SOME DISTRIBUTIONS ARISING FROM A SEQUENCE OF OBSERVATIONS AND THEIR APPLICATIONS

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TESTS OF HOMOGENEITY AND RANDOMNESS

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ABSTRACT

This thesis is an attempt to develop a test of homogeneity and randomness of two given sequences by studying the distribution of the sum of squares of the numerical differences of adjacent observations in a sequence. The various non-parametric tests that have been developed in the past are independent of the actual magnitudes of the observations and they are applied on the sample obtained by pooling the sequences in descending or ascending order. This pooling destroys the original order of the sequences. Moreover, a test which is a function of the actual numerical magnitudes of the observations is likely to give more information about the sequences.

In Chapter II of the thesis, the distribution of the sum of squares of differences between adjacent observations is studied by first evaluating its first three or four moments in sequences arising from (A) a discrete population $A_1, A_2, \ldots A_k$ with probabilities $p_1, p_2, \ldots p_k$ respectively, (B) a continuous population with frequency function $f(x)$, and (C) a discrete population with n_1 A_1's, n_2 A_2's, ... n_k A_k's. These moments are then further used to derive the asymptotic distribution for the sum of squares of differences, which has the normal form.

In Chapter III two applications are derived: a test of randomness for a given sequence, and a test of homogeneity and randomness for two given sequences. The sequences in both tests are derived from discrete populations or normal populations. However, these tests can be easily

extended to the case where the sequences are derived from any continuous population. By way of illustration, the test of homogeneity and randomness is applied in situations where the two sequences come from the same population, where they come from different populations with equal means, and from different populations with unequal means. The results are summarized in Tables III and IV.

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CHAPTER I

INTRODUCTION

The generally used test for testing homogeneity of two given samples, is the student's t-test which assumes that the two samples are random, and come from the same normal population. This test does not detect randomness, and in some experimental data the assumption of normality may not be valid. Other tests have come into use, developed by Wald and Wolfowitz (1940), Mood (1954), Dixon (1940), Wilcoxon (1945), Mann and Whitney (1947), for testing the homogeneity of two samples. These tests are independent of the nature of the distribution of the parent populations, and detect randomness of the pooled sample when the two samples are arranged in descending or ascending order. Other tests which are also independent of the distribution of the parent population of the observations and test homogeneity as well as randomness of two given samples, have been constructed by Iyer and Singh (1955). They consider all possible pairs of observations according to the order of occurrence in moving blocks of r contiguous observations, i.e.

(i) adjacent observations

(ii) adjacent and alternate observations

(iii) adjacent, alternate and others separated by two, three,... k-2 observations according as $r = 2, 3, \ldots k$.

Defining the score of the difference of two observations as +1 or -1 according to whether the difference is positive or negative, respectively, they derive the distribution of the total number of positive or negative scores of all possible pairs of observations. The distribution of functions of the actual numerical differences between the observations would be more useful for constructing statistical tests of significance of differences between two or more samples. But in view of the complexity of the problem, little work has been done along this line. However, Kamat (1953) has worked on the moments of a function based on the modulus of differences between adjacent observations, the observations coming from normal populations with equal variances but different means, with a view of estimating the standard deviation of a given sample from these populations. Von Neumann, Kent, Bellinson and Hart (1941) have given a brief historical review of the research done on the sums of squares of differences of adjacent observations coming from normal distributions, and have explained their use in ballistics and in astronomy. Iyer (1954) has discussed the probability distribution of the statistics $\Sigma(x_r-y_r)$, $\Sigma|x_r-y_r|$, and $\Sigma(x_r-y_r)^2$ arising from considerations of simple matching for two binomial sequences and has suggested their application for testing the homogeneity of two sequences.

The purpose of this thesis is to study the distribution of sums of squares of differences between adjacent observations of a sequence arising from any continuous or discrete population with infinite sampling, by evaluating the moments of this distribution. The moments for a sequence formed from a given set of values say n_1 A_1's, n_2 A_2's, ... n_k A_k's are also constructed by a special method developed by Iyer (1950).

Two tests are developed: (1) testing randomness of a given sequence of observations, and (2) testing homogeneity and randomness of two samples. In both tests (1) and (2) the observations of the samples come from a continuous or discrete distribution with infinite sampling.

CHAPTER II

DISTRIBUTION OF SUM OF SQUARES OF DIFFERENCES

We shall be concerned with the distribution of the sum of squares of differences between successive observations in a sequence of n observations for the following situations:

A. Each observation of the sequence assumes the values $A_1, A_2, \ldots A_k$ with probabilities $p_1, p_2, \ldots p_k$ respectively, subject to the condition $\sum_{r=1}^{k} p_r = 1$. This may be termed free sampling.

B. The observations of the sequence come from a continuous distribution with frequency function $f(x)$.

C. There are $n_1, n_2, \ldots n_k$ observations taking the values $A_1, A_2, \ldots A_k$ in the sequence, with the condition $\sum_{r=1}^{k} n_r = n$. This is termed non-free sampling.

Let x_i be a random variable assuming the observed value at the i^{th} place $(i = 1, 2, \ldots n)$ in the sequence. We shall refer to the value assumed by the variable x_i as an <u>element</u>; that is the value of x_i may be any of the k elements $A_1, A_2, \ldots A_k$. Let $X_i = (x_i - x_{i+1})^2$. We shall say that the two random variables X_i, X_j are <u>disconnected</u> if they do not have an observation in common, i.e. if $j > i+1$.

Let T denote the statistic

$$T = \sum_{i=1}^{n-1} X_i = \sum_{i=1}^{n-1} (x_i - x_{i+1})^2.$$

The moments of T are evaluated for the situations (A), (B) and (C), however, due to lack of time the distribution of T is studied only for the situations (A) and (B).

1. Moments of T

A. Free sampling. Let T be defined as above,

$$T = \sum_{i=1}^{n-1} X_i = \sum_{i=1}^{n-1} (x_i - x_{i+1})^2$$

where X_i assumes the value $(A_h - A_s)^2$ with probability $p_h p_s$ $(h, s = 1, 2, \ldots k)$. The expected value of T becomes

$$E(T) = E(\sum_{i=1}^{n-1} X_i) = (n-1)\, E(X_i) = (n-1)\, E(x_i - x_{i+1})^2.$$

$E(X_i)$ is obtained by considering all the possible values that x_i, x_{i+1} can assume, with their corresponding probabilities of occurrence.

Hence

$$E(X_i) = \sum_{h=1}^{k} \sum_{s=1}^{k} (A_h - A_s)^2\, p_h p_s = 2 \sum_{h>s} (A_h - A_s)^2\, p_h p_s, \qquad (1)$$

so that

$$E(T) = 2(n-1) \sum_{h>s} (A_h - A_s)^2\, p_h p_s. \qquad (2)$$

The second moment around the mean, or the variance of T is obtained as follows:

$$V(T) = E(T^2) - E^2(T)$$

$$E(T^2) = E(\sum_{i=1}^{n-1} X_i)^2 = E[\sum_{i=1}^{n-1} X_i^2 + 2\sum_{i=1}^{n-2} X_i X_{i+1} + 2\sum_{j>i+1} X_i X_j]$$

$$= (n-1)\ E(X_i^2) + 2(n-2)\ E(X_i X_{i+1}) + 2\binom{n-2}{2} \underset{j>i+1}{E} (X_i X_j).$$

Hence

$$V(T) = (n-1)\ E(X_i^2) + 2(n-2)\ E(X_i X_{i+1}) + (n-2)(n-3) \underset{j>i+1}{E} (X_i X_j)$$

$$- (n-1)^2\ [E(X_i)]^2. \qquad (3)$$

For $j>i+1$, the random variables X_i and X_j are disconnected, and therefore are independent. Moreover, by our model, $E(X_i) = E(X_j)$, $(j = 1, 2, \ldots k)$.

Hence

$$\underset{j>i+1}{E} (X_i X_j) = [E(X_i)]^2. \qquad (4)$$

We can rewrite $(n-2)(n-3)$ as

$$(n-1)^2 - 2(n-2) - (n-1). \qquad (5)$$

Substituting (4) and (5) in (3), we obtain

$$V(T) = (n-1)[E(X_i^2) - \{E(X_i)\}^2] + 2(n-2)[E(X_i X_{i+1}) - \{E(X_i)\}^2]. \qquad (6)$$

We now evaluate $E(X_i^2)$, $E(X_i X_{i+1})$, and $[E(X_i)]^2$.

The detailed expressions of expected values in the moments of T will be denoted by paragraphs a, b, ... i.

a) $E(X_i^2)$ is easily obtained,

$$E(X_i^2) = 2\sum_{h>s} (A_h - A_s)^4\ p_h p_s. \qquad (7)$$

b) For $E(X_iX_{i+1}) = E\left\{(x_i-x_{i+1})^2(x_{i+1}-x_{i+2})^2\right\}$ let each of x_i, x_{i+1}, and x_{i+2} assume all possible values in the expression $(x_i-x_{i+1})^2(x_{i+1}-x_{i+2})^2$, then multiplying each resulting expression by its probability of occurrence, form the sum of such products. The sum thus obtained is $E(X_iX_{i+1})$.

Note that $X_i = 0$ whenever the two adjacent observations x_i and x_{i+1} assume the same value, so that such cases do not contribute in the computations of expected values. Hence we need consider only the following cases:

(i) All three observations x_i, x_{i+1}, x_{i+2} assume different values, say $A_h\ A_s\ A_r$ with probability $p_hp_sp_r$. The three consecutive elements can be permuted in $3! = 6$ ways:

$$A_hA_sA_r \quad , \qquad A_hA_rA_s \quad , \qquad A_rA_hA_s \quad ,$$

and their respective inverses

$$A_rA_sA_h \quad , \qquad A_sA_rA_h \quad , \qquad A_sA_hA_r \quad .$$

The value obtained for the product X_iX_{i+1} is the same for a permutation and its inverse, since $(x_i-x_{i+1})^2(x_{i+1}-x_{i+2})^2 = (x_{i+2}-x_{i+1})^2(x_{i+1}-x_i)^2$ for any triplet of values of x_i, x_{i+1} and x_{i+2}.

Note that there are $\binom{k}{3}$ possible triplets $A_hA_sA_r$ for a set of 3 consecutive observations x_i, x_{i+1}, x_{i+2}.

(ii) Two observations out of x_i, x_{i+1}, x_{i+2} assume identical values: $A_hA_sA_h$ with probability $p_h^2p_s$, or $A_sA_hA_s$ with probability $p_s^2p_h$. There are $\binom{k}{2}$ possible pairs of elements A_hA_s for each arrangement of three consecutive elements of two kinds.

To simplify the expressions for expected values in the following discussion, we introduce the

<u>Notation</u>: Σ_r = sum over all $\binom{k}{r}$ possible choices of r-tuplets from $A_1, A_2, \ldots A_k$.

With this notation the expression for $E(X_i X_{i+1})$ becomes

$$E(X_i X_{i+1}) = 2\Sigma_3\Big[(A_h-A_s)^2(A_s-A_r)^2 + (A_h-A_r)^2(A_r-A_s)^2 + (A_r-A_h)^2(A_h-A_s)^2\Big]p_h p_s p_r + \Sigma_2(A_h-A_s)^4(p_h^2 p_s + p_h p_s^2) \qquad (8)$$

c) For $[E(X_i)]^2 = E(x_i-x_{i+1})^2(x_j-x_{j+1})^2$ let each of the observations of the disconnected pairs $x_i x_{i+1}$ and $x_j x_{j+1}$ assume all possible values in the expression $(x_i-x_{i+1})^2(x_j-x_{j+1})^2$, then multiplying each resulting expression by its probability of occurrence, form the sum of such products. The sum thus obtained is $[E(X_i)]^2$.

We have the following cases:

(i) Each of the two pairs x_i, x_{i+1} and x_j, x_{j+1} assume the values A_h, A_s with probability $p_h^2 p_s^2$. The elements $A_h A_s$ can be permuted in 2! ways within each pair, giving rise to 2! x 2! = 4 permutations for the observations of the two pairs together.

(ii) Each observation in a pair assumes one of the values A_h, A_s, A_r, as in the following table.

x_i	x_{i+1}	x_j	x_{j+1}	Probability of occurrence
A_h	A_s	A_h	A_r	$p_h^2\, p_s\, p_r$
A_h	A_s	A_s	A_r	$p_h\, p_s^2\, p_r$
A_h	A_r	A_s	A_r	$p_h\, p_s\, p_r^2$

As in (i) above, each of the three possibilities in the table gives rise to $2! \times 2! = 4$ permutations of the elements within the 2 pairs. The two pairs can then be permuted, making a total of 8 permutations for each of the three possibilities in the table. Clearly, all 8 permutations of the same 2 pairs of elements yield identical values for the expression $(x_i-x_{i+1})^2(x_j-x_{j+1})^2$.

(iii) All four observations assume different values say A_h, A_s, A_r, A_t with probability $p_h p_s p_r p_t$. The four elements A_h, A_s, A_r, A_t can be combined in 3 ways into 2 groups of 2 observations each by taking any one of the four elements and combining it with each of the remaining elements as in the following table:

x_i	x_{i+1}	x_j	x_{j+1}
A_h	A_s	A_r	A_t
A_h	A_r	A_s	A_t
A_h	A_t	A_s	A_r

For a given combination there will be $2! \times 2! = 4$ permutations of the elements within the 2 pairs and two permutations of the pairs

x_i, x_{i+1} and x_j, x_{j+1}, all 8 permutations yielding identical values for $(x_i-x_{i+1})^2(x_j-x_{j+1})^2$.

Noting that there are $\binom{k}{2}$ possible choices of elements in case (i), $\binom{k}{3}$ possible choices in case (ii), and $\binom{k}{4}$ possible choices in case (iii), we can now compute $[E(X_i)]^2$ as

$$[E(X_i)]^2 = 4\Sigma_2(A_h-A_s)^4\ p_h^2p_s^2 + 8\Big\{\Sigma_3[(A_h-A_s)^2(A_h-A_r)^2\ p_h^2p_sp_r + (A_h-A_s)^2(A_s-A_r)^2\ p_hp_s^2p_r + (A_h-A_r)^2(A_s-A_r)^2\ p_hp_sp_r^2]\Big\} + 8\Big\{\Sigma_4[(A_h-A_s)^2(A_r-A_t)^2+(A_h-A_r)^2(A_s-A_t)^2+(A_h-A_t)^2(A_s-A_r)^2]p_hp_sp_rp_t\Big\} \quad .... \quad (9)$$

Substituting expressions (1), (7) and (8) in (6), we obtain

$$V(T) = (n-1)[2\Sigma_2(A_h-A_s)^4\ p_hp_s - \Big\{2\Sigma_2(A_h-A_s)^2\ p_hp_s\Big\}^2] + 2(n-2)[2\Sigma_3\Big\{(A_h-A_s)^2(A_s-A_r)^2 + (A_h-A_r)^2(A_r-A_s)^2+(A_r-A_h)^2(A_h-A_s)^2\Big\}p_hp_sp_r + \Sigma_2(A_h-A_s)^4(p_h^2p_s+p_hp_s^2) - \Big\{2\Sigma_2(A_h-A_s)^2\ p_hp_s\Big\}^2]. \quad (10)$$

In (10) we have expressed $[E(X_i)]^2$ by expression (1) instead of the expanded expression (9).

The third moment of T is, by definition,

$$\mu_3(T) = \mu_3(\sum_{i=1}^{n-1}X_i) = E\left\{\sum_{i=1}^{n-1}X_i - (n-1)E(X_i)\right\}^3 = E\Big\{\Sigma[X_i-E(X_i)]\Big\}^3.$$

Let us write $E(X_i)$ as $\underline{a}$ for short $(i = 1, 2, \ldots n-1)$, then

$$\mu_3(T) = E\left\{\sum_{i=1}^{n-1}(X_i-a)\right\}^3 = E\left\{\sum_{i=1}^{n-1}(X_i-a)^3 + \frac{3!}{2!1!}\Big[\sum_{i\neq j}(X_i-a)^2(X_j-a) + \sum_{i\neq j}(X_i-a)(X_j-a)^2\Big] + \frac{3!}{1!1!1!}\sum_{i\neq j\neq k}(X_i-a)(X_j-a)(X_k-a)\right\}.$$

But for any set of powers s_1, s_2, s_3, $E(X_i-a)^{s_1}(X_j-a)^{s_2}(X_k-a)^{s_3} = 0$ if $k > j+1$ or $j > i+1$.

Hence,

$$\mu_3(T) = E\Big\{\sum_{i=1}^{n-1}(X_i-a)^3 + 3\sum_{i=1}^{n-2}[(X_i-a)^2(X_{i+1}-a) + (X_i-a)(X_{i+1}-a)^2]$$

$$+ 3!\sum_{i=1}^{n-3}(X_i-a)(X_{i+1}-a)(X_{i+2}-a)\Big\}$$

$$= (n-1)E(X_i-a)^3 + 3(n-2)[E(X_i-a)^2(X_{i+1}-a) + E(X_i-a)(X_{i+1}-a)^2]$$

$$+ 3!(n-3)E(X_i-a)(X_{i+1}-a)(X_{i+2}-a). \qquad (11)$$

Expanding the expected value expressions in equation (11), we obtain

$$E(X_i-a)^3 = E(X_i^3) - 3a\,E(X_i^2) + 3a^2\,E(X_i) - a^3$$

$$= E(X_i^3) - 3a\,E(X_i^2) + 2a^3.$$

$$E(X_i-a)^2(X_{i+1}-a) = E(X_i^2X_{i+1}) - a\,E(X_i^2) - 2a\,E(X_iX_{i+1}) + 2a^2\,E(X_i) + a^2E(X_{i+1})$$

$$- a^3 = E(X_i^2X_{i+1}) - aE(X_i^2) - 2aE(X_iX_{i+1}) + 2a^3.$$

$E(X_i-a)(X_{i+1}-a)^2 = E(X_i-a)^2(X_{i+1}-a)$, since X_i, X_{i+1} are identically distributed.

$$E(X_i-a)(X_{i+1}-a)(X_{i+2}-a) = E(X_iX_{i+1}X_{i+2}) - aE(X_iX_{i+1}) - aE(X_iX_{i+2})$$

$$+ a^2E(X_i) - aE(X_{i+1}X_{i+2}) + a^2E(X_{i+1}) + a^2E(X_{i+2}) - a^3$$

$$= E(X_iX_{i+1}X_{i+2}) - 2aE(X_iX_{i+1}) + a^3.$$

Substituting these expanded expressions in equation (11), and collecting terms, we obtain

$$\mu_3(T) = (n-1)\ E(X_i^3) - 3(3n-5)a\ E(X_i^2) + 6(n-2)\ E(X_i^2X_{i+1})$$
$$- 12(2n-5)\ a\ E(X_iX_{i+1}) + 6(n-3)\ E(X_iX_{i+1}X_{i+2}) + 4(5n-11)a^3. \quad \ldots\ldots \quad (12)$$

We now evaluate the expectations that appear in equation (12), in terms of the elements $A_1, A_2, \ldots A_k$ and their probabilities of occurrence $p_1, p_2, \ldots p_k$.

d) $E(X_i^3)$ is easily obtained from the definition of X_i as

$$E(X_i^3) = 2 \sum_{h>s} (A_h-A_s)^6\ p_hp_s\ . \quad (13)$$

e) $a\ E(X_i^2) = E(X_j)\ E(X_i^2) = E(x_j-x_{j+1})^2\ E(x_i-x_{i+1})^4$ is obtained by first letting each of $x_i, x_{i+1}, x_j, x_{j+1}$ assume all possible values in the expression $(x_j-x_{j+1})^2(x_i-x_{i+1})^4$, then multiplying each resulting expression by its probability of occurrence, and forming the sum of such products.

The same cases will arise as in (c), namely

(i) Each of the two pairs x_i, x_{i+1} and x_j, x_{j+1} assume the values A_h, A_s with probability $p_h^2p_s^2$, with four permutations of the elements of the two pairs.

(ii) Each observation in a pair assumes one of the values A_h, A_s, A_r, with 4 permutations of the elements within the 2 pairs, and 2 permutations of the two pairs. The two permutations of the two pairs yield distinct values for the expression $(x_j-x_{j+1})^2(x_i-x_{i+1})^4$, because $(x_i-x_{i+1})^2(x_j-x_{j+1})^4 \neq (x_j-x_{j+1})^2(x_i-x_{i+1})^4$.

(iii) All four observations assume different values, say, A_h, A_s, A_r, A_t with probability $p_hp_sp_rp_t$. There will be 4 permutations of the elements within the two pairs, yielding identical values for the expression $(x_j-x_{j+1})^2(x_i-x_{i+1})^4$, and two permutations of the two pairs, yielding distinct values for the expression $(x_j-x_{j+1})^2(x_i-x_{i+1})^4$.

We can now evaluate $E(X_j)\,E(X_i^2)$ analogously to $E(X_i)\,E(X_j)$ in equation (9), as

$$\begin{aligned} E(X_j)\,E(X_i^2) = 4[&\Sigma_2(A_h-A_s)^6\, p_h^2p_s^2 + \Sigma_3\{[(A_h-A_s)^4(A_h-A_r)^2+(A_h-A_s)^2(A_h-A_r)^4]p_h^2p_sp_r \\ &+ [(A_h-A_s)^4(A_s-A_r)^2 + (A_h-A_s)^2(A_s-A_r)^4]\,p_hp_s^2p_r \\ &+ [(A_h-A_r)^4(A_s-A_r)^2 + (A_h-A_r)^2(A_s-A_r)^4]\,p_hp_sp_r^2\} \\ &+ \Sigma_4\{(A_h-A_s)^4(A_r-A_t)^2 + (A_h-A_s)^2(A_r-A_t)^4 \\ &+ (A_h-A_r)^4(A_s-A_t)^2 + (A_h-A_r)^2(A_s-A_t)^4 \\ &+ (A_h-A_t)^4(A_s-A_r)^2 + (A_h-A_t)^2(A_s-A_r)^4\}p_hp_sp_rp_t]. \qquad (14) \end{aligned}$$

f) $E(X_i^2X_{i+1}) = E(x_i-x_{i+1})^4(x_{i+1}-x_{i+2})^2$ is obtained by first letting each of x_i, x_{i+1}, x_{i+2} assume all possible values in the expression $(x_i-x_{i+1})^4(x_{i+1}-x_{i+2})^2$, then multiplying each resulting expression by its probability of occurrence, and forming the sum of such products.

The same cases will arise as in (b), namely

(i) All 3 observations x_i, x_{i+1}, x_{i+2} assume different values, say $A_hA_sA_r$ with probability $p_hp_sp_r$, with 6 permutations each yielding a distinct value for the expression $(x_i - x_{i+1})^4(x_{i+1} - x_{i+2})$.

(ii) Two observations out of x_i, x_{i+1}, x_{i+2} assume identical values: $A_hA_sA_h$ with probability $p_h^2p_s$, and $A_sA_hA_s$ with probability $p_hp_s^2$.

We now evaluate $E(X_i^2X_{i+1})$ analogously to $E(X_iX_{i+1})$ in equation (8), as

$$E(X_i^2X_{i+1}) = \Sigma_3\Big\{(A_h-A_s)^4(A_s-A_r)^2 + (A_h-A_s)^2(A_s-A_r)^4 + (A_h-A_r)^4(A_r-A_s)^2 + (A_h-A_r)^2(A_r-A_s)^4 + (A_r-A_h)^4(A_h-A_s)^2 + (A_r-A_h)^2(A_h-A_s)^4\Big\}p_hp_sp_r + \Sigma_2(A_h-A_s)^6\,(p_h^2p_s + p_hp_s^2). \tag{15}$$

g) $aE(X_iX_{i+1}) = E(X_j)\,E(X_iX_{i+1}) = E(x_j-x_{j+1})^2\,E(x_i-x_{i+1})^2(x_{i+1}-x_{i+2})^2$ is obtained by first letting each of the five observations x_j, x_{j+1}, x_i, x_{i+1}, x_{i+2} arranged into two groups of two and three consecutive observations respectively, assume all possible values in the expression $(x_j-x_{j+1})^2(x_i-x_{i+1})^2(x_{i+1}-x_{i+2})^2$, then multiplying each resulting expression by its probability of occurrence, and forming the sum of such products. The sum thus obtained is $E(X_j)\,E(X_iX_{i+1})$.

Let the group containing the two observations x_j, x_{j+1} be called the first group, and the group containing the three observations x_i, x_{i+1}, x_{i+2} the second group. We then have the following cases:

(i) Each of the observations x_j, x_{j+1}, x_i, x_{i+1}, x_{i+2} assumes the value A_h, A_s or A_r with the condition that x_i, x_{i+1}, x_{i+2} assume different values as in the table

x_j	x_{j+1}	x_i	x_{i+1}	x_{i+2}	Probability of occurrence
A_h	A_s	A_h	A_s	A_r	$p_h^2\, p_s^2\, p_r$
A_h	A_r	A_h	A_s	A_r	$p_h^2\, p_s\, p_r^2$
A_s	A_r	A_h	A_s	A_r	$p_h\, p_s^2\, p_r^2$

The elements of the first group can be permuted in $2!$ ways both yielding the same value for $(x_j-x_{j+1})^2$ at the same time, the elements of the second group can be permuted in $3! = 6$ ways of which every pair, consisting of a permutation and its inverse, yield the same values for $(x_i-x_{i+1})^2(x_{i+1}-x_{i+2})^2$.

(ii) Each observation x_j, x_{j+1}, x_i, x_{i+1}, x_{i+2} assumes the value A_h, A_s, A_r, A_t with the condition that x_i, x_{i+1}, x_{i+2} assume different values. There are $\frac{4!}{1!3!} = 4$ different groupings of 4 distinct elements A_h, A_s, A_r, A_t into two groups of one and three elements, respectively. For each such grouping, there are 3 different ways of adding to the single element of the first group one of the elements of the second group, as in the following table:

x_j	x_{j+1}	x_i	x_{i+1}	x_{i+2}	Probability of occurrence
A_t	A_h	A_h	A_r	A_s	$p_h^2\, p_s\, p_r\, p_t$
A_t	A_r	A_h	A_r	A_s	$p_h\, p_s\, p_r^2\, p_t$
A_t	A_s	A_h	A_r	A_s	$p_h\, p_s^2\, p_r\, p_t$
A_h	A_t	A_t	A_r	A_s	$p_h\, p_s\, p_r\, p_t^2$
A_h	A_r	A_t	A_r	A_s	$p_h\, p_s\, p_r^2\, p_t$
A_h	A_s	A_t	A_r	A_s	$p_h\, p_s^2\, p_r\, p_t$
A_s	A_t	A_t	A_h	A_r	$p_h\, p_s\, p_r\, p_t^2$
A_s	A_h	A_t	A_h	A_r	$p_h^2\, p_s\, p_r\, p_t$
A_s	A_r	A_t	A_h	A_r	$p_h\, p_s\, p_r^2\, p_t$
A_r	A_t	A_t	A_h	A_s	$p_h\, p_s\, p_r\, p_t^2$
A_r	A_h	A_t	A_h	A_s	$p_h^2\, p_s\, p_r\, p_t$
A_r	A_s	A_t	A_h	A_s	$p_h\, p_s^2\, p_r\, p_t$

The elements of the first group can be permuted in 2! ways both yielding the same value for $(x_j-x_{j+1})^2$ at the same time, the elements of the second group can be permuted in $3! = 6$ ways, of which every pair, consisting of a permutation and its inverse, yield the same values for $(x_i-x_{i+1})^2(x_{i+1}-x_{i+2})^2$.

(iii) The observations x_j, x_{j+1}, x_i, x_{i+1}, x_{i+2} all assume different values. Any five different elements can be grouped in $\frac{5!}{2!3!} = 10$ different ways into two groups of two and three elements respectively as in the following table,

x_j	x_{j+1}	x_i	x_{i+1}	x_{i+2}	Probability of occurrence
A_h	A_s	A_r	A_t	A_u	$p_h\ p_s\ p_r\ p_t\ p_u$
A_h	A_r	A_s	A_t	A_u	$p_h\ p_s\ p_r\ p_t\ p_u$
A_h	A_t	A_r	A_s	A_u	$p_h\ p_s\ p_r\ p_t\ p_u$
A_h	A_u	A_r	A_s	A_t	$p_h\ p_s\ p_r\ p_t\ p_u$
A_s	A_r	A_h	A_t	A_u	$p_h\ p_s\ p_r\ p_t\ p_u$
A_s	A_t	A_h	A_u	A_r	$p_h\ p_s\ p_r\ p_t\ p_u$
A_s	A_u	A_h	A_r	A_t	$p_h\ p_s\ p_r\ p_t\ p_u$
A_r	A_t	A_h	A_s	A_u	$p_h\ p_s\ p_r\ p_t\ p_u$
A_r	A_u	A_h	A_s	A_t	$p_h\ p_s\ p_r\ p_t\ p_u$
A_t	A_u	A_h	A_s	A_r	$p_h\ p_s\ p_r\ p_t\ p_u$

In each of these groupings, there are two permutations for the elements of the first group, yielding the same values for $(x_j-x_{j+1})^2$, and $3!$ permutations for the elements of the second group with every pair, consisting of a permutation and its inverse, yielding the same values for $(x_i-x_{i+1})^2(x_{i+1}-x_{i+2})^2$.

(iv) Each observation x_j, x_{j+1}, x_i, x_{i+1}, x_{i+2} assumes the value A_h or A_s as in the following table,

x_j	x_{j+1}	x_i	x_{i+1}	x_{i+2}	Probability of occurrence
A_h	A_s	A_h	A_s	A_h	$p_h^3\, p_s^2$
A_h	A_s	A_s	A_h	A_s	$p_h^2\, p_s^3$

The elements of the first group can be permuted in $2!$ ways yielding the same expression for $(x_j-x_{j+1})^2$.

(v) Each of the observations x_j, x_{j+1}, x_i, x_{i+1}, x_{i+2} assumes the value A_h, A_s or A_r with the condition that the observations of the second group do not all assume different values. There will be $\frac{3!}{2!1!} = 3$ ways of assigning two of the three elements A_h, A_s, A_r to the first group, and the remaining element to the second group. The two other components of the second group can be chosen in one of the two ways: first, choosing one element of the first group twice; second, choosing one element of the first group and repeating the original element of the second group, as in the following table:

x_j	x_{j+1}	x_i	x_{i+1}	x_{i+2}	Probability of occurrence
A_h	A_s	A_h	A_r	A_h	$p_h^3\, p_s\, p_r$
A_h	A_s	A_s	A_r	A_s	$p_h\, p_s^3\, p_r$
A_h	A_s	A_r	A_h	A_r	$p_h^2\, p_s\, p_r^2$
A_h	A_s	A_r	A_s	A_r	$p_h\, p_s^2\, p_r^2$
A_h	A_r	A_h	A_s	A_h	$p_h^3\, p_s\, p_r$
A_h	A_r	A_r	A_s	A_r	$p_h\, p_r^3\, p_s$
A_h	A_r	A_s	A_h	A_s	$p_h^2\, p_s^2\, p_r$
A_h	A_r	A_s	A_r	A_s	$p_h\, p_r^2\, p_s^2$
A_r	A_s	A_s	A_h	A_s	$p_h\, p_s^3\, p_r$
A_r	A_s	A_r	A_h	A_r	$p_h\, p_s\, p_r^3$
A_r	A_s	A_h	A_s	A_h	$p_h^2\, p_s^2\, p_r$
A_r	A_s	A_h	A_r	A_h	$p_h^2\, p_s\, p_r^2$

For each of the above groupings there are 2! permutations of the elements of the first group, yielding identical values for $(x_j-x_{j+1})^2$.

(vi) Each of the observations x_j, x_{j+1}, x_i, x_{i+1}, x_{i+2} assumes the value A_h, A_s, A_r or A_t with the condition that the observations x_j, x_{j+1} assume values different from those assumed by x_i, x_{i+1}, x_{i+2} which do not all assume different values. There will be $\frac{4!}{2!2!}$ groupings of four different elements into two pairs. One of these pairs becomes the first group, while the second group is found from the other pair by repeating one of its elements, as in the table below.

x_j	x_{j+1}	x_i	x_{i+1}	x_{i+2}	Probability of occurrence
A_h	A_s	A_r	A_t	A_r	$p_h\ p_s\ p_r^2\ p_t$
A_h	A_s	A_t	A_r	A_t	$p_h\ p_s\ p_r\ p_t^2$
A_h	A_r	A_s	A_t	A_s	$p_h\ p_s^2\ p_r\ p_t$
A_h	A_r	A_t	A_s	A_t	$p_h\ p_s\ p_r\ p_t^2$
A_h	A_t	A_r	A_s	A_r	$p_h\ p_s\ p_r^2\ p_t$
A_h	A_t	A_s	A_r	A_s	$p_h\ p_s^2\ p_r\ p_t$
A_r	A_s	A_t	A_h	A_t	$p_h\ p_s\ p_r\ p_t^2$
A_r	A_s	A_h	A_t	A_h	$p_h^2\ p_s\ p_r\ p_t$
A_r	A_t	A_s	A_h	A_s	$p_h\ p_s^2\ p_r\ p_t$
A_r	A_t	A_h	A_s	A_h	$p_h^2\ p_s\ p_r\ p_t$
A_s	A_t	A_r	A_h	A_r	$p_h\ p_s\ p_r^2\ p_t$
A_s	A_t	A_h	A_r	A_h	$p_h^2\ p_s\ p_r\ p_t$

For each of the above groupings there are 2! permutations of the elements of the first group yielding the same value for $(x_j - x_{j+1})^2$.

Noting that there are $\binom{k}{3}$ possible choices of elements in case (i), $\binom{k}{4}$ possible choices in case (ii), $\binom{k}{5}$ possible choices in case (iii), $\binom{k}{2}$ possible choices in case (iv), $\binom{k}{3}$ possible choices in case (v), and $\binom{k}{4}$ possible choices in case (vi), we can now compute $E(X_j)\, E(X_i X_{i+1})$ as

$$
\begin{aligned}
E(X_j)\, E(X_i X_{i+1}) = \\
4\, \Sigma_3 [& \{(A_h-A_s)^4(A_s-A_r)^2+(A_h-A_r)^2(A_r-A_s)^2(A_h-A_s)^2+(A_s-A_h)^4(A_h-A_r)^2\} p_h^2 p_s^2 p_r \\
& + \{(A_h-A_s)^2(A_s-A_r)^2(A_h-A_r)^2+(A_h-A_r)^4(A_r-A_s)^2+(A_s-A_h)^2(A_h-A_r)^4\} p_h^2 p_s p_r^2 \\
& + \{(A_h-A_s)^2(A_s-A_r)^4+(A_h-A_r)^2(A_r-A_s)^4+(A_s-A_h)^2(A_h-A_r)^2(A_s-A_r)^2\} p_h p_s^2 p_r^2] \\
+4\, \Sigma_4 \{ [& \{(A_h-A_s)^2(A_s-A_r)^2+(A_h-A_r)^2(A_r-A_s)^2+(A_s-A_h)^2(A_h-A_r)^2\}(A_t-A_h)^2 p_h^2 p_s p_r p_t \\
& + \{(A_h-A_s)^2(A_s-A_r)^2+(A_h-A_r)^2(A_r-A_s)^2+(A_s-A_h)^2(A_h-A_r)^2\}(A_t-A_r)^2 p_h p_s p_r^2 p_t \\
& + \{(A_h-A_s)^2(A_s-A_r)^2+(A_h-A_r)^2(A_r-A_s)^2+(A_s-A_h)^2(A_h-A_r)^2\}(A_t-A_s)^2 p_h p_s^2 p_r p_t] \\
& + [\{(A_s-A_r)^2(A_r-A_t)^2+(A_s-A_t)^2(A_t-A_r)^2+(A_r-A_s)^2(A_s-A_t)^2\}(A_h-A_t)^2 p_h p_s p_r p_t^2 \\
& + \{(A_s-A_r)^2(A_r-A_t)^2+(A_s-A_t)^2(A_t-A_r)^2+(A_r-A_s)^2(A_s-A_t)^2\}(A_h-A_r)^2 p_h p_s p_r^2 p_t \\
& + \{(A_s-A_r)^2(A_r-A_t)^2+(A_s-A_t)^2(A_t-A_r)^2+(A_r-A_s)^2(A_s-A_t)^2\}(A_h-A_s)^2 p_h p_s^2 p_r p_t] \\
& + [\{(A_h-A_t)^2(A_t-A_r)^2+(A_t-A_h)^2(A_h-A_r)^2+(A_h-A_r)^2(A_r-A_t)^2\}(A_s-A_t)^2 p_h p_s p_r p_t^2 \\
& + \{(A_h-A_t)^2(A_t-A_r)^2+(A_t-A_h)^2(A_h-A_r)^2+(A_h-A_r)^2(A_r-A_t)^2\}(A_s-A_h)^2 p_h^2 p_s p_r p_t \\
& + \{(A_h-A_t)^2(A_t-A_r)^2+(A_t-A_h)^2(A_h-A_r)^2+(A_h-A_r)^2(A_r-A_t)^2\}(A_s-A_r)^2 p_h p_s p_r^2 p_t
\end{aligned}
$$

$$+ [\{(A_h-A_s)^2(A_s-A_t)^2+(A_s-A_h)^2(A_h-A_t)^2+(A_h-A_t)^2(A_t-A_s)^2\}(A_r-A_t)^2 p_h p_s p_r p_t^2$$

$$+\{(A_h-A_s)^2(A_s-A_t)^2+(A_s-A_h)^2(A_h-A_t)^2+(A_h-A_t)^2(A_t-A_s)^2\}(A_r-A_h)^2 p_h^2 p_s p_r p_t$$

$$+\{(A_h-A_s)^2(A_s-A_t)^2+(A_s-A_h)^2(A_h-A_t)^2+(A_h-A_t)^2(A_t-A_s)^2\}(A_r-A_s)^2 p_h p_s^2 p_r p_t]\}.$$

$$+ 4\ \Sigma_5[\{(A_u-A_t)^2(A_t-A_r)^2+(A_u-A_r)^2(A_r-A_t)^2+(A_t-A_u)^2(A_u-A_r)^2\}(A_s-A_h)^2$$

$$+\{(A_u-A_s)^2(A_s-A_t)^2+(A_u-A_t)^2(A_t-A_s)^2+(A_s-A_u)^2(A_u-A_t)^2\}(A_h-A_r)^2$$

$$+\{(A_u-A_s)^2(A_s-A_r)^2+(A_u-A_r)^2(A_r-A_s)^2+(A_s-A_u)^2(A_u-A_r)^2\}(A_h-A_t)^2$$

$$+\{(A_t-A_s)^2(A_s-A_r)^2+(A_t-A_r)^2(A_r-A_s)^2+(A_s-A_t)^2(A_t-A_r)^2\}(A_u-A_h)^2$$

$$+\{(A_h-A_u)^2(A_u-A_t)^2+(A_h-A_t)^2(A_t-A_u)^2+(A_u-A_h)^2(A_h-A_t)^2\}(A_s-A_r)^2$$

$$+\{(A_h-A_u)^2(A_u-A_r)^2+(A_u-A_h)^2(A_h-A_r)^2+(A_u-A_r)^2(A_r-A_h)^2\}(A_s-A_t)^2$$

$$+\{(A_h-A_t)^2(A_t-A_r)^2+(A_h-A_r)^2(A_r-A_t)^2+(A_t-A_h)^2(A_h-A_r)^2\}(A_s-A_u)^2$$

$$+\{(A_h-A_s)^2(A_s-A_u)^2+(A_h-A_u)^2(A_u-A_s)^2+(A_s-A_h)^2(A_h-A_u)^2\}(A_r-A_t)^2$$

$$+\{(A_h-A_s)^2(A_s-A_t)^2+(A_h-A_t)^2(A_t-A_s)^2+(A_t-A_h)^2(A_h-A_s)^2\}(A_r-A_u)^2$$

$$+\{(A_h-A_s)^2(A_s-A_r)^2+(A_h-A_r)^2(A_r-A_s)^2+(A_s-A_h)^2(A_h-A_r)^2\}(A_t-A_u)^2] p_h p_s p_r p_t p_u$$

$$+ 2\ \Sigma_2\ (A_h-A_s)^6(p_h^3 p_s^2 + p_h^2 p_s^3)$$

$$+ 2\ \Sigma_3[\ (A_h-A_r)^4(A_h-A_s)^2(p_h^3 p_s p_r + p_h^2 p_s p_r^2)+(A_r-A_s)^4(A_h-A_s)^2(p_h p_s^2 p_r^2 + p_h p_s^3 p_r)$$

$$+ (A_h-A_s)^4(A_h-A_r)^2(p_h^3 p_s p_r + p_h^2 p_s^2 p_r)+(A_r-A_s)^4(A_h-A_r)^2(p_h p_s p_r^3 + p_h p_s^2 p_r^2)$$

$$+ (A_h-A_s)^4(A_r-A_s)^2(p_h^2 p_s^2 p_r + p_h p_s^3 p_r)+(A_r-A_h)^4(A_r-A_s)^2(p_h p_s p_r^3 + p_h^2 p_s p_r^2)]$$

$$+ 2\ \Sigma_4[\{(A_r-A_t)^4(A_h-A_s)^2+(A_r-A_s)^4(A_h-A_t)^2+(A_r-A_h)^4(A_s-A_t)^2\} p_h p_s p_r^2 p_t$$

$$+ \{(A_t-A_r)^4(A_h-A_s)^2+(A_t-A_s)^4(A_h-A_r)^2+(A_t-A_h)^4(A_r-A_s)^2\} p_h p_s p_r p_t^2$$

$$+ \{(A_s-A_t)^4(A_h-A_r)^2+(A_s-A_r)^4(A_h-A_t)^2+(A_s-A_h)^4(A_r-A_t)^2\} p_h p_s^2 p_r p_t$$

$$+ \{(A_h-A_t)^4(A_r-A_s)^2+(A_h-A_s)^4(A_r-A_t)^2+(A_h-A_r)^4(A_s-A_t)^2\} p_h^2 p_s p_r p_t].$$

... (16)

h) $E(X_i X_{i+1} X_{i+2}) = E(x_i-x_{i+1})^2(x_{i+1}-x_{i+2})^2(x_{i+2}-x_{i+3})^2$ is obtained by first letting each of the observations x_i, x_{i+1}, x_{i+2}, x_{i+3} assume all possible values in the expression $(x_i-x_{i+1})^2(x_{i+1}-x_{i+2})^2(x_{i+2}-x_{i+3})^2$, then multiplying each resulting expression by its probability of occurrence, and forming the sum of such products.

The following cases will arise:

(i) Each observation x_i, x_{i+1}, x_{i+2}, x_{i+3} assumes the value A_h or A_s as in the following table,

x_i	x_{i+1}	x_{i+2}	x_{i+3}	Probability of occurrence
A_h	A_s	A_h	A_s	$p_h^2\ p_s^2$
A_s	A_h	A_s	A_h	$p_h^2\ p_s^2$

(ii) Each observation x_i, x_{i+1}, x_{i+2}, x_{i+3} assumes the value A_h, A_s or A_r. Therefore one of the values A_h, A_s, A_r will be assumed twice. Let A_h be assumed twice, then the sequence of four consecutive elements $A_h A_h A_s A_r$ may (1) begin but not end with A_h, (2) begin and end with A_h, or (3) end but not begin with A_h. For each of these three cases (1), (2) and (3), the elements A_r and A_s can be permuted twice. Thus, when A_h is assumed twice, the four elements have 6 distinct permutations, with every pair, consisting of a permutation and its inverse, yielding the same values for $(x_i-x_{i+1})^2(x_{i+1}-x_{i+2})^2(x_{i+2}-x_{i+3})^2$. The same number of permutations exist when A_s or A_r is assumed twice, as in the table below:

x_i	x_{i+1}	x_{i+2}	x_{i+3}	Probability of occurrence
A_h	A_s	A_h	A_r	$p_h^2\ p_s\ p_r$
A_h	A_r	A_h	A_s	$p_h^2\ p_s\ p_r$
A_h	A_s	A_r	A_h	$p_h^2\ p_s\ p_r$
A_h	A_r	A_s	A_h	$p_h^2\ p_s\ p_r$
A_r	A_h	A_s	A_h	$p_h^2\ p_s\ p_r$
A_s	A_h	A_r	A_h	$p_h^2\ p_s\ p_r$
A_s	A_h	A_s	A_r	$p_h\ p_s^2\ p_r$
A_s	A_r	A_s	A_h	$p_h\ p_s^2\ p_r$
A_s	A_h	A_r	A_s	$p_h\ p_s^2\ p_r$
A_s	A_r	A_h	A_s	$p_h\ p_s^2\ p_r$
A_r	A_s	A_h	A_s	$p_h\ p_s^2\ p_r$
A_h	A_s	A_r	A_s	$p_h\ p_s^2\ p_r$

x_i	x_{i+1}	x_{i+2}	x_{i+3}	Probability of occurence
A_r	A_s	A_r	A_h	$p_h\ p_s\ p_r^2$
A_r	A_h	A_r	A_s	$p_h\ p_s\ p_r^2$
A_r	A_s	A_h	A_r	$p_h\ p_s\ p_r^2$
A_r	A_h	A_s	A_r	$p_h\ p_s\ p_r^2$
A_h	A_r	A_s	A_r	$p_h\ p_s\ p_r^2$
A_s	A_r	A_h	A_r	$p_h\ p_s\ p_r^2$

(iii) All the four observations x_i, x_{i+1}, x_{i+2}, x_{i+3} assume different values. Let these values be A_h, A_s, A_r, A_t, then there will be $4! = 24$ permutations of the 4 consecutive elements $A_h\ A_s\ A_r\ A_t$, with probability $p_h\ p_s\ p_r\ p_t$, as in the table below:

x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
A_h	A_s	A_r	A_t	A_s	A_t	A_h	A_r	A_r	A_h	A_t	A_s
A_h	A_r	A_s	A_t	A_s	A_h	A_t	A_r	A_r	A_t	A_h	A_s
A_h	A_r	A_t	A_s	A_s	A_t	A_r	A_h	A_t	A_r	A_s	A_h
A_h	A_t	A_r	A_s	A_s	A_r	A_t	A_h	A_t	A_s	A_r	A_h
A_h	A_t	A_s	A_r	A_r	A_s	A_h	A_t	A_t	A_r	A_h	A_s
A_h	A_s	A_t	A_r	A_r	A_h	A_s	A_t	A_t	A_h	A_r	A_s
A_s	A_h	A_r	A_t	A_r	A_s	A_t	A_h	A_t	A_s	A_h	A_r
A_s	A_r	A_h	A_t	A_r	A_t	A_s	A_h	A_t	A_h	A_s	A_r

These 24 permutations include for each permutation of the four elements, the inverse-permutation, and each permutation and its inverse, yield the same value for $(x_i-x_{i+1})^2(x_{i+1}-x_{i+2})^2(x_{i+2}-x_{i+3})^2$.

Noting that there are $\binom{k}{2}$ possible choices for case (i), $\binom{k}{3}$ possible choices for case (ii), and $\binom{k}{4}$ possible choices for case (iii), we can now compute $E(X_iX_{i+1}X_{i+2})$ as

$$E(X_iX_{i+1}X_{i+2}) = 2\Sigma_2(A_h-A_s)^6\, p_h^2\, p_s^2$$
$$+ 2\Sigma_3[\{(A_h-A_s)^4(A_h-A_r)^2+(A_h-A_r)^4(A_h-A_s)^2+(A_h-A_s)^2(A_s-A_r)^2(A_r-A_h)^2\}p_h^2p_sp_r$$
$$+ \{(A_s-A_h)^4(A_s-A_r)^2+(A_s-A_r)^4(A_s-A_h)^2+(A_s-A_h)^2(A_h-A_r)^2(A_r-A_s)^2\}p_hp_s^2p_r$$
$$+ \{(A_r-A_s)^4(A_r-A_h)^2+(A_r-A_h)^4(A_r-A_s)^2+(A_r-A_s)^2(A_s-A_h)^2(A_h-A_r)^2\}p_hp_sp_r^2]$$
$$+ 2\Sigma_4[\ (A_h-A_s)^2(A_s-A_r)^2(A_r-A_t)^2+(A_h-A_r)^2(A_r-A_s)^2(A_s-A_t)^2+(A_h-A_r)^2(A_r-A_t)^2(A_t-A_s)^2$$
$$+ \ (A_h-A_t)^2(A_t-A_r)^2(A_r-A_s)^2+(A_h-A_t)^2(A_t-A_s)^2(A_s-A_r)^2+(A_h-A_s)^2(A_s-A_t)^2(A_t-A_r)^2$$
$$+ \ (A_s-A_h)^2(A_h-A_r)^2(A_r-A_t)^2+(A_s-A_r)^2(A_r-A_h)^2(A_h-A_t)^2+(A_s-A_t)^2(A_t-A_h)^2(A_h-A_r)^2$$
$$+ (A_s-A_h)^2(A_h-A_t)^2(A_t-A_r)^2+(A_r-A_s)^2(A_s-A_h)^2(A_h-A_t)^2+(A_r-A_h)^2(A_h-A_s)^2(A_s-A_t)^2]p_hp_sp_rp_t$$

..... (17)

i) For $a^3 = [E(X_i)]^3 = E(x_i-x_{i+1})^2(x_j-x_{j+1})^2(x_k-x_{k+1})^2$ let each observation of the three disconnected pairs of observations x_i, x_{i+1}; x_j, x_{j+1}; x_k, x_{k+1}; assume all possible values in the expression $(x_i-x_{i+1})^2(x_j-x_{j+1})^2(x_k-x_{k+1})^2$, then multiplying each resulting expression by its probability of occurrence, form the sum of such products. The sum thus obtained is $[E(X_i)]^3$.

The following cases arise:

(i) The observations in each pair assume the values A_h, A_s, with probability $p_h^3\, p_s^3$. The six elements in the three groups can be permuted in $[2!]^3 = 8$ ways, all the 8 permutations yielding the same value for $(x_i-x_{i+1})^2(x_j-x_{j+1})^2(x_k-x_{k+1})^2$.

(ii) Three elements, A_h, A_s and A_r, appear as values of the six observations x_i, x_{i+1}, x_j, x_{j+1}, x_k, x_{k+1}, such that one element is the value of three observations, another is the value of two other observations, and the third is the value of the remaining observation. These conditions require that of the three pairs, two be identical in value. There are $\frac{3!}{2!1!} = 3$ ways of choosing two pairs out of three. For any chosen two pairs the following arrangements are possible:

x_i	x_{i+1}	x_j	x_{j+1}	x_k	x_{k+1}	Probability of occurrence
A_h	A_s	A_h	A_s	A_h	A_r	$p_h^3\, p_s^2\, p_r$
A_h	A_s	A_h	A_s	A_s	A_r	$p_h^2\, p_s^3\, p_r$
A_h	A_r	A_h	A_r	A_h	A_s	$p_h^3\, p_s\, p_r^2$
A_h	A_r	A_h	A_r	A_r	A_s	$p_h^2\, p_s\, p_r^3$
A_r	A_s	A_r	A_s	A_r	A_h	$p_h\, p_s^2\, p_r^3$
A_r	A_s	A_r	A_s	A_s	A_h	$p_h\, p_s^3\, p_r^2$

For each of these arrangements there are 8 permutations of the elements, yielding the same value for $(x_i-x_{i+1})^2(x_j-x_{j+1})^2(x_k-x_{k+1})^2$.

(iii) Four elements A_h, A_s, A_r, A_t appear as values of the 6 observations x_i, x_{i+1}, x_j, x_{j+1}, x_k, x_{k+1}. Of these four elements two appear twice, and the other two appear once only. Moreover, two pairs of observations have identical values. For any chosen two pairs out of 3 possible choices satisfying the above conditions the following arrangements are possible:

x_i	x_{i+1}	x_j	x_{j+1}	x_k	x_{k+1}	Probability of occurrence
A_h	A_s	A_h	A_s	A_r	A_t	$p_h^2\, p_s^2\, p_r\, p_t$
A_h	A_r	A_h	A_r	A_s	A_t	$p_h^2\, p_s\, p_r^2\, p_t$
A_h	A_t	A_h	A_t	A_s	A_r	$p_h^2\, p_s\, p_r\, p_t^2$
A_s	A_r	A_s	A_r	A_h	A_t	$p_h\, p_s^2\, p_r^2\, p_t$
A_s	A_t	A_s	A_t	A_h	A_r	$p_h\, p_s^2\, p_r\, p_t^2$
A_r	A_t	A_r	A_t	A_h	A_s	$p_h\, p_s\, p_r^2\, p_t^2$

For each of these arrangements there are 8 permutations of the 6 elements, yielding the same value for $(x_i-x_{i+1})^2(x_j-x_{j+1})^2(x_k-x_{k+1})^2$.

(iv) The 6 observations x_i, x_{i+1}, x_j, x_{j+1}, x_k, x_{k+1}, satisfy the conditions of (iii) above, but no two pairs of observations are identical in value. The 3 different pairs can be permuted in 3! ways, and for any given permutation the following arrangements are possible:

x_i	x_{i+1}	x_j	x_{j+1}	x_k	x_{k+1}	Probability of occurrence
A_h	A_s	A_h	A_r	A_s	A_t	$p_h^2\, p_s^2\, p_r\, p_t$
A_h	A_s	A_h	A_t	A_s	A_r	$p_h^2\, p_s^2\, p_r\, p_t$
A_h	A_r	A_h	A_s	A_r	A_t	$p_h^2\, p_s\, p_r^2\, p_t$
A_h	A_r	A_h	A_t	A_r	A_s	$p_h^2\, p_s\, p_r^2\, p_t$
A_h	A_t	A_h	A_s	A_t	A_r	$p_h^2\, p_s\, p_r\, p_t^2$
A_h	A_t	A_h	A_r	A_t	A_s	$p_h^2\, p_s\, p_r\, p_t^2$
A_s	A_r	A_s	A_h	A_r	A_t	$p_h\, p_s^2\, p_r^2\, p_t$
A_s	A_r	A_s	A_t	A_r	A_h	$p_h\, p_s^2\, p_r^2\, p_t$
A_s	A_t	A_s	A_r	A_t	A_h	$p_h\, p_s^2\, p_r\, p_t^2$
A_s	A_t	A_s	A_h	A_t	A_r	$p_h\, p_s^2\, p_r\, p_t^2$
A_r	A_t	A_r	A_h	A_t	A_s	$p_h\, p_s\, p_r^2\, p_t^2$
A_r	A_t	A_r	A_s	A_t	A_h	$p_h\, p_s\, p_r^2\, p_t^2$

For each of these arrangements there are 8 permutations of the six elements, yielding the same value for $(x_i-x_{i+1})^2(x_j-x_{j+1})^2(x_k-x_{k+1})^2$.

(v) Four elements A_h, A_s, A_r, A_t appear as values of the six observations x_i, x_{i+1}, x_j, x_{j+1}, x_k, x_{k+1}, of these four elements, one appears three times, and each of the remaining three elements appears once only. There are $3!$ permutations of the 3 pairs, and for any given permutation of the pairs, the following arrangements are possible:

x_i	x_{i+1}	x_j	x_{j+1}	x_k	x_{k+1}	Probability of occurrence
A_h	A_s	A_h	A_r	A_h	A_t	$p_h^3\, p_s\, p_r\, p_t$
A_s	A_r	A_s	A_t	A_s	A_h	$p_h\, p_s^3\, p_r\, p_t$
A_r	A_h	A_r	A_s	A_r	A_t	$p_h\, p_s\, p_r^3\, p_t$
A_t	A_h	A_t	A_s	A_t	A_r	$p_h\, p_s\, p_r\, p_t^3$

For each of these arrangements there are 8 permutations of the 6 elements, yielding the same value for $(x_i-x_{i+1})^2(x_j-x_{j+1})^2(x_k-x_{k+1})^2$.

(vi) Five elements A_h, A_s, A_r, A_t, A_u appear as values of the six observations x_i, x_{i+1}, x_j, x_{j+1}, x_k, x_{k+1}, of these five elements, one appears twice, and each of the remaining 4 elements appears once only. For any of the 3! permutations of the three pairs, the following arrangements are possible:

x_i	x_{i+1}	x_j	x_{j+1}	x_k	x_{k+1}	Probability of occurrence
A_h	A_s	A_r	A_t	A_u	A_h	$p_h^2\, p_s\, p_r\, p_u\, p_t$
A_h	A_s	A_r	A_u	A_t	A_h	$p_h^2\, p_s\, p_r\, p_u\, p_t$
A_h	A_u	A_r	A_s	A_t	A_h	$p_h^2\, p_s\, p_r\, p_u\, p_t$
A_h	A_r	A_s	A_u	A_t	A_h	$p_h^2\, p_s\, p_r\, p_u\, p_t$
A_h	A_r	A_s	A_t	A_u	A_h	$p_h^2\, p_s\, p_r\, p_u\, p_t$
A_h	A_r	A_u	A_t	A_s	A_h	$p_h^2\, p_s\, p_r\, p_u\, p_t$

x_i	x_{i+1}	x_j	x_{j+1}	x_k	x_{k+1}	Probability of occurrence
A_s	A_u	A_h	A_r	A_t	A_s	$p_h\ p_s^2\ p_r\ p_t\ p_u$
A_s	A_u	A_h	A_t	A_r	A_s	$p_h\ p_s^2\ p_r\ p_t\ p_u$
A_s	A_r	A_h	A_u	A_t	A_s	$p_h\ p_s^2\ p_r\ p_t\ p_u$
A_s	A_h	A_r	A_t	A_u	A_s	$p_h\ p_s^2\ p_r\ p_t\ p_u$
A_s	A_h	A_r	A_u	A_t	A_s	$p_h\ p_s^2\ p_r\ p_t\ p_u$
A_s	A_h	A_t	A_u	A_r	A_s	$p_h\ p_s^2\ p_r\ p_t\ p_u$
A_r	A_u	A_h	A_s	A_t	A_r	$p_h\ p_s\ p_r^2\ p_u\ p_t$
A_r	A_t	A_h	A_u	A_s	A_r	$p_h\ p_s\ p_r^2\ p_u\ p_t$
A_r	A_s	A_h	A_t	A_u	A_r	$p_h\ p_s\ p_r^2\ p_u\ p_t$
A_r	A_h	A_s	A_u	A_t	A_r	$p_h\ p_s\ p_r^2\ p_u\ p_t$
A_r	A_h	A_s	A_t	A_u	A_r	$p_h\ p_s\ p_r^2\ p_u\ p_t$
A_r	A_h	A_u	A_t	A_s	A_r	$p_h\ p_s\ p_r^2\ p_u\ p_t$
A_t	A_r	A_h	A_s	A_u	A_t	$p_h\ p_s\ p_r\ p_t^2\ p_u$
A_t	A_s	A_h	A_r	A_u	A_t	$p_h\ p_s\ p_r\ p_t^2\ p_u$
A_t	A_r	A_h	A_u	A_s	A_t	$p_h\ p_s\ p_r\ p_t^2\ p_u$
A_t	A_h	A_s	A_r	A_u	A_t	$p_h\ p_s\ p_r\ p_t^2\ p_u$
A_t	A_h	A_s	A_u	A_r	A_t	$p_h\ p_s\ p_r\ p_t^2\ p_u$
A_t	A_h	A_r	A_u	A_s	A_t	$p_h\ p_s\ p_r\ p_t^2\ p_u$

x_i	x_{i+1}	x_j	x_{j+1}	x_k	x_{k+1}	Probability of occurrence
A_u	A_r	A_h	A_s	A_t	A_u	$p_h\ p_s\ p_r\ p_t\ p_u^2$
A_u	A_t	A_h	A_r	A_s	A_u	$p_h\ p_s\ p_r\ p_t\ p_u^2$
A_u	A_s	A_h	A_t	A_r	A_u	$p_h\ p_s\ p_r\ p_t\ p_u^2$
A_u	A_h	A_s	A_r	A_t	A_u	$p_h\ p_s\ p_r\ p_t\ p_u^2$
A_u	A_h	A_s	A_t	A_r	A_u	$p_h\ p_s\ p_r\ p_t\ p_u^2$
A_u	A_h	A_r	A_t	A_s	A_u	$p_h\ p_s\ p_r\ p_t\ p_u^2$

The enumeration of the above arrangements follows the rule: First, any one of the 5 elements A_h, A_s, A_r, A_t, A_u may be the repeated element. Thus there are 5 choices to begin with. Then, if A_h is the repeated element, the pair not containing A_h will contain any two of the remaining 4 elements A_s, A_r, A_t, A_u. There are $\binom{4}{2} = 6$ ways of choosing these two elements out of the 4. Hence there will be a total of 5x6 = 30 possible arrangements in pairs as expressed above. For each of these arrangements there are 8 permutations of the elements, yielding the same value for $(x_i-x_{i+1})^2(x_j-x_{j+1})^2(x_k-x_{k+1})^2$.

(vii) Out of the 6 observations x_i, x_{i+1}, x_j, x_{j+1}, x_k, x_{k+1}, each two assume the same value A_h, A_s, or A_r, such that no two pairs are identical in value. For any of the 3! permutations of the 3 pairs only the following arrangement is possible.

x_i	x_{i+1}	x_j	x_{j+1}	x_k	x_{k+1}	Probability of Occurrence
A_h	A_s	A_h	A_r	A_s	A_r	$p_h^2\ p_s^2\ p_r^2$

There will be 8 permutations of the 6 elements, yielding the same value for $(x_i-x_{i+1})^2(x_j-x_{j+1})^2(x_k-x_{k+1})^2$.

(viii) All the 6 observations x_i, x_{i+1}, x_j, x_{j+1}, x_k, x_{k+1}, assume different values. For any of the 3! permutations of the three pairs, the following arrangements are possible:

x_i	x_{i+1}	x_j	x_{j+1}	x_k	x_{k+1}	Probability of occurrence
A_h	A_u	A_r	A_t	A_v	A_s	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$
A_h	A_v	A_r	A_u	A_t	A_s	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$
A_h	A_u	A_r	A_v	A_t	A_s	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$
A_h	A_v	A_t	A_u	A_r	A_s	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$
A_h	A_u	A_t	A_v	A_r	A_s	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$
A_h	A_r	A_u	A_v	A_t	A_s	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$
A_h	A_v	A_r	A_t	A_u	A_s	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$
A_h	A_t	A_r	A_u	A_v	A_s	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$
A_h	A_t	A_r	A_v	A_u	A_s	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$
A_h	A_r	A_t	A_u	A_v	A_s	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$
A_h	A_r	A_t	A_v	A_u	A_s	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$

x_i	x_{i+1}	x_j	x_{j+1}	x_k	x_{k+1}	Probability of occurrence
A_h	A_t	A_u	A_v	A_r	A_s	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$
A_h	A_s	A_r	A_t	A_u	A_v	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$
A_h	A_s	A_r	A_u	A_t	A_v	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$
A_h	A_s	A_r	A_v	A_t	A_u	$p_h\ p_s\ p_r\ p_t\ p_u\ p_v$

The above arrangements are enumerated according to the rule: Take any two elements out of the 6 elements, say A_h and A_s. Let A_h be the first element of the first pair, and A_s be the second element of the third pair. The two elements of the second pair can then be chosen in $\binom{4}{2}$ ways out of the remaining elements A_r, A_t, A_u, A_v, and the remaining two elements can be assigned to the second position of the first pair, and the first position of the third pair, respectively, in 2 ways. Then, let A_h and A_s be the two elements of the first pair and permute the remaining 4 elements in the second and third pairs, as in the last 3 arrangements given above. For each of the 15 arrangements above, there are 8 permutations of the 6 elements, yielding the same value for $(x_i-x_{i+1})^2(x_j-x_{j+1})^2(x_k-x_{k+1})^2$.

Cases (i) to (viii) amount for all possible non-zero terms in the expansion of $[E(X_i)]^3$. Noting that there are $\binom{k}{2}$ possible choices of elements in case (i), $\binom{k}{3}$ possible choices in case (ii), $\binom{k}{4}$ possible choices in cases (iii), (iv), and (v), $\binom{k}{5}$ possible choices for case (vi), $\binom{k}{3}$ possible choices for case (vii), and $\binom{k}{6}$ possible choices for case (viii), we can now compute $[E(X_i)]^3$ as

$$[E(X_i)]^3 = 8\{\Sigma_2(A_h-A_s)^6\ p_h^3\ p_s^3$$

$$+ \frac{3!}{2!1!}\ \Sigma_3[(A_h-A_s)^4(A_h-A_r)^2p_h^3p_s^2p_r+(A_h-A_s)^4(A_s-A_r)^2p_h^2p_s^3p_r+(A_h-A_r)^4(A_h-A_s)^2p_h^3p_sp_r^2$$

$$+\ (A_h-A_r)^4(A_r-A_s)^2p_h^2p_sp_r^3+(A_r-A_s)^4(A_r-A_h)^2p_hp_s^2p_r^3+(A_r-A_s)^4(A_s-A_h)^2p_hp_s^3p_r^2]$$

$$+ \frac{3!}{2!1!}\ \Sigma_4[(A_h-A_s)^4(A_r-A_t)^2p_h^2p_s^2p_rp_t+(A_h-A_r)^4(A_s-A_t)^2p_h^2p_sp_r^2p_t$$

$$+\ (A_h-A_t)^4(A_s-A_r)^2p_h^2p_sp_rp_t^2+(A_s-A_r)^4(A_h-A_t)^2p_hp_s^2p_r^2p_t$$

$$+\ (A_s-A_t)^4(A_h-A_r)^2p_hp_s^2p_rp_t^2+(A_r-A_t)^4(A_h-A_s)^2p_hp_sp_r^2p_t^2]$$

$$+\ 3!\ \ \Sigma_4[(A_h-A_s)^2\{(A_h-A_r)^2(A_s-A_t)^2+(A_h-A_t)^2(A_s-A_r)^2\}p_h^2p_s^2p_rp_t$$

$$+\ (A_h-A_r)^2\{(A_h-A_s)^2(A_r-A_t)^2+(A_h-A_t)^2(A_r-A_s)^2\}p_h^2p_sp_r^2p_t$$

$$+\ (A_h-A_t)^2\{(A_h-A_s)^2(A_t-A_r)^2+(A_h-A_r)^2(A_t-A_s)^2\}p_h^2p_sp_rp_t^2$$

$$+\ (A_s-A_r)^2\{(A_s-A_h)^2(A_r-A_t)^2+(A_s-A_t)^2(A_r-A_h)^2\}p_hp_s^2p_r^2p_t$$

$$+\ (A_s-A_t)^2\{(A_s-A_r)^2(A_t-A_h)^2+(A_s-A_h)^2(A_t-A_r)^2\}p_hp_s^2p_rp_t^2$$

$$+\ (A_r-A_t)^2\{(A_r-A_h)^2(A_t-A_s)^2+(A_r-A_s)^2(A_t-A_h)^2\}p_hp_sp_r^2p_t^2]$$

$$+\ 3!\ \ \Sigma_4[(A_h-A_s)^2(A_h-A_r)^2(A_h-A_t)^2p_h^3p_sp_rp_t+(A_s-A_r)^2(A_s-A_t)^2(A_s-A_h)^2p_hp_s^3p_rp_t$$

$$+\ (A_r-A_h)^2(A_r-A_s)^2(A_r-A_t)^2p_hp_sp_r^3p_t+(A_t-A_h)^2(A_t-A_s)^2(A_t-A_r)^2p_hp_sp_rp_t^3]$$

$$+\ 3!\ \ \Sigma_5[\{(A_h-A_s)^2(A_r-A_t)^2(A_u-A_h)^2+(A_h-A_s)^2(A_r-A_u)^2(A_t-A_h)^2+(A_h-A_u)^2(A_r-A_s)^2(A_t-A_h)^2$$

$$+ (A_h-A_r)^2(A_s-A_u)^2(A_t-A_h)^2+(A_h-A_r)^2(A_s-A_t)^2(A_u-A_h)^2+(A_h-A_r)^2(A_u-A_t)^2(A_s-A_h)^2\}p_h^2p_sp_rp_up_t$$

$$+\{(A_s-A_u)^2(A_h-A_r)^2(A_t-A_s)^2+(A_s-A_u)^2(A_h-A_t)^2(A_r-A_s)^2+(A_s-A_r)^2(A_h-A_u)^2(A_t-A_s)^2$$

$$+ (A_s-A_h)^2(A_r-A_t)^2(A_u-A_s)^2+(A_s-A_h)^2(A_r-A_u)^2(A_t-A_s)^2+(A_s-A_h)^2(A_t-A_u)^2(A_r-A_s)^2\}p_h p_s^2 p_r p_t p_u$$

$$+\{(A_r-A_u)^2(A_h-A_s)^2(A_t-A_r)^2+(A_r-A_t)^2(A_h-A_u)^2(A_s-A_r)^2+(A_r-A_s)^2(A_h-A_t)^2(A_u-A_r)^2$$

$$+ (A_r-A_h)^2(A_s-A_u)^2(A_t-A_r)^2+(A_r-A_h)^2(A_s-A_t)^2(A_u-A_r)^2+(A_r-A_h)^2(A_u-A_t)^2(A_s-A_r)^2\}p_h p_s p_r^2 p_u p_t$$

$$+\{(A_t-A_r)^2(A_h-A_s)^2(A_u-A_t)^2+(A_t-A_s)^2(A_h-A_r)^2(A_u-A_t)^2+(A_t-A_r)^2(A_h-A_u)^2(A_s-A_t)^2$$

$$+ (A_t-A_h)^2(A_s-A_r)^2(A_u-A_t)^2+(A_t-A_h)^2(A_s-A_u)^2(A_r-A_t)^2+(A_t-A_h)^2(A_r-A_u)^2(A_s-A_t)^2\}p_h p_s p_r p_t^2 p_u$$

$$+\{(A_u-A_r)^2(A_h-A_s)^2(A_t-A_u)^2+(A_u-A_t)^2(A_h-A_r)^2(A_s-A_u)^2+(A_u-A_s)^2(A_h-A_t)^2(A_r-A_u)^2$$

$$+ (A_u-A_h)^2(A_s-A_r)^2(A_t-A_u)^2+(A_u-A_h)^2(A_s-A_t)^2(A_r-A_u)^2+(A_u-A_h)^2(A_r-A_t)^2(A_s-A_u)^2\}p_h p_s p_r p_t p_u^2]$$

$$+ 3!\ \Sigma_3(A_h-A_s)^2(A_h-A_r)^2(A_s-A_r)^2 p_h^2 p_s^2 p_r^2$$

$$+ 3!\ \Sigma_6[(A_h-A_u)^2\{(A_r-A_t)^2(A_v-A_s)^2+(A_r-A_v)^2(A_t-A_s)^2+(A_t-A_v)^2(A_r-A_s)^2\}$$

$$+ (A_h-A_v)^2\{(A_r-A_u)^2(A_t-A_s)^2+(A_t-A_u)^2(A_r-A_s)^2+(A_r-A_t)^2(A_u-A_s)^2\}$$

$$+ (A_h-A_r)^2\{(A_u-A_v)^2(A_t-A_s)^2+(A_t-A_u)^2(A_v-A_s)^2+(A_t-A_v)^2(A_u-A_s)^2\}$$

$$+ (A_h-A_t)^2\{(A_r-A_u)^2(A_v-A_s)^2+(A_r-A_v)^2(A_u-A_s)^2+(A_u-A_v)^2(A_r-A_s)^2\}$$

$$+ (A_h-A_s)^2\{(A_r-A_t)^2(A_u-A_v)^2+(A_r-A_u)^2(A_t-A_v)^2+(A_r-A_v)^2(A_t-A_u)^2\}]\ p_h p_s p_r p_t p_u p_v\}. \qquad \ldots \qquad (18)$$

On substituting expressions (13) to (18) in (12), we obtain the third moment of T.

The second and third moments as well as the cumulants of T can also be obtained by making use of a result given by Iyer (1952), namely

$$K_r\left(\sum_{i=1}^{n} X_i\right) = r! \sum_{t=0}^{r} (n-t) \sum \frac{k'_{s_1 s_2 \ldots s_t}}{s_1! s_2! \ldots s_t!} \tag{19}$$

where $s_1 + s_2 + \ldots + s_t = r$ and $k'_{s_1 s_2 \ldots s_t}$ stand for the joint product cumulant of $[X_1-E(X_1)]^{s_1}[X_2-E(X_2)]^{s_2} \ldots [X_t-E(X_t)]^{s_t}$.

B. <u>Continuous or free sampling</u>. A more general method for obtaining the moments or cumulants of T, which will hold for any continuous or discrete distribution with infinite sampling, is the following: Let $x_1, x_2, \ldots x_n$ be independently and identically distributed with mean m, and second, third and fourth moments around the mean μ_2, μ_3 and μ_4, respectively. Let $x_i - m = y_i$ $(i = 1,2, \ldots n)$.

Then

$$E(T) = E\left(\sum_{i=1}^{n-1} X_i\right) = (n-1)\, E(x_i - x_{i+1})^2.$$

But

$$E(x_i - x_{i+1})^2 = E[(x_i - m) - (x_{i+1} - m)]^2 = E(y_i - y_{i+1})^2$$

$$= E(y_i^2 + y_{i+1}^2 - 2y_i y_{i+1}) = 2\mu_2.$$

Therefore

$$E(X_i) = 2\mu_2 \tag{20}$$

and

$$E(T) = 2(n-1)\mu_2. \tag{21}$$

The second moment of T is obtained from expression (6), namely

$$V(T) = (n-1)[E(X_i^2) - \{E(X_i)\}^2] + 2(n-2)[E(X_i X_{i+1}) - \{E(X_i)\}^2].$$

Here,

$$E(X_i^2) = E(x_i - x_{i+1})^4 = E[(x_i - m) - (x_{i+1} - m)]^4 = E(y_i - y_{i+1})^4$$

$$= E(y_i^4 - 4y_i^3 y_{i+1} + 6y_i^2 y_{i+1}^2 - 4y_i y_{i+1}^3 + y_{i+1}^4)$$

$$= 2\mu_4 + 6\mu_2^2 , \qquad (22)$$

and

$$E(X_i X_{i+1}) = E(x_i - x_{i+1})^2 (x_{i+1} - x_{i+2})^2 = E[(x_i - m) - (x_{i+1} - m)]^2 [(x_{i+1} - m) - (x_{i+2} - m)]^2$$

$$= E(y_i - y_{i+1})^2 (y_{i+1} - y_{i+2})^2 = E(y_i^2 y_{i+1}^2 + y_i^2 y_{i+2}^2 - 2y_i^2 y_{i+1} y_{i+2} + y_{i+1}^4$$

$$+ y_{i+1}^2 y_{i+2}^2 - 2y_{i+1}^3 y_{i+2} - 2y_i y_{i+1}^3 - 2y_i y_{i+1} y_{i+2}^2 + 4y_i y_{i+1}^2 y_{i+2})$$

$$= 3\mu_2^2 + \mu_4 . \qquad (23)$$

Substituting expressions (20), (22) and (23) in (6) and simplifying, we obtain the variance of T,

$$V(T) = 2(2n-3)\, \mu_4 + 2\mu_2^2 . \qquad (24)$$

The third moment of T is obtained from expression (12), namely

$$\mu_3(T) = (n-1)E(X_i^3) - 3(3n-5)\, a\, E(X_i^2) + 6(n-2)\, E(X_i^2 X_{i+1})$$

$$- 12(2n-5)\, a\, E(X_i X_{i+1}) + 6(n-3)\, E(X_i X_{i+1} X_{i+2}) + 4(5n-11)\, a^3 .$$

Here,

$$E(X_i^3) = E(x_i - x_{i+1})^6 = E[(x_i - m) - (x_{i+1} - m)]^6 = E(y_i - y_{i+1})^6$$

$$= E(y_i^6 - 6y_i^5 y_{i+1} + 15y_i^4 y_{i+1}^2 - 20y_i^3 y_{i+1}^3 + 15y_i^2 y_{i+1}^4 - 6y_i y_{i+1}^5 + y_{i+1}^6)$$

$$= 2\mu_6 + 30\mu_2\mu_4 - 20\mu_3^2 . \qquad (25)$$

Here,

$$E(X_i^2) = E(x_i - x_{i+1})^4 = E[(x_i - m) - (x_{i+1} - m)]^4 = E(y_i - y_{i+1})^4$$

$$= E(y_i^4 - 4y_i^3 y_{i+1} + 6y_i^2 y_{i+1}^2 - 4y_i y_{i+1}^3 + y_{i+1}^4)$$

$$= 2\mu_4 + 6\mu_2^2 , \qquad (22)$$

and

$$E(X_i X_{i+1}) = E(x_i - x_{i+1})^2 (x_{i+1} - x_{i+2})^2 = E[(x_i - m) - (x_{i+1} - m)]^2 [(x_{i+1} - m) - (x_{i+2} - m)]^2$$

$$= E(y_i - y_{i+1})^2 (y_{i+1} - y_{i+2})^2 = E(y_i^2 y_{i+1}^2 + y_i^2 y_{i+2}^2 - 2y_i^2 y_{i+1} y_{i+2} + y_{i+1}^4$$

$$+ y_{i+1}^2 y_{i+2}^2 - 2y_{i+1}^3 y_{i+2} - 2y_i y_{i+1}^3 - 2y_i y_{i+1} y_{i+2}^2 + 4y_i y_{i+1}^2 y_{i+2})$$

$$= 3\mu_2^2 + \mu_4 . \qquad (23)$$

Substituting expressions (20), (22) and (23) in (6) and simplifying, we obtain the variance of T,

$$V(T) = 2(2n-3)\, \mu_4 + 2\mu_2^2 . \qquad (24)$$

The third moment of T is obtained from expression (12), namely

$$\mu_3(T) = (n-1)E(X_i^3) - 3(3n-5)\, a\, E(X_i^2) + 6(n-2)\, E(X_i^2 X_{i+1})$$

$$- 12(2n-5)\, a\, E(X_i X_{i+1}) + 6(n-3)\, E(X_i X_{i+1} X_{i+2}) + 4(5n-11)\, a^3 .$$

Here,

$$E(X_i^3) = E(x_i - x_{i+1})^6 = E[(x_i - m) - (x_{i+1} - m)]^6 = E(y_i - y_{i+1})^6$$

$$= E(y_i^6 - 6y_i^5 y_{i+1} + 15y_i^4 y_{i+1}^2 - 20y_i^3 y_{i+1}^3 + 15y_i^2 y_{i+1}^4 - 6y_i y_{i+1}^5 + y_{i+1}^6)$$

$$= 2\mu_6 + 30\mu_2\mu_4 - 20\mu_3^2 . \qquad (25)$$

$$E(X_i^2 X_{i+1}) = E(x_i - x_{i+1})^4 (x_{i+1} - x_{i+2})^2 = E[(x_i - m) - (x_{i+1} - m)]^4 [(x_{i+1} - m) - (x_{i+2} - m)]^2$$

$$= E(y_i - y_{i+1})^4 (y_{i+1} - y_{i+2})^2 = E(y_i^4 - 4y_i^3 y_{i+1} + 6y_i^2 y_{i+1}^2 - 4y_i y_{i+1}^3 + y_{i+1}^4)(y_{i+1} - y_{i+2})^2$$

$$= 9\mu_2\mu_4 - 4\mu_3^2 + 6\mu_2^3 + \mu_6. \qquad (26)$$

$$E(X_i X_{i+1} X_{i+2}) = E(x_i - x_{i+1})^2 (x_{i+1} - x_{i+2})^2 (x_{i+2} - x_{i+3})^2 = E[(x_i - m) - (x_{i+1} - m)]^2$$

$$\times [(x_{i+1} - m) - (x_{i+2} - m)]^2 [(x_{i+2} - m) - (x_{i+3} - m)]^2$$

$$= E(y_i - y_{i+1})^2 (y_{i+1} - y_{i+2})^2 (y_{i+2} - y_{i+3})^2$$

$$= E(y_i^2 y_{i+1}^2 + y_i^2 y_{i+2}^2 - 2y_i^2 y_{i+1} y_{i+2} + y_{i+1}^4 + y_{i+1}^2 y_{i+2}^2 - 2y_{i+1}^3 y_{i+2} - 2y_i y_{i+1}^3$$

$$- 2y_i y_{i+1} y_{i+2}^2 + 4y_i y_{i+1}^2 y_{i+2}) \times (y_{i+2}^2 + y_{i+3}^2 - 2y_{i+2} y_{i+3})$$

$$= 4\mu_2^3 + 4\mu_2\mu_4 - 2\mu_3^2 . \qquad (27)$$

From (20) and (22), we have

$$a\ E(X_i^2) = E(X_j)\ E(X_i^2) = 4\mu_2\mu_4 + 12\mu_2^3 , \qquad (28)$$

while from (20) and (23) we obtain

$$a\ E(X_i X_{i+1}) = 2\mu_2\mu_4 + 6\mu_2^3 . \qquad (29)$$

Substituting expressions (25), (26), ... (29) in (12), and simplifying, we have the third moment of T as

$$\mu_3(T) = 2(4n-7)\mu_6 + 6(4n-5)\mu_2\mu_4 - 4(14n-26)\mu_3^2 - 4(8n-11)\mu_2^3 . \qquad (30)$$

Equation (30) also gives the third cumulant of T, K_3.

To obtain the fourth cumulant of T, we make use of formula (19). Letting $r = 4$, we have

$$K_4(T) = (n-1)K_4' + \frac{4!}{3!1!}(n-2)(K_{31}' + K_{13}') + \frac{4!}{2!2!}(n-2)\,K_{22}'$$

$$+ \frac{4!}{2!1!1!}(n-3)(K_{211}' + K_{121}' + K_{112}') + 4!(n-4)K_{1111}' \,. \qquad (31)$$

The evaluation of the K's in equation (31) will now be made through equating coefficients of $t_1^{s_1}\, t_2^{s_2}\, t_3^{s_3}\, t_4^{s_4}$ in the expanded form of the identity

$$\exp.\left\{ \sum_{s_1,s_2,s_3,s_4=1}^{\infty} K_{s_1 s_2 s_3 s_4} \frac{t_1^{s_1}\, t_2^{s_2}\, t_3^{s_3}\, t_4^{s_4}}{s_1!\, s_2!\, s_3!\, s_4!} \right\} = M(t_1,t_2,t_3,t_4),$$

where $M(t_1,t_2,t_3,t_4)$ is the joint moment generating function of $X_1X_2X_3X_4$.

For $s_1=4$, $s_2 = s_3 = s_4 = 0$,

$$K_4' = E(X_i-a)^4 - 3\{E(X_i-a)^2\}^2$$

$$= E(X_i^4)-4a\;E(X_i^3)+6a^2E(X_i^2)-4a^3\;E(X_i)+a^4-3\{[E(X_i^2)]^2-2a^2E(X_i^2)+a^4\}$$

$$= E(X_i^4)-4a\;E(X_i^3)+12a^2E(X_i^2)-6a^4-3[E(X_i^2)]^2, \qquad (32)$$

since $X_1, X_2, \ldots X_n$ are identically distributed.

With a similar choice of appropriate values of s_1, s_2, s_3, s_4, we also obtain

$$K'_{31} = K'_{13} = E(X_i-a)^3(X_{i+1}-a) - 3\{E(X_i-a)^2\}\{E(X_i-a)(X_{i+1}-a)\}$$

$$= E(X_i^3X_{i+1})-a\ E(X_i^3)-3a\ E(X_i^2X_{i+1})+3a^2E(X_i^2)+3a^2E(X_iX_{i+1})-3a^3E(X_i)$$

$$-a^3E(X_{i+1})+a^4-3\{E(X_i^2)E(X_iX_{i+1})-a^2E(X_i^2)-a^2E(X_iX_{i+1})+a^4\}$$

$$= E(X_i^3X_{i+1})-3a\ E(X_i^2X_{i+1})+6a^2E(X_iX_{i+1})-a\ E(X_i^3)+6a^2E(X_i^2)$$

$$-6a^4-3E(X_i^2)\ E(X_iX_{i+1}) \tag{33}$$

$$K'_{22} = E(X_i-a)^2(X_{i+1}-a)^2-[E(X_i-a)^2][E(X_{i+1}-a)^2]-2[E(X_i-a)(X_{i+1}-a)]^2$$

$$= E(X_i^2X_{i+1}^2)-2a\ E(X_i^2X_{i+1})+a^2E(X_i^2)-2a\ E(X_iX_{i+1}^2)+4a^2E(X_iX_{i+1})+a^2E(X_{i+1}^2)$$

$$-3a^4-[E(X_i^2)E(X_{i+1}^2)-2a^2E(X_i^2)+a^4]-2[\{E(X_iX_{i+1})\}^2-2a^2E(X_iX_{i+1})+a^4]$$

$$= E(X_i^2X_{i+1}^2)-4aE(X_i^2X_{i+1})+4a^2E(X_i^2)+8a^2E(X_iX_{i+1})-6a^4$$

$$-E(X_i^2)E(X_{i+1}^2)-2\{E(X_iX_{i+1})\}^2 \tag{34}$$

$$K'_{211} = K'_{112} = E(X_i-a)^2(X_{i+1}-a)(X_{i+2}-a)-E(X_i-a)^2E(X_i-a)(X_{i+1}-a)$$

$$= E(X_i^2X_{i+1}X_{i+2})-a\ E(X_i^2X_{i+1})-a\ E(X_i^2X_{i+2})+a^2E(X_i^2)-2a\ E(X_iX_{i+1}X_{i+2})$$

$$+2a^2E(X_iX_{i+1})+2a^2E(X_iX_{i+2})-2a^3E(X_i)+a^2E(X_{i+1}X_{i+2})-a^3E(X_{i+1})$$

$$-a^3E(X_{i+2})+a^4-[E(X_i^2)E(X_iX_{i+1})-a^2E(X_i^2)-a^2E(X_iX_{i+1})+a^4]$$

$$= E(X_i^2X_{i+1}X_{i+2})-a\ E(X_i^2X_{i+1})-2a\ E(X_iX_{i+1}X_{i+2})+4a^2E(X_iX_{i+1})$$

$$-2a^4-E(X_i^2)E(X_iX_{i+1})+a^2E(X_i^2) \tag{35}$$

$$K'_{121} = E(X_i-a)(X_{i+1}-a)^2(X_{i+2}-a) - 2\{E(X_i-a)(X_{i+1}-a)\}^2$$

$$= E(X_iX_{i+1}^2X_{i+2})-a\ E(X_iX_{i+1}^2)-2a\ E(X_iX_{i+1}X_{i+2})+2a^2E(X_iX_{i+1})+a^2E(X_iX_{i+2})$$

$$- a^3E(X_i)-a\ E(X_{i+1}^2X_{i+2})+a^2E(X_{i+1}^2)+2a^2E(X_{i+1}X_{i+2})-2a^3E(X_{i+1})$$

$$- a^3E(X_{i+2})+a^4- 2\{[E(X_iX_{i+1})]^2 - 2a^2E(X_iX_{i+1})+a^4\}$$

$$= E(X_iX_{i+1}^2X_{i+2}) - 2a\ E(X_iX_{i+1}^2) - 2a\ E(X_iX_{i+1}X_{i+2}) + 8a^2E(X_iX_{i+1})+a^2E(X_{i+1}^2)$$

$$- 4a^4 - 2[E(X_iX_{i+1})]^2. \qquad (36)$$

$$K'_{1111} = E(X_i-a)(X_{i+1}-a)(X_{i+2}-a)(X_{i+3}-a) - \{E(X_i-a)(X_{i+1}-a)\}^2$$

$$= E(X_iX_{i+1}X_{i+2}X_{i+3})-a\ E(X_iX_{i+1}X_{i+3})-a\ E(X_iX_{i+1}X_{i+2})+a^2E(X_iX_{i+1})$$

$$- a\ E(X_iX_{i+2}X_{i+3})+a^2E(X_iX_{i+3})+a^2E(X_iX_{i+2})-a^3E(X_i)-a\ E(X_{i+1}X_{i+2}X_{i+3})$$

$$+ a^2E(X_{i+1}X_{i+3})+a^2E(X_{i+1}X_{i+2})-a^3E(X_{i+1})+a^2E(X_{i+2}X_{i+3})-a^3E(X_{i+3})$$

$$- a^3E(X_{i+2})+a^4 - \{[E(X_iX_{i+1})]^2 - 2a^2E(X_iX_{i+1})+a^4\}$$

$$= E(X_iX_{i+1}X_{i+2}X_{i+3})-2a\ E(X_iX_{i+1}X_{i+2})+3a^2E(X_{i+2}X_{i+3})$$

$$- \{E(X_iX_{i+1})\}^2 - a^4 . \qquad (37)$$

Substituting (32), (33), ..., (37) in (31) and simplifying, we obtain the fourth cumulant of T as

$$K_4 = (n-1)E(X_i^4)+8(n-2)E(X_i^3X_{i+1})+6(n-2)E(X_i^2X_{i+1}^2)+4!(n-3)E(X_i^2X_{i+1}X_{i+2})$$

$$+ 12(n-3)E(X_iX_{i+1}^2X_{i+2})+4!(n-4)E(X_iX_{i+1}X_{i+2}X_{i+3})-4(3n-5)a\ E(X_i^3)$$

$$+ 4!(5n-11)a^2E(X_i^2)-48(2n-5)a\ E(X_i^2X_{i+1})+4!(15n-44)a^2E(X_iX_{i+1})$$

$$- 4!(5n-17)a\ E(X_iX_{i+1}X_{i+2})-3(3n-5)\{E(X_i^2)\}^2- 4!(2n-5)E(X_i^2)E(X_iX_{i+1})$$

$$- 12(5n-16)\{E(X_iX_{i+1})\}^2 -6(35n-93)\ a^4 . \qquad (38)$$

The fourth cumulant, k_4, can also be expressed in terms of the given moments μ_2, μ_3 and μ_4 as follows:

$$E(X_i^4) = E(x_i-x_{i+1})^8 = E\{(x_i-m)-(x_{i+1}-m)\}^8 = E(y_i-y_{i+1})^8$$

$$= E(y_i^8-8y_i^7y_{i+1}+28y_i^6y_{i+1}^2-56y_i^5y_{i+1}^3+70y_i^4y_{i+1}^4-56y_i^3y_{i+1}^5+28y_i^2y_{i+1}^6-8y_iy_{i+1}^7+y_{i+1}^8)$$

$$= 2\mu_8 + 56\mu_2\ \mu_6 - 112\mu_3\ \mu_5 + 70\mu_4^2. \qquad (39)$$

Similarly

$$E(X_i^3X_{i+1}) = 18\mu_6\mu_2 - 26\mu_3\mu_5 + 15\mu_4^2 + 30\mu_4\mu_2^2 - 20\mu_3^2\mu_2 + \mu_8 \qquad (40)$$

$$E(X_i^2X_{i+1}^2) = 3\mu_4^2 + 48\mu_4\mu_2^2 - 8\mu_3\mu_5 - 32\mu_2\mu_3^2 + 12\mu_2\mu_6 + \mu_8 \qquad (41)$$

$$E(X_i^2X_{i+1}X_{i+2}) = 22\mu_2^2\mu_4 + 2\mu_4^2 - 12\mu_3^2\mu_2 + 6\mu_2^4 + 2\mu_6\mu_2 - 2\mu_5\mu_3 \qquad (42)$$

$$E(X_iX_{i+1}^2X_{i+2}) = 16\mu_2^2\mu_4 - 8\mu_2\mu_3^2 + 4\mu_2\mu_6 + 6\mu_2^4 - 8\mu_3\mu_5 + 6\mu_4^2 \qquad (43)$$

$$E(X_iX_{i+1}X_{i+2}X_{i+3}) = 5\mu_2^4 + 10\mu_2^2\mu_4 - 4\mu_2\mu_3^2 + \mu_4^2 . \qquad (44)$$

Substituting expressions (39) to (44), and expressions (20), (22), (23), (25), (26) and (27) in (38) and simplifying, we obtain

$$K_4 = 2(8n-15)\mu_8 + 8(16n-27)\mu_2\mu_6 - 32(16n-33)\mu_3\mu_5 + 10(16n-31)\mu_4^2$$

$$- 24(28n-51)\mu_4\mu_2^2 + 64(14n-33)\mu_2\mu_3^2 + 12(28n-53)\mu_2^4 . \qquad (45)$$

Formulas (21), (24), (30) and (45) give the first four cumulants of T for any distribution of the observations $x_1, x_2, \dots x_n$. For example, for the case where the observations come from a discrete population with infinite sampling, formula (2) reduces to (21). This may be verified as follows:

For a discrete population

$$\mu_2 = \sum_{h=1}^{k} A_h^2 p_h - \left(\sum_{h=1}^{k} A_h p_h\right)^2 .$$

Hence

$$E(T) = 2(n-1) \sum_{h>s} (A_h - A_s)^2 p_h p_s = (n-1) \sum_{h=1}^{k} \sum_{s=1}^{k} (A_h - A_s)^2 p_h p_s$$

$$= (n-1) \sum_{h=1}^{k} \sum_{s=1}^{k} (A_h^2 + A_s^2 - 2A_h A_s) p_h p_s$$

$$= (n-1) \left\{ \sum_{h=1}^{k} A_h^2 p_h + \sum_{s=1}^{k} A_s^2 p_s - 2 \sum_{h=1}^{k} A_h p_h \sum_{s=1}^{k} A_s p_s \right\}$$

$$= (n-1) \left\{ 2\mu_2' - 2(\mu_1')^2 \right\} = 2(n-1)\mu_2 .$$

Similarly formula (10) reduces to (24), and the formula for the third moment of T, obtained by substituting expressions (13) to (18) in (12), reduces to (30).

For the case where the observations $x_1, x_2, \dots x_n$ come from a normal population with mean m and variance σ^2, the first four cumulants of T, namely

$$K_1 = E(T) = 2(n-1)\,\sigma^2 \tag{46}$$

$$K_2 = V(T) = 4(3n-4)\,\sigma^4 \tag{47}$$

$$K_3 = \mu_3(T) = 32(5n-8)\ \sigma^6 \tag{48}$$

$$K_4 = 96(35n-64)\ \sigma^8 \tag{49}$$

are obtained by substituting

$$\mu_2 = \sigma^2, \quad \mu_3 = \mu_5 = \mu_7 = 0, \quad \mu_4 = 3\sigma^4, \quad \mu_6 = 15\sigma^6, \quad \mu_8 = 105\sigma^8,$$

in formulas (21), (24), (30) and (45).

C. Non-free sampling. We recall that in the case of non-free sampling n_i of the n observations $x_1, x_2, \ldots x_n$ have the value A_i $(i = 1,2,\ldots k)$. Thus the probability that any single observation has the value A_i is $\frac{n_i}{n}$, while the joint probability that 2 observations have the respective values A_hA_s is $\frac{n_hn_s}{n(n-1)}$, the probability of observing $A_hA_sA_r$ is $\frac{n_hn_sn_r}{n(n-1)(n-2)}$, and so on.

Iyer has shown (1950) that the moments of T in this case, can be obtained from the moments around zero in the case of free sampling by the simple substitution of the above probabilities in terms of the n_i for the probabilities in terms of p_i.

Hence, the first moment of T for non-free sampling, is obtained by substituting the quantity $\frac{n_hn_s}{n(n-1)}$ for p_hp_s in equation (2). This gives

$$E(T) = 2(n-1) \sum_{h>s} (A_h-A_s)^2 \frac{n_hn_s}{n(n-1)} = \frac{2}{n} \sum_{h>s} (A_h-A_s)^2\, n_hn_s. \tag{50}$$

To find the variance of T, we first express the variance in terms of the moments around zero.

Thus

$$V(T) = \mu_2'(T) - \{\mu_1'(T)\}^2. \tag{51}$$

Now

$$\mu_2'(T) = E(\sum_{i=1}^{n-1} X_i)^2 = E\left\{\sum_{i=1}^{n-1} X_i^2 + 2! \sum_{j\neq i} X_i X_j\right\}$$

$$= (n-1)E(X_i^2) + 2(n-2)E(X_i X_{i+1})$$

$$+ (n-2)(n-3) \underset{j\neq i+1}{E} (X_i X_j). \tag{52}$$

The expectations in equation (52) can now be obtained from equations (7), (8) and (9) by making the appropriate substitutions for the probabilities in terms of the n_i. The substitutions yield the following values:

$$E(X_i^2) = 2 \sum_{h>s} (A_h - A_s)^4 \frac{n_h n_s}{n(n-1)} \tag{53}$$

$$E(X_i X_{i+1}) = 2\Sigma_3\{(A_h-A_s)^2(A_s-A_r)^2+(A_h-A_r)^2(A_r-A_s)^2$$

$$+ (A_r-A_h)^2(A_h-A_s)^2\} \frac{n_h n_s n_r}{n(n-1)(n-2)}$$

$$+ \Sigma_2(A_h-A_s)^4\left\{\frac{n_h(n_h-1)n_s}{n(n-1)(n-2)} + \frac{n_h n_s(n_s-1)}{n(n-1)(n-2)}\right\} \tag{54}$$

$$\underset{j\neq i\neq 1}{E}(X_i X_j) = 4\Sigma_2(A_h-A_s)^4 \frac{n_h(n_h-1)n_s(n_s-1)}{n(n-1)(n-2)(n-3)}$$

$$+ 8\{\Sigma_3[(A_h-A_s)^2(A_h-A_r)^2 \frac{n_h(n_h-1)n_s n_r}{n(n-1)(n-2)(n-3)}$$

$$+ (A_h-A_s)^2(A_s-A_r)^2 \frac{n_h n_s(n_s-1)n_r}{n(n-1)(n-2)(n-3)}$$

$$+ (A_h-A_r)^2(A_s-A_r)^2 \frac{n_h n_s n_r(n_r-1)}{n(n-1)(n-2)(n-3)}]\}$$

$$+ 8\{\Sigma_4[(A_h-A_s)^2(A_r-A_t)^2+(A_h-A_r)^2(A_s-A_t)^2$$

$$+ (A_h-A_t)^2(A_s-A_r)^2] \frac{n_h n_s n_r n_t}{n(n-1)(n-2)(n-3)}\} . \qquad (55)$$

By virtue of equations (53), (54) and (55), the second moment around zero, of T in equation (52) becomes

$$\mu_2'(T) = \frac{2}{n} \sum_{h>s} (A_h-A_s)^4 n_h n_s + \frac{2}{n(n-1)}\{ 2\Sigma_3[(A_h-A_s)^2(A_s-A_r)^2+(A_h-A_r)^2(A_r-A_s)^2$$

$$+ (A_r-A_h)^2(A_h-A_s)^2]n_h n_s n_r + \Sigma_2(A_h-A_s)^4 n_h n_s(n_h+n_s-2)$$

$$+ 2\Sigma_2(A_h-A_s)^4 n_h(n_h-1)n_s(n_s-1)+4\Sigma_3[(A_h-A_s)^2(A_h-A_r)^2(n_h-1)$$

$$+ (A_h-A_s)^2(A_s-A_r)^2(n_s-1)+(A_h-A_r)^2(A_s-A_r)^2(n_r-1)]n_h n_s n_r$$

$$+ 4\Sigma_4[(A_h-A_s)^2(A_r-A_t)^2+(A_h-A_r)^2(A_s-A_t)^2+(A_h-A_t)^2(A_s-A_r)^2]n_h n_s n_r n_t\} .$$

$$\ldots \qquad (56)$$

The variance of T can now be obtained by substituting expressions (56) and (50) in (51).

Notice that the substitutions for the p_i had to be made in the non-reduced form of $V(T)$ rather than in the reduced form given by equation (6). This is due to a dependence introduced between disconnected variates X_i and X_j by the restriction of non-free sampling. For both types of sampling, the variance of T may be expressed as

$$V(T) = (n-1)E(X_i^2)+2(n-2)E(X_i X_{i+1})+(n-2)(n-3) \underset{i\neq j}{E} (X_i X_j)$$

$$- (n-1)^2 \left[E(X_i)\right]^2 \qquad (57)$$

by equations (51) and (52). In the case of free sampling, since $\underset{j>i+1}{E} (X_i X_j) = [E(X_i)]^2$, we were able to cancel out some of the terms in expression (57) to obtain the form (6). Such a cancellation is not possible in the case under discussion.

The third moment of T around the mean is obtained by first expressing it in terms of the moments around zero and then evaluating these moments in terms of the n_i. We have

$$\mu_3(T) = \mu_3'(T) - 3\mu_1'(T)\mu_2'(T) + 2\{\mu_1'(T)\}^3, \qquad (58)$$

and

$$\mu_3'(T)=E\left(\sum_{i=1}^{n-1} X_i\right)^3=E\left\{\sum_{i=1}^{n-1} X_i^3+ \frac{3!}{2!1!} \left(\sum_{i\neq j} X_i^2 X_j + \sum_{i\neq j} X_i X_j^2\right)+ \frac{3!}{1!1!1!} \sum_{j\neq i\neq k} X_i X_j X_k \right\}$$

$$=(n-1)E(X_i^3)+ \frac{3!}{2!1!}(n-2)\left\{E(X_i^2 X_{i+1})+E(X_i X_{i+1}^2)\right\}$$

$$+ \frac{3!}{2!1!}\binom{n-2}{2}\left\{\underset{j>i+1}{E} (X_i^2 X_j)+E(X_i X_j^2)\right\}+3!(n-3)E(X_i X_{i+1} X_{i+2})$$

$$+ 3!\binom{n-3}{2}\left\{ \underset{j>i+1}{E} (X_i X_{i+1} X_j)+E(X_i X_j X_{j+1})\right\} +3!\binom{n-3}{3} \underset{\substack{j>i+1\\ k>j+1}}{E} (X_i X_j X_k).$$

$$\ldots\ldots \qquad (59)$$

Since $X_1, X_2, \ldots X_{n-1}$ are identically distributed, we may write the third moment about zero as

$$\mu_3'(T)=(n-1)E(X_i^3)+3!(n-2)E(X_i^2X_{i+1})+3(n-2)(n-3)E(X_i^2)E(X_j)$$

$$+ 3!(n-3)E(X_iX_{i+1}X_{i+2})+3!(n-3)(n-4)E(X_iX_{i+1})E(X_j)$$

$$+ (n-3)(n-4)(n-5)\{E(X_i)\}^3 . \qquad (60)$$

The expectations in equation (60) can be obtained from equations (13) to (18) by making the appropriate substitutions for the probabilities in terms of the n_i. The substitutions yield the following values:

$$E(X_i^3) = 2\Sigma_2(A_h-A_s)^6 \frac{n_hn_s}{n(n-1)} \qquad (61)$$

$$E(X_j)E(X_i^2)= \frac{4}{n(n-1)(n-2)(n-3)}[\Sigma_2(A_h-A_s)^6 \; n_h(n_h-1)n_s(n_s-1)$$

$$+\Sigma_3\{[(A_h-A_s)^4(A_h-A_r)^2+(A_h-A_s)^2(A_h-A_r)^4](n_h-1)$$

$$+[(A_h-A_s)^4(A_s-A_r)^2+(A_h-A_s)^2(A_s-A_r)^4](n_s-1)$$

$$+[(A_h-A_r)^4(A_s-A_r)^2+(A_h-A_r)^2(A_s-A_r)^4(n_r-1)\}n_hn_sn_r$$

$$+\Sigma_4\{(A_h-A_s)^4(A_r-A_t)^2+(A_h-A_s)^2(A_r-A_t)^4+(A_h-A_r)^4(A_s-A_t)^2+(A_h-A_r)^2(A_s-A_t)^4$$

$$+(A_h-A_t)^4(A_s-A_r)^2+(A_h-A_t)^2(A_s-A_r)^4\}n_hn_sn_rn_t]. \qquad (62)$$

$$E(X_i^2X_{i+1})= \frac{1}{n(n-1)(n-2)}[\Sigma_3\{(A_h-A_s)^4(A_s-A_r)^2+(A_h-A_s)^2(A_s-A_r)^4$$

$$+(A_h-A_r)^4(A_r-A_s)^2+(A_h-A_r)^2(A_r-A_s)^4+(A_r-A_h)^4(A_h-A_s)^2$$

$$+(A_r-A_h)^2(A_h-A_s)^4\}n_hn_sn_r +\Sigma_2(A_h-A_s)^6 \; n_hn_s(n_h+n_s-2)] . \qquad (63)$$

$E(X_j)E(X_iX_{i+1}) =$

$$\frac{2}{n(n-1)(n-2)(n-3)(n-4)}[2\Sigma_3\{[(A_h-A_s)^4(A_s-A_r)^2+(A_h-A_r)^2(A_r-A_s)^2(A_h-A_s)^2$$

$$+(A_s-A_h)^4(A_h-A_r)^2](n_h-1)(n_s-1)+[(A_h-A_s)^2(A_s-A_r)^2(A_h-A_r)^2+(A_h-A_r)^4(A_r-A_s)^2$$

$$+(A_s-A_h)^2(A_h-A_r)^4](n_h-1)(n_r-1)+[(A_h-A_s)^2(A_s-A_r)^4+(A_h-A_r)^2(A_r-A_s)^4$$

$$+(A_s-A_h)^2(A_h-A_r)^2(A_s-A_r)^2](n_s-1)(n_r-1)\}\,n_hn_sn_r$$

$$+\ 2\Sigma_4\Big\{[\{(A_h-A_s)^2(A_s-A_r)^2+(A_h-A_r)^2(A_r-A_s)^2+(A_s-A_h)^2(A_h-A_r)^2\}(A_t-A_h)^2(n_h-1)$$

$$+\ \{(A_h-A_s)^2(A_s-A_r)^2+(A_h-A_r)^2(A_r-A_s)^2+(A_s-A_h)^2(A_h-A_r)\ \}(A_t-A_r)^2(n_r-1)$$

$$+\ \{(A_h-A_s)^2(A_s-A_r)^2+(A_h-A_r)^2(A_r-A_s)^2+(A_s-A_h)^2(A_h-A_r)^2\}(A_t-A_s)^2(n_s-1)]n_hn_sn_rn_t$$

$$+[\{(A_s-A_r)^2(A_r-A_t)^2+(A_s-A_t)^2(A_t-A_r)^2+(A_r-A_s)^2(A_s-A_t)^2\}(A_h-A_t)^2(n_t-1)$$

$$+\ \{(A_s-A_r)^2(A_r-A_t)^2+(A_s-A_t)^2(A_t-A_r)^2+(A_r-A_s)^2(A_s-A_t)^2\}(A_h-A_r)^2(n_r-1)$$

$$+\ \{(A_s-A_r)^2(A_r-A_t)^2+(A_s-A_t)^2(A_t-A_r)^2+(A_r-A_s)^2(A_s-A_t)^2\}(A_h-A_s)^2(n_s-1)]n_hn_sn_rn_t$$

$$+[\{(A_h-A_t)^2(A_t-A_r)^2+(A_t-A_h)^2(A_h-A_r)^2+(A_h-A_r)^2(A_r-A_t)^2\}(A_s-A_t)^2(n_t-1)$$

$$+\ \{(A_h-A_t)^2(A_t-A_r)^2+(A_t-A_h)^2(A_h-A_r)^2+(A_h-A_r)^2(A_r-A_t)^2\}(A_s-A_h)^2(n_h-1)$$

$$+\ \{(A_h-A_t)^2(A_t-A_r)^2+(A_t-A_h)^2(A_h-A_r)^2+(A_h-A_r)^2(A_r-A_t)^2\}(A_s-A_r)^2(n_r-1)]n_hn_sn_rn_t$$

$$+[\{(A_h-A_s)^2(A_s-A_t)^2+(A_s-A_h)^2(A_h-A_t)^2+(A_h-A_t)^2(A_t-A_s)^2\}(A_r-A_t)^2(n_t-1)$$

$$+\ \{(A_h-A_s)^2(A_s-A_t)^2+(A_s-A_h)^2(A_h-A_t)^2+(A_h-A_t)^2(A_t-A_s)^2\}(A_r-A_h)^2(n_h-1)$$

$$+\ \{(A_h-A_s)^2(A_s-A_t)^2+(A_s-A_h)^2(A_h-A_t)^2+(A_h-A_t)^2(A_t-A_s)^2\}(A_r-A_s)^2(n_s-1)]n_hn_sn_rn_t\}$$

$$+2\Sigma_5\{[(A_u-A_t)^2(A_t-A_r)^2+(A_u-A_r)^2(A_r-A_t)^2+(A_t-A_u)^2(A_u-A_r)^2](A_s-A_h)^2$$

$$+\ [(A_u-A_s)^2(A_s-A_t)^2+(A_u-A_t)^2(A_t-A_s)^2+(A_s-A_u)^2(A_u-A_t)^2](A_h-A_r)^2$$

$$+\ [(A_u-A_s)^2(A_s-A_r)^2+(A_u-A_r)^2(A_r-A_s)^2+(A_s-A_u)^2(A_u-A_r)^2](A_h-A_t)^2$$

$$+\ [(A_t-A_s)^2(A_s-A_r)^2+(A_t-A_r)^2(A_r-A_s)^2+(A_s-A_t)^2(A_t-A_r)^2](A_u-A_h)^2$$

$$+\ [(A_h-A_u)^2(A_u-A_t)^2+(A_h-A_t)^2(A_t-A_u)^2+(A_u-A_h)^2(A_h-A_t)^2](A_s-A_r)^2$$

$$+\ [(A_h-A_u)^2(A_u-A_r)^2+(A_u-A_h)^2(A_h-A_r)^2+(A_u-A_r)^2(A_r-A_h)^2](A_s-A_t)^2$$

$$+\ [(A_h-A_t)^2(A_t-A_r)^2+(A_h-A_r)^2(A_r-A_t)^2+(A_t-A_h)^2(A_h-A_r)^2](A_s-A_u)^2$$

$$+\ [(A_h-A_s)^2(A_s-A_u)^2+(A_h-A_u)^2(A_u-A_s)^2+(A_s-A_h)^2(A_h-A_u)^2](A_r-A_t)^2$$

$$+\ [(A_h-A_s)^2(A_s-A_t)^2+(A_h-A_t)^2(A_t-A_s)^2+(A_t-A_h)^2(A_h-A_s)^2](A_r-A_u)^2$$

$$+\ [(A_h-A_s)^2(A_s-A_r)^2+(A_h-A_r)^2(A_r-A_s)^2+(A_s-A_h)^2(A_h-A_r)^2](A_t-A_u)^2\}\ n_hn_sn_rn_tn_u$$

$$+\Sigma_2\ \ (A_h-A_s)^6\ n_hn_s\ (n_h-1)(n_s-1)(n_h+n_s-4)$$

$$+\Sigma_3[\ (A_h-A_r)^4(A_h-A_s)^2(n_h-1)(n_h+n_r-3)+(A_r-A_s)^4(A_h-A_s)^2(n_s-1)(n_r+n_s-3)$$

$$+ (A_h-A_s)^4(A_h-A_r)^2(n_h-1)(n_h+n_s-3)+(A_r-A_s)^4(A_h-A_r)^2(n_r-1)(n_r+n_s-3)$$

$$+ (A_h-A_s)^4(A_r-A_s)^2(n_s-1)(n_h+n_s-3)+(A_r-A_h)^4(A_r-A_s)^2(n_r-1)(n_r+n_h-3)]\ n_h n_s n_r$$

$$+\Sigma_4 \Big[\{(A_r-A_t)^4(A_h-A_s)^2+(A_r-A_s)^4(A_h-A_t)^2+(A_r-A_h)^4(A_s-A_t)^2\}(n_r-1)$$

$$+\{(A_t-A_r)^4(A_h-A_s)^2+(A_t-A_s)^4(A_h-A_r)^2+(A_t-A_h)^4(A_r-A_s)^2\}(n_t-1)$$

$$+\{(A_s-A_t)^4(A_h-A_r)^2+(A_s-A_r)^4(A_h-A_t)^2+(A_s-A_h)^4(A_r-A_t)^2\}(n_s-1)$$

$$+\{(A_h-A_t)^4(A_r-A_s)^2+(A_h-A_s)^4(A_r-A_t)^2+(A_h-A_r)^4(A_s-A_t)^2\}(n_h-1)\Big]\ n_h n_s n_r n_t].$$

$$\ldots\ldots \qquad (64)$$

$$E(X_i X_{i+1} X_{i+2}) = \frac{2}{n(n-1)(n-2)(n-3)} \Big\{\Sigma_2(A_h-A_s)^6\ n_h(n_h-1)n_s(n_s-1)$$

$$+\Sigma_3 \Big[\{(A_h-A_s)^4(A_h-A_r)^2+(A_h-A_r)^4(A_h-A_s)^2+(A_h-A_s)^2(A_s-A_r)^2(A_r-A_h)^2\}(n_h-1)$$

$$+\{(A_s-A_h)^4(A_s-A_r)^2+(A_s-A_r)^4(A_s-A_h)^2+(A_s-A_h)^2(A_h-A_r)^2(A_r-A_s)^2\}(n_s-1)$$

$$+\{(A_r-A_s)^4(A_r-A_h)^2+(A_r-A_h)^4(A_r-A_s)^2+(A_r-A_s)^2(A_s-A_h)^2(A_h-A_r)^2\}(n_r-1)\Big]\ n_h n_s n_r$$

$$+\Sigma_4 \Big[\ (A_h-A_s)^2(A_s-A_r)^2(A_r-A_t)^2+(A_h-A_r)^2(A_r-A_s)^2(A_s-A_t)^2+(A_h-A_r)^2(A_r-A_t)^2(A_t-A_s)^2$$

$$+ (A_h-A_t)^2(A_t-A_r)^2(A_r-A_s)^2+(A_h-A_t)^2(A_t-A_s)^2(A_s-A_r)^2+(A_h-A_s)^2(A_s-A_t)^2(A_t-A_r)^2$$

$$+ (A_s-A_h)^2(A_h-A_r)^2(A_r-A_t)^2+(A_s-A_r)^2(A_r-A_h)^2(A_h-A_t)^2+(A_s-A_t)^2(A_t-A_h)^2(A_h-A_r)^2$$

$$+ (A_s-A_h)^2(A_h-A_t)^2(A_t-A_r)^2+(A_r-A_s)^2(A_s-A_h)^2(A_h-A_t)^2+(A_r-A_h)^2(A_h-A_s)^2(A_s-A_t)^2\Big]$$

$$\times(n_h n_s n_r n_t)\Big\} \qquad (65)$$

$$[E(X_i)]^3 = \frac{8}{n(n-1)(n-2)(n-3)(n-4)(n-5)} [\Sigma_2 (A_h-A_s)^6 n_h(n_h-1)(n_h-2)n_s(n_s-1)(n_s-2)$$

$$+ \frac{3!}{2!1!} \Sigma_3 \{(A_h-A_s)^4(A_h-A_r)^2(n_h-1)(n_h-2)(n_s-1)+(A_h-A_s)^4(A_s-A_r)^2(n_h-1)(n_s-1)(n_s-2)$$

$$+ (A_h-A_r)^4(A_h-A_s)^2(n_h-1)(n_h-2)(n_r-1)+(A_h-A_r)^4(A_r-A_s)^2(n_h-1)(n_r-1)(n_r-2)$$

$$+ (A_r-A_s)^4(A_r-A_h)^2(n_s-1)(n_r-1)(n_r-2)+(A_r-A_s)^4(A_s-A_h)^2(n_s-1)(n_s-2)(n_r-1)\} n_h n_s n_r$$

$$\frac{3!}{2!1!} \Sigma_4 \{(A_h-A_s)^4(A_r-A_t)^2(n_h-1)(n_s-1)+(A_h-A_r)^4(A_s-A_t)^2(n_h-1)(n_r-1)$$

$$+ (A_h-A_t)^4(A_s-A_r)^2(n_h-1)(n_t-1)+(A_s-A_r)^4(A_h-A_t)^2(n_s-1)(n_r-1)$$

$$+ (A_s-A_t)^4(A_h-A_r)^2(n_s-1)(n_t-1)+(A_r-A_t)^4(A_h-A_s)^2(n_r-1)(n_t-1)\} n_h n_s n_r n_t$$

$$+ 3!\Sigma_4 \{ (A_h-A_s)^2[(A_h-A_r)^2(A_s-A_t)^2+(A_h-A_t)^2(A_s-A_r)^2](n_h-1)(n_s-1)$$

$$+ (A_h-A_r)^2[(A_h-A_s)^2(A_r-A_t)^2+(A_h-A_t)^2(A_r-A_s)^2](n_h-1)(n_r-1)$$

$$+ (A_h-A_t)^2[(A_h-A_s)^2(A_t-A_r)^2+(A_h-A_r)^2(A_t-A_s)^2](n_h-1)(n_t-1)$$

$$+ (A_s-A_r)^2[(A_s-A_h)^2(A_r-A_t)^2+(A_s-A_t)^2(A_r-A_h)^2](n_s-1)(n_r-1)$$

$$+ (A_s-A_t)^2[(A_s-A_r)^2(A_t-A_h)^2+(A_s-A_h)^2(A_t-A_r)^2](n_s-1)(n_t-1)$$

$$+ (A_r-A_t)^2[(A_r-A_h)^2(A_t-A_s)^2+(A_r-A_s)^2(A_t-A_h)^2](n_r-1)(n_t-1)\} n_h n_s n_r n_t$$

$$+ 3!\Sigma_4 \{ (A_h-A_s)^2(A_h-A_r)^2(A_h-A_t)^2(n_h-1)(n_h-2)+(A_s-A_r)^2(A_s-A_t)^2(A_s-A_h)^2(n_s-1)(n_s-2)$$

$$+ (A_r-A_h)^2(A_r-A_s)^2(A_r-A_t)^2(n_r-1)(n_r-2)+(A_t-A_h)^2(A_t-A_s)^2(A_t-A_r)^2(n_t-1)(n_t-2)\}$$

$$\times (n_h n_s n_r n_t)$$

$$+ 3!\Sigma_5 \Big\{ [(A_h-A_s)^2(A_r-A_t)^2(A_u-A_h)^2+(A_h-A_s)^2(A_r-A_u)^2(A_t-A_h)^2+(A_h-A_u)^2(A_r-A_s)^2(A_t-A_h)^2$$

$$+ (A_h-A_r)^2(A_s-A_u)^2(A_t-A_h)^2+(A_h-A_r)^2(A_s-A_t)^2(A_u-A_h)^2+(A_h-A_r)^2(A_u-A_t)^2(A_s-A_h)^2](n_h-1)$$

$$+ [(A_s-A_u)^2(A_h-A_r)^2(A_t-A_s)^2+(A_s-A_u)^2(A_h-A_t)^2(A_r-A_s)^2+(A_s-A_r)^2(A_h-A_u)^2(A_t-A_s)^2$$

$$+ (A_s-A_h)^2(A_r-A_t)^2(A_u-A_s)^2+(A_s-A_h)^2(A_r-A_u)^2(A_t-A_s)^2+(A_s-A_h)^2(A_t-A_u)^2(A_r-A_s)^2](n_s-1)$$

$$+ [(A_r-A_u)^2(A_h-A_s)^2(A_t-A_r)^2+(A_r-A_t)^2(A_h-A_u)^2(A_s-A_r)^2+(A_r-A_s)^2(A_h-A_t)^2(A_u-A_r)^2$$

$$+ (A_r-A_h)^2(A_s-A_u)^2(A_t-A_r)^2+(A_r-A_h)^2(A_s-A_t)^2(A_u-A_r)^2+(A_r-A_h)^2(A_u-A_t)^2(A_s-A_r)^2](n_r-1)$$

$$+ [(A_t-A_r)^2(A_h-A_s)^2(A_u-A_t)^2+(A_t-A_s)^2(A_h-A_r)^2(A_u-A_t)^2+(A_t-A_r)^2(A_h-A_u)^2(A_s-A_t)^2$$

$$+ (A_t-A_h)^2(A_s-A_r)^2(A_u-A_t)^2+(A_t-A_h)^2(A_s-A_u)^2(A_r-A_t)^2+(A_t-A_h)^2(A_r-A_u)^2(A_s-A_t)^2](n_t-1)$$

$$+ [(A_u-A_r)^2(A_h-A_s)^2(A_t-A_u)^2+(A_u-A_t)^2(A_h-A_r)^2(A_s-A_u)^2+(A_u-A_s)^2(A_h-A_t)^2(A_r-A_u)^2$$

$$+ (A_u-A_h)^2(A_s-A_r)^2(A_t-A_u)^2+(A_u-A_h)^2(A_s-A_t)^2(A_r-A_u)^2+(A_u-A_h)^2(A_r-A_t)^2(A_s-A_u)^2]$$

$$\cdot (n_u-1) \Big\} n_h n_s n_r n_t n_u$$

$$+ 3!\Sigma_3 \quad (A_h-A_s)^2(A_h-A_r)^2(A_s-A_r)^2 n_h(n_h-1)n_s(n_s-1)n_r(n_r-1)$$

$$+ 3!\Sigma_6 \Big\{ (A_h-A_u)^2[(A_r-A_t)^2(A_v-A_s)^2+(A_r-A_v)^2(A_t-A_s)^2+(A_t-A_v)^2(A_r-A_s)^2]$$

$$+ (A_h-A_v)^2[(A_r-A_u)^2(A_t-A_s)^2+(A_t-A_u)^2(A_r-A_s)^2+(A_r-A_t)^2(A_u-A_s)^2]$$

$$+ (A_h-A_r)^2[(A_u-A_v)^2(A_t-A_s)^2+(A_t-A_u)^2(A_v-A_s)^2+(A_t-A_v)^2(A_u-A_s)^2]$$

$$+ (A_h-A_t)^2[(A_r-A_u)^2(A_v-A_s)^2+(A_r-A_v)^2(A_u-A_s)^2+(A_u-A_v)^2(A_r-A_s)^2]$$

$$+ (A_h-A_s)^2[(A_r-A_t)^2(A_u-A_v)^2+(A_r-A_u)^2(A_t-A_v)^2+(A_r-A_v)^2(A_t-A_u)^2]\Big\} n_h n_s n_t n_r n_u] \;. \quad \ldots\ldots \quad (66)$$

By virtue of equations (61) to (66) the third moment of T around zero in equation (60) can now be expressed in terms of the n_i. Substituting this expression of $\mu_3'(T)$, and equations (56) and (50) in (58) we obtain the third moment of T around the mean. Higher moments of T can be found by the above method but are not worked out in this thesis.

2. Asymptotic Distribution of T

Sections (A), (B) and (C) above give the moments of T for the situations: free sampling, sampling from a continuous distribution, and non-free sampling, respectively. Situation (C) reduces to (A) if n_i $(i = 1,2,\ldots k)$ approach infinity at the same rate as n. Moreover, formula (19) shows that for situations (A) and (B), the cumulants of T of any order are linear functions of n. We thus conclude that for any one of the situations (A), (B) or (C), the distribution of T tends to the normal form as n and all n_i $(i = 1,2,\ldots k)$ approach infinity at the same rate.

This conclusion is based on the fact that the distribution of the sum of a number, n, of dependent or independent variables, tends to the normal form as n approaches infinity, if the cumulants of the sum of the variates are linear functions of n. Its proof can be summarized as follows:

Let $U = \sum_{i=1}^{n} u_i$, and let

$k_1, k_2, k_3, \ldots k_r \ldots$

be the first, second, third, and r^{th} cumulants of U, respectively, with each k_i $(i = 1,2,\ldots)$ a linear function of n.

The cumulants of $\frac{U-E(U)}{\sigma(U)} = \frac{U-k_1}{k_2^{\frac{1}{2}}}$ then become

$$\frac{0}{k_2^{\frac{1}{2}}}, \quad \frac{k_2}{k_2}, \quad \frac{k_3}{k_2^{3/2}}, \ldots \quad \frac{k_r}{k_2^{r/2}}, \ldots$$

or, in terms of n, the cumulants of the standardized U become

$$0,\ 1,\ O\left(\frac{1}{n^{\frac{1}{2}}}\right), \ldots\ O\left(\frac{1}{n^{r/2-1}}\right), \ldots \quad . \tag{67}$$

As n tends to infinity, the cumulants in (67) approach those of the standardized normal variable, hence the moments of the standardized U coincide in the limit with the moments of the standardized normal variate. Applying the Inversion Theorem, we arrive at the conclusion that the distribution of U is asymptotically normal.

CHAPTER III

APPLICATIONS

The statistic T can be used in large samples for testing the randomness of a given sequence, or testing the homogeneity and randomness of two given samples, from

(a) a discrete population with infinite sampling and uniform probabilities,

(b) a discrete population with infinite sampling and non-uniform probabilities, and

(c) a normal population.

1. Test of Randomness of a Given Sequence

To perform this test, evaluate T from the sequence, and transform it into the standardized variate

$$Z = \frac{T - E(T)}{\sigma(T)} .$$

If the sequence comes from populations (a) or (b), then $E(T)$ and $\sigma^2(T)$ will be obtained by formulas (21) and (24) respectively, but if the sequence comes from population (c), $E(T)$ and $\sigma^2(T)$ will be obtained by formulas (46) and (47) respectively.

Since, under the null hypothesis of randomness, z is approximately normally distributed, the hypothesis is accepted at the 5 per cent level of significance if $|z| < 1.96$.

By way of illustration, this test has been performed on four random samples from discrete populations with uniform probabilities, three random samples from discrete populations with non-uniform probabilities and four random samples from normal populations. The results of the computations are shown in Table III. In each case, the hypothesis of randomness was accepted.

2. Test of Homogeneity and Randomness of Two Given Sequences

Given the sequences of observations, $\{x_i\}$ and $\{y_i\}$ $(i = 1,2,\ldots n)$, pool the two samples together by alternating x and y observations in their order of occurrence. This can be done in two ways: either by inserting each y_i $(i = 1,2,\ldots n-1)$ between x_i and x_{i+1}, the sequence ending with y_n, or by inserting each x_i $(i = 1,\ldots n-1)$ between y_i and y_{i+1}, the sequence ending with x_n. Thus the combined sequence is either

(1) $x_1\ y_1\ x_2\ y_2\ \ldots\ x_n y_n$

or (2) $y_1\ x_1\ y_2\ x_2\ \ldots\ y_n x_n$.

Considering the combined samples as sequences of $2n$ observations from a population of $\{x_i\}$ and $\{y_i\}$, let T_1 and T_2 be the values of the statistic T for samples (1) and (2) respectively. Further let the corresponding standardized variates be

$$z_1 = \frac{T_1-E(T_1)}{\sigma(T_1)} \quad \text{and} \quad z_2 = \frac{T_2-E(T_2)}{\sigma(T_2)} ,$$

where the quantities $E(T_1)$, $E(T_2)$, $\sigma(T_1)$ and $\sigma(T_2)$ are functions of second and fourth moments of the observations around their mean, and will be evaluated below.

Since under the null hypothesis of homogeneity and randomness, z_1 and z_2 are approximately normally distributed for sufficiently large n, the hypothesis is accepted at the 5 per cent level of significance if $|z_1|$ and $|z_2|$ are smaller than 1.96 simultaneously.

To evaluate the expected value and standard deviation of T_1 and T_2, we first find the population yielding the combined sample of 2n observations. This will be done for both the discrete and continuous cases.

In the discrete case, let the sequence $\{x_i\}$ xome from the population of elements $A_1, A_2, \ldots A_k$ with probabilities $p_1, p_2, \ldots p_k$ respectively, such that $\sum_{i=1}^{k} p_i=1$, and the sequence $\{y_i\}$ come from the population of elements $B_1, B_2, \ldots B_k$ with probabilities $q_1, q_2, \ldots q_k$ respectively, $\sum_{i=1}^{k} q_i=1$. Then the combined sequences (1) or (2) originate in the population $A_1, A_2, \ldots A_k$, $B_1, B_2, \ldots B_k$ with probabilities $\frac{p_1}{2}, \frac{p_2}{2}, \ldots \frac{p_k}{2}, \frac{q_1}{2}, \frac{q_2}{2}, \ldots \frac{q_k}{2}$, where, if $A_i = B_j$ $(i,j= 1,2,\ldots k)$, only A_i or B_j will be included in the population with probability $\frac{p_i+q_j}{2}$. Denoting an observation in the combined sample by u,

$$E(u) = \sum_{i=1}^{k}[A_i \frac{p_i}{2} + B_i \frac{q_i}{2}]; \quad V(u)= \sum_{i=1}^{k}[A_i^2 \frac{p_i}{2} + B_i^2 \frac{q_i}{2}] - [\sum_{i=1}^{k}(A_i\frac{p_i}{2} + B_i\frac{q_i}{2})]^2$$

$$\mu_4(u) = \sum_{i=1}^{k}[A_i^4 \frac{p_i}{2} + B_i^4 \frac{q_i}{2}] - 4\sum_{i=1}^{k}[A_i\frac{p_i}{2} + B_i \frac{q_i}{2}] \sum_{i=1}^{k}[A_i^3 \frac{p_i}{2} + B_i^3 \frac{q_i}{2}]$$

$$+ 6[\sum_{i=1}^{k}(A_i \frac{p_i}{2} + B_i \frac{q_i}{2})]^2 \sum_{i=1}^{k}[A_i^2 \frac{p_i}{2} + B_i^2 \frac{q_i}{2}] -3[\sum_{i=1}^{k}(A_i \frac{p_i}{2} + B_i \frac{q_i}{2})]^4.$$

On the other hand, if the sequence $\{x_i\}$ comes from a normal population with density function

$$f_1(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2\sigma_1^2}(x-m_1)^2},$$

and the sequence $\{y_i\}$ comes from a normal population with density function

$$f_2(y) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2\sigma_2^2}(y-m_2)^2},$$

then the combined sample comes from a population with density function $\frac{f_1+f_2}{2}$. Hence,

$$E(u) = \int u \frac{f_1+f_2}{2}\, du = \frac{m_1+m_2}{2} \tag{68}$$

$$V(u) = E(u^2) - E^2(u) = \frac{\sigma_1^2+\sigma_2^2}{2} + \left(\frac{m_1-m_2}{2}\right)^2 \tag{69}$$

and $$\mu_4(u) = E(u^4) - 4E(u)\, E(u^3) + 6[E(u)]^2 E(u^2) - 3[E(u)]^4 . \tag{70}$$

Evaluating the expectations in (70) in terms of the means and variances of the observations x_i and y_i, and simplifying, we arrive at

$$\mu_4(u) = \frac{3}{2}(\sigma_1^4 + \sigma_2^4) + \frac{3}{4}(\sigma_1^2+\sigma_2^2)(m_1-m_2)^2 + \frac{1}{16}(m_1+m_2)^4 - \frac{1}{2}\, m_1 m_2 (m_1^2 + m_2^2) . \tag{71}$$

We now evaluate the expected value and variance of the statistic T from the combined population with mean m, variance $\mu_2(u)$, and fourth moment $\mu_4(u)$.

Let $T_1 = \sum_{i=1}^{n} X_i + \sum_{i=1}^{n-1} Y_i$, where $X_i=(x_i-y_i)^2$ and $Y_i=(y_i-x_{i+1})^2$,

then $$T_2 = \sum_{i=1}^{n-1} X_i + \sum_{i=1}^{n} Y_i.$$

$$E(T_1) = E[\sum_{i=1}^{n} X_i + \sum_{i=1}^{n-1} Y_i] = n\ E(X_i) + (n-1)\ E(Y_i).$$

Since x_i and y_i $(i = 1,2,\ldots n)$ are observations of the random variable u,

$$E(X_i)=E(Y_i)=E(y_i-x_{i+1})^2=E[(y_i-m)-(x_{i+1}-m)]^2=2\mu_2(u).$$

Hence $$E(T_2) = E(T_1) = 2(2n-1)\mu_2(u). \qquad (72)$$

$$V(T_1) = V[\sum_{i=1}^{n} X_i+\sum_{i=1}^{n-1} Y_i] = E[\sum_{i=1}^{n} X_i+\sum_{i=1}^{n-1} Y_i-2(2n-1)\mu_2(u)]^2$$

$$= E\left\{\sum_{i=1}^{n}[X_i-2\mu_2(u)] + \sum_{i=1}^{n-1}[Y_i-2\mu_2(u)]\right\}^2.$$

Let $X_i-2\mu_2(u) = z_i$ and $Y_i-2\mu_2(u) = w_i$, $(i = 1,2,\ldots n)$,

then $$V(T_1) = E[\sum_{i=1}^{n} z_i+\sum_{i=1}^{n-1} w_i]^2 = E[\sum_{i=1}^{n} z_i^2] + E[\sum_{i=1}^{n-1} w_i^2] + 2E[\sum_{i=1}^{n} z_i][\sum_{i=1}^{n-1} w_i].$$

Since z_i and w_i, and w_i and z_{i+1}, $(i = 1,2,\ldots n-1)$, have non zero covariance,

$$E[\sum_{i=1}^{n} z_i][\sum_{i=1}^{n-1} w_i] = E\sum_{i=1}^{n-1} z_i w_i + E\sum_{i=1}^{n-1} z_{i+1}w_i.$$

Hence,

$$V(T_1) = n\ E(z_i^2)+(n-1)E(w_i^2)+2(n-1)E(z_iw_i)+2(n-1)E(z_{i+1}w_i)$$

$$= n\ E[X_i-2\mu_2(u)]^2+(n-1)E[Y_i-2\mu_2(u)]+2(n-1)E[X_i-2\mu_2(u)][Y_i-2\mu_2(u)]$$

$$+\ 2(n-1)E[X_{i+1}-2\mu_2(u)][Y_i-2\mu_2(u)]$$

$$= n\ E(X_i^2)+(n-1)E(Y_i^2)+2(n-1)E(X_iY_i)+2(n-1)E(X_{i+1}Y_i)$$

$$-\ 4(6n-5)\mu_2^2(u). \tag{73}$$

Evaluating the expectations in (73), we obtain

$$E(Y_i^2) = E(y_i-x_{i+1})^4 = E[(y_i-m)-(x_{i+1}-m)]^4 = 2\mu_4(u)+6\mu_2^2(u) \tag{74}$$

$$E(X_{i+1}Y_i) = E(x_{i+1}-y_{i+1})^2(y_i-x_{i+1})^2 = E[(x_{i+1}-m)-(y_{i+1}-m)]^2[(y_i-m)-(x_{i+1}-m)]^2$$

$$= 3\mu_2^2(u) + \mu_4(u). \tag{75}$$

Since x_i, y_i $(i = 1,2,\ldots n)$ are observations of the random variable u,

$$E(X_i^2) = E(Y_i^2) \quad \text{and} \quad E(X_{i+1}Y_i) = E(X_iY_i)\ . \tag{76}$$

By virtue of formulas (74), (75) and (76), the variance of T_1 or T_2 in equation (73) becomes

$$V(T_2) = V(T_1) = 2(4n-3)\mu_4(u) + 2\mu_2^2(u)\quad . \tag{77}$$

We can now test the homogeneity and randomness of two sequences from discrete populations by using the standardized tests z_1 and z_2, where $E(T_1)$ and $\sigma(T_1)$ are given by formulas (72) and (77) respectively. If the sequences come from normal populations with different means and variances, $\mu_2(u)$ and $\mu_4(u)$ are given by (69) and (71) respectively, while for the case where $m_1 = m_2 = m$, formulas (69) and (71) reduce to

$$V(u) = \frac{\sigma_1^2 + \sigma_2^2}{2} \tag{78}$$

and

$$\mu_4(u) = \frac{3}{2}(\sigma_1^4 + \sigma_2^4). \tag{79}$$

By virtue of formulas (78) and (79), $E(T_1)$ and $V(T_1)$ in equations (72) and (77) become in this special case

$$E(T_1) = (2n-1)(\sigma_1^2 + \sigma_2^2) \tag{80}$$

$$V(T_1) = 3(4n-3)(\sigma_1^4 + \sigma_2^4) + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)^2. \tag{81}$$

If further, $\sigma_1^2 = \sigma_2^2 = \sigma^2$, formulas (69) and (71) give the variance and fourth moment of a normal variable with mean m and variance σ^2. For this case $E(T_1)$ and $V(T_1)$ become

$$E(T_1) = 2(2n-1)\sigma^2 \tag{82}$$

$$V(T_1) = 8(3n-2)\sigma^4. \tag{83}$$

The test of homogeneity and randomness is illustrated with reference to the following 3 situations in the populations (a), (b) and (c):

1. The two sequences come from the same population.

2. The two sequences come from different populations with equal means.

3. The two sequences come from different populations with unequal means.

The results of the computations are shown in Table IV. The following notation is used: a_i denotes the i^{th} discrete population $(i = 1, 2, 3)$, with uniform probabilities, and a_{is} denotes the s^{th} sample $(s = 1$ or 1 and $2)$, from population a_i. b_j denotes the j^{th} discrete population $(j = 1, 2)$, with non-uniform probabilities, and b_{jr} is the r^{th} sample $(r = 1$ or 1 and $2)$, from population b_j. c_k denotes the k^{th} normal population $(k = 1, 2, 3)$, and c_{kh} is the h^{th} sample $(h = 1$ or 1 and $2)$ drawn from population c_k.

For situation (1), the hypothesis is accepted for samples a_{11} and a_{12}, but it is rejected for samples b_{11} and b_{12}, and c_{22} and c_{21}. The latter results are explained by the fact that the standardized deviate z, in Table III, is 1.482 and -0.913 for b_{11} and b_{12} respectively. This implies that sample b_{12} has an ascending or descending trend, since T is smaller than its expected value. At the same time b_{11} has a maximum-minimum trend which results in T exceeding its expected value. The combination of the two samples b_{11} and b_{12} will therefore give a poor random sample which is detected by test 2. The situation is the same for samples c_{22} and c_{21}. The combined test of randomness, therefore seems to be more sensitive than the separate tests of randomness.

For situation (2), the hypothesis is accepted for both pairs of samples a_{21}, a_{12} and c_{21}, c_{11}. These results are as expected, because all four samples a_{21}, a_{12}, c_{21} and c_{11} are random; moreover, the difference between variances in each pair of samples are not significant at the 5 per cent level of significance.

For situation (3), the hypothesis is accepted for samples a_{12} and a_{31}, and is rejected for both pairs of samples b_{11} and b_{21}, and c_{31} and c_{21} at the 5 per cent level of significance. The results of the tests are as expected, because for samples a_{12} and a_{31} there are no significant differences between their variances or their means at the 1 per cent level of significance. On the other hand, there is a significant difference between the means of samples b_{11} and b_{21} at the 5 per cent level of significance, and there are significant differences between the variances and the means of samples c_{31} and c_{21} at the 1 per cent level of significance.

Due to lack of time, the powers of tests 1 and 2 were not studied. We hope to study them in the future and improve the tests by taking sums of squares of differences of all possible pairs of observations considered according to the order of occurrence in moving blocks of r contiguous observations $(r = 3, 4, \ldots k)$. The extension of the test of homogeneity and randomness to more than two sequences may be useful. For this case the introduction of a unique test would be desirable. A unique test for the case of two sequences combining the tests based on z_1 and z_2, was suggested by P.V.K. Iyer(1). The test would be based on the statistic

$$T = \sum_{i=1}^{n} X_i + \sum_{i=1}^{n-1} Y_i + \sum_{i=1}^{n-1} z_i,$$

where $X_i = (x_i - y_i)^2$, $Y_i = (y_i - x_{i+1})^2$ and $z_i = (x_i - y_{i+1})^2$. This line, however, has not been followed yet.

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(1) Personal communication.

TABLE I

DESCRIPTION OF POPULATIONS

FROM WHICH THE SAMPLES ARE CHOSEN

	Population					μ_1'	μ_2	μ_4
a_1	$A_1=3$ $p_1=.2$	$A_2=6$ $p_2=.2$	$A_3=9$ $p_3=.2$	$A_4=4$ $p_4=.2$	$A_5=7$ $p_5=.2$	5.8	4.56	35.79
a_2	$A_1=9$ $p_1=.2$	$A_2=6$ $p_2=.2$	$A_3=4$ $p_3=.2$	$A_4=2$ $p_4=.2$	$A_5=8$ $p_5=.2$	5.8	6.56	69.47
a_3	$A_1=2$ $p_1=.2$	$A_2=3$ $p_2=.2$	$A_3=4$ $p_3=.2$	$A_4=6$ $p_4=.2$	$A_5=8$ $p_5=.2$	4.6	4.64	37.97
b_1	$A_1=5$ $p_1=.15$	$A_2=7$ $p_2=.15$	$A_3=8$ $p_3=.2$	$A_4=6$ $p_4=.4$	$A_5=9$ $p_5=.1$	6.7	1.51	4.75
b_2	$A_1=2$ $p_1=.2$	$A_2=3$ $p_2=.3$	$A_3=4$ $p_3=.15$	$A_4=5$ $p_4=.10$	$A_5=6$ $p_5=.25$	3.9	2.19	7.81
c_1	Normal					10	2.25	
c_2	Normal					10	4	
c_3	Normal					15	9	

a_i denotes a discrete population (i=1,2,3) with uniform probabilities.

b_j denotes a discrete population (j=1,2) with non-uniform probabilities.

c_r denotes a normal population (r=1,2,3).

TABLE II

DESCRIPTION OF RANDOM SAMPLES

DERIVED FROM THE POPULATIONS LISTED IN TABLE I

a_{11}	a_{12}	a_{21}	a_{31}	b_{11}	b_{12}	b_{21}
6	4	4	2	6	6	2
4	9	2	6	9	6	4
6	9	4	4	9	5	2
4	4	9	8	8	6	3
4	6	2	4	5	6	5
4	6	6	4	6	8	2
9	9	6	2	5	6	3
7	9	2	2	8	6	3
9	7	6	6	8	5	5
9	9	9	3	9	5	3
4	7	4	6	6	7	2
3	7	4	4	5	5	2
6	7	2	6	6	9	3
6	3	4	4	5	6	5
6	4	8	6	6	6	3
6	7	9	8	8	5	2
9	4	8	6	9	6	3
6	9	2	4	5	6	6
3	7	6	2	5	6	4
7	4	9	4	8	7	6
9	9	9	4	6	6	6
6	3	4	4	7	6	3
7	6	9	8	6	5	5
4	9	8	4	8	6	2
4	4	8	8	8	7	3
4	6	2	4	7	6	6
4	3	6	6	6	6	2
7	3	4	8	9	6	3
3	7	4	6	6	6	5
3	9	4	2	8	8	6
6	7	4	4	7	6	6
3	7	8	8	7	6	3
7	3	2	8	5	5	6
9	6	4	8	8	8	4
6	6	8	4	5	8	6
4	6	6	4	8	6	4
9	3	8	2	9	5	3
7	9	4	4	6	9	2
7	7	6	8	8	8	5
7	4	4	6	6	9	2

TABLE II (Continued)

c_{11}	c_{21}	c_{22}	c_{31}
9.751	8.029	8.659	16.46
11.42	9.132	8.262	17.48
10.75	7.180	10.61	12.73
9.549	9.709	8.841	17.57
9.530	11.96	13.15	18.71
11.53	10.72	8.082	8.736
8.918	9.173	9.323	10.66
9.165	8.133	12.44	14.22
10.29	14.51	9.046	13.67
8.293	7.410	6.220	10.15
11.64	9.272	10.68	18.70
10.96	10.66	10.83	12.36
10.40	11.22	9.466	10.84
10.73	10.02	8.489	13.15
10.76	9.049	6.826	15.55
9.711	11.58	13.58	21.64
10.32	11.06	8.500	14.21
12.42	8.586	12.18	13.98
10.62	9.230	11.37	16.90
11.94	11.30	15.27	21.26
11.08	10.10	8.624	16.77
9.354	8.225	6.342	15.47
9.219	9.430	9.447	16.41
12.11	11.70	7.884	11.79
9.358	8.049	8.537	14.93
10.14	10.69	10.91	17.35
9.847	10.85	11.98	14.02
10.04	9.901	7.074	11.80
6.670	10.67	10.57	15.64
9.930	12.22	9.949	11.23

TABLE III

EXAMINATION FOR RANDOMNESS OF THE SAMPLES

Sample	n	E(T)	σ(T)	Observed T	$\frac{T - E}{\sigma}$	Remarks
a_{11}	40	355.68	74.52	251	-1.405	Sequence a_{11} is random with a descending or ascending trend.
a_{12}	40	355.68	74.52	372	0.219	Sequence a_{12} is random.
a_{21}	40	511.68	103.84	462	-0.478	Sequence a_{21} is random.
a_{31}	40	361.92	76.75	282	-1.041	Sequence a_{31} is random with a descending or ascending trend.
b_{11}	40	117.78	27.13	158	1.482	Sequence b_{11} is random with a Max--Min trend.
b_{12}	40	117.78	27.13	93	-0.913	Sequence b_{12} is random with ascending or descending trend.
b_{21}	40	170.82	34.83	164	-0.196	Sequence b_{21} is random.
c_{11}	30	130.50	41.72	86.14	-1.063	Sequence c_{11} is random with a descending or ascending trend.
c_{21}	30	232.00	74.20	173.95	-0.782	Sequence c_{21} is random.
c_{22}	30	232.00	74.20	315.48	1.125	Sequence c_{22} is random with a Max-Min trend.
c_{31}	30	522.00	166.93	533.37	0.0681	Sequence c_{31} is random.

TABLE IV

SIGNIFICANCE OF DIFFERENCES BETWEEN THE VARIOUS SAMPLES

Sample	n	E(T)	σ(T)	Observed T_1	$\frac{T_1 - E}{\sigma}$	Observed T_2	$\frac{T_2 - E}{\sigma}$	Remarks
Same pop. $m_1 = m_2$ a_{11}, a_{12}	80	729.60	106.21	720	-0.0904	599	-1.230	The two sequences are random and come from same population.
Diff. pop. $m_1 = m_2$ a_{21}, a_{12}	80	878.48	128.79	910	0.245	902	0.183	The two sequences are random. Their variances are not significantly different.
Diff. pop. $m_1 \neq m_2$ a_{12}, a_{31}	80	783.68	117.38	934	1.281	954	1.451	The two sequences are random. Their variances and means are not significantly different.
Same pop. $m_1 = m_2$ b_{11}, b_{12}	80	238.58	38.68	335	2.493	320	2.105	Sequences with ascending or descending trend, and max-min trend will not be random.
Diff. pop. $m_1 \neq m_2$ b_{11}, b_{21}	80	601.98	99.87	1138	5.367	1120	5.187	The means of the two samples are significant.
Same pop. $m_1 = m_2$ c_{22}, c_{21}	60	632.00	122.90	445.10	-1.521	304.39	2.666	Sequences with ascending trend, and max-min trend will not be random.
Diff. pop. $m_1 = m_2$ c_{21}, c_{11}	60	368.16	86.09	263.34	-1.218	239.30	-1.497	Two sequences are random, and their variances are not significantly different.
Diff. pop. $m_1 \neq m_2$ c_{31}, c_{21}	60	1504.50	317.10	2258.74	2.379	2017.24	1.617	The variances of the two samples are highly significant.

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