THE INVENTION OF DECIMAL FRACTIONS
IN THE EAST AND IN THE WEST

by

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P R E F A C E

Until recently, it was generally agreed that decimal fractions first appeared in Europe. The idea of such fractions, according to Sarton [1], "was in the air in the sixteenth century". While he recognizes the existence of "many examples of it before 1585", he claims that "nobody had become aware of the single validity which they represented", and that it was not until 1585, when Simon Stevin of Bruges published his De Thiende that the idea was "set forth completely and with the utmost lucidity".

In 1948, however, the matter was placed in a new light by the work of the late Paul Luckey of Gönningen, Germany. Luckey announced that some 160 years before Stevin, the Iranian mathematician, Ghiāth al-Dīn Jamshīd al-Kāshī, had discovered decimal fractions, explained them fully and applied them extensively.

Luckey's recognition of al-Kāshī's work is the starting point of this thesis. For the benefit of historians of mathematics in particular, and for the interest of the general reader, we go back to the original source, al-Kāshī's Miftāḥ al-Ḥisāb (The Key to Reckoning), in which his theory and use of the system were expounded. The portions of this work here presented and discussed amply confirm the validity of Luckey's assertion.

In paving the way for the discussion of al-Kāshī's work, a considerable amount of material is first given about the historical background which led to the emergence of decimal fractions. Chapter I is therefore devoted to a description of the development of different number systems which culminated in the use of decimal numbers and fractions.

Chronologically, the earliest users of a place-value system were the Babylonians, whose sexagesimal numbers had become common by about 2100 B.C. But the Hindus, probably at the beginning of the Christian era, were the first to use the decimal place-value system. Their mathematics came within reach of the Muslims during the Abbasid rule in Baghdad and these, in turn, propagated the knowledge of the decimal place-value system in the Middle East and the Mediterranean basin. From there the system was transmitted to Europe, and in Spain found wide acceptance, especially in the

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1. Sarton, 2, p. 174.
2. Luckey, 2, p. 201.

twelfth century, when translations from Arabic were made on a large scale.

Chapter II deals with the invention of decimal fractions in Europe. Equipped with the knowledge of decimals, Western mathematicians made various approaches to the invention of a fractional decimal system. Among those who came close to developing this latter was Bonfils of Tarascon (c. 1340-1377) who gave correct definitions and rules for decimal fractions, but neither illustrated them with examples nor applied them in computations. The real inventor, however, was Stevin who introduced decimal fractions in his De Thiende, defining, explaining and applying them fully.

Chapter III describes the life and time of al-Kāshī including the political and scientific environment in which he flourished. Also his various works are listed here. But the largest part is devoted to the description of his Miftāḥ al-Ḥisāb, with which our study is primarily concerned, and in which al-Kāshī gives a complete explanation of his invention.

In Chapter IV of this thesis a facsimile of the Sixth Chapter of the Arabic source itself is reproduced. In this portion of al-Kāshī's work, definitions of decimal fractions, together with their rules and different applications are found. It is accompanied by a translation into English and is followed by a commentary on some points in it which may not be clear to the non-specialist reader.

The thesis concludes that, although Stevin contributed greatly to the development of mathematics, al-Kāshī's work on the same subject is equally advanced, if not superior. Stevin's fame, in connection with the invention of decimal fractions, is perhaps due to the fact that his book was translated into several languages. The possibility of al-Kāshī as having influenced Stevin's work is considered extremely unlikely.

Synopsis of the Thesis on

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AND IN THE WEST

by Naila Leyla Rajai

Until recently, it was generally agreed that decimal fractions first appeared in Europe. The idea of such fractions, according to Sarton, "was in the air in the sixteenth century". While he recognizes the existence of "many examples of it before 1585", he claims that "nobody had seen the inner relationship of these things, nobody had become aware of the single validity which they represented", and that it was not until 1585, when Stevin of Bruges published his De Thiende that the idea was "set forth completely and with the utmost lucidity".

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Chronologically, the earliest users of a place-value system were the Babylonians, whose sexagesimal numbers had become common by about 2100 B.C. But the Hindus, probably at the beginning of the Christian era, were the first to use the decimal place-value system. Their mathematics came within reach of the Muslims, during the Abbasid rule in Baghdad, and these, in turn, propagated the knowledge of the decimal place-value system in the Middle East and the Mediterranean basin. From there the system was transmitted to Europe, and in Spain found wide acceptance, especially in the twelfth century, when translations from Arabic were made on a large scale.

Chapter II deals with the invention of decimal fractions in Europe. Equipped with the knowledge of decimals, Western mathematicians made various approaches to the invention of a fractional decimal system. Among those who came close to developing this latter was Bonfils of Tarascon (c. 1340-1377) who gave correct definitions and rules for decimal fractions, but neither illustrated them with examples nor applied them in computations. The real inventor, however, was Stevin who introduced decimal fractions in his De Thiende, defining, explaining and applying them fully. With his use of the same figures to denote integers, units and fractions, Stevin employed a special symbolism for writing fractions. He placed small circles either above or to the right of the ones' digit in the integer part of the number, and likewise for every digit in its fractional part. Inside these circles different numbers were inserted: zero for the ones and 1, 2, 3, etc. for the tneths, hundredths, thousandths, ... etc. A shorter notation, however, was also used by Stevin and this was to put on the right of the whole number the circle referring to its lowest digit. The number 234.25, for example, would be written by him in any of the following ways:

234 ⓪ 2 ① 5 ② ,

or 234⓪ 2① 5② ,

or 23425 ② .

Chapter III describes the life and time of al-Kāshī -- including the political and scientific environment in which he flourished. Also his various works are listed here. But the largest part is devoted to the description of his Miftāḥ al-Ḥisāb, with which our study is primarily concerned, and in which al-Kāshī gives a complete explanation of his invention. His method of denoting the place-value of every digit was analogous to that of Stevin, but he expressed the place of each digit in words instead of figures. His version of 234.25 would appear as:

الصحاح	الأعشار	ثاني الأعشار		units	tenths	2nd tenths	
٢٣٤	٢	٥	(i.e.	234	2	5	),

or as ٢٣٤٢٥ ثاني الأعشار (i.e. 23425 2nd tenths).

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CHAPTER I

HISTORICAL DEVELOPMENT OF THE DECIMAL PLACE-VALUE SYSTEM

1. Introduction

How can numbers be represented and recorded ? This was a problem that faced each people of the human race at the threshold of its civilization.

The primitive answer to this question was different among different peoples, Babylonians, Egyptians, Phoenicians, Greeks, Romans, and Mayas. But a principle common to all of these could be stated (in modern terminology) as follows:

To denote any positive integer or to record the number of a class of objects, use a group of marks, all of the same kind, in one-to-one correspondence with the set of objects. Thus, if a person wanted, for instance, to remember that he had bought three books, he might draw three similar marks in any position he wanted.

Of course, it is natural that the type of mark adopted should be affected by the different writing materials available to different cultures. The people of the Tigris-Euphrates region, for example, whose commonest writing material was clay, found a convenient mark to be that formed by pressing a cylindrical stylus sideways into the soft clay.[1] The Egyptians, who used papyrus, used vertical strokes in

1. Neugebauer, 1, p. 51.

designating their numbers[1]. For similar reasons the Greek acrophonic[2] notation, as well as the notations of the Romans, Phoenicians[3], and Hindus[4], used vertical strokes, while the ancient Maya made use of a dot notation.

For small integers, this type of notation has the great advantage of simplicity. Also, addition and subtraction were easily performed on numbers thus expressed. The addition of three and four, say, would simply be the drawing of a group of three marks along with a group of four similar ones. Neither would it be difficult to multiply or divide numbers expressed in this fashion.

But if the notation was easy to use in operating with small numbers, it would quickly become prohibitively cumbersome if integers of any size were involved. It would be inconvenient, for instance, to mark 235 marks. This disadvantage led to more concise notations using a base.

In all the cultures mentioned above, people eventually realized that the notation could be made more compact by introducing a new symbol, this to stand for a fixed number of marks, the fixed number being the base.

In Sumerian clay texts (c. 3000 B.C.)[5], the inclined strokes previously described were used in writing numbers from one to nine. By a vertical pressure of the stylus a round mark was produced, and

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1. Neugebauer, 1, p. 82.
2. Tod, 1, p. 27.
3. Kenrich, p. 229.
4. Datta & Singh, p. 20.
5. Neugebauer, 1, p. 51.

this came to be adopted as the symbol for ten. Later, when the triangular prismatic stylus was used for all writing, forms of the symbols were changed accordingly.

The Egyptians, who used vertical bars in expressing numbers between one and nine inclusive, symbolized ten by a mark resembling an inverted capital U[1]. Similarly, the Phoenicians used vertical bars for the same purpose as the Egyptians[2]. But for the number ten they used horizontal bars, either simple or hooked at the right end.[2]

The Greek aerophonic notation, instead of writing five of the strokes denoting units, introduced the symbol Γ [3], the initial letter of the word for five, pente. In the Roman notation, also, the five strokes were replaced by a new symbol, V or Λ .[4]

Again, the base five is asserted to have been used by the Maya Indians of Yucatan. But here the five dots were replaced by a horizontal bar, and this symbol, in varying combinations with dots, was used to express all higher numbers.[5]

In the cases cited above, we notice that the bases were either ten or five. Doubtless this was because the five fingers on each hand, ten in all, made up the most convenient and available set for counting other sets of objects. It is interesting to note that the symbol for five in the Roman notation may perhaps have been derived from a conventionalized picture of the open hand, in which the thumb is held apart from the four joined fingers.[6]

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1. Chace, plates II & VI.
2. Rawlinson, p. 383; Kenrick, p. 229.
3. Heath, pp. 29-30.
4. Sanford, p. 83.
5. Morley, p. 279.
6. Smith, 1, vol. II, p.58.

Having introduced a new symbol to stand for groups of the marks which denote units, it was an easy next step to repeat the process and set up a third symbol to stand for groups of the second one, then a fourth symbol denoting groups of the third, and so on. The Egyptians, for instance, instead of drawing ten inverted U's representing a hundred, replaced them by a spiral[1]. Then they combined ten spirals and introduced a new symbol picturing an upright lotus plant. Following the same principle, they chose for ten thousand a picture of a man holding up his arms as if in astonishment, the hieroglyph of the god Ḥḥ[2].

Altogether, this method is simple, the representations are easy to learn, and computations are easily carried out. It has, however, two disadvantages. First, the writing of large numbers still required a large amount of space, and secondly, with larger numbers, ever newer groups had to be formed, and therefore additional symbols had to be invented to represent the new grouping. That is, apart from the five symbols used for numbers up to 100,000, a new symbol would be required for 100,000, another for a million, and so on.

The Greek alphabetic notation described below overcame one of these disadvantages. The Greeks, in the eighth or ninth century B.C. took the twenty-two letters of the Semitic alphabet and adapted it to the writing of their own language. They added to it the five letters Υ, Φ, Χ, Ψ, and Ω, whose origin is either obscure or unknown, and which were not written in the Semitic languages.[3] As time went by, three

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1. Heath, p. 27; Chace, plates II & VI. 2. Heath, p. 27.
3. Heath, pp. 31-32; Encyclopaedia Britannica, vol. I, pp. 680-682.

of the twenty-seven letters, i.e., Ϡ (sampi), Ϝ (digamma) and Ϙ (qoppa), were eventually discarded. After this had occured, the Greeks began utilising their whole alphabet for representing numbers. The first nine letters were taken to stand for the numbers 1, 2,...9; a second set of nine letters stood for 10, 20,......90. And coming to the representation of the hundreds, they realised their need of three letters more. So they reinserted the three archaic letters mentioned above[1]. When the Greeks reached the counting of thousands, they resorted to the same letters but with inclined strokes to differentiate them from numbers below a thousand. The letter A, for instance, which normally represents the number one, would read 1000 if written combined with a stroke ʹA; ͵A . Tens of thousands[2] were represented by the letter M with the unit, so that MB, BM, would read two ten thousands, or 20,000.

The advantages of this alphabetic notation over the group symbols method are that it is more concise and simple. The number 999, for example, which would be written with twenty-seven Egyptian symbols or with fifteen acrophonic ones, could be represented by the three letters ϠϘΘ only. Moreover, the writing of letters is easier than the drawing of ideographs. Also the use of such numerals is not difficult for computing purposes, once their meanings and values are automatically recalled. But, compensating for those advantages, the alphabetic system has two disadvantages. First, it shares the second

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1. Heath, pp. 32-34; Encyclopaedia Britannica, Vol. 1, pp. 680-682; Tod, 2, pp. 126-37.
2. Heath, p. 39; Tod, 2, p. 126-7.

disadvantage of the Egyptian system in that it needs additional symbols for writing larger numbers. Secondly, it involves twenty-seven different symbols for numbers below 1000, whose values have to be learned by heart and whose use requires a great effort of memory.

This method of using letters of the alphabet to formulate a number system was apparently originated by the Greeks[1]. Later, the Arabs adopted it and used the 28 letters of their abjad alphabet, in the same way, forming the so-called jummal system. Out of these letters nine were used for the units, nine for the tens, nine for the hundreds and one for the 1000 [2]. But the system which eliminated all the disadvantages of the preceding ones is the decimal place-value system. Before attempting to describe the development of this latter, let us digress temporarily from the strictly historical account in order to give the theoretical characteristics of place-value systems in general.

2. Place-Value Systems in General

To simplify the matter for the reader, it is convenient to begin with a familiar number expressed in our present decimal system. Take, for instance, the number N = 7096 in which 7, 0, 9, 6 are what we call digits. It is to be noticed that each digit (1) has an intrinsic value less than ten, the base, and (2) a positional value depending on

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1. Heath, p. 32.

2. Heath, p. 32; Tod, 2, p.126.

its place in the sequence. This sequence really stands for the finite series

$$N = 7 \cdot 10^3 + 0 \cdot 10^2 + 9 \cdot 10^1 + 6 \cdot 10^0 ,$$

that is, a polynomial in ten, the coefficients of which are the positive integers 7, 0, 9 and 6. In general, using for similar coefficients the symbols: d_k, d_{k-1}, ..., d_1, d_0, any positive integer may be expressed as follows:

$$N = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \ldots\ldots\ldots\ldots + d_0 \cdot 10^0$$

or, more compactly, as

$$d_k d_{k-1} \ldots\ldots d_2 d_1 d_0 \quad ,$$

where d_k is a symbol for a digit in the $(k + 1)$ th place. That is, d_0 stands for the first digit, d_1 stands for the second digit, and so on. There is no possibility of ambiguity, because the proper power of ten corresponding to any particular digit is given by the place of the digit in the sequence, provided that whenever a power of ten fails to appear in the series the empty place is occupied by a special symbol, the zero. The necessity of having a symbol in every place makes the zero a prerequisite to the decimal place-value system, hence its importance. This need did not exist in the systems previously described.

The role played by the number ten in this system is not unique. Any positive integer $b > 1$ can also serve as a base. Paralleling what has been said above, a positive integer can be expressed in such a system by means of an ordered set of at most b different symbols called digits. The set of digits is:

$$0, 1, 2, \ldots\ldots\ldots, b-2, b-1 .$$

The number will be expressed by $d_k d_{k-1} \ldots\ldots d_1 d_0$, understood as the polynomial in b,

$$d_k \cdot b^k + d_{k-1} \cdot b^{k-1} + \ldots\ldots + d_0 \cdot b^0$$

or, more compactly, by $\sum_{i=0}^{k} d_i b^i$;

d_i is the coefficient of b^i, and its position in the ordered set determines the power of b and therefore determines its contribution to the value of the number.

In general, for a fixed b any positive integer N can be uniquely represented by such a sequence. Divide N by b to obtain a quotient q_1 and a remainder $d_0 < b$. Now divide q_1 by b to get the quotient q_2 with remainder d_1, and so on until a quotient $q_{k+1} = 0$ is reached. Then the remainders $d_k, d_{k-1}, \ldots\ldots, d_0$ are the digits of the desired representation, $d_k d_{k-1} \ldots\ldots d_1 d_0$. This set is unique, since all the digits resulted from a series of divisions, each of which gives a unique result.

To show that the process is valid, express it symbolically as:

$$\frac{N}{b} = q_1 + d_0$$

$$\frac{q_1}{b} = q_2 + d_1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{q_{k-3}}{b} = q_{k-2} + d_{k-3}$$

$$\frac{q_{k-2}}{b} = q_{k-1} + d_{k-2}$$

$$\frac{q_{k-1}}{b} = q_k + d_{k-1}$$

$$\frac{q_k}{b} = d_k$$

If one solves for q_k in the last step and substitutes its value in the step above it he gets

$$q_k = bd_k$$
$$q_{k-1} = b^2d_k + bd_{k-1} \quad .$$

Repeating the process successively there results

$$q_{k-2} = b^3d_k + b^2d_{k-1} + bd_{k-2} \quad ,$$

....

....

and finally, $N = b^kd_k + b^{k-1}d_{k-1} + \dots\dots\dots + d_0$,

a polynomial of the required type.

Conversely, any polynomial in b written in the fashion mentioned above has a unique value for a fixed b, its coefficients also being fixed.

It is to be pointed out that any integer, no matter how large, can be represented by a finite number of different symbols, that is, the symbols for the integers less than b. This was not a property of any system heretofore described.

In all the preceding it is supposed that the exponents of the base b are either zeros or positive integers. Hence the characteristics stated above apply to integers only. But the inclusion of the negative exponents enables us to represent fractions as well as integers. For example, the expression $N = 7096.23$ is the decimal representation of the rational mixed fraction whose value is

$$7 . 10^3 + 0 . 10^2 + 9 . 10^1 + 6 . 10^0 + 2 . 10^{-1} + 3 . 10^{-2} \quad .$$

In general, the expression $d_kd_{k-1}\dots\dots d_1d_0.\dots\dots d_{-m}$ stands for

$$N = d_k.b^k + d_{k-1}.b^{k-1} + \dots\dots + d_0b^0 + \dots\dots + d_{-m}b^{-m}$$

$$= \sum_{i=-m}^{k} d_i b^i \quad ,$$

where m and k are positive integers or zero and N is a rational number. It is to be noted here that in any representation of a fraction a mark of some sort is necessary for separating the integer part of the mixed number from its fractional part, that is for showing the end of the positive powers of the base and the beginning of the negative ones. In decimal fractions this mark is the familiar decimal point. It can be shown that, given a base b, there is a unique place-value representation for any real number, except that the number of digits may be infinite. Moreover, some sort of convention must be adopted to prohibit the use of such representations as 1.999 ..... to represent the same number as 2.000 .... .

In the previous illustrative example we used the decimal scale with which the reader is already acquainted. It may be of interest also to give an illustration with a different base, for example seven, the base of the so-called septimal system.

According to the method of representing numbers in such a system, if we were to express the integer $N = 500$ in the septimal system, we divide successively by seven and get the corresponding remainders: 3, 1, 3, 1, the digits of the equivalent septimal integer. Now, reversing the order of these digits we get $N = 1313$, the desired answer. This, of course, stands for

$$1 . 7^3 + 3 . 7^2 + 1 . 7^1 + 3 . 7^0 \quad ,$$

a polynomial in seven.

Also, because of its great historical interest, and in connection with our present work, we give an example in the place-value system with base sixty, known as the sexagesimal system. In representing its digits, which run from zero to fifty-nine inclusive, one would expect to have sixty different symbols. But to avoid learning a large number of words and symbols, the decimal representation of numbers up to fifty-nine will be used. This follows the practice of the Babylonians, Greeks, and Muslims in their use of the sexagesimal system. To eliminate the ambiguity caused by two-number digits, a comma will be inserted between each two successive digits. A semicolon will be used to indicate the sexagesimal point. Thus the decimal number given in the first example, 7096.23 , can be expressed in the sexagesimal system as 1,58,16;13,48 , where 1, 58, 16, 13, and 48 are the digits, each less than sixty.

By way of comparison between systems with different bases, we notice that the bigger the base the more compact is the expression of a given integer in that system. The integer 32406, for example, may be stated in the dyadic system (b = 2) as 1 110 010 111 111 , while in the sexagesimal system it is 9,0,6. The number of symbols used in the first expression are two (i.e. for zero and one), while in the second a greater number of symbols is needed (i.e. for sixty digits). But compensating for this, the system with the larger base has the disadvantage of requiring a more elaborate multiplication table. For operations with base sixty, for instance, one needs a table containing products up to 59 x 59 , while for those with base two the multiplication table contains only the products 0 . 0 = 0, 1 . 0 = 0, and 1 . 1 = 1 .

3. The Development of the Sexagesimal Place-Value System

We now resume the historical discussion interrupted in the section above. While this study is primarily concerned with the decimal system, it should be remarked that several other place-value systems have in fact come into use in the development of mathematics. One of these may have been the vigesimal system with base twenty, allegedly invented in the fourth century A.D. by the Mayas of Central America.[1] This system had no effect on the development of the decimal place-value system, and it disappeared with the Maya civilization a long time ago. But of much greater interest for our purposes is the sexagesimal system of the Babylonians, because of its probable influence on the Hindu development of the decimal system. Moreover, the first place-value system to emerge was that of the Babylonians, whose oldest known texts date back to 2100 B.C.[2]. An outline of its development resolves itself into an attempt at answering two basic questions:

(1) Why was sixty, of all numbers, chosen as a base ?

(2) How did the idea of position and place-value come out of the primitive Babylonian methods of indicating numbers ?

The reader should be cautioned that, in the following paragraphs, the essential arguments have been greatly simplified. The conclusions, however, are not purely conjectures, but were based in the first place on the reading of actual inscriptions.

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1. Morley, pp. 274-281.

2. Struik, p. 24.

The Sumerians, the first civilized people in the Tigris-Euphrates valley, used their numerals for expressing measures of length, area and weight.[1] For any one of these measures they had a set of natural and standardized, but independent and differently symbolized units. Some of these units would be large for measuring large quantities, others much smaller. For example, the unit of weight used by a jeweller might not be the same as that used by a grocer. As another instance we have the <u>cubit</u>, an arm's length, as well as the width of the <u>finger</u>, both used as measures of length. The only fractions of these units that were used, at least at the time considered, were $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$ and sometimes $\frac{5}{6}$. These had also different symbols, and constituted a category separate from the integers.

At first the Babylonians recorded their measures in terms of each unit by using the individual symbols for $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{2}{3}$, and by the repetition of the symbols for 1 and 10. But a system like that of the Babylonians, whose scope increased with the development of their civilization, could not remain stationary in its primitive form. So, when the larger units were overlapped by the smaller ones, the Babylonians realized the necessity of <u>normalizing</u> the different systems used with the same kind of measures.[2] To use again the hypothetical example above, a jeweller whose business became sufficiently large would find that he had quantities of gold, say, heavy enough to be weighed in the

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1. Neugebauer, 1, p. 90.
2. Neugebauer, 1, pp. 90-96.

grocer's units. This meant that, in every set of units, the larger unit had to be defined in terms of an integer number of the smaller one so that the former could easily be converted into the latter. And it was natural that the relation between the two units should be influenced by the primitive Babylonian system of whole numbers and fractions.[1]

In order to have a unit measure $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{2}{3}$ of a larger unit easily, it is necessary that $\frac{1}{2}$ and $\frac{1}{3}$ of the larger be an integer number of the smaller. In other words, the number of smaller units contained in the larger should have the factors two and three, i.e., it should be divisible by six. Moreover, since ten was used as the largest natural number to be represented by a single symbol, it was desirable that the coefficient of the larger unit also contain the number ten as a factor. And with these factors $2 \times 3 \times 10 = 60$ the sexagesimal base probably arose.[2] It is to be noticed that the coefficient thirty also satisfies the conditions and was used for converting some units.

As for the principle of place-value, "it seems to have resulted from the fact that the Sumerians, in contrast to other peoples, used symbols of the same shape, but made larger, for denoting larger units of the same type of measure. For example, in one system, a small semicircular impression in the clay denoted one of a certain unit while a larger semicircle stood for sixty of the same units. Then 123 of

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1. Neugebauer, 1, pp. 90-96.
2. Neugebauer, 1, p. 107.

these units would be indicated by two large and three small semicircles, a composite symbol having no place-value. But in the course of time, the scribes came to make all the individual symbols of the same size, remembering however that the group of two semicircles to the left, say, still stood for large semicircles, i.e., sixties. As soon as this convention had been adopted a place-value system had been invented." [1]

At this stage, the system lacked two things. First, it did not contain a sexagesimal point, for which the Babylonians never had a special symbol.[2] Secondly, the system needed an additional symbol to show the absence of a digit. In the old Babylonian period no sign for zero was in use except, perhaps, occasionally a blank space was employed.[3] In these texts[2], for instance, the writing of six symbols to the left of nine similar ones was used for 6,0,9 but involving no sign to indicate that this symbol does not mean 6,9. The exact number had to be decided by other means in the text. Such a representation had another kind of ambiguity, for it was possible to read it 15, since it did not involve a sign for separating two adjoining digits from each other. In general, to eliminate this ambiguity, they introduced the expedient of joining together in a group all the symbols making up a single digit, and if the groups of symbols did not actually touch each other, two digits were indicated.[2] But in later times they solved this problem of separating two successive digits by using a new symbol, a pair of angular marks.[4]

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1. Lecture Notes, Maths. 216: History of Mathematics, A.U.B., 1950-51.
2. Neugebauer, 1, p.5.
3. Neugebauer, and Sachs, pp. 34-35.
4. Neugebauer, 2, pp. 213-15, and 3, p. 267.

The same symbol was eventually used for nonexisting inner digits. This use of the mark as a zero symbol is found in the texts of the oldest Persian times, that is between 600 and 300 B.C.[1], and in Seleucid texts somewhat more recent than 300 B.C.[2]

So the Babylonian place-value system reached its highest development 3000 years after being invented[3]. It had the advantage of having the notion of place-value in it and a very flexible numeral notation[4]. This advantage helped in the development of Babylonian astronomy which was, with the sexagesimal system, adopted by the Greek astronomers. Through their astronomy, these latter transported the system afterwards to India, preparing the way for the invention of our decimal place-value system there.[4]

4. Invention of the Decimal Place-Value System in India

Although there is little doubt that the decimal place-value system originated in India, details of its emergence cannot be given. This is in contrast to the Babylonian development, but it is typical of the history of Hindu mathematics in general, and results from the fact that available sources are hard to read and have mixtures of materials of different dates. Moreover, few historians of mathematics are

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1. Neugebauer, 1, p.5, and 2, pp. 213-15.
2. Neugebauer, 2, pp. 213-15, and 3, p. 267.
3. Neugebauer, 3, p. 267.
4. Neugebauer, 2, p. 213-15.

Sanskrit scholars, hence they must depend on secondary materials, always an unsatisfactory situation.

Before the invention of the decimal place-value system, the Hindus were using primitive systems of the types mentioned above in article 1. In inscriptions of King Asoka dated between 257-232 B.C.[1], for example, the numeral symbols were merely vertical marks[2]. In the Śaka inscriptions, possibly of the first century B.C., numerals employing mixtures of the bases 4, 10 and 20 are found[2]. It is to be assumed that the decimal system was invented a long time before the inscriptions were made.

The use of very large numbers became, for some reason, popular in connection with the Hindu religion[3], and this may have stimulated progress in representing numbers. The base ten appears most frequently in Hindu sources, as it does with many others. But one peculiarity of the Indian usage was that, eventually, it gave individual names to all the successive powers of ten up to the eighteenth power[4], whereas the people of other countries were satisfied with finding names for only the first few powers of ten — the Romans reaching up to 10^3 and the Greeks using 10^4 as well.

This kind of numeration, the allotment of a different name for every power of ten, can bring out the place idea, while the Greek myriads and the Roman thousands tend to lead away from the decimal

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1. Smith, 2, p. 145.
2. Smith and Karpinski, p. 20; Smith, 2, p. 145.
3. Datta and Singh, p. 38.
4. Datta and Singh, p. 9.

notation. This is also true of the modern English manner of reading numbers. For instance, the number 52,356,189, which is read in English: fifty-two *million*, three *hundred* fifty-six *thousand*, one *hundred* eighty-nine, would be read by the Hindus somewhat as: pañca (five) *kōtis*, devi (two) *prayutas*, tri (three) *lakṣas*, pañca (five) *ayutas*, ṣaṭa (six) *sahasra*, eka (one) *śata*, astra (eight) *daśan*, nava (nine).[1] And by deleting the numeral denominations for powers of ten and replacing the remaining words by symbols, the Hindus reading can easily be transformed into a place-value representation, while the same operation with the English reading does not give a similar result. Thus, the Hindu method of numeration leads easily into the decimal place-value idea, and it is possible that it influenced the Hindu invention. But this is a conjecture.

Another conjecture is that the system evolved from the use of the abacus, a board or frame divided into columns, each supplied with pebbles, disks or other counters, and representing one of the different decimal orders. The first column, for example, represents the units, the second the tens, the next the hundreds, and so on. This instrument can be used for calculations as well as for the representation of numbers. The number 125, for example, can be represented by separating one, two, and five counters each from the other counters of the hundreds', tens' and units' column respectively.

Thus the idea of a base and of place-value is inherent in the

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1. Datta and Singh, p. 9; Smith and Karpinski, p. 42.

abacus, and by deleting the columns and substituting symbols for numbers of counters, taking into consideration the successive powers of ten related to the columns, a place-value representation can be reached. It is only necessary to indicate empty columns by a zero symbol to complete the transition.

This characteristic of the abacus has led Gandz and others to claim that the use of the abacus in India, "historically the home of the abacus", promoted the development of the decimal place-value system[1]. The theory is attractive; it needs only to be supported by concrete evidence. But as to this, Datta, a historian of Hindu mathematics, says:

"Until now no mention of the existence of the abacus, direct or indirect, has been traced in any Indian literature so that it may be taken without any fear of contradiction that if an abacus was ever in use among the learned men in India, it was discarded long ago." [2]

Substantially the same position is taken by Kaye, another student of Hindu mathematics whose opinions in general have been violently opposed by Datta. So the theory of abacus influence must be regarded as not proved. In fact, Sarton, for example, not only denies the influence of the abacus, but claims that it inhibits the invention of such a system[3], on the ground that, if a people had a convenient instrument such as the abacus, they would be satisfied with its use and would never think of replacing it by an equivalent means of reckoning.

The question cannot be finally resolved. It is relevant to

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1. Gandz, 1, p. 308.
2. Datta and Singh, p. 309.
3. Sarton, 2, pp. 163-67.

remark, however, that in spite of the masses of Mesopotamian archeological material available, there is no reason for thinking that the Babylonians used the abacus. Nevertheless, they invented a place-value system. Moreover, the Romans[1] and Greeks[2], whose use of the abacus is undoubted, failed to develop a place-value system of their own.

Leaving the question of how the decimal place-value system was invented, the following facts strengthen the assertion that, however it came about, the Hindus were the actual inventors. First, the earliest inscriptions containing numerals expressed in the decimal place-value system are of Indian origin[3]. Secondly, the shape of the digit symbols used in India with the place-value idea are clearly derived from number symbols used there with more primitive number systems.[4] Thirdly, the Hindu numeral symbols 1, 2, 3, ... 9, used with the place-value and the zero symbol were adopted as the standard symbols for those numerals in regions taking over the decimal place-value system from them.[5]

The question of when the invention was made raises another problem. Documents containing the "earliest" surviving use of the new system in India are dated differently by different writers. Bühler refers to the Gujara grant plate (from Sankheda) dated Cedi Samvat 346 which corresponds to 594 A.D.[6] The claim has been made that a much

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1. Smith, 1, vol-1, p. 165; Cajori, p. 64.
2. Smith, 1, p. 161; Cajori, p. 52.
3. Datta and Singh, p. 48.
4. Datta and Singh, p. 49.
5. Archibald, p. 49.
6. Datta and Singh, p. 40; Smith and Karpinski, p. 46.

earlier reference to place-value was given in literature in 100 B.C. Datta refers to the Bakhshâlî manuscript of the second century A.D., where both the place-value system and the zero are used in calculations.[1] The date of this manuscript, however, is strongly questioned by Smith and Karpinski[2], and Cajori[3]. In fact, Thibaut[4] places it between 700 and 900 A.D. Smith, Karpinski and Cajori claim that the zero was not used before 500 A.D. and each emphasizes that its "earliest undoubted occurence was in an inscription at Gwalior, dated Samvat 933 (876 A.D.)."[5]

One might expect the above as definite evidences of general use of the system, and yet hold that the actual invention was much earlier. Datta and Singh estimate the time of invention by the following method. They assume that in any culture the lapse between the invention of a number system and its coming into general use is approximately constant. They ascertain this constant by observation of systems in which both dates are known. Then, by subtracting that time lapse from the Hindu date of general adoption, they obtain an approximate date for the invention.

This method is crude at best. Taking the example of the Greek alphabetic notation, they note that, although it was invented as early as the seventh century B.C., it did not come into common use before the second century A.D., or after the lapse of some eight

1. Datta and Singh, p. 86.
2. Smith and Karpinski, p. 43.
3. Cajori, p. 89.
4. Smith and Karpinski, p. 43.
5. Smith and Karpinski, p. 52; Cajori, p. 89; Archibald, p. 29.

centuries. Similarly, they assert that the introduction of the previously invented Hindu numerals into the Near East took place in the eighth century A.D., when the Muslims were at the height of their intellectual activity, but that these numerals did not receive common acceptance for five or six centuries. If one allows an additional two or three hundred years for the period of development of the system, the resulting eight hundred years is the same total time as was noted above for the Greeks. The same remarks apply also to the spread of the Hindu system in Europe. Now, applying this to the Hindus, they subtract eight centuries from their date for the common adoption of the decimal place-value system in India, the eighth century A.D., and arrive at a date for the invention of the system as having taken place at about the beginning of the Christian era. The estimate made by Sarton[1] agrees with this date.

Another question in connection with the decimal place-value system is that of whether or not the already existing Babylonian place-value system influenced the Hindu invention. There can be no doubt of extensive general scientific influence of Mesopotamia on India, principally through the medium of Greek science. Of this there is no lack of proof embedded in Hindu scientific works themselves. Following are some examples.

Most important of all for our purposes, the Hindus used the sexagesimal system, and they must have acquired it from the Babylonians

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1. Sarton, 2, p. 164.

(Sumerians), its inventors[1]. In the "middle period" of Hindu astronomy (c. 400 B.C. to 400 A.D.)[2] the ratio between the length of the longest day and the shortest day was taken as 18:12, the same ratio used by the Babylonians[3]. This is a poor approximation for the parts of India in which it was used. Hence it is to be presumed that the ratio was borrowed from the Babylonians.

In addition to the purely Babylonian borrowings there are many of Greek origin. For example, the length of the anomalistic month taken by the Greeks as 27;33,20 days was used by the Hindus.[4] Also, Hindu astronomy used the same length of year as the Greek Hipparchus of the second century B.C.[5]. The theory of epicycles, basic in Greek astronomy, was taken over by the Hindus.[6]

This influence may have been transmitted in two ways. First, trade which was carried on between Mesopotamia and India from the earliest times, may have promoted the passage of scientific facts. Secondly, a much more direct influence was begun by the Indian campaign of Alexander the Great in 326-5 B.C.[7] After his death in Babylon in 323 B.C., his empire broke up into three parts: Egypt, the Seleucid Empire made of Mesopotamia and Syria, and Macedonia[8]. In India, the Greeks could not maintain themselves as conquerors beyond the middle

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1. Kaye, p. 19; Struik, p. 24.
2. Kaye, p. 7.
3. Kaye, p. 9,19; Schmidt, p. 210.
4. Kaye, p. 59.
5. Kaye, p. 61.
6. Kaye, pp. 9 & 83.
7. Encyclopaedia Britannica, vol. I, p. 568-9.
8. Struik, p. 56.

of the first century B.C.[1] But in spite of that the indirect influence of those who remained in the neighborhood of India, especially in the Seleucid Empire, still continued for a long time. This period was long enough and the contact sufficiently close for the transplanting of many Greek and Babylonian ideas.

Thus, the period of intensive Greek influence coincides approximately with the probable period of invention of the decimal place-value system by the Hindus. Hence, the Hindus, knowing the sexagesimal place-value system, may well have consciously adopted the basic idea to set up their own place-value system.

5. Transmission to the Middle East and the Mediterranean Basin

Even long before Islam, the Arabs came into contact with the Hindus when their merchants met others in India, in the Arab peninsula, and in other places. In the trading city of Alexandria and other Mediterranean towns, the Arabs met European merchants as well. In their exchange of goods there naturally arose the need for maintaining crude accounts and in this way the Arabs apparently got acquainted with the Hindu methods of counting and their number symbols.[2]

In this connection Smith and Karpinski claim that Hindu numerals without the zero were brought to Alexandria by 450 A.D. and

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1. Encyclopaedia Britannica, vol II, p. 910.
2. Smith and Karpinski, pp. 64, 82; Smith, 1, vol. I, p. 73.

from there they were transmitted to other places of the Mediterranean basin, where the Arabs got acquainted with them and afterwards used them in their computations. This adoption of Hindu numerals was called <u>ghobār</u> or <u>dust</u> computation because of the practice of writing these numerals on sand-strewn reckoning boards. These ghobār forms are much closer to the Hindu and modern European numerals than to the Arabic ones.[1]

But because the ghobār system was made of only nine characters (the powers of ten being indicated by dots written above the symbols), and had no zero, it did not include the place-value notion. Hence, regardless of how or when these symbols were transmitted they are of little importance for the present study.[2]

In a book written in 662 is found the first definite trace of the use of the Hindu place-value system outside of India. The writer of this book, Severus Sebokht, was a Syrian scholar and monk born in Nisibis but who flourished in the middle of the seventh century in Qen-Neshrê (Kennesré, modern Qunnisrīn, thirty-three kilometers south of Aleppo) which under his leadership became the main centre of Greek learning in Syria. Sebokht asserted that not all science was due to the Greeks, and in trying to show this he speaks of the Hindu computations "which excel the spoken words and ..... are done with nine symbols". In spite of his appreciation of these numerals, we have no evidence that he made any contribution in transmitting them into the Arab world or even that he was able to us them.[3]

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1. Smith and Karpinski, pp. 64; Struik, p. 68.
2. Smith and Karpinski, pp. 65,66; Cajori, p. 89: Datta and Singh, p. 91.
3. Smith and Karpinski, p. 91; Karpinski, p. 47; Datta & Singh, p. 88; Cajori, p. 89; Sarton, 1, vol. I, p. 493.

In the first half of the sixth century Jundishāpūr, a city in the south-west of Persia, was famous for its school, where many Greek scientific works were translated into Syriac and later into Arabic.[1] Here at the beginning of the seventh century, the "Royal Astronomical Tables" (Zik-i Shatro-ayār) were compiled in Pahlevi (Middle-Persian). These were later translated into Arabic and called Zīj al-Shāh or Zīj al-Shahryār and used by the Muslims in the ninth century[2].

All the above, in addition to the extensive subsequent work of Iranian Muslims with the Hindu arithmetic, lead to the suspicion that the pre-Islamic Iranians and perhaps others may possibly have been acquainted with the decimal place-value system. But this cannot as yet be proved.[3]

As for Islamic times, within a short period following the flight of Muhammad from Mecca to Medina in 622, Islam and the Arab empire spread east and west of Arabia until it covered the whole area from India to Spain. In 755 the dominion was cleft into two, each part ruled by a caliph, one at Baghdād and one at Cordova in Spain.[4]

The most brilliant period in Islamic culture occured during the Abbasid rule in the eighth century and centered at the Abbasid capital, Baghdād. Scholars and scientists were encouraged by the court, especially during the days of Harūn al-Rashīd and al-Ma'mūn.[5] The first

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1. Sarton, 1, vol. I, p. 382; Hitti, pp. 307-9, 371.
2. Encyclopaedia of Islam, vol. I, p. 498; Sarton, 1, vol. I, p. 435.
3. Luckey, 2, p. 247.
4. Hitti, p. 286.
5. Hitti, pp. 297-307.

But a fact mentioned in al-Nadīm's Fihrist, that he worked in the library of the caliph al-Ma'mūn who reigned in 813-833, indicates the period of his literary activity. His work on the Hindu astronomical tables brought him almost immediate fame, but the height of his literary activity may be placed about 825, when he was looked upon as one of the celebrated mathematicians of his time.[1] And algorism, the Latin transliteration of his name, was given afterwards to the early arithmetic of Hindu numerals.[2]

In his book , Indian Arithmetic (al-Ḥisāb al-Hindī) al-Khwārizmī appreciates the Hindu principle of position and methods of computation. His work "excels", said an Arabic writer, "all others in brevity and easiness and exhibits the Hindu intellect and sagacity in the grandest inventions."[3]

After al-Khwārizmī's time, there continued to be a great deal of Arabic scientific activity. And as Hindu arithmetic occupied one of the high places among the sciences, many books were written on it and on computational methods using it. The earliest manuscripts which contain the Hindu notation in Arabic date back to 874-888. These numerals appear also in a work dated 970 written in Shīrāz in Persia. Another use of positional numerals was found on a dated inscription on a church pillar a short distance from the Jeremias Monastery in Egypt. The date reads 349 A.H. (i.e. 961 A.D.).[5]

Many Arabic writers on Hindu arithmetic followed al-Khwārizmī's

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1. Cajori, p. 121; Smith and Karpinski, p. 138.
2. Smith and Karpinski, p. 97; Datta, p. 48.
3. Cajori, pp. 102, 121.

lead. Below are listed some of these scholars together with their works.

Al-Kindī (d. 873) wrote a Treatise on the Use of Hindu Arithmetic (Risāla fī Istiʿmāl al-Ḥisāb al-Hindī).[1]

Abu Ḥanīfa al-Dinawarī (d. 895) wrote A Table On Hindu Computational Methods (Jadwal fī Ṭuruq al-Ḥisāb al-Hindī).[2]

Kushyār bin Labban al Jubailī (971-1029) wrote on the Elements of The Computation of the Hindus (Mabādiʾ al-Ḥisāb al-Hindī)[2].

Al-Antākī (d. c. 987) wrote The Great (Computing) Board for Computation (al-Takht al-Kabīr fil-Ḥisāb al-Hindī) and Computation On the Board Without Erasing (al-Ḥisāb ʿalā al-Takht Bilā Maḥū)[3].

Sinān Ibn al-Fatiḥ wrote The Book Of The (Computing) Board For Hindu Arithmetic (Kitāb al-Takht fil Ḥisāb al-Hindī)[4].

Abū-Naṣr, Muhammad bin ʿAbdallāh al-Kalwazānī (c. 982) wrote The (Computing) Board for Hindu Arithmetic (al-Takht fil-Ḥisāb al-Hindī)[4].

It is worth noting that the sexagesimal system, which had been in use before the introduction of the Hindu numerals, also maintained itself for a long time alongside the new system, being used mainly by the astronomers.

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1. Al-Fihrist, p. 375; Smith and Karpinski, p. 92.
2. Luckey, 2, p. 246.
3. Luckey, 2, p. 246; Al-Fihrist, p. 395; Smith & Karpinski, p. 92.
4. Al-Fihrist, pp. 392-6.

6. The Transmission to Europe

There were many possibilities by which the decimal system could reach Europe. First, we have the traders who had come in contact with these numerals through association with learned people. Abūl-Ḥasan al-Mas'ūdī (d. 956), for instance, travelled to the East as far as the China Sea, and to the West as far as the Atlantic and had some knowledge of the Hindu numerals used in reckoning.[1]

Then, there were the ambassadors sent by rulers from the West to the East. For instance, Charlemagne (743 - 814), as Alcuin (766 - 780) says, sent emissaries to Baghdād during the opening period of mathematical activity there.

Occasionally genuinely learned men, like Constantine the African, followed the routes of tradesmen to seek better scientific centers. At the beginning of the eleventh century the latter spent thirty-nine years in travelling with the purpose of learning the Oriental sources. He passed through Italy, where the numerals had become partially established, India, Africa, and Asia.

Christian pilgrims to the Holy Places may also have contributed something to the dissemination of the Hindu numerals.[2]

But the most important means of transmission of the decimal place-value system to Spain and from there to Christian Europe were the schools that the Moors maintained in Spain for five hundred years

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1. Smith & Karpinski, p. 101; Datta & Singh, p. 93.
2. Smith & Karpinski, pp. 103-5.

after their conquest. These were established at Cordova, Granada and Toledo, and, being the most famous seats of learning in the twelfth century, attracted many youths from other European cities.[1] These students automatically took back with them whatever knowledge they may have gained[2]. The schools of Toledo were also the centre of translation from Arabic into Latin, or into Hebrew then into Latin. Among the famous translators from Arabic into Latin were Adelard of Bath, the translator of al-Khwārizmī's astronomical tables;[3] Robert of Chester, the translator of al-Khwārizmī's algebra[3]; and John of Seville, the translator of elaborate arithmetical works.[3] As for translations from Arabic into Hebrew, we have the mathematician Rabbi ibn Ezra, a native of Toledo, who studied the Hindu arithmetic and used with the decimal place-value system the first nine letters of the Hebrew alphabet and the circle for zero[4]. In 1160, he translated a commentary, by al-Bīrūnī, upon al-Khwārizmī's tables. This commentary is extant only in Hebrew[5].

After all that precedes, one would expect to see the European peoples appreciate the advantages of the newly transmitted place-value system and replace the ghobār symbols which they had been using with the new forms. But this was not the case. For the Arabs of Spain found it difficult to follow the lead of those of the East, their enemies.

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1. Smith & Karpinski, p. 100; Datta & Singh, p. 94; Smith, 1, vol. 1, p. 201.
2. Datta & Singh, p. 94.
3. Smith & Karpinski, p. 125; Suter, p. 10; Cajori, p. 118; Smith, 1, vol I, p. 203; Sarton, 2, p. 164.
4. Smith & Karpinski, p. 127; Karpinski, p. 49.
5. Sarton, 1, vol. II, p. 116, and vol. III, part 1, p. 132.

But, once the mathematical works were translated, the people of Spain began to understand al-Khwārizmī's arithmetic and to appreciate the Hindu art of reckoning. Hence they kept their own ghobār forms for the numerals and adopted only the zero with the place-value idea. This may explain the difference between the forms of the Arabic and European numerals.[1]

In the twelfth century, aside from Rabbi Ibn Ezra, the only mathematician who wrote on Hindu numerals and their use was the Jewish scholar Samuel bin Abbas, the author of al-Qiwāmī.[2] This shows the previously mentioned reluctance of the Western peoples to adopt the Hindu numerals.

But by the beginning of the thirteenth century the number of mathematicians who worked on this subject was gradually increasing. The most influential European writer in medieval times was the Italian Leonardo Fibonacci (c. 1175-1250), a native of Pisa[3]. He made a tour along the Mediterranean Sea, visiting Egypt, Syria, Greece, Sicily and Provence. He met there many merchants and scholars, and became acquainted with the number systems of different lands. He counted all these as poor compared with the decimal one[4]. In 1202, he wrote his Liber Abaci, a manual of business arithmetic.[4] This work did not gain any popularity because, on one hand, it was too advanced for the mercantile class, and too new for the conservative university circles on the other hand.[5]

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1. Datta & Singh, p. 89; Cajori, p. 100.
2. Smith, 1, vol. I, p. 209.
3. Smith & Karpinski, p. 128; Smith, 1, vol. I, p. 214.
4. Smith & Karpinski, p. 131; Smith, 1, vol. I, p. 215.
5. Smith & Karpinski, p. 131; Sarton, 2, p. 165.

About the time of Leonardo of Pisa, the German monk Jordanus Nemorarius (c. 1236) wrote a famous work on the theory of numbers entitled De Arithmetica decem libris demonstrata. His work De Ponderibus Propositiones, containing a brief treatment of statics, is also based on the Hindu notation[1].

The works which were more widely used in Europe are the Carmen Algorismo (Song of al-Khwārizmī's Arithmetic Methods) written in 1240 by the French mathematician Alexandre de Villedieu[2], and the Algorismus (al-Khwārizmī's Methods) written in 1250 by John of Halifax[3]. The latter contains lines quoted from the Carmen de Algorismo and was used as a textbook for university instruction. It may well be that because of its wide use and because it attributes the invention of the Hindu numerals to the Arabs, that the decimal place-value system, together with the term "Arabic numerals" became common.[3] Later in the fourteenth and fifteenth centuries the spread of the system was rapid.

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1. Smith, 1, vol. I, p. 227; vol. II, p. 384; Cajori, p. 127.
2. Smith & Karpinski, p. 133; Smith, 1, vol. I, p. 228.
3. Smith & Karpinski, pp. 133-4; Datta & Singh, pp. 94-5.

CHAPTER II

THE INVENTION OF DECIMAL FRACTIONS IN EUROPE

7. Approaches Toward the Invention

Once the decimal notation had been invented, one might expect to find some ingenious mind, who had understood it thoroughly, deducing the decimal fractions as an immediate extension. But this was not the case; a thousand years passed between the invention of the decimal system and that of decimal fractions. Until recently it had been generally accepted that the invention of the decimal fractions first occured in Europe in the sixteenth century. But lately Luckey[1] has announced that the western invention was anticipated in Central Asia in the fifteenth century. This latter, however, probably had no influence on the European invention, and although the full description of the Eastern invention is the basic objective of this work, it will be delayed to the next chapter. A sketch of the European invention is given below.

It should be stated that the mere exhibition of a fraction expressed in decimals does not constitute the invention. Nor does it suffice merely to give an exposition of the ideas involved without actual application to computation. Thus the steps made in different ages before the present form of the decimal fract ions was arrived at

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1. Luckey, 1, pp. 199-204.

will be mentioned as approaches to the real invention.

The fact that, in my culture, the development of most subjects goes from particular to general is realized in mathematics in general and in our system of representing numbers in particular. Integers, for example, were originally in a category different from that of fractions, and although the decimal system and the sexagesimal fractions were parallel developments, people did not understand the decimal idea fully. Integers were looked upon as decimals, but the fractions were commonly expressed in the sexagesimal system. And even so, sexagesimal fractions were not always considered as fractions. Ptolemy (fl. 127-141), for instance, in his Almagest "used the sexagesimal system", as he stated, "to avoid fractions".[1]

Up to the fourteenth century the method of extraction of the square root of the decimal number N was to multiply it by 10^{2n}, take the square root of the product, write the result as $\sqrt{N} = \frac{1}{10^n}\sqrt{N.10^{2n}}$, then reduce the answer to sexagesimal fractions. About 1343, for example, John of Meurs wrote the square root of two as $\frac{1}{1000}\sqrt{2,000,000} = \frac{1}{1000}\,1414$, expressing the result as 1.4.1.4 and saying that the digits are units, tens, tens of tens, and tens of tens of tens[2]. This is a close approach to the idea of decimal fractions. But then he stated the final result in sexagesimal form as $\dot{1}\ 24'\ 50''\ 24'''$.[2] Even a century later, mathematicians were confused between the decimal and sexagesimal expressions, combining them into mixed forms. The German

1. Almagest, vol. I, Book I, p. 26.
2. Karpinski, p. 127; Sarton, 2, p. 169.

Johann von Gemunden (d. 1442), for instance, used the following method. He began by expressing a sexagesimal number a, say, in terms of the lowest sexagesimal units as $\frac{1}{60^n}\sqrt{a.\ 60^{2n}}$, then magnifying[1] it on a decimal base, he got $\sqrt{a} = \frac{1}{60^n.10^n}\sqrt{a.60^{2n}.10^{2n}}$. This last result is retransformed into a sexagesimal form[2]. But a later mathematician, Paul of Middelburg (1445-1533), occasionally used the same method except that the result was expressed in decimal terms.[2]

The division of any integral multiple of the power of ten was also worked on.[3] Pietro Borghi, for example, in his arithmetic (printed in 1484), gave the rule of dividing any number by $a.10^n$ by cutting off n figures from the right of the given number and then dividing the remaining part by a. The number 2345678 being divided by 3000 gave the result $781\frac{2078}{3000}$. In 1492 Pellas in his Art de Mathematicha and in 1525 Christoph Rudolff in his Coss used the same method.[4]

Trigonometric tables also had some influence in that they gradually passed through three stages each one approaching closer to a purely decimal presentation. First, the base sixty and the sexagesimal fractions were used in expressing the entries. Ptolemy, for example, computed a table of chords in a circle with radius 60 (i.e. 1,0) and used the primes, seconds, etc., (i.e. the 60th, 60^2th, etc. of the sexagesimal system)[5]. Secondly, the product of the base sixty with a

1. This process is what we translate as "to elevate". For the word elevate see Note 12 in the Commentary.
2. Sarton, 2, p. 170.
3. Karpinski, p. 127.
4. Sarton, 2, p. 172.
5. Almagest, Book 1, vol. I, pp. 38-45.

power of ten was used with decimal integers to display the result. Georg Peurbach (1423-1461), computing a table of sines, took the radius of the fundamental circle as 60,000 or 600,000 to avoid fractions.[1] Thirdly, a power of ten was used for the radius of that circle. Peurbach's student, Regiomontanus[2], followed his teacher's method, taking the radius as 600,000,000. Then he adopted a decimal base 100,000, and finally extended the radius to 60,000,000 and to 10,000,000. This latter needed only to be unity to be used in our tables.[3]

We notice that, although none of these approaches is an invention of decimal fractions, yet they added to the accumulation of material which culminated eventually in the actual invention.

8. Bonfils' Contribution

Among the mathematicians of the fourteenth century who came close to the basic invention of the decimal fractions was Immanuel bin Jacob Bonfils. He lived (c. 1340-1377) mainly in Tarascon and also in Avignon and Orange. He was a mathematician, astronomer, astrologer, and translator from Latin into Hebrew. Although he made astronomical observations, we are more interested in two elements of his work: decimal fractions, and the exponential calculus.[4]

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1. Sarton, 2, p.171.
2. Karpinski, p. 128; Zeller, p. 19.
3. Zeller, pp. 20,34.
4. Gandz, 2, pp. 17,21.

The main interest of Bonfils was to demonstrate that multiplication and division of the decimal powers can be performed by addition and subtraction of the exponents. In order to obtain easily a unified scale of powers he described how decimal fractions can be introduced.[1] Then, to arrange the decimal powers systematically, so that the positive powers of integers correspond with the negative powers of fractions, he classified all numbers as: integers, units, and fractions, the units being an intermediate boundary between the integers and fractions. He called the degree[2] of the units, units, after their name. To the fractions obtained by dividing the unit into ten parts and each of these into ten parts and so on, he gave the degrees: primes, seconds, thirds, etc. To the tens, hundreds, and so on, he attributed the degrees: prime integers, second integers, third integers, etc. As to what we call the exponent of a place, Bonfils gave the name "denominator of a degree" (of course this nomenclature has no relation to our use of the word denominator in connection with fractions). He said, for instance, "add the denominator of the seconds to the denominator of the thirds, so that the result will be fifths" where we would say, "add the exponents of 10^{-2} and 10^{-3}, i.e. of the two mentioned places, to get 10^{-5}."

Also Bonfils did not recognize negative numbers. So, to differentiate the degrees of the integers from those of fractions, he attached to the former a direction opposing that of the latter[3]. All

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1. Gandz, 2, p. 21.
2. Bonfils uses the word degree for al-Kāshī's word martaba, which we translat e as the place of a digit. For the meaning of place see Note Ten in the Commentary.
3. cf. Notes 12 and 13 in the Commentary.

these definitions in addition to the rules of multiplication and division of two exponential digits are indicated in the first section of his manuscript.[1]

The methods of multiplying and dividing any two exponential numbers are mentioned in Sections Two and Three respectively of Bonfils' manuscript.[1] According to the previous definitions, his rules are correct. In multiplication, for example, Bonfils says that to multiply two integers or two fractions, add the denominators of the degrees of the two multipliers, and this will be the denominator of the product. This latter will be an integer if the two multipliers are integers and a fraction if they are fractions. He mentioned also the fact that when one of the factors is an integer and the other is a fraction, the denominator of the smaller should be subtracted from that of the bigger and theproduct falls in the "degree of the units" if they have the same denominator, or among the integers if the denominator of the integer is higher, and among the fractions if otherwise. His rule for division of exponential numbers is also complete.

Theoretically it seems that Bonfils has a correct, if somewhat vague, understanding of the decimal fractions, but because he did not give any application he cannot be credited with the invention. Moreover, "his work exercised no influence on his contemporaries or successors, though it seems that he was well-known to Christian scholars."[2]

It should be pointed out that Bonfils' work contains another important step toward the invention of decimal fractions. Gandz states

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1. See the facsimile in Gandz, 2, pp. 39,40.
2. Gandz, 2, p. 31.

that before Bonfils' time the mathematicians were using, in operating with integers, Archimedes' rule of ascribing one to the units' degree. As a result the exponential rules of multiplication and division as used were $a^n . a^m = a^{n+m-1}$ and $a^n : a^m = a^{n-m+1}$. In multiplication and division of fractions the correct rule of al-Khwārizmī, i.e. $\frac{1}{a^n} \cdot \frac{1}{a^m} = \frac{1}{a^{n+m}}$, and that of al-Karajī* (c. 1010), i.e. $\frac{1}{a^n} : \frac{1}{a^m} = \frac{1}{a^{n-m}}$. were followed. But the units were not counted among the fractions. Gandz also claims that Ibn Ezra (c. 1150) and Levi bin Gershon (1288-1344) were the first to ascribe the degree one to tens.[1] But among the mathematicians who appeared after them, the only one who followed their lead and applied their rule was Bonfils.[1] In fact, although Bonfils did not ascribe explicitly the degree zero to units, his attribution of degree one to the tens and tenths, two to the hundreds and hundredths, etc., made the use of the correct exponential laws of multiplication and division possible with integers as well as with fractions. All that he had to do was to give the degree zero to the units as it was done after him by al-Kāshī and Stevin.

9. Rudolff's Contribution

The German mathematician Christoph Rudolff worked also in this field. But because we do not have access to his work, the best we can do is to repeat the conflicting judgments of other historians. Smith looks

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1. Gandz, 2, pp. 25-29.

* The name appears in the European literature as al-Karkhī.

upon him as an inventor of decimal fractions because these latter appear in his work Exempel Büchlin written in 1530. He also claims that Rudolff knows how to operate with decimal fractions and how to write them, but that his work was not appreciated and understood until 1585. In contrast to this, Sarton claims that Rudolff is not an inventor because he does not justify his work nor does he prove his full comprehension of it or of its implications.[1]

10. Stevin's Invention of the Decimal Fractions In Europe

The mathematician who is the inventor of decimal fractions in Europe is Simon Stevin (1548-1620), a native of Bruges in Belgium.[2] He was the first Western writer to give a systematic treatment of decimal fractions[3]. In 1585 he wrote, in Flemish, his book entitled De Thiende. This was printed in 1585 and reprinted after Stevin's death. It was also translated by Stevin himself into French and translated again by others into Dutch, French, Latin and English.[4]

At the very beginning of his work Stevin declares that he invented the decimal fractions to remove the difficulty of the methods of multiplication and division used by the astronomers, merchants, and others. His work contains four definitions, four operations, and an appendix. The four definitions are the following:[5]

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1. Sarton, 2, p. 173.
2. Cajori, p. 147; Sanford, p. 111.
3. Cajori, p. 147; Sanford, p. 111; Karpinski, p. 128.
4. Sarton, 2, p. 159.
5. The facsimile in Sarton, 2, p. 234.

1 - "Le Disme" is a kind of arithmetic used in computing with all numbers as if they were integers.

2 - The unit "Le Commencement" refers to the units' digit in a decimal number and has the symbol ⓪. The zero is equivalent to our exponent zero. For example, the integer 364 is symbolised in the book as 364 ⓪.

3 - The successive powers of ten are given different names and symbols: primes and the symbol ① for the tens, seconds and the symbol ② for the hundreds, etc. This definition is explained by two examples one of which (in translation) is: " 8 ⓪ 9 ① 3 ② 7 ③ is equivalent to $8\frac{9}{10}\frac{3}{100}\frac{7}{1000}$, which altogether makes $8\frac{937}{1000}$."

4 - All the numbers defined above are called decimal numbers (Nombres de Disme).

The four fundamental operations explained in Stevin's work are carried out in the modern way. The only difference is that his symbolism is awkward. In addition and subtraction, for example, the given numbers are arranged vertically according to their places, and every digit of the result retains the symbol under which it falls. To add the number 8 ⓪ 5 ① 6 ② to the number 5 ⓪ 0 ① 7 ②, for example, the result is obtained thus:[1]

	⓪	①	②
	8	5	6
	5	0	7
1	3	6	3

and symbolized as 13 ⓪ 6 ① 3 ②.

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1. The facsimile in Sarton, 2, p. 235.

An example involving the addition of mixed numbers is written[1]:

```
    ⓪ ① ② ③
  2 7 8 4 7
  3 7 6 7 5
8 7 5 7 8 2
-----------
9 4 1 3 0 4 ,
```

the sum being 941⓪ 3① 0② 4③.

In multiplication the result has the symbol whose number is the sum of the numbers of the symbols of the two multipliers. The product of 2① and 3②, for instance, is given as 6③. Similarly, the product of the two mixed numbers 32⓪ 5① 7② and 89⓪ 4① 6② is obtained as follows:[1]

```
              ⓪ ① ②
        3 2 5 7
        8 9 4 6
        -------
      1 9 5 4 2
    1 3 0 2 8
  1 9 3 1 3
2 6 0 5 6
---------------
2 9 1 3 7 1 2 2
      ⓪ ① ② ③ ④ ,
```

the product being 2913⓪ 7① 1② 2③ 2④

At the end of the multiplication rule Stevin gives in a note the product of the numbers 3④ 7⑤ 8⑥ and 54② thus[1]:

```
    ④ ⑤ ⑥
    3 7 8
      5 4 ②
    -----
  1 5 1 2
1 8 9 0
---------
2 0 4 1 2
④ ⑤ ⑥ ⑦ ⑧ .
```

1. The facsimiles in Sarton, 2, pp. 235, 236, 237.

In division, the difference between the numbers inside the two small circles referring to the digits to be divided is used as the number of the symbol of the result.

In the appendix, the decimal idea is extended to monetary problems, weights, and measures.

In connection with the different kinds of notation used with the numbers of the last example mentioned above, it should be stated that Stevin used his symbolism in three ways. The number 3.14, for example, can be written

a - as 3⓪ 1① 4② ,

b - or ⓪ ① ② over 3 1 4 ,

c - or 314 ② .

It should be noted also that the first two forms are completely analogous with al-Kāshī's method of calling the *place* of the *individual digits* of a number different names[1]. The last form[2], which is more advanced than the former two, is used in Stevin's *Geometry* written in 1605. In it the symbol has the same meaning as the *place of a number* used by al-Kāshī [3]. Stevin's symbolism probably originated from the numerical tables of his time. In those tables the names of the places are written at the head of the corresponding columns.

From all that is mentioned above, it is clear that Stevin saw the inner relationships and appreciated the significance of decimal fractions better than any one preceding him in Europe. He defined them

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1. Cf. Note ten in the Commentary on the first meaning of *place*.
2. Sarton, 2, p. 176.
3. Cf. Notes 11 and 12 in the Commentary on the second meaning of *place*.

clearly and fully and gave reasons for their validity. The only defect in his work was his awkward symbolism, though really it is not difficult to understand and improve. His last notation, however, is short of one step of being the modern decimal point. In 314 ②, for instance, if one puts a dot after the second digit from the right one obtains 3.14, the modern notation.

11. The Invention of the Decimal Point

Although this last notation was so close to the decimal point, the latter developed only a long time after. In 1592 some mathematicians, like Rudolff and Thomas Masterson[1], started using bars to separate fractions from integers, while the Swiss Joost Bürgi[2] wrote 141̥4 for 141.4. Others, like Pitiscus in 1612, used the bar and the dot for the same purpose[3]. In 1603 Kepler used parentheses to separate fractions from integers, and Beyer, using only the decimal point, assumed himself to be the inventor[4]. But the first to use the decimal point with actual decimal fractions in operations including multiplication and division is Napier (1550 - 1617) as appears in his Rabdologia in 1616. He wrote[5] the number 3.14 as 3, 14 and 3 . 1' 4'' .

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1. Sarton, 2, pp. 172-7; Smith, 1, Vol. II, p. 242.
2. Sarton, 2, p. 177; " " "
4. Cajori, p. 148; Smith, 1, Vol. II, p. 244,
3. Smith, 1, Vol. II, p. 244.
5. Sanford, p. 113; Cajori, p. 148; Smith, 1, Vol. II, p. 244.

Now, having discussed the time of invention of decimal fractions and that of the decimal point, we can conclude that the end of the sixteenth century or the beginning of the seventeenth century is the time in which the decimal fractions were completely available in Europe.

CHAPTER III

THE LIFE AND WORK OF AL-KĀSHĪ

12. Biographical Sketch (including background)

In article seven above it was stated that decimal fractions were invented by the Muslims in the Middle East before their appearance in Europe. In fact in the last period of Islamic scientific activity a distinguished Iranian mathematician, Ghiyāth al-Dīn Jamshīd ibn Mas'ūd Al-Kāshī, not only fully comprehended the notion of decimal fractions but explained their use and applied them extensively in computation.

Since such inventions are usually made in a social and political environment appropriate to their development, it may be of value to give a short sketch of al-Kāshī's background from these two points of view.

In the previous chapter it has been mentioned that the golden age of Islamic intellectual activity took place during the Abbasid empire and centered in Baghdād, the capital. The knowledge which had been taken from three main sources, Semitic, Hindu and Greek, was enlarged and applied there. But this period did not last longer than the end of the tenth century when the Caliphs became too weak to maintain their dominions, and many dynasties began to rise in the different parts of the empire. But in spite of the numerous wars which troubled the Islamic world, research in science was maintained, especially in mathematics and astronomy, and scholars were supported by many of these local dynasties.

In Spain, for example, after the death of ʿAbd-al-Raḥmān III, the Omayyad Caliph of 929, the Muslims' weakness began to increase. And at the eleventh century a number of warring petty dynasties known as the "Party Kings" arose[1]. But, nevertheless, the "so-called Toledo Planetary Tables" were computed in this time by the important astronomer Al-Zarqālī (c. 1029 - 1087).[2]

In Egypt the greatest of medieval dynasties was that of the Fatimid Caliphs (909 - 1171). Well-known scientists who were supported by them were Ibn Yūnis (d. 1008) and Ibn al-Haitham (d. 1078).[3]

In Iran the dynasties were founded, in general, either by families of aristocratic local princes who wanted to reassert their authority or by invading Turkish tribes from Central Asia. Among the dynasties of Iranian origin were the Ziyarids[4] who were great protectors of le arning and the Buwayhids[4] whose power extended to southern Persia. Qābūs (976 - 1012), a ruler of the Ziyarids, was al-Bīrūnī's patron, to whom his "Chronology of Ancient Nations" was dedicated. The astronomer Abu al-Wafā' (940 -998) and a group of colleagues were maintained by the Buwayhids.

Among the dynasties of Turkish origin is that of the Ghaznavids (962 - 1186)[4]. It is interesting to mention here Sultan Maḥmūd of Ghazna (d. 1030), the great supporter of scientists. His request to

1. Lane-Poole, pp. 20-27.
2. Struik, p. 95.
3. Lane-Poole, pp. 67-8.
4. Sykes, pp. 23-6; Lane-Poole, pp. 136-9, 285.

Ma'mūn, prince of Khwārizm, to send al-Bīrūnī and Ibn Sīnā to Ghazna is a good instance. Al-Bīrūnī accompanied his patron to India, and, spending many years there, he became acquainted with Hindu knowledge. After his patron's death he published the second of his great works, the Indica.[1]

In 1077 another Turkish dynasty, the Seljuks, were ruling the whole of Western Asia from the border of present Afghānistān to the frontier of the Greek Empire in Asia Minor and of the Fatimid Caliphate of Egypt. These also were upholders of scientists, among whom was the mathematician ʿOmar al-Khayyām of Khorāsān (1040 - 1123).[2]

Later in the thirteenth century new invaders, the Mongols, appeared under Jingīz-Khān (Genghis Khan) and his grandson Hūlagū, drove away the Seljuks, and ruled a large empire. In spite of the tremendous destruction caused by Jingīz-Khān and his descendants, the Mongols were supporters of science. Hūlagū, for example, who was master of all the provinces of Iran and Asia Minor from India to the Mediterranean, encouraged science[3]. He treated Naṣīr-ad-Dīn of Ṭūs with much respect and founded the celebrated observatory at Marāgha[4] in Āzarbaijān.

At the end of the fourteenth century another powerful conqueror, a descendant of a minister of Jingīz, overran many Muslim countries. This man was Timūr (Tamurlane, 1335 - 1405), a native of

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1. Sykes, pp. 59-60.
2. Lane-Poole, pp. 149-52.
3. Lane-Poole, pp. 217-9.
4. Struik, p. 94; Sykes, p. 146; Hitti, p. 290.

Transoxiana, who was appointed by Tughā-Timūr, (the Mongolian, 1359-61) as the governer of Kāsh. Between the years 1380-1397 Timūr, during a long series of campaigns, defeated the petty dynasties of Iran: the Kurts who governed Herat from 1245-1389; the Serbaderids who governed Khorāsān between 1337-1381; the Muzafferids who kept Fārs, Isfahān, Shirāz and Khorāsān; and the Jalayers who ruled ʿIrāq, Āzirbaijān, Tabrīz, Mūsal and Diyār Baker for a long time.[1] In 1397 he entered northern India, in 1402 invaded Asia Minor, and in 1403 subdued Syria. Thus, having Samarqand (in modern Turkistān) as capital, his empire stretched from Delhi to Damascus and from the Sea of Aral to the Persian Gulf. In spite of all these wars, science was looked upon as something of great dignity. Many scholars were brought by Timūr to Samarqand. And, being himself fond of mathematics and philosophy, he made of his capital a center for science.[2] But his greatness and power did not last longer than 1405 when he died in Samarqand leaving his fourth son Shāh-Rukh to succeed him.[3]

This description brings us down to al-Kāshī's time and place, concerning which we wish to give considerable detail. The picture is by no means clear. Generally the Timūrids were the most powerful rulers in the whole area of Turkistān, Irān, Afgānistān, etc., but they exercised their power apparently not by means of a strong central government, but by a loose feudal organization of local princes. Moreover, the conquerers' sons were assigned provinces to rule. Shāh-Rukh, for

1. Lane-Poole, pp. 246-265.
3. Sédillot, p. c.
   Lane-Poole, p. 266.

example, ruled in his birthplace, Khorāsān, for some years during his father's lifetime[1]. After Timūr's death, each of his descendants wanted to be the successor. But Shāh-Rukh emerged successful and became the king of Samarqand as well as of most parts of his father's kingdom. But at that time the Ottomans, Jalayers, and Turkomāns wanted to recover their lost provinces. Thus Shāh-Rukh was continuously engaged with them in wars during which Qara Yūsuf, the second leader of a tribe of the Turkomāns known as the Black Sheep, was exiled three times. But nevertheless, Qara Yūsuf with his son Iskander was able to conquer the regions of Armenia, Vān, Mārdīn, Diyār Bekir, Tabrīz, and Mūṣil. After his father's death, Iskandar was reduced by Shāh-Rukh to the position of tributary ruler of Āzarbaijān.[1] In 1411 Iskander ruled ʿIrāq. In 1437 he died and his descendants added to their kingdom Georgia, Fārs, Kermān, Khorāsān and Herāt.

While Shāh-Rukh was busy with the Turkmāns, his son Ulugh-Beg (1393-1449) was ruling in Samarqand for twenty-eight years. After his father's death in 1447 Ulugh Beg's real reign began. But strangely enough, two years later he was murdered by his son ʿAbdul-Laṭīf. The Timūrids afterwards became very weak and when a half century had passed after Timūr's death, they were not able to maintain their kingdom, so that after 1450 the dynasty of the Black Sheep emerged from the west and began to conquer larger parts of Irān.[2]

It is worth noting that in spite of the numerous wars in

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1. Sykes, p. 137.
2. Encyclopaedia of Islam, vol. III, p. 1045.

which Shāh-Rukh and Ulugh Beg were engaged, they followed Tīmūr's steps in the scientific field. In fact, the former showed his interest in science in founding a large library and the latter, being fond of science and especially mathematics and astronomy, ordered in 1420 the building of an observatory which excelled any other one of its kind. One hundred persons were employed in it, and the celebrated Zīj-i Sultānī, the astronomical tables, known also as Zīj-i Ulugh Beg, were compiled by Ulugh Beg with the help of his associates and completed in 1437.[1]

It was in this scientific and political environment that al-Kāshī was living. Facts available for his biography are very few, and most of them are obtained from his works. Even the dates of his birth and death are not known exactly but informed guesses can be made about both. One of his works, Nuzhat al-Ḥadā'iq, is ended with a sentence which may be translated as: "Let this be the last of what I produced (brought forward) in this treatise, and praise unto the Lord of Worlds and prayers and peace unto his prophet Mohammad and his family, the virtuous ones, the pure ones, and I finished from writing on the Day of Sacrifice (10 Dhū-al-Ḥijja), eight hundred eighteen of the Hejra (i.e. 10 February 1416) in the city of Kāshān ....." This sentence does not state that his birthplace was Kāshān. But because his name is an abbreviation of it, it is fairly safe to assume that he is a native of Kāshān. It is claimed[2] that there is, in the India

1. Sédillot, p.C.
2. Knoble, p. 5.

Office, another of al-Kāshī's works, a commentary on the Ilkhanic tables composed by Naṣīr ed-Dīn al-Ṭūsī in 1269. This work of al-Kāshī is known as the Zīj-i Khaqānī, and is dated eight hundred sixteen of the Hejra (i.e. about 1413-14), but the place in which it was written is unknown. This means that this work was written about three years before the Nuzhat al-Ḥadā'iq, hence probably also written in Kāshān. Moreover, knowing that the composition of such a set of astronomical tables involves an enormous amount of computation and a deep knowledge of astronomical theory, one can presume that al-Kāshī was at least thirty years old when he finished it. Thus a safe upper limit for his birth date is 1380.

Another of al-Kāshī's works, although it is dated 818 A.H. (1416 A.D.), it does not refer to the place in which it was written. But, because its date is the same as that of the Nuzhat al-Ḥadā'iq, which was written in Kāshān, doubtless it was written there too. This work is dedicated to a prince called Iskandar. Al-Kāshī did not say anything more to identify this patron.[1] But we know of no one ruling in the vicinity of Kāshān called Iskandar except Iskandar ibn Qara Yūsuf of the Black Sheep Turkoman tribe mentioned above. Therefore we have good grounds to suppose that this prince is the one to whom the work was dedicated. To dedicate this work to Iskandar, it may well be that al-Kāshī met him somewhere. But there is no evidence for supposing that al-Kāshī worked at Iskandar's court in Āzarbaijān. Neither, on the other hand, do we have any reason for thinking that Iskandar ever controlled

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1. From the work itself.

the region of Kāshān.

The next item of information about al-Kāshī is more definite. It is contained in Ulugh Beg's Zīj and in the commentary to it written by Sédillot[1]. The latter, while speaking about Ulugh Beg and his observatory, gave an Arabic sentence which can be translated as: "The one who first began the work was the virtuous Ghiāth-al-Dīn Jamshīd, then he died before it was finished and was buried near the above-mentioned observatory."

In the introduction to his Zīj, Ulugh Beg mentioned al-Kāshī among those who helped him in its writing and praised him in an Arabic passage translated thus: "Our great master, the pride of all wise men in the world, who had completed all the knowledge of the ancient peoples, the solver of the complexities of problems, our master Ghiāth al-Milla wal-Dīn Jamshīd, may God cool his resting place .....". Right after these words a Persian sentence is given. Its translation is:"The religion-server Jamshīd, God bless him, answered the call of God and left this mortal world of perversity for the celestial immortal world. Meanwhile, before this important event was achieved (i.e. the observations for the zīj) and took place, the master-teacher, thanks to God for his services, joined the blessed of the Lord, and he did so (i.e. he worked on the zīj) together with his dear son." Doubtless this last reference is to his adopted son, al-Qushchi.

The only other extant dated work of his is the <u>Miftāh al-Hisāb</u>, the work on which this study is based, and which was completed,

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1. Sédillot, p. CXXVj.

Luckey[1] claims, in 1427 A.D. It is not stated where it was written. But because it was dedicated to Ulugh Beg and because the presumption is that at that time al-Kāshī was working for him at Samarqand, we can deduce that the writing also was in Samarqand.

The date of al-Kāshī's death can be estimated with more accuracy than that of his birth. In fact, it is possible to establish a fairly narrow span in which this occured. Sédillot claims that he died about 1430[2]. And we know from the previous information that he was alive in 1427 and that he died before the completion of Ulugh Beg's zīj, i.e. before 1436. Also, in the Encyclopaedia of Islam the date is given as about 1436/7, but the authority for this statement is not known. Therefore a lower limit for his death is 1436.

The only information we have about his character is that "mentioned in the Haft Iqlīm under Kāshān where he said: '.... the former (Kāshī) was ignorant of the etiquette of courts, but that Ulugh Beg was obliged to put up with his boorish manners because he could not dispense with his assistance.'" [3]

1. Luckey, 1, p.199-204.
2. Sédillot, p. CLV.
3. Rieu, vol. II, p. 456.

13. His Works

Al-Kāshī is the author of a number of works. These are listed below:

1 - Risālat al-watar wal-jaib wastikhrāj thulth al-qaws al-maʿlūmat al-watar wal-jaib (A Treatise on the Chord and the Sine and on Finding the Third of an Arc Whose Chord and Sine Are Known)[1].

2 - Zīj-i Khāqānī (The Khāqānī Tables), which were written in Persian as a supplement to the Ilkhānī Tables compiled by Naṣīr al-Dīn al-Ṭūsī in 1269 A.D. In it al-Kāshī gave geometrical proofs of some astronomical rules.[2]

3 - Zīj al-tashīlāt (The Simplifying Tables)[3].

4 - Al-risāla al-kamāliya (The Perfect Treatise) or Sullam al-Samā' (The Ladder to Heaven) which deals with the sizes and distances of the heavenly bodies.[4]

5 - Risāla fī istikhrāj jaib daraja wāḥida (An Essay on the Calculation of the Sine of One Degree) in which al-Kāshī gave a solution for an equation of the third degree by an interesting process of approximation.[5]

6 - Al-risāla al-muḥīṭīa (The Treatise on the Circumference) in which

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1. Woepcke, p. 113.
2. Encyclopaedia of Islam, vol. II, p. 789; Sédillot, p.C; Suter, p. 173; Woepcke, p. 113;
   Page 5 of the Princeton MS of the Miftāḥ al-Ḥisāb, hereafter referred to as P. See [article] 14 below.
3. P , p. 5; Suter, p. 173.
4. P , p. 5; Encyclopaedia of Islam, p. 789, vol. II; Suter, p. 173; Woepcke, p. 113; Haji Khalfa, vol. III, p. 610.
5. Encyclopaedia of Islam, vol. II, p. 789; Woepcke, p. 113.

al-Kāshī extracted the ratio of the circumference to the diameter of a circle. It gives the value of 2π correct to sixteen decimal places, i.e. $2\pi = 6.2831\ 8530\ 7179\ 5865$. This treatise is of special interest because it contains decimal fractions, but it was unavailable to the author at the time of writing.[1]

7 - Nuzhat al-hadā'iq (Delight of the Gardens) which was mentioned above[2]. Al-Kāshī wrote it to describe the construction and use of two instruments:

a- Tabaq al-manāṭiq (The Plate of Zones), a kind of plotting-board, used in solving graphically problems dealing with the distances of the planets from the earth, their latitudes and longitudes, the solar and lunar eclipses, and many other problems.[3]

b- Lauḥ-i ittesālāt (The Plate of Conjunctions) which was invented by al-Kāshī for the purpose of computing mechanically the time of day at which expected planetary conjunctions will occur.[4]

8 - A short treatise was composed by al-Kashi in Dhū al-Qa'da 818 A.H. (1416 A.D.) on the construction of five observational instruments. This was dedicated to Sultan Iskandar.[5]

9 - Miftāḥ al-Ḥisāb (The Key to Reckoning) which contains the sexagesimal and decimal fractions with common applications.

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1. P , p. 5; Luckey, 1, p.200; Suter, p. 173; Woepcke, p. 113.
2. The work itself, P , p. 5; Suter, p. 173; Woepcke, p. 114.
3. P , p. 5; Woepcke, p. 114; Kennedy, 2, 3.
4. Kennedy, 1.
5. This MS is in the Library of the University of Leiden, but it is not listed in the Leiden Catalogue. Thanks are due to Dr. P. Voorhoeve of the Biblioteek der Rijksuniversiteit te Leiden, who has kindly supplied a microfilm of it.

14. MSS. of the Miftāḥ

Of this last of al-Kāshī's works there are many copies extant. Below are the ones which are listed in Brockelmann[1]:

MS. 5992 in Ahlwardt, Verzeichniss der arabischen HSS. der Königl. Bibliothek zu Berlin, Bd. 1 - 8, Berlin 1887.

MS. 131 in Catalogue des Mss et xylographes orientaux de la Bibliothèque impér. publ. de St. Petersbourg, 1852.

MS. 1036 (Cod. Or. 185) in Bibliotheca Academiae Lugduno - Batava, (Leiden).

MS. 419 in Catalogue codd. MSS. qui in museo Britannico asservantur, pars II, codd. ar. amplectens, 3 vol. London, 1846-79.

MS. 756 Loth O., Catalogue of the Arab MSS in the Library of the India Office, London, 1877.

MS. 804 in Jeni Gâmi Kutubhâne defteri, Stambul, 1300.

Other copies are listed in Brockelmann, Second Supplementary Volume[2]. These are the following:

MS. 5020 in Catalogue des MSS arabes par le Baron de Slane, in Bibliothèque Nationale, Département des Manuscrits, Paris, 1883-95.

MS. 2967 in Nūri Osmānīye Kütübhāne defteri Stambul.

MS. XVII, 54, 163 in (Oktai) Fihrist kütübhāne i mubāraka āsitānie-i riḍawī, Meshed, 1345.

MS. 1687 in Lubāb al-ma'ārif al-'ilmīya fī maktabat dār al-'ulūm al-Islāmīya, Pešaverki fihristi Kutub, Pešaver.

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1. Brockelmann, 1, vol. II, p. 211.
2. Brockelmann, 2, p. 295.

MS. 413, 652 in Fihrist Kitāb ʿArabī. Catalogue of Arabic Books in the Rāmpūr State Library, 1902.

MS. 830/2 . 3, 1427 in Čām (?)

MS. 131 in Dom (?)

MS. 26 in Stockholm (?)

MS. 3479 in Serāi (?)

MS. 798 in Bankipore (?)

MS. 1189 of the uncatalogued Garrett (Yahuda) collection at Princeton University, Princeton, New Jersey.

There is also a summary of the Miftāḥ al-Ḥisāb, known as Talkhīṣ al-Miftāḥ, and copies of this are listed in Brockelmann[1] as:

MS. 757 in Loth. O., Catalogue of the Arab. MSS. in the Library of the India Office, London, 1877.

and in the Second Supplementary Volume[2], as:

MSS. 132, 186, 32, 274, 50, 2 in Dāʾūd al-Čelebī al-Mauṣilī, K. Makhṭūṭāt al-Mauṣil, Baġdād, 1927.

MS. 25 in Stockholm (?)

MS. 1460 in Čārullāh (?)

Of these copies of the Miftāḥ al-Ḥisāb, microfilms of three have been used in the present study. They are[3]:

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1. Brockelmann, 1, vol. II, p. 211.
2. Brockelmann, 2, vol. II, p. 295.
3. Grateful acknowledgement is made of the assistance given by Dr. P. Voorhoeve of the Bibliotheek der Rijksuniversiteit te Leiden; by the authorities of the India Office; and by Prof. P.K. Hitti of Princeton University, who facilitated the filming of these manuscripts.

1 - MS.1036 (Leiden)

Copied in Qazvīn; 2 Shaᶜbān, 965 A.H. (20 May, 1558 A.D.); by Saᶜdullāh bin Amānillāh bin ᶜAlī; 292 pages; 21 lines to page; in Fārisī; clear and fair hand; some diacritical marks lacking; large margins; marginal notes; glosses and explanations; interlinear notes; some numerals are lacking and some others are in error; headings and numerals are written in red ink; with catchwords; figures are well done; complete; the most accurate of the three MSS. used.

2 - MS.765 (India Office)

Bound with the manuscript is a 26-page treatise in Persian ending with an Arabic sentence which may be translated as follows:

"Finished in 10 Rabīᶜ al-Awwal 1041 A.H. (1631 A.D.) by Muḥammad Anwar Mīrzā - Muḥammad Faḍlullāh Kāmil".

The MS. is not dated; 216 pages; 18 lines to page; in Naskhī; the hand is not clear; some diacritical marks and numerals are lacking; large margins; many marginal notes; glosses and examples, some written in Arabic and some in Persian; some numerals are written with errors; headings and numerals are written in red ink; with catchwords; figures are well done; complete; less accurate than MS.1036.

3 - MS. ELS 1189 (Princeton, Yahūda)

Not dated; 224 pages; 25 lines to page; in Fārisī; clear and fair hand; some diacritical marks and many numerals are lacking; large margins; few marginal notes, glosses and explanations; lacking are many phrases, figures, sentences, and some parts of the tables; many numerals are written with errors; from time to time the scribe has neglected th write in the red ink section-headings and some numerals; with some catchwords; figures are well done; incomplete (the last 22 examples of the last treatise are lacking); not accurate.

From the description of the three copies mentioned above, it becomes apparent that the greatest number of defects lie in the last one. And upon comparing them one is struck by the great similarity between the Leiden and the India Office copies in the occurence of some material not found in the Princeton copy, and the lack of other phrases, such as "God knows better", present in the latter.

15. Table of Contents of the Miftāḥ

Page

At the very beginning of the Miftāḥ al-Ḥisāb, the author names his other works and says that he invented the decimal fractions to facilitate computations for people ignorant of the sexagesimal system. He adds that he has put seven tables in it to ease the work of the computers and geometers (engineers). 5

Then follows the table of contents and a complimentary dedication to his patron, Ulugh Beg. The table of contents, including an introduction and 5 treatises, is translated as follows: 6

CHAPTER IV

TEXT AND TRANSLATION OF THE SOURCE

16. Remarks Concerning the Translation

All the definitions of the decimal fractions, rules for multiplying and dividing them, and the different methods of transformation of decimals into sexagesimals and vice versa are given in Chapter Six of the Third Treatise of the text. A facsimile of this chapter as it appears in the Princeton manuscript, together with its translation, is given below. The arrangement is made in such fashion that every page of the text is on a left-hand page of this thesis with its translation on the right-hand page opposite. And in order to simplify reference, the lines of the translation are numbered to indicate corresponding lines of the text. The next chapter is devoted to a commentary on the translation.

An ideal translation retains literal accuracy of the original while express-ing it in good style in the language of translation. Frequently, however, not both of these can be realised. And if, in the present translation, either one has to be sacrificed, the policy has been to preserve the spirit of the original at the expense of style.

In the translation, the following abbreviations and conventions have been adopted. Phrases inserted in parentheses, (....), are added by the translator for the completion of the meaning in gaps resulting from literal translation of the text. Square brackets, [.......], enclose restorations of textual passages or entries in numerical tables not present or falsely copied in the Princeton manuscript but taken from one or both of the other copies.

The manuscripts used here are referred to by the first letters of the names of libraries where they are found. Thus the letters "L", "I", and "P" stand for the Leiden, India Office and Princeton copies respectively. Footnotes indicated by letters have to do with the text itself and are used for denoting the variants in the copies of the text or for referring to the source from which a missing or an erroneous entry has been restored. Numerical subscripts refer the reader to sections given in the commentary.

17. Facsimile and Translation

الباب السادس في تحويل الارقام الستينية الى الهندية وبالعكس صحاحا وكسورا وتحويل كسورها الى مخرج آخر ومعرفة الكسور التي وضعناها على قياس الكسور الستينية ولنقدم هذا لما استخرجنا نسبة المحيط الى القطر في رسالتنا المسماة بالمحيطية وبلغنا الكسور الى التاسعة اردنا ان نحولها الى الارقام الهندية لئلا يعجز الحاسب الذي لم يعرف حساب المنجمين اخذنا كسر المحيط من مخرج هو عشرة الالاف مكررة خمس مرات وهذا عدد مجرد فكانا قسمنا الواحد الصحيح عشرة اقسام وقسمنا كل عشرة اقسام ثم كل قسم منها عشرة اقسام هكذا بالغا ما بلغ فسمينا الاقسام الاولى اعشارا لكونها كذلك والثانية ثاني الاعشار والثالثة ثالث الاعشار وهكذا بالغا ما بلغ ليكون مراتب الكسور والصحاح على نسبة واحد على قياس حساب المنجمين وسميناها بالكسور الاعشارية وينبغي ان يكتب الاعشار في يمين الاحاد وثالث الاعشار في يمين ثانيها هكذا الى حيث بلغ فيكون الصحاح والكسور في سطر واحد والعمل به في الضرب والقسمة واستخراج الضلع الاول من المضلعات وغيرها على قياس حساب المنجمين كما اوردنا بعضها فيما سبق وكذا يكون معرفة جنس المراتب على قياس معرفة جنسية مراتب حسابهم اعني يكون مرتبة عدد الاحاد صفرا وعدد للعشرات والاعشار واحدا وللمئات وثاني الاعشار اثنين وللالوف وثالث الاعشار ثلاثة ولعشرات الالوف ورابع الاعشار اربعة وهلم جرا ومجموع عددي مرتبتي المضروب والمضروب فيه ان كانا في طرف واحد من الاحاد او التفاضل بينهما ان اختلفا فهو عدد مرتبة الحاصل

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Line

1 Chapter Six:
On Transforming Sexagesimal Numerals into Hindu Numerals and
2 Conversely, Integers and Fractions, and On Transforming Their
Fractions into Different Units[1], and the Knowledge of
3 the Fractions Which We Invented[2] Corresponding to the Sexagesimal
Fractions: Let us introduce this.
4 When we extracted the ratio of the circumference to the diameter
in our work called
5 "(On) the Circumference"[3], and we carried the fractions to the
ninth[4] (place), we wanted to
6 transform them into the Hindu numerals so that the computer who
does not know the reckoning of the astronomers[5], would not be
helpless.
7 We have taken the fraction of the circumference (as a fraction)
with denominator ten thousands repeated five[6]
8 times and this is a pure[7] number. It is as though we divided
the unit into ten
9 parts and we divided every one [of them][a] into ten parts, then
every one of these
10 into ten parts and so on. Then we called the first parts
11 tenths because they are so; the second (set), the second tenths;
the third,
12 the third tenths and so on, in order that the place of the
fractions
13 and the integers be in one ratio, in the fashion of the astronomers'
reckoning and we called them
14 decimal fractions[8]. It is necessary to write the first tenth to
the right of [the units and the second tenths to the right of][a]
15 the first tenths, the third tenths to the right of the second
tenths and so on as far as it reaches. Thus the
16 integers and the fractions are in the same line. Operation with
them in multiplication,
17 division, and extraction of the first (n-th) root[9] of the powers
(of a number) and other (operations)
18 is in the same style as the astronomers' reckoning some of which
we have previously shown
19 and so, the knowledge of the kind of places[10] is like the know-
ledge of the kind of
20 places (in) their reckoning. I mean the place of the units' number
is zero[11]
21 and that of the tens and tenths is one; of the hundreds and the
second tenths,
22 two; of the thousands and the third tenths, three; of
23 ten thousands and the fourth tenths, four, and so on[12].
24 (In multiplying,) the sum of the place numbers (i.e. of the
exponents of ten) of two one-digit multipliers (is taken) if the
two (multipliers) are on the same side of the units' place,
25 or the difference between them (is taken) if they differ (in sides)
and it (the sum or difference) is the place number (i.e. the
exponent of ten) of the result,

a. Restored from L and I.

من طرف المجموع او من طرف الفضل ويكون التفاضل بين عددى مرتبتى المقسوم بين المفرد بن ان كانا في طرف واحد من الاحاد ومجموعهما ان اختلفا فهو عدد مرتبة الخارج من القسمة من سلسلة الصعود ان كانت مرتبة المقسوم فوق مرتبة المقسوم عليه والا من سلسلة النزول واما تحويل الارقام الصحاح الستينية الى الهندية فبان يضرب ما في اعلى المراتب في ستين بالرقوم الهندية ونضع نزيد على الحاصل ما في المرتبة التي يليها ونضرب المجموع في ستين ونزيد عليه ما في المرتبة التي يليها وهكذا الى ان ينتهى الى مرتبة الدرج ليحصل المطلوب طريق اخر نا خذ احاد ما في مرتبة الدرج فهو احاد المطلوب وان لم يكن في تلك المرتبة احاد فنضع صفرا مكان الاحاد ثم نقسم الباقي على العشرة الباقي على في جدول الستين فما خرج نا خذ من الدرج احادها ونضع مكان العشرات ثم نقسم الباقي على عشرة في جدول الستين فما خرج نا خذ احاد الدرج ونضع مكان المئات وقس عليه واما تحويل الارقام الصحاح الهندية الى الستينية فبان نقسمها على ستين فما بقي فهو الدرج وما خرج من القسمة نقسم ثانيا على ستين فما بقي فهو المرفوع مرة ونقسم ما خرج من القسمة نقسم ثانيا على ستين فما بقي فهو المرفوع وما خرج من القسمة الثاني وهلم جرا طريق اخر يضرب ما في اعلى المراتب في عشرة بجدول الستين ليحصل بالرقوم الستينية ونزيد على الحاصل ما في المرتبة يليها ونضرب

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Line

1 on the side of the sum or on the side of the difference[13]. (In dividing,) it will be the

2 difference of the two place numbers of the two one-digit numbers to be divided if they are

3 on the same side of the units' (place), but their sum if they differ. It (the difference or the sum) is the place number of

4 the quotient, being of the increasing series of numbers if

5 the place (numbers) of the dividend is higher than that of the divisor, otherwise it is of the

6 decreasing series of numbers[14]. As for the transformation of sexagesimal integers into Hindu numerals, it is by

7 multiplying that which is in the highest place by sixty in Hindu numerals,

8 then we increase the result by that which is in the next (lower) place and we multiply

9 the sum by sixty and we increase it by that which is in the next place,

10 and so on until it reaches the degrees' place so that which is desired may result.[15]

11 <u>Another Method</u>: [We take the ones' (digit) of the degrees' place][a] and it is the ones' (digit) of what is

12 required and if there is no [digit][b] in the units' (place) we put

13 zero in the units' (place). Then we divide the remainder by ten

14 in the sexagesimal (multiplication) table, and of that which comes out, [we take from the degrees' place

15 its units][c] and put it in the tens' place. Then we divide the remainder

16 by ten in the sexagesimal (multiplication) table, and of that which comes out we take the units of the degrees' (place)

17 and put it in the hundreds' place and so on[16]. As for the transformation of Hindu

18 [integers][d] into sexagesimals, we divide them (i.e. the numbers to be transformed) by sixty, and that which

19 remains is the degrees' (digit). We divide the quotient a second time

20 by sixty, the remainder is the first elevate[17]. Then we divide

21 the quotient by sixty, and what remains

22 [is][d] the second elevate, and [so on][d] [18].

23 <u>Another Method</u>: That which is in the highest place is multiplied by ten

24 in the sexagesimal table to get it in the sexagesimal numerals,

25 and we increase the result by that which is in the next place and

---

a. I has اخذ (ما) ما في ...

b. I and L have احاد

c. I and L have ناخذ من احاد الدرج

d. Missing in L also.

ونضرب المجموع في عشرة بجدول الستين بجدول الستين ونزيد على هذا الحاصل ما في المرتبة التي يليها وهكذا الى ان ينتهى الاحاد يحصل المطلوب وقد وضعنا جدولا يحصل منه تحويل الارقام الصحاح الهندي الى الستينة بالعكس والجدول في صفحة الاتية وطريق العمل عنه ظاهر واما تحويل الكسور المذكورة بعضها الى بعض فاثنى عشر لان الكسور المذكورة اعنى المستعملة اربعة انواع المفرد والستينة والاعشاري والدوانيق مع كسورها وتحويل كل واحد منها الى الثلاثة الباقية يكون اثنى عشر وقد ذكرنا في الباب الحادي عشر من المقالة الثانية اثنين منها وهما تحويل الكسر المفرد الى الدوانيق والطساسيج ونذكر العشرة الباقية منها الاول اذا اردنا تحويل الكسور بالارقام الستينة الى الارقام الهندية اى الى الكسور الاعشارية نضرب الكسور بالارقام الستينة في عشرة فان كان اول مراتب الحاصل اجزاء اعنى درجا فهي الاعشار وان لم يكن اجزاء فنضع مكان الاعشاري صفرا ثم نضرب كسور الحاصل اعنى غير الاجزاء في عشرة فان كان اول مراتب الحاصل اجزاء نضعها في المرتبة التي تلينا ها هكذا صورته

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Line

1 [we multiply][a] the sum by ten [in the sexagesimal table][a]
2 and [we increase this result by
3 that which is in the next place][b] and so on until the
4 units' digit [is reached][c], the desired (thing) results[19] . We have set down a table
5 by which Hindu integers can be transformed into sexagesimal integers
6 and vice versa[20] . The table is on the following page. And the method
7 of using it is obvious[d]

---

a. P has them repeated.
b. Missing in I also.
c. Restored from I and L.
d. P has lines 7 - 25 repeated.

صورته

المفردات	الآحاد	العشرات	المئات	الآلاف	عشرات الألوف	مئات الألوف	ألوف الألوف	عشرات ألوف الألوف	مئات ألوف الألوف	ألوف ألوف الألوف
١	ا	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
٢	ب	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
٣	ج	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
٤	د	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
٥	ه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
٦	و	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
٧	ز	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
٨	ح	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
٩	ط	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

صورة

الآحاد		مضروبات …
١	ا	[illegible]
٢	ب	[illegible]
٣	ج	[illegible]
٤	د	[illegible]
٥	ه	[illegible]
٦	و	[illegible]
٧	ز	[illegible]
٨	ح	[illegible]
٩	ط	[illegible]

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Digits	Units	Tens		Hundreds		Thousands			Ten Thousands			Hundred Thousands			
		1st Elevate	Parts	1st Elevate	Parts	2nd Elevate	1st Elevate	Parts	2nd Elevate	1st Elevate	Parts	3rd Elevate	2nd Elevate	1st Elevate	Parts
1	1	0	10	1	40	0	16	40	2	46	40	0	27	46	40
2	2	0	20	3	20	0	33	20	5	33	20	0	55	33	20
3	3	0	30	5	0	0	50	0	8	20	0	1	23	20	0
4	4	0	40	6	40	1	6	40	11	6	40	1	51	6	40
5	3	0	50	8	20	1	23	20	13	53	20	2	18	53	20
6	6	1	0	10	0	1	40	0	16	40	0	2	46	40	0
7	7	1	10	11	40	1	56	40	19	26	40	3	14	26	40
8	8	1	20	13	20	2	13	20	22	13	20	3	[12]	13	20
9	9	1	30	15	0	2	30	0	25	0	0	4	10	0	0

Digits	Thousand Thousands				Ten Thousand Thousands					Hundred Thousand Thousands					
	3rd Elevate	2nd Elevate	1st Elevate	Parts	4th Elevate	3rd Elevate	2nd Elevate	1st Elevate	Parts	5th Elevate	4th Elevate	3rd Elevate	2nd Elevate	1st Elevate	Parts
1	4	57	46	40	0	46	17	46	40	0	7	42	57	46	40
2	9	15	33	20	1	32	55	33	20	0	15	25	55	33	20
3	13	53	20	0	2	18	53	20	0	0	23	8	53	20	[0]
4	18	31	6	40	3	5	11	6	40	0	30	[31]	51	6	40
5	23	8	53	20	3	51	28	53	20	0	38	34	48	53	20
6	[27]	46	40	0	4	57	46	40	0	0	46	17	46	40	0
7	[32]	24	26	40	5	24	4	26	40	0	54	0	44	20	40
8	57	2	13	20	6	10	22	13	20	1	1	43	42	13	20
9	41	40	0	0	6	56	40	0	0	1	9	26	40	0	0

Digits	Thousand Thousand Thousands						Ten Thousand Thousand Thousands						
	5th Elevate	4th Elevate	3rd Elevate	2nd Elevate	1st Elevate	Parts	6th Elevate	5th Elevate	4th Elevate	3rd Elevate	2nd Elevate	1st Elevate	Parts
1	1	17	9	37	46	40	0	12	51	36	17	46	40
2	2	34	19	15	33	20	0	25	43	12	35	33	20
3	3	51	28	53	20	0	0	33	34	48	53	20	0
4	5	8	38	31	6	40	0	51	26	25	11	6	40
5	6	25	48	8	53	20	1	4	18	1	28	53	20
6	7	42	57	46	40	0	1	17	9	37	46	40	0
7	9	0	7	24	26	40	1	30	1	14	4	26	40
8	10	17	17	2	13	20	1	42	52	50	22	13	20
9	11	34	26	40	0	0	1	55	44	26	40	0	0

واما تحويل الكسور المذكوره بعضها الى البعض فاننا نشر لان الكسور المذكوره اعنى المستعملة اربعة انواع المفرده والستينة والاعشارى والدوانيق مع كسورها وتحويل كل واحد منها الى الستة الباقى يكون اثنى عشر وقد يكون فى الباب الحادى عشر من المقاد الثانى اثنين منها وهما تحويل الكسر المفرد الى الدوانيق والطساسيج وبالعكس فنذكر العشرة الباقى ههنا الاول لا تحويل الكسور بالارقام الستينة الى الارقام الهندية اى الكسور الاعشارى بضرب الكسور بالارقام الستينة فى عشرة فان كان اول مراتب الحاصل اجزاء اعنى درجا فهى الاعشار وان لم يكن اجزاء فنضع مكان الاعشار صفرا ثم بضرب كسور الحاصل اعنى من الاجزاء فى عشرة فان كان اول مراتب الحاصل اجزاء فنضعها فى المرتبة التى سمينا ها ثانى الاعشار وان لم يكن اجزاء فنضع مكان ثانى الاعشار صفرا ثم نضرب هذا الحاصل عن الاجزاء فى عشرة ونضع اجزاء الحاصل مكان ثالث الاعشار ان رفع بالاجزاء وعلى هذا القياس [illegible] اردنا ان نحول [illegible] ثالثة الى الكسور الاعشارية وضعنا شرح العمل فى جدول ليكون هكذا [illegible]

	[illegible]	[illegible]	[illegible]	[illegible]
ضربنا [illegible] فى عشر حصل	.	كد	مه	[illegible]
ثم ضربنا [illegible] غير الاجزاء فى عشرة حصل	د	ط	[illegible]	[illegible]
ثم ضربنا [illegible] فى عشرة حصل	ا	له	[illegible]	[illegible]
ثم ضربنا [illegible] فى عشرة حصل	.	[illegible]	[illegible]	[illegible]
ثم ضربنا [illegible] فى عشرة حصل	ط	[illegible]	[illegible]	[illegible]
ثم ضربنا [illegible] فى عشرة حصل	[illegible]	له	[illegible]	[illegible]

p.89

Line

1 As for transforming the (above) mentioned (kinds of) fractions from any kind into another there are twelve (ways)

2 because of the (above) mentioned (fractions). I mean those used, there are four kinds: common,

3 sexagesimal, decimal, and dawāniq[21] with their fractions. For transforming

4 each one of them into the other three there are twelve ways. We have [mentioned][a]

5 in the Eleventh Chapter of the Second [Treatise two][b] of them.

6 They are the transformation of common fractions into dawāniq and [ṭasūj][c][22] and vice versa[23].

7 We are now going to mention the remaining ten of them. <u>The first</u>: if we wish to transform

8 sexagesimal fractions into Hindu numerals, i.e., (into) decimal fractions,

9 the fractions, in sexagesimal numerals, are multiplied by

10 ten. If the first of the places of the result is part, i.e.,

11 degrees, then they are the tenths (of the result), and if there are no parts we put in the place

12 of the tenths, zero. Then fractions of the result i.e. (the remainder) from the (previous) parts are multiplied

13 by ten. If the first of the resulting places is parts, we put them

14 in the place which we called the second tenths. But if there are no parts

15 we put, in the place of the second tenths, zero. Then we multiply this result (remaining) from the parts

16 by ten and put the parts of the result in the third tenths' place

17 if they are elevated to parts and so on[24].

<u>Example</u>: we want to transform

18 8 , 29 , 44 thirds into decimal fractions. We have put the explanation

19 of the operation in a table [to be a rule and the table is this][d][25].

[The Explanation of the Operation]	Parts	Minutes	Seconds	Thirds
We multiply 8,29,44, by ten there results	1	24	57	20
We multiply 24,57,20, by ten, there results	4	9	33	20
We multiply 9,33,20, by ten, there results	1	35	33	20
We multiply 35,33,20, by ten, there results	5	55	33	20
We multiply 55,33,20 by ten, there results	9	15	33	20
We multiply 15,33,20, by ten, there results	2	35	33	20

---

a. Restored from page 87 of P and from I and L.

b. P has اطار

c. Missing in I also.

d. Restored from I and L.

p.90

Line

1 Since the minutes of the product i.e. 35,33,20 are more
2 than half (of a unit), we raised them by one, and the parts became three[26] and they are
3 the sixth tenths. Then we wrote the numerals in the table
4 of parts in Hindu (numerals), one after the other, and (the result) became thus 141593,
5 and this is the desired (answer) [and the place at the extreme right is][a] the sixth tenths. The second (Example):
6 If we want to transform decimal fractions into sexagesimal (fractions), we multiply them
7 by sixty, and that (part) of the result which is raised to the integers' (place) is the minutes' (digit),
8 and if nothing [of it][a] is raised to the integers' (place) we put in the minutes' place
9 zero. Then we multiply the fractions of the result by sixty, and that (part) of the result which is raised
10 to the integers' (place) is the seconds' (digit). But if
11 nothing is raised to the integers' (place), we put in the seconds' place zero,
12 and we do likewise for the [remainders][b] [27]. We have set down a form for this operation
13 which [exemplifies][c] the above, i.e., we multiplied the fractions by sixty,
14 put the result under it, [then (we multiplied) the fractions of the result by
15 sixty and put the result under it][d], and so on to the extent we desire.
16 We drew a line between the integers, resulting from the multiplication, and the fractions[28].
17 For Example: We wanted to transform 376 third tenths into
18 sexagesimal numerals. We operated thus:
19 We wrote the numbers which are in
20 the integers' table,[29] in sexagesimal numerals
21 one after another,
22 i.e., 22,33,36,
23 thirds, and this is the desired (answer).

fractions	Integers	[The Explanation of the Operation]
560	[22]	We multiply 576 third tenths by sixty, there results
600	[33]	We multiply the fractional part of the result which is 560 by sixty, there results
000	[36]	We multiply the fractional part of the result which is 600 by sixty, there results

24 We have brought out a table from which there results the transformation
25 of sexagesimal fractions into decimal (fractions) and vice versa. The table is this[30]: [And the use of this table is obvious to the intelligent.][f]

---

a. Restored from I and L.

b. P has it as الثراني ; restored from I and L.

c. P has بين.

d. Missing in I.

e. Restored from I and L.

f. P has it above the table.

ولما كانت دقايق حاصل الضرب اعني ٤٠ وهي اكثر من النصف رفعناها بواحد فصارت الاجزاء ثلثة وهي سادس الاعشار ثم كتبنا الارقام في جدول الاجزاء بالهندسية على الولاء صار هكذا ٤١٥٦٢ ١ ٠ ٠ وهو المطلوب والمراد من مراتب سادس الاعشار الثاني اذا اردنا تحويل الكسور الاعشارية الى الستينية فضربها في ستين فما رفع من الحاصل الى الصحاح فهو الدقايق وان لم يرفع شيء من الى الصحاح فنضع مكان الدقايق صفرا ثم نضرب كسور الحاصل في ستين فما رفع من هذا الحاصل الى الصحاح فهو الثواني وان لم يرفع شيء الى الصحاح فنضع مكان الثواني صفرا وقس عليه الثوالث وقد وضعنا دستورا لهذا العمل بمثل ما سبق وهو ان ضربنا الكسور في ستين ووضعنا الحاصل تحته ثم كسور الحاصل في ستين ووضعنا الحاصل تحته وهكذا الى حيث شئنا وخططنا بين الصحاح الحاصلة عن الضرب والكسور خطا مثلا اردنا ان نحول ٦ ٧ ٣ ثالث الاعشار الى الرقوم الستينية عملنا هكذا فكتبنا الاعداد التي في جدول الصحاح بالرقوم الستينية على التوالي وهو ٤ ٢ ٢ ثالثة وهو المطلوب وقد اوردنا جدولا تحصيل منه تحويل الكسور الستينية الى الاعشارية وبالعكس والجدول هذا

ضربنا ٦ ٧ ٣ ثالث الاعشار في ستين حصل ٥٢٠
ضربنا كسور الحاصل وهو ٢٠ في ستين حصل ٦٠٠
ثم ضربنا كسور الحاصل وهو ٠٠٠ في ستين حصل

والعمل بهذا الجدول لا يخفى على الفطن

			[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	الاعشار	دقيقه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	ثاني الاعشار	دقيقه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
		ثانيه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	ثالث الاعشار	ثانيه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		ثالثه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	رابع الاعشار	ثانيه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		ثالثه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		رابعه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	خامس الاعشار	ثالثه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		رابعه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		خامسه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	سادس الاعشار	ثالثه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		رابعه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		خامسه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		سادسه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	سابع الاعشار	رابعه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		خامسه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		سادسه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		سابعه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	ثامن الاعشار	رابعه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		خامسه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		سادسه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		سابعه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
		ثامنه	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	تاسع الاعشار	خامسه						
		سادسه						
		سابعه						
		ثامنه						
		تاسعه						
	عاشر الاعشار	سادسه						
		سابعه						
		ثامنه						
		تاسعه						
		عاشره						

الثالث اذا اردنا افراد الكسور الستينية اعني اخذها من مخرج واحد نضرب الدقايق في ستين و نزيد على الحواصل الثواني ونضرب المجموع في ستين ونزيد على الحاصل الثوالث وهكذا الى المرتبة التي نريد فيكون الحاصل

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	Tenths	2nd Tenths	3rd Tenths	4th Tenths	5th Tenths	6th Tenths	7th Tenths	8th Tenths	9th Tenths	10th Tenths
	Minutes	Minutes Seconds	Seconds Thirds	Seconds Thirds Fourths	Thirds Fourths Fifths	Thirds Fourths Fifths Sixths	Fourths Fifths Sixths Sevenths	Fourths Fifths Sixths Sevenths Eighths	Fifths Sixths Sevenths Eighths Ninths	Sixths Sevenths Eighths Ninths Tenths
1	6	0,36	3,36	0,21,36	2,9,36	0,12,57,36	1,17,45,36	0,7,46,33,36	0,0,27,39,36	47,51,41,45,36
2	12	1,12	[7],12	0,43,12	4,19,12	0,25,55,12	2,35,31,12	0,[15],33,7,12	1,35,18,43,12	9,19,52,19,12
3	18	1,48	10,48	1,4,48	6,28,48	0,38,52,48	3,53,16,48	0,23,19,40,48	2,19,38,4,48	13,59,48,28,48
4	24	2,24	14,24	1,[26],24	8,[38],24	0,51,50,24	5,11,2,24	0,31,6,14,24	3,6,51,26,24	18,39,44,38,24
5	30	3,0	18,0	1,48,0	10,48,0	1,4,48,0	6,28,48,0	0,38,52,48,0	3,52,16,48,0	23,19,40,48,0
6	36	3,36	21,36	2,9,36	12,59,56	1,17,47,36	8,46,33,36	0,46,39,21,36	4,39,56,9,36	27,56,36,57,36
7	[4]2	4,12	25,12	2,[31],12	15,[7],12	1,30,43,12	9,4,19,12	0,54,[25],55,12	5,26,35,31,12	32,39,33,7,12
8	48	4,48	27,48	2,52,48	17,16,48	1,43,40,48	10,22,4,48	1,2,12,28,48	6,13,14,52,48	37,19,41,16,48
9	54	5,24	32,24	3,14,24	19,[26],24	1,56,38,24	11,39,50,24	1,9,59,0,24	6,59,54,14,24	41,59,25,26,24

<u>Line</u>

1 <u>The Third Example</u>: If we want to unify the sexagesimal fractions,
2 i.e.,to express them with one denominator, we multiply the minutes by sixty and we add to
3 the result the seconds; we multiply the sum by sixty, and we add to
4 the result the thirds, and so on to the place we want. The last

الاخر كسر و مخرج ثلث للاثنية مخرجاله ثم نرد حاالى اقل عدد بين ما نسبتها ان لم يكونا منه ومخارج المراتب حى سنون ومضلعاته على التوالى اوردناها فى الجدول وهو هذا الرابع ان اردنا تحويل كسر مفرد الى الكسور الستينية نحول كل واحد من الكسر ومخرجه الى الرقوم الستينة على تقدير انه صحاح كما ذكرنا ثم نقسم رقوم الكسر على رقوم

الدقائق	٠	٦															
الثواني	٠	٠	٢	٣													
الثوالث	٠	٠	٠	٦	١	٣											
الروابع	٠	٠	٠	٠	٢	٩	٢	١									
الخوامس	٥	٠	٠	٠	٠	٦	٧	٧	٧								
السوادس	٠	٠	٠	٠	٠	٠	٦	٢	٤								
السوابع	٠	٠	٠	٠	٠	٠	٠	٦	٩	٩	٧	٢					
الثوامن	٠	٠	٠	٠	٠	٠	٠	٠	٢	٥	٥	٩	٧	٦	١		
التواسع	٠	٠	٠	٠	٠	٠	٠	٠	٥	٢	٩	٥	٧	٧	٥	٥	١
العواشر	٠	٠	٠	٠	٠	٠	٠	٠	٠	٦	٧	٥	٢	٢		٥	٦
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

المخرج فهو المطلوب مثاله اردنا ان نحول هذا الكسر ١٢٥/١٢٣ الى الكسور الستينية حولنا كل واحد منها الى الرقوم الستينية حصل رقوم الكسر ٥ و رقوم المخرج … لو قسمنا الاول على الثاني خرج من القسمة ٥٥ … او خامسه وباقي تركنا الخامس مراتب الكسور وواحد على يمين الاعشار فهو مخرج لذلك الكسر وهو عدد مجرد السادس ان اردنا بالعكس اى تحويل الكسر المفرد الى الاعشارى فنقسم الكسر على المخرج فاخرج فهو المطلوب مثاله اردنا ان نحول هذا الكسر ٢٢/٨٥ الى الاعشارى قسمنا الكسر وهو ٢٢ على المخرج وهو ٨٥ كما ذكرنا فى الباب الرابع من المقالة الاولى خرج من القسمة ٨٨ ٥ ٢ رابع الاعشار

P. 22

Line

1 result will be a fraction whose denominator is that of the last place [31]. Then we reduce it (the denominator) into the lesser of the two numbers
2 according to their (two) ratios if they (the two) were not of it (i.e., the lesser) [32]. The denominators of the places, which are sixty
3 and its powers in succession we set them down in a table and it is this: The Fourth (Example):
4 If we want [the contrary i.e. if we want][a] to transform a fraction of one denomination
5 into a sexagesimal fraction we transform (its numerator) and
6 denominator each into sexagesimal numbers as if
7 they were integers, as we have mentioned (before),
8 then we divide the numerals
9 of the fraction ('s numerator) by
10 the numerals of the
11 denominator [with the sexagesimal table, and that which results][a'] is what is required. Example: We wanted to transform the fraction $\frac{125}{1256}$ into a
12 sexagesimal fraction. We transformed each one of its (parts) into sexagesimal numerals.

																0	0	Minutes' denomination
														3	6	0	0	Seconds' denomination
												2	1	6	0	0	0	Thirds' denomination
										1	2	9	6	0	0	0	0	Fourths'
								7	7	7	6	0	0	0	0	0	0	Fifths'
							[4]	[5]	[6]	[5]	[6]	0	0	0	0	0	0	Sixths'
					[2]	[7]	[9]	[9]	[3]	6	0	0	0	0	0	0	0	Sevenths'
			1	6	7	9	[6]	[1]	6	0	0	0	0	0	0	0	0	Eigths'
	1	0	0	7	7	[6]	9	6	0	0	0	0	0	0	0	0	0	Ninths'
6	0	[4]	6	[6]	[1]	7	6	0	0	0	0	0	0	0	0	0	0	Tenths'
Hundreds	Tens	[Units]	Hundreds	Tens	[Units]	Hundreds	Tens	Units	Hundred Thousand Thousands	Ten Thousand Thousands	Thousand Thousands	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Units	The Names of the Places
Repeated Five Times			Repeated Four Times			Thousand Thousand Thousands												

13 There resulted 2,5 (for) the numerals of the numerator, and (for) the numerals of the denominator 20,36. We divided the first
14 by the second; there resulted from the division 6,4,4,39,36 fifths and that which remained we discarded.
15 The Fifth (Example): [If we want (to transform) a decimal fraction (into) a fraction of one denomination we put it ('s digits) in the place of the numerator itself and we put under it zeros to the number][b] of the places of the fractions and to their [right][c] we put one,
16 and this is the denominator. This fraction is a pure number. The Sixth (Example): If we want
17 conversely to transform a fraction of one denomination into a decimal (fraction), we divide
18 the numerator by the denominator and that which results is the desired (answer). Example: We wanted to transform
19 this fraction $\frac{22}{85}$ into decimals, we divided the numerator which is
20 22 by the denominator which is 85 as we have mentioned in the Fourth Chapter of
21 the First Treatise, the quotient was 2588 fourth tenths,

---

a. Restored from L and I.

b. Here a line and a half missing in P have been restored from L, which reads: ان اردنا افراد الكسور الاعشاري نضع موضع الكسر بعينه ونضع تحتها اصفاراً بعدة ---

c. It should be said to its left.

P.93

Line:

1 and we discarded what came after that and we determined the place as we mentioned in the beginning of this
2 chapter[33]. The Seventh and the Eighth (Examples): If we want to transform the decimal
3 fractions [and the sexagesimal fractions][a] into dawānīq, ṭasāsīj, and barley grains,
4 we multiply them (the fractions) by six, which is the denominator of the dawānīq, and that which is elevated to integers
5 is the number of dawānīq. Then we multiply the remainder by four and that which is elevated
6 [to integers is the number of ṭasāsīj. Then we multiply the remainder][b] by the four,
7 and that which is elevated (to integers) is the barley grains. [We do likewise if we want the fractions of the barley grains][c]. Example: We want to transform
8 22,18,44 thirds into dawānīq and ṭasāsīj and barley grains
9 and its fractions. We operated thus:
10 so that which is found in the table
11 of integers successively is
12 the number of dawānīq and ṭasāsīj
13 and their fractions. Thus we have
14 two dawānīq, [three][d] barley grains,
15 five dawānīq from
16 a barley grain and four fifths of a barley grain approximately. An Example: Transformation of Decimal
17 Fractions into Dawānīq and Ṭasāsīj: If we want to transform 8495
18 fourth tenths into dawānīq and their fractions we operate thus:
19 [and God knows better][e].
20 The Ninth and
21 The Tenth (Examples):
22 If we want to transform
23 the dawānīq and ṭasāsīj
24 and barley grains one into
25 the other we unify them as we mentioned in the Eleventh Chapter of the
26 Second Treatise[34], then we transform that [unified][f] [fraction][g] into whichever we desire of (these units as we explained), in the Fourth and Seventh (Examples) [of the Third Treatise][g].

[The Explanation of the Operation]	Integers	Fractions
We multiply 22,18,44. thirds by six, there results	2	13,52,24
We multiply 13,52,24, by four, there results	0	55,29,36
We multiply 55,29,36 by four, there results	[3]	[41,58,24]
We multiply 41,58,24 by six, there results	[4]	[11,50,24]
We multiply 11,50,24, by four, there results	[0]	[47,21,36]
We multiply 47,21,36 by four, there results	[3]	[9,26,24]

[The Explanation of the Operation]	Fractions	Integers
We multiply 8495 fourth tenths by six, there results	097	5
We multiply 097 third tenths by four, there results	388	5
We multiply 388 by four, there results	552	1
We multiply 552 by six to elevate dawānīq of barley grains, there results	258	2
We multiply 258 by four, there results	832	0
We multiply 832 by four, there results	322	3

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a. Missing in P.
b. Missing in I.
c. Restored from I and L.
d. All three copies have one.
e. Missing in I and L.
f. Missing in I.
g. Restored from I and L.

وتركنا ما بعده وعرفنا المراتب كما ذكرنا في اوائل هذا
الباب السابع والثامن ان اردنا تحويل الكسور ا
والاعشاري الى الدوانيق والطاسيج والشعيرات
نضربها في الستة التي هي مخرج الدوانيق فما رفع الى الصحاح
فهو عدد الدوانيق ثم نضرب الباقي في اربعة فما رفع
الى الصحاح فهو عدد الطاسيج ثم نضرب الباقي في الاربعة
فما رفع فهو عدد الشعيرات مثاله اردنا ان نحول كـ
ع مه ثالثة الى الدوانيق والطاسيج والشعيرات
وكسورها عملنا هكذا
فما وقع في جدول
الصحاح على التوالي هو
اعداد الدوانيق والطاسيج
وكسورها وذلك دا
دانقان وشعير واحد
وحصة دوانيق من
شعيرة واربعة اخماس شعيرة تقريبا مثال التحويل الكسور

	[illegible]	[illegible]
ضربنا كد ع ب ثالثة في ستة حصل	ب	ع سه كد
ثم ضربنا ب سه كد في اربعة حصل	ع	ند كط مو
ثم ضربنا نه كط لو في اربعة حصل	ا	لح ط سه
ثم ضربنا لح مط سه في ستة حصل	ه	ود سه
ثم ضربنا ود سه في اربعة حصل	٢	ما كد
ثم ضربنا ما كد في اربعة حصل	٤	بركا لو

الاعشاري الى الدوانيق والطاسيج اردنا ان ٥ ٩ ٤ ٨
رابع الاعشار الى الدوانيق وكسرها عملنا هكذا و
والله اعلم
التاسع و
والعاشر
اذا اردنا تحويل
الدوانيق والطاسيج
والشعرات الى

	[illegible]	[illegible]
ضربنا ٥ ٩ ٤ ٨ رابع الاعشار في ستة حصل	٥٩٧	٥
ثم ضربنا ٧ ٩ ٥ ثالث الاعشار في اربعة حصل	٣٨٨	٥
ثم ضربنا ٨ ٨ ٣ في اربعة حصل	٥٥٢	١
ثم ضربنا ٢ ٥ ٥ في ستة لترفع دوانيق الشعير حصل	٥٨ ٢	٣
ثم ضربنا ٨ ٥ ٢ في اربعة حصل	٨٣٢	٠
ثم ضربنا ٢ ٣ ٨ في اربعة حصل	٣٣٢	٣

احد منها فنفردها كما ذكرنا في الباب الحادي عشر من المقالة
الثانية ثم نحول ذلك المفرد الى ابها اردنا في الرابع والتاسع

18. Commentary on the Text

It is assumed that the reader will work primarily either from the facsimile Arabic text reproduced in the preceding article, or from the English translation. Only those parts of the text which, in the opinion of the translator, are not self-explanatory are commented on below.

p.85

1. (line 2): By "Transforming Fractions into Different Units" is meant transforming a group of decimal or sexagesimal fractions of different denominations into a group of fractions of the same denomination. For more detail see note 32 to page 92.
2. (line 3): The phrase "The Fractions Which We Invented" shows that al-Kāshī regarded himself as the inventor of decimal fractions.
3. (line 5): See p. 56.
4. (line 5): Luckey states that al-Kāshī's computation of 2π was to nine sexagesimal places all accurate, i.e.,

$$2\pi = 6;\ 16,\ 59,\ 28,\ 1,\ 34,\ 51,\ 46,\ 14,\ 50^{1}.$$

5. (line 6): He means sexagesimal computation.
6. (line 7): What the author intends here is not clear. It may be that he wants to say that he divided the circumference of the circle into ten parts and every one of those into ten parts and so on until the circumference is divided into 10^{20} parts. Then each one of the parts will be $\frac{1}{10^4}$ multiplied by itself five times i.e. $\frac{1}{10^{20}}$.

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1. Luckey, 1, p. 199.

7. (line 8): The meaning of "pure number" is not clear from the text. It is probable that the author wanted to say that it is a number which is not a mixed one.

8. (line 14): It is curious that al-Kāshī calls the decimal fractions _al-kusūr al aʿshārī_, a name very similar to the modern Arabic one, _al-kusūr al ʿasharīya_, although al-Kashi's nomenclature probably had no influence on the modern choice. Stevin, the inventor of decimal fractions in Europe, did not call them fractions at all. He wanted to "complete all the accounts and measurements without fractions, i.e., with integers only".[1]

9. (line 17): See Dakhīl in the Bibliography.

10. (line 19): Al-Kāshī uses the Arabic word _martaba_, translated here as _place_, in two senses having related but not identical meanings. In this passage it means the place of a particular digit in a sequence representing a number. In the number 579, for example, nine is in the place of the units, _martabat al-āḥād_, whose place number is zero; seven is in the place of the tens, _martabat al-aʿshār_, whose place number is one; and five is in that of the second tens, _martabat thānī al-aʿshār_, whose place number is two. Hence it is clear that the _place_ is in fact the exponent of the base which is multiplied by the digit in question. For further discussion and for the second meaning of _martaba_ see notes 11 and 12 below.

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1. Sarton, 2, p.161. Also see p. 42 above.

11. (line 20): The use of zero for the units' place number is of considerable importance. The medieval practise was to call it one. For more detail see p. 40 above.

12. (line 23): Al-Kāshī's nomenclature "first tenths", "second tenths", "third tenths", etc., is different from our "tenths", "hundredths", "thousandths", etc., and has been taken over from the sexagesimal nomenclature "minutes", "seconds", "thirds", etc.

It is useful at this stage to compare the characteristics of the sexagesimal system with those of the decimal system as used by al-Kāshī.

There are in the sexagesimal system three points to be discussed. First, the word degree (daraja) has the meaning, as in the case with us, of an angular unit equal to $\frac{1}{360}$th of the circumference of the circle. But in medieval times it had also the general meaning of a unit of any kind of measurement, lengths, weights, areas, etc., or as the name of the units' digit in any sexagesimal number.

Secondly, there are different categories of terms used for the places, (marātib), in the sexagesimal system. Other than the degrees' place, (martabat al-daraj), there are places associated, in our terminology, with positive exponents on the one hand, and with negative exponents on the other. Consider the familiar sexagesimal number:

$$\left| s_k, \quad s_{k-1}, \ldots\ldots, s_1 \right|, s_0 \quad , \left| s_{-1}, \ldots\ldots, s_{-m}, \right|$$

which represents the finite power series

$$\left| s_k \cdot 60^k + s_{k-1} \cdot 60^{k-1} + \ldots + s_1 \cdot 60^1 \right| + s_0 \cdot 60^0 + \left| s_{-1} \cdot 60^{-1} + \ldots + s_{-m} \cdot 60^{-m} \right|$$

and notice the three classes into which the number is divided by means of vertical bars. The s's are associated with positive and negative exponents. Al-Kāshī calls the right-hand class the increasing series of numbers, (al-mutaṣā'ida); the left-hand class the decreasing series of numbers, (al-mutanāzila). The single digit s_0 is said to be in the degrees' place. The elements of the increasing series of numbers have individual names. The place of the digit s_1, for example, which we translate as the first elevate, is called al-marfū' marra, the place of the digit s_2, which we translate as the second elevate, is called al-marfū' marratain, and so on. Synonymously with the successive elevates, al-Kāshī uses mathānī, mathālith, etc. The elements of the decreasing series have also individual names. The place of s_{-1}, translated as minutes, is called daqā'iq; that of s_{-2}, translated as seconds, is called thawānī, and so on. Some such terminology as increasing and decreasing series of numbers was necessary because negative numbers were as yet unknown.

Thirdly, no such device as the "sexagesimal point" was used by the medieval mathematicians[1], and in order to fix the value of a number it is not sufficient to give only its digits. For

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1. For the method of separating integers from fractions see p.15 above.

example, 20,0,52 might mean 20; 0, 52 or 0; 0, 0, 20, 0, 52 or, in general, $(20,0,52).60^k$, where k is any integer. In order to obviate this difficulty a place, (martaba), was written with every such representation, it being the place of the last digit. Thus the number which we write as 0; 50, 13, 20 would be written as 50, 13, 20 thirds, and the number which we write as 35, 60,0,0;0 would be written as 35,6 second elevates.

This is the second sense in which al-Kāshī used the word martaba, for which the first meaning was given in Note 10 (line 19) above.

The word elevate can be used not only in the sense mentioned above, but also to indicate the operation of multiplying a number by some power of the base of the system in use. For example, the first elevate, second elevate, third elevate, etc., of the number 2, 15 thirds are 2, 15 seconds (i.e. 0; 2, 15); 2, 15 minutes (i.e. 2; 15); 2, 15 degrees (i.e. 2,15); etc. respectively. This corresponds, in modern terminology, to shifting the decimal or the sexagesimal point to the left.

Kāshī's terminology of the decimal system is analogous to that of the sexagesimal system except for the names of places. Thus the minutes, seconds, thirds, etc., of the sexagesimal system are replaced by first, second, third, etc., tenths; and the first, second, third, etc., elevates are replaced by the first, second, third, etc., tens. Here again al-Kāshī's terminology is different from the modern which uses tens, hundreds, thousands, etc., respectively.

p. 86

13. (line 1): Here is a verbal statement of the familiar law of exponents, $a^m . a^n = a^{m+n}$, where the exponents m and n may be positive, negative, or zero. But here again, as it was mentioned before, since al-Kāshī does not have negative numbers, he has to make up for this lack by using the phrases "on the same side" or "on different sides" where we would say the numbers have either positive or negative exponents. The statement that these "are added or subtracted" is like our statement, "their absolute values are to be added if the numbers have the same sign and subtracted if the signs of the numbers differ". Thus far al-Kāshī's rule is correct. It remains to determine the side (sign) of the resulting exponent (place). Al-Kāshī only says that "the side of the result is that of the sum or the difference". As for the sum, his statement is correct because there is a common side for the two added numbers. But for the difference, the statement is not clear because the two numbers to start with lie on different sides. To clarify the matter, let us suppose that μ and ν are the absolute values of the exponents m and n of the two numbers to be multiplied, a^m and a^n, m and n being of different signs. Then the rule can be written fully, using al-Kāshī's terminology, as follows:

1- If μ is ascending and $> \nu$, then the side of the place of the result is increasing

2- If μ is ascending and $< \nu$, then the side of the place of the result is decreasing

3- If μ is descending and $> \nu$, then the side of the place of the result is decreasing

4- If μ is descending and $< \nu$, then the side of the place of the result is increasing.

14. (line 6): This is the rule $a^m/a^n = a^{m-n}$, where m and n may be positive, negative or zero. Al-Kāshī's method of subtraction, namely "to add the places of the two numbers if they are on different sides and subtract them if they are on the same side", is the same as our modern rule for the subtraction of signed numbers: the absolute values of the exponents are added if they are of different signs and subtracted if they are of like ones. Moreover, his statement about the sign of the result as being of the increasing series of numbers (positive) if the place of the dividend is higher and of the decreasing series of numbers (negative) if otherwise, causes the suspicion that he unconsciously used the notion of the set .... , -3, -2, -1, 0, 1, 2, 3, ... in which any number is "higher" than that lying on its left. If this was indeed the case, al-Kāshī's rule covers all the possibilities and is completely correct.

15. (line 10): To explain this method, let us first give a numerical example. Assume that we want to transform the sexagesimal integer 2, 13, 4 into a decimal. Multiply two by sixty (in the decimal system), add thirteen to the product, multiply this sum by sixty, and add four to this result. Or, symbolically, it may be written as: $\{(2.60) + 13\}.60 + 4 = 7984$.

In general, consider the sexagesimal integer

$$a_k, a_{k-1}, \ldots, a_0 \ ,$$

in which a_i is the sexagesimal digit in the i-th place. This representation is an abbreviation of:

$$a_k.60^k + a_{k-1}.60^{k-1} + \ldots + a_0.60^0 \ .$$

According to al-Kāshī, its transformation into a decimal is:

$$(((\ldots(a_k.60 + a_{k-1}).60 + \ldots).60 + a_2.60 + a_1).60 + a_0,$$

where all the multiplications are in the decimal system. Hence the result is a decimal. But it is seen that removing the brackets gives the sexagesimal representation shown above for the given integer in the decimal system.

16. (line 17): Before explaining this method the reader should be reminded that the author uses the jummal system, whose characteristics differ, in minor respects, from those of ours.

In our system, the order of the increasing series of numbers is from right to left, while in the jummal system it is from left to right. For example, on page 89, line 18, of the text the number which would be written in our way as 0; 8, 29, 44 appears in the jummal system in the opposite order.

Every sexagesimal digit which is greater then ten in the jummal system is represented by two alphabetic letters joined together as if they are one number. The sexagesimal digit 25, for example, expressed by the use of our symbols, would be written by al-Kāshī as كه .

While al-Kāshī uses with the sexagesimal system the alphabetic numbers, he uses with the decimal system the modern Arabic symbols. On page 89, mentioned above, for instance, al-Kāshī's sexagesimal example is written with letters while its value after being transformed into a decimal is written in Hindu-Arabic symbols. In this respect al-Kāshī's method of representing numbers is different from ours since in this paper Hindu-Arabic symbols are used for both the decimal and the sexagesimal systems.

Now returning to al-Kāshī's second method of transformation, we notice that it is a special case of the method mentioned on p. 8, where N is a sexagesimal number and b = 10. The only difference is that the author here notices that because sixty is divisible by ten, each sexagesimal digit contains a part which is a multiple of ten and a remainder less than ten. The sexagesimal digit 42, for example, when divided by ten, gives a quotient four and a remainder two. So in order to avoid one operation of division he says: "the ones' digit of the degrees' place (i.e. the smaller part of the last sexagesimal digit of the dividend) is the ones' digit of the answer" and used this definition in every step of division.

For further clarification, let us give a numerical example. Suppose we want to transform the sexagesimal integer 3, 21, 42 degrees into a decimal. As it is shown in the table given here, put two, the right-hand part of the digit 42, in the degrees' place,

4th tens	3rd tens	2nd tens	1st tens	units
1	2	1	0	2

under the units' place of the answer. Divide 3, 21, 40 by ten, using the sexagesimal multiplication table, to get the quotient 20, 10, and put the zero under the tens' place. Divide 20, 10 by ten sexagesimally to get 2, 1 and put one under the second tens' place. Then divide 2, 0 by ten to get 12 and put two under the third tens' place. Lastly divide ten by ten to get one and put it under the fourth tens' place. The result 12102 is a decimal integer.

17. (line 19): For the word elevate see Note 12 above.

18. (line 22): This is a special case of the method mentioned on p. 8 above, where N is a decimal integer and b = 60.

For further clarification suppose the decimal integer 14069 is to be transformed into a sexagesimal one. Divide it first by sixty to obtain the quotient 234 and the remainder 29, which is the units' digit. Then divide 234 by sixty to obtain a quotient of 3 and a remainder 54; the former is the second elevates' digit and the latter is the first elevates' digit. Thus the sexagesimal integer becomes 3, 54, 29.

p. 87

19. (line 9): The method and proof are the same as those mentioned in Note 15 above. The only difference here is that now the multiplication is by ten (in the sexagesimal multiplication table) instead of by sixty (in the decimal system). Thus in general if we want to transform the decimal integer $d_k\, d_{k-1} \,.....\, d_0$ into a sexagesimal we write it sexagesimally according to the given rule, as: $(\,(\,(\,....(d_k.10 + d_{k-1}).10 + ...).10 + d_1).10 + d_0$, where d_i is a decimal digit in the i-th place.

20. (line 6): This table is a rectangular array bounded on top by two headings containing the names of the decimal places, and those of the sexagesimal units and elevates, and on the left by a column containing the decimal digits 1, 2, 3, ... 9. Sexagesimal numbers are entered in the remaining blanks.

Consider only the rows and columns in which the sexagesimal numbers are found. The principle upon which this table is built is the following:

The element $a_{i,j}$ in the i-th row and the j-th column is the sexagesimal representation of $i \cdot 10^{j-1}$.

In order to transform a decimal integer into a sexagesimal it is necessary only to find in the table the sexagesimal integers corresponding to the positional values of the decimal digits respectively, then to add the resulting sexagesimals.

To transform the decimal integer 32563, for example, into a sexagesimal, we look for the sexagesimal integers corresponding to 30000, 2000, 600, 60 and 3, and we get 8,20,0; 33,20; 8,20; 1,0; and 3 respectively. Addition of these gives 9,2,43, the result.

Now to transform a sexagesimal integer into a decimal, first find two successive elements of the same column, say $a_{i,j}$ and $a_{i+1,j}$ such that the given number N satisfies the inequality $a_{i,j} \leqslant N < a_{i+1,j}$. Then i is the j-th digit of the result. If $a_{i,j} = N$, the process is finished. Otherwise take the difference $N - a_{i,j}$, and, dealing with it in the same fashion as with N, we obtain the $(j - 1)$th digit of the result. The operation is repeated until the last column is reached, and hence the whole result is obtained. It is to be noticed that, in this process, whenever a column is skipped a zero must be inserted in the answer.

Suppose the sexagesimal integer 32, 29, 0, 53 is to be transformed into a decimal. Here $a_{i,j}$ is $a_{7,7}$ = 32, 24, 26, 40. Hence the first digit of the result is 7 and t he first difference is 4, 34, 13. Repeating the process with this difference and the ones following it, we get 1, 6, 4, and 53 as the successive values of i. And because we skipped the sixth column a zero should be put in the sixth place of the result, and this latter becomes 7016453.

p. 69

21. (line 3): Dawānīq is the plural of dāniq, a unit of weight equal to two carob-grains, and this in turn equals one sixth of a dashim.

22. (line 6): The ṭassūj is a unit of weight equal to four dawānīq.

23. (line 6): Methods of transforming these units will be explained in Note 34 below.

24. (line 17): In general, according to this method, to transform a sexagesimal fraction S into a decimal we multiply sexagesimally by ten as follows:

$$S.10 = d_{-1} + F_1$$
$$F_1.10 = d_{-2} + F_2$$
$$\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots$$
$$F_{(k-1)}.10 = d_{-k} + F_k$$

where d_{-i}, the integer part of the i-th product, is the decimal digit in the (-i)th place of the result, F_i is the corresponding fractional part, and F_k is the fractional part remaining after k steps. F_k may be zero, in which case the fraction terminates and is represented as $.d_{-1}\,d_{-2}\,\dots\dots\, d_{-k}$.

It is possible, however, that $F_k \neq 0$ for any k, and the fraction is non-terminating. But this distinction is not made by al-Kāshī. In this case, if we want to stop the operation after the k-th step, we obtain the decimal fraction $A = .d_{-1}\,d_{-2}\,\dots\,d_{-k}$ with a final sexagesimal remainder F_k. Now, if F_k is less than

0; 30, we discard it, but if it is bigger than 0; 30 we add one unit to the digit in the (-k)th place and the result becomes $B = .d_{-1}\, d_{-2} \ldots\ldots (d_{-k} + 1)$.

Every digit in these results is less than ten. In fact, s, being a proper fraction, satisfies the inequality $0 < s < 1$. Multiplication by ten gives $0 < 10\,s < 10$. And it is clear that if $10\,s < 10$, then its integer part is also less than ten.

Now, if one takes the value of s from the equality of the first step in the transformation method given above, puts instead of F_1 its value from the next step, then repeats the process, substituting always for the F's their values, one gets the equality: $s = 10^{-1}(d_{-1} + 10^{-1}(d_{-2} + 10^{-1}(d_{-3}+\ldots+10^{-1}(d_{-k}+F_k))))$

or $s = d_{-1}.10^{-1} + d_{-2}.10^{-2} + \ldots. + d_{-k}.10^{-k} + F_k.10^{-k}$

Discarding F_k we get :

$$s \approx d_{-1}.10^{-1} + d_{-2}.10^{-2} + \ldots. + d_{-k}.10^{-k} = A$$

Discarding F_k after adding one to the (-k) place, we get:

$$s \approx d_{-1}.10^{-1} + d_{-2}.10^{-2} + \ldots. + (d_{-k} + 1).10^{-k} = B$$

These latter approximations are the decimal representations of the results A and B.

The result of this method is completely correct if the fraction terminates and we stop the operation when $F_k = 0$ is obtained. But whether the fraction terminates or not, if we want to stop the operation at the (-k)th place, then the result will be correct to the (-k)th place and with an error less than 10^{-k}.

For, comparing the approximated results A and B with the original fraction s, we get:

$$R_A = s - A = (d_{-1}.10^{-1}+d_{-2}.10^{-2}+\dots+d_{-k}.10^{-k}+F_k.10^{-k})-(d_{-1}.10^{-1}+\dots+d_{-k}.10^{-k})$$
$$= F_k.10^{-k}.$$

and

$$R_B = B - s = (d_{-1}.10^{-1}+\dots\dots+d_{-k}.10^{-k}+1.10^{-k})-(d_{-1}.10^{-1}+\dots+d_{-k}.10^{-k}+F_k.10^{-k})$$
$$= (1 - F_k).10^{-k}$$

But F_k is a fraction, hence it is always less than one. Therefore each of R_A and R_B is less than 10^{-k}, and the error is always less than 10^{-k}.

25. (line 19): In this numerical example the final remainder is $F_k = 0; 36, 33, 20$, and because it is more than 0; 30 the author adds one unit to two, the last digit.

p. 90

26. (line 2): The addition of one to the digit in the (-k)th place is the rounding-off process customary in computations with modern decimal fractions,ie., to discard the final remainder F_k after adding one to the (-k)th digit or without the addition of anything depending on whether F_k is greater or less than half the base of the system used. The discussion is given in Note 24 above.

This careful attitude of al-Kāshī toward remainder terms is in contrast to that of Ptolemy and other ancient computers who are careless in such matters.

In connection with this process it is interesting to mention the theorem by means of which one can know whether or not the representation of a given rational fraction in any system terminates. For a necessary and sufficient condition that the representation of the rational fraction $\frac{p}{q}$ be terminating is that q shall contain no prime factors except those of b, the base of the system in which the fraction is to be represented.

Let us assume without loss of generality that p and q are relatively primes. For, if they are not, their common factors can be cancelled.

1- Necessary Condition: If the fraction is terminating, then it can be represented as $\frac{p}{q} = \frac{d_1}{b} + \frac{d_2}{b^2} + \cdots\cdots + \frac{d_k}{b^k}$, where d_i is a positive integer and less than b. Addition of the fractions on the right-hand side gives:

$\frac{p}{q} = \frac{D}{b^k}$, where $D = d_1.b^{(k-1)} + d_2.b^{(k-2)} + \ldots\ldots + d_k$ or:

$D = \frac{p}{q} \cdot b^k$, But D is an integer, hence $\frac{p}{q}$ is also an integer. Therefore q, having no common factors with p, should contain only the factors of b^k, the base of the system used.

2- Sufficient Condition: Either q contains only factors of b and the theorem is proved or it does not. If q contains factors other than those of b, then $\frac{p}{q}.b^k$ and hence D are not integers. This means that the representation given above is not terminating, which contradicts what is given. Therefore q should contain only factors of b.

27. (line 12): The method is like that explained in Note 24 above, except that here b = 60 .

28. (line 16): Al-Kāshī uses a bar to separate integers from fractions. This is a near approach to the use of a decimal point.

29. (line 20): What the author means here is that after writing all the fractions in one column in the table and all the integers in another one, he uses for the result those digits in the latter column successively and writes the answer as 0; 22, 33, 36.

30. (line 25): This table is laid out in the same fashion as that given for transforming integers. The only difference is that in the heading, here, the upper row contains the names of places of decimal fractions and in the second row the names of sexagesimal fractions are inserted.

The principle upon which this table is built is: the element $a_{i,j}$ in the i-th row and the j-th column is the sexagesimal representation of $i \cdot 10^{-j}$.

The method of transformation of decimal fractions into sexagesimals is the same as that used for integers. To transform the decimal fraction 0.258, for example, we look for the sexagesimal fractions corresponding to 0.2 ; 0.05 ; 0.008 . We get 12; 3,0; 27,48. Adding, there results 15,27,48, the answer.

Also the transformation of sexagesimal fractions into decimals is the same as that used for sexagesimal integers. The only difference is that if $a_{i,j}$ and $a_{i+1,j}$ are the two successive elements such that $a_{i+1,j} \geqslant N > a_{i,j}$ we take the difference $N - a_{i,j}$ and we repeat the process.

To transform the sexagesimal fraction 0; 22, 33, 36, for example, into a decimal we notice that the sexagesimal fraction $a_{i,j}$ is $a_{3,1}$ = 18, 0, 0, whence i = 3, and the first difference is 4, 33, 36. Repetition of the process with this difference and with the one which follows gives 7 and 6 as values for i and hence 0,376, the result.

<u>p. 91</u>

31. (<u>line 1</u>): The aim here is to reduce a sexagesimal fraction, without changing its value, into a proper fraction whose numerator is a decimal integer and whose denominator is a power of sixty. To clarify the matter let us give a numerical example explaining the method in modern terminology.

Suppose we have the sexagesimal fraction 0; 25, 2, 37. Multiplying its numerator and denominator by sixty successively we get:

$$\frac{60\,(0;25,2,37)}{60} = \frac{25;2,37}{60}$$

$$\frac{60(25;2,37)}{60^2} = \frac{60\times 25 + 2;37}{60}$$

$$\frac{60(60\times 25 + 2;37)}{60^3} = \frac{60^2\times 25 + 60\times 2 + 37}{60^3}$$

This may be written as $\dfrac{(60\times 25 + 2)\times 60 + 37}{60^3} = \dfrac{90157}{60^3}$

The left-hand side of the last step represents al-Kāshī's rule which may be expressed as follows: take the highest sexagesimal digit 25, multiply it by sixty in the decimal system, add to the result the next sexagesimal digit 2, multiply the result by sixty, and add to the product the next digit 37. This gives $\frac{90157}{60^3}$, a proper fraction whose numerator is a decimal integer and denominator is the same as that of the thirds, the lowest digit in the given sexagesimal fraction.

<u>p. 92</u>

32. <u>(line 8)</u>: The language of this passage is not clear. But apparently the author wants to say that, in general, to compare two fractions of different places, they have to be reduced to the same place. Hence the numerator and denominator of the smaller should be multiplied by the ratio between the denominator of its place and that of the place of the larger.

For example, to compare a fraction in the thirds' place $\frac{a_1}{60^3}$, say, with another in the sevenths' place $\frac{a_2}{60^7}$, we take the ratio $\frac{1}{60^3} / \frac{1}{60^7}$ and get 60^4. Then we multiply $\frac{a_1}{60^3}$ by $\frac{60^4}{60^4}$ and obtain $\frac{60^4 \cdot a_1}{60^7}$, a fraction in the sevenths' place.

p. 93

33. (line 1): See Note 11 above.

34. (line 26): The Eleventh Chapter of the Second Treatise deals with transforming fractions from one kind into another. Knowing that the "denominator" [1] of *dawāniq*, *ṭasāsīj* and *barley grains* are six, four, and four respectively, we transform *dawāniq* into *ṭasāsīj* and *ṭasāsīj* into *barley grains* by multiplying them by four. Conversely, we transform *barley grains* into *ṭasāsīj* and *ṭasāsīj* into *dawāniq* by dividing them by four. Fractions of these three units of weight are also dealt with in the same fashion.

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1. Al-Kashi uses the word *makhraj*, usually translated as denominator, for the denominator of the fraction representing the ratio between two units. For example, since 4 dawāniq = 1 ṭasūj , we have the ratio dāniq/ṭasūj = $\frac{1}{4}$. Here four is the denominator mentioned by al-Kāshī.

19. Conclusion

Having presented al-Kāshī's work in detail, we are now in a position to conclude the present study by comparing his work with that of other contributors to the same subject and by discussing the possibilities of mutual influece.

We can trace no evidence of Bonfils' influence on his contemporaries. For his work, which is listed only in Zotenberg's catalogue of the Hebrew manuscripts of Paris [1], is not known to have been translated, either in his time or in the following period, into any other language. The English translation used in this study is a very recent one.

Stevin, on the contrary, had great influence on his contemporaries as well as on his successors. The printing of his work and its translation into different languages show how much importance was attached to it.

In connection with al-Kāshī, we can say that his manuscript was not only popular but also much more so than any of his other works. In fact, copies of the manuscript are listed in seventeen different catalogues, while al-Risāla al-Kamāliya, which is the next most widespread in catalogued collections, is listed only in three catalogues. The cause of this popularity, however, may be its other contents and not the decimal fractions. In reality, decimal fractions

1. Gandz, 2, p. 19.

did not spread into wide use among the Arabs before they were imported from Europe. Besides, there is no available evidence of a direct influence of al-Kāshī's manuscript on other Western or Eastern mathematics.

Bonfils, al-Kāshī, and Stevin were not contemporaries, but they lived in the same general period, and their works have a great deal in common. None of them, for example, used a decimal point to indicate a distinction between fractions and whole numbers. Also, the rules they gave for multiplication and division are very similar. They all express the fact that to obtain the place of the product of two numbers one adds the places of the two multipliers, if they are both either integers or fractions, or subtracts them if only one is a fraction. In the second case, the product is in the units', fractions' or integers' place depending on whether the place of the fraction is equal to, larger than, or smaller than that of the integer.

Other points of resemblance can be detected between the work of any two of them, such as the meaning of numbers' places. They all recognized three classes of numbers — integers, units, and fractions — and gave individual names to the places of all their digits, except that Stevin did not name the different integers' places as distinct from each other. None of the three dealt with negative numbers, but Bonfils and al-Kāshī showed some implicit recognition of their existence. The former referred to fractions as having a "direction" opposing that of the integers, while the latter spoke of "increasing series of numbers" and of "decreasing series of numbers" instead of positive and negative exponents, respectively. Again, Stevin and al-Kāshī both employ the same basic principle in their use of symbols for integers, units, and fractions. This usage doubtless originated in the sexagesimal system, where the digits were sometimes written in columns headed by the respective names of their places, and sometimes not tabulated but each number accompanied by the place-name of the lowest digit in it. Stevin adopted the first method by writing the place of each digit (except the integers) above or to the right of it in a small circle. Al-Kāshī wrote the "martaba" of every digit at the head of the column in which it stands. The second method was also adopted by both of them, for Stevin sometimes wrote, in a little circle to the right of the number, the place-number of the lowest digit in it, and al-Kāshī wrote the place-name of the lowest digit beside the number. Bonfils, on the other hand, employed no symbols at all. Bonfils' work is also different from those of the other two in its lack of illustrative examples and application to practical problems.

Chronologically, there are three possibilities of influece between these three mathematicians: possible influence of Bonfils on al-Kāshī and on Stevin, and of al-Kāshī on Stevin. Bonfils' work in astronomy, his book entitled "Eagle Wings", was carried by the Qaraites (a migrating Jewish sect of the Middle Ages) to Constantinople[1]. But,

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1. Sarton, 1, vol. III, part II, p. 1117.

this being the only one of his works mentioned by them, we can practically rule out the possibility of his work on decimal fractions having influenced either Stevin or al-Kāshī. Of Al-Kāshī having influenced Stevin, there is a bare possibility. His work on decimal fractions may have been transmitted to Turkey by his adopted son, Ali ibn Muhammad al-Qushchī, or by others. Al-Qushchī, the third director of Ulugh Beg's observatory, went to Constantinople after the death of Ulugh Beg and died there in 1474.[1] To support the claim that al-Kāshī's work was not unknown in Turkey, we remark that al-Kāshī's instrument known as the Ṭabaq al-Manāṭiq was made the subject of a Persian treatise by some unknown person, who dedicated the book to Sultan Bayazid II (1447-1512) of Turkey.[2] From Turkey, al-Kāshī's work might have been carried to Europe through the influence of the two Jewish mathematicians, Elyah ben Ibrahim Mizrahī,[3] who was born in Constantinople in 1455 and died there in 1525, and Mordecai Comtino, who flourished in Constantinople[4] at the end of the 15th century[5]. They both helped in the advancement of European mathematics and might possibly have transmitted to Europe some inkling of al-Kāshī's results. But this possibility is practically negligible.

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1. Sarton, 1, vol. III, part II, p. 1120.
2. Kennedy, 1,
3. Sarton, 2, p. 173.
4. Sarton, 1, p. 1117, vol. III, part II; Gandz, 1, p. 16-17.
5. His death date is given by Gandz as 1525 and by Sarton as c. 1480-87.

Thus it appears that Bonfils, Stevin and al-Kāshī have had many points in common but these resemblances do not prove the existence of mutual influence among them. It is most likely that they all deduced their rules from the common fund of mathematics available in their time rather than that they adopted them from each other. Such resemblances are not uncommon among mathematicians living in the same period. Many mathematical discoveries, such as the infinitesimal calculus and non-Euclidean geometry, were made from an accumulation of similar ideas thought out independently by different persons at different places. The invention of the decimal fractions apparently originated in the same way.

Bibliography

1. Archibald, R.C. - Outline of the History of Mathematics - The American Mathematical Monthly, Vol. 56, sixth edition, 1949.
2. Brockelmann, 1 - Geschichte der arabischen Litteratur - Berlin, 1902.
3. Brockelmann, 2 - Geschichte der arabischen Litteratur, Supplementband II - Leiden, 1938.
4. Cajori, F. - A History of Mathematics - London, 1931.
5. Chace, Bull, Manning and Archibald - The Rhind Mathematical Papyrus - Mathematical Association of America, 1929.
6. Childe, G. - What Happened in History - London, 1942.
7. Dakhel, A. - The Extraction of the n-th Root in the Sexagesimal Notation - Thesis, American University of Beirut, 1950-51.
8. Datta, B. and Singh, A.N. - History of Hindu Mathematics - Part I, Motilal Banarsi Das, Lahore, 1935.
9. Encyclopaedia Britannica, A New Survey of Universal Knowledge, Vols. I, II - Edited by Yust, W., University of Chicago, 1929-1944.
10. Encyclopaedia of Islam, A Dictionary of the Geography, Ethnography, and Biography of the Muhammadan Peoples - Vol. I, Edited by Houtsma, M.Th., and T.W. Arnold, 1913; Vol. II, Edited by Houtsma, M.Th., and Wensinck, A.J., 1927; Vol. III, Edited by Houtsma, M.Th., and Wensinck, A.J., 1936.
11. Fihrist Ibn al-Nadim - Edited by G. Flugel, J. Roediger and A. Muller, Leipzig, 1871.
12. Gandz, S., 1 - The Invention of Decimal Fractions and the Application of the Exponential Calculus by Immanuel Bonfils of Tarascon (c.1350) - Isis (25, 132) 1936, pp. 16-45.
13. Gandz. S., 2 - Did the Arabs Know the Abacus ? - The American Mathematical Monthly, Vol. XXXIV, 1927, pp. 308-16.
14. Halma, M. - Composition Mathématique de Claude Ptolémée - Vol. I Paris, 1813-16.
15. Heath, T.L., - A History of Greek Mathematics - Vol. I Oxford, 1921.
16. Hajji Khalfa (or Khalifa) - Vol. III, Edited by G. Flügel, London, 1835.
17. Hitti, P.K. - History of the Arabs - London, 1940.
18. Karpinski, L.C. - The History of Arithmetic - New York, 1925.
19. Kaye, G.R. - Hindu Astronomy, Memoirs of the Archaeological Survey of India - No. 18, Calcutta, 1924.

20. Kennedy, E.S., 1 - Al-Kāshī's "Plate of Conjunctions" -
Isis, vol. 38, pp. 56-9, 1947-48.
21. Kennedy, E.S., 2 - A Fifteenth-Century Planetary Computer: al-Kāshī's "Ṭabaq al-Manāṭeq" -
Isis, vol. 41, pp. 180-3, 1950.
22. Kennedy, E.S., 3 - An Islamic Computer for Planetary Latitudes -
Journal of the American Oriental Society, vol. 71, 1951, pp. 13-21.
23. Kenrick, G.M.A. - Phoenicia -
London, 1855.
24. Knobel, E.B. - Ulugh Beg's Catalogue of Stars -
London, 1916.
25. Luckey, P. 1 - "Zur Islamischen Rechenkunst und Algebra des Mittelalters". Forschungen und Fortschritte -
Heft 17/18, 1948, pp. 199-204.
26. Luckey, P. 2 - "Die Ausziehung der n-ten Wurzel und der binomische Lehrsatz in der Islamischen Mathematik" -
Mathematische Annalen, Berlin, 120. Band, 2. Heft, 1948, pp. 217-274.
27. Lane-Poole, S. - The Mohammedan Dynasties -
Westminster, 1894.
28. Morley, S.G. - The Ancient Maya -
Stanford University, 1946.
29. Nallino, C.A. - Al-Khuwārizmī e il suo rifacimento della Geografia di Tolomeo -
(Memorie della R. Accademia dei Lincei, Classe di scienze morali, storiche e filologiche, serie V, Vol. II, parte 1a) , pp. 7 - 14.
30. Neugebauer, O. 1 - Vorlesungen über Geschichte der antiken mathematischen Wissenschaften -
Erster Band, Vorgriechische Mathematik, Berlin, 1934.
31. Neugebauer, O. 2 - On a Special Use of the Sign "zero" in Cuneiform Astronomical Texts -
Journal of the American Oriental Society, vol. 61, 1941, pp. 213-215.
32. Neugebauer, O. 3 - Review of Datta-Singh, Hindu Mathematics -
Quellen und Studien zur Geschichte der Mathematik, B3, Berlin, 1936.
33. Neugebauer, O. and Sachs, A. - Mathematical Cuneiform Texts -
New Haven, 1945.
34. Rawlinson, G.M.A. - History of Phoenicia -
London, 1889.
35. Rieu, C.- Catalogue of the Persian Manuscripts in the British Museum -
Vol. II, 1879-83, p. 456.
36. Sanford, V. - A Short History of Mathematics -
New York, 1930.
37. Sarton, G. 1 - Introduction to the History of Science -
Vol. I, 1927; Vol. II, part I, 1931; Vol. III, part I, 1947, Washington.

38. Sarton, G., 2 - The first explanation of decimal fractions and measures (1585). Together with a history of the decimal idea and a facsimile (No. XVII) of Stevin's Disme -
Isis 21, pp. 153-244, 1934.
39. Sédillot, I.P.E. - Prolégomènes des Tables Astronomiques d'Oloug-Beg -
Paris, 1847.
40. Smith, D.E., 1 - History of Mathematics - 2 vols.
New York, 1925.
41. Smith, V.A., 2 - Asoka, the Buddhist Emperor of India -
Oxford, 1919.
42. Smith and Karpinski - The Hindu-Arabic Numerals -
London, 1911.
43. Struik, D.J. - A Concise History of Mathematics - Vol. I
New York, 1948.
44. Suter, H. - Die Mathematiker und Astronomen der Araber und ihre Werke -
Leipzig, 1900.
45. Schmidt, O.H. - The Computation of the Length of Daylight in Hindu Astronomy -
Isis, Vol. XXXV, pp. 205-211, 1944.
46. Sykes, P. - A History of Persia -
London, 1930.
47. Tod, M.N., 1 - The Greek Numeral Systems -
Journal of Hellenic Studies, No. XXXIII, 1913, p.27.
48. Tod, M.N., 2 - The Alphabetic Numeral System in Attica -
Annual of the British School at Athens, No. XLV, 1950, pp. 126-138.
49. Woepcke, M.F. - Passages Relat ifs a des Sommations de Series de Cubes -
Journal de Mathematiques Pures et Appliquées, pp.112-116, Paris.
50. Zambaur, E. - Manuel de Genealogie et de Chronologie pour l'Histoire de l'Islam -
Hanovre, Lafaire, 1927.
51. Zeller, M.C. - The Development of Trigonometry From Regiomontanus to Pitiscus -
University of Michigan, 1944.