THE "JADWAL AL-TAQWĪM" OF ḤABASH AL-ḤĀSIB

by

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OF

ḤABASH

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ABSTRACT

The purpose of this thesis is to clarify some of the obscure parts of the history of Islamic trigonometric functions and tables. The study concentrates on the "Table of Rectification" of Ḥabash al-Ḥāsib. We have used the three main sources of this table, namely the published Rasā'il Abū Naṣr (No. 4) and two manuscripts, the Damascene Zīj of Ḥabash, and the so-called Berlin Copy of Ḥabash's Zīj.

In Chapter I we describe how Moslem writers became acquainted with ancient mathematics and astronomy, and especially with that of the Indians and Greeks. Then we sketch briefly the scientific life of the Abbasid period and the life history of the four individuals mainly concerned with our study.

In Chapter II we give a brief discussion of the Sources, and we show there that the Berlin copy of the Ḥabash Zīj cannot be relied on because it contains some work of later writers, whereas the Damascene Zīj is the reliable work of Ḥabash.

We explain in Chapter III the sexagesimal place-value system which was used by the Babylonians, the Greeks, and then by the Arabs in writing their tables and performing their operations. This is followed by an English-Arabic, Arabic-English technical glossary of terms used in the sources. We

conclude the technical introduction with a discussion of the medieval trigonometric functions.

Chapter IV is devoted to discussion of the tables themselves. Facsimiles of the source tables are given, together with opposite page transcriptions. We then define the functions contained in the tables of Ḥabash. They are

$$f_1(\theta) = \delta_2(\theta) = \mathrm{Tan}^{-1}\left[\frac{\mathrm{Tan}\ \epsilon \cdot \mathrm{Sin}\ \theta}{R}\right],$$

$$f_2(\theta) = \mathrm{Cos}\ \delta\ (\bar{\theta}) = \mathrm{Cos}\ \left[\mathrm{Sin}^{-1}\left(\frac{\mathrm{Cos}\ \Theta \cdot \mathrm{Sin}\ \epsilon}{R}\right)\right],$$

$$f_{3a}(\theta) = \frac{R \cdot \mathrm{Cos}\ \theta}{\mathrm{Cos}\ \delta\ (\bar{\theta})} = \frac{R \cdot \mathrm{Cos}}{f_2(\theta)},$$

$$f_{3b}(\theta) = \frac{R \cdot \mathrm{Cos}\ \theta}{\mathrm{Cos}\ \epsilon},$$

$$f_{4a}(\theta) = \frac{\mathrm{Tan}\ \theta \cdot \mathrm{Sin}\ \epsilon}{R},$$

$$f_{4b}(\theta) = \mathrm{Tan}\ \theta,$$

where the capital letters indicate ordinary functions multiplied by a suitable constant R. The accuracy obtained is then discussed briefly. It is shown that the error in computing these tables did not exceed 0.148 per cent.

In Chapter V we give the operations of Ḥabash and other mathematicians who participated in solving spherical astronomical problems. These problems give ways of determining such things as the arc of daylight, right ascension, declination, longitude and latitude, the transit, the ascendant, and the latitude of visible climate. The proof of Abū Naṣr usually follows each problem. We keep as much as possible to the original solutions as given in the text, adding notes to clarify the procedure.

The conclusion of the study appears in Chapter VI. We point out here that the purpose of Ḥabash in composing his tables is to find combinations of trigonometric functions which will be useful for solving a great variety of astronomical problems, but which, in general, may be unimportant in themselves. We have shown also that other mathematicians followed Ḥabash in writing similar tables to serve the same purpose. Finally we have shown that Ḥabash (to our present knowledge) is the inventor of the tangent function and that he is the first to use it in the solution of problems.

TABLE OF CONTENTS

Section Page

LIST OF TABLES

CHAPTER I

THE HISTORICAL BACKGROUND

1. Pre-Islamic Science

This study is primarily concerned with the Islamic period. Nevertheless some remarks on the mathematics of much earlier times are in order, because of its influence on the material of our problem.

We know for instance, that as early as the twenty-first century B. C. the Babylonians were using a place-value system for representing numbers, the base being sixty.[1] This sexagesimal system has many advantages over earlier methods of displaying numbers, and it does not differ essentially from our present practice except that now we commonly use ten as the base. The importance of the place-value concept can be observed if we attempt arithmetical operations in a system which lacks it, Roman numerals, for example. The significance of the invention is by some regarded as comparable to that of the alphabet.[2]

From the Babylonians the use of sexagesimals passed on to the Greeks, thence to the Arabs. In medieval astronomy it was never displaced by the decimal system and in fact was called in Arabic ḥisāb al-munajjimīn,[3] "the astronomer's

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1. Neugebauer (1), pp. 28-35.
2. Neugebauer (2), p. 12.
3. Kashi, "Miftāḥ al-Ḥisāb," Princeton manuscript ELS 1189 (Yahuda), folio 85.

arithmetic". It is therefore not surprising that the "Table of Rectification" on which this study is based is written in the then customary alphabetical sexagesimal system.

However the influence of Babylonian science was exerted indirectly on the Arab world; it reached Baghdād through the Indian and particularly through the Hellenistic sciences.

India acted as an early source of inspiration in mathematics and astronomy to the Arab world. About 771 A.D. a Hindu astronomer came to Baghdād in connection with a political embassy, bearing a treatise on astronomy, probably the Brahmasiddhānta of Brahmagupta (ca. 628). By order of the Caliph al-Manṣūr this was translated into Arabic by Muḥammad al-Fazārī, son of the first Moslem to construct astrolabes. He transliterated Siddhānta as Sindhind and it was by this name that the translation became famous. It had a great influence on Islamic astronomy and was used as a reference even after the appearance of the Almagest in Arabic.[4]

Another Sanscrit work which was early translated and widely used is al-Arkand. Some think that it is a translation of the Āryakhaṇḍa, others that it is the Khaṇḍakhādyaka.[5]

For our present purpose, the most significant invention of Hindu mathematics is the sine function, corresponding to half the chord subtended by a central angle, and in contrast to Ptolemy's use of the whole chord. The sine function

4. Nallino (1), pp. 150-151.
5. Kennedy, under al-Arkand.

quickly adopted by the Arabs, was of great assistance in the basic advances in trigonometry made in the work of such mathematicians as Ḥabash al-Ḥāsib, Abū-l-Wafā', Abū Naṣr and Naṣīr al-Dīn al-Ṭūsī.

Very little is definite about the influence of Persian science on early Islamic mathematics and astronomy, probably because very little is known about science in Persia, in the Sassanian period. A work called the Shāh Zīj, probably compiled during the reign of Khusrū Anūsharwān was known too and widely used by many Arab mathematicians. Its influence on Islamic astronomy is comparable with that of the Sindhind.[6]

It was Hellenistic mathematics and astronomy which had the greatest influence on the science of the Islamic period. The work of Euclid, Archimedes, and Apollonios in mathematics, and that of Hipparchos, Ptolemy, and Theon in astronomy may be considered as the foundation of the Islamic exact sciences.

The Almagest of Ptolemy was the prime book of reference for many Arab astronomers. According to Ibn al-Qifṭī (p. 69) it was first translated into Arabic by order of the well-known Abbasid minister Yaḥyā ibn Khālid ibn Barmak (d. 807). Thābit ibn Qurra revised the translation made in the time of al-Ma'mūn (813-833). Many Arab mathematicians such as al-Nairīzī, wrote commentaries on it, and a multitude of others including Ḥabash al-Ḥāsib, Abu Naṣr Manṣūr, and al-Bīrūnī used it as a model for the astronomical handbooks

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6. Ibid., under Shah Zīj.

they wrote.

In the ninth and tenth chapters of the first book of the Almagest, Ptolemy shows how to compute a table of chords. He gives such a table having entries at intervals of half a degree from 0 to 180°. This table is made to serve all the purposes for which later astronomers used the trigonometric functions. For solving figures on the sphere the one fundamental proposition known and used by him was the Menelaus Theorem. Strictly speaking the work is hardly trigonometrical at all, because it deals with properties of a complete quadrilateral rather than those of a triangle.

The Alexandrian school, after Diophantos (ca. 250 A.D.), Pappos (end of the Third Century), and Theon (end of the Fourth Century) gradually dies, and in 630 Alexandria was taken by the Arabs. Islam spread quickly all over the Near East and North Africa and became one of the greatest powers for many centuries.

Very little is known about science during the time of the first four Caliphs, which we can consider to be the period of conquest, expansion and colonisation. The same can be said of the Umayyad Caliphate. Scientific life under the Umayyads left but few sources on which we can depend.

2. The Abbasid Period

In the second half of the eighth century a new flame in the history of science began to shine in the world. The capital of learning has moved from Alexandria to Baghdād.

Scientists and translators converged on it from all neighbouring countries. The mixture of the Persian, Syrian, and Islamic cultures in the new empire gave its fruits at this time. Some of the main factors of this great growth are the wide political extent of the empire, the concentration of wealth, and the relatively peaceful conditions at this time.

In the time of al-Manṣūr many translations from Greek, Sanscrit, Syriac, and Persian into Arabic were accomplished. Many more Greek works were translated by the order of Hārūn al-Rashīd, the patron of science, art, and literature. Relations between East and West were strong at his time. In 801, Hārūn presented a water clock to Charlemagne.

In science, the ninth century was essentially an Arab century. The Moslem men of science were the real bearers of civilization in those days. The wealth of the state, the high standard of living in the capital, and the encouragement of the Caliphs made Baghdād the center of the world in literature and science.

Al-Ma'mūn was even a greater patron of science than his father. In Baghdād he founded a scientific academy (Bayt al-Ḥikma) and erected, near the Shamāsiyah gate, an astronomical observatory under the directorship of Yaḥyā ibn abī Manṣūr. Here the Caliph's astronomers compiled a great number of books and zījes in which systematic observations of the celestial movements were recorded.

To this observatory al-Ma'mūn soon added another on Mount Qāsiyūn outside of Damascus.[7] In this observatory Ḥabash al-Ḥāsib probably made his observations.

3. Ḥabash al-Ḥāsib

The full name of the inventor of the "Table of Rectification" is Aḥmad ibn ʿAbdalla al-Marwazī (i.e., from Merv, presently a city in the Turkmen Republic of the U.S.S.R.) Ḥabash[8] al-Ḥāsib (the Calculator). Al-Bīrūnī adds to his name the title al-Ḥakīm[9] (the Wise).

The three main sources of information on the life of Ḥabash and his works are al-Fihrist of Ibn al-Nadīm (d. 995), Akhbār al-ʿUlamā' of Ibn al-Qifṭī (d. 1248) and Kashf al-Ẓunūn of Ḥājjī Khalīfa (fl. 1652). In addition to these, numerous isolated items of information concerning him are to be found in various publications and manuscripts.

Here is a translation of what Ibn al-Qifṭī says about Ḥabash:

> "Habash al-Hasib, al-Marwazī by origin; it (i.e., Ḥabash) being his nickname. His name (proper) is Aḥmad ibn ʿAbdullāh, and he established himself in Baghdād. He was (living) in the time of al-Ma'mun and al-Muʿtasim after him. He made advance(s) in the computation of the motion of the planets (and he had) a reputation in this subject. He wrote three zījes. The first of them was compiled in the manner of the Sindhind; in it he disagreed with al-Fazārī and al-Khwārizmī in all the operations and in his use of the

7. Ibn al-ʿIbrī, p. 237.
8. Ḥabash is a nickname. According to Sarton (Sarton 1, p. 565), al-Ḥabash would mean the "Abyssinian". Habisha in Syriac would mean monk.
9. Al-Bīrūnī (2), p. 198.

> trepidation of the equinoxes according to the opinion of Theon of Alexandria in order to correct by it the positions of the stars in longitude. He compiled this zīj in the first part (of his career) in the days when he believed in the computational (methods) of the Sindhind. The second is known as al-Mumtaḥan and is his most famous (zīj). He compiled it after he returned to the toil of (astronomical) observations. And he included in it the motion of the planets in the manner made necessary by the tests carried out in his time. The third is the Small Zīj, known as the Shāh (Zīj). And he wrote a good book On the Use of the Astrolabe; and he reached an age of about a hundred years. And his works were: the book of the Damascene Zīj, the book of the Ma'mūnic Zīj, a book on Distances and Sizes (of the Heavenly Bodies?), a book On the Use of the Astrolabe, a book On Sundials and Measurements, a book on Tangent Circles, and one on the Method of Making the Plane, the Perpendicular, the Oblique, and the Inclined Surfaces."

The last sentence is identical, almost word by word, with the biographical note on Ḥabash in the Fihrist. It is clear from the above quotation that Ibn al-Qifṭī had two sources of reference on the life of Ḥabash and one of them is either al-Fihrist itself or the source of al-Fihrist. This follows from the fact that Ibn al-Qifṭī twice lists Ḥabash's zījes, once at the beginning of the passage, and once at its end.

We know but little about the life of Ḥabash, but Ibn Yūnis in his Great Ḥākimī Zīj mentions some facts on his life taken from Ḥabash's Arabic Mumtaḥan Zīj. The year 829 is a prominent year in his life in Baghdād. For we know that he observed in May of 829[10] a conjunction of Venus and Mars and that in the same year the star Regulus was observed

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10. Ibn Yūnis, p. 108.

in Baghdād[11] in the presence of al-Ma'mūn. He observed also from Baghdād a lunar eclipse on June 20, 829[12], and on November 30 of the same year[13] he observed an eclipse of the sun. We learn also from the introduction to his Damascene Zīj (cf. Section 4 below) that he spent at least one complete year in Damascus making observations, most probably in the time of al-Ma'mūn (813-833). From the same source we know that he was in Samarra (about sixty-five miles north of Baghdād) during the year 246[14] A.H. (860), and that he was observing the new crescent for the beginning of the month of Ramaḍan. Samarra was the capital of the Abbasid caliphs between 836 and 892. It seems that Ḥabash, like other scientists and men of letters, was living near the caliph in his capital. The last time we hear about Ḥabash is while he was observing the conjunction of Venus and Mars on Sunday, the sixth day of Ramaḍan[15], 250 A.H. (864) in the reign of al-Mustaʿīn (862-6). Suter gives the date of his death as falling between the years 864 and 874, but he does not give any reference, and this may be pure conjecture.

A son of Ḥabash called Abū Jaʿfar ibn Ḥabash was also a distinguished astronomer and instrument maker.[16]

As for Ḥabash's scientific productions, all three sources attribute to him the <u>Damascene</u> and the <u>Ma'mūnic</u> Zījes.

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11. Ibn Yūnis, p. 106.
12. *Ibid*, p. 94.
13. *Ibid*, p. 94.
14. HI, folio 222.
15. Ibn Yūnis, p. 108.
16. Al-Fihrist, p. 384.

Ibn al-Qifṭī and Ḥājjī Khalīfah add in addition three other zijes of which al-Mumtaḥan (The Tested) is one. Ibn Yūnis mentions a zīj of Ḥabash several times.[17] He names the zīj as the Arabic Zīj of Ḥabash, or the Arabic Mumtaḥan Zīj[18] of Ḥabash. In another place he says that Ḥabash called his al-Mumtaḥan Zīj the Canon. We infer that he wrote at least two different zījes namely the Arabic Mumtaḥan Zīj, which might have been called the Ma'mūnic, and the second is the Damascene Zīj. This Arabic Mumtaḥan Zīj was written in the time of al-Ma'mūn, and it is his most important work[19]; we cannot but say that he would dedicate his best work to the Caliph. The name Al-Zīj al-Mumtaḥan al-Ma'munī al-Shamāsī[20] was known to Ḥabash because it is the result of the observations made in Baghdād by Yaḥyā ibn Abī Manṣūr and other astronomers of al-Ma'mūn.

We know also that Ḥabash had a treatise called al-Risāla al-Kāmila[21] in which he discussed the observation of the new crescent.

Ḥabash's works were referred to by many Arab mathematicians, and commentaries were written on them. Abū Naṣr wrote on the astronomical problems of Ḥabash in which the "Table of Rectification" is used.[22] Apparently Ḥabash

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17. Caussin, pp. 129, 131, 133, 135, 159, 163, 165, 167, 171.
18. Ibid, p. 127.
19. Ibn al-Qifṭī, p. 117. Also p. 17, of Al-Bīrūnī Commemoration Volume, in which Al-Masū'dī is quoted (without page reference) that when one referred to Ḥabash's work (zīj) one always meant al-Mumtaḥan.
20. Ms. Escorial Ar. No. 927.
21. Rasā'il Abū Naṣr, No. 11, Maṭāli al-Samt, p. 3.
22. Ibid, No. 4.

wrote a treatise on the Ascendant of the Azimuth, which was criticized by Abū Naṣr[23]. Al-Bīrūnī in his Rasā'il mentions the Zīj of Ḥabash more than once[24], he even mentions that he himself wrote a commentary on it[25].

Al-Mas'ūdi, al-Bīrūnī and Ibn Yūnis had Ḥabash's Zīj (Arabic Mumtaḥan) as a source of reference.

It seems that Habash wrote a book in which he mentioned the observations of the astronomers of al-Ma'mūn (Aṣḥāb al-Mumtaḥan). In this book[26] he describes how al-Ma'mūn's astronomers performed one of the most important geodetic operations, namely the measuring of the length of a terrestrial degree outside Sinjār[27], they found it to be 56 1/4 miles.

Ibn Yūnis mentions a treatise written by Ḥabash in which he writes on an observation made in Damascus in A.H. 217 (832) on the star Regulus.

Nothing is extant of all the work of Ḥabash except the Damascene Zīj[28]. The Berlin Zīj of Ḥabash is believed to be a fragment of the Damascene Zīj, together with material related to other persons.

If it is true that Ḥabash lived about a century, it means that he lived in the golden age of the Abbasid Period

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23. Rasā'il Abū Naṣr, No. 11 (Maṭāli' al-Samt).
24. Al-Bīrūnī (1), No. 1, pp. 15, 131, 172, 226, etc., and al-Bīrūnī (1), No. 2, pp. 22, 63.
25. Al-Bīrūnī (1), No. 1, p. 172.
26. Ibn Yūnis, p. 81.
27. A desert between the Tigris and the Euphrates (Lat. 34°-36°).
28. See Section 10 below for a description.

and that he noticed during his life how this great empire was by then divided in the east and west among several minor dynasties.

4. The Introduction to the Damascene Zīj

In the following we give a translation of the introduction written by Ḥabash to his Damascene Zīj. It covers folios 69v - 71r of the only extant manuscript. Because the first part of the introduction (i.e., f.69v) consists only of the customary prayers to God and his Prophet, it is not translated here.

The introduction contains many interesting statements showing Ma'mūn's interest in astronomy, and how science developed in the early Islamic period.

f70r

<u>line</u>

1 ...Then they divided the parts
2 of the heaven (into) three hundred and sixty parts and called them degrees, and they called each thirty degrees of them a sign, and they divided the (sequence of) days
3 (into) years and months, (beginning) with the dates of their kings, from which (divisions) they know the positions of the sun, the moon and the planets in the ecliptic,
4 (which helps them) in fixing the best date for ploughing, progeny, pollenation of plants, change (in the lengths) of night and day
5 by increase and decrease, and eclipses of the sun and moon. And a class of people put
6 rules for that, and pretended great knowledge of the sun, the moon, and the stars for which they did not give a clear
7 proof or correct measure, although the computation of the stars is (a matter of) mighty miracles and a wonderful science. And so it attracted many
8 avaricious (individuals). As to the truth of what these (people) pretended concerning this science,

they (indulged in) watchful guarding and acquisition of it, up to believing
9 in it before they (were capable of) reading it and thus knowing its truth. This led them to pretend knowledge of the science and its practice (although)
10 wherever some fault appeared, they depended on it, and upon the expulsion of those who pointed out the fault. And this matter continued
11 in that fashion until the Caliphate arrived to Abdallāh the Leader, al-Ma'mūn, the Commander of the Faithful, may the blessings of God
12 be upon him, who possessed a skilled knowledge and was an investigator of delicate affairs and the mysteries of the sciences, and especially he had an ardent love for the science of the stars.
13 He then compared among what he found of Byzantine books such as the Canon[29] and others, and what he found of Hindu (books) such as
14 the Sindhind[29] and the Arkand[29], and what he found from the Persians in the Shāh Zīj[29] and others. He found them different,
15 each one agreeing with the right sometimes, and deviating from the way of truth occasionally. And when he understood that he ordered
16 Yaḥyā ibn Abī Manṣūr al-Ḥāsib (i.e., the Reckoner) to go back to the origin of astronimical books. And he gathered the scientists who are experts in this craft,
17 and the wise men of his time to cooperate in the investigation of the bases of this science, with the intention of correcting them. Because
18 Ptolemy of Pelusium[30] had proved that there is no end to the science of the stars.
19 Then Yaḥyā obeyed the order, and gathered the scientists in the craft of the stars and the noted wise men
20 of that time. Then he and they returned to the bases of these books and searched through them and measured what was
21 set down in them, but in all these books they found none more correct than the book of Ptolemy of Pelusium called the Almagest
22 in which Ptolemy explained the correct procedure by clear measurements
23 and geometrical proofs. And he mentioned that he observed the paths of the sun, the moon, and the planets in their positions
24 in the heaven and examined them in all their

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29. For material on these works, see Kennedy.
30. His full name is Claudius Ptolemy. His first name was apparently transliterated as <u>al-Qalūdhī</u>. Some other +

conditions. (So) that observation and examination led him to discover the error and defect which occurred

25 in the observations of the people before him, and (so) he corrected all the error and defect,

26 the which was made precise by observation and examination. Then, using the instruments (current) in his time, he fixed the positions of the stars according to what he found by observation,

f70v

1 (and) after corrections (leading to) precision, he put them into this book of his. Then they (i.e., the astronomers of al-Ma'mūn) made that book a canon for themselves.

2 Thereupon they took observational instruments, the armillary sphere and others, and indicated in their observation(s) what

3 Ptolemy described. And they examined the motion of the sun and the moon at different times in the City of Peace (i.e., Baghdād).

4 Then al-Ma'mūn, may God be gracious unto him, went to Damascus after the death of Yaḥyā ibn Abī Manṣūr. He (asked) Abū Yaḥyā

5 ibn Aktham[31], and al-ʿAbbās ibn Saʿīd al-Jawharī[32] to choose a man having keen knowledge in the art of the stars for observation

6 and (a word illegible). So they chose for him Khālid ibn ʿAbdul-Malik al Marwarūdhī[33] and he ordered him to select the most

7 precise instruments available, and to consider the motion of the stars for one complete year. Khālid did that until

8 he found the true positions of the sun and the moon in the heavens. When this was truly done, al-Ma'mūn ordered that he should make up

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scribe thought that the word was <u>al-Faludhī</u>. The word al-Qalūdhī is easily misread as <u>al-Faludhī</u> by omitting one dot of the letter <u>qāf</u> to become <u>fā</u>. The misreading of the word became permanent, because of the existance of a city in Egypt by the name Falūdha (Pelusium). See Buttman for further discussion.

31. Probably one of the notables of Damascus at that time.

32. Fl. ca. 830. He participated in the Ma'mūn observations both at Damascus and Baghdād.

33. Fl. ca. 830, he wrote a zīj now non-extant.

9 a canon (i.e., a zīj, an astronomical handbook),
to be followed by those working in this science.
He established that, as it is in this book of
mine.

10 When al-Ma'mūn died, God's Mercy be upon him,
and the observations were discontinued, I was in-
duced to consider what the
11 people examined and observed concerning the sun, the
moon, and the observation of the rest of the stars,
to establish that knowledge, (and)
12 to correct what I have of it in my chest. For it
is the duty of one who tries to study deeply in any
art or to understand
13 any science, not to be satisfied by imitation with-
out investigation, and not to accept fragments (of
knowledge) without
14 deep investigation. Ptolemy has already mentioned
in the ninth chapter of the fourth book
15 of the Almagest, after some remarks, of what was
said concerning him that he changed (some) letters,
and that it is incumbent upon the scholars of this
16 subject truly to love truth, and (upon) those who
study deeply and carefully not only to fix the
correction
17 of the old knowledge which was established by the
ancients and which they find from observations of
which there can be no doubt, but (in addition)
18 to restore and rectify any mistake which may have
occurred in what they themselves have described,
without any shame or
19 timidity. Because the affairs of this science are
mighty heavenly affairs (proceeding) by the fiat
of God the Glorified. (Thus, they need not be
ashamed)
20 even if they alone did make all the corrections
toward greater truth and veracity, but some cor-
rections were made by others.
21 He mentioned at the end of the Almagest, that what
helped him to the
22 truth of what he found, and to his corrections was
the time which elapsed between the measurements
made by the ancients and the measurements
23 of his own time. So I looked into this decisively
and made thorough investigation concerning the path
of the sun,
24 and the moon, by observation and examination made
repeatedly for al-Ma'mūn in the City of Peace
25 at (the time of) the two equinoxes, the vernal and
the autumnal, and the two solstices, the summer
and the winter. And at Damascus (I observed) for
a year

26 day by day continuously from its beginning to its end, with an armillary sphere and other instruments. And in what

f71r

1 I attempted I followed in my corrections the measurements of Ptolemy, which he verified by certain observations of which he himself took charge,
2 and preserved in his book, the Almagest, (wherever) these observations were different from the observations of the ancients who were smitten of this science, such as Hipparchus[34]
3 and others. Until (finally) I corrected the solar and lunar eclipses according to their observations. And I alone observed the rest of the stars (i.e., the planets)
4 at the times of their arrival in the positions mentioned by Ptolemy as being those in which he made his measurements. And I repeated the observation of them several times,
5 until I knew the exact cycles of the stars as well as time permitted without (using) the observations of Khālid and others
6 who observed and examined the sun and the moon for al-Ma'mūn, (in the time) between me and Ptolemy. And in my explanation of what I considered necessary I made no changes
7 in Ptolemy's adjustments for (the determination of) the true positions of the stars (or the planets) except in the case of expressions which I suspected would not be understood
8 by individuals who have not immersed themselves in this science. And so nothing is left of the (possible) situation of the sun and the moon on the ecliptic which I have not ascertained
9 by observation and examination. Until finally I have reached a state of confidence in my corrections (to the extent) that if the true positions of
10 the sun, the moon, and the planets are determined according to this book of mine, they will be found to occupy the (same) places as those obtained by observation
11 and examination, but God is (the one to be) praised.

12 Thereupon I viewed it well to compile this book on the motion of the sun, the moon, and the planets, and to make it as simple

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34. Fl ca. 140 B.C.

13 as possible. For the ancients (who labored) in this science and laid down its principles left no work
14 to those coming after, except for nearer (approach) to a meaning, simplification of a source, or correction of a mistake in one place, the (need for) the correction of which is indicated by
15 their operations in another place, or the correction of a mistake introduced into the affair after them. And I established its date
16 in Hijra years and lunar months, which no doubt are the same for all people observing the (new) crescent and the mansions
17 of the moon, and I present the chapters (in the order) which the scholar needs, in order to progress step by step,
18 one by one, if God will.

5. Decline of the Abbasid Empire

The first to establish an almost independent state east of Baghdād was al-Ma'mūn's general Ṭāhir ibn al-Ḥusayn of Khurāsān. He was rewarded in 820 by al-Ma'mūn with the governership of all lands east of Baghdād, and before his death he omitted to mention the Caliph's name in the Friday prayer. Ṭāhir's successors extended their dominion as far as the Indian frontier and moved their capital to Naysābūr and remained in power till 872, when they were superseded by the Saffārids.

Ya'qūb al-Ṣaffār, the founder of this dynasty, was a commander of the troops of the Caliph in Sijistān who succeeded in ruling all Persia. The Saffārids remained in power from 867-903, until they were followed by the Sāmānids.

The Sāmānids who ruled over Persia and Turkistān from 874-999 were descendants of Sāmān, a Zoroastrian noble of Balkh. But the one who made the state powerful was Ismāʿīl, who ruled when al-Muʿtaḍid (892-902) was the caliph in Baghdād. To this same caliph was dedicated a book on atmospheric phenomena by a scientist who plays a minor part in our study.

6. Al-Nairīzī

This individual is Abu-l-ʿAbbās al-Faḍl ibn Ḥātim al-Nairīzī (from Nairīz, near Shirāz in Persia). As usual we know but little about the life and works of al-Nairīzī. In fact we are not even completely certain about his name.

Recently Rosenfeld and Yushkevish[35] have claimed that this scientist's name is really "al-Tabrīzī" i.e., of Tabrīz, the principal city of Azarbeijan. They assert that some scribe miscopied the word as "al-Nairīzī", and that this mistake became established in the literature. In fact, if one diacritical point above the letter tā in the word Tabrīzī is placed near the dot under the letter bā, the word becomes Nairīzī. But these writers gave no evidence for their statement, nor did they cite any manuscript source in which the name appears as "Tabrīzī".

In al-Bīrūnī's "Chronology of Ancient Nations" the name of our scientist is given as "al-Tabrīzī"[36] in the only passage in which he is mentioned. Sachau, the editor and translator of the work, makes no statement regarding the

35. ʿUmar al-Khayyām, pp. 69, 145.
36. Al-Bīrūnī (2), p. 142.

name. In the Leiden copy of the Ḥakimite Zīj of Ibn Yūnis[37] no diacritical points appear on the letters of al-Nairīzī's name in the many places where he is mentioned; even the letter zāy is occasionally miscopied as dāl in the manuscript. However Caussin[38] mentions in his notes that the word should be read as "al-Nairīzī" in conformance with the usage of a manuscript found in the Escorial Library of Madrid (Manuscript Catalogue t. I, p. 421).

On the other hand the word "al-Nairīzī" is found six times on folios 82, 83, and 84 of the Berlin copy of Ḥabash's Zīj, in connection with al-Nairīzī's "Universal Table", and in none of these places is there any possibility of reading the word as "al-Tabrīzī". The manuscript is not dated, but it gives the impression of being an old copy. In the published version of Abū Naṣr Manṣūr's Rasā'il[39] to al-Bīrūnī he is mentioned clearly as "al-Nairīzī" in several places, and there is no reason for assuming that the word was consistently misread in preparing the book for the press. The same is true when this word is mentioned by al-Bīrūnī in his Rasā'il. Above all, Ibn al-Nadīm in his Fihrist and Ibn al-Qifṭī speak of him as "al-Nairīzī". With the single exception noted above, all modern authors, such as Suter, Brockelmann, Schoy, and Sarton refer to him always as al-Nairīzī.

We have therefore sound evidence to assume that unless an old manuscript can be found, dated earlier than

37. Cf. Caussin, pp. 65, 69, 71, 73, 121, 229.
38. The Ḥakimite Zīj, p. 60.
39. No. 4, pp. 30, 32, 36, 56, 57, etc.

the above sources, and in which we find the name as "al-Tabrīzī" we may continue to use the customary form.

As to his life, it is known that al-Nairīzī flourished under al-Muʿtaḍid and that he was living in Baghdād. Suter[40] mentions that al-Nairīzī died in c. 922, but with no indication of how he obtained this information.

Both al-Fihrist[41] and Ibn al-Qifṭī[42] mention that that al-Nairīzī wrote two zījes, the so-called Large Zīj, using the Sindhind methods, and another one, the Small Zīj, neither of which is extant. Both sources attribute to him a book in which he explains the Book of Euclid (probably the Elements) and another on the Tetrabiblos of Ptolemy. But Ibn al-Qifṭī adds that he wrote explanatory notes on the Almagest. According to Brockelmann[43] five of his works are still extant.

We learn from Abū Naṣr[44] that al-Nairīzī copied the "Table of Rectification" from Ḥabash's Zīj and added it to his own, calling it "The Universal Table" (Al-Jadwal al-Jāmiʿ). By use of this table he was able to solve some astronomical problems similar to those solved by Ḥabash.

We have it also from Abū Naṣr that al-Nairīzī's Zīj contained a relatively long chapter on lunar parallax on

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40. Suter (1), p. 45.
41. P. 389.
42. P. 168.
43. Suppl. I, pp. 386-387.
44. N, pp. 30, 32, 36, 56.

which Abū Naṣr wrote a commentary.[45] Al-Bīrūnī in his Rasā'il[46] mentions al-Nairīzī's zīj more than once.

Abū Naṣr respects al-Nairīzī, and whenever he refers to a mistake in the latter's zīj, he says this must be a scribal error, for it could not have been made by al-Nairīzī. The latter is similarly praised by Ibn Yūnis[47].

7. The Buwayhids and Abū Jaʿfar al-Khāzin

In the period following the death of al-Nairīzī, the Sāmānids gradually extended their kingdom to include most of Persia, Khurāsān and Transoxiana. Their capital, Bukhārā, and their leading city, Samarqand, became the centers of learning instead of Baghdād. Al-Rāzi called his book on medicine "Al-Mansūrī" in honour of his patron the the Sāmānid prince Abu-Ṣāliḥ Manṣūr. Ibn-Sīnā (Avicenna, 980-1037) while still young, visited Bukhārā to study in the rich royal library there, and Firdawsī, the epic poet of Iran, began to flourish at this time also.

As the Sāmānid dynasty began to wane, another power began to rise. Aḥmad ibn-Buwayh was given the title of Muʿizz al-Dawlah and was made the "Prince of Princes" by the Caliph al-Mustakfī (944-6). The sons of Aḥmad ruled, in addition to the southern shores of the Caspian Sea, the cities of Isfahān, Shīrāz, al-Ahwāz (Khūzistān) and Karmān. Shīrāz was chosen as capital of the new dynasty. Baghdād

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45. Ibid, pp. 36-56.
46. No. (2), p. 53 as an example.
47. Ibn Yūnis, pp. 65, 69.

was no longer the focus of the Moslem world, for not only Shīrāz but Ghazna, Cairo and Cordova were now sharing its reputation. Baghdād was governed now from the new capital, Shīrāz.

The Buwayhids (945-1055) reached their zenith in the times of Rukn al-Dawlah (932-76) and his son ʿAḍud al-Dawlah (949-83). These sultans erected numerous palaces, hospitals, and mosques. ʿAḍud built the famous hospital in Baghdad, al-Bīmāristān al-ʿAḍudī, and to this sultan the famous poet, al-Mutanabbī sang his poems.

The Buwayhids instituted an observatory in their Baghdād palaces. And in the Court of Rukn al-Dawlah of al-Rayy flourished Abū Jaʿfar al-Khāzin, who wrote some notes on the "Table of Rectification", the subject of this study.

His full name is Abū Jaʿfar Muḥammad ibn abū l-Ḥasan (Mūsā) al-Khāzin (the treasurer, or the librarian). He was born in Khurāsān and died, as Brockelmann[48] mentions, between 961 and 971, but the source of this information is not given.

Al-Bīrūnī mentions in his Kitāb Taʿdīd Nihāyāt al-Amākin li Taṣḥīḥ Masāfāt al-Masākin[49] (MS. 3386, Fātiḥ, Istambul) that Abū Faḍl al-Harawī observed the altitude of the sun in Rayy (Near Teheran) on June 22, 959 A.D., in the presence of Abū Jaʿfar al-Khāzin. This is the latest date

48. Suppl. I, pp. 386, 387.
49. Krause (2), p. 33; ff. 88, 89 of the ms.

available to this writer for which it is known that al-Khāzin was still living.

According to the Ibn-al-Qifṭī[50], al-Khāzin is his nickname, it being better known than his name proper. Ibn-al-Qifṭī adds that he was an expert in arithmetic, geometry, and astronomy, and that he wrote two books; Zīj al-Ṣafā'iḥ[51] (Plates) and al-Masā'il al-ʿAdadiyyah (Numerical Problems).

Al-Khāzin wrote also a commentary on the Tenth Book of Euclid[52]. But he is famous for solving the cubic equation[53] of the type $x^3 + a = cx^2$, known as al-Māhānī's equation. He solved this equation by means of the intersections of the conics, the parabola and the equilateral hyperbola.

This equation was discussed by many Greek and Arab mathematicians such as Archimedes, Dionysodorus, al-Māhānī, Ibn-al-Haitham, abu-l-Jūd and completely solved by ʿUmar al-Khayyām.

Abū Jaʿfar's extant writings are still unpublished[54]. But Abū Naṣr Manṣūr (ca. 1000), wrote a treatise[55], correcting his Ṣafā'iḥ Zīj. From this treatise we know that al-Khāzin wrote a commentary on Menelaus' Elements of Geometry, and that the Ṣafā'iḥ Zīj contained many astronomical problems

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50. P. 259.
51. Ms. No. 5857, Berlin.
52. Brockelmann, Suppl. I, p. 387.
53. Woepcke, p. 3, and Kasir, p. 43.
54. Such as Ms. 5857, Berlin; Ms. 968/9, Leiden; 2467,17, Paris; and 5924, Berlin.
55. Rasā'il Abū Naṣr, No. 3.

solved by use of spherical trigonometry. We know from Abū Naṣr also that al-Khāzin wrote another commentary on Menelaus' Spherica[56], and that he wrote a supplement to his Ṣafā'iḥ Zīj concerned with the properties of the spherical triangle[57]. From Abū Naṣr also we learn that Abū-Jaʿfar wrote explanatory notes on the Almagest.

Although Abū Naṣr finds some mistakes in the Ṣafā'iḥ Zīj, he praises al-Khāzin and says that these mistakes are just slips.

Al-Bīrūnī mentions in his Rasā'il[58] and his al-Athār[59] the work of al-Khāzin in astronomy.

8. Abū Naṣr Manṣūr

The last mathematician in whom we are interested is Abū Naṣr Manṣūr ibn ʿAlī ibn ʿIrāq; the title al-Jīlī (from Jīl (Gilan), one of the Caspian provinces of Iran) is sometimes mentioned[60]. He is also given two other titles: al-'Amīr (the prince) and Mawlā 'Amīr al-Mu'minīn (The Servant of the Commander of the Believers, i.e., the Caliph's Servant). This same title was adopted also before by some rulers of the Ṭāhirid and Sāmānid dynasties[61]. The word ʿIrāq mentioned

56. Ibid, p. 45.
57. Ibid, p. 43.
58. No. I, pp. 41, 129, 170, and No. III, p. 78.
59. Al-Bīrūnī (2), pp. 258, 326.
60. Rasā'il Abū Naṣr, the preface pp. 2, 6, 8.
61. Spuler pp. 358-360, Sachau, Biruni (2), p. 33 of the preface.

in his title is not only the name of one of his ancestors but also is his family name. Al-Bīrūnī mentions twenty-two Shāhs of the ʿIrāq family in the Afriq-Sīyāwush dynasty[62] of Khwārizm.

Al-Bīrūnī praises the House of ʿIrāq in a poem[63] saying: "The family of ʿIrāq has nourished me by their money, and one of them, Manṣūr, took care of my education". Abū Naṣr was the teacher of al-Bīrūnī; this is evident when the latter says in the Chronology[64]: "My teacher Abū Naṣr Manṣūr - - -".

Direct information concerning the life of Abū Naṣr has not reached us, so we are left with what we can infer about him from occasional remarks, chiefly made by al-Bīrūnī.

Let us agree with Sachau[65] and Krause[66] that al-Bīrūnī probably left his fatherland (Khwārizm) after the revolution which took place in it, when the power was transferred to the Banū Ma'mūn in 995[67]. So the teacher-student connection between Abū Naṣr and al-Bīrūnī perhaps existed between 990 and 995. Therefore when al-Bīrūnī (972-1048) was twenty-three years old, Abū Naṣr may have been thirty to thirty-five years of age, in which case his date of birth would fall between 961 and 965.

62. Krause (1), p. 110, footnote 2.
63. Rasā'il Abū Naṣr, preface p. 8.
64. Al-Bīrūnī (2), p. 184.
65. Ibid, preface p. 19.
66. Krause (2), p. 110.
67. Encyclopedia of Islam, v. 3, under Ma'mūn.

Following the conquest of Khwārizm by Sultan Maḥmūd al-Ghaznawī in 1016, he took Abū Naṣr and al-Bīrūnī back to Ghazna[68] with him in 1018.

In the titles of most of the treatises of Abū Naṣr to al-Bīrūnī, it is mentioned that Abū Naṣr died in the decade of 430 A.H. (say ca. 1029-1038).

In the year 427 A.H. (1036) al-Bīrūnī mentioned in his Fihrist the name of Abū Naṣr followed by the sentence, "May God enlighten his Proof"[69], (Anār Allāh burhānahu!). This sentence probably implies that Abū Naṣr was deceased by that time. From all this we can assume that his death falls somewhere between 1018 and 1036.

From his writings known to us, only his work on the Sphaerica of Menelaus[70] is dated (1007/8), while his book al-Majisṭī al-Shāhī[71], probably dedicated to Abū-l-ʿAbbās ʿAli ibn Ma'mūn ibn Muḥammad Khwārizmshāh[72], should have been written between 997 and 1010. The rulers of the dynasty of Ma'mūns were patrons of learning. In the court of Abū-l-ʿAbbās flourished Abū Naṣr, al-Bīrūnī, and the two great physicians Ibn Sīnā (Avicenna, d. 1037) and Abūl-Khair ibn Khammār[73].

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68. Rasā'il Abū Naṣr, Introduction, p. 9.
69. Al-Bīrūnī (2), pp. 34, 47.
70. Ms. No. 989, Leiden and Rasā'il Abū Naṣr No. 12.
71. This treatise is non-extant except for a short extract, No. 734, 2°, India Office.
72. Kennedy, p. 43.
73. Encyclopedia of Islam, v. 3, under Ma'mūn.

In the Rasā'il of Abū Naṣr to al-Bīrūnī[74], we find fifteen different astronomical and mathematical works, including the proofs given by him to Ḥabash's problems, using the "Table of Rectification". These treatises were written at the request of al-Bīrūnī himself to his teacher.[75] In treatise No. 4 (p. 14) Abū Naṣr mentions that he wrote a work on spherical triangles, and on page 58 he wrote that he compiled a book called Tahdhīb al-Taʿālīm, in which he discussed some famous zījes.

Krause[76] lists six mathematical and seventeen astronomical works of Abū Naṣr and Brockelmann[77] mentions twenty-two different works.

As for Abū Naṣr's accomplishments, certainly the discovery of the Sine Law relative to the spherical triangle is his most important. Two other contemporary Moslem mathematicians to whom this discovery is attributed are Abu-l-Wafā' al-Būzjānī, who flourished in 940 in Baghdād and died in 997/8, and al-Khujandī, who died ca. 1000. This law displaced the Menelaus Theorem as the principal law in spherical trigonometry.

Which of the three mathematicians was the first to discover the sine law is not certain. But Naṣīr al-Dīn al-Ṭūsī (d. 1274) mentions[78] that al-Bīrūnī thinks that Abū Naṣr

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74. According to Krause (2), p. 112, these Rasā'il were scribed in 1233 A.D.
75. Rasā'il, Introduction p. 7, No. 4, p. 71 etc.
76. Krause (2), pp. 111-115.
77. Brockelmann, v. 1, pp. 472 and 511, and Suppl. I, p. 368.
78. Nallino (1), p. 245.

was most probably the first to use this law in all places, and that both Abu-l-Wafā' and al-Khujandī pretended to be the first to use it.

In fact Abū Naṣr uses extensively the sine law for right and oblique triangles in solving astronomical problems in his Rasā'il[79].

Suter has published a German translation of Abū Naṣr's treatise on the proof of the sine law according to a manuscript in the Leiden Library.[80] In one of his Rasā'il to al-Bīrūnī (No. 8),[81] Abū Naṣr proves clearly and simply the sine law for any triangle.

In addition, we know that Abū Naṣr used the properties of polar triangles[82], the Rule of Four relating two spherical triangles[83], and the definition of the tangent function[84].

Abū Naṣr was much respected by his student al-Bīrūnī, who says[85] that the works of his teacher are as necklaces of jewels.

From the way he deals with the astronomical problems, we can say that Abū Naṣr is one of the great scientists of his age and of the Islamic period.

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79. Cf. 5, 6, 18, 21, 25, 31, 56, 59, 60, etc.
80. Suter (2), pp. 156-160.
81. P. 5.
82. N, p. 28.
83. *Ibid*, pp. 4, 5, 25.
84. *Ibid*, pp. 25, 35, 71.
85. Al-Bīrūnī (2), preface, p. 47.

CHAPTER II

THE SOURCES

9. Abū Naṣr's Work on the Table of Rectification (N)

The main source for this study is a treatise (hereafter referred to as N) by Abū Naṣr ibn Manṣūr addressed to his eminent sometime student, Abū Raiḥān al-Bīrūnī and entitled "On the Proofs of the Operations (connected with) the 'Table of Rectification' in the Zīj of Ḥabash al-Ḥāsib".

The only extant manuscript of this work is Part 8 of Arabic manuscript 2468 in the collection of the Oriental Public Library, Bankipore, India. The same manuscript is mentioned by Krause (2, p. 113), who gives its number as 2519.

The manuscript itself, however, has not been accessible to the present writer, who has used the printed version, Rasā'il. This contains fifteen such tracts, all by Abū Naṣr and addressed to al-Bīrūnī. The tract in which we are interested takes up some seventy-one pages and is the fourth in the printed volume. The reader should note that the latter is not paginated from beginning to end. The pages are numbered beginning again with each treatise. Our page number references to N are restricted to the fourth tract only.

We should be grateful to the Osmania Oriental Publications Bureau, Hyderabad, Deccan, for publishing the manuscript. By so doing they have made available for study a text which would otherwise be completely inaccassible to most scholars. Since this edition is still in print and can easily be obtained and referred to, we reproduce no part of the original here.

The reader should be warned that the printed text is in no sense a critical edition. It contains many errors due either to scribes who made successive copies or to misreadings in preparing the Bankipore version for the press. No apparent effort has been made by the publishers to restore scribal errors in the light of Abū Naṣr's original intent. Thus, for example, where the word mail (declination) should appear the word mithl (like) is frequently printed. Doubtless the two dots under the medial letter for the yā were omitted in some manuscript version and mistakenly replaced by three dots over the same character to give a thā.

The thirty-six figures inserted between the pages of the printed version are out of place. For example, the figure opposite page twenty-eight belongs to the problem beginning on page fifteen, and so on. Many letters referring to the figures are misread, such as Ḥ instead of J, B instead of N, and R for Z or vice versa.

An incomplete version of the "Table of Rectification" appears in N on page 3 of the printed text; this will be discussed in Section 16 below.

10. The Damascene Zīj of Ḥabash (HI)

The second source for this study is the so-called Damascene Zīj (al-Zīj al-Dimishqī) of Ḥabash (hereafter referred to as HI). The only extant copy[86] of this work is contained in (Istanbul) Yenicāmi Ms. 784, 2°.

The zīj proper begins on folio 69V with an introduction. It contains 162 folios with twenty-six lines per page. The script is naskhī, and the manuscript is not dated. Krause ((1), p. 446) indicates that the copy was made in about the sixth century A.H. (twelfth century A.D.). However on folio 74, a regnal list of Caliphs appears giving the date of termination of their caliphate in years, months and days, together with the number of years, months and days each ruled. The last Caliph on the list is al-Muṭīʿlillāh al-Faḍl ibn al-Muqtadir. It is indicated that he ruled twenty-nine years and six months, and the date of termination of his caliphate is given as the eleventh month of the year 362 A.H. These dates are easily confirmed by reference to a number of standard works[87]. The entries for the last few Caliphs are not interpolations by a later scribe, for the entire list is written in the hand of the individual who copied the rest of the manuscript. This date (363 A.H. = 973 A.D.) is about one century after the death of Ḥabash. It can therefore be assumed with a fair degree of probability

86. Krause (1), p. 446; Brockelmann, Suppl. I, p. 393.
87. Cf., for example Encyclopedia of Islam, article on Abbasids.

that the date of copying of the manuscript is the latter part of the tenth century A.D.

On folio 222V Ḥabash mentions that he observed the new crescent of the month of Ramaḍān in the year 246 A.H. (860 A.D.) in Sāmarra[88] which means that he was still compiling this zīj at the end of his life. (He died between 846 and 874).

The "Table of Rectification" appears twice, namely on folios 147r-148v and on folios 226rv-227r. The zīj contains also the other tables customary in such collections. These include a sine table for 15' intervals, a tangent[89] table for 30' intervals, tables of right and oblique ascensions, and a declination table. On folios 132, 169, 190, 225 and 226 are discussed the problems to which Ḥabash applies the "Table of Rectification."

11. The Berlin Habash Zīj (HB)

A third source is the undated Berlin Arabic Ms. WE.I. 90(5750), comprising 169 folios. The handwriting is naskhī.

Although this work is entitled "The Zīj of Ḥabash al-Ḥāsib" it is evidently not due to Ḥabash alone, but is a mixture of material from many sources including Ḥabash.

88. The capital of the Abbasid Caliphs from 836-892 A.D. This observation was taken at the time of al-Mutawakkil.
89. Folios 227v and 228r.

Sections probably due to Ḥabash are:

1. A good part of the introduction to HI appears on folios one and two of HB, word by word. Of the ninety-six lines taken up by the introduction to HI, twenty-three are reproduced in this document. Details are given in Section 4 above. But in general the historical parts have been left out of HB.

2. A part of the "Table of Rectification" as is found in HI appears also here on folios 82v, 83v, 84v, among other tables called the "Universal Tables".

3. The basic lunar mean motion parameters are the same in both HI and HB.

4. Tables of maximum and minimum daily motions of planets according to Ḥabash are given in an anonymous zīj, Bibliotheque National (Paris) Ms. Arabe 2513 on folio 82v. These same limiting speeds appear also in HB, scattered through the mean motion tables.

But on the other hand the following sections are clearly the work of others:

1. The mean motion tables of HB have as epoch the year 511 A.H. (1117). Another table for the mean motion of Saturn appears on folio 41r, and has as epoch the year 878 A.H. (1473).

2. The HB zīj contains also a table of the planetary apogees for the year 876 A.H. (1471), a date after the death of Ḥabash by about six centuries.

3. On folio 53v a table of the anomaly of Venus is given; on the lower margin there is a note given, in the same hand as the table itself, noting a parameter used by Abū-l-Wafā' who lived from 940 to 997/8.

4. Two tables attributed to al-Nairīzī are given among other tables on folios 82v, 83v, and 84v. Al-Nairīzī lived about half a century after Ḥabash.

A collection of trigonometric and astronomical tables appear on folios 82-84; the sine, the versed sine, the declination, and the sine and cosine of the declination are among other tables. All these tables are given to three sexagesimal places.

On folio 85r the tangent table is given, among other tables called the "Proportion Tables". This table which is called by Ḥabash and others *Al-Ẓill al-Maʿkūs* or *Umbra Versa* is correct to three sexagesimal places.

12. The Escorial "Mumtaḥan" Zīj

In an undated manuscript of the Escorial Library in Madrid, (Codex Arabe, 927) there are two sections (on ff. 92v and 93r) describing the solution of problems 13:12 and 15:14 (Cf. Section 19 below) by use of the "Table of Rectification".

The methods used in the solution are the same as those of N, but the wording is different.

We know that Yaḥyā ibn abī Manṣūr (died ca. 831) and other astronomers of al-Ma'mūn compiled a zīj called al-Zīj al Mumtaḥan (The Tested Tables). But this manuscript is definitely not the work of Yaḥyā in its original form because it contains many sections attributed to later writers in the text itself (Cf. Kennedy). This is not the case with the passages referred to above, but neither can they be assumed to be the work of Yaḥyā.

CHAPTER III

TECHNICAL INTRODUCTION

13. The Arabic Sexagesimal Numerals

The "Table of Rectification", the subject of this study, is written in Arabic characters of the alphabet, using the sexagesimal system, where the base is sixty. A brief statement of this system is given here to help the reader in understanding the subject.

The origin of the place-value system and use of base 60 is due to the Babylonians. This system appears on cuneiform tablets of ca. 2100 B.C.[90] The Greek mathematicians and astronomers took over the sexagesimal system but used the characters of their alphabet to denote the different digits. The Arabs, similarly utilized the same system, their letters serving as numerals. They called the system Ḥisāb al-Jummal or Ḥisāb Abjad[91], because of the use of letters to denote the different digits. Thus

1	2	3	4	5	6	7	8	9	10	20	30	40	50
ا	ب	ج	د	ه	و	ز	ح	ط	ي	ك	ل	م	ن

with the zero symbol usually of the form: (ة) are the digits used to form any sexagesimal number.

In principle a system with base 60 needs sixty different symbols. The Arabs used combinations of two letters

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90. Struik, v. 1, p. 24.
91. Irani, p. 2.

to represent a single sexagesimal digit. For instance the digit 49 is represented by مط, the connected letters 40 = م and 9 = ط.

Note that the ten's symbol is to the right of the unit's symbol as a matter of custom, and the two letters making up a single digit must be joined. م and ط written separately constitute two digits. Thus ط, م = 40, 9.

As an example

نب كج يح ه ز لو مه مرابع

means 52,23,18,0,7,36,45 fourths

or 52,23,18;0,7,36,45.

Note that we use a semicolon to denote the "sexagesimal point", and a comma to separate sexagesimal digits.

The three previous representations of the same number are abbreviated forms of the expression

$$52 \times 60^2 + 23 \times 60^1 + 18 \times 60^0 + 0 \times 60^{-1} + 7 \times 60^{-2} + 36 \times 60^{-3} + 45 \times 60^{-4},$$

just as 3409.68 is a short way of indicating

$$3 \times 10^3 + 4 \times 10^2 + 0 \times 10^1 + 9 \times 10^0 + 6 \times 10^{-1} + 8 \times 10^{-2}$$

It will be noticed that in the Arabic sexagesimal system the digits of higher order are written from right to left, contrary to modern usage. It should also be noted that the forms in which these letters are written are slightly modified from that customary in modern Arabic writing.. For a fuller discussion of the sexagesimal place value system, see Raja'i, Irani, and Struik, v. I.

14. Technical Glossary

In the following we give English translations together with definitions of some of the medieval Arabic astronomical terms which are used in N. An acquaintance with spherical astronomy is assumed on the part of the reader.

There are two arrangements, English-Arabic and Arabic-English. In the English-Arabic set the terms are followed by their Arabic equivalents, thence by the definitions. The passage in N where the word occurs is shown between parentheses, the first number indicating the page, the second the line.

The underlined words used in the definitions proper denote that their definitions appear in another part of this glossary.

The Arabic-English portion is a list only of the Arabic terms followed by their English equivalents. The Arabic words are written in Latin characters, and arranged in the order of the Latin alphabet. The system of transliteration used is that set forth in the _Catalog of the American University of Beirut_ for the year 1955-56, page 159, with the exception of the letter _qāf_ which we transliterate as q. The article _al_ before _qamarī_ and _shamsī_ letters is written _al_ in small letters followed by a hyphen.

English-Arabic

Altitude (h, al-irtifāʿ)

angular distance above the horizon. (N, 19:9).

Altitude, circle of (dāyirat al-irtifāʿ)

any great circle passing through the zenith. (N, 51:15).

Altitude, meridian (irtifāʿ al-kawkab fī falak niṣf al-nahār)

of a star, altitude at culmination, i.e., altitude at meridian passage. (N, 19:6).

Ascendant (H, al-ṭāliʿ)

the eastern point of intersection of the ecliptic with the local horizon at any instant. (N, 24:1).

Argument of the ascendant (ḥiṣṣat al-ṭāliʿ)

the oblique ascension of the ascendant. (N, 61:4).

Ascendant, equation of (taʿdīl al-ṭāliʿ)

arc of the ecliptic between the circle of the latitude of visible climate and the meridian (de Vaux, p. 428, N, 30:40).

Ascension, oblique (al-maṭāliʿ fil-balad)

of any point on the ecliptic, is defined as follows: consider the point on the equator which crosses the eastern half of the local horizon simultaneously with the given point. The arc of the equator measured eastward from the vernal equinoctial point to the equatorial point defined is the required oblique ascension. (N, 24:2).

Ascension, right (α, al-Maṭāli' fil-falak al-mustaqīm)

of a point on the ecliptic, is the distance, measured eastward on the equator, from the vernal point to the projection of the given point on the equator. In modern usage the point need not be on the ecliptic. (N, 4:13).

Ascension, right for a star not on the ecliptic (al-daraja al-latī tatawasaṭ al-samā' ma' al-kawkab)

see right ascension. (N, 15:14).

Azimuth (al-samt)

the arc of the horizon betwen the east point and the intersection of the horizon with the circle of altitude passing through the star. (N, 58:15).

Azimuth, ascension of (maṭāli' al-samt)

the right ascension (in the modern sense) of the projection on the horizon of any point. (N, 61:13).

Circle of latitude of visible climate (dāyirat 'arḍ iqlīm al-ru'ya)

the circle passing through the poles of the horizon and the ecliptic. (De Vaux, p. 428; N, 28:9).

Complement (tamām)

of an angle θ or an arc is $90° - \theta = \bar{\theta}$. Sometimes the word tamām is also used to denote $180° - \theta$. If tamām bi-dawr is used it means $360° - \theta$. (N, 33:2).

Daylight, arc of (D, qaws al-nahār)

that part of a parallel circle which is above the horizon. $D = 180° + 2d$. Cf. the next definition. (N, 3:16).

Daylight, equation of (d, ta'dīl al-nahār)

half the difference between the arc of daylight and 180°, i.e., $d = 1/2(D - 180)$. (N, 3:18).

Daylight, excess of (faḍl al-nahār)

the difference between the arc of daylight of a given point and 180°; it is twice the equation of daylight. (N, 32:17).

Daylight, maximum equation of (max d, ta'dīl al-nahār al-a'ẓam)

occurs when the length of daylight is a maximum or a minimum. (N, 6:7).

Daylight, mean length of (al-nahār al-mu'tadil)

equals twelve mean hours. (N, 5:4).

Declination (δ or δ_1, al-mayl, al-mayl al-awwal)

of a point on the ecliptic, its distance to the equator. More generally, the declination of any number θ is the number $\sin^{-1}(\sin\epsilon \sin\theta)$, so that θ need not be along the ecliptic at all, or on any other circle, for that matter. (Cf. N, 13:8, for example.) (bu'd, or bu'd al-kawkab 'an mu'addal al-nahār), the declination in the modern sense; i.e., the distance from any point to the celestial equator.

Declination, maximum (ϵ, al-mayl al-aʿzam)

the inclination of the ecliptic. (N, 8:19).

Declination, second (∂_2, al-mayl al-thānī)

of a point on the ecliptic is the length of the perpendicular erected from it to the celestial equator. (N, 12:2).

East point (al-mashriq)

the intersection of the celestial equator and the horizon.

Ecliptic (falak al-burūj, or manṭiqat al-burūj)

the annual path of the sun on the celestial sphere, a great circle. (N, 4:6).

Equation (taʿdīl)

here used in the astronomical (rather than in the usual mathematical) sense to denote a variable which, when added algebraically to another, makes the sum equal to a third variable. (N, 9:1).

Equation of upper midheaven (taʿdīl wasaṭ al-samaʾ).

arc of the ecliptic included between upper mid-heaven and the foot of the perpendicular dropped from the zenith to the ecliptic. (N, 24:6).

Equinoctial point (al-iʿtidāl)

either one of the points of intersection of the ecliptic with the equator. (N, 5:3).

Equator, celestial (dāyirat muʿaddal al-nahār, or muʿaddal al-nahār) the great circle whose pole is the north celestial pole. (N, 4:8).

Horizon, local (ufq al-balad)

for any point on the earth's surface, is the great circle in which the tangent plane to the earth at that point intersects the celestial sphere. (N, 26:17).

Horoscope (ṭāliʿ)

same as ascendant.

Latitude, adjusted (al-ʿarḍ al muʿaddal)

= latitude, corrected. (N, 11:12).

Latitude, corrected (ʿarḍ al-balad al-muṣaḥḥaḥ)

the distance on the meridian from the zenith to upper mid-heaven. (N, 11:13).

Latitude, terrestrial (φ, ʿarḍ al-balad)

the distance in degrees from the geographical location to the terrestrial equator (equals the altitude of the north pole above the local horizon). (N, 5:1).

Latitude of the visible climate ($\bar{\alpha}_e$, ʿarḍ iqlīm al-ru'ya)

the great circle distance from the zenith to the ecliptic; the complement of the acute angle between the ecliptic and the horizon. (N, 10:10).

Longitude, celestial (λ, darajat al-kawkab)

the distance measured eastward on the ecliptic from the vernal equinox to the projection of the given point on the ecliptic. (N, 16:3).

Latitude, celestial (β, al-ʿarḍ)

distance from the star to the ecliptic. (N, 14:15).

Meridian (dāyirat or falak niṣf al-nahār)

the great circle passing through the north pole and the zenith. (N, 19:9).

Midheaven, upper (M, wasaṭ al-samā')

at any given instant, is the upper intersection of the ecliptic with the meridian. (N, 11:1).

Ortive amplitude (siʿat al-mashriq)

for any star, the horizon distance from the east point to the point where the star rises. (N, 27:11).

Parallel circle of a star (majra al-kawkab)

any small circle having the north celestial pole as its pole.

Parallel circle, distance of (buʿd majra-al-kawkab)

the distance between the celestial equator and the parallel circle of the star, in modern terminology, the declination. (N, 32:14).

Pole (quṭb)

of a circle in a sphere is one of the two end points of the diameter of the sphere perpendicular to that circle. (N, 13:3).

Position of the star (mawḍiʿ al-kawkab min falak al-burūj)

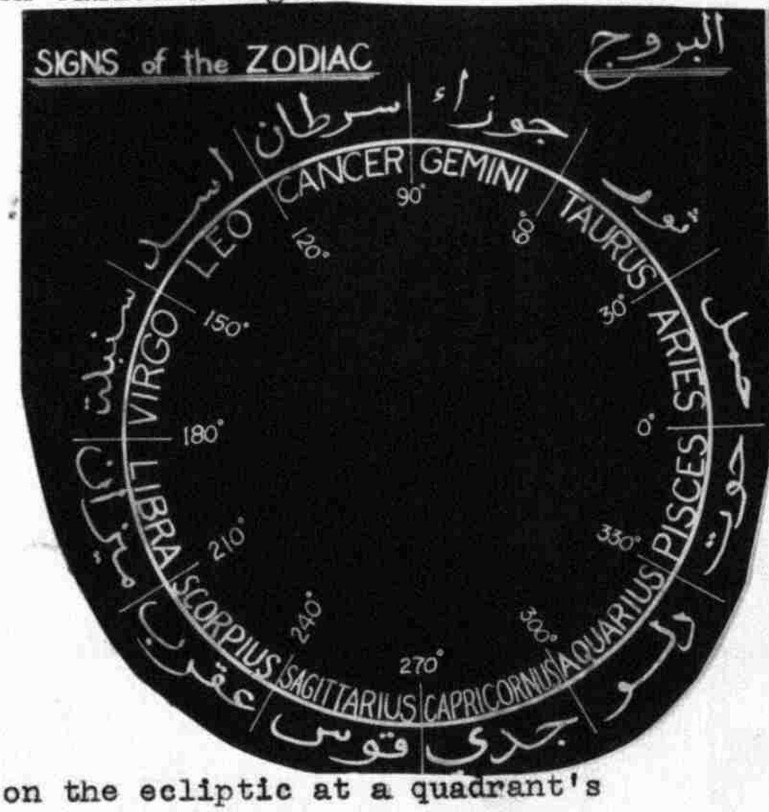
point of intersection of the circle of declination of the star with the ecliptic (de Vaux, p. 430).

Revolution, arc of (r, al-dāyir min al-falak, or simply al-dāyir)

with respect to a point, is the arc of its parallel between its rising and the point itself. (de Vaux, p. 429; N, 63:10).

Signs of the zodiac (al-burūj)

the zone of the heavens within which lie the paths of the sun, moon, and planets. It is divided into twelve equal parts each called a sign.



Solsticial point (al-munqalab)

either of the points on the ecliptic at a quadrant's distance from the equinoctial points. (N, 3:16).

Sphere, the right (al-kura al-mustaqīma or al-falak al-mustaqīm)

the celestial sphere as viewed by an observer located on the terrestrial equator. From this terminology is the modern usage, "right ascension". (N, 4:6 and 4:14).

Transit (al-mamarr)

passage through the meridian. (N, 23:11).

Transit, degree of (darajat al-mamarr)

the point on the ecliptic which culminates or transits with the star. (de Vaux, p. 430; N, 15: 17).

Vernal point (al-iʿtidāl al-rabīʿī, awwal al-ḥamal)

that point of intersection of the ecliptic and the celestial equator through which the sun passes going north.

Zenith (samt al-ra's)

the pole of the horizon. (N, 20:11).

Arabic-English

ʿArḍ al-balad, terrestrial latitude.

ʿArḍ al-balad al-muʿaddal, corrected latitude.

al-ʿArḍ al-muṣaḥḥaḥ, latitude corrected or adjusted.

ʿArḍ iqlīm al-ruʾya, latitude of the visible climate.

al-ʿArḍ, celestial latitude.

al-Buʿd, declination of a star.

Buʿd al-kawkab ʿan muʿaddal al-nahār, declination.

Buʿd majrā al-kawkab, declination.

al-Daraja allatī tatawasaṭ al-samā'maʿ al-kawkab, right ascension.

Darajat al-kawkab, longitude.

Darajat al-mamarr, degree of transit.

Darajat al-ṭāliʿ, ascendant.

al-Dāyir, arc of revolution.

al-Dāyir min al-falak or Dāyir, arc of revolution.

Dāyirat or falak niṣf al-nahār, meridian.

Dāyirat ʿarḍ iqlīm al-ru'ya, circle of latitude of visible climate.

Dāyirat al-irtifāʿ, circle of altitude.

Faḍl al-maṭāliʿ, difference of ascension.

Faḍl al-nahār, excess of daylight.

Faḍl niṣf al-nahār, equation of daylight.

Falak al-burūj, ecliptic.

Ikhtilāf darajat al-kawkab, variation of the degree (of transit).

al-Irtifāʿ, altitude.

Irtifāʿ al-kawkab fī falak niṣf al-nahār, meridian altitude.

al-Iʿtidāl, equinoctial point.

al-Iʿtidāl al-rabīʿī, vernal point.

al-Kura al-mustaqīma or al-falak al-mustaqīm, the right sphere.

Majrā al-kawkab, parallel circle of a star.

Mamarr, transit.

Manṭiqat al-burūj, ecliptic.

al-Mashriq, east point.

Maṭāliʿ al-samt, ascadant of azimuth.

al-Maṭāliʿ fī al-balad, oblique ascension.

al-Maṭāliʿ fī al-falak al-mustaqīm, right ascension.

Maṭlaʿ, rising or rising point.

Mawḍiʿ al-kawkab min falak al-burūj, position of the star.

al-Mayl, or al-Mayl al-awwal or al-mayl al-juzʾī, declination or first declination.

al-Mayl al aʿẓam or al-mayl kulluhū, inclination of the ecliptic, lit., the maximum declination.

Mayl majrā al-kawkab, declination.

al-Mayl al-thānī, second declination.

Muʿaddal al-nahār, celestial equator.

al-Munqalab or al-Inqilāb, solsticial point.

al-Nahār al-muʿaddal or muʿtadil, mean length of daylight.

Qaws al-nahār, arc of daylight

Quṭb, pole.

Samt, azimuth.

Samt al-raʾs, zenith.

Samt al-ṭāliʿ, azimuth of the ascendant point.

Siʿat al-mashriq, ortive amplitude.

Taʿdīl, equation.

Taʿdīl al-ʿarḍ, equation of latitude.

Taʿdīl maṭāliʿ al-samt, the equation of ascendant of the

Taʿdīl al-nahār, equation of daylight.

Taʿdīl al-nahār al a ẓam, maximum equation of daylight.

Taʿdīl al-ṭāliʿ, equation of the ascendant.

Taʿdīl wasaṭ al-samāʾ, equation of upper mid-heaven.

al-Ṭāli', ascendant, horoscope.

Tamām, complement.

Ufq al-balad, terrestrial horizon.

al-Ṭul, longitude.

Wasaṭ al-samā', upper mid-heaven.

15. Medieval Trigonometric Functions

The trigonometric functions used in the "Table of Rectification" differ somewhat from the corresponding modern ones. We distinguish between the medieval and modern functions by using an initial capital letter for the former. (Thus we write Sin, for example, read "cap sine"). The Arabs, like the Greeks, put the angle whose trigonometric function is required as a central angle of a circle whose radius is some constant other than unity. If the radius of the circle is R, then the following identities relate the modern and medieval functions:

$$\mathrm{Sin}\ \theta \equiv R \cdot \sin\theta$$

$$\mathrm{Cos}\ \theta \equiv R \cdot \cos\theta$$

$$\mathrm{Tan}\ \theta \equiv R \cdot \tan\theta$$

$$\mathrm{Cot}\ \theta \equiv R \cdot \cot\theta$$

The most common value for R was 60, i.e., 1,0, a natural choice if the sexagesimal system is in use.

Some Hindu works have R = 150; this is probably due to the fact[92] that $\frac{60^2}{\mathrm{Sin}\,\epsilon} \approx \frac{60 \times 60}{24} = 150$; other values of R

92. See Section 35 below.

have been noted[93].

If R = 1,0 = 60, we have Sin 30° = 30;0, Cos 30° = 51;57,36, Tan 45° = 1,0;0,0, more often written as 60;0,0. It should be observed that to convert the ordinary sin θ (for example) to Sin θ, (if the former is given in the sexagesimal system), it suffices to advance the "sexagesimal point" one place ahead. For example.

sin 60° = 0.8660 = 0;51,57,36,

while Sin 60° = 51;57,36.

The operation corresponding to the motion of the sexagesimal point forward or backward was termed by Arab mathematicians al-Raf' (elevation) and al-Haṭṭ[94] (depression), respectively.

It should be noted that

Sin 90° = Tan 45° = R,

so long as the same defining circle is used for both defining functions. Abū Naṣr called Sin 90° al-Jaib Kulluhū[95] (all the sine), which means the greatest value of the sine.

The inverse of any trigonometric function will be indicated by the superscript -1. Thus $y = \sin^{-1} x$, means $x = \sin y$.

93. Kennedy and Transue, p. 80.
94. N, pp. 11, 14, 69, etc.
95. Ibid, pp. 5, 21, 25, 31, 32, etc.

CHAPTER IV

THE TABLES

16. The Tables, Facsimile and Transcription

The different functions (f_1, f_2, f_{3a}, f_{3b}, f_{4a}, and f_{4b})[96] of the "Table of Rectification" are displayed on pages 52, 54, 56, 58, 60, and 62 as facsimiles of the tables in the sources. In addition, pages 64 thru 67 are a facsimile and transcription of the tangent table described in Section 42 below. A transcription appears opposite each facsimile page. To the transcription proper have been added two other columns for each function, namely the first and second differences (Δ_1 and Δ_2). To distinguish between the material actually appearing in the text and the columns of differences, the former are written in larger characters and bold face.

An incomplete version of the "Table of Rectification" appears in N (p. 3). It contains the four functions for values of the argument between 1° and 30°, and from 61° to 90° only. These tables are full of scribal errors, undoubtedly due to the carelessness of the scribe who copied them from the original manuscript. These tables are identical with those in HI_1, and therefore we found no reason to reproduce them, since we have reproduced the tables from

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96. Details of our notation are given in Section 17 below.

other sources.

The definitions given to the different functions in Section 17 below are so formulated that they yield the values actually shown in most of the tables, where the respective columns are headed _degrees_, _minutes_, and _seconds_. However, we see that the columns of digits for f_2, f_3 and f_{4b} in HI_2 are headed _minutes_, _seconds_, and _thirds_. This means that in order to apply to these particular tables, our definitions of the corresponding functions should be modified by being divided by R = 1,0.

Abū Naṣr[97] criticises both Ḥabash and Abū Jaʿfar al-Khāzin more than once, when they usually "elevate" their values one place (i.e., multiply by 1,0). This point will be discussed in Section 38 below.

The tabular entries are given to three sexagesimal places for each integer degree of the argument θ from 1° to 90°, and there are thirty entries per page. For all the functions

$$|f(\theta)| \equiv |f(180° - \theta)|.$$

Utilizing this fact, the supplement of the argument, 180°-θ, appears in HI_1, alongside θ itself.

Square brackets are used in the transcription to indicate restorations of the text. Such restorations have been made only where it is impossible to give a correct reading.

97. N, pp. 4, 68, etc.

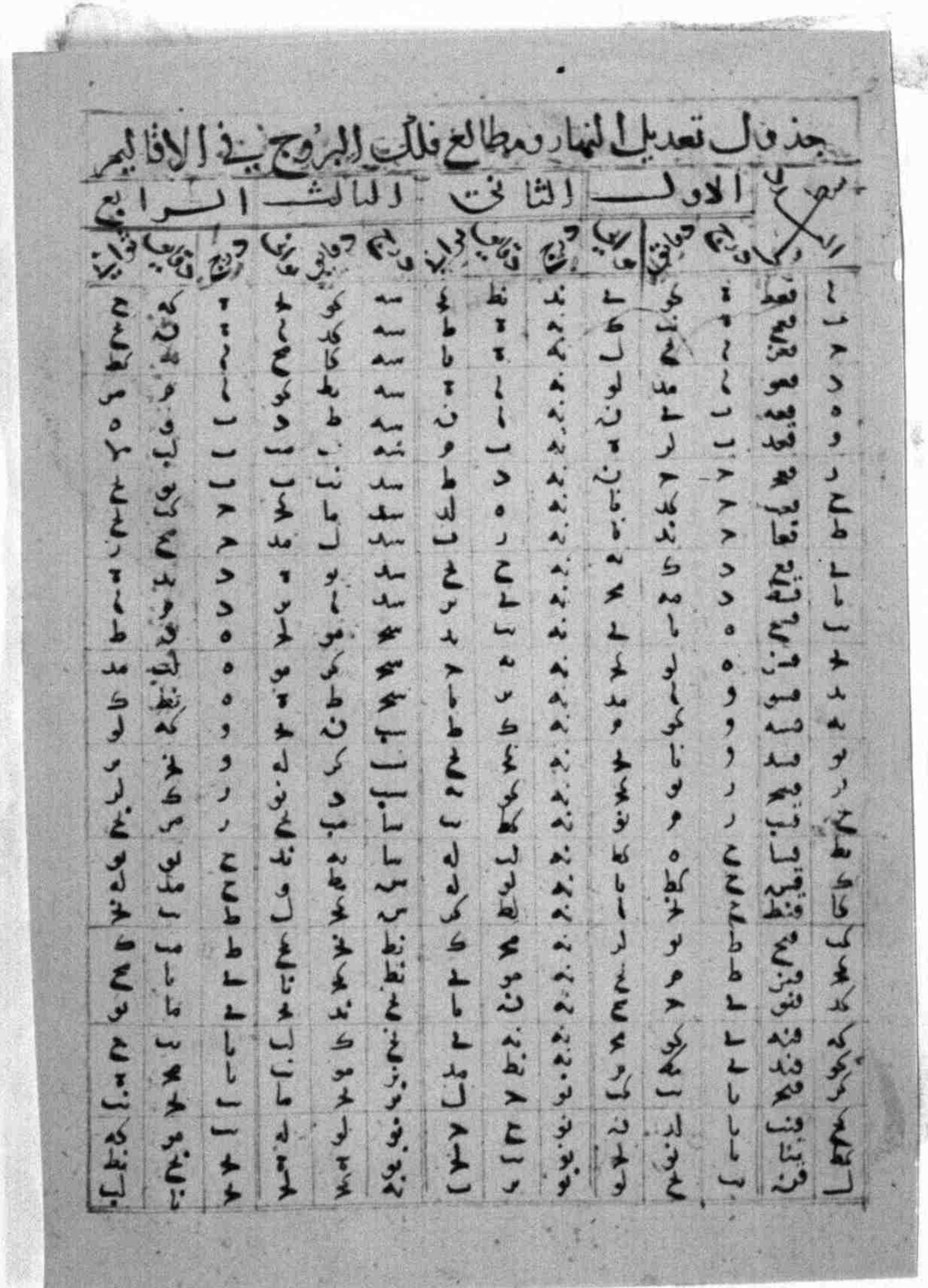
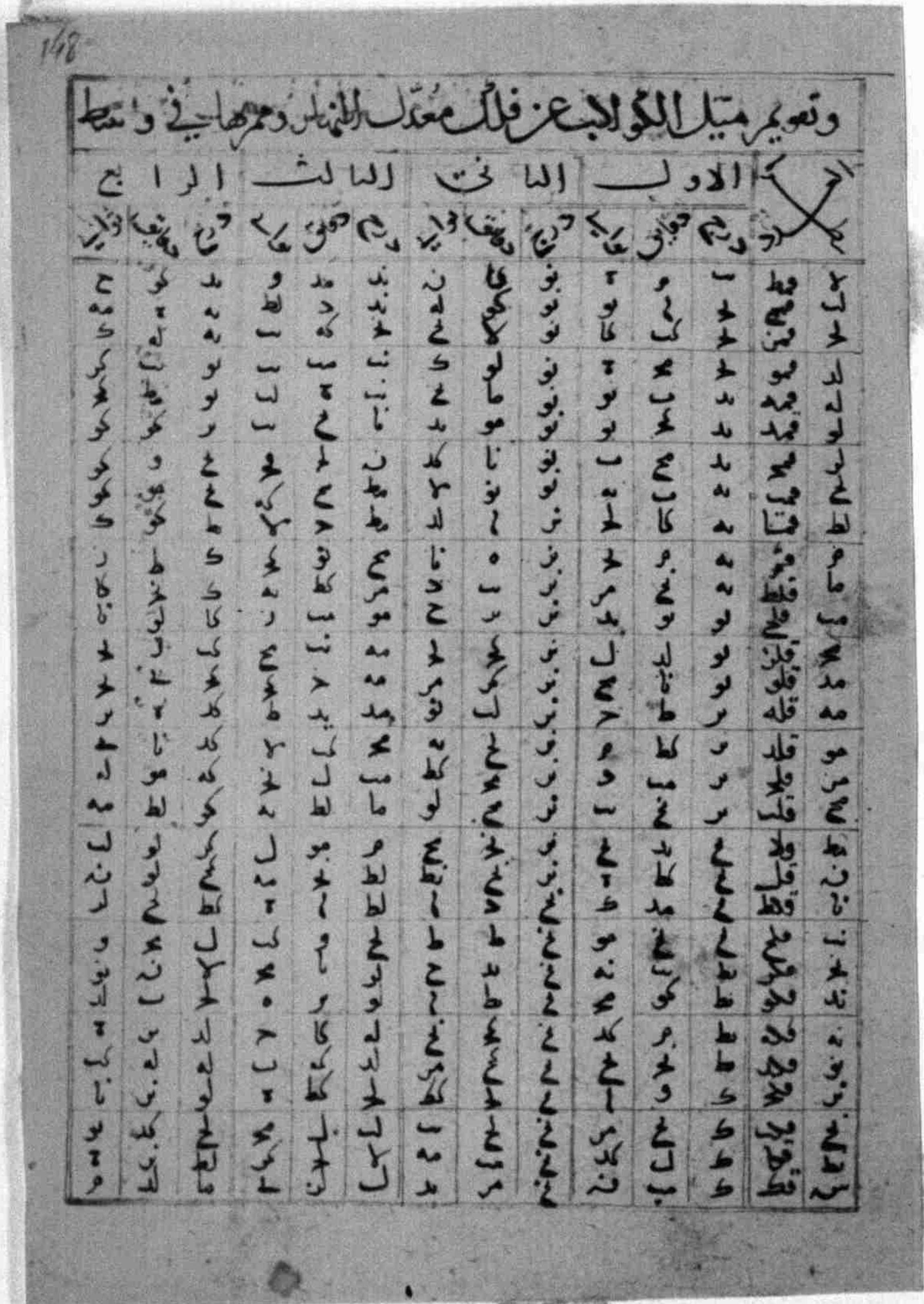


Table of the Equation of Daylight and the Ascension of the Ecliptic for the Climates

		The First (f 1)	Δ1	Δ2	The Second (f 2)	Δ1	Δ2	The Third (f 3a)	Δ1	Δ2	The Fourth (f 4a)	Δ1	Δ2
1	179	0,26,10	0,26,10		54,59,23	0,0,46		65,26,13	-0,2,12		0,25,8	0,25,10	
2	178	0,52,20	0,26,10		55,0,9	0,0,32	-14	65,24,1	-0,2,13		0,50,18	0,24,11	
3	177	1,18,30	0,26,10		55,0,41	0,0,19	-13	65,21,48	-0,6,22	-4,9	1,15,29	0,25,18	7
4	176	1,44,[40]	0,26,10		55,1,0	0,0,50	31	65,[5],26	-0,6,22		1,40,47	0,25,18	
5	175	2,10,50	0,26,10		55,1,50	0,0,50	0	65,9,4	-0,6,22		2,6,5	0,25,26	8
6	174	2,37,0	0,26,5	-5	55,2,40	0,1,27	37	65,2,42	-0,10,40	-4,18	2,3[1],31	0,25,27	
7	173	3,3,[5]	0,25,46	-19	55,4,9	0,1,29	2	64,52,2	-0,10,39		2,56,58	0,25,35	6
8	172	3,2[8],51	0,26,0	14	55,5,34	0,1,25	-4	64,41,23	-0,10,39		3,22,3[3]	0,25,34	
9	171	3,54,51	0,25,24	-36	55,7,2	0,1,16	-9	64,30,44	-0,14,44	-4,5	3,48,7	0,25,53	19
10	170	4,20,15	0,25,26	4	55,8,18	0,1,59	43	64,16,0	-0,14,43		4,14,0	0,26,1	
11	169	4,45,43	0,25,27	-1	55,10,17	0,1,57	-2	64,1,17	-0,14,44		4,40,1	0,26,8	
12	168	5,11,10	0,25,13	-14	55,12,14	0,2,49	52	63,46,33	-0,18,47	-4,3	5,6,9	0,26,35	27
13	167	5,36,23	0,25,21	8	55,15,3	0,2,8	-41	63,27,46	-0,18,46		5,32,44	0,26,36	
14	166	6,1,44	0,24,56	-25	55,17,11	0,2,58	50	63,9,0	-0,18,47		5,59,20	0,26,36	
15	165	6,26,40	0,24,33	-23	55,20,9	0,3,9	11	62,50,13	-0,22,38	-3,51	6,25,[56]	0,27,20	44
16	164	6,51,13	0,25,10	37	55,23,18	0,2,57	-12	62,27,35	-0,22,39		6,53,16	0,27,21	
17	163	7,16,23	0,24,33	-37	55,26,15	0,2,57	0	62,4,56	-0,22,38		7,20,37	0,27,21	
18	162	7,40,56	0,24,33	0	55,29,12	0,3,23	26	61,42,18	-0,26,24	-3,46	7,47,58	0,28,18	57
19	161	8,5,29	0,23,42	-51	55,32,35	0,3,40	17	61,15,54	-0,26,24		8,16,16	0,28,19	
20	160	8,29,11	0,23,50	6	55,36,[15]	0,3,8	-32	60,49,[30]	-0,26,24		8,44,35	0,28,18	
21	159	8,53,1	0,23,36	-14	55,39,[23]	0,3,37	29	60,23,[6]	-0,29,38	-3,14	9,12,53	0,29,27	1,9
22	158	9,16,37	0,23,41	5	55,43,[0]	0,3,10	-27	59,53,28	-0,29,37		9,42,20	0,29,28	
23	157	9,40,18	0,23,30	-11	55,46,10	0,4,1	51	59,23,51	-0,29,37		10,11,48	0,29,28	
24	156	10,3,48	0,22,55	-35	55,50,11	0,3,59	-2	58,54,13	-0,33,41	-3,4	10,41,16	0,30,52	1,24
25	155	10,26,43	0,22,57	2	55,54,10	0,4,34	35	58,20,32	-0,33,40		11,12,8	0,30,52	
26	154	10,49,40	0,22,42	-15	55,[56],44	0,4,46	12	57,46,52	-0,33,41		11,43,0	0,30,52	
27	153	11,12,22	0,22,28	-14	56,3,30	0,4,33	-13	57,13,[11]	-0,36,36	-2,55	12,13,52	0,32,33	1,41
28	152	11,34,50	0,21,43	-55	56,8,3	0,4,30	-3	56,36,35	-0,36,35		12,46,25	0,32,34	
29	151	11,56,33	0,22,3	20	56,12,33	0,4,29	-1	56,0,0	-0,36,27		13,18,59	0,32,33	
30	150	12,18,36	0,21,24	-39	56,17,2	0,4,28	19	55,23,33	-0,39,27	-2,52	13,51,32	0,34,36	2,3

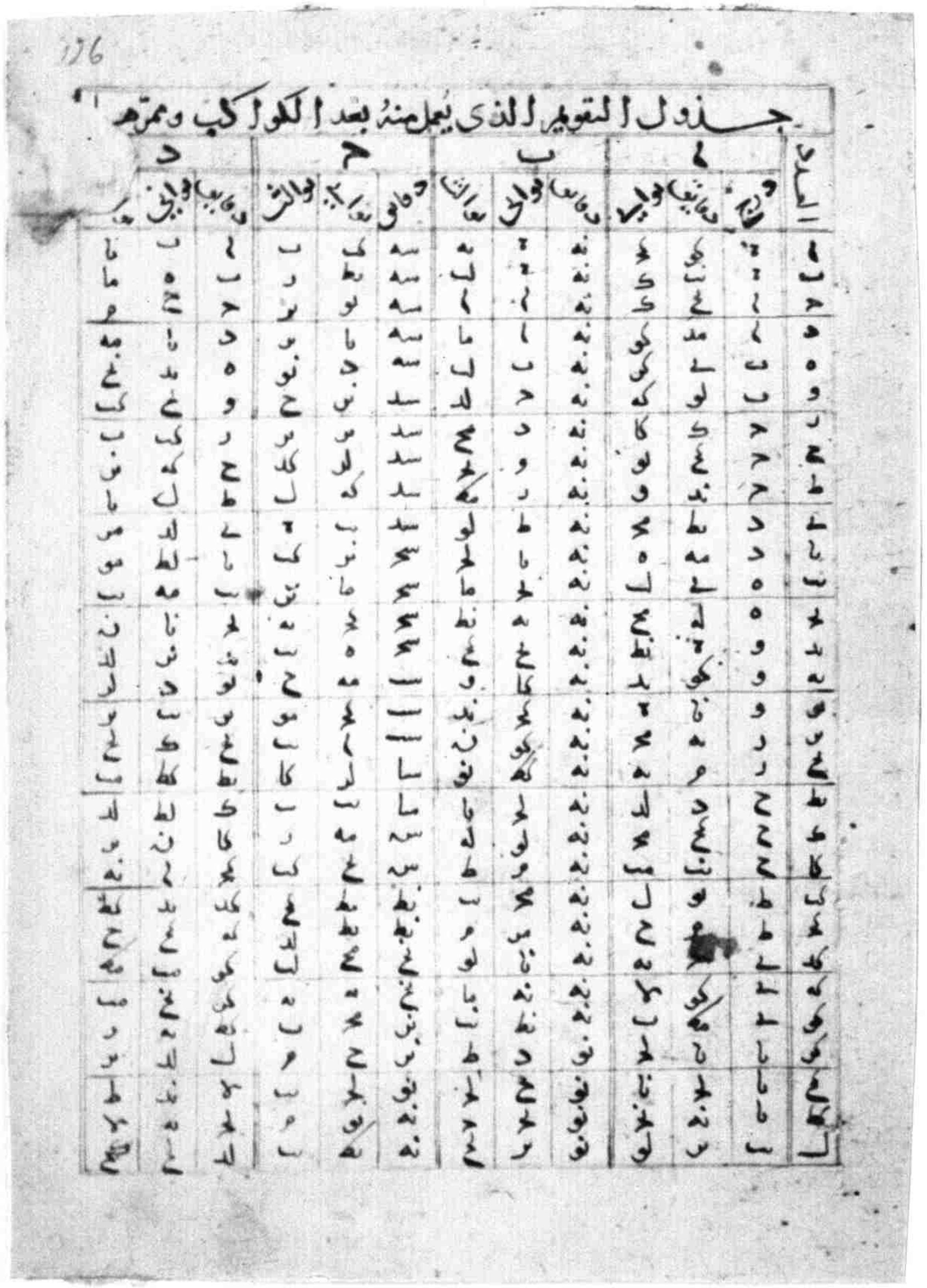


and the Rectification of the Declination of the Planets from the Celestial Equator and their Transit in Mid_

		The First (f 1)	Δ1	Δ2	The Second (f 2)	Δ1	Δ2	The Third (f 3a)	Δ1	Δ2	The Fourth (f 4a)	Δ1	Δ2
31	149	12,40,0	0,21,16	-8	56,21,50	0,4,45	3	54,44,6	-0,39,27		14,26,8	0,34,3	
32	148	13,1,16	0,21,5	-11	56,26,35	0,4,43	-2	54,4,39	-0,39,27		15,0,45	0,34,35	
33	147	13,22,21	0,20,39	-26	56,31,18	0,5,2	19	53,25,12	-0,42,20	-2,53	15,35,20	0,37,2	2,26
34	146	13,43,0	0,19,56	-43	56,36,20	0,4,58	-4	52,42,52	-0,42,20		16,12,2[2]	0,37,1	
35	145	14,2,56	0,20,20	24	56,41,18	0,4,56	-2	52,0,32	-0,42,20		16,49,23	0,37,3	
36	144	14,23,16	0,19,46	-34	56,46,14	0,5,10	14	51,18,12	-0,44,54	-2,34	17,26,26	0,40,0	2,57
37	143	14,4[3],2	0,19,13	-33	56,51,24	0,5,7	-3	50,33[18]	-0,44,53		18,6,26	0,40,0	
38	142	15,2,15	0,19,18	5	56,56,31	0,5,3	-4	49,48,25	-0,44,54		18,46,26	0,40,0	
39	141	15,21,33	0,18,50	-28	57,1,34	0,5,17	14	49,3,31	-0,47,8	-2,14	19,26,2[6]	0,43,24	3,24
40	140	15,40,13	0,18,34	-16	57,[6],51	0,5,13	-4	48,[16],23	-0,47,8		20,9,50	0,43,31	
41	139	15,58,47	0,17,57	-37	57,12,4	0,5,4	-9	47,29,15	-0,47,8		20,53,21	0,43,10	
42	138	16,16,44	0,17,46	-11	57,17,8	0,5,25	21	46,42,7	-0,49,19	-2,11	21,36[3]1	0,43,42	
43	137	16,34,30	0,17,18	-28	57,2[2],33	0,5,24	-1	45,52,48	-0,49,25		22,[20],13	0,50,0	6,18
44	136	16,51,48	0,17,15	-3	57,27,47	0,5,9	-15	45,3,23	-0,49,14		23,10,13	0,50,5	
45	135	17,9,3	0,16,33	-42	57,32,56	0,5,19	10	44,14,9	-0,51,38	-2,24	24,0,17	0,50,52	
46	134	17,25,40	0,16,24	-9	57,38,15	0,5,14	-4	43,22,31	-0,51,38		24,51,10	0,55,25	0,4,28
47	133	17,42,4	0,16,8	-16	57,43,29	0,5,7	-7	42,30,53	-0,51,38		25,46,35	0,55,10	
48	132	17,58,12	0,16,6	-2	57,48,36	0,5,12	5	41,39,15	-0,52,45	-1,7	26,[41],45	0,55,5	
49	131	18,14,18	0,15,1	-15	57,53,48	0,5,11	-1	40,46,30	-0,52,45		27,36[4]2	1,0,8	0,5,3
50	130	18,29,[19]	0,15,1	0	57,58,59	0,5,2	-9	39,53,45	-0,52,45		28,36,50	1,1,47	0,1,39
51	129	18,44,20	0,14,26	-35	58,4,1	0,5,8	-6	39,1,0	-0,54,38		29,38,37	1,5,3	0,3,16
52	128	18,58,46	0,14,9	-17	58,9,9	0,4,59	-9	38,6,22	-0,54,39		30,43,40	1,7,16	0,2,13
53	127	19,12,55	0,13,48	-21	58,14,8	0,4,53	-6	37,11,43	-0,54,38		31,50,56	1,11,38	0,4,22
54	126	19,26,43	0,13,41	-7	58,19,1	0,4,57	-4	36,17,5	-0,56,2	-1,24	33,2,34	1,14,26	0,2,48
55	125	19,40,24	0,12,54	-47	58,23,58	0,4,49	-8	35,21,3	-0,56,1		34,17,0	1,18,22	0,3,56
56	124	19,53,18	0,12,43	-11	58,28,47	0,4,42	-7	34,25,2	-0,56,2		35,35,22	1,21,29	0,3,7
57	123	20,6,1	0,12,46	3	58,33,29	0,4,43	-1	33,29,0	-0,58,17	-2,15	36,57,51	1,27,5	0,5,36
58	122	20,18,47	0,11,57	-49	58,38,12	0,4,33	-10	32,30,43	-0,58,16		38,24,56	1,32,4	0,4,59
59	121	20,30[4]4	0,11,46	-11	58,42,45	0,4,29	-4	31,33,27	-0,58,17		39,57,0	1,37,40	0,5,36
60	120	20,42,[30]	0,10,48	-56	58,47,14	0,4,24	-5	30,3[5],10	-0,58,23	-6	41,34,40	1,43,22	0,5,42

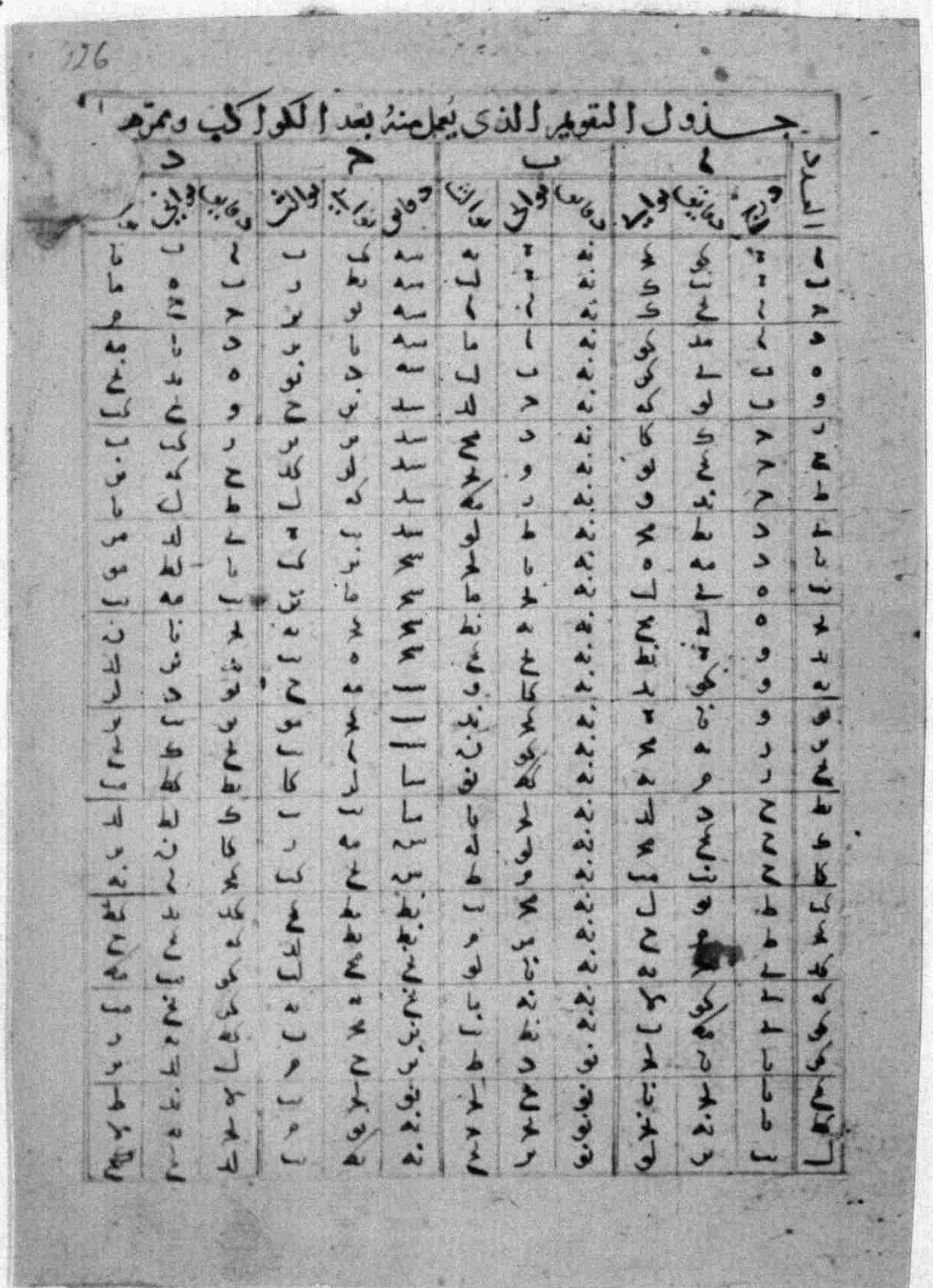
Heaven with the parts of the Ecliptic

		The First (f 1)	△1	△2	The Second (f 2)	△1	△2	The Third (f 3a)	△1	△2	The Fourth (f 4a)	△1	△2
61	119	20,53,18	0,10,49	1	58,51,38	0,4,19	-5	29,38,47	-0,58,23		43,18,22	1,50,37	0715
62	118	21,4,7	0,10,15	-34	58,55,57	0,4,10	-9	28,40,24	-0,58,23		45,8,59	1,57,46	079
63	117	21,14,22	0,10,49	34	59,0,7	0,4,7	-3	27,42,1	-0,59,20	-57	47,6,45	2,5,41	0755
64	116	21,25,11	0,9,28	-21	59,4,14	0,3,58	-9	26,42,41	-0,59,20		49,12,26	2,15,44	0102
65	115	21,34,39	0,8,51	-37	59,8,12	0,3,49	-9	25,43,21	-0,59,19		51,28,10	2,26,48	0114
66	114	21,43,3	0,8,50	-1	59,12,1	0,3,44	-5	24,44,2	-1,0,8	-1,49	53,54,58	2,37,24	01036
67	113	21,52,20	0,8,34	16	59,15,45	0,3,34	-10	23,43,54	-1,0,9		56,32,22	2,51,48	01424
68	112	22,0,54	0,9,2	28	59,19,19	0,3,26	-8	22,43,45	-1,0,9		59,24,10	3,7,56	0168
69	111	22,9,56	0,8,4	-58	59,22,45	0,3,22	-4	21,43,36	-1,0,50	-41	62,32,6	3,23,24	01528
70	110	22,18,0	0,7,30	-34	59,26,7	0,3,17	-5	20,42,46	-1,0,50		65,55,30	3,47,26	0242
71	109	22,25,30	0,6,51	-39	59,29,24	0,3,13	-4	19,41,56	-1,0,50		69,42,56	4,9,53	02627
72	108	22,32,21	0,6,54	-3	59,32[3]7	0,2,51	-22	18,41,6	-1,1,26	-36	73,52,49	4,38,11	02818
73	107	22,39,15	0,6,27	-27	59,35,18	0,2,45	-6	17,39,40	-1,1,26		78,31,0	5,11,0	03249
74	106	22,45,42	0,6,17	-10	59,38,3	0,2,35	-10	16,38,14	-1,1,26		83,42,0	5,53,15	04215
75	105	22,51,59	0,5,17	-1,0	59,40,38	0,2,30	-5	15,36,48	-1,1,54	-28	89,35,15	6,44,45	05130
76	104	22,5[7,1]6	0,5,14	-3	59,43,8	0,2,15	-15	14,34,54	-1,1,54		96,20,0	7,38,30	05345
77	103	23,2,30	0,4,26	-48	59,45,23	0,2,9	-6	13,33,0	-1,1,55		103,58,30	8,57,29	11859
78	102	23,6,56	0,4,4	-22	59,47,32	0,1,57	-12	12,31,5	-1,2,16	-22	112,55,59	10,31,1	13532
79	101	23,11,[0]	0,4,4	0	59,49,29	0,1,48	-9	11,28,49	-1,2,16		123,29,0	12,37,40	2439
80	100	23,[5],4	0,4,1	-3	59,51,17	0,1,37	-11	10,26,33	-1,2,17		136,6,40	15,27,7	24927
81	99	23,19,5	0,3,7	6	59,52,54	0,1,29	-8	9,24,16	-1,2,33	-17	151,33,47	19,14,13	3476
82	98	23,22,12	0,3,9	2	59,54,23	0,1,20	-9	8,21,43	-1,2,32		170,48,0	24,39,48	52535
83	97	23,25,21	0,2,27	46	59,55,43	0,1,8	-12	7,19,11	-1,2,33		195,27,48	32,55,50	8162
84	96	23,27,48	0,2,4	-23	59,56,51	0,0,58	-10	6,16,38	-1,2,43	-10	228,23,38	45,56,22	13032
85	95	23,29,52	0,1,42	-22	59,57,49	0,0,47	-11	5,13,55	-1,2,43		274,20,0	68,52,51	225629
86	9	23,31,34	0,1,19	-23	59,58,36	0,0,37	-10	4,11,12	-1,2,43		343,12,51	115,8,56	4616,5
87	9	23,3[2]53	0,1,18	-1	59,59,13	0,0,26	-13	3,8,29	-1,2,50	-7	458,21,47	229,13,53	114457
88	92	23,34,11	0,0,37	-41	59,59,39	0,0,16	-10	2,5,39	-1,2,49		687,35,40	687,32,45	4581852
89	91	23,34,48	0,0,12	-15	59,59,55	0,0,5	-11	1,2,50	-1,2,50		1375,8,25		
90	90	23,35,0			60,0,0			0,0,0					



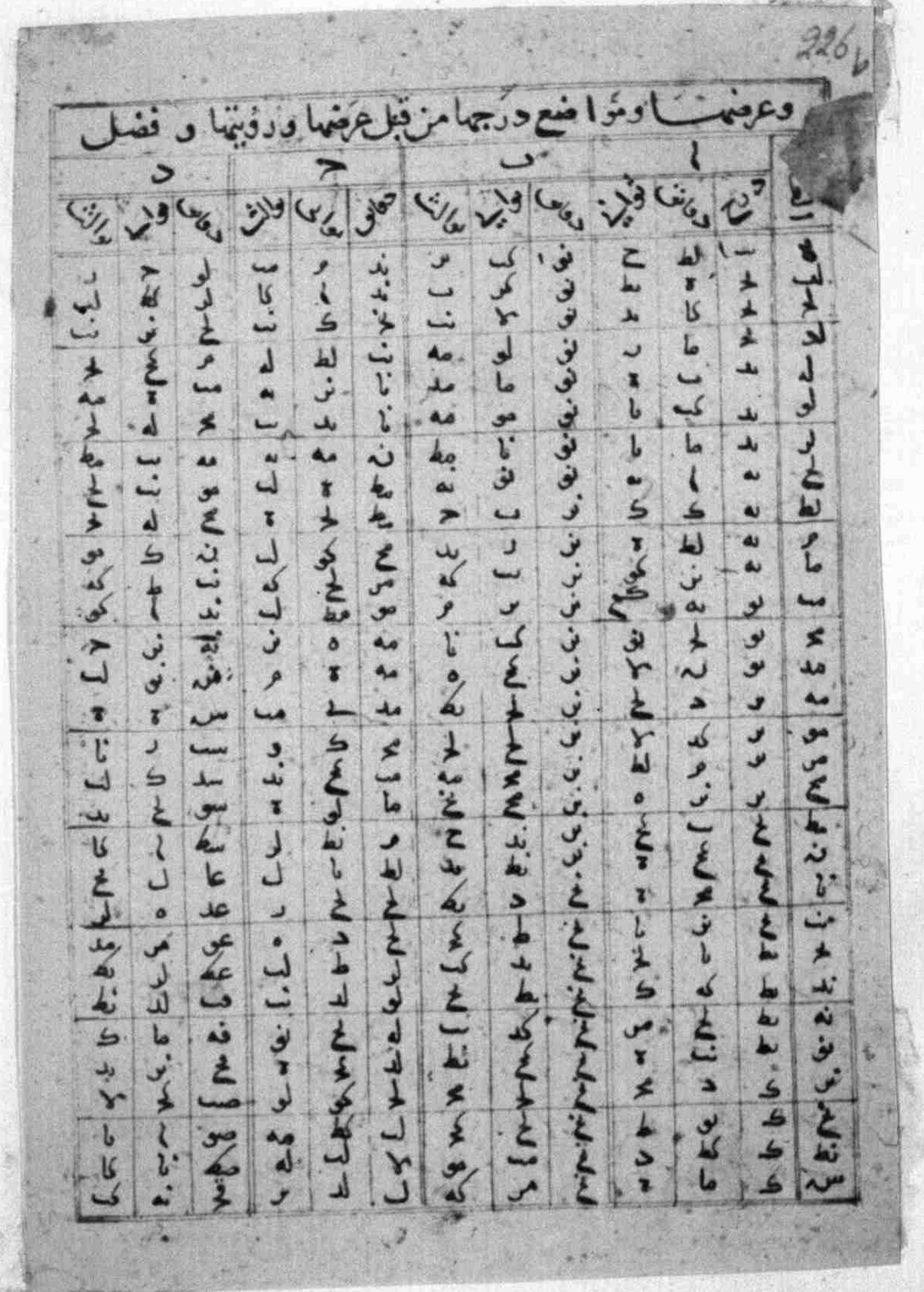
The Table of Rectification from which is Determined the Distance of the Planets and their Transit

	A (f_1)	Δ_1	Δ_2	B (f_2)	Δ_1	Δ_2	C (f_{3a})	Δ_1	Δ_2	D (f_{4b})
1	0;26,[10]	0;26,10		55;0,15	0;0,17		65;22,2	-0;2,55		1;2,51
2	0;52,20	0;26,0	-10	55;0,32	0;0,29	12	65;19,7	-0;2,50	5	2;5,41
3	1;18,20	0;26,6	6	55;1,1	0;0,40	11	65;16,17	-0;5,0	-2,10	3;8,40
4	1;44,26	0;26,1	-5	55;1,41	0;0,51	11	65;11,17	-0;6,21	-1,21	4;11,45
5	2;10,27	0;25,58	-3	55;2,32	0;1,2	11	65;4,56	-0;7,48	-1,27	5;14,58
6	2;36,25	0;26,6	8	55;3,34	0;1,14	12	64;57,8	-0;9,51	-2,3	6;18,22
7	3;[2],[3]1	0;26,5	-1	55;4,48	0;1,25	11	64;47,17	-0;9,53	-2	7;22,2
8	3;28,36	0;25,30	-35	55;6,13	0;1,36	11	64;37,24	-0;11,54	-2,1	8;25,57
9	3;54,6	0;25,37	7	55;7,49	0;1,47	11	64;25,30	-0;13,30	-1,36	9;30,11
10	4;19,43	0;25,22	-15	55;9,36	0;1,57	10	64;12,0	-0;14,28	-58	10;34,47
11	4;45,5	0;25,27	5	55;11,33	0;2,8	11	63;57,32	-0;16,35	-2,7	11;39,46
12	5;10,32	0;25,16	-11	55;13,41	0;2,18	10	63;4[0],57	-0;17,42	-1,7	12;45,12
13	5;35,48	0;25,11	-5	55;15,59	0;2,29	11	63;23,15	-0;18,3	-21	13;51,[8]
14	6;0,59	0;25,15	4	55;18,28	0;2,38	9	63;5,12	-0;20,4	-2,1	14;57,34
15	6;26,14	0;24,46	-29	55;21,6	0;2,48	10	62;45,8	-0;21,22	-1,18	16;4,37
16	6;51,0	0;24,43	-3	55;23,54	0;2,56	8	62;23,46	-0;22,34	-1,12	17;12,17
17	7;15,43	0;24,32	-11	55;26,50	0;3,6	10	62;1,12	-0;23,51	-1,17	18;20,38
18	7;40,15	0;24,19	-13	55;29,56	0;3,15	10	61;37,21	-0;25,19	-1,28	19;29,42
19	8;4,34	0;24,9	-10	55;33,11	0;3,24	9	61;12,2	-0;25,55	-36	20;39,34
20	8;28,43	0;23,59	-10	55;36,35	0;3,34	10	60;4[6],7	-0;27,45	-1,50	21;50;17
21	8;52,42	0;23,48	-11	55;40,9	0;3,43	9	60;18,22	-0;28,54	-1,9	23;1,55
22	9;16,30	0;23,38	-10	55;43,52	0;3,51	8	59;49,28	-0;29,54	-1,0	24;14,29
23	9;40,8	0;23,27	-11	55;47,4[3]	0;3,58	7	59;19,34	-0;31,2	-1,8	25;28,8
24	10;3,[3]5	0;23,16	-11	55;51,[41]	0;4,4	6	58;48,32	-0;32,17	-1,15	26;42,49
25	10;26;[5]1	0;22,21	-55	55;55,4[5]	0;4,8	4	58;[16],15	-0;33,13	-56	27;58,42
26	10;49,[1]2	0;22,21	0	55;59,5[3]	0;4,16	8	57;43,2	-0;34,22	-1,9	29;15,50
27	11;11,33	0;22,18	-3	56;4,9	0;4,24	8	57;8,40	-0;35,28	-1,6	30;34,17
28	11;33,51	0;22,2	-16	56;8,33	0;4,30	6	56;33,12	-0;36,32	-1,4	31;54,9
29	11;55,53	0;21,43	-19	56;13,3	0;4,35	5	55;56,40	-0;37,28	-56	33;15,31
30	12;17,36	0;21,32	-11	56;17,38	0;4,39	4	55;19,12	-0;38,30	-1,2	34;38,28



The Table of Rectification from which is Determined the Distance of the Planets and their Transit

	A (f_1)	Δ_1	Δ_2	B (f_2)	Δ_1	Δ_2	C (f_{3a})	Δ_1	Δ_2	D (f_{4b})
1	0;26,[10]	0,26,10		55; 0,15	0, 0,17		65;22, 2	-0, 2,55		1; 2,51
2	0;52,20	0,26, 0	-10	55; 0,32	0, 0,29	12	65;19, 7	-0, 2,50	5	2; 5,41
3	1;18,20	0,26, 6	6	55; 1, 1	0, 0,40	11	65;16,17	-0, 5, 0	-2,10	3; 8,40
4	1;44,26	0,26, 1	-5	55; 1,41	0, 0,51	11	65;11,17	-0, 6,21	-1,21	4;11,45
5	2;10,27	0,25,58	-3	55; 2,32	0, 1, 2	11	65; 4,56	-0, 7,48	-1,27	5;14,58
6	2;36,25	0,26, 6	8	55; 3,34	0, 1,14	12	64;57, 8	-0, 9,51	-2,3	6;18,22
7	3;[2], [3]1	0,26, 5	-1	55; 4,48	0, 1,25	11	64;47,17	-0, 9,53	-2	7;22, 2
8	3;28,36	0,25,30	-35	55; 6,13	0, 1,36	11	64;37,24	-0,11,54	-2,1	8;25,57
9	3;54, 6	0,25,37	7	55; 7,49	0, 1,47	11	64;25,30	-0,13,30	-1,36	9;30,11
10	4;19,43	0,25,22	-15	55; 9,36	0, 1,57	10	64;12, 0	-0,14,28	-58	10;34,47
11	4;45, 5	0,25,27	5	55;11,33	0, 2, 8	11	63;57,32	-0,16,35	-2,7	11;39,46
12	5;10,32	0,25,16	-11	55;13,41	0, 2,18	10	63;40,57	-0,17,42	-1,7	12;45,12
13	5;35,48	0,25,11	-5	55;15,59	0, 2,29	11	63;23,15	-0,18, 3	-21	13;51,[8]
14	6; 0,59	0,25,15	4	55;18,28	0, 2,38	9	63; 5,12	-0,20, 4	-2,1	14;57,34
15	6;26,14	0,24,46	-29	55;21, 6	0, 2,48	10	62;45, 8	-0,21,22	-1,18	16; 4,37
16	6;51, 0	0,24,43	-3	55;23,54	0, 2,56	8	62;23,46	-0,22,34	-1,12	17;12,17
17	7;15,43	0,24,32	-11	55;26,50	0, 3, 6	10	62; 1,12	-0,23,51	-1,17	18;20,38
18	7;40,15	0,24,19	-13	55;29,56	0, 3,15	10	61;37,21	-0,25,19	-1,28	19;29,42
19	8; 4,34	0,24, 9	-10	55;33,11	0, 3,24	9	61;12, 2	-0,25,55	-36	20;39,34
20	8;28,43	0,23,59	-10	55;36,35	0, 3,34	10	60;4[6], 7	-0,27,45	-1,50	21;50;17
21	8;52,42	0,23,48	-11	55;40, 9	0, 3,43	9	60;18,22	-0,28,54	-1,9	23; 1,55
22	9;16,30	0,23,38	-10	55;43,52	0, 3,51	8	59;49,28	-0,29,54	-1,0	24;14,29
23	9;40, 8	0,23,27	-11	55;47,4[3]	0, 3,58	7	59;19,34	-0,31, 2	-1,8	25;28, 8
24	10; 3,[3]5	0,23,16	-11	55;51,[41]	0, 4, 4	6	58;48,32	-0,32,17	-1,15	26;42,49
25	10;26;[5]1	0,22,21	-55	55;55,4[5]	0, 4, 8	4	58;[16],15	-0,33,13	-56	27;58,42
26	10;49,[1]2	0,22,21	0	55;59,5[3]	0, 4,16	8	57;43, 2	-0,34,22	-1,9	29;15,50
27	11;11,33	0,22,18	-3	56; 4, 9	0, 4,24	8	57; 8,40	-0,35,28	-1,6	30;34,17
28	11;33,51	0,22, 2	-16	56; 8,33	0, 4,30	6	56;33,12	-0,36,32	-1,4	31;54, 9
29	11;55,53	0,21,43	-19	56;13, 3	0, 4,35	5	55;56,40	-0,37,28	-56	33;15,31
30	12;17,36	0,21,32	-11	56;17,38	0, 4,39	4	55;19,12	-0,38,30	-1,2	34;38,28



... and their Latitude and the Positions of their Longitude and the Excess from their Latitude and their Visibility and ...

	A(f_1)	Δ_1	Δ_2	B(f_2)	Δ_1	Δ_2	C(f_{3a})	Δ_1	Δ_2	D(f_{4b})
31	12;39, 8	0,21,11		56;22,17	0, 4,44		54;40,42	-0,39,21		36; 3, 7
32	13; 0,19	0,20,55	-16	56;27,[1]	0, 4,51	7	54; 1,21	-0,40,29	-1,8	37;29,32
33	13;21,14	0,20,36	-19	56;31,52	0, 4,54	3	53;20,52	-0,41,17	-48	38;57,52
34	13;41,50	0,20,10	-26	56;36,4[6]	0, 4,58	4	52;39,35	-0,42,20	-1,3	40;28,13
35	14; 2, 0	0,20,11	1	56;41,44	0, 5, 1	3	51;57,15	-0,43, 3	-43	42; 0,45
36	14;22,11	0,20, 0	-11	56;46,45	0, 5, 4	3	51;14,12	-0,28,57	-46	43;35,33
37	14;4[2],11	0,19, 4	-56	56;51,49	0, 5, 6	2	50;45,15	-0,44,43	-5,40	45;12,49
38	15; 1,15	0,19, 5	1	56;56,55	0, 5, 8	2	50; 0,32	-0,46,32	-49	46;52,38
39	15;20,20	0,18,40	-25	57; 2, 3	0, 5,11	3	49;1[4], 0	-0,47,28	-56	48;35,13
40	15;39, 0	0,18,26	-14	57; 7,14	0, 5,11	0	48;26,32	-0,48, 7	-35	50;20,46
41	15;57,26	0,18, 2	-24	57;12,25	0, 5,13	2	47;38,25	-0,48,53	-46	52; 9,25
42	16;15,28	0,17,28	-34	57;17,[38]	0, 5,13	0	46;49,32	-0,49,35	-42	54; 1,26
43	16;3[2],56	0,17,27	-1	57;22,51	0, 5,14	1	45;[59],57	-0,50,17	-42	55;57, 3
44	16;50;[2]1	0,17,17	-10	57;28, 5	0, 5,14	0	45;[9],40	-0,55,58	-54	57;56,32
45	17;[7],38	0,16,53	-24	57;33,19	0, 5,14	-1	44;[13],42	-0,53,36	2,24	60; 0, 0
46	17;24,31	0,16,53	-45	57;38,33	0, 5,13	-1	43;20, 6	-0,51,12	-1,42	62; 7,51
47	17;40,39	0,16,88	18	57;43,4[6]	0, 5,12	-2	42;28,54	-0,52,54	31	64;20,32
48	17;57, 5	0,16,26	-1,3	57;48,58	0, 5,10	-4	41;36, 0	-0,52,23	16	66;38,14
49	18;12,28	0,15,23	9	57;54, 8	0, 5, 6	-2	40;[43],37	-0,52, 7	-1,16	69; 1,21
50	18;28, 0	0,15,32	-32	57;59,14	0, 5, 5	-1	39;51,30	-0,53,23	-39	71;30,18
51	18;43, 0	0,15, 0	-9	58; 4,19	0, 5, 4	-5	38;58, 7	-0,54, 2	-31	74; 5,37
52	18;57,51	0,14,51	-1,9	58; 9,23	0, 4,59	-3	38; 4, 5	-0,54,33	-1,7	76;47,44
53	19;11,33	0,13,42	5	58;14,22	0, 4,56	-2	37; 9,32	-0,55,40	-16	79;37,19
54	19;25,20	0,13,47	-20	58;19,18	0, 4,54	8	36;[13],52	-0,55,56	0	82;34,59
55	19;38,47	0,13,27	-14	58;24,12	0, 4,46	-1	35;18,56	-0,55,56	-28	85;41,20
56	19;52, 0	0,13,13	-1,0	58;28,58	0, 4,45	-5	34;23, 0	-0,56,24	-27	88;57,14
57	20; 4,[1]3	0,12,13	-17	58;33,43	0, 4,40	-4	33;26,36	-0,56,51	-29	92;23,31
58	20;16, 9	0,11,56	-1	58;38,23	0, 4,36	-7	32;29,45	-0,57,10	-18	96; 1,11
59	20;2[8], 4	0,11,55	1,1	58;42,56	0, 4,29	-6	31;32,35	-0,58,18	13	99;51,21
60	20;41, 0	0,12,56	-1,52	58;47,25	0, 4,24	-5	30;34,17	-0,57,15	-1,5	103;55,22

and their Latitude and the Positions of their Longitude
from their Latitude and their Visibility and the Excess

	A(f_1)	Δ₁	Δ₂	B(f_2)	Δ₁	Δ₂	C(f_{3a})	Δ₁	Δ₂	D(f_{4b})
31	12;39, 8	0;21,11		56;22,17	0; 4,44		54;40,42	-0;39,21		36; 3, 7
32	13; 0.19	0;20,55	-16	56;27,[1]	0; 4,51	7	54; 1,21	-0;40,29	-1,8	37,29,32
33	13;21,14	0;20,36	-19	56;31,52	0; 4,54	3	53;20,52	-0;41,17	-48	38;57,52
34	13;41,50	0;20,10	-26	56;36,4[6]	0; 4,58	4	52;39,35	-0;42,20	-13	40;28,13
35	14; 2, 0	0;20,11	1	56;41,44	0; 5, 1	3	51;57,15	-0;43, 3	-43	42; 0,45
36	14;22,11	0;20, 0	-11	56;46,45	0; 5, 4	3	51;14,12	-0;28,57	-46	43;35,33
37	14;4[2],11	0;19, 4	-56	56;51,49	0; 5, 6	2	50;45,15	-0;44,43	5,46	45;12,49
38	15; 1,15	0;19, 5	1	56;56,55	0; 5, 8	2	50; 0,32	-0;46,32	-1,49	46;52,38
39	15;20,20	0;18,40	-25	57; 2, 3	0; 5,11	3	49;1[4], 0	-0;47,28	-50	48;35,13
40	15;39, 0	0;18,26	-14	57; 7,14	0; 5,11	0	48;26,32	-0;48, 7	-39	50;20,46
41	15;57,26	0;18, 2	-24	57;12,25	0; 5,13	2	47;38,25	-0;48,53	-46	52; 9,25
42	16;15,28	0;17,28	-34	57;17,[38]	0; 5,13	0	46;49,32	-0;49,35	-42	54; 1,26
43	16;3[2],56	0;17,27	-1	57;22,51	0; 5,14	1	45;[59],57	-0;50,17	-42	55;57, 3
44	16;50;[2]1	0;17,17	-10	57;28, 5	0; 5,14	0	45; [9],40	-0;55,58	-5,41	57;56,32
45	17;[7],38	0;16,53	-24	57;33,19	0; 5,14	-1	44;[13],42	-0;53,36	2,24	60; 0, 0
46	17;24,31	0;16,53	-45	57;38,33	0; 5,13	-1	43;20, 6	-0;51,12	-1,42	62; 7,51
47	17;40,39	0;16,88	18	57;43,4[6]	0; 5,12	-2	42;28,54	-0;52,54	31	64;20,32
48	17;57, 5	0;16,26	-1,3	57;48,58	0; 5,10	-4	41;36, 0	-0;52,23	16	66;38,14
49	18;12,28	0;15,23	9	57;54, 8	0; 5, 6	-1	40;[43],37	-0;52, 7	-1,16	69; 1,21
50	18;28, 0	0;15,32	-32	57;59,14	0; 5, 5	-1	39;51,30	-0;53,23	-39	71;30;18
51	18;43, 0	0;15, 0	-9	58; 4,19	0; 5, 4	-5	38;58, 7	-0;54, 2	-31	74; 5,37
52	18;57,51	0;14,51	-1,9	58; 9.23	0; 4,59	-3	38; 4, 5	-0;54,33	-1,7	76;47,44
53	19;11,33	0;13,42	5	58;14,22	0; 4,56	-2	37; 9,32	-0;55,40	-16	79;37,19
54	19;25,20	0;13,47	-20	58;19,18	0; 4,54	-8	36;[13],52	-0;55,56	0	82;34,59
55	19;38,47	0;13,27	-14	58;24,12	0; 4,46	-1	35;18,56	-0;55,56	-28	85;41,20
56	19;52, 0	0;13,13	-1,0	58;28,58	0; 4,45	-5	34;23, 0	-0;56,24	-27	88;57,14
57	20; 4,[1]3	0;12,13	-17	58;33,43	0; 4,40	-4	33;26,36	-0;56,51	-29	92;23,31
58	20;16, 9	0;11,56	-1	58;38,23	0; 4,36	-7	32;29,45	-0;57,10	-1,8	96; 1,11
59	20;2[8], 4	0;11,55	1,1	58;42,56	0; 4,29	-6	31;32,35	-0;58,18	1,3	99;51,21
60	20;41, 0	0;12,56	-1,52	58;47,25	0; 4,24	-5	30;34,17	-0;57,15	-1,5	103;55,22

of the Ascensions and the Ascendant

	A(f_1)	Δ_1	Δ_2	B(f_2)	Δ_1	Δ_2	C(f_{3a})	Δ_1	Δ_2	D(f_{4b})
61	20;52,4	0;11,4		58;51,49	0;4,24		29;37,2	-0;57,15		108;14,31
62	21;3,0	0;10,56	-8	58;56,6	0;4,18	-5	28;38,42	-0;58,20	-1,5	112;50,38
63	21;13,27	0;10,27	-29	59;0,18	0;4,12	-6	27;40,5	-0;58,37	-17	117;45,25
64	21;23,37	0;10,10	-17	59;4,21	0;4,3	-9	26;41,7	-0;58,58	-21	123;1,6
65	21;33,13	0;9,36	-34	59;8,18	0;3,57	-6	25;41,50	-0;59,17	-19	128;40,16
66	21;42,55	0;9,46	10	59;12,9	0;3,50	-7	24;42,17	-0;59,33	-16	134;45,45
67	21;51,34	0;8,39	-17	59;15,52	0;3,44	-6	23;42,30	-0;59,47	-14	141;21,56
68	22;0,17	0;8,43	4	59;19,27	0;3,35	-9	22;42,25	-1;0,5	-18	148;30,21
69	22;8,29	0;8,12	-31	59;22,54	0;3,27	-8	21;42,5	-1;0,20	-15	156;18,17
70	22;16,19	0;7,50	-22	59;26,13	0;3,19	-8	20;41,35	-1;0,30	-10	164;50,59
71	22;24,0	0;7,41	-9	59;29,24	0;3,11	-8	19;40,45	-1;0,50	-20	174;15,16
72	22;31,0	0;7,0	-41	59;32,27	0;3,3	-8	18;39,57	-1;0,48	2	184;39,40
73	22;37,36	0;6,35	-24	59;35,21	0;2,54	-9	17;38,48	-1;1,9	-21	196;15,4
74	22;43,50	0;6,14	-22	59;38,6	0;2,45	-9	16;37,15	-1;1,33	-24	209;14,43
75	22;49,50	0;6,0	-14	59;40,41	0;2,35	-10	15;35,42	-1;1,33	0	223;55,23
76	22;55,26	0;5,36	-24	59;43,10	0;2,29	-6	14;34,1	-1;1,41	-8	240;38,49
77	23;0,35	0;5,9	-27	59;45,26	0;2,16	-13	13;32,12	-1;1,49	-8	259;53,19
78	23;5,2	0;4,27	-42	59;47,33	0;2,7	-9	12;30,12	-1;2,0	-11	282;16,40
79	23;9,17	0;4,15	-12	59;49,31	0;1,58	-9	11;28,10	-1;2,2	-2	308;40,24
80	23;14,0	0;4,43	28	59;51,20	0;1,49	-9	10;26,0	-1;2,10	-8	340;16,39
81	23;17,24	0;3,24	-1,19	59;52,58	0;1,38	-11	9;23,37	-1;2,23	-13	378;49,32
82	23;20,39	0;3,15	-9	59;54,26	0;1,28	-10	8;21,15	-1;2,22	1	426;55,11
83	23;23,34	0;2,55	-20	59;55,44	0;1,18	-10	7;18,46	-1;2,29	-7	488;39,30
84	23;26,3	0;2,29	-26	59;56,52	0;1,8	-10	6;16,12	-1;2,34	-5	570;51,57
85	23;28;9	0;2,6	-23	59;57,49	0;0,57	-11	5;13,37	-1;2,35	-1	685;48,13
86	23;30,0	0;1,51	-15	59;58,36	0;0,47	-10	4;10,57	-1;2,40	-5	858;2,28
87	23;31,14	0;1,14	-37	59;59,14	0;0,38	-9	3;8,15	-1;2,42	-2	1144;37,41
88	23;32,13	0;0,59	-15	59;59,39	0;0,25	-13	2;5,30	-1;2,45	-3	1718;14,31
89	23;32,48	0;0,35	-24	59;59,54	0;0,15	-10	1;2,45	-1;2,45	0	3437;20,25
90	23;33,0	0;0,12	-23	60;0,0	0;0,6	-9	0;0,0	-1;2,45	0	∞

of the Ascensions and the Ascendant

	A(f_1)	Δ₁	Δ₂	B(f_2)	Δ₁	Δ₂	C(f_{3a})	Δ₁	Δ₂	D(f_{4b})
61	20;52,4	0;11,4		58;51,49	0;4,24		29;37,2	-0;57,15		108;14,31
62	21;3,0	0;10,56	-8	58;56,6	0;4,18	-5	28;38,42	-0;58,20	-1,5	112;50,38
63	21;13,27	0;10,27	-29	59;0,18	0;4,12	-6	27;40,5	-0;58,37	-17	117;45,25
64	21;23,37	0;10,10	-17	59;4,21	0;4,3	-9	26;41,7	-0;58,58	-21	123;1,6
65	21;33,13	0;9,36	-34	59;8,18	0;3,57	-6	25;41,50	-0;59,17	-19	128;40,16
66	21;42,55	0;9,46	10	59;12,9	0;3,50	-7	24;42,17	-0;59,35	-18	134;45,45
67	21;51,34	0;8,39	-1,7	59;15,52	0;3,44	-6	23;42,30	-0;59,47	-14	141;21,56
68	22;0,17	0;8,43	4	59;19,27	0;3,35	-9	22;42,25	-1;0,5	8	148;30,21
69	22;8,29	0;8,12	-31	59;22,54	0;3,27	-8	21;42,5	-1;0,20	-15	156;18,17
70	22;16,19	0;7,50	-22	59;26,13	0;3,19	-8	20;41,35	-1;0,30	-10	164;50,59
71	22;24,0	0;7,41	-9	59;29,24	0;3,11	-8	19;40,45	-1;0,50	-20	174;15,16
72	22;31,0	0;7,0	-41	59;32,27	0;3,3	-8	18;39,57	-1;0,48	2	184;39,40
73	22;37,36	0;6,36	-24	59;35,21	0;2,54	-9	17;38,48	-1;1,9	-21	196;15,4
74	22;43,50	0;6,14	-22	59;38,6	0;2,45	-9	16;37,15	-1;1,33	-24	209;14,43
75	22;49,50	0;6,0	-14	59;40,41	0;2,35	-10	15;35,42	-1;1,55	0	223;55,23
76	22;55,26	0;5,36	-24	59;43,10	0;2,29	-6	14;34,1	-1;1,41	-8	240;38,49
77	23;0,35	0;5,9	-27	59;45,26	0;2,16	-13	13;32,12	-1;1,49	-8	259;53,19
78	23;5,2	0;4,27	-42	59;47,33	0;2,7	-9	12;30,12	-1;2,0	-11	282;16,40
79	23;9,17	0;4,15	-12	59;49,31	0;1,58	-9	11;28,10	-1;2,2	-2	308;40,24
80	23;14,0	0;4,43	28	59;51,20	0;1,49	-9	10;26,0	-1;2,10	-8	340;16,39
81	23;17,24	0;3,24	-1,19	59;52,58	0;1,38	-11	9;23,37	-1;2,23	-13	378;49,32
82	23;20,39	0;3,15	-9	59;54,26	0;1,28	-10	8;21,15	-1;2,22	1	426;55,11
83	23;23,34	0;2,55	-20	59;55,44	0;1,18	-10	7;18,46	-1;2,29	-7	488;39,30
84	23;26,3	0;2,29	-26	59;56,52	0;1,8	-10	6;16,12	-1;2,34	-5	570;51,57
85	23;28,9	0;2,6	-23	59;57,49	0;0,57	-11	5;13,37	-1;2,35	-1	685;48,13
86	23;30,0	0;1,51	-15	59;58,36	0;0,47	-10	4;10,57	-1;2,40	-5	858;2,28
87	23;31,14	0;1,14	-37	59;59,14	0;0,38	-9	3;8,15	-1;2,42	-2	1144;37,41
88	23;32,13	0;0,59	-15	59;59,39	0;0,25	-13	2;5,30	-1;2,45	-3	1718;14,31
89	23;32,48	0;0,35	-24	59;59,54	0;0,15	-10	1;2,45	-1;2,45	0	3437;20,25
90	23;33,0	0;0,12	-23	60;0,0	0;0,6	-9	0;0,0	-1;2,45	0	∞

THE TANGENT TABLE

θ	Tangent	θ	Tangent	θ	Tangent
0° 30'	00;31,26	15 30	16;38,26	30 30	35;20,48
1 00	01; 2,50	16 00	17;12,16	31 00	36; 3, 6
1 30	01;34,16	16 30	17;46,26	31 30	36;46,18
2 00	02; 5,40	17 00	18;20,32	32 00	37;29,32
2 30	02;37,10	17 30	18;55, 6	32 30	38;13,12
3 00	03; 8,40	18 00	19;29,44	33 00	38;57,52
3 30	03;40,12	18 30	20; 7,40	33 30	39;43, 2
4 00	04;11,44	19 00	20;39,34	34 00	40;28,13
4 30	04;43,20	19 30	21;14,54	34 30	41;14,24
5 00	05;14,54	20 00	21;50,16	35 00	42; 0,44
5 30	05;46,20	20 30	22;26, 4	35 30	42;48,13
6 00	06;18,22	21 00	23; 1,54	36 00	43;35,32
6 30	06;49,32	21 30	23;38,10	36 30	44;25,10
7 00	07;22, 2	22 00	24;14, 6	37 00	45;12,45
7 30	07;58,40	22 30	24;51,18	37 30	46; 2,42
8 00	08;25,56	23 00	25;28, 8	38 00	46;52,38
8 30	08;57,44	23 30	26; 5,24	38 30	47;43,48
9 00	09;30,18	24 00	26;42,48	39 00	48;35,12
9 30	10; 2,30	24 30	27;20, 0	39 30	49;27, 8
10 00	10;34,46	25 00	27;58,42	40 00	50;20,46
10 30	11; 7,16	25 30	28;37,16	40 30	51;15, 6
11 00	11;39,46	26 00	29;15,50	41 00	52; 9,24
11 30	12;12,30	26 30	29;55, 4	41 30	53; 2,24
12 00	12;45,12	27 00	30;34,17	42 00	54; 1,26
12 30	13;18, 8	27 30	31;15,46	42 30	55; 3,24
13 00	13;51, 6	28 00	31;56,44	43 00	55;57,22
13 30	14;24, 4	28 30	32;35,54	43 30	56;58,00
14 00	14;57,34	29 00	33;14,30	44 00	57;56,32
14 30	15;31, 4	29 30	33;57,00	44 30	58;58,16
15 00	16; 4,36	30 00	34;38,28	45 00	60;00,00

THE TANGENT TABLE

θ	Tangent	θ	Tangent	θ	Tangent
45° 30'	61; 3 ,14	60 30	106; 2 ,20	75 30	232; 0 , 6
46 00	62; 6 ,50	61 00	108;14 ,00	76 00	240;39,00
46 30	63;11,50	61 30	110;30,22	76 30	250;23, 8
47 00	64;10,32	62 00	112;49,20	77 00	259;53,30
47 30	65;32,00	62 30	115;16,00	77 30	270;38,20
48 00	66;24,15	63 00	117;43,20	78 00	282;16,20
48 30	67;38,20	63 30	120;20,30	78 30	294;54,34
49 00	68;57,20	64 00	122;59,30	79 00	308;40,30
49 30	70;36,20	64 30	125;47,30	79 30	328;49,00
50 00	71;30,12	65 00	128;38,50	80 00	340;16,20
50 30	72;47,54	65 30	131;39,00	80 30	358;32,40
51 00	74; 5 ,36	66 00	134;45,42	81 00	378;50,00
51 30	75;26,38	66 30	138; 0 ,00	81 30	401;28,00
52 00	76;47,44	67 00	141; 19, 8	82 00	426;56,00
52 30	78;11,46	67 30	144;50,56	82 30	455;45,00
53 00	79;37,18	68 00	148;28,34	83 00	488;39,00
53 30	81; 6 , 8	68 30	152;19,16	83 30	526;37,00
54 00	82;35,00	69 00	156;20,12	84 00	570;52,00
54 30	84; 8 , 8	69 30	160;28,30	84 30	623; 8 ,00
55 00	85;41,20	70 00	164;51,00	85 00	685;48,00
55 30	87;18,24	70 30	169;24,20	85 30	723;24,00
56 00	88;57,14	71 00	174;15,14	86 00	858; 1 ,00
56 30	90;40,22	71 30	179;19,16	86 30	986; 00,00
57 00	92;23,30	72 00	184;40,00	87 00	1144;12,00
57 30	94;12,20	72 30	190;18,00	87 30	1374;12,00
58 00	96; 1 ,10	73 00	196;15,00	88 00	1718;14,00
58 30	97;54,40	73 30	202;33,20	88 30	2291;20,00
59 00	99;52,00	74 00	209;14,00	89 00	3436;18,00
59 30	101;52,00	74 30	216;20,00	89 30	
60 00	103;55,24	75 00	223;57,00	90 00	

17. The Functions

In the sources, the functions whose tables make up the "Table of Rectification" are referred to as <u>the first</u> (al-awwal), <u>the second</u>, and so on, or denoted by the Arabic letters corresponding to A, B, J, and D. In accordance with this we denote these functions by f_1, f_2, f_3, and f_4. For the function referred to in N as the third, two non-equivalent definitions are given. We distinguish between them by the symbols f_{3a} and f_{3b}. The fourth function is defined in N in one way only. Moreover the tables corresponding to it in N, HI and B have been computed on the basis of this definition. In another part of HI, however, the fourth function tabulated is not identical with the above. We call these two functions f_{4a} and f_{4b} respectively.

When necessary we will use the additional subscripts N, HI and HB (Cf. Sections 9, 10, 11) to denote the source of a particular table. The subscripts HI_1 and HI_2 will distinguish respectively between the two places in the Damascene Zīj, folios 147r - 148v, and folios 226r - 227r, where the two versions of the "Table of Rectification" appear.

(a) The <u>first function</u>, f_1, is as Abū Naṣr mentions, the second declination[98] (∂_2). We obtain an expression for f_1 in the ordinary trigonometric functions by applying the spherical right triangle formula

98. N, p. 3.

$$\sin b = \cot A \cdot \tan a,$$

where $b = \theta$ $A = \epsilon$, $a = \partial_2$. We find

$$\sin \theta = \cot \epsilon \cdot \tan \partial_2(\theta)$$

or

$$\tan \partial_2 (\theta) = \tan \epsilon \cdot \sin \theta$$

or

$$\partial_2(\theta) = \tan^{-1} (\tan \epsilon \cdot \sin \theta) = f_1(\theta)$$

If cap functions are used, then

$$\partial_2(\theta) = \mathrm{Tan}^{-1} \left(\frac{\mathrm{Tan}\, \epsilon \cdot \mathrm{Sin}\, \theta}{R}\right) = f_1(\theta)$$

This function is used in N, HI_1 and HI_2, but not in HB.

(b) The <u>second function</u> (f_2) is also defined by Abū Naṣr[99] as <u>jaib tamām mayl tamām al-daraja</u> (Cos $[\partial(\bar{\theta})]$). By using the sine law, we have

$$\frac{\mathrm{Sin}\, \partial(x)}{\mathrm{Sin}\, \epsilon} = \frac{\mathrm{Sin}\, x}{\mathrm{Sin}\, 90^\circ}$$

or

$$\mathrm{Sin}\, \partial(x) = \frac{\mathrm{Sin}\, x \cdot \mathrm{Sin}\, \epsilon}{R}$$

Substituting $\bar{\theta} = x$, we have

$$\mathrm{Sin}\, \partial(\bar{\theta}) = \frac{\mathrm{Sin}\, \bar{\theta} \cdot \mathrm{Sin}\, \epsilon}{R} = \frac{\mathrm{Cos}\, \theta \cdot \mathrm{Sin}\, \epsilon}{R}$$

or

$$\partial(\bar{\theta}) = \mathrm{Sin}^{-1}\left(\frac{\mathrm{Cos}\, \theta \cdot \mathrm{Sin}\, \epsilon}{R}\right)$$

and

$$\mathrm{Cos}\, \partial(\bar{\theta}) = \mathrm{Cos}\, \left[\mathrm{Sin}^{-1}\left(\frac{\mathrm{Cos}\, \theta \cdot \mathrm{Sin}\, \epsilon}{R}\right)\right] = f_2(\theta)$$

---

99. N, p. 3.

(c) We observe in N, (10:4) that $f_{3a}(\theta)$ is defined as

$$\frac{R \operatorname{Cos} \theta}{\operatorname{Cos} \partial (\bar{\theta})}$$

From the foregoing this is seen to be the same as $\frac{R \operatorname{Cos} \theta}{f_2(\theta)}$.

In another place (N, 69:10) the same function is defined as

$$\frac{\operatorname{Cos} \theta \cdot \operatorname{Cos} f_1 (\theta)}{\operatorname{Cos} \epsilon}$$

These two forms are equivalent. For it is proved in N (68:11) that

$$\frac{R \operatorname{Cos} \epsilon}{\operatorname{Cos} f_1 (\theta)} = f_2 (\theta).$$

By substitution, we have

$$\frac{R \operatorname{Cos} \theta}{\operatorname{Cos} \partial (\bar{\theta})} = \frac{R \operatorname{Cos} \theta}{f_2 (\theta)} = \frac{R \operatorname{Cos} \theta \cdot \operatorname{Cos} f_1 (\theta)}{R \operatorname{Cos} \epsilon}$$

$$= \frac{\operatorname{Cos} \theta \cdot \operatorname{Cos} f_1 (\theta)}{\operatorname{Cos} \epsilon}$$

f_{3a} was used by Abū Naṣr in proving problems 9:18; 16:1; 19:9; 19:11; 23:8; 63:6; 69:7.

In N (68:3) a totally different definition for the third function is given by Abū Naṣr, as

$$f_{3b} = \frac{R \operatorname{Cos} \theta}{\operatorname{Cos} \epsilon}$$

Al-Khāzin used another definition of f_{3b},

$$\frac{R \cdot \operatorname{Sin} \partial (\bar{\theta})}{\operatorname{Cos} \epsilon \cdot \operatorname{Sin} \epsilon} .$$

This form is easily changed to

$$\frac{R \operatorname{Cos} \theta}{\operatorname{Cos} \epsilon}$$

as is explained in N (67:15).

f_{3b} is merely a constant multiple of Cos θ, while f_{3a} can be expressed as a fraction whose numerator is indeed Cos θ, but whose denominator varies with θ.

Although f_{3a} and f_{3b} are not equivalent, we note that they are close to each other, coinciding at the end-points $\theta = 0$ and $\theta = 90°$. To evaluate their difference, put

$$f_{3a}(\theta) - f_{3b}(\theta) \equiv d_1(\theta)$$

or by substitution

$$\frac{\cos\theta}{\operatorname{Cos}[\sin^{-1}(\sin\epsilon\cos\theta)]} - \frac{\cos\theta}{\cos\epsilon} = \frac{d_1(\theta)}{R} = d(\theta):$$

To find the maximum value of d, we differentiate and set the derivative equal to zero. This gives

$$(\sin^4\epsilon - \sin^6\epsilon)\cos^4\theta + (3\sin^2\epsilon - 2\sin^4\epsilon)\cos^2\theta - \sin^2\epsilon = 0.$$

Solving for $\cos^2\theta$, we have

$$\cos^2\theta = \frac{2\sin^2\epsilon - 3 \pm \sqrt{9 - 8\sin^2\epsilon}}{1/2\sin^2 2\epsilon}$$

If we consider $\sin\epsilon \approx 0.4$[100], we have

$$\cos^2\theta = 0.36686,$$

or $$\theta = 52;43°.$$

At this angle

$$f_{3a} - f_{3b} = R(.6243 - .6609) = -0.0366R,$$

so the maximum difference is $-2;11,52°$.

Hence the percentage of maximum difference between f_{3a} and f_{3b} is

$$5.86\% \text{ at } \theta \approx 52°43'.$$

---

100. N, p. 60:13.

The table on the following page will show a comparison between these two functions, using $\epsilon = 23,35°$.

We will show in the following that the responsibility for thus treating two different functions as though they were one and the same must be shared by both Ḥabash and Abū Naṣr.

Although it is true that Ḥabash, to the best of our knowledge, neither defines any functions, nor anywhere tabulates f_{3b}, he implicitly uses both f_{3a} and f_{3b}. For, in a number of problems his directions implicitly demand f_{3a}, while in another problem his solution will be incorrect unless f_{3b} is used.

Abū Jaʿfar al-Khāzin defined the third function as f_{3b}, but we cannot be sure that he confused it with f_{3a}.

Abū Naṣr clearly defined both functions and he fails to point out that they are different, although this fact should have been well understood by a mathematician of his ability.

Apparently Ḥabash was in need of f_{3b} to solve some problems, and because of the close values of f_{3a} and f_{3b}, he thought, without careful investigation, that the two functions were identical.

(d) The following relations between the first three functions may be of interest. In N (68:11) it was proved that

$$\cos f_1(\theta) \cdot f_2(\theta) = R \cos \epsilon \text{, a constant}$$

Solving for $\cos f_1(\theta)$, we have

$$\cos f_1(\theta) = \frac{R \cos \epsilon}{f_2(\theta)}.$$

	(1) The Text	(2) Computed as f_{3a}	(3) Computed as f_{3b}	(1) - (2)	% Difference	(1) - (3)	% Differences
1°	65;26,13	65;27,25	65;27,27	-0;1,12	0.03%	-0;1,14	0.03%
15	62;50,13	62;50,24	63;14,13	-0;0,11	0.004%	-0;24,0	0.5%
30	55;23,33	55;23,34	56;41,49	-0;0,1	0.0005%	-1;8,16	2%
45	44;14,9	44;14,1	46;17,31	+0;0,8	0.005%	-2;3,22	4.6%
53	37;11,43	37;27,29	39;38,14	-0;15,46	0.66%	-2;26,31	6.5%
60	30;35,10	30;37,7	32;44,2	-0;1,57	0.106%	-1;8,52	3.7%
75	15;36,48	15;36,47	16;56,41	+0;0,1	0.002%	-1;19,53	9.6%
90	0	0	0	0	0%	0	0%

But it was shown above that

$$f_{3a}(\theta) = \frac{\cos\theta \cdot \cos f_1(\theta)}{\cos\epsilon}$$

or

$$\cos f_1(\theta) = \frac{f_{3a}(\theta) \cdot \cos\epsilon}{\cos\theta}$$

$\therefore$

$$\frac{R\cos\epsilon}{f_2(\theta)} = \frac{f_{3a}(\theta) \cdot \cos\epsilon}{\cos\theta} .$$

Therefore

$$f_{3a}(\theta) \cdot f_2(\theta) = R\cos\theta,$$

or, by substitution

$$\frac{\cos f_1(\theta)}{f_{3a}(\theta)} = \frac{\cos\epsilon}{\cos\theta}$$

(e) The fourth function is, as defined by Abū Naṣr (N, 3:9)

$$f_{4a},\ N,\ HI,\ B = \frac{\sin\theta}{\cos\theta} \cdot \mathrm{Sin}\,\epsilon$$

$$= \frac{\mathrm{Tan}\,\theta \cdot \mathrm{Sin}\,\epsilon}{R}$$

But we find in HI_2 (folios 226-227) that instead of f_{4a}, a table of Tan θ is found, which, because of its position, we call f_{4b}. The latter was compared with the tangent table of Ulugh Beg's[101], and the difference in most cases is negligible.

18. Accuracy

The chart on page 75 shows the degree of accuracy attained in the sources. Check computations have been made

101. Bodleian Library, Oxford, Arabic Ms. LXX (Pocock 226).

SPOT CHECK CHART OF TABULAR VALUES

θ	ε	$f_1(\theta)$ SOURCES	$f_1(\theta)$ VALUES	$f_2(\theta)$ SOURCES	$f_2(\theta)$ VALUES	$f_{3a}(\theta)$ SOURCES	$f_{3a}(\theta)$ VALUES	$f_{4a}(\theta)$ SOURCES	$f_{4a}(\theta)$ VALUES	
1°	23;35°	N, HI_1	0;26,10 0;26,11 0.064	N, HI_1	54;59,23 54;59,25 0.001	N, HI_1, HB	65;26,13 65;27,25 0.030	N, HI_1, HB	0;25,8 0;25,8 0	(1) (2) (3)
	23;33°	HI_2	0;26,[10] 0;26,9 -0.064	HI_2, HB	55;0,15 55;0,15 0	HI_2	65;22,2 65;26,22 0.110			
15°	23;35	N, HI_1	6;26,40 6;26,46 0.026	N, HI_1	55;20,9 55;20,19 0.005	N, HI_1, HB	62;50,13 62;50,24 0.004	N, HI_1, HB	6;25,[5]16 6;25,59 0.013	
	23;33	HI_2	6;26,14 6;26,12 -0.009	HI_2, HB	55;21,6 55;21,6 0	HI_2	62;45,8 62;49,20 0.111			
30°	23;35	N, HI_1	12;18,36 12;18,46 0.023	N, HI_1	56;17,2 56;17,0 -0.001	N, HI_1, HB	55;23,33 55;23,34 0.0005	N, HI_1, HB	13;51,32 13;51,33 0.002	
	23;33	HI_2	12;17,36 12;17,48 0.027	HI_2, HB	56;17,38 56;17,37 -0.0005	HI_2	55;19,12 55;22,56 0.112			
45°	23;35	HI_1	17;9,3 17;9,16 0.021	HI_1	57;32,56 57;32,56 0	HI_1, HB	44;14,9 44;14,1 -0.005	HI_1, HB	24;0,17 24;0,18 0.001	
	23;33	HI_2	17;[7],38 17;7,34 0.009	HI_2, HB	57;33,19 57;33,20 0.0005	HI_2	44;1[3],42 44;13,42 0			
60°	23;35	HI_1	20;42,50 20;42,34 -0.021	HI_1	58;47,14 58;47,14 0	HI_1, HB	30;37,10 30;37,7 0.002	HI_1, HB	41;34,40 41;34,43 0.002	
	23;33	HI_2	20;41,0 20,40,46 -0.019	HI_2, HB	58;47,25 58;47,26 0.0005	HI_2	30;34,17 30;37,1 0.148			
75°	23;35	N, HI_1	22;51,59 22;51,49 -0.012	N, HI_1	59;40,38 59;40,38 0	N, HI_1, HB	15;36,48 15;36,47 -0.002	N, HI_1, HB	89;35,15 89;35,16 0.0003	
	23;33	HI_2	22;49,50 22;49,51 0.001	HI_2, HB	59;40,41 59;40,41 0	HI_2	15;35,42 15;36,46 0.114			
90°	23;35	N, HI_1	23;35	N, HI_1	60;0,0 60;0,0 0	N, HI_1, HB	0 0	N, HI_1, HB	∞ ∞	
	23;33	HI_2	23;33	HI_2, HB	60;0,0 60;0,0 0	HI_2	0 0 0			

1. TEXT. 2. CHECK COMPUTATION. 3. % DIFFERENCE.

by the present author for the values of the argument θ shown in the first column.

The second column records the value of the inclination of the ecliptic (ϵ), used in computing the tabular values in that row. Two different ϵ's are used in the versions of these tables: 23;35° and 23;33°. The first, that of Theon of Alexandria and accepted by most Moslem astronomers[102], was used by Ḥabash throughout in computing the first version of the "Table of Rectification" (HI_1) which is found in his Damascene Zīj. It is easy to show that there 23;35° was used in f_1 if one puts $\theta = 90°$. Then

$$f_1(90°) = \mathrm{Tan}^{-1}(R \cdot \tan\epsilon) = \epsilon$$

or

$$\delta_2(90°) = \epsilon .$$

And in fact the corresponding value of f_1 in the text is 23;35°. The use of $\quad$ = 23;35° in the other functions of HI_1 was shown by direct substitution in the course of computing the chart.

A second version of the "Table of Rectification", which we call HI_2, appears also in the Damascene Zīj. The value of ϵ used in computing these tables is 23;33°. Ibn Yūnis mentions in his Ḥākimite Zīj[103] that this latter parameter was found by al-Ma'mūn's astronomers; so it is not surprising that it should have been known to Ḥabash. What is more to the point, however, is that in another place[104] Ibn

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102. E.G., al-Battānī and Ibn Yūnis. It is stated in the Sanjarī Zīj (Vatican, Arabic Ms. 761) that those who +

Yūnis says "...Ḥabash.....fixed it (ϵ) in his Mumtaḥan Zīj, which he called 'The Canon', in two places: in the declination table as 23;35°, but in the 'Table of Rectification' as 23;33° .....". In the Damascene Zīj itself no explanation is given as to why the same tables should appear twice but with different ϵ. At least two conjectures may be made. The first is that perhaps Habash first computed the "Table of Rectification" (HI_1) when he was acquainted only with the old value, ϵ = 23;35°. However, when he acquired (from the Ma'mūnic observations) the second determination (ϵ = 23;33°), he computed the second version (HI_2). A second possibility is that Ḥabash composed his two versions of the "Table of Rectification" in order to offer users of his zīj a choice between the two determinations of the parameter. Knowing that these tables do not appear adjoining each other in the Damascene Zīj, we infer that the first possibility is the more acceptable.

In HB, which contains only f_2, f_3 and f_{4a}, we find that while ϵ = 23;35° was used in computing f_3 and f_{4a}, the second value (ϵ = 23;33°) was used for f_2. This is further evidence that HB was put together from different sources.

Having disposed of the first two columns of the chart, the reader should note that the remainder is divided into four

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wrote the Mumtaḥan Zīj used this value of ϵ.

103. Caussin, p. 56.

104. P. 223 line 10 of (Leiden) Cod. Or. 143, Bibl. Acad., Lugduno-Batava.

main columns for each function. Each main column is in turn divided into two sub-columns, the first for noting the sources, the second for comparisons of tabular values. For the latter purpose, three numbers appear in each rectangle: (1) the value as given in the sources, followed by (2) our computed value and then by (3) the percentage difference between (1) and (2).

It should be observed that in general the tabulated values differ from the accurate values in at most the third sexagesimal place. In f_1 and for the given arguments the maximum error detected is only 0;0,21 at $\theta = 45°$. For f_2 the error is even less than with f_1, nowhere exceeding 0;0,10. In most cases, in fact, the error is zero. The error in the fourth function is also very small, and it does not exceed 0;0,3.

The only function which exhibits an unusual error is f_{3a} for ϵ = 23;33° (HI_2). The possible reason for such large errors in HI_2 is that a table of f_{3a} for ϵ = 23;33° is found only once. If two or more manuscript copies had been available it would have been easy to compare the values in the different versions for restoration. The tables which are identical are:

$$f_2\ HI_2 = f_2\ HB$$
$$f_{3a}\ HI_1 = f_3\ HB$$
$$f_{4a}\ HI_1 = f_{4a}\ HB,$$

while

$$f\ N = f\ HI_1$$

for all the functions.

CHAPTER V

THE PROBLEMS

19. Notations and Conventions

This chapter is devoted to the discussion of the astronomical problems appearing in N and solvable by application of the "Table of Rectification". Although such a discussion was the main object of Abū Naṣr in writing N, nevertheless problems occur in N in which this table was not used. Examples are the problems appearing in Sections 29, 32, 33, and 34 below.

In addition to problems solved by Ḥabash we find in N problems originating from al-Nairīzī, Abū Jaʿfar, and Abū Naṣr himself.

The following remarks may help the reader in following the exposition below:

(a) It is to be understood throughout that all sequences of operations are attributed to Ḥabash and all proofs to Abū Naṣr unless there is a statement to the contrary.

(b) Anything in square brackets is a restoration of the text; material in round brackets indicates clarifying additions made by the present writer.

(c) Note that the original has no symbolism, and all operations are written out as verbal statements. But for the sake of compactness, we use modern symbols.

(d) The notation 6:4, for example, means that the statement is given on page 6, line 4 of the printed text N. This same notation may refer also to the statement or expression itself.

(e) As noted in Section 15 above, operations are indicated by the superscript -1. Thus if α is the right ascension of an ecliptic point of longitude λ we may write $\alpha = A_o(\lambda)$ or equivalently $\lambda = A_o^{-1}(\alpha)$. The medieval Arabic idiom for performing such an inverse operation is qawwasa (qaws = arc) i.e., to find the corresponding arc.

(f) The symbol $\bar{\theta}$ means the complementary angle or arc of θ, i.e., $90° - \theta$, and therefore $\sin \bar{\theta} = \cos \theta$, etc.

(g) Note our use of the notation

$$\frac{a}{b} = \frac{c = e}{d = f},$$

to express such parenthetical statements as " a is to b as c (which is equal to e) is to d (which is equal to f)". This type of expression is very common in the text and in medieval Islamic mathematics generally.

(h) For the convenience of the reader we give below the standard symbols used for various technical terms:

α = Right ascension

β = Latitude of the star

D = Arc of daylight

d = Equation of daylight

δ = Declination

δ_2 = Second declination

ϵ = Inclination of the ecliptic

φ = Terrestrial latitude

λ = Longitude of the star

M = Upper midheaven

The accompanying figure illustrates some of these terms as well as those listed below.

A = East point

CD = Small circle parallel to the celestial equator

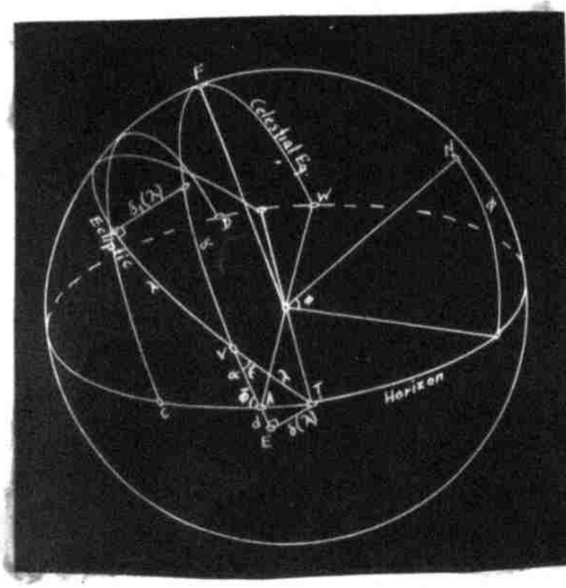
N = North pole of the celestial equator

Ṭ = Ascendant

ṬA = Ortive amplitude

V = Vernal equinoctial point

W = West point



20. Knowledge of the Arc of Daylight

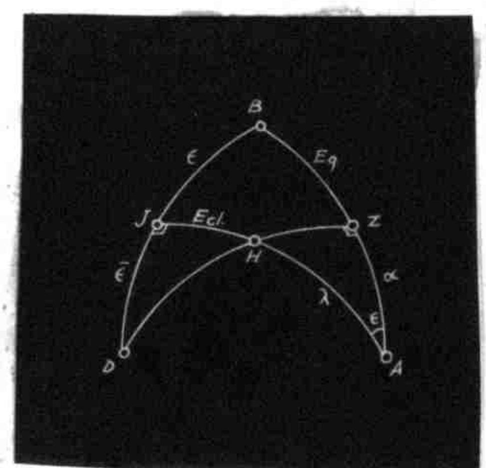
(The methods of solution do not yield D directly; they give d. But since D = 180° + 2d, the one leads immediately to the other.

Abū Naṣr's object in this whole section is to prove Ḥabash's statement

3:16 $$d = [\mathrm{Sin}^{-1}](f_{3b}(\bar{\alpha})\cdot f_{4a}(\varphi)).$$

He first proves

4:5 $$\frac{\mathrm{Sin}\,\lambda}{\mathrm{Sin}\,\alpha} = \frac{\mathrm{Cos}\,\delta\,(\lambda)}{\mathrm{Cos}\,\epsilon}$$



He next obtains an expression for d in terms of the trigonometric functions of $\delta(\lambda)$ and φ, namely

4:19 $$\frac{\frac{\sin\delta(\lambda)\cdot\sin\varphi}{\cos\varphi}}{\cos\delta(\lambda)} = \frac{\sin d}{R}\ .$$

Thirdly he shows, by two independent methods that

6:3 $$\frac{\sin \max d}{\sin d} = \frac{R}{\sin\alpha}$$

By manipulating these three expressions and applying the definitions of f_2 and f_4 he obtains the desired)

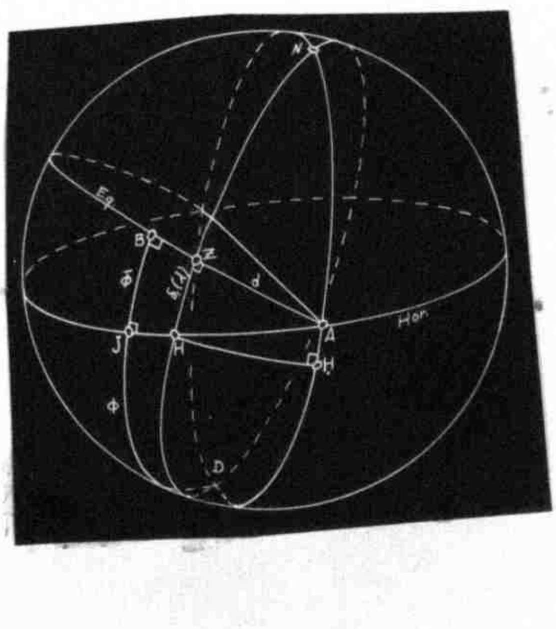
3:16 $$d = \sin^{-1}\left(f_3(\bar{\alpha})\cdot f_4(\varphi)\right).$$

(The text has $d = f_3(\bar{\alpha})\cdot f_4(\varphi)$, an obvious error. All the following statements are due to Abū Naṣr).

4:5 $$\frac{\sin\lambda}{\sin\alpha} = \frac{\cos\delta(\lambda)}{\cos\epsilon}.$$

(For)

4:15 $$\frac{\sin AH}{\sin AZ} = \frac{\sin DH}{\sin JD},$$



(by the Rule of Four Quantities (Cf. Section 38 below). Substituting λ for AH, α for AZ, $\delta(\lambda)$ = HZ for DH, and $\bar{\epsilon}$ for JD, we have

$$\frac{\sin\lambda}{\sin\alpha} = \frac{\overline{\sin\delta(\lambda)}}{\sin\bar{\epsilon}},$$

which is equivalent to (4:5)).

(Abū Naṣr now writes)

$$4{:}19 \quad \frac{\dfrac{\operatorname{Sin} \delta(\lambda) \cdot \operatorname{Sin}(\varphi)}{\operatorname{Cos}(\varphi)}}{\operatorname{Cos} \delta(\lambda)} = \frac{\operatorname{Sin}(d)}{R}.$$

(To prove this he puts)

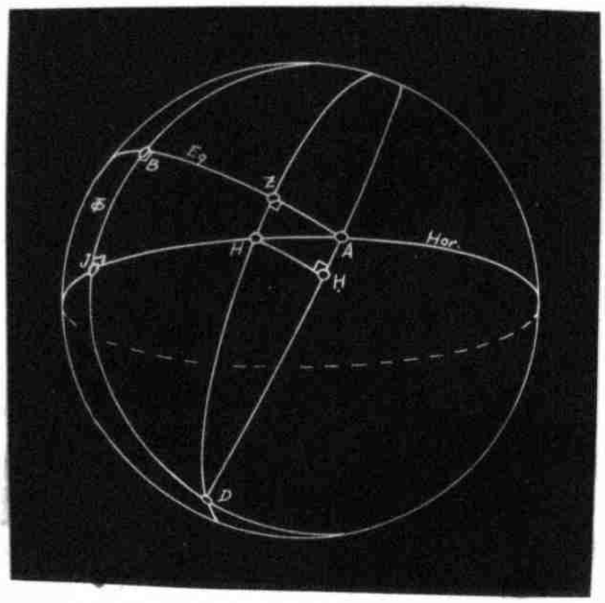
$$5{:}11 \quad \frac{\operatorname{Sin} HZ}{\operatorname{Sin} AH} = \frac{\operatorname{Sin} BJ}{R},$$

(true by the Rule of Four Quantities. Then)

$$5{:}12 \quad \frac{\operatorname{Sin} AH}{\operatorname{Sin} H\underset{\cdot}{H}} = \frac{R = \operatorname{Sin} \underset{\cdot}{H}}{\operatorname{Sin} DAH = \operatorname{Sin} \varphi},$$

(by the Sine Law. Now by multiplication of 5:11 and 5:12)

$$5{:}15 \quad \frac{\operatorname{Sin} HZ}{\operatorname{Sin} H\underset{\cdot}{H}} = \frac{\operatorname{Sin} BJ}{\operatorname{Sin} JD}$$



(solving this for Sin $H\underset{\cdot}{H}$),

$$5{:}16 \quad \operatorname{Sin} HH = \frac{\operatorname{Sin} HZ \cdot \operatorname{Sin} JD}{\operatorname{Sin} BJ}$$

and by the Rule of Four

$$5{:}17 \quad \frac{\operatorname{Sin} H\underset{\cdot}{H}}{\operatorname{Sin} HD} = \frac{\operatorname{Sin} AZ}{\operatorname{Sin} ZD}$$

Now $HD = \delta(\bar{\lambda})$ and Sin ZD = R, (whence 4:19 follows, for, from 5:16 and 5:17 we get

$$\frac{\operatorname{Sin} H\underset{\cdot}{H}}{\operatorname{Sin} HD} = \frac{\dfrac{\operatorname{Sin} HZ \cdot \operatorname{Sin} JD}{\operatorname{Sin} BJ}}{\operatorname{Sin} HD} = \frac{\operatorname{Sin} AZ}{\operatorname{Sin} ZD},$$

whence by substitution from the figure,

$$\frac{\dfrac{\operatorname{Sin} \delta(\lambda) \cdot \operatorname{Sin} \varphi}{\operatorname{Sin} \bar{\varphi}}}{\operatorname{Sin} \delta_1(\lambda)} = \frac{\operatorname{Sin} d}{R}$$

which is equivalent to 4:19. This is indeed an expression for d, and immediately reduces to

$$\sin d = \tan \delta \cdot \tan \phi$$

a classical expression (Cf., for example, al-Khwārizmī and Nallino (2)).

(At this stage Abū Naṣr digresses to prove 6:3.

By using the formula

$$\sin b = \cot A \cdot \tan a$$

we have

$$\tan \epsilon \cdot \sin \alpha = \tan \delta$$

Therefore

$$\sin d = \tan \phi \tan \epsilon \cdot \sin \alpha$$

another expression for d.)

6:3 $\frac{\text{Sin max } d}{\text{Sin } d} = \frac{R}{\text{Sin } \alpha}$

(That is)

6:12 $\frac{\text{Sin } AD}{\text{Sin } AS} = \frac{R}{\text{Sin } (B'S = \alpha)}$

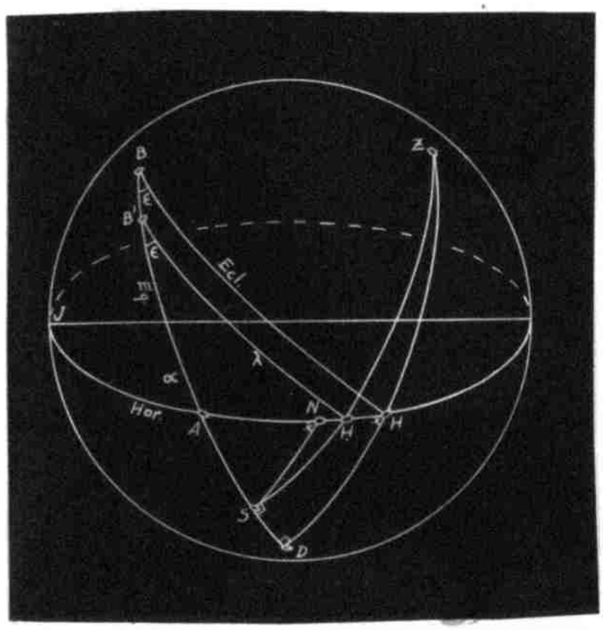
We make SNA = DHA,

(and by the Sine Law for oblique triangles:)

6:16 $\frac{\text{Sin } DH}{\text{Sin } SN} = \frac{\text{Sin } DA}{\text{Sin } AS}$

because HAD is common to the two triangles (ADH and SNA).

(By the Rule of Four:)



6:18 $$\frac{\text{Sin DH}}{\text{Sin SḤ}} = \frac{R}{\text{Sin } (\text{BḤ} = \lambda)},$$

(applied to the triangles BDH and B'SḤ).

6:19 $$\text{DH} = \epsilon, \quad \text{SḤ} = \delta_1(\text{B'H} = \lambda).$$

(Applying the Rule of Four to the triangles ḤSN and ḤZH, since)

$$\text{SNḤ} = \text{ḤHZ}$$

7:2 $$\frac{\text{Sin SḤ}}{\text{Sin SN}} = \frac{\text{Sin ḤZ}}{\text{Sin [Z]H}}.$$

(Also:)

7:5 $$\frac{\text{Sin [ḤZ]}}{\text{Sin [Z]H}} = \frac{\text{Sin } \lambda}{\text{Sin } \alpha}.$$

(For, from (4:5): $\frac{\text{Sin}\lambda}{\text{Sin}\,\alpha} = \frac{\text{Cos } \delta_1(\lambda)}{\text{Cos } \epsilon} = \frac{\text{Cos SḤ}}{\text{Cos DH}} = \frac{\text{Sin ḤZ}}{\text{Sin ZH}}$)

(By multiplying the left-hand sides of 6:18 and 7:2 and their right-hand sides, and remembering that the right-hand side of 7:2 equals the right-hand side of 7:5. we get 6:12, which is equivalent to 6:3.)

(Abū Naṣr gives here another method of proving 6:3:)

7:14 We make HAḤ = $\bar{\Phi}$, ḤL = JH, ZṬ = BD.

7:18 (It is necessary to show that) AṬ = d(B)

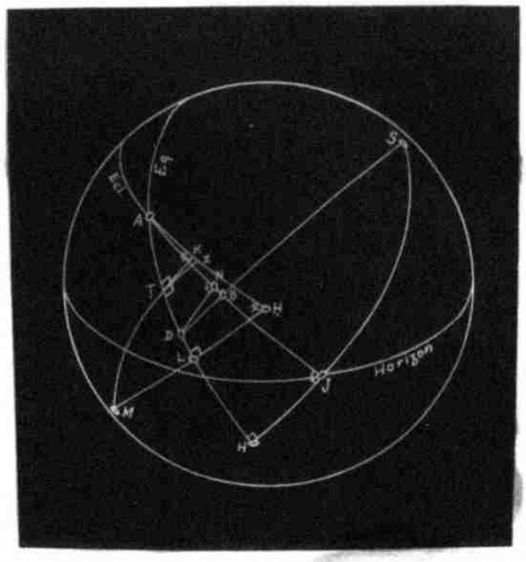
A[L] = d(J)

7:19 $$\frac{\text{Sin AL}}{\text{Sin AṬ}} = \frac{\text{Sin HA}}{\text{Sin AD}}$$

7:20 We make AND = AJH,

AKṬ = AḤL

8:2 We produce HJ and DB to meet at S. Similarly LḤ and ṬZ meet at M.



8:4 Because SJ = MḤ

and SB = MZ

(for SH = ML = 90°, and ḤL = JH. Similarly SB = MZ)

8:5 $$\frac{\text{Sin SJB}}{\text{Sin SBJ}} = \frac{\text{Sin MHZ}}{\text{Sin MZḤ}},$$

(by application of the Sine Law to triangles SBJ and MḤZ and application of 8:4)

8:7 $$\frac{\text{Sin SJB}}{\text{Sin SBJ}} = \frac{\text{Sin BD}}{\text{Sin DN}},$$

and

$$\frac{\text{Sin MḤZ}}{\text{Sin MZH}} = \frac{\text{Sin ZṬ}}{\text{Sin ṬK}}$$

(For in the pairs of triangles SJB and BDN, MḤZ and ZKṬ, two pairs of angles are equal in each.)

8:10 But it is given that ZṬ = BD,

(then) DN = ṬK.

8:11 $$\frac{\text{Sin JH}}{\text{Sin DN}} = \frac{\text{Sin LḤ}}{\text{Sin ṬK}}$$

(for JH = LḤ and DN = ṬK).

8:12 $$\frac{\text{Sin JH}}{\text{Sin DN}} = \frac{\text{Sin HA}}{\text{Sin A[D]}},$$

(by application of the Rule of Four to triangles AJH and ADN which have A in common, and J = N. Similarly)

8:13 $$\frac{\text{Sin } L\d{H}}{\text{Sin } \d{T}K} = \frac{\text{Sin } AL}{\text{Sin } A\d{T}}$$

8:14 Then $$\frac{\text{Sin } HA}{\text{Sin } AD} = \frac{\text{Sin } AL}{\text{Sin } A\d{T}}$$

(by application of 8:10, 8:12, and 8:13. This, says Abū Naṣr, is what we wanted to prove, presumably 6:3, which indeed follows from 8:14, provided we make HA a quadrant).

8:16 $$f_4(\varphi) = \frac{\text{Sin } \epsilon . \text{Sin } \varphi}{\text{Cos } \varphi}$$

8:17 $$\frac{\text{Sin } \varepsilon . \text{ Sin } \varphi}{\text{Cos } \varphi} \cdot \frac{R}{\text{Cos } \varepsilon} = \text{Sin max d}$$

(For in 4:19 $\text{Sin d} = \frac{\text{Sin } \partial(\lambda) \cdot \text{Sin } \varphi}{\text{Cos } \varphi} \cdot \frac{R}{\text{Cos } \delta(\lambda)}$,

but when d is a maximum, δ becomes ε, hence 8:17).

9:1 $$\frac{\text{Sin max d}}{\text{Sin d}} = \frac{R}{\text{Sin } \alpha},$$

(as proved in 6:3)

9:4 $$\frac{\text{Sin } \epsilon . \text{ Sin } \varphi}{\text{Cos } \varphi} \cdot \frac{\text{Sin } \alpha}{\text{Cos } \epsilon} = \text{Sin d},$$

(For $\text{Sin d} = \frac{\text{Sin } \alpha . \text{ Sin max d}}{R}$ in 9:1, and, substituting for Sin max d from 8:17, we get 9:4).

9:9 $$\frac{\text{Sin } \alpha}{\text{Cos } \epsilon} = \frac{\text{Sin } \lambda}{\text{Cos } \delta},$$

(as proved in 4:5).

9:11 The work of Ḥabash in finding the arc of daylight is correct and clear.

(For from 9:4 $\frac{\text{Sin } \epsilon . \text{ Sin } \varphi}{\text{Cos } \varphi} = f_4(\varphi)$; $\frac{\text{Sin } \alpha}{\text{Cos } \epsilon} = \frac{\text{Cos } \bar{\alpha}}{\text{Cos } \epsilon} =$

$f_3(\bar{\alpha})$,

and $\text{Sin } d = f_4(\varphi) \cdot f_3(\bar{\alpha})$; equivalent to 3:16).

21. Knowledge of the Right Ascension (α)

(The following statements are due to Abū Naṣr, who says that Ḥabash made no mention of this problem)

9:18 $$f_{3a}(\bar{\lambda}) \cdot f_{4a}(\bar{\epsilon}) = \text{Sin } \alpha$$

10:4 $$f_{3a}(\theta) = \frac{\text{Cos } \theta}{\text{Cos } \delta(\bar{\theta})},$$

(the definition of f_3 which is that used for the computation of the table. Strangely enough, in another place, 68:3, he gives a different definition using it instead of this one for proving all the other methods in which f_3 is used.)

10:7 $$\frac{\text{Sin } \lambda}{\text{Cos } \delta(\lambda)} = \frac{\text{Sin } \alpha}{\text{Cos } \epsilon},$$

(proved on 4:5).

10:9 $$f_3(\bar{\lambda}) \cdot \text{Cos } \epsilon = \text{Sin } \alpha.$$

(From 10:7, $\text{Sin } \alpha = \frac{\text{Sin } \lambda}{\text{Cos } \delta(\lambda)} \cdot \text{Cos } \epsilon = f_3(\bar{\lambda}) \cdot \text{Cos } \epsilon$.)

10:11 $$f_4(\bar{\epsilon}) = \text{Cos } \epsilon,$$

because $$f_4(\theta) = \frac{\text{Sin } \theta \cdot \text{Sin } \epsilon}{\text{Cos } \theta}$$

$$f_4(\bar{\epsilon}) = \frac{\text{Cos } \epsilon \cdot \text{Sin } \epsilon}{\text{Sin } \epsilon} = \text{Cos } \epsilon$$

(From 10:9 and 10:11 we get 9:18.)

22. Work of Ḥabash on the Latitude of Visible Climate by the Table of Rectification

Find $f_1(\alpha_M)$ and $f_2(\alpha_M)$ (where α_M is the right ascension of upper midheaven).

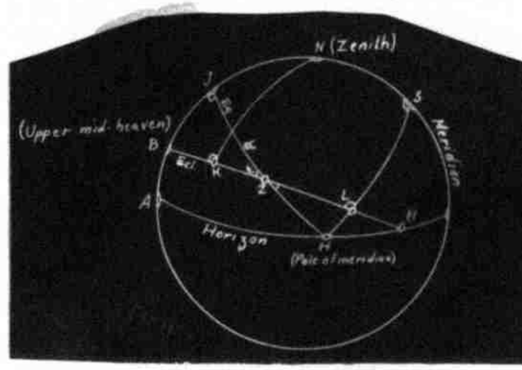
Then the adjusted terrestrial latitude is

11:2 $$\phi - f_1(\alpha_M) = \phi_a,$$

(where distances to the north are measured positive.)

11:4 $$\text{Sin}\,\phi_a \cdot f_2(\alpha_M) = \text{Sin}\,\bar{a}_e.$$

(The remainder of the article is devoted to a proof of this statement. By the Sine Law)



11:14 $$\frac{\text{Sin NK}}{\text{Sin BN}} = \frac{\text{Sin}\,\bar{a}_e}{\text{Sin}\,\phi_a} = \frac{\text{Sin NBK}}{R}$$

(We prove first that)

11:17 $$\text{NBK} = \delta(\overline{JZ})$$

(By definition)

12:1 $$f_2(\theta) = \text{Cos}\,\delta(\bar{\theta}) \text{ and } f_1(\theta) = \delta_2(\theta)$$

or

12:15 $$f_1(JZ) = BJ,$$

and

12:17 $$BJ + JN = NB = \phi_a.$$

(For since $JN = \phi$, the above expression is equivalent to 11:2.)

12:18 $$\frac{\text{Sin BN}}{\text{Sin [NJ]K}} = \frac{R}{\text{Sin NBK}}$$

(is equivalent to 11:14. Or)

12:19 $$\frac{\text{Sin }(BN = \phi_a) \cdot \text{Sin NBK}}{R} = \text{Sin NK} = (\text{Sin}\,\bar{a}_e).$$

But

13:8 $HL = \delta(ZH) = \delta(\overline{ZJ})$. ($\delta_1$ of a point not of the ecliptic.)

Therefore

$$\overline{LS} = \delta(\overline{JZ}), \text{ or } LS = \overline{\delta(\overline{JZ})}.$$

13:9 But $LS = NBK$

or

$$NBK = \overline{\delta(\overline{JZ})} = \overline{\delta(\overline{\alpha}_M)},$$

(which is 11:17).

(Substituting in 12:19 the value of NBK, we have)

$$\frac{\text{Sin}\,\phi_a \cdot \text{Sin}\,\overline{\delta(\overline{\alpha})}}{R} = \text{Sin}\,\overline{a}_e$$

or

$$\frac{\text{Sin}\,\phi_a \cdot \text{Cos}\,\delta(\overline{\alpha})}{R} = \text{Sin}\,\overline{a}_e$$

or

$$\text{Sin}\,\phi_a \cdot f_2(\alpha) = \text{Sin}\,\overline{a}_e, \text{ which is 11:4)}$$

23. Ḥabash's Operation for Finding the Declination of a Star by the Table of Rectification

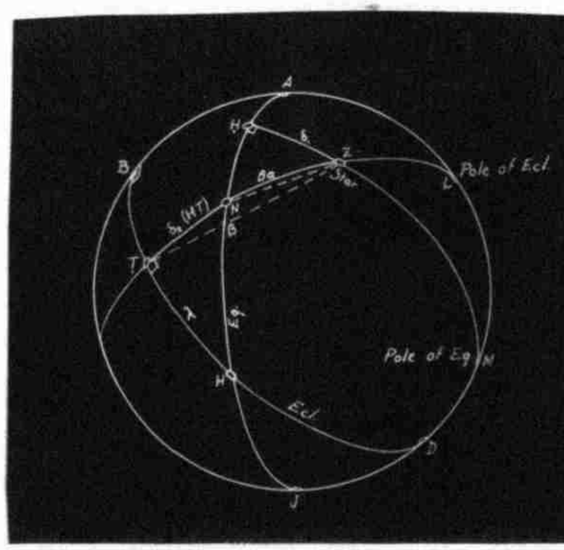
Find $f_1(\lambda)$ and f_2, and call

13:17 $\beta - f_1(\lambda) = \beta_a$,

(then)

14:2 $\text{Sin}\,\beta_a \cdot f_2(\lambda) = \text{Sin}\,\delta(z)$.

(This solves the problem. The rest of the passage is a proof of 14:2. By definition)



14:15 ZṬ = β and NṬ = δ_2(HṬ)

Therefore

$$\beta_a = ZN = \beta - f_1(\lambda)$$

14:18 Find f_2(HT); it is equal to

Sin HNṬ, because HNṬ = $\overline{\delta\ \overline{\text{HṬ}}}$,

(proved in 11:17)

(By the Sine Law)

15:1 $\dfrac{\text{Sin ZN}}{\text{Sin ZḤ}} = \dfrac{R}{\text{Sin (ZNH = HNT)}}$

or

15:5 Sin ZḤ = $\dfrac{\text{Sin ZN} \cdot \text{Sin HNṬ}}{R}$

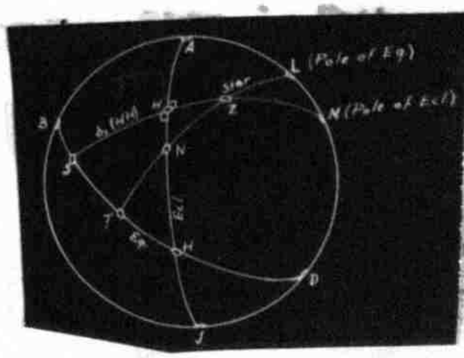
(But Sin HNṬ = Cos δ ($\overline{\text{HT}}$) = $f_2(\lambda)$, therefore
Sin ZḤ = Sin δ (z) = Sin$\beta_a \cdot f_2(\lambda)$, which is 14:2).

(Abū Naṣr makes a special demonstration for the case where the ecliptic is nearer the star than the equator.)

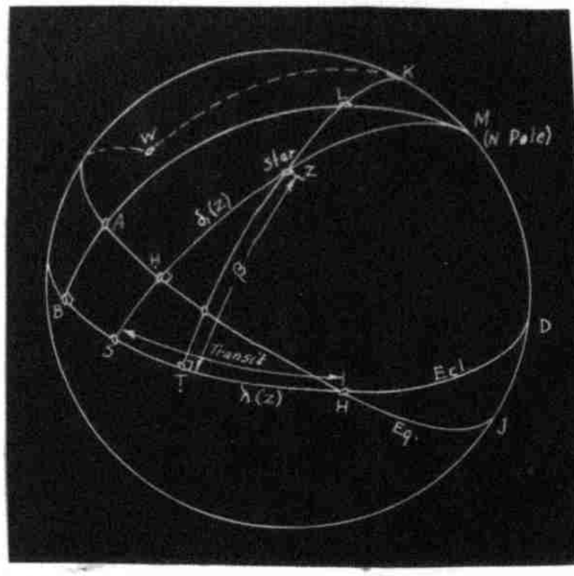
ZḤ = β, ZS = β_a = β + f_1(HḤ)

15:10 (As in 14:18) HSḤ = $\overline{\delta(\overline{\text{HḤ}})}$

(The expression and proof for finding the declination δ(z) is as above.)



24. Habash's Operation for finding the Degree of Transit



(Two main steps are used to find the degree of transit, HS in the figure. The first is to find the equation of the arc, $N\underset{\bullet}{H}$, and to show that:)

16:1 $\text{Sin } N\underset{\bullet}{H} = f_{4a}[\delta(Z)] \cdot f_{3a}(\lambda)$,

where $\delta(Z)$ denotes the declination of the star.

(It is assumed that $\delta(Z)$, $\lambda(Z)$ and a table of right ascension are given, a method for finding $\delta(Z)$ was discussed in 14:2. By the Rule of Four:)

17:3
$$\frac{\text{Sin } ZN}{\text{Sin } N\underset{\bullet}{H}} = \frac{\text{Sin } ZM}{\text{Sin } MK},$$

or

$$\text{Sin } N\underset{\bullet}{H} = \frac{\text{Sin } ZN \cdot \text{Sin } MK}{\text{Sin } ZM}.$$

Also, (by the Sine Law:)

17:7
$$\frac{\text{Sin } Z\underset{\bullet}{H}}{\text{Sin } ZN} = \frac{\text{Sin } (N = KW)}{R}$$

(By substitution from 17:3 and 17:7 we get:)

17:8
$$\frac{\dfrac{\text{Sin } Z\underset{\bullet}{H} \cdot \text{Sin } MK}{\text{Sin } ZM = \text{Cos } Z\underset{\bullet}{H}}}{\text{Sin } N\underset{\bullet}{H}} = \frac{\text{Sin } KW}{R} = \frac{\text{Sin } Z\underset{\bullet}{H}}{\text{Sin } ZN}$$

(Solving for Sin $\underset{\bullet}{H}$N, we get:)

$$\frac{\text{Sin } Z\underset{\bullet}{H} \cdot \text{Sin } MK}{\text{Cos } Z\underset{\bullet}{H}} \cdot \frac{R}{\text{Sin } KW} = \text{Sin } N\underset{\bullet}{H}$$

17:14 But $$f_4[ZH = \delta(z)] = \frac{\text{Sin } Z\underset{\bullet}{H}}{\text{Cos } Z\underset{\bullet}{H}} \cdot \text{Sin } \epsilon$$

(Remembering that Sin KW = Sin N = Sin $\overline{\delta(\overline{H\underset{\bullet}{T}})}$ = Cos δ ($\overline{HT}$),

we get

$$\frac{\text{Sin } Z\underset{\bullet}{H}}{\text{Cos } Z\underset{\bullet}{H}} \cdot \text{Sin } \epsilon \cdot \frac{\frac{R \text{ sin } MK}{\text{Sin } \epsilon}}{\text{Cos } \delta(H\underset{\bullet}{T})} = \text{Sin } N\underset{\bullet}{H}$$

(The published text contains some scribal errors on 17:11 and is garbled at the end. The problem is now reduced to that of showing that

$$\frac{R \cdot \text{sin } MK}{\text{Sin } \epsilon} = \text{Cos } H\underset{\bullet}{T}$$

This is not done by Abū Naṣr. We complete the proof by applying to the triangle H$\underset{\bullet}{T}$N the formula

$$\text{Cos } A = \text{Sin } B \cdot \text{Cos } a,$$

a relation certainly known to the Islamic mathematicians by the 15th century (e.g., the Khāqānī Zīj, India Office copy, f. 334.) and perhaps before. This gives for the right-hand side

$$\text{Cos } N = \text{Cos } KW = \text{Sin } MK$$

which equals to the left-hand side, hence

$$\text{Sin } N\underset{\bullet}{H} = f_4(Z\underset{\bullet}{H}) \cdot f_3(H\underset{\bullet}{T}) \ ,$$

which is equal 16:1.

(The second step is)

16:14 Find α of λ (z) = HN

and

16:15 HN + NḤ + HḤ

16:19 α^{-1}(HḤ) = HS,

the degree of transit.

(A shorter method is now given here by Abū Naṣr for finding the transit:)

18:4 BLṬ is known, because BLṬ = BṬ = $\overline{\text{HṬ}}$, and HṬ = λ which is given.

18:5 LZ is known because it is equal $\bar{\beta}$, similarly MZ is known because it is equal to $\overline{\delta(z)}$.

(By the Sine Law for oblique triangles:)

18:7 $$\frac{\text{Sin MZ}}{\text{Sin LZ}} = \frac{\text{Sin BLṬ}}{\text{Sin (AMḤ = AḤ)}}$$

From this AḤ is determined. Find $\overline{\alpha(\text{AḤ})}$ = (HS) = the transit.

(A similar short method is given next by Abū Naṣr, assuming AHJ to be the ecliptic and that the known angle is AMḤ).

25. Ḥabash's Operation for Finding the Latitude and Longitude of a Star by the Use of the Table of Rectification

(α_{M}, the altitude of the star (NA) and the altitude of upper midheaven (BA) are given.)

19:6 NA - BA = NB = Δh.

(It is required to show that)

19:9 $\sin \Delta h \cdot f_2(\alpha_M) = \text{Sin}\, \beta$

and that

19:11 $f_{4a}(\beta) \cdot f_{3a}(\alpha_M) = \text{Sin}\, KB$,

where KB = the arc of equation, and finally

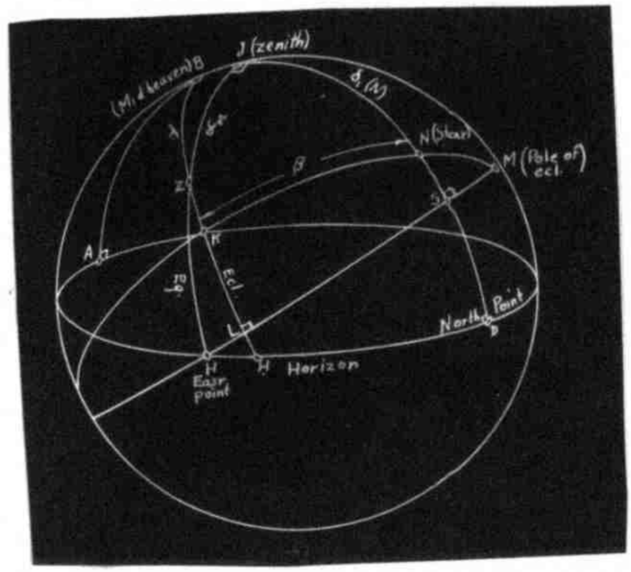
KB - BZ = ZK = $\lambda(N)$

20:14 $f_1(JZ) = \delta_2(JZ) = BJ$

(Here Abū Naṣr uses the general meaning of δ_2)

20:17 BJ + JN = BN,

(where JN = $\delta(N)$ was found in 13:12)



But

20:18 $f_2(JZ) = \text{Cos}\, \delta(\overline{JZ}) = \text{Cos}\, \delta(ZH) - \text{Cos}\, (HL) = \text{Sin}\, (SL)$

also MS = HL

(By the Rule of Four, we have)

21:1 $$\frac{\text{Sin NB}}{\text{Sin BK}} = \frac{\text{Sin [NM]}}{\text{Sin MS}} .$$

(Similarly by the Sine Law and noting that B = SL)

21:2 $$\frac{\text{Sin NK}}{\text{Sin BN}} = \frac{\text{Sin SL}}{R} .$$

(Solving for Sin NK, and noting that Sin SL = $f_2(\alpha_M)$, we have)

21:3 $$\frac{\text{Sin NB} \cdot f_2(\alpha_M)}{R} = \text{Sin NK}$$

or

$$\frac{\text{Sin } (\Delta h) \cdot f_2(\alpha_M)}{R} = \text{Sin } \beta$$

(which is 19:9)

(To show 19:11, we have by the Rule of Four)

21:8 $$\frac{\text{Sin NB}}{\text{Sin KB}} = \frac{\text{Sin [NM]}}{\text{Sin MS}}$$

or

21:9 $$\text{Sin BK} = \frac{\text{Sin NB} \cdot \text{Sin MS}}{\text{Sin MN}}$$

(By using 21:2 and 21:9 we write)

21:10 $$\frac{\frac{\text{Sin NK} \cdot \text{Sin MS}}{\text{Sin MN}}}{\text{Sin BK}} = \frac{\text{Sin NK}}{\text{Sin NB}} = \frac{\text{Sin SL}}{R},$$

(which leads to)

21:13 $$\frac{\text{Sin NK} \cdot \text{Sin MS}}{\text{Sin MN} = \text{Cos KN}} \cdot \frac{R}{\text{Sin SL}} = \text{Sin KB}.$$

But

21:15 $$f_4(\text{NK}) = \frac{\text{Sin NK}}{\text{Cos NK}} \cdot \text{Sin } \epsilon,$$

or

21:16 $$\frac{f_4(\text{NK})}{\frac{\text{Sin NK} \cdot \text{Sin MS}}{\text{Cos NK}}} = \frac{\text{Sin } \epsilon}{\text{Sin (MS = LH)}} = \frac{R}{\text{Sin ZH}}$$

(By using 21:10 and 21:16 we have:)

22:1 $$f_4(\text{NK}) \cdot \frac{\text{Sin ZH}}{\text{Sin SL}} = \text{Sin BK}$$

(But $\frac{\text{Sin ZH}}{\text{Sin SL}} = \frac{\text{Cos JZ}}{\text{Sin B}} = \frac{\text{Cos JZ}}{\text{Cos } \delta \text{ (JZ)}} = f_3(\text{JZ})$, or

$$f_4(\beta) \cdot f_5(\alpha_M) = \text{Sin BK}, \text{ equivalent to 19:11.}$$

Finally $$KB - BZ(\lambda_M) = ZK(= \lambda_N)$$

26. Finding the Latitude of a Star by the Degree of Transit

(Abū Naṣr says that the scribes make mistakes by copying manuscripts without understanding the meaning. In the following problem, for example, the longitude of the star is tacitly assumed as given, wheras what was originally intended was that the right ascension should be given. The method is valid only for the latter. References are to the figure of Section 24.)

22:16 Given the degree of transit, HS.

Find β (= ZṬ).

(Abū Naṣr mentions that most of the manuscripts of the Ḥabash Zīj state that)

22:17 $$f_1(HS) \text{ and } f_2(HS)$$

should be used, but what corresponds to HS on the equator is the proper arc to be used.

22:17 $$f_1(HḤ) = ḤS \text{ , and } f_2(HḤ) = \text{Cos } \delta(\overline{HḤ}),$$

then

22:19 $$ḤS + [ZḤ = \delta(Z)] = ZS.$$

(It is necessary to show that:)

23:1 $$\text{Sin ZS} \cdot f_2(HḤ) = \text{Sin ZṬ} = \text{Sin } \beta.$$

(The method of proof is not mentioned in the text, but by the Sine Law

$$\frac{\text{Sin ZṬ}}{\text{Sin S}} = \frac{\text{Sin ZS}}{R}$$

or

$$\frac{\text{Sin ZS} \cdot \text{Sin S}}{R} = \text{Sin Z}\underset{\bullet}{\text{T}} = \text{Sin } \beta$$

The recollection that Sin S = Cos δ($\overline{\text{H}\underset{\bullet}{\text{H}}}$) shows that 23:1 is correct except for the factor 1/R on the left-hand side, the need for which Abū Naṣr remarks.)

27. Finding the Longitude of a Star by its Latitude

(It is also assumed that the transit of the star (HS) is given. Here again Abū Naṣr corrects the mistakes found in some manuscripts to the effect that α_H should be used and not λ_H. The figure of Section 24 is again used.)

To find λ_Z

23:8 $$f_4(\beta) \cdot f_3(\alpha_T) = \text{Sin } d_1$$

(Abū Naṣr does not specify what this "first distance", d_1, is. In fact, as we demonstrate below it is TS.)

23:8(a) $$\frac{\text{Sin Z}\underset{\bullet}{\text{T}}}{\text{Cos Z}\underset{\bullet}{\text{T}}} \cdot \text{Sin } \epsilon \cdot \frac{\text{Cos H}\underset{\bullet}{\text{H}}}{\text{Cos } \delta(\overline{\text{H}\underset{\bullet}{\text{H}}})} .$$

(Using the formula Cos a · Sin B = Cos A, and putting H$\underset{\bullet}{\text{H}}$ = a, $\beta = \epsilon$, we have

$$\text{Cos H}\underset{\bullet}{\text{H}} \cdot \text{Sin } \epsilon = \text{Cos S},$$

and 23:8(a) is reduced to

$$\text{Tan Z}\underset{\bullet}{\text{T}} \cdot \text{Cos S},$$

remembering that

$$\text{Cos } \delta(\overline{\text{H}\underset{\bullet}{\text{H}}}) = \text{Sin S}.$$

But

$$\tan b \cdot \cot B = \text{Sin a},$$

therefore

$$\tan Z\underset{\cdot}{T} \cdot \cot S = \sin \underset{\cdot}{T}S$$
$$HS - \underset{\cdot}{T}S = H\underset{\cdot}{T} = \lambda(Z).)$$

28. Habash's Operation for Finding the Ascendant (λ_H) by the Table of Rectification. (Without Using Oblique Ascensions).

(The first step is to find the equation of upper midheaven, BK.)

Find

24:3 $\sin f_1(90^\circ - \lambda_M)$,

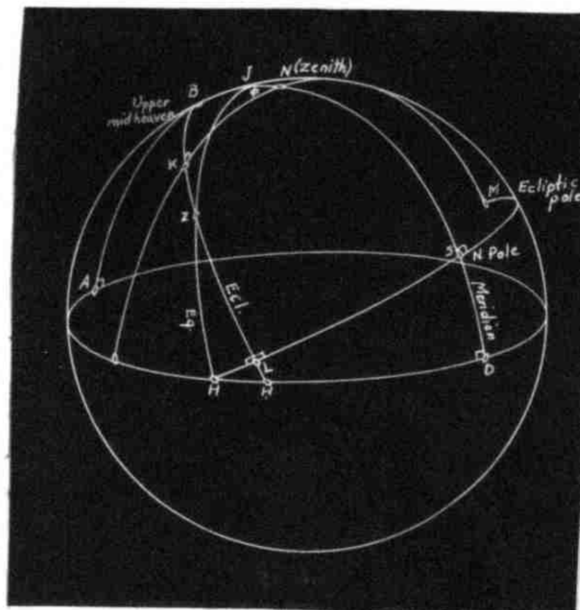
Where λ_M = degree of upper midheaven

and

24:5 $\varphi + f_1(\lambda_M)$

$= JN + \delta_2(BZ)$

$= JN + BJ = BN = \varphi_a$.



Then

24:6 $\tan BK = \sin f_1(\overline{\lambda}_M) \cdot \tan \varphi_a$

Finally: $\lambda_H = Z\underset{\cdot}{H} = 90^\circ - \lambda_M + BK$

(Abū Naṣr proves 24:6 as follows: by definition of the tangent of an angle:)

25:2 $$\frac{\text{Tan BK}}{\text{R}} = \frac{\text{Sin BK}}{\text{Cos BK}(= \text{Sin KL})},$$

(and by the Rule of Four)

25:4 $$\frac{\text{Sin KL}}{\text{Sin S[N]}} = \frac{\text{R}(= \text{Sin MK})}{\text{Sin MN}} .$$

(By substitution in 25:2 and 25:4, we have;)

25:6 $$\frac{\text{Tan BK}}{\text{Sin MN}} = \frac{\text{Sin BK}}{\text{Sin SN}},$$

or

25:7 $$\frac{\text{Tan BK}}{\text{Sin BK}} = \frac{\text{Sin MN}}{\text{Sin SN}},$$

(and by the Rule of Four)

25:8 $$\frac{\text{Sin BK}}{\text{Sin NB}} = \frac{\text{Sin SM}}{\text{Sin MN}}.$$

(By multiplying the left-hand side of 25:7 and 25:8, and similarly their right-hand sides we get:)

25:10 $$\frac{\text{Tan BK}}{\text{Sin BN}} = \frac{\text{Sin SM}}{\text{Sin SN}},$$

But SN = $\overline{\text{BN}}$, and $\frac{\text{Sin BN}}{\text{Cos BN}}$ = Tan BN; we have:

25:11 $$\frac{\text{Sin BN}}{\text{Tan BN}} = \frac{\text{Sin}(\text{SN} = \overline{\text{BN}})}{\text{R}},$$

and by multiplying 25:10 and 25:11):

25:14 $$\frac{\text{Tan BK}}{\text{Tan [B]N}} = \frac{\text{Sin SM}}{\text{R}} ,$$

(But MS = HL, and Sin MS = Sin HL = Sin δ_2 (ZL), and Sin δ_2 (ZL) = Sin δ_2 (90 - BZ) = Sin $f_1(\overline{\lambda}_M)$, and by solving for Tan BK, we have

$$\text{Tan } BK = \frac{\text{Sin } f_1(\overline{\lambda_M}) . \text{ Tan } \varphi_a}{R}$$

which is 24:6. Finally, the longitude of the ascendant,

$$Z\underset{\bullet}{H} = 90 - BZ + L\underset{\bullet}{H},$$

and since $L\underset{\bullet}{H} = BK$

or $$Z\underset{\bullet}{H} = 90 + BK - \lambda_M .)$$

29. Another Method for Finding the Ascendant (λ_H).

(Use the figure of 24:3).

(By the Sine Law we have:)

28:8 $$\frac{\text{Sin } SM}{\text{Sin } MN} = \frac{\text{Sin } N}{R},$$

in which SM is known (SM is the distance between the ecliptic pole and the north pole = ϵ) and MN = a_e, which is easy to find by these tables. (See 11:4).

28:11 Therefore N is known.

(Since $\triangle$ NMS is the polar $\triangle$ to $ZH\underset{\bullet}{H}$, and $N + H\underset{\bullet}{H} = 180^\circ$)

28:11 $N = H\underset{\bullet}{H}$ = The ortive amplitude of the ascendant.

(Also by the Sine Law):

28:13 $$\frac{\text{Sin } N (= \text{Sin } H\underset{\bullet}{H})}{\text{Sin } KB} = \frac{R}{\text{Sin } BN},$$

28:15 $$BN = \varphi_a,$$

(as 24:5) If $\lambda_M = BZ$ is known, then h_M is known.

(This is because $h_M = BA = \overline{BN}$, and BN was found in 24:5).

28:17 Therefore $DS = \overline{h_M}$ is known,

(and because H is the pole of SD, we have:)

28:18 $DS = SHD$,

(and because $\triangle$ ZḤH is polar to $\triangle$ NMS, we have MN + LḤH = 180° and therefore:)

28:19 $MN = L\dot{H}H = \alpha_e = \overline{KN}$

(By using 28:19 and the Sine Law, we have:)

29:1 $$\frac{\text{Sin } HL \;(= f_1(ZL))}{\text{Sin } L\dot{H} = \text{Sin } KB} = \frac{\text{Sin } \dot{H} \;(= \text{Sin } \alpha_e)}{\text{Sin } H \;(= \text{Sin } DS = \text{Sin } h_M}$$

(from which KB is found.

$$Z\dot{H} = \lambda_H = ZL + L\dot{H} = \overline{BZ} + KB = \lambda_M + KB).$$

30. Nairīzī's Operation for Finding the Ascendant (ḤZ) by the "University Table."

(Without Using Oblique Ascension).

(Abū Naṣr mentions here that Nairīzī copied these tables into his zīj and called them the Universal Table. The figure of Section 28 should be used. It is assumed that φ and λ_M are given.)

(The first step is to find the equation of the ascendant = BK, for which we need φ_a;)

30:9 $$\varphi + f_1(\lambda_M = BZ) = NJ + BJ = BN = \varphi_a,$$

(and by the Rule of Four:)

31:10 $$\frac{\text{Sin } NB}{\text{Sin } BK} = \frac{\text{Sin } [N]M}{\text{Sin } MS}$$

But

31:12 $$MS = HL = \delta_2(ZL) = \delta_2(\overline{\lambda}_M),$$

(which means that MS is known, and by using 31:10 we have)

31:14 $$\frac{\text{Sin } BN.\ \text{Sin } \epsilon}{\text{Sin } BK.\ \text{Sin } \epsilon} = \frac{\text{Sin } BN}{\text{Sin } BK} = \frac{\text{Sin } NM}{\text{Sin } MS},$$

(But by 25:14, we have)

32:2 $$\frac{f_4(BN)}{f_4(BK)} = \frac{\text{Sin } ML\ (= R)}{\text{Sin } MS}$$

or

32:4 $$f_4(BK) = \frac{f_4(BN).\ \text{Sin } MS}{R}$$

(All the quantities on the right hand side are now known, which means that BK is known.)

(But)

30:14 $$\lambda_H = (Z\dot{H}) = 90^\circ + BK - \lambda_M$$

(Here Abū Naṣr mentions another method for finding BK as follows:)

32:6 $$\frac{f_2(\alpha_M).\ \text{Sin } BN}{R} = \text{Sin } NK.$$

(Abū Naṣr does not prove 32:6. But we know that

$f_2(\alpha_M) = \cos \delta_1 (\overline{JZ}) = \sin B$. If this is substituted in the above, 32:6 is seen to be valid by the Sine Law and NK is then known).

32:9 $$NM = \overline{NK},$$

(and BK will be found from the proportion, after using the Rule of Four)

32:9 $$\frac{\sin BN}{\sin BK} = \frac{\sin NM}{\sin MS},$$

(where BN, NM and MS are known. The application of 30:14 follows.)

31. Nairīzī's Operation for Finding the Arc of Daylight (D) by the Universal Table

(The following method is due to Nairīzī, but Abū Naṣr says, "thus I found it in the copy, but it is wrong, (however) it is not Nairīzī's (mistake), for such would be far from him, rather it is one of the misdemeanors of the copyists".)

If δ_s is less than ϵ, then

32:14 $$f_2(\sin \bar{\delta}_s) . f_3(\delta_s) . f_4(\varphi) = \sin d.$$

If δ_s is greater than ϵ, then

33:5 $$\frac{f_4(\varphi) . \sin \delta_s}{\cos \delta_s} = \sin d.$$

But Abū Naṣr states that the correct expression is

33:9 $$\frac{\frac{f(\varphi)\cdot \operatorname{Sin}\delta_s}{4}}{\operatorname{Cos}\delta_s}\cdot\frac{5}{2} = \operatorname{Sin} d,$$

because

34:9 $$\frac{5}{2} \approx \frac{R}{\operatorname{Sin}\epsilon}\ .$$

(This is a fact, since sin 23; 34, 38° = 0.4000, and Abū Naṣr's value of ϵ is 23; 35°. If this substitute is made in 33:9 it becomes

34:12 $$\frac{\frac{f(\varphi)\cdot \operatorname{Sin}\delta_s}{4}}{\operatorname{Cos}\delta_s}\cdot\frac{R}{\operatorname{Sin}\epsilon} = \operatorname{Sin} d.$$

Abū Naṣr now discusses the matter saying that it has previously been explained. In fact if $\frac{R}{\operatorname{Sin}\epsilon}$ is substituted for $\frac{5}{2}$, then 33:9 reduces to sin d = tan δ. tan φ.).

(The above-mentioned method is the same as that of Ḥabash given in 4:19, except that $\frac{R}{\operatorname{Sin}\epsilon}$ is replaced by $\frac{5}{2}$. Abū Naṣr does not complete the problem by stating that D = 180 + 2d.).

32. Ḥabash's Operation for Finding the Arc of Daylight (D) by the "Table of Rectification"

(Abū Naṣr says that) Ḥabash has a method exactly similar (to Nairīzī's, e.g. 33:5)

If $\delta_s > \epsilon$, Ḥabash said that:

34:17 $$\frac{\text{Sin}\,\delta_s \cdot \text{Tan}\,\varphi}{\text{Cos}\,\delta_s} = [\text{Sin}]\,d$$

(The difference between 34:17 and the expression given previously, 4:19, is in the use of Tan φ here instead of $\frac{\text{Sin}\,\varphi}{\text{Cos}\,\varphi}$.)

We explained in 4:19

35:2 $$\frac{\frac{\text{Sin}\,\delta_s \cdot \text{Sin}\,\varphi}{\text{Cos}\,\varphi}}{\text{Sin}\,d} = \frac{\text{Cos}\,\delta_s}{R}.$$

But

35:5 $$\frac{\text{Tan}\,\theta}{R} = \frac{\text{Sin}\,\theta}{\text{Cos}\,\theta},$$

(where Abū Naṣr calls R the <u>gnomon</u>, (<u>miqyas</u>).) Therefore

35:6 $$\frac{\text{Sin}\,\delta_s \cdot \text{Tan}\,\varphi}{R} = \frac{\text{Sin}\,\delta_s \cdot \text{Sin}\,\varphi}{\text{Cos}\,\varphi}$$

(Abū Naṣr mentions that 34:17 is equivalent to 35:2).

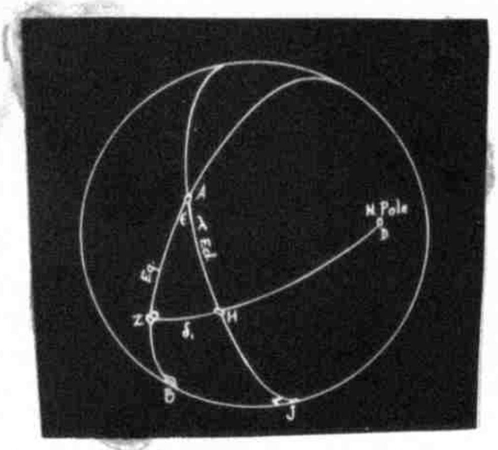
(Note that this is fairly conclusive evidence that Ḥabash's tangent function is not 12 tan θ, but 1,0. tan θ.)

33. Nairīzī's Operation for Finding ϵ In Terms of δ and Vice Versa

56:7 $$\frac{R^2}{\text{Sin}\,\lambda} \cdot \frac{\text{Sin}\,\delta(\lambda)}{R} = [\text{Sin}]\,\epsilon$$

(It is assumed that λ and δ are given.)

"Look", says Abū Naṣr sarcastically, "how much difference there is between this and simply taking the quotient of the sines and elevating the quotient!"



To prove 56:7 he puts

56:16 $$\frac{\text{Sin } HZ}{\text{Sin } AH} = \frac{\text{Sin } \lfloor BJ \rfloor \ (=\epsilon)}{AJ\,(=R)},$$

or

56:18 $$\text{Sin } \epsilon = \frac{R \text{ Sin } HZ}{\text{Sin } AH},$$

(which is 56:7)

(The solution of the inverse problem is not given. It is, of course, a matter of solving 56:7 for Sin $\delta\,(\lambda)$.)

34. NairīzI's Operation for Finding the Equation of Daylight (d) In Terms of the Ortive Amplitude (Oa).

(It is required to show that)

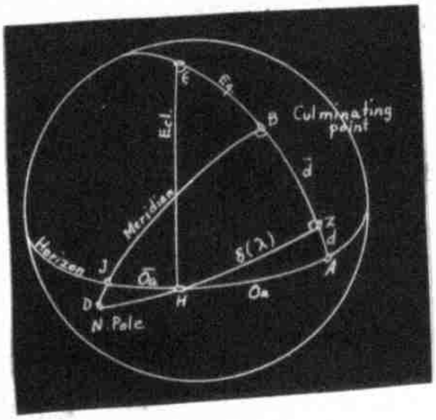
57:10a $$\frac{R^2 \cdot \text{Sin } \overline{\max Oa}}{R \text{ Cos } \epsilon} = \text{Sin } \overline{\max d},$$

and

57:10b $$\frac{R^2 \cdot \text{Sin } \overline{Oa}}{R \text{ Cos } \delta\,(\lambda)} = \text{Sin } \overline{d}$$

(Here again Abū Naṣr criticises NairīzI for multiplying by R^2 and dividing by R. Since DZ = 90° and Z = 90°, therefore D is the pole of the equator, and then B = 90°. AB = 90°, therefore B is the culminating point (the highest point from the horizon), hence $J = 90^\circ$, $D = BZ$. By the Sine Law)

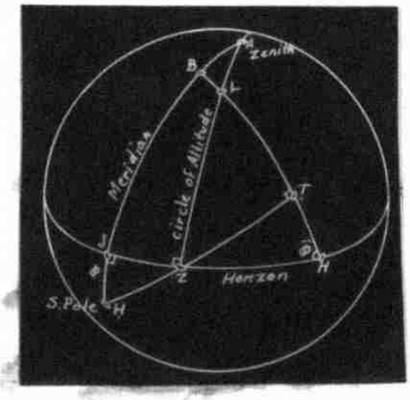
58:4 $$\frac{\mathrm{Sin}(HJ = \overline{AH})}{\mathrm{Sin}(BZ = \overline{AZ} = \overline{d} = D)} = \frac{\mathrm{Sin}(HD = \overline{\delta}(\lambda))}{R}$$



(by substituting the equivalent values, we will have 57:10b. Abū Naṣr did not show 57:10a, but it is clear that if Oa is a maximum then d is also a maximum.)

35. Ḥabash's Operation for Finding the Right Ascension of the Azimuth of a Star

(Although formal hypotheses are not given it is apparent from the text that, in addition to the local latitude φ and the azimuth of the star ZH, the degree of the equatorial rising point H, is also assumed as known.



As usual the text is set up in two parts, the operation as prescribed by Ḥabash, followed by a proof by Abū Naṣr. The first part is reasonably intact, and the proof can be read as it stands in the published text, but only after extensive restoration of miscopied letters from the figure.

Ḥabash's solution is as follows: Form)

58:19 $$\frac{\text{Sin}\,\phi}{\text{Cos}\,\phi}\cdot 150 = m.$$

(To obtain ZṬ use the relation)

59:1 $$\text{Sin ZH}.\ \text{Cos}\,\phi = \text{R Sin ZṬ}$$

59:3 $$f_4\,(\text{ZṬ}).\ m = \text{Sin HṬ}$$

(Here the procedure stops, although as Abū Naṣr remarks, (59:7) HṬ is not the right ascension of the azimuth. But the difference between HṬ and the degree of the equatorial rising point is the required right ascension. The proof now commences. Applying the Rule of Four to triangles ZJḤ and ZṬH, we have)

59:15 $$\frac{\text{Sin Ṭ[H]}}{\text{Sin(ḤJ} = \phi)} \qquad \frac{\text{Sin[ZH]}}{\text{Sin ḤZ}},$$

or

60:2 $$\text{Sin ṬH} = \frac{\text{Sin ḤJ. Sin ZH}}{\text{Sin ḤZ}}$$

(By the Sine Law

$$\frac{\text{Sin ZH}}{\text{R}} = \frac{\text{Sin ZṬ}}{\text{Sin}\,\bar{\phi}}$$

or $$\text{R sin ZṬ} = \text{Sin ZH. Cos}\,\phi\ ,$$

equivalent to 59:1.

From 59:15, we have)

60:8 $$\frac{\dfrac{\text{Sin [Z] Ṭ . Sin}\,\phi}{\text{Sin Z [Ḥ]}}}{\text{Sin HṬ}} = \frac{\text{Sin [Z] Ṭ}}{\text{Sin H [Z]}}$$

or

$$\frac{\dfrac{\text{Sin } Z\underset{\bullet}{T} \cdot \text{Sin } \epsilon}{\text{Sin } Z\underset{\bullet}{H}}}{\text{Sin } H\underset{\bullet}{T}} = \frac{\text{Sin } \epsilon \cdot \text{Sin } ZT}{\text{Sin } \phi \cdot \text{Sin } HZ}$$

(and from 59:1)

$$= \frac{\text{Sin } \epsilon \cdot \text{Cos } \phi}{R \cdot \text{Sin } \phi}$$

$$= \frac{R}{\dfrac{R^2 \cdot \text{Sin } \phi}{\text{Sin } \epsilon \cdot \text{Cos } \phi}}$$

(But)

60:13 $$150 = \frac{60^2}{24} = \frac{R^2}{\text{Sin } \epsilon}$$

(Therefore)

$$\frac{\dfrac{\text{Sin } Z\underset{\bullet}{T} \cdot \text{Sin } \epsilon}{\text{Sin } Z\underset{\bullet}{H} = \text{Cos } ZT}}{\text{Sin } H\underset{\bullet}{T}} = \frac{R}{150 \cdot \dfrac{\text{Sin } \phi}{\text{Cos } \phi}}$$

or

$$\frac{f_4(Z\underset{\bullet}{T})}{\text{Sin } H\underset{\bullet}{T}} = \frac{R}{150 \cdot \dfrac{\text{Sin } \phi}{\text{Cos } \phi}}$$

Solving for

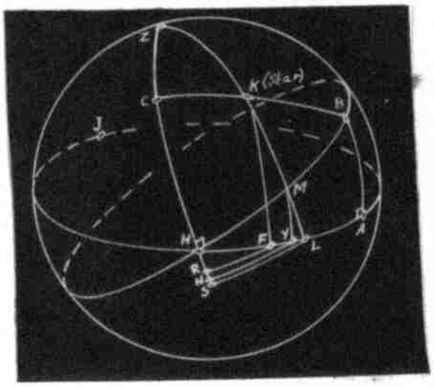
$$\text{Sin } H\underset{\bullet}{T} = \frac{f_4(Z\underset{\bullet}{T}) \cdot \dfrac{\text{Sin } \phi}{\text{Cos } \phi} \cdot 150}{R}$$

(which is 59:3.) $$= \frac{f_4(Z\underset{\bullet}{T}) \cdot m}{R},$$

<u>36. Ḥabash's Operation for Finding the Ascendant by the Table of Rectification (Without Using Tables of Oblique Ascensions.)</u>

(Given are the arc of revolution r (CR in the figure on the following page), ϕ, λ_K, and a table of right ascensions.

The text gives Ḥabash's procedure for solving the problem, followed by a proof by Abū Naṣr. The former part is garbled, nevertheless we give it below substantially as it appears. A basic technique used in it is the method of finding the equation of daylight of a point on the ecliptic having right ascension α by writing, as proved in Section 20 above,



$$\text{Sin } d = \text{Cos } \overline{\alpha} \tan \varphi \tan \epsilon = f_{3b}(\overline{\alpha})\, f_4(\varphi).$$

We have reconstructed the method of proof, supplying missing steps where needed.

To find the ascendant, Ḥabash proceeds as follows:)

From

63:10 $$r + \alpha_k - d_k$$

or

$$CR + ZC - HR = ZH$$

Then

63:12 $$A_0^{-1}(ZH) = ZM$$

63:13 $$f_3(\overline{ZH}) \cdot f_4(\varphi) = m$$

63:15(a) $$\overline{A_0^{-1}(ZC)} = W$$

63:15(b) $$f_3(w) \cdot f_4(\varphi) = V$$

63:16 $$1,0 - V = U$$

63:17 $$\sqrt{m^2 + U^2} = r$$

63:18 $$\sin^{-1}\left(\frac{1,0 \cdot m}{r}\right) = s$$

64:1 $$A_0^{-1}(s + ZC) = ZL = \lambda_H,$$

the required ascendant. (We postpone discussion of the above until after the demonstration immediately following.)

$$\sin J_K = f_3(\overline{ZC}) \cdot f_4(\varphi) = \sin HR$$

is known, also

64:8 $$ZC + CR - RH = ZH$$

is known.

64:15 $$A_0^{-1}(ZH) = ZM$$

Similarly

64:19 $$\sin d_M = f_3(\overline{ZH}) \cdot f_4(\varphi) = \sin HN$$

(By application of the Rule of Four to the triangles ZHM and ZSL, and also to the triangles HYN and HLS, and noting that NY = HM, we have:)

65:10 $$\frac{\sin ZS}{\sin HS} = \frac{\sin ZH}{\sin NH} = \frac{\sin ZH}{\sin d_M}$$

Thus the ratio

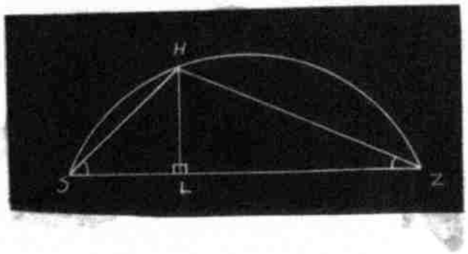
$$\frac{\sin ZS}{\sin HS}$$

is known as well as the difference between the same two arcs, for

$$ZS - HS = ZH.$$

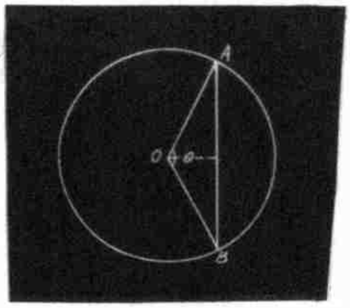
(Apparently Ḥabash sought to utilize these two facts in order to find $\widehat{ZS}$.

Consider the inscribed plane triangle HZS. In it ZH, hence also angle S is known.



For any two points on a circle of radius R = 1,0, the fundamental relation between the chord and the Sine function of a given arc is

$$\text{Crd } AB = 2 \text{ Sin } \frac{\widehat{AB}}{2} .$$



But the inscribed angle such as S,

$$S = \frac{1}{2}\widehat{ZH},)$$

therefore

65:16 $\text{Crd } ZH = 2 \text{ Sin } \widehat{ZH}$

and

65:16 $\text{Crd } ZS = 2 \text{ Sin } \widehat{ZS}$

also

65:17 $\text{Crd } HS = 2 \text{ Sin } \widehat{HS}$

Therefore

$$\frac{\text{Crd } HS}{\text{Crd } ZS} = \frac{\text{Sin } \widehat{HS}}{\text{Sin } \widehat{ZS}}$$

is known, by 65:10

(We define $\left(\frac{SL}{SH=60}\right)$ as "find SL in units such that SH = 60 units".)

But $\underset{(SH=60)}{SL} = \text{Cos } S$, therefore

$$\underset{(ZS=60)}{SL} = \left(\frac{SH}{ZS}\right) \cdot \underset{(SH=60)}{SL}$$

or $\underset{(ZS=60)}{SL}$ is known, and

66:12 $$1{,}0 - \underset{(ZS=60)}{\lfloor SL \rfloor} = \underset{(ZS=60)}{LZ}$$

is also known. Moreover

$$\underset{(ZS=60)}{HL} = \underset{(SH=60)}{HL}\left(\frac{SH}{ZS}\right) = (\text{Sin } S)\left(\frac{SH}{ZS}\right):$$

But

66:14 $$\underset{(ZS=60)}{HZ} = \sqrt{\underset{(ZS=60)}{SL^2} + \underset{(ZS=60)}{LZ^2}}$$

therefore $\underset{(ZS=60)}{HZ}$ is known, but

$$\widehat{SZ} = \underset{(SZ=60)}{HZ}\left(\frac{1{,}0}{HZ}\right),$$

therefore $\widehat{SZ}$ is known, but

$$\widehat{SZ} = \text{CRD}^{-1}(ZS) = 2\ \text{Sin}^{-1}\left(\frac{SZ}{2}\right),$$

or SZ is known, and

$$\text{Ao}^{-1}\ (\widehat{ZS}) = \widehat{ZL} = \lambda_H \,.$$

(Some steps given by Ḥabash are identical with the steps given by Abū Naṣr or given by us and which we think to be the original solution of Ḥabash, however Ḥabash's method was garbled during transcription.

In this connection Abū Naṣr admits that the method of Ḥabash contains mistakes. (cf. 67:6.)

The following pairs of steps are identical:

Ḥabash: 63:10 ; 63:12 ; 63:13 ; 63:16 ; 63:17

Abū Naṣr: 64:8 ; 64:15 ; 64:19 ; 66:12 ; 66:14.

The steps 63:18 and 64:1 should be modified as they appear in the solution.)

37. The Work of Abū-Jaʿfar al-Khāzin on the Rectification Table.

1. The Third Function.

Abū Naṣr mentions that the reasoning of al-Khāzin in his commentary on the Almagest about the third table is correct, but is long.

(It is required to show that):

67:15 $$\frac{\mathrm{Sin}\,\delta(\bar{\theta})\cdot 60}{\mathrm{Cos}\,\epsilon}\cdot\frac{60\cdot\mathrm{Cos}\,\theta}{\mathrm{Cos}\,\epsilon}=\frac{\mathrm{Sin}\,\delta(\theta)}{\mathrm{Sin}\,\theta}=\frac{\mathrm{Sin}\,\epsilon}{R},$$

or

67:19 $$\frac{\mathrm{Sin}\,\delta(\bar{\theta})\cdot 60}{\mathrm{Cos}\,\epsilon}\cdot\frac{R}{\mathrm{Sin}\,\epsilon}=\frac{60\cdot\mathrm{Cos}\,\theta}{\mathrm{Cos}\,\epsilon}.$$

Abū Naṣr states here that he has said that

68:3 $$f_{3b}(\theta)=\frac{\mathrm{Cos}\,\theta}{\mathrm{Cos}\,\epsilon},$$

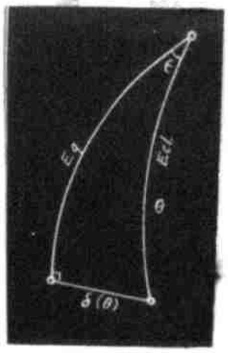
while Ḥabash and al-Khāzin state that

$$f_3(\theta)=\frac{60\,\mathrm{Cos}\,\theta}{\mathrm{Cos}\,\epsilon},$$

so that the minutes become degrees.

(Abū Naṣr does not show in the text the proof of 67:15, but by using the Sine Law:

$$\frac{\mathrm{Sin}\,\delta(\theta)}{\mathrm{Sin}\,\theta}=\frac{\mathrm{Sin}\,\epsilon}{R},$$



but

$$\frac{\text{Sin}\,\delta(\bar{\theta})}{\text{Sin}\,(\bar{\theta})} = \frac{\text{Sin}\,\delta(\theta)}{\text{Sin}\,\theta}$$

or

$$\frac{\text{Sin}\,\delta(\bar{\theta})}{\text{Cos}\,\theta} = \frac{\text{Sin}\,\delta(\theta)}{\text{Sin}\,\theta} = \frac{\text{Sin}\,\epsilon}{R}$$

or

$$\frac{60\ \text{Sin}\,\delta(\bar{\theta})}{\text{Cos}\,\epsilon} \cdot \frac{60\ \text{Cos}\,\theta}{\text{Cos}\,\epsilon} = \frac{\text{Sin}\,\delta(\theta)}{\text{Sin}\,\theta} = \frac{\text{Sin}\,\epsilon}{R},$$

which is 67:15. By arranging the terms in 67:15, we have

$$\frac{60\ \text{Sin}\,\delta(\bar{\theta})}{\text{Cos}\,\epsilon} \cdot \frac{R}{\text{Sin}\,\epsilon} = \frac{60\ \text{Cos}\,\theta}{\text{Cos}\,\epsilon} = f_3(\theta),$$

as al-Khāzin and sometimes Abū Naṣr think. In all the problems where $f_3(\theta)$ is used, it is defined as

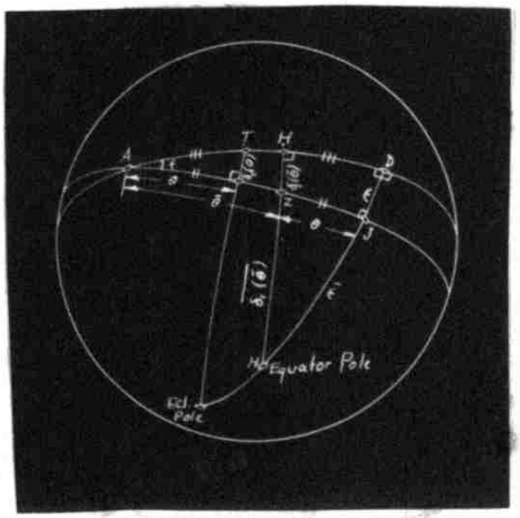
$$f_{3a}(\theta) = \frac{\text{Cos}\,\theta}{\text{Cos}\,\delta(\bar{\theta})} = \frac{\text{Cos}\,\theta}{f_2(\theta)},$$

except in one problem, namely 3:16, where

$$f_{3b}(\theta) = \frac{\text{Cos}\,\theta}{\text{Cos}\,\epsilon}$$

is used. As has been shown, in the actual tables only f_{3a} appears.

2. The Second Table.



Al-Khāzin states that Ḥabash was right when he mentioned that

68:11 $\frac{R\ \text{Cos}\,\epsilon}{\text{Cos}\ f_1(\theta)} = f_2(\theta)$

68:17 Construct ṬB perpendicular to AB, and take

68:18 ZJ = AB.

(Then the first step is to show that)

68:19 $$[\text{DḤ}] = \text{AṬ} = \overline{\text{AḤ}}$$

and

69:1 $$Z = \overline{\delta(\overline{\text{AḤ}})} = \overline{\delta(\text{AṬ})} = \overline{\delta(\text{DḤ})}$$

(Abū Naṣr does not prove 68:19 and 69.1. In triangle AṬB and by using the relation

$$\cos A = \tan b \cdot \cot c$$

we have

$$\cos \epsilon = \tan \theta \cdot \cot \text{AṬ}$$

In the triangle AḤZ, we have

$$\cos \epsilon = \tan \text{AḤ} \cdot \cot \overline{\theta} = \tan \text{AḤ} \cdot \tan \theta \, .$$

Therefore

$$\cot \text{AṬ} = \tan \text{AḤ}$$

or

$$\text{AṬ} = \overline{\text{AḤ}}$$

but

$$\overline{\text{AḤ}} = \text{ḤD}$$

or

$$\text{DḤ} = \text{AṬ} = \overline{\text{AḤ}},$$

equivalent to 68:19.

For proving 69:1, we apply formula sin b = cot A • tan a

$$R \cdot \sin \theta = \cot \epsilon \cdot \tan \text{ṬB}$$

or

$$R \cdot \cot \overline{\text{ṬB}} = \sin \theta \cdot \tan \epsilon$$

In triangle HZJ, and by applying the same formula we have

$$R \cdot \sin ZJ = \cot Z \cdot \tan \overline{\epsilon}$$

or

$$R \cdot \operatorname{Sin} \theta = \operatorname{Cot} Z \cdot \operatorname{Cot} \in ,$$

then

$$\operatorname{Cot} \in \cdot \operatorname{Tan} \underset{\bullet}{T}B = \operatorname{Cot} Z \cdot \operatorname{Cot} \in$$

therefore

$$Z = \overline{\underset{\bullet}{T}B}$$

But

69:2 $$\underset{\bullet}{T}B = \delta(A\underset{\bullet}{T}) = \delta_2(AB)$$

or

$$Z = \overline{\delta(A\underset{\bullet}{T})} = \overline{\delta(A\underset{\bullet}{H})}$$

which is 68:19.

Therefore

69:2 $$Z = \overline{\delta_2(AB)} = \overline{f_1(AB)}.$$

By the Sine Law

69:3 $$\frac{\operatorname{Sin} HJ}{\operatorname{Sin} ZH} = \frac{\operatorname{Sin} Z}{R} .$$

Solving for Sin ZH and substituting for the other terms of 69:3, we have:

69:5 $$\operatorname{Sin} ZH = \frac{R \operatorname{Cos} \in}{\operatorname{Cos} f_1(\theta)} .$$

But

$$ZH = \overline{\delta(\overline{\theta})}.$$

Therefore

$$\operatorname{Cos} \delta(\overline{\theta}) = \frac{R \operatorname{Cos} \in}{\operatorname{Cos} f_1(\theta)}$$

or

$$f_2(\theta) = \frac{R \operatorname{Cos} \in}{\operatorname{Cos} f_1(\theta)} ,$$

which is 68:11

(In 68:1 we proved that:

$$f_2(\theta) \cdot \text{Cos } f_1(\theta) = \text{R Cos } \epsilon = \text{A constant.}$$

An important relation is given in this problem:

AJ and AD are two intersecting quadrants.

$$ZJ = AB$$

From B a perpendicular is erected to cut AD at Ṭ

and from Z a perpendicular is dropped to cut AD at Ḥ,

then

$$AṬ = ḤD.$$

Abū Naṣr mentions here that instead of using

$$\frac{\text{R cos } \epsilon}{\text{Cos } f_1(\theta)} = f_2(\theta)$$

for finding $f_2(\theta)$, we can find $f_2(\theta)$ by use of the relation:

$$\text{Cos } \delta(\bar{\theta}) = f_2(\theta).$$

38. Abū Naṣr's Trigonometry

We here summarize the trigonometry used by Abū Naṣr in solving the preceeding problems. He states[105] that: "Everything we mentioned in the proofs is (based) on our Elements[106].... It is clear that the previous men did not have these elements as a foundation..., but they (used) the regula sex quantitatum (Menelaus' Theorem)".

Abū Naṣr uses the following theorems in his proofs.

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105. N, 70:16.
106. Probably a book written by him, perhaps his version of the Menelaus Sphaerica, Krause (2) in the bibliography.

(a) The <u>Sine Law</u> for the right spherical triangle, namely

$$\frac{\text{Sin } A}{\text{Sin } a} = \frac{\text{Sin } C \ (= 90^\circ = 1.0 = R)}{\text{Sin } c}$$

This law appears extensively in most of the problems.

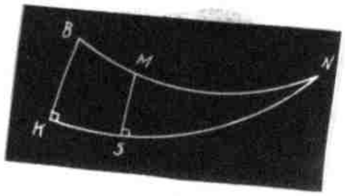
(b) The <u>Sine Law</u> for <u>oblique triangles</u>:

$$\frac{\text{Sin } A}{\text{Sin } a} = \frac{\text{Sin } B}{\text{Sin } b} = \frac{\text{Sin } C}{\text{Sin } c}.$$

He used this law in 6:16 and 18:7 of N.

(c) <u>The Rule of Four</u> (Regula Quatuor Quantitatum) for sines only:

$$\frac{\text{Sin } SM}{\text{Sin } BK} = \frac{\text{Sin } MN}{\text{Sin } BN},$$



This rule was used in 4:15, 5:11, 25:4 and 25:8.

(d) <u>The properties of polar triangles</u>: If two spherical triangles are polar to each other an angle of one triangle is supplementary to the opposite side of the other triangle. This property was used on 28:18.

(e) In Section 36 he makes use of the following theorem, treating it as though it were well known. In the birectangular triangle AJD (see figure of Section 36) if ṬB is perpendicular to AB, ZJ = AB, and ZḤ is perpendicular to AD, then

DḤ = AṬ.

PROBLEMS SOLVED BY USE OF THE "TABLE OF RECTIFICATION."

N				HI	HB	ESCORIAL MUMTAḤAN	ABU NAṢR'S MINUTES TABLE	
GIVEN	REQUIRED	DUE TO	PROBLEM	FOLIO	FOLIO	FOLIO	PROBLEM	FUNCTIONS USED
α, φ	d, D	ḤABASH	3:16	190:39 227 134:24	99 90 97			f_{3b}, f_{4a}
$\frac{\sin\lambda}{\sin\alpha} = \frac{\cos\delta(\lambda)}{\cos\epsilon}$		ABŪ NAṢR	4:5					
λ, φ	d	ABŪ NASR	4:19				7:14	
$\frac{\sin \max d}{\sin d} = \frac{R}{\sin\alpha}$		ABŪ NAṢR	6:3					
λ	α	ABŪ NAṢR	9:14				6:7	f_{3a}, f_{4a}
α_M, φ	$\bar{a}\,e$	ḤABASH	10:17	134:32				f_1, f_2
β, λ	δ_s	ḤABASH	13:12	226		f. 92 V		f_1, f_2
δ, λ	Transit	ḤABASH	15:14	226	99	f. 93 r		f_{3a}, f_{4a}
β, λ	Transit	ABŪ NAṢR	18:4					
λ_M, h_s, h_M	β	ḤABASH	19:5	169:34				f_2, f_{3a}
α_M, β	λ_s	ḤABASH	19:16	169:34				f_{3a}, f_{4a}
Transit, δ_s, α	β	ḤABASH	23:1	226	99			f_1, f_2
$\alpha_{Tr}, \lambda_{Tr}, \beta$	λ_s	ḤABASH	23:8	227	99			f_{3a}, f_{4a}
λ_M, φ	λ_H	ḤABASH	24:1					f_1, f_{4b}
a_e, λ_M, φ	λ_H	ḤABASH	28:7					f_1, f_2
φ, λ_M	λ_H	NAIRĪZĪ	30:5					f_1, f_2, f_{4a}
φ, α_M	λ_H	ABŪ NAṢR	32:6					f_2
φ, δ_s	d, D	NAIRĪZĪ	32:11					f_2, f_{4a}
φ, δ_s	d, D	ḤABASH	34:9		99			f_{4b}
$\begin{cases}\delta\\ \epsilon\end{cases}$	$\begin{cases}\epsilon\\ \delta\end{cases}$	NAIRĪZĪ	56:3					
$O_a, \delta(\lambda)$	d	NAIRĪZĪ	57:6					
φ	Ascensions of AZIMUTH	ḤABASH	58:14		93			f_{4a}
α, d, φ	λ_H	ḤABASH	63:6		99			f_{3a}, f_{4a}
$\sin\theta$	$f_3(\theta)$	ABŪ-JA'FAR AL-KHĀZIN	67:12					
$f_1(\theta)$	$f_2(\theta)$	ABŪ-JA'FAR AL-KHĀZIN	68:11					f_1, f_2
$f_1(\theta)$	$f_{3a}(\theta)$	?	69:10					f_1, f_{3a}

CHAPTER VI

THE CONCLUSION

39. The Development of Tables of Functions

The use of tables for solving mathematical and astronomical problems is as old as mathematics itself. The Babylonians used such tables in multiplication, in finding roots, and in numerous other operations. Ptolemy in his Almagest had tables of declinations, right and oblique ascensions, planetary mean motions, and many others. His table of chords may be considered as the earliest surviving attempt at constructing a trigonometric table.

The Indians also used tables in their astronomical books. Implicit in the Khaṇḍakhādyaka of Brahmagupta are several astronomical tables, including one of the sine function.

Thus, when the Arabs also began their scientific activity at the end of the eighth century, they knew the Hindu and Greek astronomical tables.

Al-Khwārizmī's Zīj, written ca. 840, contains astronomical and trigonometric tables. These are the earliest Moslem tables which have survived and they contain not only the sine function, but also a table of 12 cot θ.

Al-Battānī (d.929) also compiled many astronomical tables, such as declination, mean motion, and planetary equation tables. The third chapter of his zīj is devoted to trigonometry. In addition to the sine function he, like al-Khwārizmī, has a table of the function 12 cot θ. These two examples are given because they are the only astronomers contemporary with Ḥabash whose tables have survived. Probably they are typical of the many which have disappeared.

But all these were individual tables of individual functions which in many cases served one purpose only. A table of declination, for example, exists essentially for finding the distance of a given point on the ecliptic from the equator, and is not used for any other purpose.

But the "Table of Rectification" of Ḥabash was of another type; and has been set up for another reason. The idea of Ḥabash was to assemble a set of a few functions, each function perhaps of little use in itself, but so chosen that by combining them or by performing successive operations with them the user is enabled to solve a wide range of practical problems in spherical astronomy. In the time of Ḥabash this notion seems to have been unique, and he was apparently the first to work out such a set of tables.

With the exception of $f_1(\theta)$ which is $\partial_2(\theta)$, Ḥabash's other functions are artificial and do not have any immediate physical interpretation. The entries in a table of right ascensions, for instance, give the lengths of actual arcs on the celestial sphere. But a table of cos $\partial(\overline{\theta})$, Ḥabash's $f_2(\theta)$ does not give any such arc.

It is clear from the solutions of the problems of Ḥabash in N that he is trying to show the wide range of application of his "Table of Rectification". Abū Naṣr also noticed this (in N 63:8-9) and remarks that "Habash... refers all operations to the Table of Rectification."

Let us examine if Ḥabash was in fact able to restrict himself to his table, and to what extent he succeeded in doing this.

Ḥabash is credited with the solution of thirteen problems in N (Cf. the chart on page 121.) In all of these he used, in addition, a table of sines. Beyond this he made use of a table of right ascensions in three problems and a tangent table in three more problems. Ḥabash has a problem for finding the right ascensions in N (Section 21 above), and f_{4b}, in fact, is a tangent table. Therefore it is not surprising that he uses these two functions in the solution of his problems.

Thus, with the exception of the sine table, we find that Ḥabash was able to restrict himself to his "Table of Rectification" in this considerable array of problems.

Let us compare the methods of solution used by Ḥabash with procedures applied when additional tables are available. Of the thirteen problems, Ḥabash's solutions for seven are no longer than solutions made without the "Table of Rectification". For the remaining six problems, however, his work is more lengthy than would be the case if other tables were allowed. In particular, in the problem of Section 36, Ḥabash uses an extremely long and complicated method for finding the ascendant, just because he confined himself to his tables.

40. The Emulators of Ḥabash

It is clear that Ḥabash's idea made a favourable impression on at least a few other mathematicians. The first of these was al-Nairīzī.

We know from Abū Naṣr that al-Nairīzī added the "Table of Rectification" to his Zīj and called it the "Universal Table"[107] (al-Jadwal al-Jāmiᶜ). But we also find that al-Nairīzī constructed at least two other tables to serve the same general purpose as the "Table of Rectification". These two tables of al-Nairīzī form the ninth and the tenth functions of a set of tables which appears in HB, between ff. 82r and 84v. This set is called the "Universal Tables" (al-Jadāwil al-Jāmiᶜa). This writer believes that these tables are the same which are mentioned by Abū Naṣr to be Nairīzī's tables.

Al-Nairīzī was sufficiently interested in applying the "Table of Rectification" that he solved some of the problems of Ḥabash by alternative methods. These problems are four in number and appear in Sections 30, 31, 33, and 34 above. In all these problems he utilized the Sine Law. In two of them he made no use of the "Table of Rectification".

Another man also influenced by Habash is Abū Naṣr. Not only did he supply proofs for the operations of Ḥabash

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107. N, p. 30.

together with commentaries on the methods, but he himself wrote similar tables which he called the "Table of Minutes" (Jadwal al-Daqā'iq), and which contain five functions.

Apparently Abū Naṣr had two reasons for computing his own set of tables. The first may have been to show that he also was able to originate tables of a type similar to those of Ḥabash and to serve the same general purpose. The second is probably connected with the fact that these tables are in minutes. The use of minutes instead of degrees in tables simplifies the computations considerably. This point will be discussed in the following section.

The following are the five functions of Abū Naṣr in his table[108].

$$f_1(\varphi) = \frac{\sin\epsilon}{\cos\varphi}$$

$$f_2(\varphi) = \frac{\tan\varphi}{\cot\epsilon}$$

$$f_3(\lambda) = \cot\epsilon \cdot \tan\partial(\lambda)$$

$$f_4(\lambda) = \sin\lambda$$

$$f_5(\lambda') = f_4[\beta(\lambda')].$$

The operations used by Abū Naṣr are more straightforward and easier than the operations of Ḥabash; but Abū Naṣr does not give proofs for his own methods.

108. The information regarding these functions was kindly supplied by Mr. Muhammad Agha.

Abū Naṣr himself has in N six problems which are in general alternative methods to those of Ḥabash. He used the "Table of Rectification" in two of them, while the remaining four were solved by use of the sine table alone.

A third scholar, Abū Jaʿfar al-Khāzin (cf. Section 7 above) had an interest in the "Table of Rectification". He has two problems which appear in N and in which he utilized the sine function in addition to direct application of the "Table of Rectification". These two problems are discussed in Section 37 above.

41. Emergence of the Modern Trigonometric Functions

We have seen that, in general, all trigonometric tables of the Islamic period are tables of what we have called "cap functions" (cf. Section 15 above).

The main advantage of the modern functions over these is that if a pair of the modern trigonometric functions is combined to give a third similar function, the radius of the defining circle, R, does not enter explicitly. But the corresponding combination with the cap functions requires a manipulation of R. For instance

$$\tan \theta = \frac{\sin \theta}{\cos \theta},$$

whereas

$$\mathrm{Tan}\ \theta = R \cdot \frac{\mathrm{Sin}\ \theta}{\mathrm{Cos}\ \theta}.$$

Abū Naṣr mentions more than once in N[109] that Ḥabash has two copies of his tables. In the first copy, says Abū Naṣr, he has "the third function multiplied by sixty, while in the second copy it is not". Thus in the second copy the function tabulated is $\cos \theta / \cos \partial(\bar{\theta})$ rather than $R \operatorname{Cos} \theta / \operatorname{Cos} \partial(\bar{\theta})$. (Cf. Section 17 above.) Abū Naṣr prefers the use of the second copy. He states clearly that use of it eliminates the need for constant manipulation of sixty, and that the results become "more correct and easier"[110].

Abū Naṣr not only criticised Ḥabash for introducing R in his results, but he himself compiled tables where R is not used. He, as mentioned before, called his tables the "Table of Minutes", perhaps to emphasize the fact that no R is involved. They appear in the Rasā'il of Abū Naṣr to al-Bīrūnī, next to N.

The remarks of Abū Naṣr mentioned above and his construction of the "Table of Minutes" constitute two steps in the direction of the modern trigonometric functions.

42. Invention of the Tangent Function

There is a certain amount of confusion in the published literature with regard to the first use of the tangent function. In this section we will review this material and show that some of the statements made cannot be supported on

109. For example 3:13, 4:2, 11:5, 14:3, 23:7, 30:14, and 70:4-10.
110. N, 70:10.

the basis of available evidence, and that other claims can be established, but only by use of material not available to the scholars who made the claims.

Sarton makes[111] the following statement concerning the use of the six trigonometric functions: "Suter says that the six lines were already known by Ḥabash al-Ḥāsib?". Sarton does not give any reference indicating how he obtained this opinion from Suter. On the basis of a remark by Suter[112] and a similar statement by Smith[113], it seems clear that this opinion was derived from Nallino.

Nallino himself writes that[114]

> "At the end of the third century A.H. or the beginning of the fourth (end of ninth century A.D.), the Arabs had arrived at the knowledge of all these principles relating to right spherical triangles, because I found it used for the solution of the problems of spherical astronomy in the unique manuscript of the Zīj of Aḥmad ibn ʿAbdallah, known as Ḥabash the Calculator, which is kept in the Berlin Library (HB). This zīj was composed very few years after the third century, as I concluded by different reasons. Naṣīr al-Dīn al-Ṭusī[115] (d. 1274) was mistaken in attributing the invention of the use of tangents in the solution of right spherical triangles to Abū-al-Wafa' al-Būzjani (died 998)."

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111. Sarton, V. 1, p. 667, footnote two.
112. Suter (1), p. 209.
113. Smith, V. 2, p. 620.
114. Nallino (1), p. 248-249.
115. "Kitāb al-Shakl al-Qitaʿ", the Book of the Regula Sex Quantitatum, Constantinople, 1309 A.H., p. 126. Naṣīr al-Dīn's source is Abū-al-Riḥān al-Bīrunī who died in 1048".

A third writer, Björnbo[116] states that Ḥabash was the first mathematician to compose a table of 1,0 cot θ.

We believe that all the above writers depend ultimately on HB as a basis for their assertions. This is certain in the case of all except Björnbo, whose original publication we have not seen. Nallino thought that HB was entirely the work of Ḥabash, and Nallino's opinion was followed by all later writers. But in this connection we refer the reader to Section 11 above where we have shown that this zīj (HB) is a mixture of material from many sources including Ḥabash himself. Therefore HB cannot be used as an authority for the statement that Ḥabash knew the six trigonometric functions nor that he used either a tangent or a cotangent table.

It is true that on f. 85r tables of 1,0 tan θ and 1,0 cot θ make up part of the "Proportion Tables" (Jadāwil al-Nisab), but there is no essential reason for thinking that these are the work of Ḥabash.

Abū-al-Wafa' may very well have been the first to use the tangent function in the solution of right spherical triangles, as mentioned by Naṣīr al-Dīn on the authority of al-Bīrūnī. For Nallino, in denying the statement, again assumes that all of HB was written by Ḥabash.

Nevertheless there is strong reason for thinking that many of Nallino's claims are correct.

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116. Björnbo and Suter, p. 76.

We find in HI, for instance, two tables of 1,0 tan θ (see Section 16 above). We have every reason for thinking that HI is entirely the work of Ḥabash. It contains an introduction written by him (see Section 4 above), and it has most of the problems found in N, together with his own dated observations. Moreover Abū Naṣr mentions clearly that Ḥabash compiled a tangent table when he says (N, 70:19 - 71:1): "But the copy of the 'Table of Rectification' which has the tangent of the argument as the fourth (function)...". Hence we can conclude that these tangent tables are due to Ḥabash.

Furthermore Ḥabash actually makes use of this tangent table in the solution of the problem on f. 156r line 1 of HI, the problems 24:1 and 34:18 of N, and the problem on f. 99 of HB:

The tangent table appears in HI twice; the first is as a part of the "Table of Rectification" as function f_{4b} on ff. 226r - 227r, and the second as a separate table on ff. 227v - 228r.

The facsimile of the separate table of HI together with its transcription appear on pages 64 - 67. The table gives 1,0 tan θ for intervals of 30' between 30' and 90°. The functional values are given to three sexagesimal places, and when compared with the entries of Ulugh Beg's tangent table[117], most entries are identical except for a few where the maximum difference is 0;0,4.

117. Bodleian Library, Oxford, Arabic Ms. LXX (Pocock 226).

In estimating the significance of Ḥabash's work we should state that al-Khwārizmī and al-Battānī, both contemporaries of Ḥabash, give a table of 12 cot θ in their respective zījes. In fact this function, called the "second shadow" (<u>al-ẓill al-thānī</u>) or the "horizontal shadow" (<u>al-ẓill al-mabsūṭ of al-mustawī</u>) is found in most of the zījes written down to the fifteenth century. The name indicates the origin of the function; it is the horizontal shadow cast by a vertical <u>gnomon</u> (i.e., stick, <u>miqyās</u> or <u>shakhṣ</u>). For instance, the gnomon of al-Khwārizmī and al-Battānī for the cotangent function was divided into twelve parts. The division of the gnomon into twelve parts is of Indian origin. The cotangent function was applied in sundial theory.

On the other hand the tangent function was called the "first shadow" (<u>al-ẓill al-awwal</u>), or the "vertical shadow" (<u>al-ẓill al qā'im</u> or <u>muntaṣib</u>), or the "inverted shadow (<u>al-ẓill al-ma'kūs</u>). The name also indicates the origin of the function, it is the vertical shadow cast by a horizontal gnomon.

Hence it is clear that Ḥabash can hardly be given credit for the introduction of the cotangent function. But <u>no tangent function was, to our knowledge, used by any mathematician before Ḥabash, and hence, (to our present knowledge) he is the inventor of the tangent function and the first to use it in the solution of problems</u>.

Moreover, his choice of R as 1,0 rather than 12 or some other constant deserves much credit. This fact is stated by Abū Naṣr (N, 25:5 and 25:13) thus: "... his (Ḥabash's) gnomon is divided by the divisions of the total sine (<u>sinus totus</u>, i.e., Sin 90° = 1,0)".

It should be emphasized that Ḥabash's 1,0 tan θ and the modern tan θ, if the latter is expressed in the sexagesimal system have identical digits and differ only in the position of the "sexagesimal point".

For example,

Habash's 1,0 tan 30° = 34;38,28

while tan 30° = 0;34,38,28.

Manipulation of the R in Ḥabash's cap tangent is simply a matter of shifting the sexagesimal point, hence his function is easier to use than, say 12 cot θ.

<u>Therefore the cap tangent of Ḥabash is a significant step in the direction of trigonometry as we know it.</u>

BIBLIOGRAPHY

(In the alphabetical listing abū or ibn is considered part of the name; the article al- is not.)

Abū Naṣr Manṣūr, Rasā'il Abī Naṣr ila al-Bīrūnī (Fifteen treatises), Osmania Oriental Publications Bureau, Hyderabad-Deccan, 1948.

Almagest: See Halma.

Battānī, Zīj of: See Nallino (2).

Berlin: Ahlwardt, W., Verzeichniss der arab. Hss. der Königlichen Bibl. Zu Berlin, v.s., Berlin, 1893.

Al-Bīrūnī, Abul-Rīḥān Muḥammad (1), Rasā'il al-Bīrūnī (Four treatises: 1. Istikhrāj al-Awtār fil-Dā'irah. 2. Ifrād al-Maqāl fī 'Amr al-Ẓilāl. 3. Tamhīd al-Mustaqarr li Ma'na al-Mamarr. 4. Rāshikāt al-Hind), Osmania Oriental Publications Bureau, Hyderabad-Deccan, 1948.

——— (2) Al-'Āthār al-Bāqiyah 'An al-Qurūn al-Khāliyah (Chronology of Ancient Nations), ed. by E. Sachau, Leipzig, 1878.

Al-Bīrūnī Commemoration Volume, 362-1362 A.H., Iran Society, Calcutta, 1362 A.H.

Björnbo and Suter, Die astronomischen Tafeln des ... al-Khwārizmī ..., Copenhagen, 1914.

Brockelmann, C., Geschichte der arabischen Litteratur, 2 vs. (2nd ed.) and 3 suppl. vs., Leiden, 1943.

Buttmann, "Ueber Klaudius Ptolemäus", Museum der Alterthum-Swissenschaft, v. 2, (1810), pp. 483-485.

Caussin de Perceval, Le Livre de la grand table Hakémite, Paris, 1804.

Derenbourg, Les mss. arabes de l'Escurial, Tome II, fasisc. 2, Paris, 1941.

Encyclopedia Britannica, Great Britain, 1955 edition.

Encyclopaedia of Islam, A Dictionary of the Geography, Ethnography, and Biography of the Muhammadan Peoples. 3 vs., Leyden, 1913-1936.

Escorial: see Derenbourg.

al-Fihrist: see Ibn al-Nadīm.

Hajji Khalifeh, Kashf al-Ẓunūn ʿan Asāmī al-Kutub wal-Funūn, 2 vs., Istanbul, 1313 A.H.

Halma, M., Composition Mathematique de Claude Ptolémée, 2 vs., Paris, 1813-16.

HB: the Berlin Ḥabash Zīj. See Section 11.

Heath, T.L., A History of Greek Mathematics, 2 vs., Oxford, 1921.

HI, the Damascene Zīj of Ḥabash. See Section 10.

Hitti, P.K., History of the Arabs, London, 1940.

Ibn al-Nadīm, Al-Fihrist, Arabic edition, Cairo, 1948.

Ibn al-Qifṭī Yūsif, Akhbār al-ʿUlamā' bi Akhbār al Ḥukamā', Cairo, 1326 A.H.

Ibn Yūnis, The Ḥakimite Zīj, or al-Zīj al-Ḥākimī, Ms. Cod. Or. 143, Leiden.

India Office: Loth, O., Cat. of the Arabic Mss. in the Library of the India Office, London, 1877.

Irani, R., "Arabic Numerical Forms", Centaurus, V. 4, 1955, No. 1, pp. 1-12.

Kasir, D., The Algebra of Omar al-Khayyam, New York, 1931.

Kennedy, E.S., "A Survey of Islamic Astronomical Tables", Transactions of the American Philosophical Society, V. 46 (1956), pp. 123-177.

Kennedy, E.S., and Transue, W.R., "A Medieval Iterative Algorism", The American Mathematical Monthly, V. 63 (1956), pp. 80-83.

Khwārizmī, Zīj of: see Björnbo and Suter.

Krause, M. (1), "Stambuler Hss. Islamischer Mathematiker", Quellen und Studien zur Geschichte der Mathematik Astronomie und Physik, Abt. B., Bd. 3., pp. 437-532, Berlin, 1936.

——— (2), Die Sphärik von Menelaos aus Alexandrien in der Verbesserung von Abū Naṣr Manṣūr b. ʿAlī b. ʿIrāq, Berlin, 1936.

Leiden: Dozy, de Jong, de Goeje et Houtsma, Catalogus codicus orientalium bibliothecae academiae Lugduno Batavae, 6 vs., Lugd. Bat. 1851-1877.

N, Rasāʾil Abū Naṣr No. 4. See a description in Section 9.

Nallino, C. (1), ʿIlm al-Falak, Tārīkhuhū ʿInd al-ʿArab fil-Qurūn al-Wusṭā, Rome, 1911. This has been translated into Italian and published as "Storia dell' astronomia, presso gli Arabi nel Medio Evo", in V. 5 of Raccolta di Scritti ..., Rome, 1944, pp. 88-329.

——— (2), Al-Battani sive Albatenii Opus Astronomicum, V. 3, Milan, 1899-1907.

Neugebauer, O. (1), The Exact Sciences in Antiquity, Copenhagen, 1951.

——— (2), "The History of Ancient Astronomy", Journal of Near Eastern Studies, 4 (1945), pp. 1-38.

Paris: Catalogue des Mss. arabes par M. le Baron de Slane, Paris, 1883-1895.

Rajā'ī, N.L., "The Invention of Decimal Fractions in the East and in the West", Thesis, American University of Beirut, 1951.

Rasā'il: See Abū Naṣr or al-Bīrūnī.

Rozenfeld and Yushkevich, "The Mathematical Tracts of Omar Khayyam" (in Russian), Istoriko-matematicheskie Issledovaniya, V. 6., Moscow, 1953, pp. 1-172.

Sarton, G., Introduction to the History of Science, V. I, 1927; V. II part I, 1951; V. III, part I, Washington, 1947.

Sengupta, P., The Khaṇḍakhādyaka, an Astronomical Treatise of Brahmagupta, Calcutta, 1934.

Smith, D.E., History of Mathematics, 2 vs., New York, 1925.

Spuler, B., Iran in früh-islamischer Zeit, Weisbaden, 1952.

Struik, D., A Concise History of Mathematics, 2 vs., New York, 1948.

Suter, H. (1), Die Mathematiker und Astronomen der Araber und ihre Werke, Abhand, zur Gesch der Math. X. Heft., Leipzig, 1900.

——— (2), "Zur Trigonometrie der Araber", (Bibliotheca Mathematica, herausgegeben von G. Eneström, 3. Folge, 10. Bd., 1910, 156-160).

De Vaux, Carra, "L'Almageste d'Abû'lwafa Albûzdjâni", Journal Asiatique, 1892, pp. 408-471.

Woepcke, F., L'algebre d'Omar Alkhayyāmi, Paris, 1851.