SOLUTION OF THREE-DIMENSIONAL ELASTICITY
PROBLEMS IN NUCLEI OF STRAIN

By

Mark Lesley

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James D. Talman

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ABSTRACT

Kelvin's solution to the equations of the linear theory of elasticity gives the displacements and stresses produced by a concentrated force acting at a point in the interior of a solid of indefinite extent. Nuclei of strain are solutions obtained by differentiation, integration, and superposition, from Kelvin's solution. A number of solutions for bounded bodies may be constructed by appropriately superposing the nuclei as unit solutions, to obtain desired conditions at the boundary of the solid.

The problems considered in this thesis concern the effects of a concentrated force acting in the interior or on the surface of a solid bounded by a plane, or by two or three perpendicular planes. For some problems of this type, specifically those Boussinesq, Cerruti, and Mindlin for a solid with a single free plane boundary, the solutions in nuclei of strain are well known. Some other problems of the type considered have been solved using Papkovitch functions; specifically, the problem of Rongved for a solid with a single fixed plane boundary, and the problems of Hijab for solids bounded by a single plane and by three perpendicular planes, on which certain mixed conditions obtain.

We present in this thesis an organization and characterization of nuclei of strain based on their known Galerkin vector and potential function representations, in which certain properties of the nuclei are clearly displayed. Using these properties, construction of solutions to many problems of the above mentioned type becomes a relatively straightforward process.

We construct the solutions for a concentrated force in the interior of a solid bounded by one, two, or three planes with two mixed boundary conditions on the planes, the solutions for a solid bounded by a single plane which may be fixed or free, and the solutions for a solid bounded by two or three planes, with mixed conditions on all planes but one, which may be either fixed or free.

Constructing solutions in nuclei reveals relationships between the various problems which are interesting in themselves, and which may prove useful in constructing solutions for more difficult problems of the same type.

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CHAPTER I

INTRODUCTION

1. Historical Background*

In 1848, Lord Kelvin presented a fundamental solution in the linear theory of elasticity. Kelvin's solution gives the displacements and stresses at any point of a homogeneous isotropic solid of indefinite extent, caused by a concentrated force acting at a point interior to the solid. An infinite class of solutions may be derived from Kelvin's by differentiation, integration, and superposition. From these, known collectively as nuclei of strain, the solutions of many problems of practical significance have in turn been obtained. For example, the solution to Lame's problem (10) of a spherical container under uniform external or internal pressure, and of Southwell's problem (11) of a cavity in the interior of an infinite solid under uniform tension can be obtained by combining suitable nuclei. In constructing the solutions, the appropriate nuclei are selected and

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*In the preparation of this section I have drawn heavily upon the introductory sections of (1, 7, 14). Here, and within the text hereafter, numbers inside parentheses refer to the correspondingly numbered entries in the Bibliography.

superposed so as to obtain the desired conditions of displacement or stress at the boundary of the body. All the nuclei have singular lines or points, at which the stresses and displacements become infinite, which points are thus imagined to be outside the body, or within a cavity inside the body.

Among the problems which are themselves concerned with the effects of a concentrated force, Boussinesq's problem of a normal force on the plane boundary of a semi infinite solid, and Cerruti's problem of a tangential force on the same boundary, can be solved in nuclei of strain (11), though they were first solved, in 1879 and 1882 respectively, by the use of potential theory (1). Following Westergaard's interpretation (11) of these two problems, in which he also introduced the Galerkin vector representation of the nuclei, R.D. Mindlin in 1935 solved the problems of a concentrated force in the interior of a semi infinite solid (5, 7). More recently (9) Mindlin and Cheng have derived from these last two solutions a series of nuclei for the semi infinite solid, and have shown their practical importance.

It was Mindlin himself, however, who in 1953 initiated (12) a new method of attack on problems concerned with the effects of a concentrated force in the interior of a solid with specified boundary conditions. After noting that the essential procedure in solving a problem by nuclei is guessing, he showed how, by an ingenuous combination of the Papkovitch functions solution of elasticity and Green's analysis, the solutions he had previously found in nuclei, could be derived directly. The Papkovitch functions approach has been used on all concentrated force problems since this time. In 1955, L. Rongved derived (13) the solutions for a force in the interior of a semi infinite solid with a fixed plane boundary, and in 1956 W. Hijab showed (14) that the power of the method is sufficient to enable the derivation of solutions under mixed boundary conditions, and for bodies with composite boundaries, such as body bounded by three perpendicular planes.

2. Scope of the Thesis

This thesis is concerned with the solution, in nuclei of strain, of concentrated force problems in a solid of indefinite extent bounded by a plane or by two or three perpendicular planes (half-space, quarter-space and eighth-space). While the construction of solutions in nuclei does not have the certainty and straightforwardness of the more powerful techniques, still, the nuclei have some properties which can be exploited to eliminate much of the guesswork usually encountered in this procedure.

First, we take full advantage of the Galerkin vector representation of the nuclei by constructing tables from which the displacements and stresses produced by a nucleus can be written out rapidly. This eliminates much of the computational drudgery encountered when solving problems and also provides, in conjunction with the Galerkin vector, an analytical characterization of the nuclei, which possesses a clarity lacking in the intuitive characterization generally used. The tables are also useful in solving problems in that one may see in them the effects of superposition on various of the expressions which appear as displacements and stresses of nuclei, after which, consideration of the Galerkin vector will lead to the selection of a nucleus which creates the desired effect on the boundary.

In showing how this is done, we extend, and to a certain extent unify, the collection of concentrated force solutions mentioned in the last section. As it happens, Kelvin's solution and the nuclei derived from it are particularly suitable for mixed boundary condition solutions of the type considered by W. Hijab. We derive in Chapter III the solutions of all such problems that can occur in half-space, quarter-space, and eighth-space. At the end of Chapter IV, we supplement this set with solutions in quarter-space and eighth-space with one boundary either fixed or free, and the mixed condition on the others.

In Chapter IV, we show how the tables and vector representation can be used, by constructing in a relatively straightforward manner, solutions for the problems of Rongved and Mindlin. In nuclei, these solutions are more complicated than the mixed boundary solutions, and those of Mindlin more complicated than those of Rongved, but the construction of the solutions in nuclei reveals striking relationships among the problems, and shows the simpler solutions useful in constructing the more difficult ones.

Mindlin has shown how the solutions in nuclei of the problems of Boussinesq and Cerruti can be derived from his solutions. At the end of Chapter IV, we investigate the fixed boundary analogues to the solutions of Boussinesq and Cerruti, which may be obtained in the same manner from the solution in nuclei, of Rongved's problem.

The results reported here were found in the course of investigating the problem of a concentrated force in quarter space with free boundary. This seems to be an intrinsically difficult problem, having yielded as yet neither to nuclei nor to Papkovitch functions. It is hoped that the results of this work will contribute toward its solution.

3. Equations of Elasticity; Galerkin Vector and Papkovitch Functions

The fundamental equations of the linear theory of elasticity are the equilibrium equations expressing the condition that the resultant force on any element of the elastic solid is zero, and the generalized Hooke's laws, stating the relationship between the stresses and displacements in an ideally elastic, isotropic solid (10). Substitution of Hooke's laws into the equilibrium equations produces the so-called basic equation of elasticity, which may be written concisely in vector form:

$$G(\Delta + \frac{1}{1-2\nu} \text{grad div}) \underline{u} = 0, \qquad [1]$$

where G is the modulus of rigidity, ν is Poisson's ratio, Δ is Laplace's operator, and

$$\underline{u} = \underline{i}u_x + \underline{j}u_y + \underline{k}u_z$$

is the displacement vector. In deriving eq. [1] it has been assumed that body forces are negligible: we shall assume zero body forces throughout this thesis.

Solving a problem in elasticity might then be said to consist of finding a vector function (or its component scalar function) which satisfies eq. [1] throughout the solid and which produces displacements satisfying desired conditions on the surface of the solid. If the conditions are made on the displacements, the problem is said to be a first boundary value problem; if on the stresses, a second boundary value problem. Or some of conditions may be made on the displacements, and some on the stresses, in which case the problem is said to be a mixed boundary value problem.

Two forms of solutions of eq. [1] are of interest to us in this thesis. The Galerkin vector, with scalar components the Galerkin functions (2), will be essential in our work with nuclei. The displacements are derived from the Galerkin vector by the formula

$$2G\underline{u} = [2(1-\nu)\Delta - \text{grad div}]\,\underline{F}\,, \qquad [2]$$

where

$$\underline{F} = \underline{i}F_x + \underline{j}F_y + \underline{k}F_z$$

is the Galerkin vector and F_x, F_y, and F_z are the Galerkin functions. In order that the displacement vector derived by eq. [2] should satisfy eq. [1], the Galerkin vector must satisfy the biharmonic equation

$$\Delta\Delta\underline{F} = 0.$$

Westergaard has observed (11) that a single axially symmetrical component of the vector is identical with Love's strain function (1, 10). We consider the Galerkin vector in detail, in the next chapter.

The Papkovitch functions solution (3, 4), though of import for the methodical solution of concentrated force problems (14), will be employed only when we desire to compare a solution obtained here in nuclei of strain, with one given elsewhere in Papkovitch functions. For this, we make use of the relationship between the Galerkin vector and the Papkovitch functions given by Mindlin (8):

$$\begin{aligned} \underline{i}B_x + \underline{j}B_y + \underline{k}B_z &= \frac{1-\nu}{G}\,\Delta\underline{F} \\ \beta &= \frac{1-\nu}{G}\,(2\,\mathrm{div}\,\underline{F} - \underline{R}\cdot\Delta\underline{F}) \end{aligned} \qquad [3]$$

where B_x, B_y, B_z and β are the Papkovitch functions, and

$$\underline{R} = \underline{i}x + \underline{j}y + \underline{k}z$$

is the radius vector.

4. Some Well-known Nuclei of Strain

In this section we present some typical nuclei of strain in the terms commonly used for their description. It is in large part, therefore, a synopsis of Love's description (1) of the nuclei. The Galerkin vectors for the nuclei are taken from Mindlin (10), except in some cases where minor alterations have been made to conform more closely with Love's discussion.

a) Single force (Kelvin's solution). The displacements and stresses at a point (x, y, z) in the elastic solid, due to a single force acting at the origin is given in Chapter III, 1. The displacements and stresses are seen to be singular at the origin, which is thus taken to be outside the solid. Specifically, the conditions satisfied by the nucleus are that the resultant of forces on a small cavity surrounding the origin is a simple force, and that the displacements and stresses vanish at infinite distance from the origin. The Galerkin vectors for the single force are:

$\underline{F} = \underline{i}R$ (single force in x-direction),

$\underline{F} = \underline{j}R$ (single force in y-direction),

$\underline{F} = \underline{k}R$ (single force in z-direction),

where R is the distance from the point (x, y, z) to the origin.

b) Double force in z-direction. We superpose a single force in the z-direction at (0, 0, 0) and an equal but oppositely directed force at (0, 0, h). Dividing the magnitude of the force by h and diminishing h indefinitely, we obtain the double force in z-direction. The components of displacement and stress at a point of the body are then the partial derivatives with respect to z, of the corresponding displacement and stress components of a simple force in z-direction. The Galerkin vector producing these components is found by a similar differentiation of the vector for the single force:

$$\underline{F} = \underline{k}\,\frac{z}{R} \quad \text{(double force in z-direction).}$$

c) Double force in z-direction with moment about y-axis. We superpose a single force in z-direction at (0, 0, 0) and an equal but oppositely directed force at (h, 0, 0) and pass to the limit as before. The components of displacement and stress are thus the partial derivatives with respect to x of the corresponding components of the single force. The forces produced in the neighborhood of the origin are equivalent to a couple about the y axis, hence the name of the nucleus. The Galerkin vector for the nucleus is found from that for the single force in z-direction by differentiating with respect to x:

$$\underline{F} = \underline{k}\,\frac{x}{R} \quad \begin{array}{l}\text{(double force in z-direction}\\ \text{with moment about y-axis).}\end{array}$$

d) Center of compression. A double force in x-direction, a double force in y-direction, and a double force in z-direction are superposed at the origin. The effect on any spherical surface with center at the origin is a uniform normal tension. The Galerkin vector may be taken to be:

$$\underline{F} = \underline{i}\,\frac{x}{R} + \underline{j}\,\frac{y}{R} + \underline{k}\,\frac{z}{R} \quad \text{(center of compression).}$$

e) Line of double forces with moment. We may suppose nuclei of the type in (c) above, to be distributed uniformly along the negative z-axis. The components of displacement and stress are then the integrals with respect to z, between the limits z and ∞, of the corresponding components of the double force with moment. The resulting components are singular along the negative z-axis, but vanish elsewhere at infinite distance from the origin. The definite integrals just mentioned are therefore the negatives of the indefinite integrals, and so we may take for a vector the negative of the indefinite integral of the vector for the double force with moment, suitably adjusted so that the components of displacement and stress vanish at infinite distance. The vector is then:

$$\underline{F} = -\,\underline{k}\;x \log (R+z)$$

(Line, along negative z-axis,
of double forces in z-direction
with moment about x axis).

f) Line of compression. Again, we may suppose centers of compression to be uniformly distributed along the negative z-axis.

By reasoning similar to that in the last paragraph, we find the vector from that for the center of compression:

$$\underline{F} = - \underline{i}\ x \log (R+z) - \underline{j}\ y \log (R+z) - \underline{k}\ R \quad \text{(line of compression along negative z axis).}$$

An indefinite number of nuclei may be obtained by the processes illustrated above. Starting with a single force or center of compression one may differentiate an arbitrary number of times, and integrate as many times as allowed by the condition that the displacements should vanish at infinite distance from the origin. Of course, many nuclei may be formed by simple superposition of nuclei derived from single forces; the center of compression and nuclei derived from it receive special consideration because they are exceedingly useful in solving the type of problem considered in this thesis.

5. Notation for Nuclei of Strain

We will consider the single forces and the center of compression as fundamental nuclei, from which all other nuclei are derived by differentiation and integration. These nuclei will be given letter names:

X	(single force in x-direction),
Y	(single force in y-direction),
Z	(single force in z-direction),
C	(center of compression).

The name of any nucleus derived from these will be formed by prefixing to the letter name of the nucleus from which it is derived, the operators encountered in the derivation.

We will use ∂ to denote partial differentiation, with subscripts indicating number of differentiations and variables with respect to which differentiation is performed. The double forces of the previous section would then receive the names:

$\partial_z Z$ (double force in z-direction),

$\partial_x Z$ (double force in z-direction with moment about y-axis).

We will use the integral sign similarly subscripted to indicate variables of integration and number of integrations. It will be seen in the next chapter that in handling nuclei of strain, we are working with a clearly defined set of functions in which the result of integration is unique, so that the abbreviated notation does not lead to confusion.

The last two nuclei of the previous section would then receive the names

$- \int_z \partial_x Z$ (line, along negative z-axis, of double forces with moment about y-axis),

$- \int_z C$ (line of compression along negative z-axis).

The minus signs appear because we shall require that differentiation of an integral produce the integrand, while the integrals encountered in the derivation of the nuclei yield, upon differentiation, the negative of the integrand.

The justification for this somewhat elaborate system of nomenclature, which has evolved in the course of working with the nuclei, is that it has been found to facilitate thought by expressing characteristics of nuclei significant in solving problems.

6. Symbols Used in the Thesis

The symbols used in the thesis, defined also when they are first used, are collected here for reference.

G	modulus of rigidity
ν	Poisson's ratio
$\underline{F}$	Galerkin vector
F_x, F_y, F_z	components of Galerkin vector
B_x, B_y, B_z, β	Papkovitch functions
$\underline{u}$	displacement vector
u_x, u_y, u_z	components of displacement vector
σ_{xx}, σ_{yy}, σ_{zz}	normal components of stress
σ_{xy}, σ_{yz}, σ_{zx}	shearing components of stress
Δ	Laplace's operator
δ_x, δ_{xx}, δ_{xy}, etc.	partial derivative, with respect to subscript variables
$\int_x$, $\int_{xx}$, etc.	integral, with respect to subscript variables
$\frac{1}{2}\Delta F$	potential function for nuclei
ϕ	biharmonic expression in Appendix Table
$\frac{1}{2}\Delta\phi$	harmonic expression in Appendix Table
(x, y, z)	arbitrary point of elastic solid
R	distance from origin to (x, y, z)

X, Y, Z, C	single forces in x-,y-,z-directions, center of compression
$\int_z \delta_x Z$, $\int_z C$, etc.	nuclei derived from single forces and center of compression by indicated operations
(x', y', z')	point at which nucleus is located
R_{ijk}	distance from (x', y', z') to (x, y, z)
Z_{ijk}, C_{ijk}, etc.	nucleus located at (x', y', z')

where

$$i = \begin{cases} 0 \text{ if } x' = 0 \\ 1 \text{ if } x' = a \\ 2 \text{ if } x' = -a \end{cases} \qquad j = \begin{cases} 0 \text{ if } y' = 0 \\ 1 \text{ if } y' = b \\ 2 \text{ if } y' = -b \end{cases} \qquad k = \begin{cases} 0 \text{ if } z' = 0 \\ 1 \text{ if } z' = c \\ 2 \text{ if } z' = -c \end{cases}$$

$R_{.jk}$, $R_{i.k}$, $R_{ij.}$	value of R_{ijk} at $x=0$, $y=0$, $z=0$ respectively
R_k, Z_k, C_k etc.	abbreviation of R_{00k}, Z_{00k}, C_{00k} etc.
$\frac{P}{8\pi(1-\nu)}$	"force adjustment" so that solution represents force of magnitude P.

CHAPTER II

DISPLACEMENTS AND STRESSES, CHARACTERIZATION OF NUCLEI OF STRAIN

1. Stresses and Displacements of the Nuclei; Galerkin Vector and Potential Function

a) Nuclei derived from X. The displacements and stresses of X are found from the Galerkin vector

$$\underline{F} = \underline{i}R \ .$$

The vector for any nucleus derived from X, is found from the above by performing on it, the same differentiations and integrations as were performed on the displacements and stresses of X in the derivation, hence will be of the form

$$\underline{F} = \underline{i}F_x \ .$$

The components of displacement u_x, u_y, u_z are computed from eq. [2], the components of normal stress $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$, and shearing stress $\sigma_{xy}, \sigma_{yz}, \sigma_{zx}$ follow from these by Hooke's laws. They are:

$$\begin{aligned}
2Gu_x &= 2(1-\nu)\Delta F_x &&- \delta_{xx}F_x \\
2Gu_y &= &&- \delta_{yx}F_x \\
2Gu_z &= &&- \delta_{zx}F_x \\
\sigma_{xx} &= (2-\nu)\delta_x\Delta F_x &&- \delta_{xxx}F_x \\
\sigma_{yy} &= \nu\delta_x\Delta F_x &&- \delta_{yyx}F_x \\
\sigma_{zz} &= \nu\delta_x\Delta F_x &&- \delta_{zzx}F_x \\
\sigma_{xy} &= (1-\nu)\delta_y\Delta F_x &&- \delta_{xyx}F_x \\
\sigma_{yz} &= &&- \delta_{yzx}F_x \\
\sigma_{zx} &= (1-\nu)\delta_z\Delta F_x &&- \delta_{zxx}F_x
\end{aligned} \tag*{[4]}$$

b) <u>Nuclei derived from Y</u>. The vector for Y is

$$\underline{F} = \underline{j}R$$

so the vector for any nucleus derived from Y is

$$\underline{F} = \underline{j}F_y \ .$$

The displacements and stresses are:

$$\begin{aligned}
2Gu_x &= &&- \delta_{xy}F_y \\
2Gu_y &= 2(1-\nu)\Delta F_y &&- \delta_{yy}F_y \\
2Gu_z & &&- \delta_{zy}F_y \\
\sigma_{xx} &= \nu\delta_y\Delta F_y &&- \delta_{xxy}F_y \\
\sigma_{yy} &= (2-\nu)\delta_y\Delta F_y &&- \delta_{yyy}F_y \\
\sigma_{zz} &= \nu\delta_y\Delta F_y &&- \delta_{zzy}F_y \\
\sigma_{xy} &= (1-\nu)\delta_x\Delta F_x &&- \delta_{xyy}F_y \\
\sigma_{yz} &= (1-\nu)\delta_z\Delta F_y &&- \delta_{yzy}F_y \\
\sigma_{zx} &= &&- \delta_{zxy}F_y
\end{aligned} \tag*{[5]}$$

c) <u>Nuclei derived from Z</u>. The vector for Z is

$$\underline{F} = \underline{k}R$$

so the vector for any nucleus derived from z is

$$\underline{F} = \underline{k}F_z \; .$$

The displacements and stresses are:

$$\begin{aligned}
2Gu_x &= & &- \delta_{xz}F_z \\
2Gu_y &= & &- \delta_{yz}F_z \\
2Gu_z &= 2(1-\nu)\Delta F_z & &- \delta_{zz}F_z \\
\sigma_{xx} &= \nu\delta_z\Delta F_z & &- \delta_{xxz}F_z \\
\sigma_{yy} &= \nu\delta_z\Delta F_z & &- \delta_{yyz}F_z \\
\sigma_{zz} &= (2-\nu)\delta_z\Delta F_z & &- \delta_{zzz}F_z \\
\sigma_{xy} &= & &- \delta_{xyz}F_z \\
\sigma_{yz} &= (1-\nu)\delta_y\Delta F_z & &- \delta_{yzz}F_z \\
\sigma_{zx} &= (1-\nu)\delta_x\Delta F_z & &- \delta_{zxz}F_z
\end{aligned} \qquad [6]$$

d) <u>Nuclei derived from C</u>. We will take as a vector for C:

$$\underline{F} = - \frac{1}{2(1-2\nu)} \left(\underline{i}\frac{x}{R} + \underline{j}\frac{y}{R} + \underline{k}\frac{z}{R} \right),$$

a constant multiple of the vector given in Chapter I, 4(d).

This vector is the gradient of a scalar function:

$$\underline{i}\frac{x}{R} + \underline{j}\frac{y}{R} + \underline{k}\frac{z}{R} = \text{grad } R.$$

The vector for a nucleus derived from C is found from the vector for C by subjecting it to the same operations as performed on the displacements and stresses in the derivation, and this vector will be the gradient of a function found by doing the same to R.

Letting F denote this function, then the vector for any nucleus derived from C is of the form

$$\underline{F} = - \frac{1}{2(1-2\nu)} \text{grad } F.$$

Substituting in eq. [2], the displacement vector in terms of F is

$$2G\underline{u} = - \frac{1}{2(1-2\nu)}[2(1-\nu)\Delta - \text{grad div}]\text{grad } F \; .$$

Since div grad = Δ, this reduces to

$$2G\underline{u} = - \text{grad}(\tfrac{1}{2}\Delta F).$$

The function $\frac{1}{2}\Delta F$ is called the potential function.

The components of displacement and stress are:

$$\begin{aligned}
2Gu_x &= - \delta_x(\tfrac{1}{2}\Delta F)\\
2Gu_y &= - \delta_y(\tfrac{1}{2}\Delta F)\\
2Gu_z &= - \delta_z(\tfrac{1}{2}\Delta F)\\
\sigma_{xx} &= - \delta_{xx}(\tfrac{1}{2}\Delta F)\\
\sigma_{yy} &= - \delta_{yy}(\tfrac{1}{2}\Delta F)\\
\sigma_{zz} &= - \delta_{zz}(\tfrac{1}{2}\Delta F)\\
\sigma_{xy} &= - \delta_{xy}(\tfrac{1}{2}\Delta F)\\
\sigma_{yz} &= - \delta_{yz}(\tfrac{1}{2}\Delta F)\\
\sigma_{zx} &= - \delta_{zx}(\tfrac{1}{2}\Delta F)
\end{aligned} \qquad [8]$$

A one component vector producing the same displacements and stresses as [8] may be found if desired. In fact, R being biharmonic, F is also, so that

$$\Delta(\tfrac{1}{2}\Delta F) = 0.$$

Then, comparison of [8] with [4], [5] and [6] shows that any of the three vectors:

$$\underline{i}\textstyle\int_x(\tfrac{1}{2}\Delta F),\ \underline{j}\textstyle\int_y(\tfrac{1}{2}\Delta F),\ \underline{k}\textstyle\int_z(\tfrac{1}{2}\Delta F) \qquad [9]$$

will suffice.

As a representation of nuclei derived from C, both the potential function and the vectors [9] have drawbacks; the vectors obscure the symmetry and simplicity of [8], which thus hampers one in solving problems, while the potential function does not mix well with the vectors representing nuclei for which vector representation is essential.

2. Construction of Tables for Displacements and Stresses of the Nuclei

It is seen from equations [4], [5], and [6] that we may write out the displacements of any nucleus derived from X, Y, or Z if we have at hand a certain set of derivatives, the Laplacian, and derivatives of the Laplacian, of the scalar component of the vector for the nucleus. For X, Y, and Z this function is R.

It is convenient to think in three dimensions: let us imagine R located at (0,0,0). By repeated differentiation of R, we obtain functions at the lattice points in the first quadrant of a rectangular coordinate system, placing $\delta_x R$ at (1,0,0), $\delta_y R$ at (0,1,0), $\delta_z R$ at (0,0,1), $\delta_{xy} R$ at (1,1,0) etc. Supposing the process to have been continued indefinitely, we then have at the lattice points functions which are the scalar component of the vector for every nucleus derived from a single force by differentiation alone, and all the functions which appear in their displacements and stresses.

Taking ϕ to denote any function so obtained, the scheme is consistent since $\delta_{xy}\phi = \delta_{yx}\phi$, etc. For each of the ϕ, let us compute also $\frac{1}{2}\Delta\phi$, and take, as $\int_x \phi$, $\int_{xy}\phi$, etc., the function such that $\delta_x \int_x \phi = \phi$, $\delta_{xy}\int_{xy}\phi = \phi$ etc., then the operators δ, $\int$, and $\frac{1}{2}\Delta$ commute. (A sample of the ϕ and $\frac{1}{2}\Delta\phi$ is found in the Appendix Table).

Looking at the functions more closely, we see that all ϕ obtained from R by two or more differentiations are singular at the origin and vanish at infinity, and may appear in the components of displacement and stress of some nuclei, the others appearing only in the Galerkin vector for some nuclei.

We may extend the scheme to include nuclei derived by integration from X,Y,Z, and C, by integrating the ϕ already obtained. The integrals are all subject to the restriction that any ϕ which may appear in a component of displacement or stress of a nucleus must vanish at infinity. Under this restriction, any given ϕ will have been obtained from R by at most as many integrations as differentiations, and the properties of cancellation and commutivity continue to hold for δ and $\int$.

We may integrate $\frac{1}{2}\Delta R$ once more than we differentiate, to obtain a potential function for a nucleus derived from C. Then, within the region where both exist, we maintain commutivity of δ, $\int$, and $\frac{1}{2}\Delta$, and cancellation of δ and $\int$.

Omitting consideration of the $\frac{1}{2}\Delta\phi$, we may describe the resulting situation as follows: There is a ϕ at every lattice point in the region $x + y + z \geq 0$. Every ϕ in the region $x + y + z \geq 2$ vanishes at infinity; the others will appear only as components of the Galerkin vector. All the ϕ in the region $x + y + z \geq 2$ but outside the first quadrant are singular along one or more of the negative axes, and will appear in the displacements or stresses of a nucleus derived from a single force by a process including integration, the ϕ in the first quadrant being as described above. An analogous situation obtains for the potential functions, if we imagine them similarly situated in space.

3. Characterization of the Nuclei

The geometrical image of the previous section provides a characterization of the nuclei, when we associate with each lattice point the three nuclei, one each derived from X, Y, and Z, having the ϕ at that point as scalar component of their vector, with a similar arrangement for the nuclei derived from C. Such a broad view clarifies the notion of what nuclei are available, and discloses the pattern of some general characteristics, such as type of singularity.

The relationships displayed by the scheme help one to keep order in situations which can easily become tangled and confusing, particularly if one relies on the conventional terminology for the nuclei. In solving a problem one may have been led, for instance, to superpose

$$\delta_x X,\ \int_x \delta_{yy} X, \text{ and } \int_x \delta_{zz} X.$$

But, noting that $\delta_x X = \int_x \delta_{xx} X$, the Galerkin vector for the combination is

$$\underline{F} = \underline{i} \int_x (\delta_{xx} R + \delta_{yy} R + \delta_{zz} R) = \underline{i} \int_x \Delta R$$

i.e., by eq. [9] the same displacements and stresses are produced by C, apart from a constant. Such results can be anticipated in the geometrical image, increasing efficiency in solving problems. So too, the relationships are seen in our notation for the nuclei, which is essentially a symbolic counterpart of the geometrical image.

These aspects of the situation are still of somewhat peripheral importance, though they help one understand the nature of the entities he is working with. In solving problems, one derives the greatest benefit from the scheme by descending to its smallest parts: What is usually desired, in solving a problem, is to find a nucleus or a group of nuclei, which have a particular ϕ or $\frac{1}{2}\Delta\phi$ in a certain one of their displacements or stresses. This function being located in the scheme (or in its imperfect realization, the Appendix Table), then, using the forumlae [4], [5], [6] and [8], one determines the nuclei that have the desired properties. The details of this process will be seen in Chapter IV, in deriving solutions to the problems of Rongved and Mindlin.

4. Data for Displacements and Stresses of 49 Nuclei of Strain

The displacements and stresses of 10 nuclei derived from each of X, Y, and Z, and 19 from C may be found from the entries of the Appendix Table, which have been computed as described in the previous sections. The nuclei are assumed to be located at the origin or along the whole of one of the negative axes.

The appropriate entries for a given nucleus are located from its vector or potential function given below and formulae [4], [5], [6], and [8]. The tables include, for each potential function, at least one of the integrals [9], for Galerkin vector representation of nuclei derived from C.

Nucleus	$\underline{F}$	
X:	$\underline{i}R$	
Y:	$\underline{j}R$	(single force)
Z:	$\underline{k}R$	
$\delta_x X$:	$\underline{i}\,\frac{x}{R}$	
$\delta_y Y$:	$\underline{j}\,\frac{y}{R}$	(double force)
$\delta_z Z$:	$\underline{k}\,\frac{z}{R}$	

Nucleus	$\underline{F}$	
$\delta_y X$:	$\underline{i}\,\frac{y}{R}$	(double force with moment)
$\delta_z X$:	$\underline{i}\,\frac{z}{R}$	
$\delta_x Y$:	$\underline{j}\,\frac{x}{R}$	
$\delta_z Y$:	$\underline{j}\,\frac{z}{R}$	
$\delta_x Z$:	$\underline{k}\,\frac{x}{R}$	
$\delta_y Z$:	$\underline{k}\,\frac{y}{R}$	
$\int_y \delta_x X$:	$\underline{i}\ x \log(R+y)$	(line of double forces)
$\int_z \delta_x X$:	$\underline{i}\ x \log(R+z)$	
$\int_x \delta_y Y$:	$\underline{j}\ y \log(R+x)$	
$\int_z \delta_y Y$:	$\underline{j}\ y \log(R+z)$	
$\int_x \delta_z Z$:	$\underline{k}\ z \log(R+x)$	
$\int_y \delta_z Z$:	$\underline{k}\ z \log(R+y)$	
$\int_x \delta_y X$:	$\underline{i}\ y \log(R+x)$	(line of double forces with moment)
$\int_z \delta_y X$:	$\underline{i}\ y \log(R+z)$	
$\int_x \delta_z X$:	$\underline{i}\ z \log(R+x)$	
$\int_y \delta_z X$:	$\underline{i}\ z \log(R+y)$	
$\int_y \delta_x Y$:	$\underline{j}\ x \log(R+y)$	
$\int_z \delta_x Y$:	$\underline{j}\ x \log(R+z)$	
$\int_x \delta_z Y$:	$\underline{j}\ z \log(R+x)$	
$\int_y \delta_z Y$:	$\underline{j}\ z \log(R+y)$	
$\int_y \delta_x Z$:	$\underline{k}\ x \log(R+y)$	
$\int_z \delta_x Z$:	$\underline{k}\ x \log(R+z)$	
$\int_x \delta_y Z$:	$\underline{k}\ y \log(R+x)$	
$\int_z \delta_y Z$:	$\underline{k}\ y \log(R+z)$	

Nucleus	$\frac{1}{2}\Delta F$	
C:	$\frac{1}{R}$	(center of compression)
$\int_x C$:	$\log(R+x)$	
$\int_y C$:	$\log(R+y)$	(line of compression)
$\int_z C$:	$\log(R+z)$	
$\delta_x C$:	$-\frac{x}{R^3}$	
$\delta_y C$:	$-\frac{y}{R^3}$	(doublet)
$\delta_z C$:	$-\frac{z}{R^3}$	
$\int_z \delta_x C$:	$\frac{x}{(R+z)R}$	
$\int_y \delta_x C$:	$\frac{x}{(R+y)R}$	
$\int_x \delta_y C$:	$\frac{y}{(R+x)R}$	(line of doublets)
$\int_z \delta_y C$:	$\frac{y}{(R+z)R}$	
$\int_x \delta_z C$:	$\frac{z}{(R+x)R}$	
$\int_y \delta_z C$:	$\frac{z}{(R+y)R}$	
$\int_{zz} \delta_x C$:	$-\frac{x}{R+z}$	
$\int_{yy} \delta_x C$:	$-\frac{x}{R+y}$	
$\int_{xx} \delta_y C$:	$-\frac{y}{R+x}$	(line of doublets, strength proportional to distance from origin)
$\int_{zz} \delta_y C$:	$-\frac{y}{R+z}$	
$\int_{xx} \delta_z C$:	$-\frac{z}{R+x}$	
$\int_{yy} \delta_z C$:	$-\frac{z}{R+y}$	

CHAPTER III

A SERIES OF MIXED BOUNDARY VALUE PROBLEMS: HIJAB'S PROBLEMS

1. Preliminary Considerations

We will determine the components of displacement and stress produced by a concentrated force in the interior of three bodies, subject to two boundary conditions.

The bodies are:

The half-space, bounded by the plane $z = 0$ and occupying the region $z \geq 0$. The force acts at $(0,0,c)$ and (x,y,z) is an arbitrary point of the body, as in Fig. 1.

The quarter-space, bounded by the planes $z = 0$ and $x = 0$, and occupying the region $z \geq 0$, $x \geq 0$. The force acts at $(a,0,c)$ and (x,y,z) is an arbitrary point in the body, as in Fig. 2.

The eighth-space, bounded by the planes $z = 0$, $x = 0$, and $y = 0$, and occupying the region $z \geq 0$, $x \geq 0$, $y \geq 0$. The force acts at (a,b,c) and (x,y,z) is an arbitrary point in the body, as in Fig. 3.

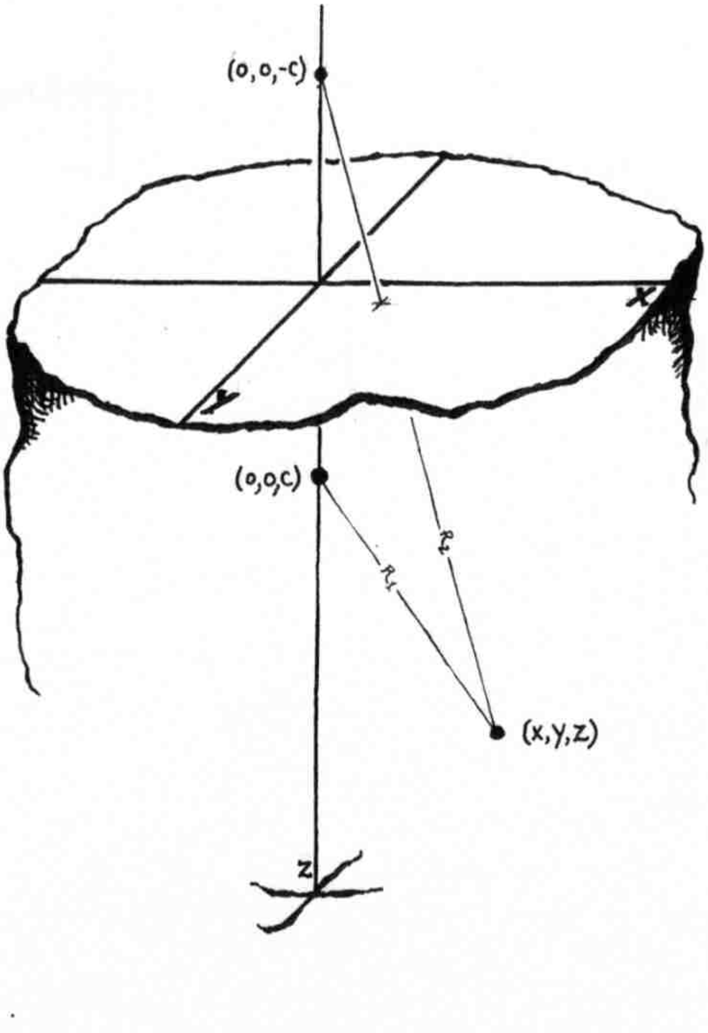


Fig. 1: The Half-space

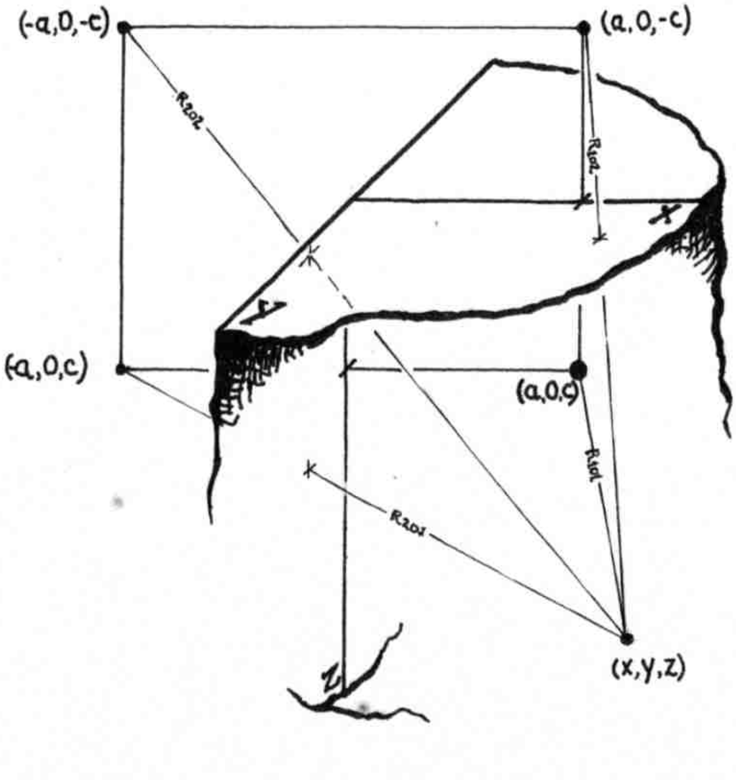


Fig. 2: The Quarter-space

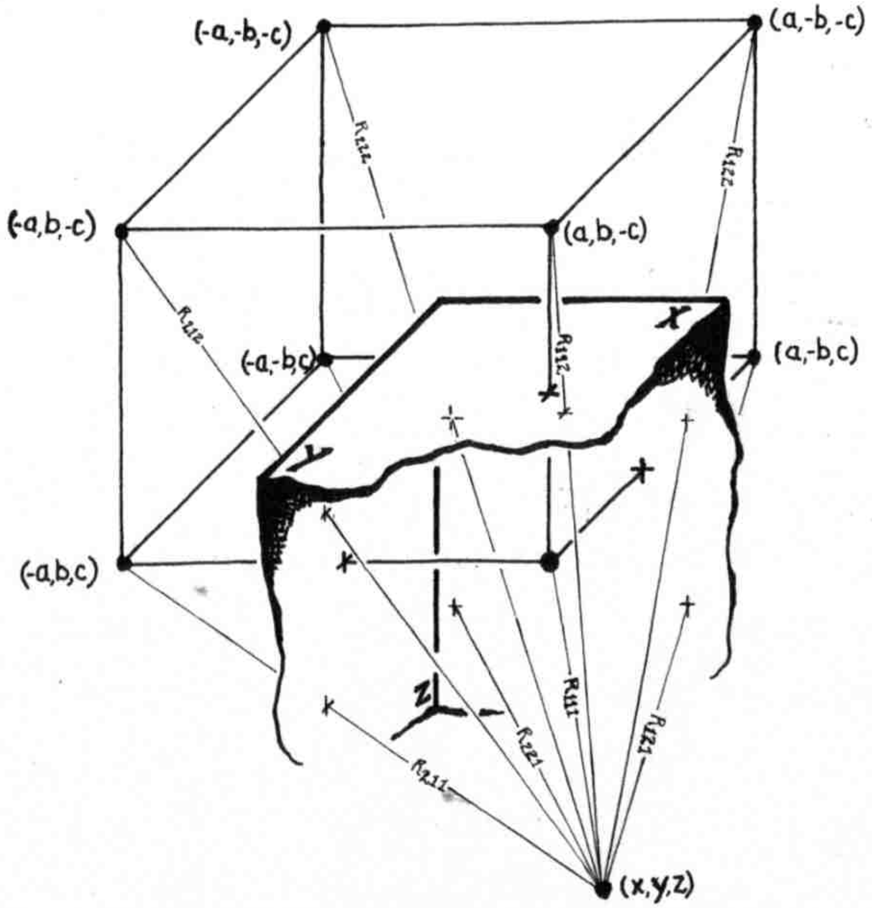


Fig. 3: The Eighth-space

The boundary conditions are:

Zero normal displacement and zero shearing stresses. Specifically,

on $z = 0$	on $x = 0$	on $y = 0$
$u_z = 0$	$u_x = 0$	$u_y = 0$
$\sigma_{zx} = 0$	$\sigma_{xy} = 0$	$\sigma_{yx} = 0$
$\sigma_{zy} = 0$	$\sigma_{xz} = 0$	$\sigma_{yz} = 0$

Zero in-plane displacements and zero shearing stresses. Specifically,

on $z = 0$	on $x = 0$	on $y = 0$
$u_x = 0$	$u_y = 0$	$u_x = 0$
$u_y = 0$	$u_z = 0$	$u_z = 0$
$\sigma_{zz} = 0$	$\sigma_{xx} = 0$	$\sigma_{yy} = 0$

Nuclei will be situated at the point where the force acts, and at points symmetrical to it, with respect to the boundaries. We will use subscripts to denote the distance from these points to the arbitrary point (x,y,z), and to indicate the location of the nuclei:

R_{ijk} is the distance from (x,y,z) to (x',y',z')

X_{ijk}, Y_{ijk}, etc. is located at (x',y',z')

where

$$i = \begin{cases} 0 \text{ if } x' = 0 \\ 1 \text{ if } x' = a \\ 2 \text{ if } x' = -a \end{cases} \qquad j = \begin{cases} 0 \text{ if } y' = 0 \\ 1 \text{ if } y' = b \\ 2 \text{ if } y' = -b \end{cases} \qquad k = \begin{cases} 0 \text{ if } z' = 0 \\ 1 \text{ if } z' = c \\ 2 \text{ if } z' = -c \end{cases}$$

When we are working with the half-space, we will use the abbreviations

$$R_k = R_{00k},\ X_k = X_{00k},\ \text{etc.}$$

The displacements and stresses produced at (x,y,z) by the Nuclei X,Y,Z, are found from their Galerkin vectors, formulae [4], [5], and [6], and the Appendix Table:

X (single force in x-direction)

$$\underline{F} = \underline{i}R$$

$$2Gu_x = \frac{3-4\nu}{R} + \frac{x^2}{R^3}$$

$$2Gu_y = \frac{xy}{R^3}$$

$$2Gu_z = \frac{xz}{R^3}$$

$$\sigma_{xx} = -\frac{(1-2\nu)x}{R^3} - \frac{3x^3}{R^5}$$

$$\sigma_{yy} = \frac{(1-2\nu)x}{R^3} - \frac{3xy^2}{R^5}$$

$$\sigma_{zz} = \frac{(1-2\nu)x}{R^3} - \frac{3xz^2}{R^5}$$

$$\sigma_{xy} = -\frac{(1-2\nu)y}{R^3} - \frac{3x^2y}{R^5}$$

$$\sigma_{yz} = -\frac{3xyz}{R^5}$$

$$\sigma_{zx} = -\frac{(1-2\nu)z}{R^3} - \frac{3x^2z}{R^5}$$

Y (single force in y-direction)

$$\begin{aligned}
\underline{F} &= \underline{j}R \\
2Gu_x &= \frac{yx}{R^3} \\
2Gu_y &= \frac{3-4\nu}{R} + \frac{y^2}{R^3} \\
2Gu_z &= \frac{yz}{R^3} \\
\sigma_{xx} &= \frac{(1-2\nu)y}{R^3} - \frac{3yx^2}{R^5} \\
\sigma_{yy} &= -\frac{(1-2\nu)y}{R^3} - \frac{3y^3}{R^5} \\
\sigma_{zz} &= \frac{(1-2\nu)y}{R^3} - \frac{3yz^2}{R^5} \\
\sigma_{xy} &= -\frac{(1-2\nu)x}{R^3} - \frac{3y^2x}{R^5} \\
\sigma_{yz} &= -\frac{(1-2\nu)z}{R^3} - \frac{3y^2z}{R^5} \\
\sigma_{zx} &= -\frac{3xyz}{R^5}
\end{aligned}$$

Z (single force in z-direction)

$$\begin{aligned}
\underline{F} &= \underline{k}R \\
2Gu_x &= \frac{zx}{R^3} \\
2Gu_y &= \frac{zy}{R^3} \\
2Gu_z &= \frac{3-4\nu}{R} + \frac{z^2}{R^3} \\
\sigma_{xx} &= \frac{(1-2\nu)z}{R^3} - \frac{3zx^2}{R^5} \\
\sigma_{yy} &= \frac{(1-2\nu)z}{R^3} - \frac{3zy^2}{R^5} \\
\sigma_{zz} &= -\frac{(1-2\nu)z}{R^3} - \frac{3z^3}{R^5}
\end{aligned}$$

$$\underline{F} = \underline{k}R$$

$$\sigma_{xy} = -\frac{3xyz}{R^5}$$

$$\sigma_{yz} = -\frac{(1-2\nu)y}{R^3} - \frac{3z^2y}{R^5}$$

$$\sigma_{zx} = -\frac{(1-2\nu)x}{R^3} - \frac{3z^2x}{R^5}$$

It may be noted that the formulae for Y and Z may be obtained from those for X by cyclic permutation of variables and subscripts. The formulae are applicable if the nuclei are located at the origin. If the nuclei are at (x',y',z') the proper formulae are obtained from those given by substituting $x-x'$, $y-y'$, $z-z'$, and R_{ijk} for $x,y,z,$ and R.

The magnitude of the above forces is $8\pi(1-\nu)$ in the positive direction. The magnitude is not affected by the superposition of nuclei at a finite distance from the point at which the force is acting, so that all our solutions will represent a force of this same magnitude. Having once obtained a solution, we may multiply throughout by

$$\frac{P}{8\pi(1-\nu)} \qquad \text{("force adjustment")}$$

to obtain the solution for force of magnitude P.

2. Half-space, Quarter-space, and Eighth-space With Zero Normal Displacement and Zero Shearing Stresses at the Boundary

a) Half-space. Hijab (14) has solved this by the Papkovitch functions:

$$B_x = 0$$

$$B_y = 0$$

$$B_z = \frac{P}{4\pi G}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\beta = -\frac{cP}{4\pi G}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

for a force in the z-direction, and

$$B_x = \frac{P}{4\pi G}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$B_y = 0$$

$$B_z = 0$$

$$\beta = 0$$

for a force in the x-direction.

Utilizing the relationships [3], we find that the Galerkin vectors corresponding to these Papkovitch functions are

$$\underline{F} = \underline{k}\frac{P}{8\pi(1-\nu)}(R_1 - R_2)$$

for a force in the z-direction, and

$$\underline{F} = \underline{i}\frac{P}{8\pi(1-\nu)}(R_1 + R_2)$$

for a force in the x-direction. That is, apart from the force adjustment, the two solutions are obtained by superposing the nuclei

$$Z_1 - Z_2$$

and

$$X_1 + X_2$$

respectively.

Investigating the reason for this striking result, we find these interesting facts: The displacements and stresses of X, Y, and Z are all even or odd in each of the variables, so that if two identical single forces are situated symmetrically with respect to a plane, the displacements and stresses on the plane are double that for one force alone, or zero.

Looking to the Appendix Table we see that the functions there possess the same property, alternate derivatives being successively even or odd in the variable of differentiation. Finally, the structure formed of the functions in the displacements and stresses of the nuclei is seen from formulae [4], [5], and [6], and this structure leads us to state the principle:

(i) Place two single forces symmetrically with respect to a plane. Then, to obtain zero normal displacements and zero shearing stresses on the plane, forces perpendicular to the plane should be in opposite directions, forces parallel to the plane in the same direction.

The half-space solution for force in the y-direction is then given by the nuclei

$$Y_1 + Y_2.$$

Hijab has given the displacements and stresses for the force in z-direction and force in x-direction, but they are given in Tables I and II (computed from the single force solutions), for later reference. Displacements and stresses for force in y-direction may be obtained from Table II by interchanging x and y subscripts and variables.

b) Quarter-space. We apply the above principle to the quarter-space, and obtain the solutions immediately. They are given by the nuclei

$$Z_{101} - Z_{102} + Z_{201} - Z_{202},$$
$$X_{101} + X_{102} - X_{201} - X_{202},$$
$$Y_{101} + Y_{102} + Y_{201} + Y_{202},$$

for forces in the z, x, and y directions. Displacements and stresses for force in the z-direction may be computed from Table I in the order:

$$(Z_{101} - Z_{102}) + (Z_{201} - Z_{202}).$$

That the condition on the plane $x = 0$ is satisfied may be seen from Table I, by noting that only odd powers of x appear in $2Gu_x$, σ_{xy}, and σ_{xz}. Similar methods of computation and verification may be used for the other solutions.

c) Eighth-space. Applying the above principle to the eighth-space, we obtain:

$$Z_{111} + Z_{121} + Z_{211} + Z_{221} - Z_{112} - Z_{122} - Z_{212} - Z_{222},$$
$$X_{111} + X_{112} + X_{121} + X_{122} - X_{211} - X_{212} - X_{221} - X_{222},$$
$$Y_{111} + Y_{211} + Y_{112} + Y_{212} - Y_{121} - Y_{221} - Y_{122} - Y_{222}.$$

Displacements and stresses for force in the z-direction could be obtained from the quarter-space solution by computing in the order:

$$[Z_{111} - Z_{112} + Z_{211} - Z_{212}] + [Z_{121} - Z_{122} + Z_{221} - Z_{212}].$$

The others would then follow from this by cyclic permutation of subscripts and variables.

TABLE I

DISPLACEMENTS AND STRESSES FOR A FORCE IN Z-DIRECTION IN HALF-SPACE WITH ZERO NORMAL DISPLACEMENT AND ZERO SHEARING STRESSES ON THE BOUNDARY

$$2Gu_x = x\left[\frac{z-c}{R_1^3} - \frac{z+c}{R_2^3}\right]$$

$$2Gu_y = y\left[\frac{z-c}{R_1^3} - \frac{z+c}{R_2^3}\right]$$

$$2Gu_z = (3-4\nu)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] + \left[\frac{(z-c)^2}{R_1^3} - \frac{(z+c)^2}{R_2^3}\right]$$

$$\sigma_{xx} = (1-2\nu)\left[\frac{z-c}{R_1^3} - \frac{z+c}{R_2^3}\right] - 3x^2\left[\frac{z-c}{R_1^5} - \frac{z+c}{R_2^5}\right]$$

$$\sigma_{yy} = (1-2\nu)\left[\frac{z-c}{R_1^3} - \frac{z+c}{R_2^3}\right] - 3y^2\left[\frac{z-c}{R_1^5} - \frac{z+c}{R_2^5}\right]$$

$$\sigma_{zz} = -(1-2\nu)\left[\frac{z-c}{R_1^3} - \frac{z+c}{R_2^3}\right] - 3\left[\frac{(z-c)^3}{R_1^5} - \frac{(z+c)^3}{R_2^5}\right]$$

$$\sigma_{xy} = -\ 3xy\left[\frac{z-c}{R_1^5} - \frac{z+c}{R_2^5}\right]$$

$$\sigma_{yz} = -\ (1-2\nu)y\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3y\left[\frac{(z-c)^2}{R_1^3} - \frac{(z+c)^2}{R_2^3}\right]$$

$$\sigma_{zx} = -\ (1-2\nu)x\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3x\left[\frac{(z-c)^2}{R_1^3} - \frac{(z+c)^2}{R_2^3}\right]$$

A factor of $\frac{P}{8\pi(1-\nu)}$ is omitted throughout

TABLE II

DISPLACEMENTS AND STRESSES FOR A FORCE IN X-DIRECTION IN HALF-SPACE WITH ZERO NORMAL DISPLACEMENT AND ZERO SHEARING STRESSES ON THE BOUNDARY

$$2Gu_x = (3-4\nu)\left[\frac{1}{R_1} + \frac{1}{R_2}\right] + x^2\left[\frac{1}{R_1^3} + \frac{1}{R_2^3}\right]$$

$$2Gu_y = xy\left[\frac{1}{R_1^3} + \frac{1}{R_2^3}\right]$$

$$2Gu_z = x\left[\frac{z-c}{R_1^3} + \frac{z+c}{R_2^3}\right]$$

$$\sigma_{xx} = -(1-2\nu)x\left[\frac{1}{R_1^3} + \frac{1}{R_2^3}\right] - 3x^3\left[\frac{1}{R_1^5} + \frac{1}{R_2^5}\right]$$

$$\sigma_{yy} = (1-2\nu)\ x\ \left[\frac{1}{R_1^3} + \frac{1}{R_2^3}\right] - 3xy^2\left[\frac{1}{R_1^5} - \frac{1}{R_2^5}\right]$$

$$\sigma_{zz} = (1-2\nu)\ x\ \left[\frac{1}{R_1^3} + \frac{1}{R_2^3}\right] - 3x\left[\frac{(z-c)^2}{R_1^5} + \frac{(z+c)^2}{R_2^5}\right]$$

$$\sigma_{xy} = -(1-2\nu)y\left[\frac{1}{R_1^3} + \frac{1}{R_2^3}\right] - 3x^2y\left[\frac{1}{R_1^5} + \frac{1}{R_2^5}\right]$$

$$\sigma_{yz} = -3xy\left[\frac{z-c}{R_1^5} + \frac{z+c}{R_2^5}\right]$$

$$\sigma_{zx} = -(1-2\nu)\left[\frac{z-c}{R_1^3} + \frac{z+c}{R_2^3}\right] - 3x^2\left[\frac{z-c}{R_1^5} + \frac{z+c}{R_2^5}\right]$$

A factor of $\frac{P}{8\pi(1-\nu)}$ is omitted throughout

3. Half-space, Quarter-space, and Eighth-space with Zero In-plane Displacement and Zero Normal Stress at the Boundary

a) *Half-space*. The investigation which led us to state principle (i) leads at the same time to

(ii) Place single forces symmetrically with respect to a plane. Then, to obtain zero in-plane displacements and zero normal stress, forces perpendicular to the plane should be in the same direction, forces parallel to the plane in opposite directions.

Thus, the nuclei which produce the desired conditions are:

$$Z_1 + Z_2$$

$$X_1 - X_2$$

$$Y_1 - Y_2$$

for the force in the z, x, and y directions, respectively. The displacements and stresses for force in the z-direction and in the x-direction are given in Tables III and IV. Those for force in y-direction are obtained from Table IV by interchanging x and y subscripts and variables.

b) *Quarter-space*. Applying principle (ii) to the quarter-space, we obtain the nuclei:

$$Z_{101} + Z_{102} - Z_{201} - Z_{202}$$

$$X_{101} - X_{102} + X_{201} - X_{202}$$

$$Y_{101} - Y_{102} - Y_{201} + Y_{202}$$

for forces in the z,x, and y directions. The solution may be verified, and the displacements and stresses computed, using Tables III and IV, as for the quarter-space solutions in the previous section.

c) <u>Eighth-space</u>. Principle (ii) applied to the eighth-space leads us to the solution, for force in the x-direction:

$$X_{111} - X_{112} - X_{121} + X_{122} + X_{211} - X_{212} - X_{221} + X_{222}.$$

This solution has been obtained by Hijab (14). The Galerkin vector (with force adjustment) for the above nuclei is

$$\underline{F} = \underline{i}\frac{P}{8\pi(1-\nu)}(R_{111} - R_{112} - R_{121} + R_{122} + R_{211} - R_{212} - R_{221} + R_{222}).$$

Then using relation [3], the Papkovitch functions are:

$$B_x = \frac{P}{4\pi G}(\frac{1}{R_{111}} - \frac{1}{R_{112}} - \frac{1}{R_{121}} + \frac{1}{R_{122}} + \frac{1}{R_{211}} - \frac{1}{R_{212}} - \frac{1}{R_{221}} + \frac{1}{R_{222}}),$$

$$B_y = 0,$$

$$B_z = 0,$$

$$\beta = -\frac{Pa}{4\pi G}(\frac{1}{R_{111}} - \frac{1}{R_{112}} - \frac{1}{R_{121}} + \frac{1}{R_{122}} - \frac{1}{R_{211}} + \frac{1}{R_{212}} + \frac{1}{R_{221}} - \frac{1}{R_{222}}),$$

which are identical (apart from notation) with those given by Hijab. Hijab has also given the stresses and displacements for the solution.

The nuclei for a force in the y and z-directions are:

$$Y_{111} - Y_{211} - Y_{112} + Y_{212} + Y_{121} - Y_{221} - Y_{122} + Y_{222},$$

$$Z_{111} - Z_{121} - Z_{211} + Z_{221} + Z_{112} - Z_{122} - Z_{212} + Z_{222}.$$

The stresses and displacements may be obtained, as usual, by cyclic permutation of those for a force in the x-direction.

TABLE III

DISPLACEMENTS AND STRESSES FOR A FORCE IN Z-DIRECTION IN HALF-SPACE WITH ZERO IN-PLANE DISPLACEMENTS AND ZERO NORMAL STRESS ON THE BOUNDARY

$$2Gu_x = x\left[\frac{z-c}{R_1^3} + \frac{z+c}{R_2^3}\right]$$

$$2Gu_y = y\left[\frac{z-c}{R_1^3} + \frac{z+c}{R_2^3}\right]$$

$$2Gu_z = (3-4\nu)\left[\frac{1}{R_1} + \frac{1}{R_2}\right] + \left[\frac{(z-c)^2}{R_1^3} + \frac{(z+c)^2}{R_2^3}\right]$$

$$\sigma_{xx} = (1-2\nu)\left[\frac{z-c}{R_1^3} + \frac{z+c}{R_2^3}\right] - 3x^2\left[\frac{z-c}{R_1^5} + \frac{z+c}{R_2^5}\right]$$

$$\sigma_{yy} = (1-2\nu)\left[\frac{z-c}{R_1^3} + \frac{z+c}{R_2^3}\right] - 3y^2\left[\frac{z-c}{R_1^5} + \frac{z+c}{R_2^5}\right]$$

$$\sigma_{zz} = -(1-2\nu)\left[\frac{z-c}{R_1^3} + \frac{z+c}{R_2^3}\right] - 3\left[\frac{(z-c)^3}{R_1^5} + \frac{(z+c)^3}{R_2^5}\right]$$

$$\sigma_{xy} = -3xy\left[\frac{z-c}{R_1^5} + \frac{z+c}{R_2^5}\right]$$

$$\sigma_{yz} = -(1-2\nu)y\left[\frac{1}{R_1^3} + \frac{1}{R_2^3}\right] - 3y\left[\frac{(z-c)^2}{R_1^5} + \frac{(z+c)^2}{R_2^5}\right]$$

$$\sigma_{zx} = -(1-2\nu)x\left[\frac{1}{R_1^3} + \frac{1}{R_2^3}\right] - 3x\left[\frac{(z-c)^2}{R_1^5} + \frac{(z+c)^2}{R_2^5}\right]$$

A factor $\frac{P}{8\pi(1-\nu)}$ is omitted throughout

TABLE IV

DISPLACEMENTS AND STRESSES FOR A FORCE IN X-DIRECTION IN HALF-SPACE WITH ZERO IN-PLANE DISPLACEMENTS AND ZERO NORMAL STRESS ON THE BOUNDARY

$$2Gu_x = (3-4\nu)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] + x^2\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right]$$

$$2Gu_y = xy\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right]$$

$$2Gu_z = x\left[\frac{z-c}{R_1^3} - \frac{z+c}{R_2^3}\right]$$

$$\sigma_{xx} = -(1-2\nu)x\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3x^3\left[\frac{1}{R_1^5} - \frac{1}{R_2^5}\right]$$

$$\sigma_{yy} = (1-2\nu)x\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3xy^2\left[\frac{1}{R_1^5} - \frac{1}{R_2^5}\right]$$

$$\sigma_{zz} = (1-2\nu)x\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3x\left[\frac{(z-c)^2}{R_1^5} - \frac{(z+c)^2}{R_2^5}\right]$$

$$\sigma_{xy} = -(1-2\nu)y\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3x^2y\left[\frac{1}{R_1^5} - \frac{1}{R_2^5}\right]$$

$$\sigma_{yz} = -3xy\left[\frac{z-c}{R_1^5} - \frac{z+c}{R_2^5}\right]$$

$$\sigma_{zx} = -(1-2\nu)\left[\frac{z-c}{R_1^3} - \frac{z+c}{R_2^3}\right] - 3x^2\left[\frac{z-c}{R_1^5} - \frac{z+c}{R_2^5}\right]$$

A factor of $\frac{P}{8\pi(1-\nu)}$ is omitted throughout

4. Quarter-space and Eighth-space with an Arbitrary Combination of the Two Boundary Conditions

By applying principles (i) and (ii) we may obtain quarter-space and eighth-space solutions with zero normal displacement and zero shearing stresses on any one or two of the plane boundaries, and zero in-plane displacements and zero normal stress on the others. As we may specify the boundary conditions in two ways for the quarter-space, and in six, for the eighth-space, for each of three directions of the force, there are 24 solutions, many of them obtainable from others by permuting variables.

Nothing is to be gained by writing out the solutions; one example will suffice. We specify zero normal displacement and zero shearing stress on the plane $z = 0$, zero in-plane displacement and zero normal stress on $x = 0$, for a force in the z-direction in quarter-space. Then we must take the forces symmetrically located with respect to the boundary planes to be oppositely directed.

The nuclei for the solution are then

$$Z_{101} - Z_{102} - Z_{201} + Z_{202}.$$

We may compute the displacements and stresses from Z directly, or use Table I by computing in the order:

$$(Z_{101} - Z_{102}) - (Z_{201} - Z_{202}).$$

By interchanging x and z, we would find the displacements and stresses for

$$X_{101} - X_{201} - X_{102} + X_{202}$$

giving a force in the x-direction in quarter-space with boundary conditions, also the interchange of those above.

CHAPTER IV

FIRST AND SECOND BOUNDARY VALUE PROBLEMS IN HALF-SPACE: RONGVED'S AND MINDLIN'S PROBLEMS

1. Preliminary Considerations

We will be concerned with:

Rongved's problem (Z), of a force in z-direction in half-space (Fig.1) with boundary fixed (a first boundary value problem: zero displacements at the boundary);

Rongved's problem (X), of a force in x-direction in half-space with boundary fixed;

Mindlin's problem (Z), of a force in the z-direction in half-space with boundary free (a second boundary value problem: zero stresses);

Mindlin's problem (X), of a force in the x-direction in half-space with boundary free.

By superposing nuclei at $(0,0,-c)$, we will remove from the plane $z=0$, the displacements and stresses of the single force at $(0,0,c)$. But before we attempt to work with the nuclei as units, it is well to consider individually the functions appearing in their stresses and displacements. These are the functions of the Appendix Table except that, the nuclei being located at $(0,0,-c)$ instead of at the origin, $z + c$ and R_2 appear in place of z and R. We denote a biharmonic expression obtained from a ϕ by this substitution as ϕ_2, and a harmonic expression obtained from a $\frac{1}{2}\Delta\phi$, as $\frac{1}{2}\Delta\phi_2$. As the value of a ϕ_2 or $\frac{1}{2}\Delta\phi_2$ at $z = 0$ is obtained from the corresponding ϕ or $\frac{1}{2}\Delta\phi$ of the Appendix Table simply by replacing z with c and R with $R_{.} = (x^2+y^2+c^2)^{\frac{1}{2}}$, we may conveniently look there to investigate what linear combinations of the ϕ_2 and $\frac{1}{2}\Delta\phi_2$ vanish at the boundary.

We see then that in many cases a combination of a given ϕ_2 with its derivative $\delta_z\phi_2$ contains terms which vanish at $z = 0$. If, for instance, we choose $\phi = \frac{z}{R}$, then $\delta_z\phi = \frac{1}{R} - \frac{z^2}{R_2^3}$, and

$$\phi_2 - c\delta_z\phi_2 = \frac{z}{R_2} - \frac{c(z+c)^2}{R_2^3}$$

has first term which vanishes at $z = 0$. We say that the latter function has annulled the former, and formulate a guide to procedure:

(iii) Annullment of a given ϕ_2 may often be accomplished by superposing $\delta_z\phi_2$ (though this may introduce nonvanishing new terms).

Another combination vanishing at $z = 0$ may be obtained from ϕ_2 and the harmonic expression $\int_z\frac{1}{2}\Delta\phi_2$. Using the same ϕ as before, then $\int_z\frac{1}{2}\Delta\phi = \frac{1}{R}$, so that

$$\phi_2 - c\int_z\tfrac{1}{2}\Delta\phi_2 = \frac{z}{R_2}$$

vanishes at $z = 0$. Again, the nonvanishing term of the previous example was introduced by $\delta_z\phi_2$, and so, finding

$$\int_z\tfrac{1}{2}\Delta(\delta_z\phi) = \tfrac{1}{2}\Delta\phi = -\frac{z}{R^3}$$

we obtain another combination,

$$\phi_2 - c\delta_z\phi_2 - c^2\tfrac{1}{2}\Delta\phi_2 = \frac{z}{R_2} + \frac{cz(z+c)}{R_2^3} ,$$

which vanishes at $z = 0$. We thus formulate another guide to procedure:

(iv) Annullment of a given ϕ_2 may often be accomplished by superposing $\int_z \tfrac{1}{2}\Delta\phi_2$.

The Galerkin vector and potential function formulae [4], [5], [6], and [8] allow us to reformulate the guides to procedure in terms appropriate for work with nuclei as units. In the problems to be considered, we have need only for nuclei derived from Z and C, and in any case, we are interested only in components of stress acting on the boundary. The pertinent information then, is:

$$\begin{aligned}
\underline{F} &= kF_z & & \\
2Gu_x &= & &- \delta_{zx}F_z \\
2Gu_y &= & &- \delta_{zy}F_z \\
2Gu_z &= 2(1-\nu)\Delta F_z & &- \delta_{zz}F_z \\
\sigma_{zx} &= (1-\nu)\delta_x\Delta F_z & &- \delta_{zzx}F_z \\
\sigma_{zy} &= (1-\nu)\delta_y\Delta F_z & &- \delta_{zzy}F_z \\
\sigma_{zz} &= (2-\nu)\delta_z\Delta F_z & &- \delta_{zzz}F_z
\end{aligned} \qquad [6']$$

$$\begin{aligned}
\underline{F} &= -\operatorname{grad}(\tfrac{1}{2}\Delta F) \\
2Gu_x &= -\delta_x(\tfrac{1}{2}\Delta F) \\
2Gu_y &= -\delta_y(\tfrac{1}{2}\Delta F) \\
2Gu_z &= -\delta_z(\tfrac{1}{2}\Delta F) \\
\sigma_{zx} &= -\delta_{zx}(\tfrac{1}{2}\Delta F) \\
\sigma_{zy} &= -\delta_{zy}(\tfrac{1}{2}\Delta F) \\
\sigma_{zz} &= -\delta_{zz}(\tfrac{1}{2}\Delta F)
\end{aligned} \qquad [8']$$

Viewing [6'] and [8'], we may say:

(v) If the nucleus producing the ϕ_2 to be annulled has Galerkin vector $\underline{F} = \underline{k}F_z$, then the $\delta_z\phi_2$ of (iii) may be obtained from the nucleus with Galerkin vector $\underline{F} = \underline{k}\delta_z F_z$.

(vi) If the nucleus producing the ϕ_2 to be annulled has Galerkin vector $\underline{F} = \underline{k}F_z$, then the $\int_z \frac{1}{2}\Delta\phi_2$ of (iv) may be obtained from the nucleus with potential function $\frac{1}{2}\Delta F = \frac{1}{2}\Delta F_z$. We note an alternative way of obtaining harmonic expressions (for whatever purpose):

(vii).A certain group of harmonic expressions, "symmetrical with respect to z" (see [6']) can be obtained from a nucleus with Galerkin vector $\underline{F} = \underline{k}F_z$.

We see from [6'] and [8'] that superposing nuclei, we may not only annull certain expressions but also, if these are harmonic, eliminate them for all values of z. In particular,

(viii) Harmonic expressions may be eliminated from a nucleus with Galerkin vector $\underline{F} = \underline{k}F_z$ by superposing a nucleus with potential function $\frac{1}{2}\Delta F = \frac{1}{2}\Delta\int_z F_z$.

The nuclei (excepting the single forces) to be used in the problems follow. Galerkin vectors are given for all, but only components of displacement and stress acting on the boundary of interest in the particular problem appear. The nuclei are assumed to be located at the origin.

$\delta_z Z$ (double force in z-direction)

$$\underline{F} = \underline{k}\,\frac{z}{R}$$

$$2Gu_x = \frac{x}{R^3} - \frac{3xz^2}{R^5}$$

$$2Gu_y = \frac{y}{R^3} - \frac{3yz^2}{R^5}$$

$$2Gu_z = -\frac{(1-4\nu)z}{R^3} - \frac{3z^3}{R^5}$$

$$\sigma_{zx} = -\frac{3(1+2\nu)xz}{R^5} + \frac{15xz^3}{R^7}$$

$$\sigma_{zy} = -\frac{3(1+2\nu)yz}{R^5} + \frac{15yz^3}{R^7}$$

$$\sigma_{zz} = -\frac{(1-2\nu)}{R^3} - \frac{6(1+\nu)z^2}{R^5} + \frac{15z^4}{R^7}$$

C (center of compression)

$$\tfrac{1}{2}\Delta F = \frac{1}{R} \qquad \underline{F} = \underline{k}\,\log(R+z)$$

$$2Gu_x = \frac{x}{R^3}$$

$$2Gu_y = \frac{y}{R^3}$$

$$2Gu_z = \frac{z}{R^3}$$

$$\sigma_{zx} = -\frac{3xz}{R^5}$$

$$\sigma_{zy} = -\frac{3yz}{R^5}$$

$$\sigma_{zz} = \frac{1}{R^3} - \frac{3z^2}{R^5}$$

$\delta_z C$ (doublet with axis parallel to x-axis)

$$\tfrac{1}{2}\Delta F = -\frac{z}{R^3} \qquad \underline{F} = \underline{k}\,\frac{1}{R}$$

$$2Gu_x = -\frac{3xz}{R^5}$$

$$2Gu_y = -\frac{3yz}{R^5}$$

$$2Gu_z = \frac{1}{R^3} - \frac{3z^2}{R^5}$$

$$\sigma_{zx} = -\frac{3x}{R^5} + \frac{15xz^2}{R^7}$$

$$\sigma_{zy} = -\frac{3y}{R^5} + \frac{15yz^2}{R^7}$$

$$\sigma_{zz} = -\frac{9z}{R^5} + \frac{15z^3}{R^7}$$

$\int_z C$ (line of compression along negative z-axis)

$$\tfrac{1}{2}\Delta F = \log(R+z) \qquad \underline{F}=\underline{k}[z \log(R+z)-R]$$

$$\sigma_{zx} = \frac{x}{R^3}$$

$$\sigma_{zy} = \frac{y}{R^3}$$

$$\sigma_{zz} = \frac{z}{R^3}$$

$\delta_x Z$ (double force in z-direction with moment about y-axis)

$$\underline{F} = \underline{k}\,\frac{x}{R}$$

$$2Gu_x = \frac{z}{R^3} - \frac{3x^2z}{R^5}$$

$$2Gu_y = -\frac{3xyz}{R^5}$$

$$2Gu_z = -\frac{(3-4\nu)x}{R^3} - \frac{3xz^2}{R^5}$$

$$\sigma_{zx} = -\frac{(1-2\nu)}{R^3} + \frac{3(1-2\nu)x^2}{R^5} - \frac{3z^2}{R^5} + \frac{15x^2z^2}{R^7}$$

$$\sigma_{zy} = \frac{3(1-2\nu)xy}{R^5} + \frac{15xyz^2}{R^7}$$

$$\sigma_{zz} = \frac{3(1-2\nu)xz}{R^5} + \frac{15xz^3}{R^7}$$

$\delta_x C$ (doublet with axis parallel to x-axis)

$$\tfrac{1}{2}\Delta F = -\frac{x}{R^3} \qquad \underline{F} = \underline{i}\,\frac{1}{R}$$

$$2Gu_x = \frac{1}{R^3} - \frac{3x^2}{R^5}$$

$$2Gu_y = -\frac{3xy}{R^5}$$

$$2Gu_z = -\frac{3xz}{R^5}$$

$$\sigma_{zx} = -\frac{3z}{R^5} + \frac{15x^2z}{R^7}$$

$$\sigma_{zy} = \frac{15xyz}{R^7}$$

$$\sigma_{zz} = -\frac{3x}{R^5} + \frac{15xz^2}{R^7}$$

$\int_z \delta_x C$ (line, along negative z-axis, of doublets with axis parallel to x-axis)

$$\tfrac{1}{2}\Delta F = \frac{x}{(R+z)R} \qquad \underline{F} = \underline{i}\,\log(R+z)$$

$$\sigma_{zx} = \frac{1}{R^3} - \frac{3x^2}{R^5}$$

$$\sigma_{zy} = -\frac{3xy}{R^5}$$

$$\sigma_{zz} = -\frac{3xz}{R^5}$$

$\int_z \delta_x Z$ (line, along negative z-axis, of double forces with moment about y-axis)

$$\underline{F} = \underline{k}\ x \log(R+z)$$

$$\sigma_{zx} = \frac{z}{R^3} - \frac{3x^2z}{R^5} + \frac{2(1-\nu)}{(R+z)R} - 2(1-\nu)x^2\left[\frac{1}{(R+z)^2R^2} + \frac{1}{(R+z)R^3}\right]$$

$$\sigma_{zy} = -\frac{3xyz}{R^5} - 2(1-\nu)xy\left[\frac{1}{(R+z)^2R^2} + \frac{1}{(R+z)R^3}\right]$$

$$\sigma_{zz} = -\frac{(3-2\nu)}{R^3} - \frac{3xz^2}{R^5}$$

$\int_{zz} \delta_x C$ (line, along negative z-axis, of doublets with axis parallel to x-axis, strength proportional to distance from origin)

$$\tfrac{1}{2}\Delta F = -\frac{x}{R+z} \qquad \underline{F} = \underline{i}[z \log(R+z)-R]$$

$$\sigma_{zx} = -\frac{1}{(R+z)R} + x^2\left[\frac{1}{(R+z)^2R^2} + \frac{1}{(R+z)R^3}\right]$$

$$\sigma_{zy} = xy\left[\frac{1}{(R+z)^2R^2} + \frac{1}{(R+z)R^3}\right]$$

$$\sigma_{zz} = \frac{x}{R^3}\ .$$

2. Half-space with Fixed Boundary and Free Boundary, Force in z-direction

a) Rongved's problem (Z). We begin with $Z_1 - Z_2$ (Table I). This produces, at the boundary:

$$2Gu_x = -\frac{2xc}{R^3}\ ,$$

$$2Gu_y = -\frac{2yc}{R^3}\ , \qquad (Z_1-Z_2)_{z=0}$$

$$2Gu_z = 0\ .$$

Annullment of these displacements may only be accomplished by superposing nuclei at $(0,0,-c)$, the c in $\underset{\cdot}{R}^3$ being obtainable in no other way. Using (v), we select $\delta_z Z$, which contains terms to annull the above, but which also introduces nonvanishing new expressions, some harmonic and some biharmonic. Using (viii), we select C to eliminate the harmonic ones, since for $\delta_z Z$, $F_z = \delta_z R$, and for C, $\frac{1}{2}\Delta F = \frac{1}{2}\Delta R = \int_z \frac{1}{2}\Delta(\delta_z R)$. Using (vi), we annull the biharmonic expressions with $\delta_z C$. These three nuclei combine to produce at the boundary:

$$\begin{aligned} 2Gu_x &= \frac{(3-4\nu)x}{\underset{\cdot}{R}^3} \quad , \\ 2Gu_y &= \frac{(3-4\nu)y}{\underset{\cdot}{R}^3} \quad , \quad (\delta_z Z_2 + 2(1-2\nu)G_2 - c\delta_z C_2)_{z=0} \\ 2Gu_z &= 0. \end{aligned}$$

The desired condition at the boundary, then, is provided by the combination:

$$Z_1 - Z_2 + \frac{2c}{3-4\nu}[\delta_z Z_2 + 2(1-2\nu)C_2 - c\delta_z C_2]. \qquad [10]$$

The Galerkin vector (with force adjustment) for these nuclei is:

$$\underline{F} = \frac{P}{8\pi(1-\nu)}\,\underline{k}\,\left\{R_1 - R_2 + \frac{2c}{3-4\nu}\left[\frac{z+c}{R^2} + 2(1-2\nu)\log(R+z+c) - \frac{c}{R^2}\right]\right\} .$$

The Papkovitch functions are found from the Galerkin vector by eq.[3]:

$$\begin{aligned} B_x &= 0 \ , \\ B_y &= 0 \ , \\ B_z &= \frac{P}{4\pi G}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) - \frac{Pc}{2\pi(3-4\nu)G}\left(\frac{z+c}{R_2^3}\right) \ , \\ \beta &= -\frac{Pc}{4\pi G}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \ , \end{aligned}$$

identical with those obtained by Rongved (13). The displacements and stresses are given in Table V.

b) <u>Mindlin's problem (Z)</u>. We begin with Z_1+Z_2 (Table III). This produces, at the boundary:

$$\begin{aligned} \sigma_{zx} &= -\frac{2(1-2\nu)x}{\underset{\cdot}{R}^3} - \frac{6xc^2}{\underset{\cdot}{R}^5}, \\ \sigma_{zy} &= -\frac{2(1-2\nu)y}{\underset{\cdot}{R}^3} - \frac{6yc^2}{\underset{\cdot}{R}^5}, \qquad\qquad (Z_1+Z_2)_{z=0} \\ \sigma_{z} &= 0. \end{aligned}$$

To annull the final terms, using (v), we again select $\delta_z Z$, then, to remove or annull the nonvanishing terms in $\delta_z Z$ we select, using (viii) and (vi), C and $\delta_z C$, exactly as before. These three nuclei combine to produce at the boundary:

$$\begin{aligned} \sigma_{zx} &= -\frac{3cx}{\underset{\cdot}{R}^5}, \\ \sigma_{zy} &= -\frac{3cy}{\underset{\cdot}{R}^5}, \qquad\qquad (\delta_z Z_2 + (1-2\nu)C_2 - c\delta_z C_2)_{z=0} \\ \sigma_{zz} &= 0. \end{aligned}$$

The initial terms are harmonic and may be eliminated. Noting the symmetry (vii), we select Z. The nonvanishing harmonic expressions introduced by Z are eliminated using (viii), with $\int_z C$, nonvanishing biharmonic ones annulled using (vi), with C. These three nuclei combine to produce at the boundary:

$$\sigma_{zx} = \frac{x}{R^3} ,$$

$$\sigma_{zy} = \frac{y}{R^3} , \qquad (Z_2 + 2(1-\nu)\int_z C_2 - cC_2)_{z=0}$$

$$\sigma_{zz} = 0.$$

The desired condition at the boundary then, is provided by the combination:

$$\begin{aligned} &Z_1+Z_2 \;+\; 2(1-2\nu)[Z_2 + 2(1-2\nu)\int_z C_2 - cC_2] \\ &\qquad - 2c[\partial_z Z_2 + (1-2\nu)C_2 - c\partial_z C_2]. \end{aligned} \qquad [11]$$

Some nuclei having been mentioned twice, we rewrite the combination as:

$$Z_1 + (3-4\nu)Z_2 - 2c\partial_z Z_2 - 4(1-2\nu)cC_2 + 4(1-\nu)(1-2\nu)\int_z C_2 + 2c^2\partial_z C_2.$$

From this latter form, we write out the Galerkin vector:

$$\begin{aligned} \underline{F} = \frac{P}{8\pi(1-\nu)}\, \underline{k}\Big\{ & R_1+(3-4\nu)R_2 - \frac{2c(z+c)}{R_2} - 4(1-2\nu)c\, \log(R_2+z+c) \\ & +4(1-\nu)(1-2\nu)[(z+c)\log(R_2+z+c) - R_2] + \frac{2c^2}{R_2}\Big\} \end{aligned}$$

given by Mindlin (7). Mindlin has given the displacements and stresses.

TABLE V

DISPLACEMENTS AND STRESSES FOR A FORCE IN Z-DIRECTION IN HALF-SPACE WITH FIXED BOUNDARY

$$2Gu_x = x(z-c)\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - \frac{6c}{3-4\nu}\left[\frac{xz(z+c)}{R_2^5}\right]$$

$$2Gu_y = y(z-c)\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - \frac{6c}{3-4\nu}\left[\frac{yz(z+c)}{R_2^5}\right]$$

$$2Gu_z = (3-4\nu)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] + \left[\frac{(z-c)^2}{R_1^3} - \frac{(z+c)^2}{R_2^3}\right] - \frac{6c}{3-4\nu}\left[\frac{z(z+c)^2}{R_2^5}\right]$$

$$\sigma_{xx} = (1-2\nu)(z-c)\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3x^2(z-c)\left[\frac{1}{R_1^5} - \frac{1}{R_2^5}\right] + \frac{2c}{3-4\nu}\left[\frac{4\nu(1-2\nu)}{R_2^3} + \frac{6\nu(z+c)^2}{R_2^5} - \frac{3z(z+c)}{R_2^5} + \frac{15x^2z(z+c)}{R_2^7}\right]$$

$$\sigma_{yy} = (1-2\nu)(z-c)\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3y^2(z-c)\left[\frac{1}{R_1^5} - \frac{1}{R_2^5}\right] + \frac{2c}{3-4\nu}\left[\frac{4\nu(1-2\nu)}{R_2^3} + \frac{6\nu(z+c)^2}{R_2^5} - \frac{3z(z+c)}{R_2^5} + \frac{15y^2z(z+c)}{R_2^7}\right]$$

$$\sigma_{zz} = -(1-2\nu)(z-c)\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3(z-c)\left[\frac{(z-c)^2}{R_1^5} - \frac{(z+c)^2}{R_2^5}\right] + \frac{2c}{3-4\nu}\left[\frac{4(1-\nu)(1-2\nu)}{R_2^3} + \frac{6(1-\nu)(z+c)^2}{R_2^5} - \frac{9z(z+c)}{R_2^5} + \frac{15z(z+c)^3}{R_2^7}\right]$$

$$\sigma_{xy} = -3xy(z-c)\left[\frac{1}{R_1^5} - \frac{1}{R_2^5}\right] + \frac{30c}{3-4\nu}\left[\frac{xyz(z+c)}{R_2^7}\right]$$

$$\sigma_{yz} = -(1-2\nu)y\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3y\left[\frac{(z-c)^2}{R_1^5} - \frac{(z+c)^2}{R_2^5}\right] + \frac{2c}{3-4\nu}\left[-\frac{6(1-\nu)y(z+c)}{R_2^5} - \frac{3yz}{R_2^5} + \frac{15yz(z+c)^2}{R_2^7}\right]$$

$$\sigma_{zx} = -(1-2\nu)x\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3x\left[\frac{(z-c)^2}{R_1^5} - \frac{(z+c)^2}{R_2^5}\right] + \frac{2c}{3-4\nu}\left[-\frac{6(1-\nu)x(z+c)}{R_2^5} - \frac{3xz}{R_2^5} + \frac{15xz(z+c)^2}{R_2^7}\right]$$

A factor of $\frac{P}{8\pi(1-\nu)}$ is omitted throughout

3. Half-space with Fixed Boundary and Free Boundary, Force in x-direction

a) Rongved's problem (X). We begin with X_1-X_2 (Table IV) which produces, at the boundary:

$$2Gu_x = 0 \,,$$

$$2Gu_y = 0, \qquad\qquad (X_1 - X_2)_{z=0}$$

$$2Gu_z = - \frac{2xc}{R^3} \,.$$

As observed in (vii), we may annull this with the harmonic expression in a nucleus with vector $\underline{F} = \underline{k}F_z$. By (iii), the appropriate expression is $\int_z \frac{1}{2}\Delta \frac{xz}{R^3} = \frac{1}{2}\Delta \frac{x}{R}$, so we select $\delta_x Z$ having $\underline{F} = \underline{k}\frac{x}{R}$. Using (vi), we annull the nonvanishing biharmonic terms introduced by $\delta_x Z$ with $\delta_x C$, to produce, at the boundary:

$$2Gu_x = 0 \,,$$

$$2Gu_y = 0 \,, \qquad\qquad (\delta_x Z_2 - c\delta_x C_2)_{z=0}$$

$$2Gu_z = - \frac{(3-4\nu)x}{R^3} \,.$$

This time, two nuclei are sufficient. (The third nucleus entering the previous combinations to eliminate the presently desired harmonic expression would have been, using (viii), $\int_z \delta_x C$.)

The desired condition at the boundary, then, is provided by the combination

$$X_1 - X_2 - \frac{2c}{3-4\nu}(\delta_x Z_2 - c\delta_x C_2). \qquad [12]$$

The Galerkin vector (with force adjustment) for these nuclei is:

$$\underline{F} = \frac{P}{8\pi(1-\nu)}\left\{\underline{i}[R_1 - R_2 + (\frac{2c^2}{3-4\nu})\frac{1}{R_2}] + \underline{k}[-(\frac{2c}{3-4\nu})\frac{x}{R_2}]\right\}.$$

The Papkovitch functions are found from the Galerkin vector by eq. [3]:

$$B_x = \frac{P}{4\pi G}(\frac{1}{R_1} - \frac{1}{R_2}) ,$$

$$B_y = 0,$$

$$B_z = \frac{Pc}{2\pi(3-4\nu)G}\ (\frac{x}{R_2^3}) ,$$

identical with those obtained by Rongved (13). The displacements and stresses are given in Table VI.

b) <u>Mindlin's problem (X)</u>. We begin with X_1+X_2 (Table II). This produces, at the boundary:

$$\sigma_{zx}\ \cancel{2Gu}_x = 0 ,$$

$$\sigma_{zy}\ \cancel{2Gu}_y = 0 , \qquad\qquad (X_1 + X_2)_{z=0}$$

$$\sigma_{zz}\ \cancel{2Gu}_z = \frac{2(1-2\nu)x}{R^3} - \frac{6xc^2}{R^5} .$$

Using (vii) and (iii) as before, we select $\delta_x Z$ to annull the final term. Some of the nonvanishing expressions introduced by

$\delta_x Z$ are harmonic. Using (viii) we select $\int_z \delta_x C$ to eliminate the harmonic expressions, and, using (vi), we select $\delta_x C$ to annull the biharmonic ones. The three nuclei combine to produce at the boundary:

$$\sigma_{zx}\ \cancel{2G}u_x = 0 \ ,$$

$$\sigma_{zy}\ \cancel{2G}u_y = 0 \ , \qquad (\delta_x Z_2 + (1-2\nu)\int_z \delta_x C_2 - c\delta_x C_2)_{z=0}$$

$$\sigma_{zz}\ \cancel{2G}u_z = \frac{3xc}{R^5} \ .$$

The initial term is harmonic and may be eliminated, noting the symmetry (vii), by a nucleus with Galerkin vector $\underline{F} = \underline{k}F_z$. The condition that $\delta_z \frac{1}{2}\Delta F_z = \frac{x}{R^3} = \delta_x \frac{1}{2}\Delta R$ is satisfied by $\int_z \delta_x Z$ with Galerkin vector $\underline{F} = \underline{k}\int_z \delta_x R$. The nonvanishing harmonic expressions introduced by $\int_z \delta_x Z$ are eliminated, using (viii), with $\int_{zz} \delta_x C$, and the nonvanishing biharmonic expressions annulled, using (vi), with $\int_z \delta_x C$. The three nuclei combine to produce at the boundary:

$$\sigma_{zx}\ \cancel{2G}u_x = 0$$

$$\sigma_{zy}\ \cancel{2G}u_y = 0 \qquad (\int_z \delta_x Z_2 + 2(1-\nu)\int_{zz} \delta_x C_2 - c\int_z \delta_x C_2)_{z=0}$$

$$\sigma_{zz}\ \cancel{2G}u_z = -\frac{x}{R^3} \ .$$

The desired condition at the boundary, then, is provided by the combination:

$$X_1 + X_2 + 2(1-2\nu)[\int_z \delta_x Z_2 + 2(1-\nu)\int_{zz}\delta_x C_2 - c\int_z \delta_x C_2]$$

$$+ 2c[\delta_x Z_2 + (1-2\nu)\int_z \delta_x C_2 - c\delta_x C_2]. \qquad [13]$$

The $\int_z \delta_x C$ cancel one another, so we rewrite the combination as:

$$X_1 + X_2 - 2c^2\delta_x C_2 + 4(1-\nu)(1-2\nu)\int_{zz}\delta_x C_2 + 2c\delta_x Z_2 + 2(1-2\nu)\int_z \delta_x Z_2 \,.$$

From this latter form, we write out the Galerkin vector:

$$\underline{F} = \frac{P}{8\pi(1-\nu)}\Big\{\underline{i}\ [R_1 + R_2 - \frac{2c^2}{R_2} + 4(1-\nu)(1-2\nu)([z+c]\log[R_2+z+c]-R_2)]$$

$$+ \underline{k}[\frac{2cx}{R_2} + 2(1-2\nu)x\ \log(R_2+z+c)]\Big\}$$

given by Mindlin (7). Mindlin has given the displacements and stresses.

TABLE VI

DISPLACEMENTS AND STRESSES FOR A FORCE IN X-DIRECTION IN HALF-SPACE WITH FIXED BOUNDARY

$$2Gu_y = (3-4\nu)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] + x^2\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - \frac{2c}{3-4\nu}\left[\frac{z}{R_1^3} - \frac{3x^2z}{R_2^5}\right]$$

$$2Gu_y = xy\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] + \frac{6c}{3-4\nu}\left[\frac{xz(z+c)}{R_2^5}\right]$$

$$2Gu_z = x(z-c)\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] + \frac{6c}{3-4\nu}\left[\frac{xz(z+c)}{R_2^5}\right]$$

$$\sigma_{xx} = -(1-2\nu)x\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3x^3\left[\frac{1}{R_1^5} - \frac{1}{R_2^5}\right] - \frac{2c}{3-4\nu}\left[\frac{6\nu x(z+c)}{R_2^5} - \frac{9xz}{R_2^5} + \frac{15x^3z}{R_2^7}\right]$$

$$\sigma_{yy} = (1-2\nu)x\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3xy^2\left[\frac{1}{R_1^5} - \frac{1}{R_2^5}\right] - \frac{2c}{3-4\nu}\left[\frac{6\nu x(z+c)}{R_2^5} - \frac{3xz}{R_2^5} + \frac{15xy^2z}{R_2^7}\right]$$

$$\sigma_{zz} = (1-2\nu)\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3x\left[\frac{(z-c)^2}{R_1^5} - \frac{(z+c)^2}{R_2^5}\right] - \frac{2c}{3-4\nu}\left[\frac{6\nu x(z+c)}{R_2^5} - \frac{3xz}{R_2^5} + \frac{15xz(z+c)^2}{R_2^7}\right]$$

$$\sigma_{xy} = -(1-2\nu)y\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3x^2y\left[\frac{1}{R_1^5} - \frac{1}{R_2^5}\right] - \frac{2c}{3-4\nu}\left[-\frac{3yz}{R_2^5} + \frac{15x^2yz}{R_2^7}\right]$$

$$\sigma_{yz} = -3xy(z-c)\left[\frac{1}{R_1^5} - \frac{1}{R_2^5}\right] - \frac{2c}{3-4\nu}\left[-\frac{6(1-\nu)xy}{R_2^5} + \frac{15xyz(z+c)}{R_2^7}\right]$$

$$\sigma_{zx} = -(1-2\nu)(z-c)\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] - 3x^2(z-c)\left[\frac{1}{R_1^5} - \frac{1}{R_2^5}\right] - \frac{2c}{3-4\nu}\left[-\frac{4(1-\nu)(1-2\nu)}{R_1^3} - \frac{6(1-\nu)x^2}{R_2^5} + \frac{15x^2z(z+c)}{R_2^7}\right]$$

A factor of $\frac{P}{8\pi(1-\nu)}$ is omitted throughout

4. Observations on the Solutions of the Four Problems

The three nuclei in each group (two for Rongved's problem (X)) bear a fixed relationship to one another, the last two having been found in each case by (viii) and (vi) to eliminate and annull the nonvanishing expressions of the first. "Factoring" the common operator, the six groups of [10], [11], [12], and [13] in the order they appear, contain the nuclei

$$\delta_z \, (Z, \int_z C, C) \qquad [10a]$$

$$(\int_z \delta_z)(Z, \int_z C, C) \qquad [11b]$$

$$\delta_z \, (Z, \int_z C, C) \qquad [11a]$$

$$\delta_x \, (Z, \qquad C) \qquad [12a]$$

$$\int_z \delta_x \, (Z, \int_z C, C) \qquad [13b]$$

$$\delta_x \, (Z, \int_z C, C). \qquad [13a]$$

Disregarding then, the steps which led to the selection of the last two of each group, we have made only six selections. These further reduce to three, [10a] and [11a] both were obtained by (v) to annull ϕ with $\delta_z \phi$ as in (iii), [12a] and [13a] both by (vii) to annull ϕ with $\int_z \frac{1}{2} \Delta \phi$ as in (iv), and [11b] and [12b] both by (vii) to eliminate harmonic expressions.

This regularity has been brought about by taking the half-space solutions of Chapter III as starting point. This important first step has also induced the similarities between fixed boundary and free boundary problems, and induced the symmetry of the conditions on the boundary that made the observation (vii) a fruitful one. It is noted that when the combinations are rewritten, combining like nuclei and dropping nuclei which cancell one another, little remains to indicate the mode of constructing the solutions.

5. Quarter-space and Eighth-space with One Fixed or One Free Boundary, and Mixed Conditions on the Other Boundaries

The properties of the functions entering in the displacements and stresses, and the structure of [4], [5], and [6] which led to the enunciation of (i) and (ii) hold not only for the single forces but for all other nuclei as well, and hence (i) and (ii) may be applied to any nuclei, if we interpret "same direction of force" to mean "same algebraic sign" and "opposite directions of force" as "opposite algebraic sign".

In particular, (i) and (ii) hold for all the nuclei in [10], [11], [12], and [13], as may be verified by looking at the displacements and stresses for Rongved's problems, Tables V and VI, and for Mindlin's problems, (7). Therefore, we may use the combination of nuclei solving these problems as units just as we used the single forces, to obtain the mixed conditions on a plane boundary in quarter-space, or two plane boundaries in eighth-space, the other being free or fixed.

One example is sufficient: let us require a force in the z-direction in quarter-space, with the boundary $z = 0$ fixed, and with zero normal displacements and zero shearing stresses on the boundary $x = 0$. Then, by (i) we must place two forces in the same direction symmetrically with respect to the plane $x = 0$; we obtain $z = 0$ fixed by using, for the two forces, the nuclei for Rongved's problem (Z).

The combination of nuclei producing the desired condition is therefore written out from [10]:

$$\left\{Z_{101} - Z_{102} + \frac{2c}{3-4\nu}[\delta_z Z_{102} + 2(1-2\nu)C_{102} - c\delta_z C_{102}]\right\}$$
$$+\left\{Z_{201} - Z_{202} + \frac{2c}{3-4\nu}[\delta_z Z_{202} + 2(1-2\nu)C_{202} - c\delta_z C_{202}]\right\} .$$

The displacements and stresses may be written out from Table V. A glance at that table shows that the desired condition on $x = 0$ is indeed obtained, since only odd powers of x appear in $2Gu_x$, σ_{xy}, and σ_{xz}.

6. Fixed Boundary Analogues to the Problems of Boussinesq and Cerruti

Mindlin has shown (7) that the solution of Boussinesq's problem of a force in the z-direction on the surface of the half-space, and of Cerruti's problem of a force in the x-direction on the surface of the half-space (11), may be obtained as limiting cases of his solutions [11] and [13], as c is indefinitely diminished. That is, the combination of nuclei solving Boussinesq's problem is, from [11]:

$$2Z + 2(1-2\nu)[Z + 2(1-\nu)\int_z C]$$

for a force of magnitude $8\pi(1-\nu)$, and the combination solving Cerruti's problem with some magnitude force is, from [13]:

$$2X + 2(1-2\nu)[\int_z \delta_x Z + 2(1-\nu)\int_{zz}\delta_x C].$$

Upon similar passage to the limit with the solutions [10] and [12] of Rongved's problems, all the nuclei disappear. But we may, as in deriving the double force from the single force, divide by c and then pass to the limit, obtaining thus an approximate solution for a force in the neighborhood of a fixed boundary of the half-space.

Rewriting [10] to include c with the force adjustment, the combination for Rongved's problem (Z) with force of magnitude P is:

$$\frac{Pc}{8\pi(1-\nu)}\left[\frac{1}{c}(Z_1-Z_2) + \frac{2}{3-4\nu}(\delta_z Z_2 + 2(1-2\nu)C_2 - c\delta_z C_2)\right].$$

As c is diminished, $\frac{1}{c}(Z_1-Z_2)$ becomes $-2\delta_z Z$, and $c\delta_z C_2$ produces negligible effect. Therefore, for small c, the solution to Rongved's problem (Z) with force of magnitude P is given approximately, by the combination:

$$\frac{(1-2\nu)Pc}{2\pi(1-\nu)(3-4\nu)}(-\delta_z Z + C).$$

The Galerkin vector for these nuclei is:

$$\underline{F} = -\frac{(1-2\nu)Pc}{2\pi(1-\nu)(3-4\nu)}\underline{k}\left[\frac{z}{R} - \log(R+z)\right].$$

The displacements and stresses are given in Table VII.

Treating [12] similarly, we rewrite it as:

$$\frac{Pc}{8\pi(1-\nu)}\left[\frac{1}{c}(X_1-X_2) - \frac{2}{3-4\nu}(\delta_x Z_2 - c\delta_x C_2)\right].$$

As c is diminished, $\frac{1}{c}(Z_1-Z_2)$ becomes $-2\delta_x Z$, and so for small c the solution of Rongved's problem (X) with force of magnitude P is given, approximately, by the combination:

$$\frac{Pc}{4\pi(1-\nu)(3-4\nu)}\left[-(3-4\nu)\delta_z X - \delta_x Z\right].$$

The Galerkin vector for these nuclei is:

$$\underline{F} = -\frac{Pc}{4\pi(1-\nu)(3-4\nu)}\left[\underline{i}\,\frac{(3-4\nu)}{R} + \underline{k}\,\frac{x}{R}\right].$$

The displacements and stresses are given in Table VIII.

TABLE VII

DISPLACEMENTS AND STRESSES FOR A FORCE IN Z-DIRECTION IN HALF-SPACE WITH FIXED BOUNDARY, WHEN THE FORCE IS VERY CLOSE TO THE BOUNDARY

$$2Gu_x = \frac{3xz^2}{R^5}$$

$$2Gu_y = \frac{3yz^2}{R^5}$$

$$2Gu_z = \frac{2(1-2\nu)z}{R^3} + \frac{3z^2}{R^5}$$

$$\sigma_{xx} = \frac{2\nu}{R^3} + \frac{3(1-2\nu)z^2}{R^5} - \frac{15z^2x^2}{R^7}$$

$$\sigma_{yy} = \frac{2\nu}{R^3} + \frac{3(1-2\nu)z^2}{R^5} - \frac{15z^2x^2}{R^7}$$

$$\sigma_{zz} = \frac{1-2\nu}{R^3} + \frac{3(1+2\nu)}{R^5} - \frac{15z^4}{R^7}$$

$$\sigma_{xy} = -\frac{15xyz^2}{R^7}$$

$$\sigma_{yz} = \frac{6\nu yz}{R^5} - \frac{15yz^3}{R^7}$$

$$\sigma_{zx} = \frac{6\nu xz}{R^5} - \frac{15xz^3}{R^7}$$

A factor of $\frac{(1-2\nu)Pc}{2\pi(1-\nu)(3-4\nu)}$ is omitted throughout

TABLE VIII

DISPLACEMENTS AND STRESSES FOR A FORCE IN X-DIRECTION IN HALF-SPACE WITH FIXED BOUNDARY, WHEN THE FORCE IS VERY CLOSE TO THE BOUNDARY

$$2Gu_x = \frac{8(1-2\nu)z}{R^3} + \frac{12x^2z}{R^5}$$

$$2Gu_y = \frac{12xyz}{R^5}$$

$$2Gu_z = \frac{12xz^2}{R^5}$$

$$\sigma_{xx} = \frac{3\nu xz}{R^5} - \frac{60x^3z}{R^7}$$

$$\sigma_{yy} = \frac{12(1-2\nu)xz}{R^5} - \frac{60xy^2z}{R^7}$$

$$\sigma_{zz} = \frac{24(1-\nu)xz}{R^5} - \frac{60xz^3}{R^7}$$

$$\sigma_{xy} = -\frac{6(1-4\nu)yz}{R^5} - \frac{60x^2yz}{R^7}$$

$$\sigma_{yz} = \frac{6xy}{R^5} - \frac{60xyz^2}{R^7}$$

$$\sigma_{zx} = \frac{4(1-2\nu)}{R^3} - \frac{6(1-4\nu)z^2}{R^5} + \frac{6x^2}{R^5} - \frac{60x^2z^2}{R^7}$$

A factor of $\frac{Pc}{4\pi(3-4\nu)}$ is omitted throughout

APPENDIX

TABLE OF ϕ AND $\frac{1}{2}\Delta\phi$: EXPRESSIONS FOR COMPUTING DISPLACEMENTS AND STRESSES OF NUCLEI OF STRAIN

Abbreviations:

$\mathrm{I}^3(t) = \frac{1}{(R+t)R}$

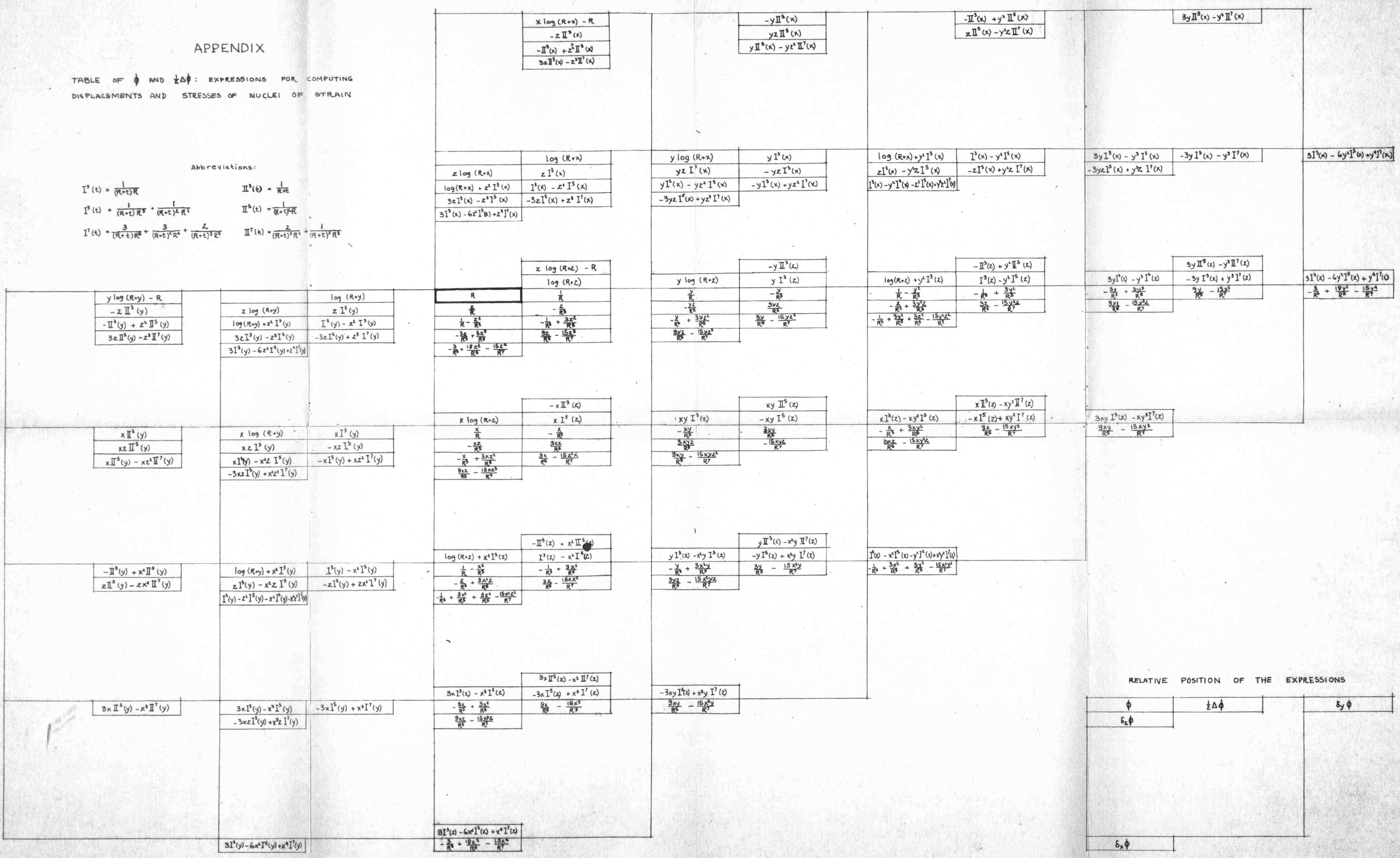
$\mathrm{I}^5(t) = \frac{1}{(R+t)R^3} + \frac{1}{(R+t)^2R^2}$

$\mathrm{I}^7(t) = \frac{3}{(R+t)R^5} + \frac{3}{(R+t)^2R^4} + \frac{2}{(R+t)^3R^3}$

$\mathrm{II}^3(t) = \frac{1}{R+t}$

$\mathrm{II}^5(t) = \frac{1}{(R+t)^2R}$

$\mathrm{II}^7(t) = \frac{2}{(R+t)^3R^2} + \frac{1}{(R+t)^2R^3}$



$x\log(R+x) - R$
$-z\,\mathrm{II}^3(x)$
$-\mathrm{II}^3(x) + z^2\mathrm{II}^5(x)$
$3z\mathrm{II}^5(x) - z^3\mathrm{II}^7(x)$

$-y\mathrm{II}^3(x)$
$yz\mathrm{II}^5(x)$
$y\mathrm{II}^5(x) - yz^2\mathrm{II}^7(x)$

$-\mathrm{II}^3(x) + y^2\mathrm{II}^5(x)$
$z\mathrm{II}^5(x) - y^2z\mathrm{II}^7(x)$

$3y\mathrm{II}^5(x) - y^3\mathrm{II}^7(x)$

	$\log(R+x)$
$z\log(R+x)$	$z\mathrm{I}^3(x)$
$\log(R+x) + z^2\mathrm{I}^3(x)$	$\mathrm{I}^3(x) - z^2\mathrm{I}^5(x)$
$3z\mathrm{I}^3(x) - z^3\mathrm{I}^5(x)$	$-3z\mathrm{I}^5(x) + z^3\mathrm{I}^7(x)$
$3\mathrm{I}^3(x) - 6z^2\mathrm{I}^3(x) + z^4\mathrm{I}^7(x)$	

$y\log(R+x)$	$y\mathrm{I}^3(x)$
$yz\mathrm{I}^3(x)$	$-yz\mathrm{I}^5(x)$
$y\mathrm{I}^3(x) - yz^2\mathrm{I}^5(x)$	$-y\mathrm{I}^5(x) + yz^2\mathrm{I}^7(x)$
$-3yz\mathrm{I}^5(x) + yz^3\mathrm{I}^7(x)$	

$\log(R+x) + y^2\mathrm{I}^3(x)$	$\mathrm{I}^3(x) - y^2\mathrm{I}^5(x)$
$z\mathrm{I}^3(x) - y^2z\mathrm{I}^5(x)$	$-z\mathrm{I}^5(x) + y^2z\mathrm{I}^7(x)$
$\mathrm{I}^3(x) - y^2\mathrm{I}^5(x) - z^2\mathrm{I}^5(x) + y^2z^2\mathrm{I}^7(x)$	

$3y\mathrm{I}^3(x) - y^3\mathrm{I}^5(x)$	$-3y\mathrm{I}^5(x) - y^3\mathrm{I}^7(x)$
$-3yz\mathrm{I}^5(x) + y^3z\mathrm{I}^7(x)$	

$3\mathrm{I}^3(x) - 6y^2\mathrm{I}^5(x) + y^4\mathrm{I}^7(x)$

$y\log(R+y) - R$
$-z\,\mathrm{II}^3(y)$
$-\mathrm{II}^3(y) + z^2\mathrm{II}^5(y)$
$3z\mathrm{II}^5(y) - z^3\mathrm{II}^7(y)$

	$\log(R+y)$
$z\log(R+y)$	$z\mathrm{I}^3(y)$
$\log(R+y) + z^2\mathrm{I}^3(y)$	$\mathrm{I}^3(y) - z^2\mathrm{I}^5(y)$
$3z\mathrm{I}^3(y) - z^3\mathrm{I}^5(y)$	$-3z\mathrm{I}^5(y) + z^3\mathrm{I}^7(y)$
$3\mathrm{I}^3(y) - 6z^2\mathrm{I}^5(y) + z^4\mathrm{I}^7(y)$	

	$z\log(R+z) - R$
	$\log(R+z)$
R	$\frac{1}{R}$
$\frac{z}{R}$	$-\frac{z}{R^3}$
$\frac{1}{R} - \frac{z^2}{R^3}$	$-\frac{1}{R^3} + \frac{3z^2}{R^5}$
$-\frac{3z}{R^3} + \frac{3z^3}{R^5}$	$\frac{9z}{R^5} - \frac{15z^3}{R^7}$
$-\frac{3}{R^3} + \frac{18z^2}{R^5} - \frac{15z^4}{R^7}$	

	$-y\mathrm{II}^3(z)$
$y\log(R+z)$	$y\mathrm{I}^3(z)$
$\frac{y}{R}$	$-\frac{y}{R^3}$
$-\frac{yz}{R^3}$	$\frac{3yz}{R^5}$
$-\frac{y}{R^3} + \frac{3yz^2}{R^5}$	$\frac{3y}{R^5} - \frac{15yz^2}{R^7}$
$\frac{9yz}{R^5} - \frac{15yz^3}{R^7}$	

	$-\mathrm{II}^3(z) + y^2\mathrm{II}^5(z)$
$\log(R+z) + y^2\mathrm{I}^3(z)$	$\mathrm{I}^3(z) - y^2\mathrm{I}^5(z)$
$\frac{1}{R} - \frac{y^2}{R^3}$	$-\frac{1}{R^3} + \frac{3y^2}{R^5}$
$-\frac{z}{R^3} + \frac{3y^2z}{R^5}$	$\frac{3z}{R^5} - \frac{15y^2z}{R^7}$
$-\frac{1}{R^3} + \frac{3y^2}{R^5} + \frac{3z^2}{R^5} - \frac{15y^2z^2}{R^7}$	

	$3y\mathrm{II}^5(z) - y^3\mathrm{II}^7(z)$
$3y\mathrm{I}^3(z) - y^3\mathrm{I}^5(z)$	$-3y\mathrm{I}^5(z) + y^3\mathrm{I}^7(z)$
$-\frac{3y}{R^3} + \frac{3y^3}{R^5}$	$\frac{9y}{R^5} - \frac{15y^3}{R^7}$
$\frac{9yz}{R^5} - \frac{15y^3z}{R^7}$	

$3\mathrm{I}^3(z) - 6y^2\mathrm{I}^5(z) + y^4\mathrm{I}^7(z)$
$-\frac{3}{R^3} + \frac{18y^2}{R^5} - \frac{15y^4}{R^7}$

$x\mathrm{II}^3(y)$
$xz\mathrm{II}^5(y)$
$x\mathrm{II}^5(y) - xz^2\mathrm{II}^7(y)$

$x\log(R+y)$	$x\mathrm{I}^3(y)$
$xz\mathrm{I}^3(y)$	$-xz\mathrm{I}^5(y)$
$x\mathrm{I}^3(y) - x^2z\mathrm{I}^5(y)$	$-x\mathrm{I}^5(y) + xz^2\mathrm{I}^7(y)$
$-3xz\mathrm{I}^5(y) + x^2z^2\mathrm{I}^7(y)$	

	$-x\mathrm{II}^3(z)$
$x\log(R+z)$	$x\mathrm{I}^3(z)$
$\frac{x}{R}$	$-\frac{x}{R^3}$
$-\frac{xz}{R^3}$	$\frac{3xz}{R^5}$
$-\frac{x}{R^3} + \frac{3xz^2}{R^5}$	$\frac{3x}{R^5} - \frac{15z^2x}{R^7}$
$\frac{9xz}{R^5} - \frac{15xz^3}{R^9}$	

	$xy\mathrm{II}^5(z)$
$xy\mathrm{I}^3(z)$	$-xy\mathrm{I}^5(z)$
$-\frac{xy}{R^3}$	$\frac{3xy}{R^5}$
$\frac{3xyz}{R^5}$	$-\frac{15xyz}{R^7}$
$\frac{3xy}{R^5} - \frac{15xyz^2}{R^7}$	

	$x\mathrm{II}^5(z) - xy^2\mathrm{II}^7(z)$
$x\mathrm{I}^3(z) - xy^2\mathrm{I}^5(z)$	$-x\mathrm{I}^5(z) + xy^2\mathrm{I}^7(z)$
$-\frac{x}{R^3} + \frac{3xy^2}{R^5}$	$\frac{3x}{R^5} - \frac{15xy^2}{R^7}$
$\frac{3xz}{R^5} - \frac{15xy^2z}{R^7}$	

$3xy\mathrm{I}^5(z) - xy^3\mathrm{I}^7(z)$
$\frac{9xy}{R^5} - \frac{15xy^3}{R^7}$

$-\mathrm{II}^3(y) + x^2\mathrm{II}^5(y)$
$z\mathrm{II}^5(y) - zx^2\mathrm{II}^7(y)$

$\log(R+y) + x^2\mathrm{I}^3(y)$	$\mathrm{I}^3(y) - x^2\mathrm{I}^5(y)$
$z\mathrm{I}^3(y) - x^2z\mathrm{I}^5(y)$	$-z\mathrm{I}^5(y) + zx^2\mathrm{I}^7(y)$
$\mathrm{I}^3(y) - z^2\mathrm{I}^5(y) - x^2\mathrm{I}^5(y) + x^2z^2\mathrm{I}^7(y)$	

	$-\mathrm{II}^3(z) + x^2\mathrm{II}^5(z)$
$\log(R+z) + x^2\mathrm{I}^3(z)$	$\mathrm{I}^3(z) - x^2\mathrm{I}^5(z)$
$\frac{1}{R} - \frac{x^2}{R^3}$	$-\frac{1}{R^3} + \frac{3x^2}{R^5}$
$-\frac{z}{R^3} + \frac{3x^2z}{R^5}$	$\frac{3z}{R^5} - \frac{15zx^2}{R^7}$
$-\frac{1}{R^3} + \frac{3x^2}{R^5} + \frac{3z^2}{R^5} - \frac{15x^2z^2}{R^7}$	

	$y\mathrm{II}^5(z) - x^2y\mathrm{II}^7(z)$
$y\mathrm{I}^3(z) - x^2y\mathrm{I}^5(z)$	$-y\mathrm{I}^5(z) + x^2y\mathrm{I}^7(z)$
$-\frac{y}{R^3} + \frac{3x^2y}{R^5}$	$\frac{3y}{R^5} - \frac{15x^2y}{R^7}$
$\frac{3yz}{R^5} - \frac{15x^2yz}{R^7}$	

$\mathrm{I}^3(z) - x^2\mathrm{I}^5(z) - y^2\mathrm{I}^5(z) + x^2y^2\mathrm{I}^7(z)$
$-\frac{1}{R^3} + \frac{3x^2}{R^5} + \frac{3y^2}{R^5} - \frac{15x^2y^2}{R^7}$

$3x\mathrm{II}^5(y) - x^3\mathrm{II}^7(y)$

$3x\mathrm{I}^3(y) - x^3\mathrm{I}^5(y)$	$-3x\mathrm{I}^5(y) + x^3\mathrm{I}^7(y)$
$-3xz\mathrm{I}^5(y) + x^3z\mathrm{I}^7(y)$	

	$3x\mathrm{II}^5(z) - x^3\mathrm{II}^7(z)$
$3x\mathrm{I}^3(z) - x^3\mathrm{I}^5(z)$	$-3x\mathrm{I}^5(z) + x^3\mathrm{I}^7(z)$
$-\frac{3x}{R^3} + \frac{3x^3}{R^5}$	$\frac{9x}{R^5} - \frac{15x^3}{R^7}$
$\frac{9xz}{R^5} - \frac{15x^3z}{R^7}$	

$-3xy\mathrm{I}^5(z) + x^3y\mathrm{I}^7(z)$
$\frac{9xy}{R^5} - \frac{15x^3y}{R^7}$

$3\mathrm{I}^3(y) - 6x^2\mathrm{I}^5(y) + x^4\mathrm{I}^7(y)$

$3\mathrm{I}^3(z) - 6x^2\mathrm{I}^5(z) + x^4\mathrm{I}^7(z)$
$-\frac{3}{R^3} + \frac{18x^2}{R^5} - \frac{15x^4}{R^7}$

RELATIVE POSITION OF THE EXPRESSIONS

APPENDIX

TABLE OF ϕ AND $\frac{1}{2}\Delta\phi$: EXPRESSIONS FOR COMPUTING DISPLACEMENTS AND STRESSES OF NUCLEI OF STRAIN

Abbreviations:

$\mathrm{I}^3(t) = \frac{1}{(R+t)R}$

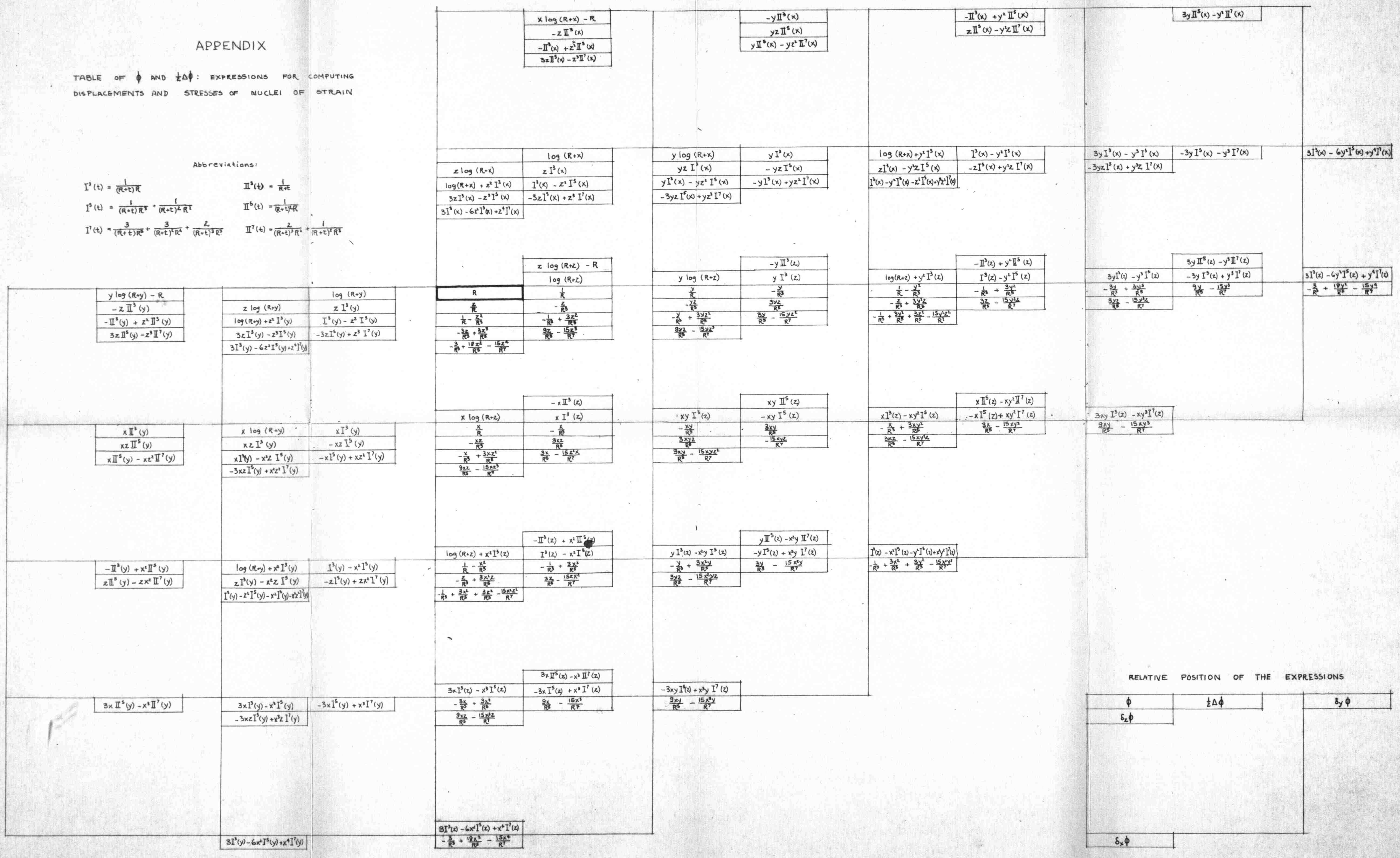
$\mathrm{I}^5(t) = \frac{1}{(R+t)R^3} + \frac{1}{(R+t)^2 R^2}$

$\mathrm{I}^7(t) = \frac{3}{(R+t)R^5} + \frac{3}{(R+t)^2 R^4} + \frac{2}{(R+t)^3 R^3}$

$\mathrm{II}^3(t) = \frac{1}{R+t}$

$\mathrm{II}^5(t) = \frac{1}{(R+t)^2 R}$

$\mathrm{II}^7(t) = \frac{2}{(R+t)^3 R^2} + \frac{1}{(R+t)^2 R^3}$



$x \log(R+x) - R$
$-z\,\mathrm{II}^3(x)$
$-\mathrm{II}^3(x) + z^2\mathrm{II}^5(x)$
$3z\mathrm{II}^5(x) - z^3\mathrm{II}^7(x)$

$-y\mathrm{II}^3(x)$
$yz\mathrm{II}^5(x)$
$y\mathrm{II}^5(x) - yz^2\mathrm{II}^7(x)$

$-\mathrm{II}^3(x) + y^2\mathrm{II}^5(x)$
$z\mathrm{II}^5(x) - y^2 z\mathrm{II}^7(x)$

$3y\mathrm{II}^5(x) - y^3\mathrm{II}^7(x)$

$z\log(R+x)$
$\log(R+x) + z^2\mathrm{I}^3(x)$
$3z\mathrm{I}^3(x) - z^3\mathrm{I}^5(x)$
$3\mathrm{I}^3(x) - 6z^2\mathrm{I}^3(x) + z^4\mathrm{I}^7(x)$

$\log(R+x)$
$z\mathrm{I}^3(x)$
$\mathrm{I}^3(x) - z^2\mathrm{I}^5(x)$
$-3z\mathrm{I}^5(x) + z^3\mathrm{I}^7(x)$

$y\log(R+x)$
$yz\mathrm{I}^3(x)$
$y\mathrm{I}^3(x) - yz^2\mathrm{I}^5(x)$
$-3yz\mathrm{I}^5(x) + yz^3\mathrm{I}^7(x)$

$y\mathrm{I}^3(x)$
$-yz\mathrm{I}^5(x)$
$-y\mathrm{I}^5(x) + yz^2\mathrm{I}^7(x)$

$\log(R+x) + y^2\mathrm{I}^3(x)$
$z\mathrm{I}^3(x) - y^2 z\mathrm{I}^5(x)$
$\mathrm{I}^3(x) - y^2\mathrm{I}^5(x) - z^2\mathrm{I}^5(x) + y^2z^2\mathrm{I}^7(x)$

$\mathrm{I}^3(x) - y^2\mathrm{I}^5(x)$
$-z\mathrm{I}^5(x) + y^2 z\mathrm{I}^7(x)$

$3y\mathrm{I}^3(x) - y^3\mathrm{I}^5(x)$
$-3yz\mathrm{I}^5(x) + y^3 z\mathrm{I}^7(x)$

$-3y\mathrm{I}^5(x) - y^3\mathrm{I}^7(x)$

$3\mathrm{I}^3(x) - 6y^2\mathrm{I}^5(x) + y^4\mathrm{I}^7(x)$

$y\log(R+y) - R$
$-z\,\mathrm{II}^3(y)$
$-\mathrm{II}^3(y) + z^2\mathrm{II}^5(y)$
$3z\mathrm{II}^5(y) - z^3\mathrm{II}^7(y)$

$z\log(R+y)$
$\log(R+y) + z^2\mathrm{I}^3(y)$
$3z\mathrm{I}^3(y) - z^3\mathrm{I}^5(y)$
$3\mathrm{I}^3(y) - 6z^2\mathrm{I}^5(y) + z^4\mathrm{I}^7(y)$

$\log(R+y)$
$z\mathrm{I}^3(y)$
$\mathrm{I}^3(y) - z^2\mathrm{I}^5(y)$
$-3z\mathrm{I}^5(y) + z^3\mathrm{I}^7(y)$

$\boxed{R}$
$\frac{z}{R}$
$\frac{1}{R} - \frac{z^2}{R^3}$
$-\frac{3z}{R^3} + \frac{3z^3}{R^5}$
$-\frac{3}{R^3} + \frac{18z^2}{R^5} - \frac{15z^4}{R^7}$

$z\log(R+z) - R$
$\log(R+z)$
$\frac{1}{R}$
$-\frac{z}{R^3}$
$-\frac{1}{R^3} + \frac{3z^2}{R^5}$
$\frac{9z}{R^5} - \frac{15z^3}{R^7}$

$y\log(R+z)$
$\frac{y}{R}$
$-\frac{yz}{R^3}$
$-\frac{y}{R^3} + \frac{3yz^2}{R^5}$
$\frac{9yz}{R^5} - \frac{15yz^3}{R^7}$

$-y\mathrm{II}^3(z)$
$y\mathrm{I}^3(z)$
$-\frac{y}{R^3}$
$\frac{3yz}{R^5}$
$\frac{3y}{R^5} - \frac{15yz^2}{R^7}$

$\log(R+z) + y^2\mathrm{I}^3(z)$
$\frac{1}{R} - \frac{y^2}{R^3}$
$-\frac{z}{R^3} + \frac{3y^2z}{R^5}$
$-\frac{1}{R^3} + \frac{3y^2}{R^5} + \frac{3z^2}{R^5} - \frac{15y^2z^2}{R^7}$

$-\mathrm{II}^3(z) + y^2\mathrm{II}^5(z)$
$\mathrm{I}^3(z) - y^2\mathrm{I}^5(z)$
$-\frac{1}{R^3} + \frac{3y^2}{R^5}$
$\frac{3z}{R^5} - \frac{15y^2z}{R^7}$

$3y\mathrm{I}^3(z) - y^3\mathrm{I}^5(z)$
$-\frac{3y}{R^3} + \frac{3y^3}{R^5}$
$\frac{9yz}{R^5} - \frac{15y^3z}{R^7}$

$3y\mathrm{II}^5(z) - y^3\mathrm{II}^7(z)$
$-3y\mathrm{I}^5(z) + y^3\mathrm{I}^7(z)$
$\frac{9y}{R^5} - \frac{15y^3}{R^7}$

$3\mathrm{I}^3(z) - 6y^2\mathrm{I}^5(z) + y^4\mathrm{I}^7(z)$
$-\frac{3}{R^3} + \frac{18y^2}{R^5} - \frac{15y^4}{R^7}$

$x\mathrm{II}^3(y)$
$xz\mathrm{II}^5(y)$
$x\mathrm{II}^5(y) - xz^2\mathrm{II}^7(y)$

$x\log(R+y)$
$xz\mathrm{I}^3(y)$
$x\mathrm{I}^3(y) - x^2z\mathrm{I}^5(y)$
$-3xz\mathrm{I}^5(y) + x^2z^2\mathrm{I}^7(y)$

$x\mathrm{I}^3(y)$
$-xz\mathrm{I}^5(y)$
$-x\mathrm{I}^5(y) + xz^2\mathrm{I}^7(y)$

$x\log(R+z)$
$\frac{x}{R}$
$-\frac{xz}{R^3}$
$-\frac{x}{R^3} + \frac{3xz^2}{R^5}$
$\frac{9xz}{R^5} - \frac{15xz^3}{R^7}$

$-x\mathrm{II}^3(z)$
$x\mathrm{I}^3(z)$
$-\frac{x}{R^3}$
$\frac{3xz}{R^5}$
$\frac{3x}{R^5} - \frac{15z^2x}{R^7}$

$xy\mathrm{I}^3(z)$
$-\frac{xy}{R^3}$
$\frac{3xyz}{R^5}$
$\frac{3xy}{R^5} - \frac{15xyz^2}{R^7}$

$xy\mathrm{II}^5(z)$
$-xy\mathrm{I}^5(z)$
$\frac{3xy}{R^5}$
$-\frac{15xyz}{R^7}$

$x\mathrm{I}^3(z) - xy^2\mathrm{I}^5(z)$
$-\frac{x}{R^3} + \frac{3xy^2}{R^5}$
$\frac{3xz}{R^5} - \frac{15xy^2z}{R^7}$

$x\mathrm{II}^5(z) - xy^2\mathrm{II}^7(z)$
$-x\mathrm{I}^5(z) + xy^2\mathrm{I}^7(z)$
$\frac{3x}{R^5} - \frac{15xy^2}{R^7}$

$3xy\mathrm{I}^5(z) - xy^3\mathrm{I}^7(z)$
$\frac{9xy}{R^5} - \frac{15xy^3}{R^7}$

$-\mathrm{II}^3(y) + x^2\mathrm{II}^5(y)$
$z\mathrm{II}^5(y) - zx^2\mathrm{II}^7(y)$

$\log(R+y) + x^2\mathrm{I}^3(y)$
$z\mathrm{I}^3(y) - x^2z\mathrm{I}^5(y)$
$\mathrm{I}^3(y) - z^2\mathrm{I}^5(y) - x^2\mathrm{I}^5(y) - x^2z^2\mathrm{I}^7(y)$

$\mathrm{I}^3(y) - x^2\mathrm{I}^5(y)$
$-z\mathrm{I}^5(y) + zx^2\mathrm{I}^7(y)$

$\log(R+z) + x^2\mathrm{I}^3(z)$
$\frac{1}{R} - \frac{x^2}{R^3}$
$-\frac{z}{R^3} + \frac{3x^2z}{R^5}$
$-\frac{1}{R^3} + \frac{3x^2}{R^5} + \frac{3z^2}{R^5} - \frac{15x^2z^2}{R^7}$

$-\mathrm{II}^3(z) + x^2\mathrm{II}^5(z)$
$\mathrm{I}^3(z) - x^2\mathrm{I}^5(z)$
$-\frac{1}{R^3} + \frac{3x^2}{R^5}$
$\frac{3z}{R^5} - \frac{15zx^2}{R^7}$

$y\mathrm{I}^3(z) - x^2y\mathrm{I}^5(z)$
$-\frac{y}{R^3} + \frac{3x^2y}{R^5}$
$\frac{3yz}{R^5} - \frac{15x^2yz}{R^7}$

$y\mathrm{II}^5(z) - x^2y\mathrm{II}^7(z)$
$-y\mathrm{I}^5(z) + x^2y\mathrm{I}^7(z)$
$\frac{3y}{R^5} - \frac{15x^2y}{R^7}$

$\mathrm{I}^3(z) - x^2\mathrm{I}^5(z) - y^2\mathrm{I}^5(z) + x^2y^2\mathrm{I}^7(z)$
$-\frac{1}{R^3} + \frac{3x^2}{R^5} + \frac{3y^2}{R^5} - \frac{15x^2y^2}{R^7}$

$3x\mathrm{II}^5(y) - x^3\mathrm{II}^7(y)$

$3x\mathrm{I}^3(y) - x^3\mathrm{I}^5(y)$
$-3xz\mathrm{I}^5(y) + x^3z\mathrm{I}^7(y)$

$-3x\mathrm{I}^5(y) + x^3\mathrm{I}^7(y)$

$3x\mathrm{I}^3(z) - x^3\mathrm{I}^5(z)$
$-\frac{3x}{R^3} + \frac{3x^3}{R^5}$
$\frac{9xz}{R^5} - \frac{15x^3z}{R^7}$

$3x\mathrm{II}^5(z) - x^3\mathrm{II}^7(z)$
$-3x\mathrm{I}^5(z) + x^3\mathrm{I}^7(z)$
$\frac{9x}{R^5} - \frac{15x^3}{R^7}$

$-3xy\mathrm{I}^5(z) + x^3y\mathrm{I}^7(z)$
$\frac{9xy}{R^5} - \frac{15x^3y}{R^7}$

$3\mathrm{I}^3(y) - 6x^2\mathrm{I}^5(y) + x^4\mathrm{I}^7(y)$

$3\mathrm{I}^3(z) - 6x^2\mathrm{I}^5(z) + x^4\mathrm{I}^7(z)$
$-\frac{3}{R^3} + \frac{18x^2}{R^5} - \frac{15x^4}{R^7}$

RELATIVE POSITION OF THE EXPRESSIONS

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