THE EXTRACTION OF THE n-th ROOT
IN THE SEXAGESIMAL NOTATION

by

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P R E F A C E

This is a detailed examination of a topic in medieval Islamic computational methods. It deals mainly with a general method of extracting the n-th root of a positive number displayed in the sexagesimal notation.

The topic was suggested by the work of the German mathematician P. Luckey (d. 1949), whose paper "Die Ausziehung der n-ten Wurzel und der binomische Lehrsatz in der islamischen Mathematik"[1] is one of a series of important contributions made by him to the history of Islamic mathematics. It is a pleasure to acknowledge also that extensive use has been made of Luckey's paper in the symbolism and presentation of the present study.

Although this work and that of Luckey have much in common they by no means cover the same ground. The scope of Luckey's is, in several senses, considerably broader than that of the present one. He compares Islamic with Hindu and European work, and compares also the so-called Ruffini-Horner method of solving polynomial equations by the binomial coefficient method. In almost all of the paper cited he confines himself to decimals.

The present study, on the other hand, deals exclusively with sexagesimals. Moreover, it is not comparative; it treats only the

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1. See Luckey in bibliography.

work of a single individual, Ghiāth ad-Dīn Jamshīd al-Kāshī, (c.1390-1430), an Iranian mathematician and astronomer [1]. One of his works, "The Key of Arithmetic" (Miftāḥ al-Ḥisāb), is composed of five treatises, each containing several chapters.[2] Chapter Five in the Third Treatise of this book is herein translated into English and commented upon in rather minute detail.

This study serves a general three-fold objective:

1. To make the source available for Arabic-speaking specialists in the history of mathematics in a form as close to the original as possible. Hence a facsimile reproduction of the source is given.

2. To present a translation of the source into a European language for the benefit of specialists who cannot read Arabic. With this in mind an opposite-page English translation is also presented.

3. To make it possible for non-specialists to understand the matherial without a deep examination of the source. It is assumed of the reader that his mathematical background is at least that of secondary-school level and that he is familiar with the sexagesimal system of writing numbers.[3]

The thesis is organised in three parts:

I. Preparation. This includes Chapters I and II.

II. Text, which is contained in Chapter III, and

III. Commentary, which is covered by Chapters IV, V, and a conclusion.

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1. See Rāja'ī, p. 82, for a biographical sketch of Kāshī .
2. Ibid,p. 61, for a more detailed description of the "Key".
3. Ibid, pp. 6 - 11, for the conventions and basic theory of the sexagesimal system.

The first chapter deals with the Jummal system of writing numbers, and with the concept of place in its two-fold meaning as was used by our author and contemporary mathematicians.

The second chapter is devoted to setting up a theoretical background, and to the presentation of the Ruffini-Horner method for approximating real roots of polynomials.

The third chapter contains a facsimile reproduction of Chapter Five in the Third Treatise of the "Key", together with its translation into English.

The fourth chapter is a kind of introduction to some concepts which our author uses either with or without definition.

And the last chapter describes the method and underlying theory of extracting the n-th root of a number.

A conclusion to the effect that the so-called Ruffini-Horner method was in use before either Ruffini or Horner lived is given at the end of the commentary.

A bibliography of the references used in this work, a vocabulary of some terms used in the translation, and a sexagesimal multiplication table are also given at the end of the thesis.

C O N T E N T S

An Abstract of a Thesis Entitled
" The Extraction of the n-th Root
in the Sexagesimal Notation "

by

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Chapter I : The Jummal system and the concept of place

<u>The Jummal system :</u> In the Semitic order of the Arabic alphabet there are twenty-eight letters given in the following passage from right to left :

ابجد هوز حطي كلمن سعفص قرشت ثخذ ضظغ

The first nine letters, from ا to ط , are used for the units. The next nine letters, from ي to ص , are used for the tens, the third nine letters, from ق to ظ , for the hundreds, and the last letter, غ , for one thousand.

These letters are used, either separate or combined in descending order, to denote all numbers. If the thousands are more than one, their number is written before the غ.

<u>The sexagesimal - Jummal system :</u> Combining the Jummal system with the sexagesimal place-value system, only 59 letters, from ا to نط , and the zero symbol, τ , are needed.

In this system we remark that :

a) Places are higher from left to right, the converse of our use in modern notation,

b) No sexagesimal point is used; thus the concept of the place of a number is necessary to eliminate this ambiguity.

<u>The concept of place :</u> There are two, not completely different, meanings of place :

a) In the first meaning <u>the places are the positions of the successive digits from right to left in the row,</u> i.e. units' place, hundreds' place, etc, in the decimal notation. The corresponding places in the sexagesimal notation are the degrees' place, first elevates' place, second elevates' place, etc. This series of places called " the ascending series " and is (in the sexagesimal-Jummal system) to the right of the degrees' place. On the other side of the degrees' place is the so-called " descending series " whose places are those of minutes, seconds, thirds, etc.

b) For the second meaning of place, we note that if in a number we know the place of one of its digits, say the lowest place, the others are determined. We speak of the place of a number as its lowest place. If the degrees' place is associated with the integer zero, the places in the ascending series associated with the positive integers, and those in the descending series associated with the negative integers, then the number whose lowest place is associated with the integer n is said to ~~the~~ have the place-number n.

This second meaning of place served the same purpose as does the sexagesimal point in modern usage.

Chapter II : Theoretical basis for the extraction of roots

First a proof is given for the legitimacy of the long-division algorism in the special case when the divisor is of the form x-m, then an explanation of how its steps can be telescoped in this special case to obtain the so-called " synthetic division " algorism.

To diminish the roots of an equation by a constant : If we have an equation $f(x) = 0$, and if r is any value of x for which $f(r) = 0$, then diminishing the roots of $f(x) = 0$ by m means to carry the transformation $x' = x - m$, or $x = x' + m$, into $f(x) = 0$ such that the transformed equation $f_1(x') = 0$ satisfies $f_1(r-m) = 0$.

The transform can be found by two methods :

a) By expanding $f(x' + m) = 0$ and collecting like terms,

b) By the use of division. The equation $f(x) = 0$ is divided by $x - m$ and the remainder taken, again the quotient is divided by $x - m$ and the second remainder taken, and so on. Thus the successive remainders are the coefficients of the transform in the reverse order. Synthetic division is usually used and the method is called " the Ruffini-Horner method ".

Horner's scheme : In the synthetic division algorism used for finding the coefficients of a transformed equation when diminishing by a constant, if the operations of multiplication and addition are performed separately and the results only written down, the so-called " triangular Horner's scheme " for the special example at hand is obtained. The numbers along the hypotenuse of the triangle are the coefficients of the transform.

The general Horner's scheme is usually displayed in a double-subscript form in which the numbers in the first row and first column are given by :

$$a_{i,-1} = 0 , \quad a_{-1,j} = a_j \quad (i, j = 0,1,2,\ldots,n),$$

where a_j are the coefficients of the given equation. The remaining numbers are computed according to the recursion expression :

$$a_{i,j} = a_{i-1,j} + m\, a_{i,j-1} \quad \left\{ \begin{array}{l} i = 1,2,3, \ldots n \\ j = 1,2,3,\ldots n-i \end{array} \right\}$$

Then the numbers $a_{i,n-i}$ $(i = 1,2,3,\ldots n)$, along the hypothenuse of the triangular scheme are the coefficients of the transformed equation.

Stretching a function f (x) by a constant factor : Analytically, this is the transformation $\bar{x} = mx$, where $m > 0$. Geometrically, it means to increase (or decrease if $0 < m < 1$) all horizontal dimensions of the graph of f (x) by the factor m, leaving vertical dimensions unchanged, hence the name.

In the equation

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_n = 0,$$

stretching f (x) by m yields an equation whose roots are m times as large as those of f(x).

Thus, if we are seeking a root, x_0, of an equation $f(x)=0$ and we know that $1{,}0^k \leq x_0 < 1{,}0^{k+1}$, then we stretch (or here compress) the function f(x) by $1{,}0^{-k}$ in order to obtain another equation $f_1(x) = 0$ whose root r_1, satisfies $1 \leq r_1 < 1{,}0$

The Ruffini-Horner method : This is a method for approximating the real roots of a polynomial equation f(x) = 0. Its essential steps are :

1. To locate a root, x_0, between successive powers of 1,0 i.e $1{,}0^k \leq x_0 < 1{,}0^{k+1}$,

2. To stretch f (x) by $1{,}0^{-k}$ to obtain $f_1(x) = 0$ whose root r_1 has its first digit in the degrees' place,

3. To locate r_1 between successive digits.

4. To diminish the roots of $f_1(x)$ by the lesser digit to obtain $f_2(x) = 0$,

5. To stretch $f_2(x)$ by 1,0 obtaining an equation $f_1(x) = 0$ whose root , r_3, has its first digit in the degrees' place,

6. To locate the root r_3 of $f_3(x) = 0$ between successive digits. The first digit of r_3 is the second digit of x_0

7. To diminish the roots of $f_3(x) = 0$ by the lesser digit, stretch the new transform by 1,0, then locate the root of the equation obtained between successive digits, thus obtaining the third digit of x_0. The cycle is repeated to find as many digits of x_0 as may be required.

Chapter III : Source and translation

A literal opposite- page translation of Chapter Five in the Third Treatise of al-Kāshī's " the Key of Arithmetic " (Miftāh al-Hisāb) is given, together with a facsimile reproduction of it taken from the Princeton manuscript. It deals with the extraction of the n-th root of a number expressed in the sexagesimal notation. The following two chapters contain a detailed commentary on this chapter.

Chapter IV : Preparatory

In this chapter are given :

a) Comments and explanations of some terms which Kāshī uses without explanation, and

b) Facts given or explained by him previously in the " Key " and which he reestablishes here.

Among the topics dealt with in a) are (i) perfect powers, (ii) perfect places, and (iii) cycles.

(i) A number q is a perfect n-th power if the n-th root of q is an integer, N ; otherwise q, as an n-th power is imperfect.

(ii) The perfect places in a number q, thought of as an n-th power, are those places whose numbers have the form kn, where k is any integer; the other places are imperfect. It is not necessary that q be a perfect n-th power.

(iii) In an n-th power, the set consisting of one perfect place and n-1 imperfect places on its left, if any, is called a cycle.

Chapter V : The extraction of roots

Here is presented an algorism of root extraction in general, together with justification of its steps.

The process is summarized as follows. Let a number q be given and it is required to extract its n-th root.

Separate the number into cycles, then find the largest digit, α, whose n-th power can be subtracted from the first cycle on the left. This is the first digit of the root.

Now, as a preparation for finding the next digit, β, of the root, certain cyclic processes are performed on the digit α, and the results of these processes are transposed by one, two, three, etc., places to the right, then the second and later digits are located subject to certain similar conditions.

Kāshī gave the description of the n-th root extraction in general and without reference to any particular example. In the commentary an example is given to clarify the explanation.

Kāshī then gives a special example to display his method. He proposes to extract the square root of 10, 9, 49, 30.

At the end of the translated extract of the " Key ", our author gives two worked examples. The one displays the extraction of the cube root of 18, 53; 59, 43, 51, 25, and the other shows the extraction of the 6th root of 43, 59, 1, 7, 14, 54, 23, 3, 47, 37; 40. The latter is the one we used in connection with the general method.

Conclusion : In the sequel, it is seen that Kāshī, when finding the n-th root of a number, followed the same steps as those of the Ruffini-Horner method.

His separation of the number into cycles and considering the first cycle on the left correspond in the Ruffini-Horner method to locating the root of the first -given equation between successive powers of 1,0 and stretching (here compressing) the related function by the lesser power.

Kāshī's finding of the digit α whose n-th power can be subtracted from the first cycle corresponds to locating the root x_1 of the stretched equation $f_1(x) = 0$ between successive digits.

Performing the set of processes on α and transposing the results correspond to diminishing the roots of $f_1(x) = 0$ by α and stretching the new transform $f_2(x) = 0$ by 1,0

Thus recalling that Kāshī (fl. 1410 A.D) used the algorism a long time before either Ruffini (1765 - 1822) or Horner (1786 - 1837) lived, we say that the latter two are not the <u>first</u> to invent the method and use it. It was certainly known and used by oriental mathematicians, among whom Kāshī is only one, at least four centuries before Ruffini and Horner.

<u>Hints to the reader.</u> In the sequel

a. Numbers inside parentheses refer to expressions: equations, inequalities, forms, tables, etc..

b. Numbers without parentheses refer usually to sections. Some of the paragraphs, however, are also numbered so that reference can be made to them where there is no need to refer to the whole section.

c. All the numbers that occur in this work, except those given in 1, are to be understood as expressed in the sexagesimal notation. In 1, however, the decimal notation is used in connection with the *Jummal* system because the latter is related in its basic idea to the former. Page-numbers and the numbers of sections and expressions are also in the decimal notation.

d. A bibliography of the works used is given at the end of the thesis. Footnote references to the bibliography will be by authors' names. The only exception is Kāshī's "The Key of Arithmetic", which is referred to by the letters P, I, or L according as it is the Princeton, the India Office, or the Leiden copy which is referred to.

PART I

P R E P A R A T I O N

CHAPTER I

THE JUMMAL SYSTEM AND THE CONCEPT OF PLACE

1 <u>The Jummal system of writing numbers.</u>[1] This section is intended for readers who will use the Arabic source or, in fact, medieval Arabic mathematical manuscripts in general. It may be omitted by any who intend to read only the English translation. Familiarity with the different forms of the Arabic characters is assumed, i.e. the form a letter takes when written separately or when attached at the beginning, in the middle, or at the end of a word.

In the Arabic alphabet there are twenty-eight letters [2] given in the sequence of the following passage read from right to left:

ابجد هوز حطي كلمن سعفص قرشت ثخذ ضظغ

The first nine letters, from ا to ط, are used to represent numbers from 1 to 9 successively:

ا	ب	ج	د	ه	و	ز	ح	ط
1	2	3	4	5	6	7	8	9 .

the numbers in the lower line correspond respectively to the

1. See P, p. 66.
2. Here the ancient Semitic order is used which has no <u>hamza</u> (هَمْزة).

letters over them in the upper line.

The next nine letters are used for the tens, thus:

ی	ک	ل	م	ن	س	ع	ف	ص
10	20	30	40	50	60	70	80	90 .

The dots of ب , ج , ز , and ي are omitted, the ج is shortened to ↄ in order to distinguish it from the ح , and ي when single is written sideways, ى.

To write numbers composed of tens and units, their corresponding letters are combined, putting the tens first. E.G. 39 is written لط in which ل means 30 and ط means 9 ; their combination means 39. Similarly 47 is مر , 64 is سد , 92 is صب , 88 is فح , 23 is كج , etc.

The third nine letters are used for the hundreds

ق	ر	ش	ت	ث	خ	ذ	ض	ظ
100	200	300	400	500	600	700	800	900 .

and the last letter غ is used for 1000.

To write numbers composed of one thousand, hundreds, tens, and units, the letters representing the thousand, hundreds, tens, and units are combined in the descending order. E.g. 438 is written تلح , 102 is قب , 901 is ظا , 1534 is غثلد , and 1020 is غك .

If the number of thousands is more than one, the letter representing their number is written in front of غ . E.g. 4632 is written دغثلب , 53418 is نجغتيح , 234619 is رلدغخيط .

The following table shows the twenty-eight Arabic letters as they are written when separate, together with their numerical values.

A Table of the Arabic Alphabet Letters
with their numerical values

1	ا	10	ى	100	ق	1000	غ
2	ب	20	ك	200	ر		
3	ج	30	ل	300	ش		
4	د	40	م	400	ت		
5	ه	50	ن	500	ث		
6	و	60	س	600	خ		
7	ز	70	ع	700	ذ		
8	ح	80	ف	800	ض		
9	ط	90	ص	900	ظ		

Here are some examples to show how these letters are combined to represent various numbers.

46	مو	103	قج	1030	غل
82	فب	231	رلا	1222	غركب
12	يب	801	ضا	2402	بغتب
11	يا	329	شكط	6008	وغح
31	لا	618	خيح	10548	يغثمح
51	نا	740	ذم	123419	قكجغتيط

2 <u>The sexagesimal-Jummal system.</u> If the Jummal system is combined with the sexagesimal place-value system, then only numbers from ا to نط, i.e. from 1 to 59, appear. For the zero the symbol (٢) is used.

3 In this new system, the sexagesimal-Jummal system, the following things are to be remembered:

a. Places are higher from left to right, the converse of our use in modern notation. E.g. 1, 12, 36 is written ا ىب لو

b. No sexagesimal point is used. Thus the number لو م ند may mean just 36,40,54, or it may mean 36;40,54, or 36,40,54,0,0,0 , or in general $1,0^{k}$. 36,40,54, where k may be say integer, positive, negative, or zero.

To eliminate this ambiguity, however, a skilful device was used. That device is based on the concept of place which will be explained in 4 - 8 below.

4 <u>The concept of place.</u> Kāshī used the term "place" (martaba) in two meanings. They are not completely different; in fact, the one is an extension of the other. We shall give the first meaning as the author <u>defines</u> it, then we give the second meaning which he <u>uses</u> without a special definition[1].

5 When the author first defines places, he does it for the Hindu-decimal system in his First Treatise. He says[2] : "The

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1. In Raja'i, pp. 79-81, there is also a description of the concept of place.
2. L, p. 6.

places are the positions of the successive digits from right to left in the row (Cf. 3). They (i.e. mathematicians) called the first position __the units' place__, the one on its left __the tens' place__, the one on the left (of this) __the hundreds' place__, etc."

Then, for the sexagesimal system in his Third Treatise he says:[1] "As there (i.e. in the Hindu system) the first places of whole numbers are called __units__, here __degrees__ is the name of the place, and as the series of places there is one[2], here are two series — the one is in the ascending side and the other is in the descending side, and the degrees' (place) is central between the two series."

6 Thus the degrees' place in the author's sexagesimal notation corresponds to the units' place in the decimal notation. The __ascending series__, in this system, is to the right of the degrees' place and its places successively he calls[3] first elevates, second elevates, third elevates, etc. These correspond respectively to the places of tens, hundreds, thousands, etc. in the usual decimal notation. The __descending series__ is to the left of the degrees' place, and its places successively are called the places of minutes, seconds, thirds, fourths, etc. These places correspond respectively to the places of tenths, hundredths, thousandths, ten thousandths, etc. in the decimal

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1. P, p. 67.
2. It is to be remarked incidentally that up to the time of Kāshī, no decimal fractions were used; so there was only one "series of places" as he calls it. He himself invented the decimal fractions; see Raja'i.
3. P, p. 67.

notation as we use it.

7 Thus we note that the places in this system increase, opposite to modern usage, from left to right (Cf. 3), and that, if in a number we know the place of one of its digits, then the places of the other digits are determined. E.g. if in the number

12,23,56,7,41,35,18,28,47,59 = يب كج نو ز ما له يح كح مز نط

we know that the degrees' place is that of 35 = له, then 59 = نط is in the place of fourths, 28 = كح is in theplace of seconds, 41 = ما is in the place of first elevates, 23 = كج is in the place of fourth elevates, etc., and the number is written in modern notation in the form 12,23,56,7,41,35;18,28,47,59.

8 We are now going to give the second meaning of place. In the above-mentioned example, the lowest place, i.e. that of 59 = نط, is the place of fourths. When this place is given, the others are then determined. We speak of the place of a number as its lowest place. Thus we say that the place of the previous number is fourths, or more simply, that the number is fourths. For example, when we say ل يز كح مه نح seconds we mean the number 30,17,28;45,58 ; when we say ج يط ما نا third elevates we mean the number 3,19,41,51,0,0,0. In general, if, with the places of ...., fourths, thirds, seconds, minutes, degrees, first elevates, second elevates, third elevates, etc., we associate the integers ...., -4, -3, -2, -1, 0, 1, 2, 3, etc., respectively, any number whose place, n, is given can be

interpreted easily in modern notation. <u>Then we say that the place-number of the given number is n.</u> E.g., 32,10 fourths is then the number $1,0^{-4}$. 32,10 and 50,31,12 fifth elevates is the number $1,0^{5}$. 50,31,12, and their place-numbers are -4 and 5 respectively.

This second meaning of place was used to fulfill the same purpose as does the sexagesimal point in modern use,

CHAPTER II

THEORETICAL BASIS FOR THE EXTRACTION OF ROOTS

In setting down the theoretical basis for Kāshī's extraction of the n-th root of a number, we need first to explain certain terms and operations. These are given in 9 - 28 below.

9 **Long division.** Following is a proof of the legitimacy of the long division algorism[1] for a special case, i.e. division of a polynomial in x by the monomial x - m, where m is a positive or negative constant.

Long division is defined as the inverse operation of long multiplication. If we have the polynomial

(1) $$D = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n$$

and we divide it by x - m, then we should get a polynomial of the form

(2) $$Q = c_0x^r + c_1x^{r-1} + c_2x^{r-2} + \ldots\ldots + c_r$$

and a remainder, R, such that when (2) is multiplied by x-m, and the result added to R, we get precisely (1). i.e. $(x-m)Q + R \equiv D$.

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1. Algorism, or algorithm, means a process of solving a certain type of problem (Cf. Jones, p.6). The term took its origin from the name of al-Khowarizmi (fl. 830 A.D.), an Islamic mathematician and astronomer.

The long division algorism is displayed below, it being assumed that the reader is familiar with it.

The Quotient $\rightarrow a_0x^{n-1} + (a_1 + ma_0)x^{n-2} + [a_2 + m(a_1 + ma_0)]\,x^{n-3} + \cdots\cdots + a_{n-1} + m\{a_{n-2} + m(a_{n-3} + \cdots + m(a_1 + ma_0)\cdots)\}]$

The Divisor ↓ | The Dividend ↓

$x - m$

$$a_0x^n \quad + a_1x^{n-1} \quad + a_2x^{n-2} \quad + a_3x^{n-3} \quad + \cdots\cdots + a_n$$

$$-a_0x^n \pm ma_0x^{n-1}$$

$$(a_1 + ma_0)\,x^{n-1} \quad + a_2x^{n-2} \quad + a_3x^{n-3} \quad + \quad + a_n$$

$$-(a_1 + ma_0)\,x^{n-1} \pm m(a_1 + ma_0)\,x^{n-2}$$

$$[a_2 + m(a_1 + ma_0)]\,x^{n-2} \quad + a_3x^{n-3} \quad + \quad + a_n$$

$$-[a_2 + m(a_1 + ma_0)]\,x^{n-2} \pm m[a_2 + m(a_1 + ma_0)]\,x^{n-3}$$

$$\{a_3 + m[a_2 + m(a_1 + ma_0)]\}\,x^{n-3} + \quad + a_n$$

The Remainder $\rightarrow a_n + m\,[a_{n-1} + m\,\{a_{n-2} + m\,(a_{n-3} + \cdots + m\,(a_1 + ma_0)\cdots)\}]$

Thus the quotient is

$$Q = a_0x^{n-1} + (a_1+ma_0)x^{n-2} + \left[a_2+m(a_1+ma_0)\right] x^{n-3} + \ldots\ldots + \left[a_{n-1}+m\left\{a_{n-2}+m\left(a_{n-3}+\ldots+m(a_1+ma_0)\ldots\right)\right\}\right]$$

and the remainder is

$$R = a_n + m\left[a_{n-1} + m\left\{a_{n-2} + m\left(a_{n-3}+\ldots + m(a_1+ma_0)\ldots\ldots\right)\right\}\right] .$$

For the division to be valid we should have $(x-m)Q+R \equiv D$, i.e.

$$(4)\qquad \begin{aligned}(x-m)\Big\{&a_0x^{n-1}+(a_1+ma_0)x^{n-2}+\left[a_2+m(a_1+ma_0)\right] x^{n-3}+\ldots\\ &\ldots+\left[a_{n-1}+m\left\{a_{n-2}+m\left(a_{n-3}+\ldots+m(a_1+ma_0)\ldots\right)\right\}\right]\Big\}\\ &+a_n+m\left[a_{n-1}+m\left\{a_{n-2}+m\left(a_{n-3}+\ldots+m(a_1+ma_0)\ldots\right)\right\}\right]\\ \equiv\ & a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n .\end{aligned}$$

The left-hand side of (4), when expanded, gives:

$$\begin{aligned}&a_0x^n - ma_0x^{n-1} + a_1x^{n-1} + ma_0x^{n-1} - m(a_1+ma_0)x^{n-2} + a_2x^{n-2} + m(a_1+ma_0)x^{n-2}\\ &- m\left[a_2 + m(a_1+ma_0)\right] x^{n-3} + a_3x^{n-3} + \ldots\\ &- m\left[a_{n-1} + m\left\{a_{n-2} + m\left(a_{n-3}+\ldots+m(a_1+ma_0)\ldots\ldots\right)\right\}\right]\\ &+ a_n + m\left[a_{n-1} + m\left\{a_{n-2} + m\left(a_{n-3}+\ldots+m(a_1+ma_0)\ldots\ldots\right)\right\}\right]\\ \equiv\ & a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n .\end{aligned}$$

which is precisely the right-hand side. Thus the correctness of the result is established.

10. <u>Synthetic division.</u> We can telescope the steps of the process of long division, in the special case given in 9, to obtain the so-called "synthetic division" algorism.

If the quotient is of the form (2), and since it is of one degree less than the dividend (1), we must have:

(5) $$(x-m)(c_0x^{n-1}+c_1x^{n-2}+c_2x^{n-3}+\dots+c_{n-1})+R \equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_n$$

i.e.

(5') $$c_0x^n+(c_1-mc_0)x^{n-1}+(c_2-mc_1)x^{n-2}+\dots+(c_j-mc_{j-1})x^j+\dots+R-mc_{n-1} \equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_jx^j+\dots+a_n .$$

A theorem in the theory of equations states that [1]: "If two polynomials are identically equal, their corresponding coefficients are equal." Thus we obtain the relations (6) and (6')

(6)
$$\begin{aligned} a_0 &= c_0 \\ a_1 &= c_1 - mc_0 \\ a_2 &= c_2 - mc_1 \\ &\dots\dots\dots \\ a_j &= c_j - mc_{j-1} \\ &\dots\dots\dots \\ a_n &= R - c_{n-1} \end{aligned}$$

i.e. (6')
$$\begin{aligned} c_0 &= a_0 \\ c_1 &= a_1 + mc_0 \\ c_2 &= a_2 + mc_1 \\ &\dots\dots\dots \\ c_j &= a_j + mc_{j-1} \\ &\dots\dots\dots \\ R &= a_n + mc_{n-1} \end{aligned}$$

11. In the above we note:

a. When the coefficients of the given polynomial are a_j, $(j = 1, 2, \dots n)$, the coefficients c_j, $(j = 1, 2, \dots, n-1)$, of the quotient polynomial are obtained by the recursion relation

(7) $$c_j = a_j + mc_{j-1} \quad .$$

b. The coefficient of the highest power of x in the quotient is equal to that of the highest power of x in the dividend, i.e. $c_0 = a_0$.

---

1. Fine, p. 90.

c. The remainder R has the form $R = a_n + mc_{n-1}$.

d. The degree of the quotient polynomial is one less than that of the dividend.

e. In performing the operation of division, the x's can be omitted without altering the legitimacy of the process; only care should be taken that the coefficients be arranged in descending order, and that zeros fill in the places of lacking powers, if any.

12. If use be made of the remarks in a – e above, the triangular array (3) can be telescoped into the 3-rowed rectangular array

$$
(7)\qquad
\begin{array}{c|cccccccc}
m & a_0 & a_1 & a_2 & \cdots\cdots & a_j & \cdots\cdots & a_n \\
 & & mc_0 & mc_1 & & mc_{j-1} & & mc_{n-1} \\
\hline
 & c_0 = a_0 & c_1 = a_1 + mc_0 & c_2 = a_2 + mc_1 \cdots & & c_j = a_j + mc_{j-1} \cdots & & \boxed{R = a_n + mc_{n-1}}
\end{array}
$$

and so we obtain the coefficients c_j of the quotient polynomial and the remainder R. The quotient polynomial can now readily be written down. E.g. to divide $4x^4 - 6x^3 + 24x^2 - x + 12$ by $x - 3$, we write

$$
\begin{array}{c|ccccc}
3 & 4 & -6 & 24 & -1 & 12 \\
 & & 12 & 18 & 2,6 & 6,18 \\
\hline
 & 4 & 6 & 42 & 2,5 & \boxed{6,27}
\end{array}
\quad .
$$

The quotient is $4x^3 + 6x^2 + 42x + 2,5$ and the remainder is 6,27.

13. <u>To diminish the roots of an equation by a constant.</u> Given the equation

(8) $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n = 0$,

it is required to transform this equation so as to diminish its roots by m, where m is a positive or negative constant.

This means that if r is any value of x for which $f(r) = 0$, then in the transformed equation $f_1(x') = 0$, where $x' = x - m$, we have $f_1(r - m) = 0$.

Since $x' = x - m$, then $x = x' + m$. Substituting this value of x in (8), we obtain:

(9) $(x' + m)^n + a_1(x' + m)^{n-1} + \ldots\ldots\ldots + a_n = 0$.

14. The transform, as a polynomial in x', can be found by two methods:

a. By expanding (9) and collecting like terms [1]. E.g. diminish the roots of the equation

(10) $$x^3 + 2x^2 + 7x - 3 = 0$$

by 2.

We substitute $x' + 2$ for x in (10) to get

$$(x' + 2)^3 + 2(x' + 2)^2 + 7(x' + 2) - 3 = 0 .$$

Expanding: $x'^3 + 6x'^2 + 12x' + 8 + 2x'^2 + 8x' + 8 + 7x' + 14 - 3 = 0$.

Collecting like terms, we get finally:

(11) $$x'^3 + 8x'^2 + 27x' + 27 = 0.$$

---

1. This method Lackey calls The "Renaissance Method", Lackey, p. 226.

15. b. The second method of obtaining the transformed equation when diminishing (8) by a constant m is by division. This method is shorter than the first (Cf. 14). It may be illustrated with the above example (in 14).

Since $x' = x - 2$, the transformed equation (11) is the same thing as

(12) $$(x - 2)^3 + 8(x - 2)^2 + 27(x - 2) + 27 = 0,$$

which is equivalent to (10).

From the form of (12) it is seen that the coefficients 27, 27, and 8 can be obtained in succession by dividing (12), or its equivalent (10), by $x - 2$ and taking the remainder, again dividing the quotient by $x - 2$ and taking the second remainder, and so on thus obtaining the coefficients of the transform in the reverse order. In this process synthetic division is usually used (Cf. 10 - 12). Thus in the example (10) we write

	2	1	2	7	-3	
			2	8	30	
		1	4	15	27	...... first remainder
(13)			2	12		
		1	6	27		.... second remainder
			2			
		1	8			.... third remainder
		1				

This second method is called "The Ruffini-Horner Method" (Cf. 23 below).

The two methods above are illustrated with a special example. In general, however, the same process can be applied for any polynomial whatever.

16. <u>Horner's scheme</u>. In (13) let us perform the operations of multiplication and addition separately and write down the results only, thus obtaining the so-called <u>triangular Horner's scheme</u> (14) for this special example.

(14)

2	1	2	7	-3
	1	4	15	27
	1	6	27	
	1	8		
	1			

The numbers along the hypotenuse of the triangle are the coefficients of the transform.

We note that each number in this scheme (14) is obtained by multiplying $m = 2$ by the number which is on its left and adding the result to the number which is over it. E.g. the 8 is $2 \cdot 1 + 6 = 8$, the 15 is $2 \cdot 4 + 7 = 15$, the upper 27 is $2 \cdot 15 + (-3) = 27$.

The Horner's scheme in general can be displayed in the following double subscript form:

(15)

	$a_{-1,0}$	$a_{-1,1}$	$a_{-1,2}$ ··	$a_{-1,j}$ ··	$a_{-1,n-1}$	$a_{-1,n}$
$a_{0,-1}$	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$ ···	$a_{0,j}$ ···	$a_{0,n-1}$	$a_{0,n}$
$a_{1,-1}$	$a_{1,0}$	$a_{1,1}$	$a_{1,2}$ ···	$a_{1,j}$ ···	$a_{1,n-1}$	
$a_{2,-1}$	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$ ···	$a_{2,j}$ ··	$a_{2,n-2}$	
....	....	....	....	....		
$a_{i,-1}$	$a_{i,0}$	$a_{i,1}$	$a_{i,2}$ ···	$a_{i,j}$ ··	$a_{i,n-i}$	
....	....	....	....	....		
$a_{n-1,-1}$	$a_{n-1,0}$	$a_{n-1,1}$				
$a_{n,-1}$	$a_{n,0}$					

where

1. The numbers outside the right triangle are given as follows:

(16) $a_{i,-1} = 0 \qquad (i = 0, 1, 2, \ldots, n)$

$a_{-1,j} = a_j \qquad (j = 0, 1, 2, \ldots, n).$

Thus in the uppermost row there stand the coefficients of the given equation, and in the first column zeros.

2. The remaining numbers are computed according to the recursion expression

(17) $a_{i,j} = a_{i-1,j} + m a_{i,j-1} \qquad \left\{ \begin{array}{l} i = 1, 2, 3, \ldots, n \\ j = 1, 2, 3, \ldots, n-1 \end{array} \right.$

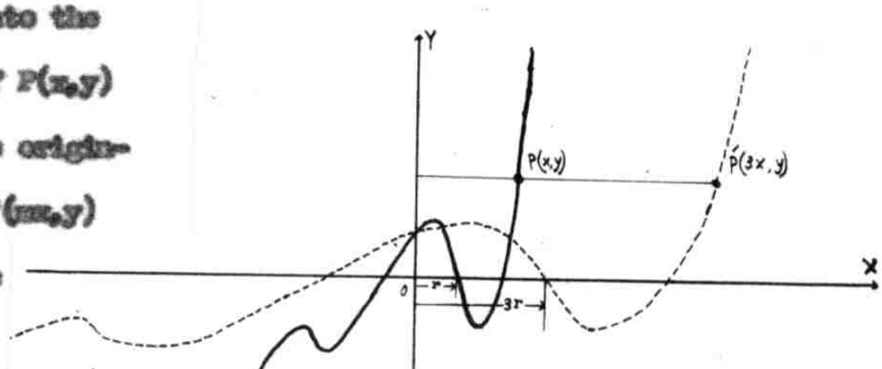
Then the numbers $a_{i,n-i}$ $(i = 1, 2, \ldots, n)$, along the hypotenuse of the triangular scheme, are the coefficients of the transformed equation.

17. Stretching a function. This section deals with a type of transformation different from that used for diminishing roots (Cf. 13). This transformation we call a stretch. The analytic definition of a stretch in the ratio m is the transformation

(18) $$x' = mx ,$$

where $m > 0$.

Consider a function $f(x)$ whose graph is the continuous curve in the figure. By the transformation $x' = mx$, this graph is transformed into the dotted graph. If $P(x,y)$ is a point on the original curve, then $P'(mx,y)$ is a point on the stretched curve.



In particular, if r is an intercept of the first graph, then mr is an intercept of the transformed graph.

Now the reason for the terminology is evident, since the effect of the transformation is to increase all horizontal dimensions by the factor m, leaving vertical dimensions unchanged.

18. On the other hand, had we started with the dotted graph, we could have transformed it into the continuous-line graph by the transformation $x = m'x' = \frac{1}{m} x'$. Thus by this type of transformation the graph is either stretched or compressed in width.

19. Now consider the polynomial equation

(19) $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n = 0.$

To stretch $f(x)$ by m is, by (18), to write $x' = mx$ or $x = \frac{1}{m}x'$ in (19); there results

$$a_0\frac{x'^n}{m^n} + a_1\frac{x'^{n-1}}{m^{n-1}} + a_2\frac{x'^{n-2}}{m^{n-2}} + \ldots\ldots + a_n = 0$$

or equivalently, after dropping the dashes,

(20) $a_0x^n + a_1mx^{n-1} + a_2m^2x^{n-2} + \ldots\ldots + a_nm^n = 0.$

The roots of (20) are then m times as large as those of (19). E.g. take the equation

(21) $f(x) = 3x^4 + 25x^3 - 1{,}2x^2 - x - 1{,}14 = 0.$

This equation has a positive root between 2 and 3, as may be verified by substitution. We stretch $f(x)$ in (21) by 1,0 obtaining:

(22) $3x^4 + 25{,}0\,x^3 - 1{,}2{,}0{,}0\,x^2 - 1{,}0{,}0{,}0\,x - 1{,}14{,}0{,}0{,}0{,}0 = 0,$

which has a positive root between 2,0 and 3,0.

More generally, stretching $f(x)$ in (21) by $1{,}0^k$ yields the equation

(23) $3x^4 + 25 \,.\, 1{,}0^kx^3 - 1{,}2 \,.\, 1{,}0^{2k}x^2 - 1{,}0^{3k}x - 1{,}14 \,.\, 1{,}0^{4k} = 0,$

whose positive root lies between $2 \,.\, 1{,}0^k$ and $3 \,.\, 1{,}0^k$.

20. Thus we note in (23), with the transformation $x' = 1{,}0^kx$, that the j-th coefficient, a_j, of the original equation (21) is multiplied by $1{,}0^{jk}$ in order to obtain the corresponding coefficient of the transform (23); in other words, the sexagesimal point of a_j should be shifted by jk places either to the right or left

according as $k > 0$, or $k < 0$. E.g. if in (23) we set $k = 2$ we obtain:

(23') $3x^4 + 25,0,0x^3 - 1,2,0,0,0,0x^2 - 1,0,0,0,0,0,0x - 1,14,0,0,0,0,0,0,0 = 0.$

Again setting $k = -1$, we obtain:

(23") $3x^4 + 0;25x^3 - 0;1,2x^2 - 0;0,0,1x - 0;0,0,1,14 = 0.$

The above conclusions have been drawn with regard to a specific equation. In fact, however, the same remarks can be applied to any polynomial equation whose coefficients are sexagesimals.

21. The motivation for introducing this kind of transformation is that Kāshī, in extracting the roots of a special case of (19), first compresses the function sufficiently that if x_1 is a root of the transformed equation, it satisfies the inequality

(24) $$1 \leqq x_1 < 1,0$$

i.e. the first digit of x_1 falls in the degrees' place.

22. Thus if an equation $f(x) = 0$ has a positive root x_0 which we seek, and which we know satisfies the relation

(25) $$1,0^k \leqq x_0 < 1,0^{k+1}$$

then we stretch (or more exactly if $k > 0$ we compress, Cf. 17) the function by $1,0^{-k}$ in order to obtain another function, $f_1(x)$, whose root, x_1, satisfies (24).

23. <u>The Ruffini-Horner method.</u> We are now ready to describe the so-called Ruffini-Horner method for approximating real roots of a polynomial equation.[1]

The essential steps of this method may best be explained in connection with an example.

<u>Example:</u> Find to two sexagesimal places the positive root x_0 of the equation

(26) $f(x) = x^3 - 2,1,31x^2 - 2,0,50x - 4,25,4,48 = 0.$

Let x_0 be composed of the sexagesimal digits $\alpha, \beta, \gamma, \ldots$ so that

(27) $$x_0 = \alpha, \beta, \gamma, \delta, \epsilon, \ldots\ldots, \kappa ; \lambda, \mu, \ldots$$

1. It can easily be verified by substitution that a positive root, x_0, lies between $1,0^2$ and $1,0^3$ i.e.

$$1,0^2 < x_0 < 1,0^3$$

or x_0 is of the form:

(27') $$x_0 = \alpha, \beta, \gamma ; \delta, \epsilon, \ldots$$

2. Thus, according to 22, stretch the function $f(x)$ by $1,0^{-2}$ to get

(28) $f_1(x) = x^3 - 2;1,31x^2 - 0;0,2,0,50x - 0;0,0,4,25,4,48 = 0,$

whose root

(29) $$x_1 = 1,0^{-2} x_0 = \alpha ; \beta, \gamma, \delta, \ldots$$

has its first digit α in the units place.

---

1. The name is misleading. The method was already used by Islamic and Chinese mathematicians long before either Ruffini or Horner lived; Cf. Luckey's paper, and Struik, p. 96.

3. Locate x_1 between two successive digits (Cf. 31 below). It is found to lie between 2 and 3. Thus the first digit of x_0 is $\alpha = 2$.

4. Now diminish the roots of $f_1(x)$ in (28) by $\alpha = 2$ (Cf. 15 - 16) to obtain:

(30) $f_2(x) = x^3 + 3;38,29x^2 + 3;53,53,59,10x - 0;6,6,6,5,4,48 = 0,$

whose root

(31) $$x_2 = x_1 - \alpha = 0;\beta,\gamma,\delta,\ldots .$$

lies between 0 and 1.

5. To bring the first digit β of x_2 to the units place, the function $f_2(x)$ is stretched by 1,0 (Cf. 22) to obtain:

(32) $f_3(x) = x^3 + 3,38;29x^2 + 3,53,53;59,10x - 6,6,6;5,4,48 = 0,$

whose root

(33) $$x_3 = 1,0\ x_2 = \beta;\gamma,\delta,\epsilon,\ldots$$

has its first digit in the units place.

6. Locate x_3 between successive digits (Cf. 31 below). It is found to lie between 1 and 2; therefore the second digit of the root x_0 of equation (26) is $\beta = 1$.

7. Diminish x_3 by $\beta = 1$ to obtain:

(34) $f_4(x) = x^3 + 4,1;29x^2 + 4,1,53;57,10x - 2,10,12;36,54,48 = 0,$

whose root

(35) $$x_4 = x_3 - \beta = 0;\gamma,\delta,\epsilon,\ldots$$

lies between 0 and 1.

8. Stretch the function $f_4(x)$ in (34) also by 1,0 so that

the first digit, γ, of the root

(36) $$x_5 = 1.0\, x_4 = \gamma ; \delta , \epsilon , \ldots\ldots$$

of the new transformed equation

(37) $$f_5(x) = x^3 + 4,1,20x^2 + 4,1,53,57,10x - 2,10,12,36,54,48 = 0$$

falls in the degrees' place.

9. Locate the root x_5, of the transform (37) between successive digits, thus obtaining the third digit, γ, of x_0. In this example γ is found to be 32 (Cf. 31 below).

24. This cycle of operations, namely locating, diminishing, and stretching, is repeated, each cycle giving an additional digit of the root x_0 of equation (26); and the process can be carried on until the desired degree of accuracy is obtained. In this example two more digits are $\delta = 0$, $\epsilon = 43$.

25. We note at this point that the first, second, and third digits of x_0, namely α, β, and γ, have been found when locating each of the roots x_1, x_3, and x_5 respectively between successive digits. In general, as may readily be generalized, the n-th digit, ν, of x_0 is found when locating, between successive digits, the root x_{2n-1} of the corresponding equation $f_{2n-1}(x) = 0$.

26. We note also, from (29), (31), (33), (35), and (36), the relations

$$
\begin{aligned}
r_1 &= 1,0^{-k}\, x_0 \\
r_2 &= r_1 - \alpha \\
r_3 &= 1,0\ r_2 \qquad\qquad (38) \\
r_4 &= r_3 - \beta \\
r_5 &= 1,0\ r_4
\end{aligned}
$$

We may then, by examining (38), generalise and give the two following recursion relations (39) for finding r_i, after having determined r_1 from x_0:

$$
\begin{aligned}
r_{2n} &= r_{2n-1} - \nu \\
r_{2n+1} &= 1,0\ r_{2n}
\end{aligned}
\qquad (39) \qquad \text{where } \begin{pmatrix} n = 1,\ 2, \ldots\ldots \\ \nu = \alpha, \beta, \gamma, \ldots \end{pmatrix}
$$

27. In the course of locating the root, r_{2n-1}, of the transformed equation, $f_{2n-1}(x) = 0$, between successive digits (Cf.25), if the root turns out to be an integer, i.e. $r_{2n-1} = \nu$, then the process is stopped forthwith, and the root of (28) has been found exactly. For then, according to (39), $r_{2n} = r_{2n-1} - \nu = 0$, $r_{2n+2} = r_{2n+3} = \ldots\ldots\ldots = 0$, and the remaining digits are all zeros.

28. Now for the sexagesimal point, it should be placed in such fashion that the inequality (25) holds, namely $1,0^{k} \leq x_0 < 1,0^{k+1}$. In the given example we have $x_0 = 2,1,32;0,43$ to two sexagesimal places.

This method also has been shown by an example. The process, however, is the same in finding a root of any polynomial equation whatever.

29. Finding the positive n-th root of a number, q, is only a special case of the problem of finding the roots of a polynomial equation of degree n. For, we may write

(40) $$x^n - q = 0,$$

and then we have to find the positive root of (40).

Kāshī, as will be seen in the sequel, when finding the n-th root of a number, followed the same steps as those of the Ruffini-Horner method as explained above. This, however, is not the only form in which the method can be displayed.[1]

30. Analytic conditions for locating roots of polynomials. Let us now set down analytic conditions for the steps involved in the Ruffini-Horner method as displayed in 23 - 28.

If the function f(x) is monotonic increasing for $x > 0$, and if $f(0) < 0$, then the equation $f(x) = 0$ has at most one positive root. This is the case in extracting the n-th positive root of a positive number q, i.e. in finding the positive root of equation (40). In this case the following inequalities (41) involve necessary conditions for locating the roots of the equations $f_{2n-1}(x) = 0$, where n = 1, 2, 3, ... (Cf. 25).

1. E.g. see Jones, p. 120; Fine, p. 453; Griffin, p. 354.

The digits $\alpha, \beta, \gamma, \ldots\ldots$ are to be found such that:

$$f_1(\alpha) \leqq 0 < f_1(\alpha + 1),$$
$$f_3(\beta) \leqq 0 < f_3(\beta + 1),$$
(41) $$f_5(\gamma) \leqq 0 < f_5(\gamma + 1),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f_{2n-1}(\nu) \leqq 0 < f_{2n-1}(\nu + 1).$$

These inequalities, when used later, will be somewhat chang ed in form in order to fit the purpose at hand.

31. Locating the next digit of a root of an equation. The usual method of locating a root, x, of an equation, f(x), between successive digits is by trial substitution. However, one does not have to try blindly all the sixty digits if he is working with sexagesimals.

In approximating an irrational root of an equation f(x), and, starting from the second digit, β, of the root (27) onwards, one can ascertain the next digit, or at least a digit not far from the required one, by the method of trial divisor. In brief the method is to divide the constant term of the last transformed equation, with its sign changed, by the coefficient of x. For example, in locating x_3 in (33), it is seen from (32) that dividing 6,8,6;5,4,48 by 3,53,53;59,10 yields a number between 1 and 2. Trying these two digits shows that they are exactly the digits between which x_3 lies, and then $\beta = 1$

(See 23 step 6). Again in locating r_5 in (36) one can see in (37) that dividing 2,10,12,36,54,48 by 4,1,53,57,10 yields a number between 32 and 33 . Trying these two numbers assures that the root lies between them, and $\gamma = 32$ (step 9 in 23). The same applies in locating later digits.

This is tantamount to assuming that the polynomial $f_{2n-1}(x)$, (Cf. 25), behaves linearly in the neighborhood of the root, and that the contribution of higher powers of x to the result is negligible.

This method cannot be trusted to give the first digit of x_0 correctly from the first transformed equation (28). It may sometimes give some indication as to what that digit is. In (28), however, no indication whatever is given.

Occasionally the method may fail to give correctly even the second digit, β , of x_0; but in this case the correct number will not be far from the one found, and so the number of trials is considerably reduced.

Such a device was known and used even in ancient times[1]; but Kāshī g ives no indication as to whether he made use of it or not.

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1. Luckey, p. 265.

PART II

TEXT

CHAPTER III

SOURCE AND TRANSLATION

In this part we give a literal opposite-page translation of Chapter Five in the Third Treatise of the "Key", together with a facsimile reproduction of pages 80-84 of the Princeton copy.

In the preparation of this study, microfilms of three manuscript copies of the "Key" have been at hand. These are:

1. The Princeton copy,
2. The India Office copy, and
3. The Leiden copy.[1]

For convenience, these copies will be referred to as P, I, and L respectively.

Some relations as to dependence and dating of these three copies, and a list of fourteen other manuscripts of the "Key" are given in Miss Raja'i 's thesis.[2]

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1. See the bibliography.

2. See Raja'i, pp. 58-61.

32. <u>Hints to the reader.</u> While reading this chapter the following things should be kept in mind:

a. Parentheses enclose an inserted word or phrase. This is done:

(i) to clarify or explain vague words or ideas,

(ii) to complete or correct the meaning when necessary,

(iii) to indicate the names to which some pronouns refer, and

(iv) to introduce some symbols to which later references are made.

b. Square brackets indicate parts lacking or incorrect in P and restored from either one or both of I and L. Footnotes are given to state from which copy it is restored.

c. The marks '......' denote that there is a variant among P, L and I. Footnotes state what the variant is.

d. Two types of superscripts are used:

(i) Alphabetical superscripts with a, b, c, ... These refer to the footnotes at the bottom of the page.

(ii) Numerical superscripts. These refer to the numbers in the commentary where the corresponding ideas are developed or commented upon.

e. In the translation, the connective article "و" joining two sentences is avoided when possible, and a new sentence is started.

f. The translation is made line-by-line to facilitate cross-reference between the Arabic and the English versions. Lines of the translation bear the same numbers as the corresponding ones of the text.

g. All the numbers are transcribed in the translation into the modern Hindu-Arabic symbols and the order of digits has been reversed. In cases where a whole array of such numbers is involved, the transcription (as to order) forms a mirror image of the original.

Because of this difference in notation, it frequently happens that where the author says "right" the correct direction in our notation is "left", and vice versa. In such cases, the passage has been translated literally without change, but a note is inserted to remind the reader to reverse his direction.

h. A vocabulary of technical terms and various other words used in the translation is given at the end of the thesis.

الباب الخامس في استخراج الضلع الاول من المضلعات

كل عدد مفرد يضرب في نفسه ثم في الحاصل الثاني (ثم في الحاصل الاول صح)
وهكذا الى ما لا نهاية له ويزاد عدد مرتبة ذالك المفرد
على نفسه ثم على المجموع ثم على المجموع الثاني وهكذا الى ما
لا نهاية له فهذه الاعداد على التوالي اعداد مراتب تلك
الحواصل على التوالي كل لنظيره على ما سبق ان عدد
مرتبة حاصل الضرب بقدر مجموع عددي مرتبتي المضروبين
المضروبين ان كانا في طرف واحد من الدرج ولا
محالة يحصل بهذه الاعداد ايضا من ضرب عدد مرتبة
ذالك المفرد في عدد منزلة كل ضلع ومن هذا علم
ان كل ضلع من المضلعات يوجد في المرتبة التي اذا
قسم عددها على عدد منزلته لم يبق شيء اي يعد عدد
منزلته عددها او يساويها ان كان لها عدد ويقال انها منطقة
بذلك المضلع فرتبة الدرج منطقة بجميع المضلعات والتي
المرفوع والدقايق بشيء منها والثاني والثواني منطقان
بجذر لا غير والثالث والثوالث بكعب والرابع والروابع
بمال مال وجذر ايضا والخامس والخوامس بمال كعب
والسادس والسوادس بكعب كعب وبجذر وكعب
ايضا وهذا على هذا القياس فاذا اردنا ان نستخرج
عدد ضلعه الاول على انه ضلع مفروض نضع العدد ونخط
فوقه خطا عرضيا وبين كل مرتبتين خطا طوليا ونعرف
المراتب المنطقة بذلك المضلع كم كانت ونجعل الخطوط
التي على يسار المراتب المنطقة مشابهة للادوار بعضها عن بعض (تمييز المراتب)
بعض ويتم الدور الايسر بالجداول ان لم يكن تاما
ولو اردنا ان نلحق ادوارا اخرا وازيد فرتبة اخر كل

33. P, p.80

1- <u>'The Fifth Chapter[a]', On Extracting the First (i.e. the n-th) Root of the Powers (of a number).</u>

2- Each 'one-digit ' number is multiplied by itself, then by the 'first ' result, then by the second result,

3- and so on <u>ad infinitum</u>, and the place-number of that one-digit (number) is added

4- to itself, 'then to the sum ', then to the second sum and so on <u>ad</u>

5- <u>infinitum</u> . Then these numbers successively are the place-numbers of those

6- results (i.e. products) successively, each to its corresponding (place-number) , according to what preceded, that the 'place-number '

7- of the product is to the amount of the sum of the two place-numbers

8- of the two multipliers if they were (both) on one side of the degrees' (place) .

9- Doubtless, these numbers also result from multiplying the place-number

10- of that one-digit (number) by the exponent of each power . It is known from this

11- that any one of the powers is located in the place which, if

12- its number is divided by its (i.e. the power's) exponent, there remains nothing, i.e. its exponent

13- counts (i.e. measures, is a factor of) its (i.e. the place-) number, or else equals it if it has a number (i.e. is non-zero , then) it is said to be perfect

14- in that power and the ones in it which are not divisible are imperfect . The quotient is the place number of the first root of that power . Then the degrees' place is perfect in all powers;

15- the first elevates' and the minutes' (places) are perfect in none of them (i.e. the powers); the second elevates' and the seconds' (places) are perfect,

16- in a root (i.e. a square and) none other; the third elevates' and the thirds' (places) in a cube; the fourth elevates' and the fourths' (places)

17- in a square-square (i.e. 4-th power) and also a root (i.e. a square); the fifth elevates' and the fifths' (places) in a square-cube (i.e. 5th power);

18- the sixth elevates' and the sixths' (places) in a cube-cube (i.e. 6th power) and in a square and a cube

19- also; and so on accordingly . Thus if we wanted to extract from

20- a number its first (i.e. n-th) root on the supposition that it is an assumed power (of the first root), we set down the number, and we draw

21- over it a transverse line, and between each two places a line lengthwise (i.e. vertically) . We ascertain

22- the perfect places in that power, how many they are, and one makes the lines

23- which are on the left (i.e. right) of the perfect places double to distinguish the cycles , the ones from

24- the others, and one completes the left (i.e. right) cycle in the column if it is not complete .

25- And if we want to adjoin to it another cycle or more , then the place at the end of each

---

a. in I.
b. Lacking in L.
c. Lacking in I and L.
d. Lacking in I.
e. From I and L.
f. in I and L; this seems to be put for
g. From L.
h. in I.
i. in I and L.
j. in I.
k. in I.
l. in I.
m. in I.

دور هي المنطقة بالمضلع المفروض والباقية احمة ونقسم
الجدول في الطول صفوفا بعدد منزلة المضلع المفروض
ونكتب اسماءها على ايمنها كما سبق في المقالة الاولى ثم
نطلب اكثر مفرد يمكن نقصان مضلعه للمفروض عما كان
في الدور الاول من العدد اعني الدور الايمن فاذا وجد
نضعه في سطر الخارج فوق المنطق الاول اي فوق الجدول
الاخير من الدور الاول وتحته في اسفل صف الضلع
ونضع مضلعاته المتوالية في اسافل الصفوف
على التوالي الى ان نضع مضلعه للمطلوب تحت العدد
بحيث يقع آخر مراتبها في جدول اخر الدور ليكون محاذيا
لما وضع في سطر الخارج وننقصه عما يحاذيه من العدد
ثم نزيد المفرد الفوقاني على التحتاني الذي في صف
الضلع مرة لصف ثاني العدد ونضربه في المجموع ونزيد الحاصل
على ما في صف المال ونضربه في هذه المجموع ونزيد على ما
فوقه وهكذا الى ان يبلغ صف ثاني العدد ثم نعمل هكذا
لصف ثالث العدد وهكذا الى ان ينتهي الى صف الضلع
فنزيد الفوقاني على ما في صف الضلع لاجله وننقل ما في
ثاني العدد بمرتبة الى اليسار وما في ثالثة بمرتبتين و
وما في رابعة بثلاث مراتب هكذا الى ان ينتهي الى
صف الضلع فننقله بعدة الصفوف التي تحت صف العدد
ثم نطلب اكثر مفرد بالصفة المذكورة فاذا وجد نضعه
فوق المنطق الثاني وتحته في صف الضلع على ايسر
ما وضع فيه ونضربه فيما وضع فيه ونزيد الحاصل على ما
فوقه ثم فيما فوقه ونزيد الحاصل على ما فوقه وهكذا الى
ان يبلغ صف ثاني العدد ونضربه فيما فيه وننقص

P, p. 81

1- cycle is the perfect (one) in the assumed power, and the others (i.e. other places) are imperfect . One divides
2- the scheme in length into rows to a number (equal to) the exponent of the assumed power,
3- and we write their (i.e. the rows') names on their right (i.e. left) as preceded (i.e. as was done) in the First Treatise . Then
4- we demand the largest one-digit (number, say α,) whose assumed (i.e. n-th) power can be subtracted from what is
5- in the first cycle of the number, I mean the right(-hand, i.e. left-hand) cycle . Then, when it is found,
6- we put it in the line of the result over the first perfect (place), i.e. over the last
7- column of the first cycle, and under it at the bottom of the row of the root .
8- We put its successive powers at the bottom of the rows
9- successively until we put its required power under the number
10- such that their (i.e. the powers') last places fall in the column at the end of the cycle, so that it is opposite to
11- what was put in the line of the result . We subtract it from what is opposite it of the number .
12- Then we add the upper one-digit (number, α,) to the lower which is in the row
13- of the root , once up to the row of the second of the number ; we multiply it (α) by the sum, and add the result
14- to what is in the row of the square; we (also) multiply it (α) by this sum, and add the result to what
15- is over it; and so on until one reaches the row of the second of the number . Then we do likewise
16- up to the row of the third of the number , and so on until one gets to the row of the root ;
17- then we add the upper to what is (already) in the row of the root in order to close it . We transpose what is in
18- the (row of the) second of the number by one place to the left (i.e. right), what is in (the row of) its third by two places,
19- what is in (the row of) its fourth by three places, (and) so on until one gets to
20- the row of the root, then we transpose it by the amount of the rows which are under the row of the number .
21- Then we demand the largest one-digit (number, say β,) with the above-mentioned quality . Then when it is found we put it
22- over the second perfect (place) and (also) under it in the row of the root (say R) on the left (i.e. right) of
23- what was put in it (R) . We multiply it (β) by what was put in it (R) and add the result to what
24- is over it (R), then (multiply β) by what (resulted) over it (R). We add the (new) result to what is over it, and so on until
25- one reaches the row of the second of the number, and (then) we multiply it (β) by what is in it (i.e. the row of the second of the number) and we subtract

---

a. in I and L.
b. Lacking in I.
c. in I and L.
d. in I.
e. in I and L.
f. in I.
g. in I there is the addition ... which gives no sense in the sentence.
h. in I and L.
i. in L.
j. in I and L.

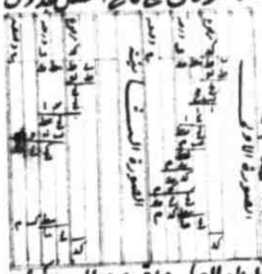
الحاصل على ما فوقه ثم فيما فوقه ونزيد الحاصل على ما فوقه
وهكذا الى ان يبلغ صف ثاني العدد ونضربه فيما فيه
وننقص الحاصل عما في صف العدد ثم ننقل بصف صف كما
ذكرنا للنقل وننقل على ما سبق وهكذا نعمل في كل دور على
قياس ما نقلنا في المقالة الاولى الى ان ينتهي العدد او
الى حيث شئنا ان نقطع العمل فما حصل في سطر الخارج
فهو الضلع الاول لذلك المضلع تحقيقا ان لم يبق في
العدد شئ والا يكون تقريبا وظاهر ان كلما نزداد مراتب
سطر الخارج في سلسلة التنزل كان ادق واذا قسم
عدد كل واحد من المراتب المنطقة على عدد منزلة المضلع
المفروض فالخارج من القسمة هو عدد مرتبة المفرد
الذي وضع على فوق تلك المرتبة فلنكتب فوقه
والدرجة تقع فوق الدرجة مثاله اردنا ان نستخرج جذر
ب ط مط ك درجة وضعناه ورسمنا الجداول الطولية
وفصلنا الادوار بالخطوط المثناة كما ذكر وطلبنا اكثر
مفرد بالصفة المذكورة فوجدناه كد وضعناه فوق
المنطق الاول وهو ط وتحتها في اسفل الجدول وضربناه
في نفسه حصل ط لو نقصناه عما يحاذيه اعني عن ب ط
بقي لج وضعناه تحت ط بعد الخط الفاصل ثم
زدنا الفوقاني اعني كد على التحتاني فصار مح نقلناه
الى اليسار بمرتبة ثم طلبنا اكثر مفرد بالصفة المذكورة
وجدناه ط وضعناه فوق منطق الدور الثاني وكتبناه
على ايسر مح وضربناه فيما هو اسفل الجدول اما في
كل واحد من مفرداته نقصنا الحاصل عما يحاذي به
كما في الصورة الاولى او فيه بطريق ما كان احد المفروضين

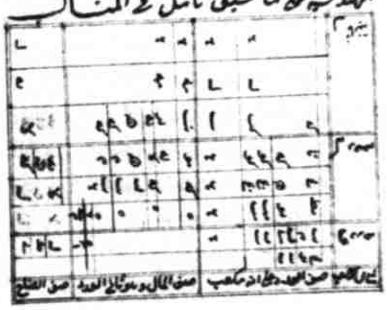
P, p. 82

1- ....
2- .... Repetition of lines 23 and 24 of the previous page ......
3- ..... the result from what is in the row of the number . Then we do likewise, row by row, as
4- we mentioned (before) , up to the transposition, and we transpose similarly to what preceded , and so we do in each cycle
5- according to what we said in the first treatise until the number ends , or
6- to where we want to cut off the process . Then what resulted in the line of the result
7- is the first root of that power, exactly if there remains
8- nothing in the row of the number, otherwise it is approximate . It is evident that the more we increase the places
9- of the result-line in the descending series the more accurate (the result) is . If one divides
10- the number of each of the perfect places by the exponent of the assumed
11- power, then the quotient is the place-number of the digit
12- which was put over that place, so let us write (its place number) over it ,
13- and the degree falls over the degree . <u>For example</u>, we wanted to extract the square root of
14- 10,9,49,10 degrees . We set it down and drew the lengthwise columns,
15- and we separated the cycles by the double lines as we mentioned (above) . We demanded the largest
16- one-digit (number) with the above-mentioned quality, then we found it (to be) 24 . We put it over
17- the first perfect (place, which) is 9, and (also) under it at the bottom of the column . We multiplied it
18- by itself; 9,36 resulted. We subtracted it from what is opposite it, I mean from 10,9;
19- there remained 33 . We put it under 9 after the separating line, then
20- we added the upper, I mean 24, to the lower; then it became 48; we transposed it
21- by one place to the left (i.e. right). Then we demanded the largest one-digit (number) with the above-mentioned quality .
22- We found it (to be) 41 . We put it over the perfect (place) of the second cycle, and (also) under it
23- to the left (i.e. right) of 48 . We multiplied it by what is at the bottom of the column, either (1) by
24- each one of its digits, and we subtracted the result from what is opposite it
25- as in the first form , or (2) by it in the (same) way (as if) one of the two multipliers were

---

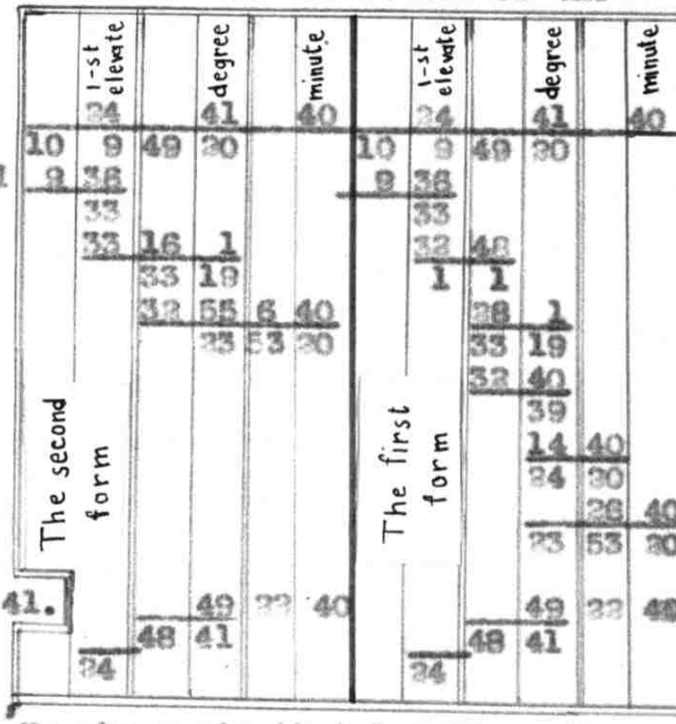
a. Not repeated in I and L. b. in L, in I. c. in I and L.
d. in I. e. in L. in I. f. From I and L.
g. in I. h. in I. i. Lacking in L.
j. in I. k. in I. l. in I and L.

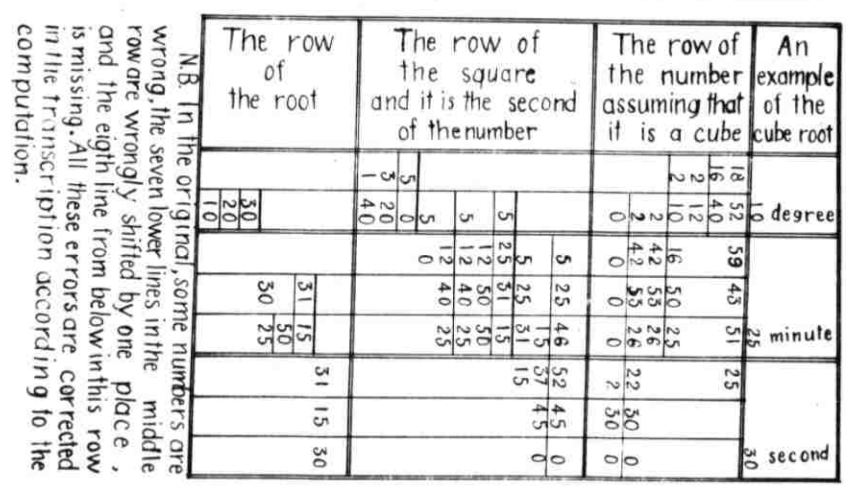
معزدا ونقصناه للحاصل هاذيه على ماسبق كما في الصورة
الثانية ثم زدنا ما الفوقاني على ما في اسفل الجدول

فصار سطـ ك
نقلناه بمرتبة
وطلبنا اكثر مفرد
اخر بالصفة المذكورة
وجدناه م ووضعناه
فوق منطق الدو
الثالث وتحته
على يمين ك وبه قطعناه العمل وبقي من العدد ... رسالة محيطية
... ثالثة كما في هاتين الصورتين وما وقع فوق الخط دسوما)
وقد اخرجنا في رسالتنا المسماة المحيط صورا كثيرة الارقام
واستعملنا فيها نكات غريبة ومن اراد ذلك فليرجع اليها
ثم اوردنا ههنا مثالا لاستخراج الكعب ومثالا اخر لاستخراج
الضلع الاول لكعب الكعب ولم نتعرض شرح العمل لئلا
يطول الكتاب وذلك سهل على من استحضر العمل بالرقوم
الهندسية على ما سبق تأمل في المثال



مثال استخراج

1- a one-digit (number), and we subtracted the result from what is opposite it according to what preceded, as in the
2- second form . Then we added 41 to what is at the bottom of the column;
3- then it became 49,22 .
4- We transposed it by one place,
5- and we demanded (another time) the largest one-digit
6- (number) with the above-mentioned quality .
7- We found it (to be) 40 , and we put it
8- over the perfect (place) of the third
9- cycle, and (also) under it
10- to the right of 22 . By it (i.e. at this point) we cut off the process, and there remained of the number 33, (53), 30
11- seconds, as in these two forms . That which befell over the degrees is degrees , and it is 41.
12- We extracted in our book "On the Circumference" roots with many digits (of numbers of many parts
13- and we employed in it odd tricks. He who wants that, let him refer to it .
14- Then we presented here an example for the extraction of the cube root, and another example for the extraction
15- of the first root of the cube-cube(i.e. 6-th power) , and(we) did not deal with the explanation" of the process lest
16- the book become long, and that is easy for he who recalls the process in Indian
17- numerals according to what preceded in the First Treatise . Ponder on the example and the two examples are these.





a. in I and L. b. in I and L. c. Lacking in I.
d. in I. e. The author should have said left in his system. In modern use it is right.
f. in I, in L.
g. From I and L. h. in L. i. in I, in L.
j. in I k. From L. l. in I and L.
m. in L. n. in I and L. o. in I and L.
p. in I and L.

مثال استخراج الضلع الاول لكعب كعب العدد الموضوع في صف العدد

صور العدد في الضرب الكعب

مال العدد وهو ضلع مال الكعب

مال العدد وهو ضلع مال المال

ربع العدد وهو ضلع الكعب

ضلع العدد وهو ضلع المال

ضلع الضلع

 L, p.88 (for P, p.84)

An example of extracting the 6-th root of the number put in the row of the number

The line of the result				1-st elevate 14						degree 0						second 30
	9-th elevates	8-th elevates	7-th elevates	6-th elevates	5-th elevates	4-th elevates	3-rd elevates	2-nd elevates	1-st elevates	degree	minute	second	third	fourth	fifth	sixth
The row of the number i.e. The row of the 6-th power	34	59	1	7	14	56	23	3	47	37	40					
	34	51	32	16												
		7	28	51												
		7	28	51	14	54	23	3	47	37	30	56	15	0	0	0
											9	3	45	0	0	0
The second of the number i.e. the row of the 5-th power			14	57	42	20	48	46	7	35	15	1	52	30	0	0
				1	20	5	40	46	7	35	15	1	52	30	0	0
			14	56	22	24										
		14	56	22	24											
	14	56	22	24												
	12	26	58	40												
	2	29	23	44												
The third of the number i.e. The row of the 4-th power					2	40	11	37	32	15	10	30	3	45	0	0
							7	37	32	15	10	30	3	45	0	0
					2	40	4	0								
			2	40	4	0										
	2	40	4	0												
	1	46	42	40												
		33	21	20												
		42	41	4												
		10	40	16												
The fourth of the number i.e. The row of the cube								15	15	4	30	21	0	7	30	0
										24	30	21	0	7	30	0
								15	14	40						
					15	14	40									
		15	14	40												
		7	37	20												
		7	37	20												
		4	54	24												
		3	2	56												
		2	17	12												
			45	44												
The fifth of the number i.e. The row of the square											49	0	42	0	15	0
							49	0								
			49	0												
			16	20												
			32	40												
			13	4												
			19	36												
			9	48												
			9	48												
			6	32												
			5	16												
The sixth of the number i.e. The row of the root													1	24	0	30
								1	24	0						
			1	24												
			1	10												
				56												
				42												
				28												
				14												

a. The numbers in P are missing, so we reproduced this page from L.

PART III

COMMENTARY

This is a commentary on Part II, the source and translation. It is made self-contained, however, so that it can be read without referring to the text. Those who wish to follow the text will find, in the translation, reference numbers to the commentary to indicate where the corresponding ideas are developed or explained.

CHAPTER IV

PREPARATORY

Chapter Five in the Third Treatise of the "Key" deals with the extraction of the n-th root of a number expressed in the sexagesimal notation.

In commencing a commentary on this work, we now give:

1. Comments and explanations of some terms which the author uses without explanation (36-39).

2. Facts given or explained by the author previously in his book and which he reestablishes in the translated extract (40-44).

36 <u>Terminology. Root and power.</u> The term <u>ḍil'</u> in Arabic means <u>side</u>, and <u>muḍalla'</u>, pl. <u>muḍalla'āt</u>, means <u>an object which has sides</u>. In plane geometry it means <u>polygon</u>.

Medieval Arab mathematicians used the expression "the first <u>ḍil'</u> of the <u>muḍalla'āt</u>" to mean "the n-th root of a number". I.e. if a number, q, is put in the form $q = x^n$, then x is the first <u>ḍil'</u> of the <u>muḍalla'</u> x^n.

Luckey translates the word <u>muḍalla'</u> into German by <u>Geseiteten</u>. In this thesis we use the term power to serve the purpose.

Evidently the Arabic nomenclature is based on a geometric concept. To find the square root of a number, q, is equivalent to finding the side of the square whose area is equal to q. Again, to find the cube root of q is equivalent to finding the edge of the cube whose volume equals q. For powers higher than the third, the geometric interpretation fails. Medieval mathematicians, it seems, generalized the nomenclature for $n > 3$. Thus one can speak of an n-dimensional space in which there is an n-dimensional regular object whose "volume" is q, then $\sqrt[n]{q}$ is the edge of that object.

35 Perfect powers and perfect places[1]. Following is the definition of (a) perfect powers, and (b) perfect places in a given power.[2]

A number, q, is a perfect (muntaq)[3] n-th power if the n-th root of q is an integer N, i.e. $\sqrt[n]{q} = N$; otherwise q, as an n-th power, is imperfect (aṣamm)[3]. E.g. 49 is a perfect square, for its square root is 7. 12,9 is a perfect cube, for its cube root is 9. 18,54,28,21,0,0,0,0,0 is a perfect 5-th power, for its 5-th root is 21,0, but none of these three numbers is a perfect 4-th power, for none of them has an integer 4-th root.

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1. P, p. 26.
2. Here place is in the first meaning, Cf. 5.
3. Luckey reads it incorrectly (muntaq) and translates it by rational. He translates (aṣamm) by irrational (Cf. Luckey, p. 235).

36 For the second meaning of perfect, i.e. <u>perfect places</u>, consider a number represented in the sexagesimal notation, and which is thought of as being an n-th power, i.e. we want to extract its n-th root, then the <u>perfect places[1] in the representation are those places whose numbers (cf. 8) have the form kn, where k is any integer; the other places are imperfect.</u> It is not necess ary that the power be perfect in the sense of 35 above. E.g. consider the number

(42) $$q = 16,9,1,6,2,41,31;4,3,17,32,53,12$$

as being a 4-th power, i.e. n = 4, then from left to right the perfect places are those of the digits 1, 31, and 32 respectively, for their place-numbers respectively are 4, 0, and -4, which are all of the form 4k, with k an integer. The remaining places are imperfect.

If the number is thought of as being a cube, i.e. n = 3, then the perfect places are those of 16, 6, 31, 17, and 12, for their place-numbers respectively are 6, 3, 0, -3, and -6, which are all multiples of 3. Again, when n = 5, the perfect places are those of 9, 31, and 53, whose place-numbers respectively are 5, 0, and -5.

37 From another point of view, a place whose number is n, is

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1. Here <u>place</u> is in the first meaning, Cf. 5.

perfect in all powers which are factors of n. Thus in the number (42), the degrees' place, that of the digit 31, whose number is zero, is perfect in all powers, i.e. for n = 2, 3, ...; the places of the first elevates and minutes, those of 41 and 4, whose numbers are 1 and -1 respectively, are perfect in no power; the places of the second elevates and seconds, those of the digits 2 and 3, whose numbers respectively are 2 and -2, are perfect in squares only, i.e. when n = 2. The place whose number is ± 12 is perfect in 12-th powers, 6-th powers, 4-th powers, cubes, and squares; and so on for other powers.

36 <u>Cycles</u>. A concept used by the author is that of <u>cycles</u> (<u>adwār</u>, sing. <u>dawr</u>).

<u>In an n-th power, the set consisting of one perfect place and n-1 imperfect places on its left, if any, is called a cycle.</u> Thus if in (42) we set n = 6, then the cycles from left to right are respectively

16 ,
9, 1, 6, 2, 41, 31 ,
and 4, 3, 17, 32, 53, 12 .

We note that the first cycle on the left consists simply of 16, for the place of 16 is perfect and there are no digits on its left.

In each cycle, the lowest place is the perfect one, and the others are imperfect.

39 However, if the number of fractional places is not a multiple of n, we fill in zeros to complete the last fractional cycle. E.g. if the number

19,14,6,31,42,58;1,53,16,22,38,12

is regarded as a 4-th power, the cycles from left to right are

19, 14 ,

6, 31, 42, 58 ,
1, 53, 16, 22 ,
and 38, 12, 0, 0 .

On the right-hand side of a number, however, we can add as many more cycles as we like, filling in their places with zeros, without changing the value of the number so long as the sexagesimal point remains in its place. E.g., if the number

16,1,43,17,28,11,50,13;3,34,12

is thought of as a 5-th power, it can be written in the form

16,1,43,17,28,11,50,13;3,34,12,0,0,0,0,0,0,0,0,0,0,0,0,

and the cycles from left to right are then:

16, 1, 43 ,

17, 28, 11, 50, 13 ,

3, 34, 12, 0, 0 ,

0, 0, 0, 0, 0,

and 0, 0, 0, 0, 0 .

40 <u>Development of certain concepts.</u> In this section, 40-44, we give the author's ideas in modern terminology, with proofs

justifying his s tatements.

Take a one-digit number, say a, of place-number p (Cf. 8), e.g. a = 24 second elevates = $24 \,.\, 1,0^2$, and write its successive powers in a line

(43) $a \quad a^2 \quad a^3 \quad \ldots\ldots\ldots \quad a^n \quad \ldots\ldots$

In the example these powers respectively are

(43') $24.1,0^2 \quad 24^2.1,0^4 \quad 24^3.1,0^6 \ldots\ldots \quad 24^n.1,0^{2n}\ldots\ldots$

Then the number, a, is called a first root (ǧaḏr) with respect to any one of these powers (Cf. 34).

41 Since the place-number of a is p, then that of $a^2 = a.a$ is $p + p = 2p$, that of $a^3 = a^2.a$ is $2p + p = 3p$, and that of $a^n = a^{n-1}.a$ is $(n-1)p + p = np$. E.g., in $a = 24.1,0^2$, the place-number of a^2 is $2 + 2 = 4$, that of a^3 is $4 + 2 = 6$, etc.

Thus the place-numbers of the successive powers of a are respectively

(44) $p \quad p + p \quad 2p + p \quad \ldots\ldots \quad (n-1)p + p \ldots\ldots$

This is in fact an application of the law of exponents

(45) $$x^l \,.\, x^m = x^{l+m} ,$$

where here l and m are any integers, positive, negative or zero, and x is any terminating sexagesimal.

42 In the law of exponents, l and m are added algebraically, i.e. their absolute values are added to each other when they are

of the same sign, and are subtracted from each other when they are of opposite signs. The author speaks of the places of the two multipliers as being on the same side of the degrees' place or on opposite sides where we would say that their place-numbers have the same or opposite signs (Cf. 8).

43 The numbers in (44) could equally well be obtained by multiplying p, the place-number of a, by the exponents of the corresponding powers, i.e. (44) may be written in the form

(46) $$p \quad 2p \quad 3p \quad \ldots\ldots \quad np \quad .$$

In fact, this is the same as (44), only multiplication t akes the place of addition. Or, we may say this is an application of the law

(47) $$(x^l)^m = x^{lm} \quad .$$

44 The author remarks also that the place-number of any one of the powers, if it is not zero, is divisible by the exponent of the power or is equal to it, i.e. np is divisible by n, and that the quotient, p, is the place-number of the first (i.e. n-th) root, a.

This is in fact true even when $np = 0$, for $n = 1, 2, 3, \ldots\ldots$

CHAPTER V

THE EXTRACTION OF ROOTS

45 In this chapter we present an algorism of root extraction in general, together with justification for its steps. Kāshī states the process for the general case, without proofs, and without reference to any particular example. He then applies the general method to an example of square root extraction. Here we explain the general algorism by means of a particular example. The one chosen is also given by the author, but without explanation, at the end of the translated part. See page 37.

46 <u>Preparation for the extraction of the n-th root.</u> <u>Example.</u> Find the 6-th root of the number

(48) $$q = 34,59,1,7,14,54,23,3,47,37;40 \quad ,$$

or, in algebraic form, find the positive root of the equation

(49) $$f(x) = x^6 - q = x^6 - 34,59,1,7,14,54,23,3,47,37;40 = 0 \quad .$$

Write the number at the top of a fairly large sheet of paper (See page 37).

If the last cycle on the right is not complete, fill it in with zeros (Cf. 39). In the example, five zeros are put (or assumed) in front of 40 to complete the last cycle; thus it becomes 40,0,0,0,0,0.

Determine the perfect places in the number (Cf.38). These are the places whose numbers are of the form nk = 6k i.e. 12, 6, 0, -6, ... In the example, these are the places of 7, 37, and the last zero on the right.

Draw a horizontal line over the number, and vertical lines along the sheet separating the digits, those on the right of the perfect places are made double in order to distinguish the cycles from one another. Thus the cycles in the example from left to right are

34, 59, 1, 7 ,

14, 54, 23, 3, 47, 37 ,

and 40, 0, 0, 0, 0, 0 .

47 Divide the sheet by horizontal lines into n = 6 parts, called rows, the given number, i.e. the radicand, being at the top of the highest one. These rows are named in two ways, as described in the First Treatise of the "Key".[1]

(i) In the first way, the lowermost row is called "the row of the root", the one over it is "the row of the square", the next "the row of the cube", and so on until the highest one is "the row of the n-th, here the 6-th, power".

(ii) In the other way, the highest row is called "the row of the number", the one below it is "the row of the second of the number", the next below "the row of the third of the number", and so

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1. P, p. 30.

on until the row of the root is "the row of the n-th, here the 6-th, of the number".

The names of the rows are written, in both ways, on their left side, as shown on page 37.

Over the number leave a line for the result and call it "the line of the result".

48 <u>Starting the extraction of the n-th root. Locating the first digit.</u> Having completed the preparations cited in the previous section, we are now ready to explain the process of the extraction of roots.

One should first have tables of squares, cubes, et c., up to the n-th, here 6-th, powers of the digits from 1 to 59 so that they can be easily looked up.

Let the digits of the root from left to right be denoted by $\alpha, \beta, \gamma, \ldots\ldots$

In the table of n-th, here 6-th, powers seek the largest digit, α, whose n-th power can be subtracted from the first cycle on the left i.e. from 34,59,1,7. To explain this we have to refer back to the idea of a <u>stretch</u> (cf. 17-22).

Stretching f(x) in (49) by $1,0^{-k} = 1,0^{-1}$ (cf. 22) we get:

(50) $f_1(x) = x^6 - q_1 = x^6 - 34,59,1,7;14,54,23,3,47,37,40 = 0,$

where (cf. 20)

(51) $q_1 = 1,0^{-6}\, q = 34,59,1,7;14,54,23,3,47,37,40\ .$

Now we note that the integer part of q_1 is precisely the first cycle on the left. Thus the statement "to seek the largest digit, α, whose 6-th power can be subtracted from the last cycle on the left" means to seek the digit which satisfies the relation

(52) $$\alpha^6 \leq q_1 < (\alpha + 1)^6 .$$

Inequality (52) is equivalent to the first condition in (41), namely $f_1(\alpha) \leq 0 < f_1(\alpha + 1)$, which is here $\alpha^6 - q_1 \leq 0 < (\alpha + 1)^6 - q_1$.

α is found to be 14.

Rasm of the first digit. Put the number $\alpha = 14$ in the line of the result just above the perfect place of the above-mentioned cycle, namely just above 7 (See p. 37).

Write also the number $= 14$ at the bottom of the lowermost row, i.e. the row of the root, and write the successive powers of α, i.e. $\alpha^2 = 3,16$, $\alpha^3 = 45,44$, ... , $\alpha^{n-1} = \alpha^5 = 2,29,23,44$, at the bottoms (See 51 below) of their corresponding rows. Then the n-th power, here $\alpha^6 = 34,51,32,16$, is put at the top of the row of the n-th power just under the number in order that it can be subtracted from it. After subtraction, the remainder

(53) $$q_2 = q_1 - \alpha^6 = 7,33,51;14,54,23,3,47,37,40$$

is obtained in the row of the number.

All these numbers are written in their corresponding rows in such fashion that their first digits on the right fall in the column of the perfect place 7, i.e. in the column where $\alpha = 14$ was put.

51 It is to be noted that the author works in the rows from the bottom upwards, where he could equally well do the opposite, i.e. write the numbers on the tops of the rows and work downwards. Why he does so we do not know. Luckey claims[1] that this is due to the influence of sand-table reckoning where the computer writes some number in the sand, performs some operation on it, then rubs it out in order to replace it by the result.

52 Cyclic operations on the first digit. Now there occur several cyclic processes of addition and multiplication. They are all similar in form and all start from the row of the root, differing only in the st age at which they are discontinued. The author uses the following terminology in naming them.

When he says "Once up to the row of the second of the number", he means that what he is going to perform, i.e. this process, will be carried on starting from the row of the root, going up until its last steps are performed in the row of the second of the number. Likewise "Once up to the row of the third of the number" means that the process will be carried on until its last steps are performed in the row of the third of the number, and so on.

The number $\alpha = 14$, put in the line of the result, we shall call "the upper", and the one put in the row of the root we call "the lower".

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1. Luckey, pp. 256-257.

53 We begin the process "Once up to the row of the second of the number", here up to the row of the 5-th power.

Add "the upper" $\alpha = 14$ to "the lower" $\alpha = 14$ which is in the row of the root. The sum $2\alpha = 28$ is written above $\alpha = 14$ in the row of the root, after drawing a stroke over the letter in order to cancel it.[1] Now multiply "the upper" $\alpha = 14$ by $2\alpha = 28$ contained in the row of the root and add the product $2\alpha^2 = 6,32$ to the $\alpha^2 = 3,16$ lying at the very bottom of the row of the square. The sum $3\alpha^2 = 9,48$ is also multiplied by "the upper" $\alpha = 14$ and their product $3\alpha^3 = 2,17,12$ is added to the $\alpha^3 = 45,44$ already at the bottom of the row of the cube. The new sum $4\alpha^3 = 3,2,56$ is also multiplied by $\alpha = 14$ and the product added to what is in the row over it. This process continues until the row of the second of the number, here the row of the 5-th power, is reached, where $5\alpha^4 . \alpha = 5\alpha^5 = 12,26,58,40$ is added to the $\alpha^5 = 2,29,23,44$ already at the bottom of the row of the 5-th power in order to obtain $6\alpha^5 = 14,56,22,24$. This process, "Once up to the row of the second of the number", is now complete.

54 Then start the process "Once up to the row of the third of the number". "The upper" $\alpha = 14$ is added to the $2\alpha = 28$ which is in the row of the root, and the sum $3\alpha = 42$ is written over the $2\alpha = 28$ after a line is drawn over the letter in order to cancel it. This $3\alpha = 42$ is now multiplied by "the upper" $\alpha = 14$ and the

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1. According to Luckey, this is the method of the sand-table computer. Cf. 51.

product $3\alpha^2$ = 9,48 is added to what is already in the row of the square, namely $3\alpha^2$ = 9,48. The new sum $6\alpha^2$ = 19,36 is also multiplied by α = 14, and the product $6\alpha^3$ = 4,34,24 is added to what is already in the row of the cube, namely $4\alpha^3$ = 3,2,56. Again, the sum $10\alpha^3$ = 7,37,20 is multiplied by α = 14 and the product added to what is already in the row above it, and so on until the row of the third of the number, here the row of the 4-th power, is reached, where $10\alpha^3.\alpha$ = $10\alpha^4$ = 1,46,42,40 is added to the $5\alpha^4$ = 53,21,20 already in the row of the 4-th power, and the result $15\alpha^4$ = 2,40,4,0 is now the highest number in the row of the 4-th power. The process "Once up to the row of the third of the number" is also complete.

56 Perform similarly the process "Once up to the row of the 4-th of the number", then "Once up to the row of the 5-th of the number", and so on until the process "Once up to the row of the (n-1)-th of the number", i.e. the row of the square, is completed. In the example, the last results in the rows of the cube and square are respectively $20\alpha^3$ = 15,14,40 and $15\alpha^2$ = 49,0. The last step is to add α = 14 to what is by now in the row of the root, which is in fact 5α = 1,10, and the last result $n\alpha$, here 6α = 1,24, is the highest number in the row of the number. We may, to generalise, consider this single step as forming the process "Once up to the row of the n-th, here 6-th, of the number".

Now the following numbers appear at the tops of the n - 1 = 5 lower rows:

$$6\alpha^5 = 14,56,22,24 \; ,$$
$$15\alpha^4 = 2,40,4,0 \; ,$$
$$20\alpha^3 = 15,14,40 \; ,$$
$$15\alpha^2 = 49,0 \; ,$$
and
$$6\alpha = 1,24 \; .$$

56 <u>Justification of the previous processes.</u> To explain why these processes are performed, let us write down the Horner's scheme for this example in the form (Cf. 16)

$\alpha=14$						
	1 0	0	0	0	0	$-q_1$
	1 $\alpha=14$	$\alpha^2=3,16$	$\alpha^3=45,44$	$\alpha^4=10,40,16$	$\alpha^5=2,29,23,44$	$-q_2$
	1 $2\alpha=28$	$3\alpha^2=9,48$	$4\alpha^3=3,2,56$	$5\alpha^4=53,21,20$	$6\alpha^5=14,56,22,24$	
	1 $3\alpha=42$	$6\alpha^2=19,36$	$10\alpha^3=7,37,20$	$15\alpha^4=2,40,4,0$		
(54)	1 $4\alpha=56$	$10\alpha^2=32,40$	$20\alpha^3=15,14,40$			
	1 $5\alpha=1,10$	$15\alpha^2=49,0$				
	1 $6\alpha=1$					
	1					

$$-q_1 = -34,50,1,7;14,54,25,3,47,37,40$$
$$-q_2 = -7,20,51;14,54,25,3,47,37,40$$

The upper line inside the right triangle in (54) above gives, except for the first number 1 and the last number $-q_2$, the successive powers of $\alpha=14$ which are put at the bottoms of the $n-1 = 5$ lower rows. The last number, $-q_2$, is, in absolute value, the remainder (53) in the row of the number.

The process "Once up to the row of the second of the number" merely brings the numbers in the n-1 = 5 lower rows to what is given in the second line of the right triangle above, namely

$2\alpha = 28 \qquad 3\alpha^2 = 9,48 \qquad 4\alpha^3 = 3,2,56 \qquad 5\alpha^4 = 53,21,20$

$6\alpha^5 = 14,56,22,24$. For, the statement "multiply the upper, α, by what is in the row of the (j - 1)-th power and add the product to what is already in the row of the j-th power" is precisely the operation used for obtaining $a_{i,j}$ in the expression (cf. (17)).

$$a_{i,j} = a_{i-1,j} + \alpha \cdot a_{i,j-1} \quad , \text{ where } \quad \begin{pmatrix} i = 1,2,3,\ldots\ldots n \\ j = 1,2,3,\ldots n-1 \end{pmatrix}$$

Only the number $6\alpha^5 = 14,56,22,24$, which is in the row of the second of the number, has reached its final form; all the other numbers in the n-2 = 4 lower rows will still be dealt with in the following processes.

Similarly, the process "Once up to t he row of the third of ~~the third of~~ the number" brings the numbers in the n-2 = 4 lower rows to what is given in the third line of the triangle (54) above, namely

$3\alpha = 42 \qquad 6\alpha^2 = 19,36 \qquad 10\alpha^3 = 7,37,20 \qquad 15\alpha^4 = 2,40,6,0.$

The last number in this row, namely $15\alpha^4 = 2,40,40,0$, will now have reached its final form; the others are still to be dealt with in later processes.

Again the process "Once up to the row of the 4-th of the number" brings the number in the n-3 = 3 lower rows to what is given in the 4-th line of the triangle (54); and so in the process

"Once up to the row of the i-th of the number", the numbers in the i-1 lower rows are brought to the values given in the corresponding i-th line of (54).

When all the n - 1 processes are complete, the numbers at the tops of the rows are precisely those along the hypotenuse of the triangular scheme (54). These are then the coefficients of the transformed equation obtained when diminishing the roots of the original equation $f_1(x) = x^n - q_1 = 0$ in (50) by α, (Cf. 13-15), i.e. the transformed equation is

$$(55) \quad f_2(x) = x^6 + 6\alpha x^5 + 15\alpha^2 x^4 + 20\alpha^3 x^3 + 15\alpha^4 x^2 + 6\alpha^5 x - q_2$$
$$= x^6 + 1{,}24x^5 + 40{,}0x^4 + 15{,}14{,}40x^3 + 2{,}40{,}4{,}0x^2$$
$$+ 14{,}56{,}22{,}24x - 7{,}28{,}51;14{,}54{,}28{,}3{,}47{,}37{,}40{,}0 = 0.$$

97 __Transpositions.__ Thus far we have found the first digit = 14 of the root. The next digit, β, when found from (55), will be in the minutes' place (Cf. 6), i.e. a fraction. To bring it to the degrees' place, stretch (55) by 1,0 (Cf. 22). Thus (55) becomes

$$(56) \quad f_3(x) = x^6 + 1{,}0.6\alpha x^5 + 1{,}0^2.15\alpha^2 x^4 + 1{,}0^3.20\alpha^3 x^3 + 1{,}0^4.15\alpha^4 x^2$$
$$+ 1{,}0^5.6\alpha^5 x - 1{,}0^6 q_2$$
$$= x^6 + 1{,}24{,}0x^5 + 40{,}0{,}0{,}0x^4 + 15{,}14{,}40{,}0{,}0{,}0x^3$$
$$+ 2{,}40{,}4{,}0{,}0{,}0{,}0{,}0x^2 + 14{,}56{,}22{,}24{,}0{,}0{,}0{,}0{,}0x$$
$$- 7{,}28{,}51{,}14{,}54{,}28{,}3{,}47{,}37;40 \quad = 0.$$

We note in this stretch that the coefficients of (55), which are precisely the numbers at the tops of the n = 6 rows, except the coefficient of x^6 which is unity, are multiplied from left to right respectively by $1,0$, $1,0^2$, $1,0^3$ .... $1,0^n = 1,0^6$ in order to obtain the coefficients of (36) (cf. 30). Thus in the scheme on page 37, in order to account for this change, the following transpositions are performed:

Displace the $6\alpha^5 = 14,56,22,34$ standing in the row of the second of the number by one place to the right, the $15\alpha^4 = 2,40,4,0$ in the row of the third of the number by two places, the $20\alpha^3 = 15,14,40$ by three places, etc., until the $6\alpha = 1,34$ is displaced by $n - 1 = 5$ places to the right such that its last digit, 34, comes in the column of the imperfect place of 47, just at the left of the perfect place of 37. (See p. 37).

The remainder in the row of the number is now

$$q_3 = 1,0 q_2 = 7,28,51,14,54,28,3,47,37;40 \quad , \tag{37}$$

whose integer digits are contained in the first two cycles on the left.

The $n - 1 = 5$ places to the right of the displaced number $6^5 = 14,56,22,34$ in the row of the 5-th power are to be considered as filled with zeros. The same thing applies to the other rows, i.e. consider that four zeros fill the four empty places to the right of the displaced $15\alpha^4 = 2,40,4,0$, that three zeros fill the empty places to the right of the displaced $20\alpha^3 = 15,14,40$, and so on.

Now, after these transpositions and filling in of zeros are performed, the first digits on the right of all the numbers in the zeros fall in the perfect place of 37, now to be considered as the degrees' place.

Insofar as the computation is concerned, all the operations can now be performed as though β were in the degrees' place. There is no necessity actually to fill in the zeros so long as digits of like places fall in the same column, and this has been taken care of by the transpositions.

In anticipation of future operations, it is mentioned at this point that, after each additional cycle of processes, an additional such set of transpositions will be carried out, the objective again being to put digits of like places under each other. The reason is similarly to bring the next digit of the root to the degrees' place, and to have the numbers which will be dealt with in the next processes in the proper cycle.

The processes can be continued through as many cycles as may be required to find the desired number of significant digits. Whether the digits were fractional or not makes no difference in the computational scheme.

38 <u>Locating the second digit of the root.</u> We now proceed to find the next digit, β, of the root.

Suppose first that β has been determined. Then add β to the $6,0\alpha = 1,24,0$ which is in the row of the root, thus obtaining the number $6,0\alpha + \beta = 1,24,\beta$. Now multiply this sum by β and add the product, $\beta(6,0\alpha+\beta)$, to the $15,0,0\alpha^2$ in the row of the square obtaining $15,0,0\alpha^2+\beta(6,0\alpha+\beta)$; multiply this by β and add the result, $\beta\left[15,0,0\alpha^2+\beta(6,0\alpha+\beta)\right]$, to the $20,0,0,0\alpha^3$ in the row of the cube; multiply the sum also by   and add the result to what is in the next row over it, and so on until the last result

$$(58)\quad \beta\left[6,0,0,0,0,0\alpha^5+\beta\left(15,0,0,0,0\alpha^4+\beta\left\{20,0,0,0\alpha^3+\beta\left[15,0,0\alpha^2+\beta\left(6,0\alpha+\beta\right)\right]\right\}\right)\right] .$$

is obtained when multiplying the sum in the row of the (n-1)-the power, i.e. the row of the second of the series, by β.

But β has not yet been determined. We have to determine it such that the number obtained from (58) can be subtracted from 7,20,51,14,54,23,3,47,37.

This is equivalent to the corresponding condition in (43), namely

$$f_5(\beta) \leq 0 < f_5(\beta+1),$$

which can be written here as

$$(59)\qquad f_5(\beta)+q_5 \leq q_5 < f_5(\beta+1)+q_5 .$$

For the left member of (59) is equivalent to (58), as may be seen by comparing it with (56). The middle member is the remainder q_5 = 7,20,51,14,54,23,3,47,37;40 in the row of the number as given in (57). The right member is what the left member becomes when β is increased by 1.

The author refers to this property as "the above-mentioned quality". It was mentioned lyhin in the First Treatise of the "Key".[1]

1. P, p. 33.

β is found to be 0. (Cf. 31).

59 <u>Process on with the second digit.</u> Having found β, we begin a second series of processes preparing for the determination of the next digit, γ.

We will explain the following processes in general, i.e. as if β were different from zero.

Write $\beta = 0$ in the line of the result just above the perfect place of the digit 37 (see p. 37). Then perform the operations cited in § 58, until the expression (58) is obtained. Subtract this from q_5 in the row of the number (see (37)) to obtain

(60) $$q_5' = q_5 - \beta\left[1{,}0^5 6\alpha^5 + \beta\left(1{,}0^4 .15\alpha^4 + \beta\left\{1{,}0^3 .20\alpha^3 + \beta\left[1{,}0^2 .15\alpha^2 + \beta(6{,}0\alpha + \beta)\right]\right\}\right)\right].$$

We note in this special example that nothing has been changed in all the rows. Only $\beta = 0$ has been written in the line of the result. This is due, evidently, to the fact that β is zero and all the products obtained when multiplying by it are also zero.

60 Now start the process "Once up to the row of the second of the number". Add β to the $6{,}0\alpha + \beta$ in the row of the root. Multiply the sum $6{,}0\alpha + 2\beta$ by β and add the product $\beta(6{,}0\alpha + 2\beta)$ to what is already in the row of the square. Multiply the new sum by β and add the product to what is in the row over it, and so on until in the row of the second of the number, here the row of the 5-th power, the expression

$$1{,}0^5 .6\alpha^5 + \beta\left(1{,}0^4 .30\alpha^4 + \beta\left\{1{,}0^4\alpha^3 + \beta\left[1{,}0^3\alpha^2 + \beta(30{,}0\alpha + 6\beta)\right]\right\}\right)$$

is obtained.

Again perform the process "Once up to the row of the third of the number", and so on for the other processes until the last one is merely the addition of β to what has become by now in the row of the root. In our example the last results obtained in the rows of the 5-th power, 4-th power, cube, square, and root are respectively:

$$1,0^5.6\alpha^5+\beta\left(1,0^4.30\alpha^4+\beta\left\{1,0^4\alpha^3+\beta\left[1,0^3\alpha^2+\beta(30,0\alpha+6\beta)\right]\right\}\right)=14,33,22,34,0,0,0,0,0\ ,$$

$$1,0^4.15\alpha^4+\beta\left\{1,0^4\alpha^3+\beta\left[1,30\alpha^2+\beta(1,0^2\alpha+15\beta)\right]\right\}=2,40,4,0,0,0,0,0\ ,$$

(61) $$1,0^3.20\alpha^3+\beta\left[1,0^3\alpha^2+\beta(1,0^2\alpha+20\beta)\right]=15,14,40,0,0,0,0\ ,$$

$$1,0^2.15\alpha^2+\beta(1,0.30\alpha+15\beta)=40,0,0,0\ ,$$

$$6,0\alpha+6\beta=1,24,0\ .$$

These, with $-q_4$ given in (60), are the coefficients of the transformed equation, $f_4(x)$, obtained when diminishing the roots of $f_3(x)$ in (56) by $\beta=0$ (Cf. 13 - 15).

We again remark that, since β happened to be zero, in this particular example, these processes changed nothing in all the rows. This means that the equation $f_4(x)$ is identical with $f_3(x)$ given in (56). However, the procedure is stated for the general case, where the processes are performed successively, similarly to 56, in order to form the lines of Horner's triangular scheme (Cf. 16).

The general scheme, i.e. when $\beta\neq 0$, is displayed on the next page in (62).

3	1	$6,0\alpha$	$15,0,0\alpha^2$	$20,0,0,0\alpha^3$	$15,0,0,0,0\alpha^4$	$6,0,0,0,0,0\alpha^5$	$-1,0^6 q_1$
	1	$6,0\alpha+\beta$	$15,0,0\alpha^2+\beta(6,0\alpha+\beta)$	$1,0^3\cdot20\alpha^3+\beta[1,0^2\cdot15\alpha^2+\beta(6,0\alpha+\beta)]$	$1,0^4\cdot15\alpha^4+\beta\{1,0^3\cdot20\alpha^3+\beta[1,0^2\cdot15\alpha^2+\beta(6,0\alpha+\beta)]\}$	T_1	T_2
	1	$6,0\alpha+2\beta$	$15,0,0\alpha^2+\beta(12,0\alpha+3\beta)$	$1,0^3\cdot20\alpha^3+\beta[1,0^2\cdot30\alpha^2+\beta(18,0\alpha+4\beta)]$	$1,0^4\cdot15\alpha^4+\beta\{1,0^3\cdot40\alpha^3+\beta[1,0^2\cdot45\alpha^2+\beta(24,0\alpha+5\beta)]\}$	T_3	
	1	$6,0\alpha+3\beta$	$15,0,0\alpha^2+\beta(18,0\alpha+6\beta)$	$1,0^3\cdot20\alpha^3+\beta[1,0^2\cdot45\alpha^2+\beta(36,0\alpha+10\beta)]$	$1,0^4\cdot15\alpha^4+\beta\{1,0^4\alpha^3+\beta[1,0^2\cdot1,30\alpha^2+\beta(1,0^2\alpha+15\beta)]\}$		
	1	$6,0\alpha+4\beta$	$15,0,0\alpha^2+\beta(24,0\alpha+10\beta)$	$1,0^3\cdot20\alpha^3+\beta[1,0^3\alpha^2+\beta(1,0^2\alpha+20\beta)]$			
	1	$6,0\alpha+5\beta$	$15,0,0\alpha^2+\beta(30,0\alpha+15\beta)$				
	1	$6,0\alpha+6\beta$					
	1						

$$T_1=1,0^5\cdot6\alpha^5+\beta\left(1,0^4\cdot15\alpha^4+\beta\{1,0^3\cdot20\alpha^3+\beta[1,0^2\cdot15\alpha^2+\beta(6,0\alpha+\beta)]\}\right)$$

$$T_2=-1,0^6q_2+\beta\left[1,0^5\cdot6\alpha^5+\beta\left(1,0^4\cdot15\alpha^4+\beta\{1,0^3\cdot20\alpha^3+\beta[1,0^2\cdot15\alpha^2+\beta(6\alpha+\beta)]\}\right)\right]$$

$$T_3=1,0^5\cdot6\alpha^5+\beta\left(1,0^4\cdot30\alpha^4+\beta\{1,0^4\alpha^3+\beta[1,0^3\alpha^2+\beta(30,0\alpha+6\beta)]\}\right)$$

The numbers along the hypotenuse of the triangular scheme (63), which are identical, except perhaps for sign, with the numbers in (61) and (60), are then the coefficients of the transformed equation

$$f_4(x) = f_3(x) = x^6 + 1,0.6\alpha x^5 + 1,0^2 15\alpha^2 x^4 + 1,0^3 30\alpha^3 x^3 + 1,0^4 15\alpha^4 x^2 + 1,0^5 . 6\alpha^5 x - 1.0^6 q_1$$

$$= x^6 + 1.34,0x^5 + 49,0,0,0x^4 + 15,14,40,0,0,0x^3 + 2,40,4,0,0,0,0,0x^2 + 14,56,33,34,0,0,0,0,0x - 7,38,51,14,54,33,3,47,37;40 = 0 .$$

Let us, for later purposes, denote the coefficients of (63) by b_1, b_2, b_3 .... b_n respectively.

61 <u>A second series of transpositions.</u> Now transpose the numbers at the tops of the rows similarly to 57, namely transpose what is in the row of the 5-th power, i.e. the row of the second of the number, by <u>one</u> place to the right, what is in the row of the 4-th power by <u>two</u> places, and so on until what is in the row of the root is transposed by <u>n-1 = 5</u> places to the right.

It is to be noted here also, as in 57, that these transpositions, if zeros are considered as filling the places before the numbers in the third cycle, are equivalent to stretching (63), namely

$$f_4(x) = x^6 + b_1x^5 + b_2x^4 + b_3x^3 + b_4x^2 + b_5x + b_6$$

by 1,0 (Cf. 17 - 22). We then obtain

$$f_5(x) = x^6 + 1,0b_1x^5 + 1,0^2b_2x^4 + 1,0^3b_3x^3 + 1,0^4b_4x^2 + 1,0^5b_5x$$
$$+ 1,0^6b_6$$

(64) $$= x^6 + 1,34,0,0x^5 + 49,0,0,0,0x^4 + 15,34,40,0,0,0,0,0,0x^3$$
$$+ 2,40,4,0,0,0,0,0,0,0,0,0x^2 + 14,58,22,24,0,0,0,0,0,00000x$$
$$- 7,28,51,14,54,25,3,47,37,40,0,0,0,0,0 = 0 .$$

The remainder in the row of the numbers is to be considered now as

(65) $$q_5 = 1,0\ q_4 = 7,28,51,14,54,25,3,47,37,40,0,0,0,0,0 .$$

62 <u>Locating the third digit of the root and generalization.</u> Now it is required to find the next digit, γ, which has the same property as that of β explained in 58; namely, it is required to find the largest digit, γ, such that the quantity

$$(66) \qquad 1,0^6 b_6 + \gamma\Big[1,0^5 b_5 + \gamma\Big(1,0^4 b_4 + \gamma\big\{1,0^3 b_3 + \gamma\big[1,0^2 b_2 + \gamma\big(1,0 b_1 + \gamma\big)\big]\big\}\Big)\Big]$$

can be subtracted from q_5 in (65).

This condition is equivalent to the corresponding equation in (41), namely

$$f_5(\gamma) \leq 0 < f_5(\gamma+1) ,$$

which can be written as

$$(67) \qquad f_5(\gamma) + q_5 \leq q_5 < f_5(\gamma+1) + q_5 .$$

For, the left member of (67) is the same as (66), as may be seen by comparing (67) with (64) and (65), and the right member is what the left member becomes when $\gamma+1$ is substituted for .

γ is found to be 30 (Cf. 31).

63 Having found three digits of the root, x_0, we stop the algorism at this point, as did the author in the example, and locate the sexagesimal point of the root such that the inequality (25) holds. This inequality in our example is

$$1,0 \leq x_0 < 1,0^2 ,$$

and the root to one sexagesimal place is

$$x_0 = 14,0;30 .$$

64 If the result is to be more accurate than this, the same procedure, i.e. the cycle of locating, performing processes, and transposing, may be repeated to find as many more digits as may be desired or until a zero remainder is obtained in the row of the number.

The author speaks of finding more places in "the descending series", meaning to find more fractional sexagesimal places.

If the root turns out to be an integer, then the number is perfect; otherwise, it is imperfect, and the root has been found approximately.

65 It is to be noted that the place-number, kn, of any perfect place in the number q, when divided by the exponent, n, gives the place-number, k, of the digit in the result line put just above it. E.g., over the 7, whose place-number is 6, comes 14 whose place-number is 1; the degrees' digit, 0, comes over the degrees' place; the digit 30 whose place-number is -1 comes over the perfect place whose number is -6; and similarly for other places.

66 In the author's system, because he used no sexagesimal point, (Cf. 3), it was necessary to mention the place-numbers of the digits in the result, or at least one of them. In modern notation there is no need for this, since the sexagesimal point tells what the places are.

67 <u>The author's example.</u> The author gave the preceding description of the n-th root extraction in general, and without referring to any particular example. The example we gave served to clarify the explanation.

Now the author gives a special example to display his method. He proposes to extract the square root of 10,0,40,30.

The procedure is the same as that of our example above.

68 Write the number as shown on top of page 36. The cycles from left to right are respectively 10,9 and 48,20. The perfect places are then 9 and 20.

Draw vertical lines to separate the digits, the lines between the cycles being doubled.

There are here only two rows: the row of the number, the square, and the row of the root. A horizontal line may be drawn to separate them (this is not done in the text; it is not necessary).

69 The author gives two forms of procedure (See top of p. 36). (a) In the one the multiplication is performed as we moderns do. E.g., he multiplies 48,41 by 41 and puts the result 33,16,1 under the number; then he subtracts. He claims that this form of procedure is his own invention[1]. (b) In the other form, the multiplication is performed as was usual with mathematicians at his time. To multiply 48,41 by 41, he first multiplies 48 by 41 and subtracts the result 32,48 fromthe number, after writing digits of like places under each other, then he multiplies 41 by 41 and subtracts the result 28,1 from what remained of the number after the previous subtraction.

70 Now, in a table of squares look up the largest digit, α, whose square is less than or equal to 10,9. It is found to be 24. This is written in the line of the result over the perfect place of the digit 9, and also in the same column all the way down in the row of the root.

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1. Luckey, p. 238.

71 Then multiply the upper α=24 by the lower α=24, and the result α^2= 9,36 is subtracted from 10,9, the first cycle on the left. The remainder 33 is to be thought of as replacing the 10,9 in the number.

72 Add the upper α =24 to the lower α=24 and transpose the result 2α=48 by one place to t he right. It falls in the column of the digit 40 in the number, and is then to be considered as having a zero in front of it such that its value is 2,0 α =48,0.

73 It is now required to find the largest digit β which has the property that, if added to the 48,0 and the sum 48,β multiplied by β, the product is less than or equal to 33,40,20. I.e., it is required to find the digit β such that

(68) $$\beta\,(48{,}0+\beta) \leq 33{,}40{,}20 < (\beta+1)(48{,}0+\beta+1).$$

By the method of trial divisor (Cf. 31), β is found to be 41.

74 Write β =41 over the perfect place of the digit 20, and also in the row of the root to the right of the already written 48.

Multiply 48,41 by β =41 and subtract the result 33,16,1 from 33,40,20. There results 24,19.

75 The author goes on to find an additional place. He adds β =41 to the 48,41 in the row of the root. The result, 49,22, he transposes by one place to the right. Then he seeks the largest digit, γ, which if put to the right of 49,22, thus forming 49,22,γ, and then the resulting number multiplied by , the product will be less than or equal to 24,19,0,0. I.e., analytically

it is required that

(69) $$\gamma(40,22,0+\gamma) \leq 33,10,0,0 < (\gamma+1)(40,22,0+\gamma+1) .$$

76 By the method of 32, γ is found to be 40.

Put $\gamma=40$ in the line of the result, in the column whose place-number is -2 as shown inthe top figure of page 56, and also in front of 40,22 to form 40,22,40. This is multiplied by 40, and the result, 32,55,6,40, subtracted from 33,10,0,0 to obtain 26,53,20.

77 The process can be repeated to find any number of new places, but the author discontinues the work, and the result to one sexagesimal place is 34,41;40. The remainder in the row of the number is 26;53,20.

78 The degrees' digit, 41, falls over the degrees' digit in the number, i.e. over 20. For the other digits of the root, the one having the place-number n falls over the digit in the number whose place-number is kn - 2n (Cf. 65).

79 After this example the author mentions his book "On the Circumference" (al-Muḥīṭiyya), which was not available to the editor at the time of this study.

80 Then he gives two worked examples. The one displays the extraction of the cube root of 18,52,59,43,51,25 fourths, i.e. of the number 18,52;59,43,51,25, and the other shows the extraction of the 6-th root of 34,59,1,7,14,54,23,3,47,37,40 minutes = 34,59,1,7,14,54,23,3,47,37;40. The latter example has been discussed in detail in connection with the n-th root extraction in general (Cf. 46), and we leave the former for those who wish to practise finding roots.

63 <u>Conclusion.</u> Having presented both the Ruffini-Horner method of locating roots of polynomial equations in 23-29, and al-Kāshī's method of extracting the n-th root of a number, in 40-62, we are now in a position to compare these two methods, at least in a special case.

As was mentioned in 29, finding the positive n-th root of a number, q, is a special case of the problem of finding the roots of a polynomial equation of degree n. In this case the equation is $f(x) = x^n - q = 0$.

Kāshī's separation of the number q into cycles and his considering first the last cycle on the left (Cf. 48) corresponds in the Ruffini-Horner method to locating the root of $f(x) = 0$ between successive powers of 1.0 and stretching (or here compressing) the function $f(x)$ by the lesser power of 1.0 (Cf. steps 1 and 2 of 23).

Kāshī's finding of the digit, α, whose n-th power can be subtracting from the last cycle corresponds to locating the root x_1 of the stretched equation $f_1(x) = 0$ between successive digits, in steps 3 of 23 of the Ruffini-Horner method.

Writing the powers of α, and then performing the set of processes given in 49 - 56 correspond to forming the Horner's scheme used in diminishing the roots of the stretched equation $f_1(x)$ by α to obtain $f_2(x)$ as in step 4 of 23.

Again, the transpositions cited in 57 with carrying the work to the next cycle of the number q, corresponds to step 5 in 23, i.e. to stretching the new transformed equation $f_2(x) = 0$ by 1,0.

And the cycle of locating, performing processes, and transposing (Cf. 64) is precisely the same thing as the cycle of locating, diminishing, and stretching in the Ruffini-Horner method (Cf. 94).

And lastly, this "above-mentioned quality", described in 58 and 62, which Kāshī states in words, corresponds to the analytic conditions (41) for locating the roots of polynomial equations (Cf. 30).

Now, recalling that Kāshī (fl. 1410 A.D.) used this method, which he did not himself invent, four centuries before either Ruffini (1765 - 1822) or Horner (1786 - 1837) lived, we are assured that, although the latter two invented the algorism independently of each other and of other previous inventors[1], they were not the <u>first</u> to work it out and to use it. In fact, Coolidge claims[2] that the method was discovered by the Chinese sage Ch'in Chiu Shao who flourished around 1247 A.D. In any event it was certainly known and used by oriental mathematicians, among whom Kāshī is only one, at least four centuries before Ruffini and Horner.

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1. Coolidge, pp. 193-196.

2. Ibid. p. 196.

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VOCABULARY

Of Some Terms Used in the Translation

abovementioned, the-	المذكور
accordingly	على هذا القياس
add, to -	زاد
ad infinitum	الى ما لا نهاية له
adjoin, to -	ألحق
amount, to the - of	بقدر ، بعدة ، بعدد
ascertain, to -	عرفه
assumed	مفروض
befall, to -	وقع
chapter	باب ج أبواب
close, in order to - it	لأجله
column	جدول ج جداول
corresponding	نظير
count, to -	عدّ
cube	كعب ، مكعب
cube-cube	كعب كعب (القوة السادسة لعدد
cut , to - off	قطع
cycle	دور ج أدوار
deal, to - with	تعرض
degree	درجة ج درج
demand, to -	طلب
descending, the - series	سلسلة التنزل
distinguish, to -	ميّز
do, to -	عمل
elevate	مرفوع
employ, to -	استعمل
end, (n. and v.)	نهاية ،انتهى
exactly	تحقيقاً
explanation	شرح
exponent	منزلة
extract, to -	استخرج
extraction	استخراج
fall, to -	وقع
form	صورة
get, to - to	انتهى الى

Vocabulary (contd.)

increase, to -	زادَ ، ازدادَ
infinitum, ad -	إلى ما لانهايةَ لهُ
imperfect	أصمّ ، أصمّة
lengthwise	بالطول ، طولي
lest	لئلّا
located, is -	يوجَد
lower, the -	التحتاني
minute	دقيقة ج دقائق
multiplier	مضروب ج مضاريب
odd	فَرِيب ، فَرِيبَة
one-digit (number)	(عدد) مفرد
opposite to	بمحاذاة
perfect	منطق ، منطقة
place	مرتبة ج مراتب
ponder, to -	تأمّل
power	مضلّع ج مضلّعات
precede, to -	سبق
present, to -	أورد
process	عمل
product	حاصل الضرب
reach, to -	بلغ
recall, to -	استحضر
refer, to -	رجع
required	مطلوب
result (n.)	خارج ،حاصل ج حواصل
result, to -	حصل
root	ضلع ج أضلاع
row	صفّ ج صفوف
second	ثانية ج ثوانٍ
section	فصل
series	سلسلة
set down, to -	وضع
side	طرف
square	مال ، مجذور (مربّع)
square-square	مال مال (القوة الرابعة لعدد)
square-cube	مال كعب (القوة الخامسة لعدد)
subtract	نقّص ، انتقص
successive	متوالٍ
supposition, on the - that it is	على أنّه .

Vocabulary (contd.)

table	جَدْوَل
transpose, to -	نَقَلَ
transposition	نَقْل
transverse	عَرْضِيّ
treatise	مَقَالَة
trick	نُكْتَة ، نُكَات
upper, the -	الفَوْقَانِي

1	59	58	57	56	55	54	53	52	51	50	49	48	47	46	45	44	43	42	41	40	39	38	37	36	35	34	33	32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	
2	1,58	1,56	1,54	1,52	1,50	1,48	1,46	1,44	1,42	1,40	1,38	1,36	1,34	1,32	1,30	1,28	1,26	1,24	1,22	1,20	1,18	1,16	1,14	1,12	1,10	1,8	1,6	1,4	1,2	1,0	58	56	54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	24	22	20	18	
3	2,57	2,54	2,51	2,48	2,45	2,42	2,39	2,36	2,33	2,30	2,27	2,24	2,21	2,18	2,15	2,12	2,9	2,6	2,3	2,0	1,57	1,54	1,51	1,48	1,45	1,42	1,39	1,36	1,33	1,30	1,27	1,24	1,21	1,18	1,15	1,12	1,9	1,6	1,3	1,0	57	54	51	48	45	42	39	36	33	30	27	
4	3,56	3,52	3,48	3,44	3,40	3,36	3,32	3,28	3,24	3,20	3,16	3,12	3,8	3,4	3,0	2,56	2,52	2,48	2,44	2,40	2,36	2,32	2,28	2,24	2,20	2,16	2,12	2,8	2,4	2,0	1,56	1,52	1,48	1,44	1,40	1,36	1,32	1,28	1,24	1,20	1,16	1,12	1,8	1,4	1,0	56	52	48	44	40	36	
5	4,55	4,50	4,45	4,40	4,35	4,30	4,25	4,20	4,15	4,10	4,5	4,0	3,55	3,50	3,45	3,40	3,35	3,30	3,25	3,20	3,15	3,10	3,5	3,0	2,55	2,50	2,45	2,40	2,35	2,30	2,25	2,20	2,15	2,10	2,5	2,0	1,55	1,50	1,45	1,40	1,35	1,30	1,25	1,20	1,15	1,10	1,5	1,0	55	50	45	
6	5,54	5,48	5,42	5,36	5,30	5,24	5,18	5,12	5,6	5,0	4,54	4,48	4,42	4,36	4,30	4,24	4,18	4,12	4,6	4,0	3,54	3,48	3,42	3,36	3,30	3,24	3,18	3,12	3,6	3,0	2,54	2,48	2,42	2,36	2,30	2,24	2,18	2,12	2,6	2,0	1,54	1,48	1,42	1,36	1,30	1,24	1,18	1,12	1,6	1,0	54	
7	6,53	6,46	6,39	6,32	6,25	6,18	6,11	6,4	5,57	5,50	5,43	5,36	5,29	5,22	5,15	5,8	5,1	4,54	4,47	4,40	4,33	4,26	4,19	4,12	4,5	3,58	3,51	3,44	3,37	3,30	3,23	3,16	3,9	3,2	2,55	2,48	2,41	2,34	2,27	2,20	2,13	2,6	1,59	1,52	1,45	1,38	1,31	1,24	1,17	1,10	1,3	
8	7,52	7,44	7,36	7,28	7,20	7,12	7,4	6,56	6,48	6,40	6,32	6,24	6,16	6,8	6,0	5,52	5,44	5,36	5,28	5,20	5,12	5,4	4,56	4,48	4,40	4,32	4,24	4,16	4,8	4,0	3,52	3,44	3,36	3,28	3,20	3,12	3,4	2,56	2,48	2,40	2,32	2,24	2,16	2,8	2,0	1,52	1,44	1,36	1,28	1,20	1,12	1,
9	8,51	8,42	8,33	8,24	8,15	8,6	7,57	7,48	7,39	7,30	7,21	7,12	7,3	6,54	6,45	6,36	6,27	6,18	6,9	6,0	5,51	5,42	5,33	5,24	5,15	5,6	4,57	4,48	4,39	4,30	4,21	4,12	4,3	3,54	3,45	3,36	3,27	3,18	3,9	3,0	2,51	2,42	2,33	2,24	2,15	2,6	1,57	1,48	1,39	1,30	1,21	
10	9,50	9,40	9,30	9,20	9,10	9,0	8,50	8,40	8,30	8,20	8,10	8,0	7,50	7,40	7,30	7,20	7,10	7,0	6,50	6,40	6,30	6,20	6,10	6,0	5,50	5,40	5,30	5,20	5,10	5,0	4,50	4,40	4,30	4,20	4,10	4,0	3,50	3,40	3,30	3,20	3,10	3,0	2,50	2,40	2,30	2,20	2,10	2,0	1,50	1,40		
11	10,49	10,38	10,27	10,16	10,5	9,54	9,43	9,32	9,21	9,10	8,59	8,48	8,37	8,26	8,15	8,4	7,53	7,42	7,31	7,20	7,9	6,58	6,47	6,36	6,25	6,14	6,3	5,52	5,41	5,30	5,19	5,8	4,57	4,46	4,35	4,24	4,13	4,2	3,51	3,40	3,29	3,18	3,7	2,56	2,45	2,34	2,23	2,12	2,1			
12	11,48	11,36	11,24	11,12	11,0	10,48	10,36	10,24	10,12	10,0	9,48	9,36	9,24	9,12	9,0	8,48	8,36	8,24	8,12	8,0	7,48	7,36	7,24	7,12	7,0	6,48	6,36	6,24	6,12	6,0	5,48	5,36	5,24	5,12	5,0	4,48	4,36	4,24	4,12	4,0	3,48	3,36	3,24	3,12	3,0	2,48	2,36	2,24				
13	12,47	12,34	12,21	12,8	11,55	11,42	11,29	11,16	11,3	10,50	10,37	10,24	10,11	9,58	9,45	9,32	9,19	9,6	8,53	8,40	8,27	8,14	8,1	7,48	7,35	7,22	7,9	6,56	6,43	6,30	6,17	6,4	5,51	5,38	5,25	5,12	4,59	4,46	4,33	4,20	4,7	3,54	3,41	3,28	3,15	3,2	2,49					
14	13,46	13,32	13,18	13,4	12,50	12,36	12,22	12,8	11,54	11,40	11,26	11,12	10,58	10,44	10,30	10,16	10,2	9,48	9,34	9,20	9,6	8,52	8,38	8,24	8,10	7,56	7,42	7,28	7,14	7,0	6,46	6,32	6,18	6,4	5,50	5,36	5,22	5,8	4,54	4,40	4,26	4,12	3,58	3,44	3,30	3,16						
15	14,45	14,30	14,15	14,0	13,45	13,30	13,15	13,0	12,45	12,30	12,15	12,0	11,45	11,30	11,15	11,0	10,45	10,30	10,15	10,0	9,45	9,30	9,15	9,0	8,45	8,30	8,15	8,0	7,45	7,30	7,15	7,0	6,45	6,30	6,15	6,0	5,45	5,30	5,15	5,0	4,45	4,30	4,15	4,0	3,45							
16	15,44	15,28	15,12	14,56	14,40	14,24	14,8	13,52	13,36	13,20	13,4	12,48	12,32	12,16	12,0	11,44	11,28	11,12	10,56	10,40	10,24	10,8	9,52	9,36	9,20	9,4	8,48	8,32	8,16	8,0	7,44	7,28	7,12	6,56	6,40	6,24	6,8	5,52	5,36	5,20	5,4	4,48	4,32	4,16								
17	16,43	16,26	16,9	15,52	15,35	15,18	15,1	14,44	14,27	14,10	13,53	13,36	13,19	13,2	12,45	12,28	12,11	11,54	11,37	11,20	11,3	10,46	10,29	10,12	9,55	9,38	9,21	9,4	8,47	8,30	8,13	7,56	7,39	7,22	7,5	6,48	6,31	6,14	5,57	5,40	5,23	5,6	4,49									
18	17,42	17,24	17,6	16,48	16,30	16,12	15,54	15,36	15,18	15,0	14,42	14,24	14,6	13,48	13,30	13,12	12,54	12,36	12,18	12,0	11,42	11,24	11,6	10,48	10,30	10,12	9,54	9,36	9,18	9,0	8,42	8,24	8,6	7,48	7,30	7,12	6,54	6,36	6,18	6,0	5,42	5,24										
19	18,41	18,22	18,3	17,44	17,25	17,6	16,47	16,28	16,9	15,50	15,31	15,12	14,53	14,34	14,15	13,56	13,37	13,18	12,59	12,40	12,21	12,2	11,43	11,24	11,5	10,46	10,27	10,8	9,49	9,30	9,11	8,52	8,33	8,14	7,55	7,36	7,17	6,58	6,39	6,20	6,1											
20	19,40	19,20	19,0	18,40	18,20	18,0	17,40	17,20	17,0	16,40	16,20	16,0	15,40	15,20	15,0	14,40	14,20	14,0	13,40	13,20	13,0	12,40	12,20	12,0	11,40	11,20	11,0	10,40	10,20	10,0	9,40	9,20	9,0	8,40	8,20	8,0	7,40	7,20	7,0	6,40												
21	20,39	20,18	19,57	19,36	19,15	18,54	18,33	18,12	17,51	17,30	17,9	16,48	16,27	16,6	15,45	15,24	15,3	14,42	14,21	14,0	13,39	13,18	12,57	12,36	12,15	11,54	11,33	11,12	10,51	10,30	10,9	9,48	9,27	9,6	8,45	8,24	8,3	7,42	7,21													
22	21,38	21,16	20,54	20,32	20,10	19,48	19,26	19,4	18,42	18,20	17,58	17,36	17,14	16,52	16,30	16,8	15,46	15,24	15,2	14,40	14,18	13,56	13,34	13,12	12,50	12,28	12,6	11,44	11,22	11,0	10,38	10,16	9,54	9,32	9,10	8,48	8,26	8,4														
23	22,37	22,14	21,51	21,28	21,5	20,42	20,19	19,56	19,33	19,10	18,47	18,24	18,1	17,38	17,15	16,52	16,29	16,6	15,43	15,20	14,57	14,34	14,11	13,48	13,25	13,2	12,39	12,16	11,53	11,30	11,7	10,44	10,21	9,58	9,35	9,12	8,49															
24	23,36	23,12	22,48	22,24	22,0	21,36	21,12	20,48	20,24	20,0	19,36	19,12	18,48	18,24	18,0	17,36	17,12	16,48	16,24	16,0	15,36	15,12	14,48	14,24	14,0	13,36	13,12	12,48	12,24	12,0	11,36	11,12	10,48	10,24	10,0	9,36																
25	24,35	24,10	23,45	23,20	22,55	22,30	22,5	21,40	21,15	20,50	20,25	20,0	19,35	19,10	18,45	18,20	17,55	17,30	17,5	16,40	16,15	15,50	15,25	15,0	14,35	14,10	13,45	13,20	12,55	12,30	12,5	11,40	11,15	10,50	10,25																	
26	25,34	25,8	24,42	24,16	23,50	23,24	22,58	22,32	22,6	21,40	21,14	20,48	20,22	19,56	19,30	19,4	18,38	18,12	17,46	17,20	16,54	16,28	16,2	15,36	15,10	14,44	14,18	13,52	13,26	13,0	12,34	12,8	11,42	11,16																		
27	26,33	26,6	25,39	25,12	24,45	24,18	23,51	23,24	22,57	22,30	22,3	21,36	21,9	20,42	20,15	19,48	19,21	18,54	18,27	18,0	17,33	17,6	16,39	16,12	15,45	15,18	14,51	14,24	13,57	13,30	13,3	12,36	12,9																			
28	27,32	27,4	26,36	26,8	25,40	25,12	24,44	24,16	23,48	23,20	22,52	22,24	21,56	21,28	21,0	20,32	20,4	19,36	19,8	18,40	18,12	17,44	17,16	16,48	16,20	15,52	15,24	14,56	14,28	14,0	13,32	13,4																				
29	28,31	28,2	27,33	27,4	26,35	26,6	25,37	25,8	24,39	24,10	23,41	23,12	22,43	22,14	21,45	21,16	20,47	20,18	19,49	19,20	18,51	18,22	17,53	17,24	16,55	16,26	15,57	15,28	14,59	14,30	14,1																					
30	29,30	29,0	28,30	28,0	27,30	27,0	26,30	26,0	25,30	25,0	24,30	24,0	23,30	23,0	22,30	22,0	21,30	21,0	20,30	20,0	19,30	19,0	18,30	18,0	17,30	17,0	16,30	16,0	15,30	15,0																						
31	30,29	29,58	29,27	28,56	28,25	27,54	27,23	26,52	26,21	25,50	25,19	24,48	24,17	23,46	23,15	22,44	22,13	21,42	21,11	20,40	20,9	19,38	19,7	18,36	18,5	17,34	17,3	16,32	16,1																							
32	31,28	30,56	30,24	29,52	29,20	28,48	28,16	27,44	27,12	26,40	26,8	25,36	25,4	24,32	24,0	23,28	22,56	22,24	21,52	21,20	20,48	20,16	19,44	19,12	18,40	18,8	17,36	17,4																								
33	32,27	31,54	31,21	30,48	30,15	29,42	29,9	28,36	28,3	27,30	26,57	26,24	25,51	25,18	24,45	24,12	23,39	23,6	22,33	22,0	21,27	20,54	20,21	19,48	19,15	18,42	18,9																									
34	33,26	32,52	32,18	31,44	31,10	30,36	30,2	29,28	28,54	28,20	27,46	27,12	26,38	26,4	25,30	24,56	24,22	23,48	23,14	22,40	22,6	21,32	20,58	20,24	19,50	19,16																										
35	34,25	33,50	33,15	32,40	32,5	31,30	30,55	30,20	29,45	29,10	28,35	28,0	27,25	26,50	26,15	25,40	25,5	24,30	23,55	23,20	22,45	22,10	21,35	21,0	20,25																											
36	35,24	34,48	34,12	33,36	33,0	32,24	31,48	31,12	30,36	30,0	29,24	28,48	28,12	27,36	27,0	26,24	25,48	25,12	24,36	24,0	23,24	22,48	22,12	21,36																												
37	36,23	35,46	35,9	34,32	33,55	33,18	32,41	32,4	31,27	30,50	30,13	29,36	28,59	28,22	27,45	27,8	26,31	25,54	25,17	24,40	24,3	23,26	22,49																													
38	37,22	36,44	36,6	35,28	34,50	34,12	33,34	32,56	32,18	31,40	31,2	30,24	29,46	29,8	28,30	27,52	27,14	26,36	25,58	25,20	24,42	24,4																														
39	38,21	37,42	37,3	36,24	35,45	35,6	34,27	33,48	33,9	32,30	31,51	31,12	30,33	29,54	29,15	28,36	27,57	27,18	26,39	26,0	25,21																															
40	39,20	38,40	38,0	37,20	36,40	36,0	35,20	34,40	34,0	33,20	32,40	32,0	31,20	30,40	30,0	29,20	28,40	28,0	27,20	26,40																																
41	40,19	39,38	38,57	38,16	37,35	36,54	36,13	35,32	34,51	34,10	33,29	32,48	32,7	31,26	30,45	30,4	29,23	28,42	28,1																																	
42	41,18	40,36	39,54	39,12	38,30	37,48	37,6	36,24	35,42	35,0	34,18	33,36	32,54	32,12	31,30	30,48	30,6	29,24																																		
43	42,17	41,34	40,51	40,8	39,25	38,42	37,59	37,16	36,33	35,50	35,7	34,24	33,41	32,58	32,15	31,32	30,49																																			
44	43,16	42,32	41,48	41,4	40,20	39,36	38,52	38,8	37,24	36,40	35,56	35,12	34,28	33,44	33,0	32,16																																				
45	44,15	43,30	42,45	42,0	41,15	40,30	39,45	39,0	38,15	37,30	36,45	36,0	35,15	34,30	33,45																																					
46	45,14	44,28	43,42	42,56	42,10	41,24	40,38	39,52	39,6	38,20	37,34	36,48	36,2	35,16																																						
47	46,13	45,26	44,39	43,52	43,5	42,18	41,31	40,44	39,57	39,10	38,23	37,36	36,49																																							
48	47,12	46,24	45,36	44,48	44,0	43,12	42,24	41,36	40,48	40,0	39,12	38,24																																								
49	48,11	47,22	46,33	45,44	44,55	44,6	43,17	42,28	41,39	40,50	40,1																																									
50	49,10	48,20	47,30	46,40	45,50	45,0	44,10	43,20	42,30	41,40																																										
51	50,9	49,18	48,27	47,36	46,45	45,54	45,3	44,12	43,21																																											
52	51,8	50,16	49,24	48,32	47,40	46,48	45,56	45,4																																												
53	52,7	51,14	50,21	49,28	48,35	47,42	46,49																																													
54	53,6	52,12	51,18	50,24	49,30	48,36																																														
55	54,5	53,10	52,15	51,20	50,25																																															
56	55,4	54,8	53,12	52,16																																																
57	56,3	55,6	54,9																																																	
58	57,2	56,4																																																			
59	58,1																																																				