}} \mu(K).$$

Hence, by inner regularity of μ we get

$$Mw(E) \geq \mu(E).$$

Therefore, we conclude that $w \in \mathcal{F}(\mu, g)$. This ends this step.

Step 4. We show that $U \in \mathcal{F}(\mu, g)$. U is convex in Ω as it is the pointwise supremum of convex functions in Ω. To see this, let $x, y \in \Omega$ and $t \in (0, 1)$. We have $v((1-t)x + ty) \leq (1-t)v(x) + tv(y)$ for any $v \in \mathcal{F}(\mu, g)$. Taking the supremum over $\mathcal{F}(\mu, g)$, we obtain

$$U((1-t)x + ty) \leq (1-t) \sup_{v\in\mathcal{F}(\mu,g)} v(x) + t \sup_{v\in\mathcal{F}(\mu,g)} v(y) = (1-t)U(x) + tU(y).$$

Moreover, $U = g$ on $\partial\Omega$ since $v = g$ on $\partial\Omega$ for every $v \in \mathcal{F}(\mu, g)$.

Besides, $U \in C(\bar{\Omega})$. Continuity on Ω follows from the convexity of U and Theorem 1.3.5. To show that U is continuous up to the boundary $\partial\Omega$ we proceed as follows. Let $\xi \in \partial\Omega$, we have $v(x) \leq W(x)$ $\forall x \in \Omega\, \forall v \in \mathcal{F}(\mu, g)$ which implies that $U(x) \leq W(x)$ $\forall x \in \Omega$. Hence we get

$$v(x) \leq U(x) \leq W(x) \quad \forall x \in \Omega.$$

But

$$\lim_{x\to\xi} v(x) = v(\xi) = g(\xi) \qquad \text{and} \qquad \lim_{x\to\xi} W(x) = W(\xi) = g(\xi)$$

since $v, W \in C(\bar{\Omega})$. Hence

$$\lim_{x\to\xi} U(x) = g(\xi) = U(\xi)$$

concluding that U is continuous at ξ with ξ arbitrary in $\partial\Omega$.

It remains to show that $MU \geq \mu$. Recall that

$$\mu = \sum_{i=1}^{N} a_i \delta_{x_i}.$$

We first claim that $MU(\{x_i\}) \geq a_i$ $\forall 1 \leq i \leq N$ with $x_i \in \Omega$ and $a_i > 0$. Fix $x_1 \in \Omega$. By Step 3, we proved that there exists a uniformly bounded sequence $v_m \in \mathcal{F}(\mu, g)$

converging uniformly on compact subsets of Ω to a function $w \in \mathcal{F}(\mu, g)$ with $w(x_1) = U(x_1)$. We have $Mw \geq \mu$, then $Mw(\{x_1\})) \geq \mu(\{x_1\}) = a_1$.

We show that $\partial w(x_1) \subseteq \partial U(x_1)$. Let $p \in \partial w(x_1)$, so

$$w(x) \geq w(x_1) + p \cdot (x - x_1) \ \ \forall x \in \Omega.$$

But $U(x) \geq w(x) \ \ \forall x \in \Omega$ and $U(x_1) = w(x_1)$ obtaining that

$$U(x) \geq U(x_1) + p \cdot (x - x_1) \ \ \forall x \in \Omega.$$

Thus $p \in \partial U(x_1)$. Therefore, we get

$$MU(\{x_1\}) = |\partial U(\{x_1\})| \geq |\partial w(\{x_1\})| = Mw(\{x_1\}) \geq a_1$$

which proves our claim.

More generally, let $E \subseteq \Omega$ any Borel subset. If $x_i \notin E$ for all $1 \leq i \leq N$,

$$\mu(E) = \mu(E \cap \{x_1, ..., x_N\}) = 0 \leq MU(E).$$

If $\exists i_0 \in \{1, .., N\}$ such that $x_i \in E$,

$$\mu(E) = \mu(E \cap \{x_1, ..., x_N\}) = a_{i_0} \leq MU(\{x_{i_0}\}) \leq MU(E).$$

Thus we complete this step.

Step 5. In this step, we conclude our work and show that U is a convex Aleksandrov solution of the problem

$$\begin{cases} \det D^2 u = \mu & in \ \ \Omega \\ u = g & on \ \ \partial\Omega \end{cases}.$$

In Step 4, we proved that $U \in \mathcal{F}(\mu, g)$. It remains to show that $MU \leq \mu$. We first prove that MU is concentrated on $\{x_1, ..., x_N\}$. To see this, we let $x_0 \in \Omega$ such that $x_0 \neq x_i \ \ \forall 1 \leq i \leq N$. Clearly, we can find $r > 0$ such that $B(x_0, r) \cap \{x_1, ..., x_N\} = \emptyset$ and $\overline{B(x_0, r)} \subseteq \Omega$. By Theorem 4.1.2, there exists a unique convex function $v \in C(\overline{B(x_0, r)})$ Aleksandrov solution of the problem

$$\begin{cases} \det D^2 v = 0 & in \ \ B(x_0, r) \\ v = g & on \ \ \partial B(x_0, r) \end{cases}.$$

We define the lifting w of U as follows

$$w(x) = \begin{cases} U(x) & in \ \ \bar{\Omega} \setminus B(x_0, r) \\ v(x) & in \ \ \overline{B(x_0, r)} \end{cases}$$

and we claim that $w \in \mathcal{F}(\mu, g)$. We have $w \in C(\bar{\Omega})$ since $U \in C(\bar{\Omega})$, $v \in C(\overline{B(x_0, r)})$, and $v = U$ on $\partial B(x_0, r)$. Also, $w(x) = U(x) = g(x) \ \ \forall x \in \partial\Omega$. Now, we show that w

is convex in Ω. We have $MU \geq 0 = Mv$ in $B(x_0, r)$. Hence by Comparison principle 3.5.3, we get

$$\min_{x \in \overline{B(x_0,r)}} \{v(x) - U(x)\} = \min_{x \in \partial B(x_0,r)} \{v(x) - U(x)\}.$$

But $v = U$ on $\partial B(x_0, r)$, thus

$$v(x) \geq U(x) \ \ \forall x \in \overline{B(x_0, r)}. \tag{4.15}$$

To study convexity of w, we will consider three cases.
Case 1: Let $x, y \in B(x_0, r)$ and $t \in (0, 1)$, then $(1-t)x+ty \in B(x_0, r)$ since $B(x_0, r)$ is convex. By convexity of v in $B(x_0, r)$, we get

$$w((1-t)x + ty) = v((1-t)x + ty) \leq (1-t)v(x) + tv(y) = (1-t)w(x) + tw(y).$$

Case 2: Let $x, y \in \Omega \setminus B(x_0, r)$. We will study convexity on two different parts of the segment between x and y. On $\{(1-t)x + ty : t \in (0, 1)\} \cap (\Omega \setminus B(x_0, r))$, we have by convexity of U in Ω

$$w((1-t)x + ty) = U((1-t)x + ty) \leq (1-t)U(x) + tU(y) = (1-t)w(x) + tw(y).$$

On $\{(1-t)x+ty : t \in (0, 1)\} \cap B(x_0, r)$, we consider ξ_1, ξ_2 the points of intersection of the segment and $\partial B(x_0, r)$. Thus we can write $(1-t)x+ty = (1-\lambda)\xi_1 + \lambda\xi_2$ with $\xi_1 = (1-\lambda_1)x+\lambda_1 y$ and $\xi_2 = (1-\lambda_2)x+\lambda_2 y$ for some $\lambda, \lambda_1, \lambda_2 \in (0, 1)$. Substituting the last two equalities in the first one gives that $t = \lambda_1 - \lambda\lambda_1 + \lambda\lambda_2$. By convexity of v in $\overline{B(x_0, r)}$ (as $v \in C(\overline{B(x_0, r)})$), convexity of U in Ω, and equality of U and v on $\partial B(x_0, r)$, we obtain

$$\begin{aligned}
w((1-t)x + ty) &= v((1-t)x + ty) = v((1-\lambda)\xi_1 + \lambda\xi_2) \\
&\leq (1-\lambda)v(\xi_1) + \lambda v(\xi_2) = (1-\lambda)U(\xi_1) + \lambda U(\xi_2) \\
&\leq (1-\lambda)((1-\lambda_1)U(x) + \lambda_1 U(y)) + \lambda((1-\lambda_2)U(x) + \lambda_2 U(y)) \\
&= (1-\lambda_1 + \lambda\lambda_1 - \lambda\lambda_2)U(x) + (\lambda_1 - \lambda\lambda_1 + \lambda\lambda_2)U(y) = (1-t)U(x) + tU(y) \\
&= (1-t)w(x) + tw(y).
\end{aligned}$$

Case 3: Let $x \in \Omega \setminus B(x_0, r)$ and $y \in B(x_0, r)$. On $\{(1-t)x + ty : t \in (0, 1)\} \cap (\Omega \setminus B(x_0, r))$, we have by convexity of U in Ω and (4.15)

$$w((1-t)x+ty) = U((1-t)x+ty) \leq (1-t)U(x)+tU(y) \leq (1-t)U(x)+tv(y) = (1-t)w(x)+tw(y).$$

On $\{(1-t)x + ty : t \in (0, 1)\} \cap B(x_0, r)$, we let ξ be the point of intersection of the segment and $\partial B(x_0, r)$. Then we write $(1-t)x + ty = (1-\lambda)\xi + \lambda y$ with $\xi = (1-\lambda_1)x + \lambda_1 y$ for some $\lambda, \lambda_1 \in (0, 1)$. Combining the latter equality with the former one, we get $t = \lambda_1 - \lambda\lambda_1 + \lambda$. Therefore, from (4.16) and convexity of v and U, we obtain

$$\begin{aligned}
w((1-t)x + ty) &= v((1-t)x + ty) = v((1-\lambda)\xi + \lambda y) \\
&\leq (1-\lambda)v(\xi) + \lambda v(y) = (1-\lambda)U(\xi) + \lambda v(y) \\
&\leq (1-\lambda)((1-\lambda_1)U(x) + \lambda_1 U(y)) + \lambda v(y) \\
&= (1-\lambda)((1-\lambda_1)U(x) + \lambda_1 v(y)) + \lambda v(y) = (1-t)U(x) + tv(y) \\
&= (1-t)w(x) + tw(y).
\end{aligned}$$

From Cases 1,2, and 3 , we get that w is convex in Ω.

To show that $w \in \mathcal{F}(\mu, g)$, it remains to show that $Mw \geq \mu$ in Ω. Let $E \subseteq B(x_0, r)$ a Borel subset, then $\partial w(E) \subseteq \partial v(E)$. In fact, let $p \in \partial w(E)$, so there exists $x \in E$ such that $p \in \partial w(x)$. Thus $w(y) \geq w(x) + p \cdot (y - x)\ \forall y \in \Omega$ which implies that $v(y) \geq v(x) + p \cdot (y - x)\ \forall y \in B(x_0, r)$. Thus $p \in \partial v(E)$. Hence

$$Mw(E) = |\partial w(E)| \leq |\partial v(E)| = Mv(E) = 0.$$

Let $E \subseteq \Omega \setminus B(x_0, r)$. We show that $\partial U(E) \subseteq \partial w(E)$. For $p \in \partial U(E)$, there exists $x \in E$ such that $p \in \partial U(x)$. Hence $U(y) \geq U(x) + p \cdot (y - x)\ \forall y \in \Omega$. This gives that $w(y) \geq w(x) + p \cdot (y - x)\ \forall y \in \Omega \setminus B(x_0, r)$ and $w(y) = v(y) \geq U(y) \geq w(x) + p \cdot (y - x)\ \forall y \in B(x_0, r)$ (from 4.16). Therefore, $w(y) \geq w(x) + p \cdot (y - x)\ \forall y \in \Omega$ obtaining $p \in \partial w(E)$. Thus

$$Mw(E) = |\partial w(E)| \geq |\partial U(E)| = MU(E).$$

Generally, let $E \subseteq \Omega$ any Borel subset. Then

$$\begin{aligned} Mw(E) &= Mw(E \cap B(x_0, r)) + Mw(E \cap B(x_0, r)^c) \\ &= 0 + Mw(E \cap B(x_0, r)^c) \\ &\geq MU(E \cap B(x_0, r)^c) \\ &\geq \mu(E \cap B(x_0, r)^c) \qquad (\text{as } U \in \mathcal{F}(\mu, g) \text{ from step 4}) \\ &= \mu(E) \qquad (\text{as } B(x_0, r) \cap \{x_1, ..., x_N\} = \emptyset). \end{aligned}$$

We conclude that $w \in \mathcal{F}(\mu, g)$. Thus, by definition of U, we have $w \leq U$ in $\bar{\Omega}$. However $w = v \geq U$ in $B(x_0, r)$, so $v = U$ in $B(x_0, r)$. Therefore, $MU = 0$ in $B(x_0, r)$ with $B(x_0, r)$ any ball in Ω such that $B(x_0, r) \cap \{x_1, ..., x_N\} = \emptyset$.

Moreover, to show that MU is concentrated on $\{x_1, ..., x_N\}$, we consider the following. First, let $K \subseteq \Omega$ compact such that $K \cap \{x_1, ..., x_N\} = \emptyset$. For each $y \in K$, $\exists\ \epsilon_y > 0$ such that $B(y, \epsilon_y) \cap \{x_1, ..., x_N\} = \emptyset$. Then we know that $\{B(y, \epsilon_y) : y \in K\}$ is an open cover of K which has finite subcover, say $K \subseteq B(y_1, \epsilon_{y_1}), ..., B(y_n, \epsilon_{y_n})$. Hence by countable subadditivity,

$$MU(K) \leq \sum_{i=1}^{n} MU(B(y_i, \epsilon_{y_i}) = 0.$$

More generally, let $E \subseteq \Omega$ any Borel subset such that $E \cap \{x_1, ..., x_N\} = \emptyset$. For each $K \subseteq E$ compact, $MU(K) = 0$ from first case. Thus by inner regularity of MU,

$$MU(E) = \sup\{MU(K) : K \subseteq E \ \ compact\} = 0$$

We conclude that MU is concentrated on $\{x_1, ..., x_N\}$. However, $MU \geq \mu$, hence we can write

$$MU = \sum_{i=1}^{N} \lambda_i a_i \delta_{x_i}$$

with $\lambda_i \geq 1$ for all $1 \leq i \leq N$.

To end this step with proving that $MU = \mu$, we claim that $\lambda_i = 1 \ \forall 1 \leq i \leq N$. Suppose $\lambda_1 > 0$ and without loss of generality consider $x_1 = 0$. Choose $r > 0$ such that $B(0,r) \cap \{x_2, ..., x_N\} = \emptyset$. We have $MU = \lambda_1 a_1 \delta_0$ in $B(0,r)$. Since $\partial U(\{0\}) = \partial U(0)$ is convex set (by Proposition 2.2.1) and $|\partial U(\{0\})| = MU(\{0\}) = \lambda_1 a_1 > 0$, then there exists $p_0 \in \partial U(\{0\})$ and $\epsilon > 0$ such that $B(p_0, \epsilon) \subseteq \partial U(\{0\})$. Define v on $\bar{\Omega}$ such that

$$v(x) = U(x) - p_0 \cdot x.$$

We have $v(x) - v(0) = U(x) - U(0) - p_0 \cdot x$, but $U(x) \geq U(0) + p \cdot x \ \ \forall x \in \Omega \, \forall p \in B(p_0, \epsilon)$. Thus $v(x) \geq v(0) + (p - p_0) \cdot x \ \ \forall x \in \Omega \, \forall p \in B(p_0, \epsilon)$. The latter inequality extends to $\overline{B(p_0, \epsilon)}$ from continuity of v on $\bar{\Omega}$. Apply above inequality for $x \in \Omega$ and $p = p_0 + \epsilon \frac{x}{|x|}$ obtaining that

$$v(x) \geq v(0) + \epsilon |x|.$$

Choose α a constant number such that $v(0) < \alpha < r\epsilon + v(0)$ and define on $\bar{\Omega}$

$$\tilde{v}(x) = v(x) - \alpha.$$

We have $\tilde{v}(0) < 0$ and

$$\tilde{v}(x) - \tilde{v}(0) = v(x) - \alpha - v(0) + \alpha = v(x) - v(0) \geq \epsilon |x| \ \ \forall x \in \Omega.$$

Hence, if $|x| \geq \frac{-\tilde{v}(0)}{\epsilon}$ then $\tilde{v}(x) \geq 0$. Thus

$$\tilde{v}(x) < 0 \implies |x| < \frac{-\tilde{v}(0)}{\epsilon} = \frac{\alpha - v(0)}{\epsilon} < r. \tag{4.16}$$

Consider now

$$w(x) = \begin{cases} \tilde{v}(x) & if \ \ \tilde{v}(x) \geq 0 \\ \lambda_1^{-\frac{1}{n}} \tilde{v}(x) & if \ \ \tilde{v}(x) < 0 \end{cases}.$$

We show that $w \in \mathcal{F}(\mu, \tilde{g})$ where $\tilde{g} = \tilde{v}|_{\partial\Omega}$. Clearly, $w \in C(\bar{\Omega})$ as $\tilde{v} \in C(\bar{\Omega})$. Also, on $\partial\Omega$, $w = \tilde{v}|_{\partial\Omega} = \tilde{g}$. Besides, w is convex in Ω. On the set $\{\tilde{v}(x) < 0\}$,

$$\lambda_1^{-\frac{1}{n}} \tilde{v}(x) > \tilde{v}(x) \tag{4.17}$$

since $\lambda_1 > 1$ and we have $\tilde{v}$ is convex in Ω as U is convex in Ω. To study convexity of w, we consider the following three cases.
<u>Case 1:</u> Let $x, y \in \{\tilde{v} < 0\}$ and $t \in (0,1)$. By convexity of $\tilde{v}$ in Ω, we have $(1-t)x + ty \in \{\tilde{v} < 0\}$ and thus

$$w((1-t)x+ty) = \lambda_1^{-\frac{1}{n}} \tilde{v}((1-t)x+ty) \leq \lambda_1^{-\frac{1}{n}}(1-t)\tilde{v}(x) + \lambda_1^{-\frac{1}{n}} t\tilde{v}(y) = (1-t)w(x) + tw(y).$$

<u>Case 2:</u> Let $x, y \in \{\tilde{v} \geq 0\}$. We work on two different parts of the segment between x and y. On $\{(1-t)x + ty : t \in (0,1)\} \cap \{\tilde{v} \geq 0\}$, we get by convexity of $\tilde{v}$ in Ω

$$w((1-t)x + ty) = \tilde{v}((1-t)x + ty) \leq (1-t)\tilde{v}(x) + t\tilde{v}(y) = (1-t)w(x) + tw(y).$$

On $\{(1-t)x+ty : t \in (0,1)\} \cap \{\tilde{v} < 0\}$, we obtain from the fact that $\lambda_1^{-\frac{1}{n}} < 1$ and convexity of $\tilde{v}$

$$w((1-t)x+ty) = \lambda_1^{-\frac{1}{n}}\tilde{v}((1-t)x+ty) < (1-t)\tilde{v}(x) + t\tilde{v}(y) = (1-t)w(x) + tw(y).$$

<u>Case 3:</u> Let $x \in \{\tilde{v} \geq 0\}$ and $y \in \{\tilde{v} < 0\}$. On $\{(1-t)x+ty : t \in (0,1)\} \cap \{\tilde{v} \geq 0\}$, we have from (4.17) and convexity of $\tilde{v}$

$$w((1-t)x+ty) = \tilde{v}((1-t)x+ty) < (1-t)\tilde{v}(x) + t\lambda_1^{-\frac{1}{n}}\tilde{v}(y) = (1-t)w(x) + tw(y).$$

On $\{(1-t)x+ty : t \in (0,1)\} \cap \{\tilde{v} < 0\}$, we get

$$w((1-t)x+ty) = \lambda_1^{-\frac{1}{n}}\tilde{v}((1-t)x+ty) < (1-t)\tilde{v}(x) + t\lambda_1^{-\frac{1}{n}}\tilde{v}(y) = (1-t)w(x) + tw(y).$$

Therefore, w is convex in Ω.

Now, it remains to show that $Mw \geq \mu$ to obtain that $w \in \mathcal{F}(\mu, \tilde{g})$. Let $E \subseteq \{\tilde{v} \geq 0\} \subseteq \Omega$ a Borel subset. We have $\partial\tilde{v}(E) \subseteq \partial w(E)$. In fact, let $p \in \partial\tilde{v}(E)$ then $\exists\, x \in E$ such that $p \in \partial\tilde{v}(x)$. Hence $\tilde{v}(y) \geq \tilde{v}(x) + p \cdot (y-x)\ \ \forall y \in \Omega$. This implies that $w(y) \geq w(x) + p \cdot (y-x)\ \ \forall y \in \{\tilde{v} \geq 0\}$ and

$$w(y) = \lambda_1^{-\frac{1}{n}}\tilde{v}(y) > \tilde{v}(y) \geq w(x) + p \cdot (y-x)\ \ \forall y \in \{\tilde{v} < 0\}.$$

Thus $w(y) \geq w(x) + p \cdot (y-x)\ \ \forall y \in \Omega$ obtaining $p \in \partial w(E)$. Note that $M\tilde{v} = MU$ by Theorem 3.2.7. Therefore

$$Mw(E) = |\partial w(E)| \geq |\partial\tilde{v}(E)| = M\tilde{v}(E) = MU(E) \geq \mu(E).$$

Let $E \subseteq \{\tilde{v} < 0\}$ then $\partial(\lambda_1^{-\frac{1}{n}}\tilde{v})(E) \subseteq \partial w(E)$. To see this, let $p \in \partial(\lambda_1^{-\frac{1}{n}}\tilde{v})(E)$, so there exists $x \in E$ such that $p \in \partial(\lambda_1^{-\frac{1}{n}}\tilde{v})(x)$. Then

$$\lambda_1^{-\frac{1}{n}}\tilde{v}(y) \geq \lambda_1^{-\frac{1}{n}}\tilde{v}(x) + p \cdot (y-x)\ \ \forall y \in \Omega.$$

Hence $w(y) \geq w(x) + p \cdot (y-x)\ \ \forall y \in \{\tilde{v} < 0\}$ and

$$w(y) = \tilde{v}(y) > \lambda_1^{-\frac{1}{n}}\tilde{v}(y) \geq w(x) + p \cdot (y-x)\ \ \forall y \in \{\tilde{v} \geq 0\}.$$

This implies that

$$Mw(E) \geq M(\lambda_1^{-\frac{1}{n}}\tilde{v})(E) = \lambda_1^{-1}M\tilde{v}(E) = \lambda_1^{-1}MU(E) = \lambda_1^{-1}\lambda_1 a_1\delta_0(E) = a_1\delta_0(E) = \mu(E)$$

since $E \subseteq \{\tilde{v} < 0\} \subseteq B(0,r)$ (see (4.16)) and using Proposition 3.2.8. Consequently $w \in \mathcal{F}(\mu, \tilde{g})$.

Moreover,

$$\begin{aligned}
\tilde{v}(x) &= U(x) - p_0 \cdot x - \alpha \\
&= \sup\{v(x) : v \in \mathcal{F}(\mu, g)\} - p_0 \cdot x - \alpha \\
&= \sup\{v(x) - p_0 \cdot x - \alpha : v \in \mathcal{F}(\mu, g)\}.
\end{aligned}$$

Define

$$v'(x) = v(x) - p_0 \cdot x - \alpha.$$

We show that $v' \in \mathcal{F}(\mu, \tilde{g}) \iff v \in \mathcal{F}(\mu, g)$. Clearly, $v' \in C(\bar{\Omega}) \iff v \in C(\bar{\Omega})$ and v' is convex in $\Omega \iff v$ is convex in Ω. Also, if $Mv' \geq \mu$ then $Mv = Mv' \geq \mu$ and if $Mv \geq \mu$ then $Mv' = Mv \geq \mu$. Let $x \in \partial\Omega$. If $v'(x) = \tilde{g}(x)$ then $v(x) = \tilde{g}(x) + p_0 \cdot x + \alpha$. But

$$\tilde{g}(x) = \tilde{v}|_{\partial\Omega}(x) = U(x) - p_0 \cdot x - \alpha,$$

so $v(x) = U(x) = g(x)$. Conversely, if $v(x) = g(x)$ then

$$v'(x) = g(x) - p_0 \cdot x - \alpha = U(x) - p_0 \cdot x - \alpha = \tilde{g}(x).$$

Therefore,

$$\tilde{v}(x) = \sup\{v'(x) : v' \in \mathcal{F}(\mu, \tilde{g})\}.$$

Thus $w(x) \leq \tilde{v}(x) \ \forall x \in \Omega$ as we proved that $w \in \mathcal{F}(\mu, \tilde{g})$. In particular, $w(0) \leq \tilde{v}(0)$ with $w(0) = \lambda_1^{-\frac{1}{n}} \tilde{v}(0)$ since $\tilde{v}(0) < 0$. Hence $\lambda_1^{-\frac{1}{n}} \tilde{v}(0) \leq \tilde{v}(0)$ which implies that $\lambda_1^{-\frac{1}{n}} \geq 1$ obtaining a contradiction to the fact that $\lambda_1 > 1$. We conclude that $\lambda_1 = 1$ and thus $\lambda_i = 1 \ \forall 1 \leq i \leq N$. Therefore

$$MU = \sum_{i=1}^{N} a_i \delta_{x_i} = \mu.$$

Step 6. We show that U is unique. Suppose there exists $V \in C(\bar{\Omega})$ convex solution to the problem

$$\begin{cases} Mu = \mu & in \ \ \Omega \\ u = g & on \ \ \partial\Omega \end{cases}.$$

By Corollary 3.5.4, we directly get that $U = V$ in $\bar{\Omega}$ and thus U is unique. $\square$

Bibliography

[1] Y. Guan Pengfei; Li, "C1,1 estimates for solutions of a problem of alexandrov," *Communications on pure and applied mathematics*, 1997.

[2] C. E. Gutiérrez, *Optimal Transport and Applications to Geometric Optics*. Springer, 2023.

[3] Y. J.-D. Benamou; Brenier, "Weak existence for the semigeostrophic equations formulated as a coupled monge-ampère/transport problem," *SIAM journal on applied mathematics*, 1998.

[4] C. E. Gutiérrez, *The Monge-Ampére Equation* (Progress in Nonlinear Differential Equations and Their Applications). Springer Science+Business Media, LLC, 2001, ISBN: 978-1-4612-6656-3.

[5] R. T. Rockafellar, *Convex Analysis*. Princeton University Press. 41 William Street. Princeton. New Jersey 08540, 1997.

[6] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford University Press. Inc. New York, 2000.

[7] A. Sard, "The measure of the critical values of differentiable maps," *Bulletin (new series) of the American Mathematical Society*, 1942.

[8] G. B. Folland, *Real analysis : modern techniques and their applications* (Pure and applied mathematics (John Wiley & Sons : Unnumbered), 0079-8185). New York : John Wiley & Sons, c1999, 1999, ISBN: 0471317160 (hc.)

[9] D. A. Harville, *Matrix Algebra From a Statistician's Perspective*. Springer-Verlag New York, Inc., 1997.

[10] J. Shurman, *Calculus and Analysis in Euclidean Space*. Springer International Publishing AG 2016, corrected publication 2019, 2016, ISBN: 978-3-319-49312-1.

[11] W. Rudin, *Principles of mathematical analysis* (International series in pure and applied mathematics). McGraw Hill, 1976, 3rd ed.

[12] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Springer Berlin, Heidelberg, 2013.