AMERICAN UNIVERSITY OF BEIRUT

MATHEMATICAL ANALYSIS AND NUMERICAL SIMULATION OF THE BIDOMAIN MODEL USED IN CARDIAC ELECTROPHYSIOLOGY

by

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AN ABSTRACT OF THE THESIS OF

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Title: Mathematical Analysis and Numerical Simulation of the Bidomain Model Used in Cardiac Electrophysiology

The bidomain model describes the cardiac electrical activity. It has been considerably used in studies investigating cardiac arrythmia such as ventricular fibrillation. Recently, the well-posedness of the model was studied by Bourgault et al.. Their analysis was based on a formulation of the problem as a system of **coupled parabolic and elliptic PDEs** for two potentials and **ODEs** representing the ionic activity. The main idea was to reformulate the parabolic and elliptic PDEs into a single parabolic PDE by the introduction of a bidomain operator. A proof of **existence, uniqueness and regularity of local in time strong solution** was obtained by a **semigroup** approach. This approach applies to fairly general ionic models. The bidomain model was then reformulated as a **parabolic variational problem**, through the introduction of a bidomain bilinear form. A proof of **existence and uniqueness of a global in time weak solution** was obtained using a **compactness** argument, this time for an ionic model reading as a single ODE but including polynomial nonlinearities. The hypotheses behind the existence of the global weak solution were verified for three commonly used ionic models namely the FitzHugh-Nagumo, Aliev-Panfilov and McCulloch models. In this thesis, we prove, using Galerkin approximations and classic regularity results on elliptic Neumann problems, and under some assumptions of regularity on the initial data and the source terms, that the weak solution is actually **uniformly bounded and regular** enough. This means that the global in time regular weak solution is indeed a **global in time strong solution**.
Moreover, we present numerical simulations of electrical wave propagation done using finite differences and finite elements in one and two dimensional spaces. In particular, we generate, using the monodomain and bidomain models, spiral waves that model electrical disorder in cardiac activity.

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Notations

Ω is an open subset of $\mathbb{R}^n$.

- $C(\Omega) = \{u : \Omega \to \mathbb{R} | u \ \text{ continuous}\}$

- $C(\bar{\Omega}) = \{u \in C(\Omega) | u$ is uniformly continuous on bounded subsets of $\Omega\}$

- $C^k(\Omega) = \{u : \Omega \to \mathbb{R} | u$ is k-times continuously differentiable$\}$

- $C^k(\bar{\Omega}) = \{u \in C^k(\Omega) | D^\alpha u$ is uniformly continuous on bounded subsets of Ω,

  for all $|\alpha| \leq k\}$.

- $C^\infty(\Omega) = \{u : \Omega \to \mathbb{R} | u$ is infinitely differentiable$\} = \cap_{k=0}^{\infty} C^k(\Omega)$.

- $C^\infty(\bar{\Omega}) = \cap_{k=0}^{\infty} C^k(\bar{\Omega})$.

- $\mathcal{D}(\Omega) = \{u \in C^\infty(\Omega) | \text{supp } u$ is compact$\}$.

- $L^p(\Omega) = \{u : \Omega \to \mathbb{R} | u$ is Lebesgue measurable, $\|u\|_{L^p(\Omega)} < \infty\}$, where

  $$\|u\|_{L^p(\Omega)} = \Big(\int_\Omega |u|^p \, dx \Big)^{1/p} \quad (1 \leq p < \infty)\}.$$

  $L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} | u$ is Lebesgue measurable, $\|u\|_{L^\infty(\Omega)} < \infty$, where

  $$\|u\|_{L^\infty(\Omega)} = \text{ess sup}_\Omega |u|\}.$$

  $L^p_{loc}(\Omega) = \{u : \Omega \to \mathbb{R} | u \in L^p(V) \ \text{ for each } \ V \subset \bar{V} \subset U\}$.

Introduction

The "bidomain" model has been recently used in advanced electrocardiology studies investigating the electrical behavior of the anisotropic cardiac tissue. Despite its discrete structure, the tissue is represented at a macroscopic level by a continuous model [5]. The proof of the well-posedness of the model was presented mainly in two references: Colli-Franzone and Savaré's paper [5] and Veneroni's report [17]. However these approaches were restricted to particular cases of ionic models that do not include the widely used Aliev-Panfilov [13] and McCulloch [15] models. An interesting result of existence and uniqueness of a global strong solution of a simplified version of the bidomain model called the monodomain model has been obtained by Coudière et al. [6] using a famous technique based on invariant regions.

The aim of our thesis is to study the results published in a very recent paper of Bourgault et al. [3] where the existence and uniqueness of the solution of the bidomain model have been proven under some assumptions on the ionic models which are satisfied by both Aliev-Panfilov and McCulloch models. Indeed, the proof has been based on a formulation of the model as a system of coupled parabolic and elliptic PDEs for two potentials and ODEs representing the ionic activity. The parabolic and elliptic PDEs are reformulated into a single parabolic PDE by the introduction of a bidomain operator, which has been properly defined and analyzed. Then a proof of **existence, uniqueness and regularity of a local in time strong solution** is presented based on a semigroup approach. The strength of this proof is that it applies to general ionic models. Next, the problem is formulated in a variational form and a proof of **existence and uniqueness of global in time weak solution** is obtained using a Faedo-Galerkin technique, some energy-like estimates and a compactness result.

Although there are numerical experiments and intuitions that the solutions are bounded functions, the problem of regularity of the solutions has not been addressed in Bourgault et al. paper [3]. In this thesis, we prove, using Galerkin approximations and classic regularity results on elliptic Neumann problems, and under some assumptions of regularity on the initial data and the source terms, that the solution is actually uniformly bounded and regular enough $(u(\cdot, t) \in H^2(\Omega), \quad \forall t)$ to get a **global in time strong solution**.

The thesis is sketched as follows:

- Chapter 1 presents an overview of cardiac electrophysiology and modeling. It also includes the derivation of the bidomain model and some ionic models.

- Chapter 2 presents the mathematical tools used in the different proofs of Chapters 3 and 4.
- Existence and uniqueness of local in time strong solution and global in time weak solution, along with some examples, are presented in Chapter 3. This chapter is mainly based on Bourgault et al. paper [3]. However, a proof of stability of the solutions with respect to the data is added.
- Chapter 4 presents our contribution in proving regularity of the weak solution that leads to the existence and uniqueness of the global strong solution.
- Chapter 5 presents numerical simulations of electrical wave propagation done using finite differences and finite elements in one and two dimensional spaces. Also, we simulate the generation of spiral waves that lead to ventricular fibrillation.

Chapter 1

Physiology and Mathematical Modeling

1.1 Introduction

Sudden cardiac death is the number one health problem in the developed countries, as announced by the World Health Organization report in 1985. Most of these deaths are caused by electrical activity disorders, visible through the mechanical deficiency of the heart. This organ is divided into two halves (left and right) by the interventricular septal wall. It consists of four major chambers (two in each half) which are the left and right ventricles and the left and right atria [16].

Mechanical contraction of the heart is caused by the electrical activation of myocardial cells. The beats are initiated by the heart itself on a regular basis. In other words, the heart is self contained and can continue to beat even after

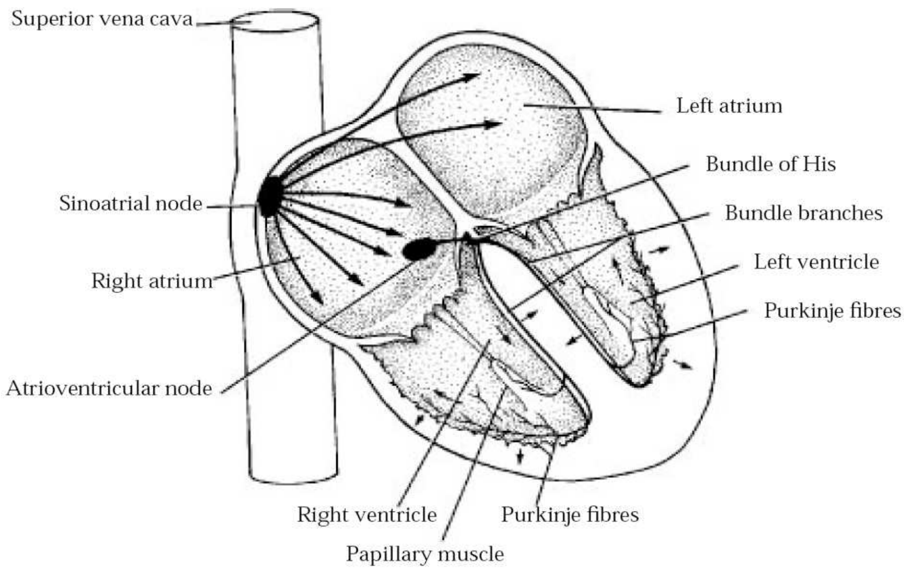

Figure 1.1: Schematic of the activation sequence (Berne and Levy, 1988).

being removed from the body. Actually, the initiation of electrical activity is accomplished by the pacemaker cells which exist in various locations throughout the heart. The sinoatrial (SA) node contains the pacemaker cells with fastest rate of electrical activity. Hence they control the activity of the entire heart. Action potentials, generated in the SA node, propagate from cell to cell through firstly the right atrium then closely to the left atrium, until they reach the atrioventricular (AV) node. The slower conduction rate in the AV node gives enough time for the atria to contract and pump blood into the ventricles. From the AV node the electrical propagation continues through the bundle of His which divides into left and right bundle branches. The branches continue to subdivide into a complex network of Purkinje fibers spreading through the ventricular myocardium. The bundle and the Purkinje fibers are fast conducting so that the entire myocardium is excited simultaneously [16].

At the level of the cell, the plasma membrane is viewed as a capacitor. It separates the intracellular and the extracellular electrolytic solutions and serves as a permeability barrier that allows the cell to maintain an interior composition different from the composition of the extracellular fluid. The potential difference across the membrane is known as the *transmembrane potential* V_m. Changes in this quantity are given by

$$\frac{dV_m}{dt} = -\frac{\Sigma I_{ion}}{C_m}$$

where C_m is the *membrane capacitance*, and the I_{ion} are the various ionic currents flowing across the membrane. The latter are mainly caused by the flow of sodium (Na^+), potassium (K^+), and calcium (Ca^{2+}) through individual ion channels in the membrane. These channels have been profoundly studied by molecular biologists and mathematical models have been formulated. The first description of ion channels was developed by Hodgkin and Huxley (1952) for the squid axon [16].

In the resting state, the transmembrane potential is about $-80mv$. This is the phase during which the heart is passively filling with blood. Once activated, the cell membrane becomes rapidly depolarized (phase 0) due to the opening of sodium channels and the resulting inward sodium current. Then a short period of repolarization (phase 1), largely due to the closure of sodium channels, is followed by a plateau (phase 2) which in turn is maintained by the inward calcium current. Finally, the potential decreases again (repolarization - phase 3) until the resting state is achieved. This sequence of changes in potential from the activation point

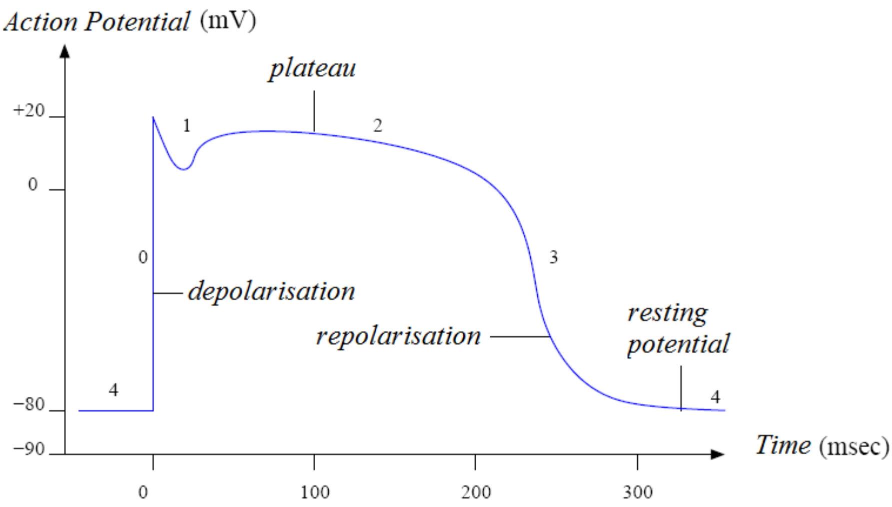

Figure 1.2: Stages of the ventricular action potential (Katz, 1992).

to the resting state constitutes the *action potential*. Clearly, the action potential is due to the superposition of many ionic currents [16].

1.2 Cardiac activation modeling

More than a 100 years ago, the electrical activity in the torso was directly associated to the heart beat [6]. The entire cardiac electrical state was first represented as a dipole in an infinite homogeneous medium. This simple representation does not model the propagation of an action potential, but it describes the integrated effect of cardiac electrical activity by interpreting voltage-time diagrams (which show the difference in potential between various extremities as a function of time). And these are still in use as the basis of standard electrocardiogram (ECG) analysis. Later models, known as empirical models describe localized electrical activity by discretizing the heart into a large number of *cells* (not the same as the biological cells). Each cell has a number of properties which describe the conductivity, fiber direction, the transmembrane potential, and other static or dynamic properties. The activation process at each point is modeled, and the state of the heart can be defined at a given time [16]. But these models are no longer appropriate because they disregard cellular processes.

As experimental techniques and computer power have improved, cellular electrical activity has been better understood and more detailed models have been made computationally tractable. However, due to computing resources limitations, a particular level of detail has to be considered sufficiently accurate. The subsequent section discusses the development of the *Bidomain model*, which flexibly includes any given model of the ionic processes.

1.3 The bidomain model: Mathematical derivation

The bidomain model describes current flow through the cell membrane in a volume-averaged approach. It averages the electrical properties over a length scale which is appropriately chosen to ignore the effect of cell junctions on propagation [16]. Since the resistance of these junctions is comparable to that of the intracellular medium, the cardiac tissue can be considered as a continuum, see [5]. In this framework, two domains are defined: the *intracellular domain* (given the subscript "i") is the region inside the cell, and the *extracellular domain* (given the subscript "e") is the region between the cells. These two domains coexist at every point in space, i.e. the properties and the state of the tissue have separate components related to each domain (e.g. conductivity in each of the domains).

At a point, let ϕ_i and ϕ_e represent the volume-averaged macroscopic potentials in the intracellular and extracellular spaces respectively. The transmembrane potential V_m is the potential difference across the cell membrane

$$V_m = \phi_i - \phi_e. \tag{1.1}$$

All the values are measured in mV.

There is a local material coordinate system defined at every point with axes aligned with each of the local fiber, cross-fiber (sheet), and cross-sheet directions. These material axes are defined to be orthogonal, and they are used to determine the principal directions of propagation [16]. In such a coordinate system, the conductivity tensors are diagonal. As previously stated, there are two conductivity tensors corresponding to the intracellular and extracellular domains which are Λ_i and Λ_e respectively, with units $(\Omega m)^{-1}$. The conductivity tensors in the global coordinates are denoted σ_i, and σ_e respectively. The latter have the same eigenbasis $Q(x) = (q_1(x), ..., q_d(x))$, d =1,2,or 3 in $\mathbb{R}^d$, which reflects the organization of the muscle in fibers, [3]. Therefore we have

$$\sigma_{i,e} = Q(x)\Lambda_{i,e}Q(x)^T$$

where

$$\Lambda_{i,e} = diag(\lambda^1_{i,e}(x), ..., \lambda^d_{i,e}(x)).$$

The intracellular and extracellular current densities J_i and J_e (with units Am^{-1})

are derived by Ohm's law as

$$J_i = -\Lambda_i \nabla \phi_i \tag{1.2}$$

$$J_e = -\Lambda_e \nabla \phi_e \tag{1.3}$$

The change in current density between the two domains is equal and opposite, since the current leaving one domain crosses the membrane to the other domain. Hence

$$-\nabla \cdot J_i = A_m I_m - I_s = \nabla \cdot J_e \tag{1.4}$$

where A_m (unit m^{-1}) is the surface-to-volume ratio of the cell membrane, I_m (unit Am^{-2}) is the transmembrane current density per unit area and I_s (unit Am^{-3}) is an externally imposed source current per unit volume. Ignoring I_s, we get

$$\nabla \cdot (\Lambda_i \nabla \phi_i) = A_m I_m \tag{1.5}$$

$$\nabla \cdot (\Lambda_e \nabla \phi_e) = -A_m I_m \tag{1.6}$$

From equations (1.5) and (1.6), and by using equation (1.1), we write

$$\nabla \cdot (\Lambda_i \phi_i) = -\nabla \cdot (\Lambda_e \nabla \phi_e) \tag{1.7}$$

$$\nabla \cdot (\Lambda_i \nabla \phi_i - \Lambda_i \nabla \phi_e) = -\nabla \cdot (\Lambda_e \nabla \phi_e) - \nabla \cdot (\Lambda_i \nabla \phi_e). \tag{1.8}$$

We get the following conservation of current equation

$$\nabla \cdot (\Lambda_i \nabla V_m) = -\nabla \cdot ((\Lambda_i + \Lambda_e) \nabla \phi_e) \tag{1.9}$$

The transmembrane current I_m is given by the sum of a capacitive current due to the change in transmembrane potential and an ionic current governed by ionic models for cardiac tissue thus

$$I_m = C_m \frac{\partial V_m}{\partial t} + I_{ion} \tag{1.10}$$

where C_m is the transmembrane capacitance per unit area (unit $\mu F.mm^{-2}$), and I_{ion} is the nonlinear function representing the transmembrane ionic currents (unit Am^{-2}). There are many possibilities to define I_{ion} as will be seen in the next section.

Now, multiply (1.10) by A_m and use (1.5) to get

$$\nabla \cdot (\Lambda_i \nabla \phi_i) = A_m (C_m \frac{\partial V_m}{\partial t} + I_{ion}) - I_s \tag{1.11}$$

and use (1.1):

$$\nabla \cdot (\Lambda_i \nabla V_m) + \nabla \cdot (\Lambda_i \nabla \phi_e) = A_m (C_m \frac{\partial V_m}{\partial t} + I_{ion}) - I_s. \tag{1.12}$$

Equations (1.9) and (1.12) are the bidomain equations.

If the extracellular space is assumed to be highly conducting (i.e. Λ_e is effectively infinite) or if $\Lambda_i = c\Lambda_e$ where c is a constant (i.e. the domains are equally anisotropic), we get from (1.9) and (1.12) a single equation known as the *monodomain equation*:

$$\nabla \cdot (\Lambda \nabla V_m) = A_m (C_m \frac{\partial V_m}{\partial t} + I_{ion}) - I_s$$

where $\Lambda = \Lambda_i$.

Since the intracellular domain is self-contained, no flux boundary condition is assumed at all points where it is required i.e.

$$\frac{\partial \phi_i}{\partial n} = 0$$

where n is the outward unit normal to the domain boundary [16].

1.4 Ionic current models

The framework of the bidomain model is based on the existence of mathematical models describing the flow of ionic currents across the membrane. Ideally, these models would describe each of the individual ionic currents whose sum defines the ionic current I_{ion}.

There are two main approaches to the construction of an ionic current model. The first is to build a *biophysical model* which attempts to describe specific actions within the cell membrane. Such exact models are derived either by fitting the parameters of an equation to match experimental data or by defining equations that were confirmed by later experiments. Moreover, they are based on the cell membrane formulation developed by Hodgkin and Huxley for nerve fibers [16].

The second approach consists of producing simpler models which replicate certain key features of activation and recovery. They can be used in large problems because they are typically small and fast to solve, although they are less flexible in their response to variations in cellular properties such as concentrations or cell size [16].

1.4.1 Beeler-Reuter model

As an example of a biophysical model, we consider the Beeler-Reuter model. It was developed in 1977 to describe the mammalian ventricular action potential. It considers only four ionic currents: a sodium inward current (i_{Na}), a calcium-based inward current (i_s), a background potassium current (i_{K_1}), and a plateau potassium current (i_{χ_1}) [1]. The result is the following ionic current:

$$I_{ion} = i_{Na} + i_s + i_{K_1} + i_{\chi_1}.$$

There are two main problems with the biophysical approach. Firstly, the ionic processes are not fully understood. Secondly, the models produced are large and complex, making prohibitive the computational time required to solve the resulting system of equations [16].

1.4.2 FitzHugh-Nagumo model

One of the most popular simple models of activation-recovery was developed by FitzHugh, Nagumo and Bronhoffer and it has become known as the FHN model. In this model, the transmembrane potential is normalized using the relation

$$u = \frac{V_m - V_{rest}}{V_{plateau} - V_{rest}}$$

where u is the normalized potential (this potential will be adopted in the rest of the report), V_{rest} is the potential at rest ($-80mv$), $V_{plateau}$ is the plateau potential and V_m is the transmembrane potential.

The ionic current is given by

$$I_{ion(FHN)} = c_1 u(u - \alpha)(1 - u) - c_2 w$$

where c_1 and c_2 are the excitation rate and excitation decay constants respectively, α is the activation threshold value ($0 < \alpha \leq 1/2$).

The recovery variable is governed by the equation

$$\frac{dw}{dt} = b(u - dw)$$

where b and d are the recovery rate and recovery decay constants respectively [8].

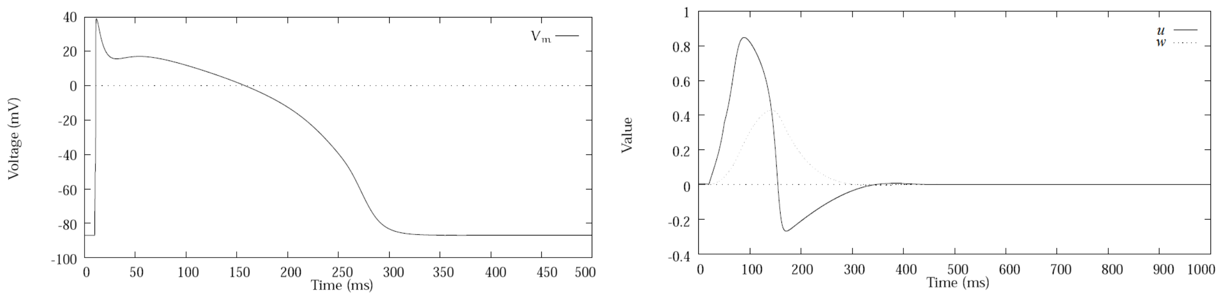

Figure 1.3: Left: Beeler-Reuter action potential. Right: Action potential generated by the FHN model and shape of the recovery variable time course (Sands, 1998).

Other models have evolved from the FHN model in order to represent more realistic shape of the cardiac ventricular action potential. Rogers and McCulloch, [15], have extended the model by rewriting the ionic current as:

$$I_{ion(RC)} = c_1 u(u - \alpha)(1 - u) - c_2 uw.$$

Aliev and Panfilov, [13], have defined I_{ion} by:

$$I_{ion(AP)} = -ku(u - a)(u - 1) - uw.$$

They have also updated the recovery variable as

$$\frac{dw}{dt} = \epsilon(u, w)(-w - ku(u - a - 1))$$

where

$$\epsilon(u, w) = \epsilon_0 + \mu_1 \frac{w}{u + \mu_2}$$

with ϵ_0,μ_1,μ_2 and k are constants.

1.4.3 Karma model

This model was proposed by Karma in 1993 and it has several properties that are not in the FHN model. For instance, it generates a repolarization period which is much longer than the fast depolarization period.

The ionic current in this model is given by

$$I_{ion} = -V_m + \left[A - \left(\frac{n}{n_B}\right)^M\right]\left[1 - \tanh(V_m - 3)\right]\frac{{V_m}^2}{2}$$

where $A = 1.5451$, M and n_B are constants with typical values $M = 30$ and $n_B = 0.507$. Actually, M controls the wavefront sensitivity and n_B controls the action potential duration of an isolated pulse. The change in the recovery variable n is given by

$$\frac{\partial n}{\partial t} = H(V_m - 1) - n$$

where $H(x)$ is the standard Heaviside step function [16].

Chapter 2

Preliminaries; Tools from Functional Analysis

2.1 Sobolev spaces

Notation: We will call a function ϕ belonging to $\mathcal{D}(\Omega)$ a *test funtion.*

Definition: Suppose $u, v \in L^1_{loc}(\Omega)$, and α is a multi-index. We say that v is the α^{th}-weak partial derivative of u, written

$$D^\alpha u = v,$$

provided

$$\int_\Omega u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega v\phi \, dx \tag{2.1}$$

for all test functions $\phi \in \mathcal{D}(\Omega)$.

Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer.

Definition. The Sobolev space

$$W^{k,p}(\Omega)$$

consists of all locally summable functions $u : \Omega \mapsto \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$.

Remark: If $p = 2$, we usually write

$$H^k(\Omega) = W^{k,2}(\Omega) \quad (k = 1, ...).$$

Definition: If $u \in W^{k,p}(\Omega)$, we define its norm to be

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \, dx \right)^{1/p} & (1 \leq p < \infty) \\ \sum_{|\alpha| \leq k} \text{ ess sup}_\Omega |D^\alpha u| & (p = \infty). \end{cases}$$

Theorem 1(Sobolev spaces as function spaces) [7]. *For each $k = 1, ...$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Theorem 2(Trace Theorem) [7]. *Let $1 \leq p < \infty$. Assume Ω is bounded and $\partial\Omega$ is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \mapsto L^p(\partial\Omega)$$

such that

1. *$Tu = u_{|\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$, and*

2. *$\|Tu\|_{L^p(\partial\Omega)} \leq \|u\|_{W^{1,p}(\Omega)}$, for each $u \in W^{1,p}(\Omega)$, with the constant C depending only on p and Ω.*

Definition: We call Tu the trace of u on $\partial\Omega$.

Proposition. *Let $H^{1/2}(\partial\Omega)$ denote the image of $H^1(\Omega)$ by the trace map T. Then*

$H^{1/2}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$.

Theorem 3 (Poincaré-Wirtinger Inequality) [4]. *Let Ω be be an open connected subspace of $\mathbb{R}^n$ with C^1 boundary, and let $1 \le p \le \infty$. Then there exists a constant C such that*

$$\|u - \bar{u}\|_{L^p} \le C\|\nabla u\|_{L^p} \ \ \forall u \in W^{1,p}(\Omega) \quad \textit{with } \bar{u} = \frac{1}{|\Omega|}\int_\Omega u.$$

2.1.1 Sobolev inequalities and embeddings

Theorem 4 (Gagliardo-Nirenberg-Sobolev) [7]. *Assume $1 \le p < n$. Then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p*}(\mathbb{R}^n) \quad \textit{where} \quad p* = \frac{np}{n-p},$$

and there exists a constant $C = C(n,p)$ such that

$$\|u\|_{L^{p*}(\mathbb{R}^n)} \le C\|\nabla u\|_{L^p(\mathbb{R}^n)} \ \ \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corollary 1. [4] *Let $1 \le p < n$. Then*

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \quad \forall q \in [p, p*]$$

with continuous injection.

Corollary 2. [4] *We have*

$$W^{1,n}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \quad \forall q \in [n, +\infty)$$

with continuous injection.

Theorem 5 (Morrey). *Let $p > n$. Then*

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$$

with continuous injection.

Moreover, $\forall u \in W^{1,p}(\mathbb{R}^n)$ *we have*

$$|u(x) - u(y)| \leq C|x-y|^{\alpha}\|\nabla u\|_{L^p} \quad a.e.\ x, y \in \mathbb{R}^n$$

with $\alpha = 1 - \frac{n}{p}$ *and* $C = C(n,p)$.

Corollary 3. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$*, and suppose* $\partial\Omega$ *is* C^1*. Let* $1 \leq p \leq \infty$*. We have*

$$\begin{aligned} &if \quad 1 \leq p < n, \quad then \quad W^{1,p}(\Omega) \hookrightarrow L^{p*}(\Omega) \ \ where \ \ p* = \tfrac{np}{n-p}, \\ &if \quad p = n, \quad then \quad W^{1,p}(\Omega) \hookrightarrow L^{q}(\Omega) \ \ \forall q \in [p, +\infty), \\ &if \quad p > n, \quad then \quad W^{1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega), \end{aligned}$$

with continuous injections.

Also, if $p > n$ *we have* $\forall u \in W^{1,p}(\Omega)$

$$|u(x) - u(y)| \leq C\|u\|_{W^{1,p}}|x-y|^{\alpha} \quad a.e.\ x, y \in \Omega$$

with $\alpha = 1 - \frac{n}{p}$ *and* $C = C(\Omega, p, n)$*. In particular,* $W^{1,p}(\Omega) \subset C(\bar{\Omega})$*. [4]*

Theorem 6 (Rellich-Kondrachov). *Suppose that* Ω *is bounded with* C^1 *boundary. We have*

$$\begin{aligned} &if \quad p < n, \quad then \quad W^{1,p}(\Omega) \subset L^{q}(\Omega) \ \ \forall q \in [1, p*) \ \ where \ \ p* = \tfrac{np}{n-p}, \\ &if \quad p = n, \quad then \quad W^{1,p}(\Omega) \subset L^{q}(\Omega) \ \ \forall q \in [1, +\infty), \\ &if \quad p > n, \quad then \quad W^{1,p}(\Omega) \subset C(\bar{\Omega}), \end{aligned}$$

with **compact** *injections* [4].

2.1.2 Spaces involving time

We introduce now some other sorts of Sobolev spaces which comprise functions mapping time into Banach spaces. These are essential in the construction of weak solutions to parabolic PDEs as will be seen in Chapter 3.

Let X denote a real Banach space with norm $\| \ \|$.

Definition: The space

$$L^p(0,T;X)$$

consists of all measurable functions $\mathbf{u} : [0,T] \to X$ with

1. $$\|\mathbf{u}\|_{L^p(0,T;X)} := \Big(\int_0^T \|\mathbf{u}(t)\|^p \, dt \Big)^{1/p} < \infty$$

   for $1 \le p < \infty$, and

2. $$\|\mathbf{u}\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{0 \le t \le T} \|\mathbf{u}(t)\| < \infty.$$

Definition: The space

$$C([0,T];X)$$

comprises all continuous functions $\mathbf{u} : [0,T] \to X$ with

$$\|\mathbf{u}\|_{C([0,T];X)} := \max_{0 \le t \le T} \|\mathbf{u}(t)\| < \infty.$$

Definition: Let $\mathbf{u} \in L^1(0,T;X)$. We say $\mathbf{v} \in L^1(0,T;X)$ is the weak derivative of $\mathbf{u}$, written

$$\mathbf{u}' = \mathbf{v},$$

provided

$$\int_0^T \phi'(t)\mathbf{u}(t)\, dt = -\int_0^T \phi(t)\mathbf{v}(t)\, dt$$

for all test functions $\phi \in \mathcal{D}(0,T)$.

Definition: The Sobolev space

$$W^{1,p}(0,T;X)$$

consists of all functions $\mathbf{u} \in L^p(0,T;X)$ such that $\mathbf{u}'$ exists in the weak sense and belongs to $L^p(0,T;X)$. Furthermore,

$$\|\mathbf{u}\|_{W^{1,p}(0,T;X)} := \begin{cases} \left(\int_0^T \|\mathbf{u}(t)\|^p + \|\mathbf{u}'(t)\|^p\, dt \right)^{1/p} & (1 \le p < \infty) \\ \operatorname*{ess\,sup}_{0 \le t \le T} (\|\mathbf{u}(t)\| + \|\mathbf{u}'(t)\|) & (p = \infty). \end{cases}$$

Remark: $H^1(0,T;X) = W^{1,2}(0,T;X)$.

Theorem 7. *Let* $\mathbf{u} \in W^{1,p}(0,T;X)$ *for some* $1 \le p \le \infty$. *Then*

$$\mathbf{u} \in C([0,T];X)$$

(after possibly being redefined on a set of measure zero) [7].

2.2 Duality

Let X denote a real Banach space.

Definitions:

1. A bounded linear operator $u^* : X \to \mathbb{R}$ is called a *bounded linear functional* on X.

2. X^* denotes the set of all bounded linear functionals on X. It is the *dual space* of X.

3. If $u \in X$, $u^* \in X^*$, the symbol $< \, , \, >$ denotes the pairing of X^* and X. Also, $< u^*, u >$ denotes $u^*(u)$.

4. A Banach space is reflexive if $(X^*)^* \equiv X$.

Theorem 8. *Every Hilbert space is reflexive.*

Theorem 9. *Every L^p space ,with $1 < p < \infty$, is reflexive.*

Theorem 10.

$$(L^p)^* \equiv L^q, \quad \forall 1 < p < +\infty \quad \text{with } \tfrac{1}{p} + \tfrac{1}{q} = 1,$$
$$(L^1)^* \equiv L^\infty.$$

Theorem 11. *X is reflexive iff $X^*$is reflexive.*

Definition: A metric space is separable if it has a countable dense subset.

Theorem 12. *If $(X^*, \|\cdot\|_*)$ is separable then $(X, \|\cdot\|)$ is separable.*

Theorem 13. *$L^p(\Omega)$ is separable $\forall \quad 1 \leq p < +\infty$.*

<u>Note</u>: The converse of Theorem 12 is not true. For instance, L^1 is separable but $(L^1)^* \equiv L^\infty$ is not separable.

Theorem 14. *$(X, \|\cdot\|)$ is reflexive and separable iff $(X^*, \|\cdot\|_*)$ is reflexive and separable.*

Definition: We say a sequence $\{u_k\}_{k=0}^{\infty} \subset X$ converges weakly to $u \in X$ (written $u_k \rightharpoonup u$), if

$$< u^*, u_k > \rightarrow < u^*, u >$$

for each bounded linear functional $u^* \in X^*$.

Theorem 15 (Weak Compactness) [7]. *Let X be a reflexive Banach space and suppose the sequence $\{u_k\}_{k=1}^{\infty} \subset X$ is bounded. Then there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and $u \in X$ such that*

$$u_{k_j} \rightharpoonup u.$$

The following proposition will be used in Chapters 3 and 4. For the convenience of the reader we include a proof.

Proposition. *Let Ω be an open subset of $\mathbb{R}^n$. Q_T denotes $(0,T)\times\Omega$ and V denotes the space $H^1(\Omega)$. Then*

$$\Big[L^p(Q_T) \cap L^2(0,T;V)\Big]' \equiv L^{p'}(Q_T) + L^2(0,T;V').$$

Proof.

- Let $u \in L^{p'}(Q_T)+L^2(0,T;V')$, so $u = u_1+u_2$, where $u_1 \in L^{p'}(Q_T)$, and $u_2 \in L^2(0,T;V')$.

  So $u_1 : L^p(Q_T) \to \mathbb{R}$ and $u_2 : L^2(0,T;V) \to \mathbb{R}$ are linear and continuous.

  Therefore, $u : L^p(Q_T) \cap L^2(0,T;V) \to \mathbb{R}$ is linear and continuous and $u \in \Big[L^p(Q_T) \cap L^2(0,T;V)\Big]'$, i.e. $L^{p'}(Q_T) + L^2(0,T;V') \subset \Big[L^p(Q_T) \cap L^2(0,T;V)\Big]'$.

- Let $u \in \Big[L^p(Q_T) \cap L^2(0,T;V)\Big]'$, i.e. $u : L^p(Q_T) \cap L^2(0,T;V) \to \mathbb{R}$ is linear and continuous.

  By continuity of u and density of $L^p(Q_T) \cap L^2(0,T;V)$ in $L^p(Q_T)$, u can be uniquely extended to $L^p(Q_T)$. Let u_1 be its extension. Similarly, let u_2 be its extension to $L^2(0,T;V)$.

Now let $\tilde{u} = \frac{u_1+u_2}{2} \in L^{p'}(Q_T) + L^2(0,T;V')$. We have $\tilde{u}|_{L^p(Q_T)\cap L^2(0,T;V)} = u$.

Thus, $u \equiv \tilde{u} \in L^{p'}(Q_T) + L^2(0,T;V')$.

Therefore, $\Big[L^p(Q_T) \cap L^2(0,T;V)\Big]' \subset L^{p'}(Q_T) + L^2(0,T;V')$. $\square$

2.3 Spectral theory

Definition: A linear operator $A : X \to Y$ is called closed if whenever $u_k \to u$ in X and $Au_k \to v$ in Y, then

$$Au = v.$$

Definitions: Let $A : X \to X$ be a bounded linear operator.

1. The *resolvent set* of A is

$$\rho(A) = \{\eta \in \mathbb{R} | (A - \eta I) \text{ is one-to-one and onto}\}.$$

2. The *spectrum* of A is

$$\sigma(A) = \mathbb{R} - \rho(A).$$

Let H denote a Hilbert space, with inner product $(\ , \ )$.

Definitions:

1. If $A : H \to H$ is a bounded linear operator, its *adjoint* $A^* : H \to H$ satisfies

$$(Au, v) = (u, A^*v)$$

for all $u, v \in H$.

2. A is self-adjoint if $A^* = A$.

Definition: A bounded linear operator

$$K : X \to Y$$

is called *compact* provided for each bounded sequence $\{u_k\}_{k=1}^{\infty} \subset X$, there exists a subsequence $\{u_{k_j}\}_{k=1}^{\infty}$ such that $\{Ku_{k_j}\}_{j=1}^{\infty}$ converges in Y.

Theorem 16 (Eigenvectors of a compact, self-adjoint operator). *Let H be a separable Hilbert space, and suppose $K : H \to H$ is a compact and self-adjoint operator. Then there exists a countable orthonormal basis of H consisting of eigenvectors of K* [7].

Theorem 17. [4] *Let V, H be two Hilbert spaces such that $V \subset H$ with compact injection. And let $a(\cdot, \cdot)$ be a symmetric and cœrcive bilinear form. Then the eigenvalues of of the problem*

$$\forall v \in V, \quad a(u, v) = \lambda(u, v)$$

form an increasing sequence

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...,$$

with $\lambda_n \to +\infty$. Moreover, the corresponding eigenvectors ψ_n form an orthonormal Hilbert basis for H such that

$$\forall v \in V, \quad a(\psi_n, v) = \lambda_n(\psi_n, v), \quad n = 1, 2, ...$$

2.4 Parabolic problems

In this section, we investigate the existence, uniqueness and regularity of the solution of the nonlinear equation

$$\frac{du}{dt} + Au = f(t, u), \qquad t > t_0$$

$$u(t_0) = u_0.$$

Hence we introduce the notion of sectorial operators and fractional powers of operators as in Dan Henry's monograph [10].

2.4.1 Sectorial operators and analytic semigroups

Definition: Let X be a Banach space. The linear operator $A : D(A) \subseteq X \to X$ is called a *sectorial operator* if it is a **closed, densely defined** operator such that, for some ϕ in $(0, \pi/2)$ and some $M \geq 1$ and real a, the sector

$$S_{a,\phi} = \{\lambda | \phi \leq |arg(\lambda - a)| \leq \pi, \ \ \lambda \neq a\}$$

is in the resolvent set of A ($\rho(A)$) and

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - a|} \quad \forall \lambda \in S_{a,\phi}.$$

Note: The angle opening of the section $S_{a,\phi}$ is $2\pi - 2\phi > \pi$.

Proposition: If A is a bounded linear operator on a Banach space, then A is sectorial.

Proposition: If A is a self-adjoint densely defined operator in a Hilbert space, and if A is bounded below, then A is sectorial.

Proposition: If A is sectorial in X, B is sectorial in Y, then $A \times B$, is sectorial in $X \times Y$, where $(A \times B)(u, v) = (Au, Bv)$ for $u \in D(A)$, $v \in D(B)$.

Definition: An *analytic semigroup* on a Banach space X is a family of continuous linear operators on X, $\{T(t)\}_{t \geq 0}$, satisfying

1. $T(0) = I$, $T(t)T(s) = T(t+s)$ for $t \geq 0$, $s \geq 0$,
2. $T(t)u \to u$ as $t \to 0^+$, for each $u \in X$,
3. $t \to T(t)u$ is real analytic on $0 < t < \infty$ for each $u \in X$.

The infinitesimal generator L of this semigroup is defined by

$$Lu = \lim_{t \to 0^+} \frac{T(t)u - u}{t},$$

and its domain $D(L)$ is defined by

$$D(L) = \{u \in X | \lim_{t \to 0^+} \frac{T(t)u - u}{t} \ \text{ exists in } X\}.$$

We usually write $T(t) = e^{Lt}$.

2.4.2 Fractional powers of operators

In what follows, A is a sectorial operator on the Banach space X.

Definition: Suppose $\operatorname{Re} \sigma(A) > 0$, then for any $\alpha > 0$

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} \, dt.$$

Examples:

1. If $A \in \mathbb{R}^+$, then $A^{-\alpha}$ is the usual $(-\alpha)$ power of A.

2. A^{-1} is the inverse of A.

Definition: A^α =inverse of $A^{-\alpha}$ $(\alpha > 0)$, $D(A) = R(A^{-\alpha})$; A^0 =identity on X.

Proposition: If $\alpha > 0$, A^α is closed and densely defined.

Proposition: If $\alpha \geq \beta$ then $D(A^\alpha) \subset D(A^\beta)$.

Definition: For each $\alpha \geq 0$,

$$X^\alpha = D((A + aI)^\alpha)$$

with the graph norm

$$\|u\|_\alpha = \|(A + aI)^\alpha u)\|, \quad u \in X^\alpha,$$

where a is chosen so Re $\sigma((A + aI)^\alpha) > 0$.

Theorem 18. *X^α is a Banach space in the norm $\|\cdot\|_\alpha$ for $\alpha \geq 0$, $X^0 = X$, and for $\alpha \geq \beta \geq 0$, X^α is a dense subspace of X^β with continuous inclusion. If A has compact resolvent, the inclusion $X^\alpha \subset X^\beta$ is compact when $\alpha > \beta \geq 0$* [10].

Theorem 19. [10] *Suppose $\Omega \subset \mathbb{R}^n$ is an open set having the C^m extension property, $1 \leq p < \infty$, and A is a sectorial operator in $X = L^p(\Omega)$ with $D(A) = X^1 \subset W^{m,p}(\Omega)$ for some $m \geq 1$. Then for $0 \leq \alpha \leq 1$,*

$$X^\alpha \subset W^{k,p}(\Omega) \quad \textit{when} \quad k - n/q < m\alpha - n/p, \quad q \geq p,$$

$$X^\alpha \subset C^\nu(\Omega) \quad \textit{when} \quad 0 \leq \nu < m\alpha - n/p.$$

2.4.3 Local existence, uniqueness and regularity

Now we consider the nonlinear equation

$$(*) \quad \begin{cases} \frac{du}{dt} + Au = f(t,u), \quad t > t_0, \\ u(t_0) = u_0' \end{cases}$$

where A is a sectorial operator so that the fractional powers of $A + aI$ are well defined and the spaces $X^\alpha = D((A+aI)^\alpha)$ with the graph norm $\|u\|_\alpha$ are defined for $\alpha \geq 0$. Let $U \subset \mathbb{R} \times X^\alpha$,. We assume $f : U \to X$, $0 \leq \alpha < 1$, is locally Hölder continuous in t and locally Lipschitzian in u on U. In other words, $\forall (t_1, u_1) \in U$, $\exists$ a neighborhood $V \subset U$ such that for $(t,u) \in V$, $(s,v) \in V$,

$$\|f(t,u) - f(s,v)\| \leq L(|t-s|^\theta + \|u-v\|_\alpha),$$

for some constants $L > 0$, $\theta > 0$.

Definition: A solution of the Cauchy problem on (t_0, t_1) is a continuous function $u : [t_0, t_1) \to X$ such that $u(t_0) = u_0$ and on (t_0, t_1) we have $(t, u(t)) \in U$, $u(t) \in D(A)$, $\frac{du}{dt}(t)$exists, $t \to f(t, u(t)$is locally Hölder continuous, and $\int_{t_0}^{t_0+\rho} \|f(t,u(t))\|\, dt < \infty$ for some $\rho > 0$, and the differential equation $(*)$ is satisfied on (t_0, t_1).

Theorem 20. *Assume A is a sectorial operator, $0 \leq \alpha < 1$, and $f : U \to X$, U an open subset of $\mathbb{R} \times X^\alpha$, $f(t,u)$ is locally Hölder continuous in t, locally Lipschitzian in u; then for any $(t_0, u_0) \in U$ there exists $T = T(t_0, u_0) > 0$ such that $(*)$ has a unique solution u on $(t_0, t_0 + T)$ with initial value $u(t_0) = u_0$* [10].

It has been shown that a certain degree of smoothing occurs; if the solution is bounded in X^α, then it is bounded in X^β with $\alpha < \beta < 1$. Also with initial value in $X^\alpha = D((A + aI)^\alpha)$, $0 < \alpha < 1$, the solution is in $D(A)$ at any later time. We state the following theorem in order to show the expressions of this smoothing action precisely.

Theorem 21. *Assume A is sectorial, $f : U \to X$ is locally Lipschitzian on an open set $U \subset \mathbb{R} \times X^\alpha$, for some $0 \le \alpha < 1$. Suppose $u(\cdot)$ is a solution on (t_0, t_1) of*

$$\frac{du}{dt} + Au = f(t, u), \quad u(t_0) = u_0$$

and $(t_0, u_0) \in U$.

Then if $\nu < 1$, $t \to \frac{du}{dt}(t) \in X^\nu$ is locally Hölder continuous for $t_0 < t < t_1$, with

$$\|\frac{du}{dt}\|_\nu \le C(t - t_0)^{\alpha - \nu - 1}$$

for some constant C [10].

2.5 Elliptic problems

In this section, we investigate the solvability and regularity of uniformly elliptic, second order partial differential equations of the form

$$Lu = f, \quad \text{in } \Omega$$

where $Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u$ and $a^{ij} = a^{ji}$.

H denotes a Hilbert space with norm $\|\ \|$ and inner product $(\ ,\ )$.

Theorem 22 (Lax-Milgram) [7]. *Let*

$$B : H \times H \to \mathbb{R}$$

be a bilinear, bicontinuous and cœrcive mapping. And let $f : H \to \mathbb{R}$ *be a bounded linear functional on* H*. Then there exists a unique element* $u \in H$ *such that*

$$B(u, v) =< f, v >$$

for all $v \in H$*.*

Theorem 23 (Regularity for the Neumann problem) [4]. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set with bounded,* C^2 *boundary* Γ*. Let* $f \in L^2(\Omega)$ *and* $u \in H^1(\Omega)$ *verify*

$$\int_\Omega \nabla u \nabla \phi \; + \int_\Omega u\phi = \int_\Omega f\phi \quad \forall \phi \in H^1(\Omega).$$

then $u \in H^2(\Omega)$ *and* $\|u\|_{H^2} \leq C\|f\|_{L^2}$ *where* C *is a constant depending only on* Ω*.*

Moreover, if Ω *is* C^{m+2} *and* $f \in H^m(\Omega)$*, then*

$$u \in H^{m+2}(\Omega) \quad \textit{and} \quad \|u\|_{H^{m+2}} \leq C\|f\|_{H^m}.$$

In particular, if $m > \frac{n}{2}$*, then* $u \in C^2(\bar{\Omega})$*.*

Finally, if Ω *is of class* C^∞ *and* $f \in C^\infty(\bar{\Omega})$*, then* $u \in C^\infty(\bar{\Omega})$*.*

We have the same conclusions for a general elliptic second order operator, in other words if $u \in H^1(\Omega)$ verifies:

$$\int \sum_{i,j} a_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_j} = \int f\phi \quad \forall \phi \in H^1(\Omega).$$

In this case, we get

$$f \in L^2(\Omega) \quad \text{and} \quad a_{i,j} \in C^1(\bar{\Omega}) \quad \Rightarrow \quad u \in H^2(\Omega),$$
$$f \in H^m(\Omega) \quad \text{and} \quad a_{i,j} \in C^{m+1}(\bar{\Omega}) \quad \Rightarrow \quad u \in H^{m+2}(\Omega).$$

Theorem 24 (Schauder). *Let L be a uniformly elliptic operator with $c \leq 0$ and coefficients in $C^\alpha(\bar{\Omega})$ in a $C^{2,\alpha}$ domain Ω. Let $Nu \equiv \gamma u + \beta \cdot Du$ define a boundary operator on $\partial\Omega$ such that $\gamma(\beta \cdot \nu) > 0$ on $\partial\Omega$ if ν is the outward unit normal on $\partial\Omega$. Assume that $\gamma,\ \beta \in C^{1,\alpha}(\partial\Omega)$. Then the problem*

$$Lu = f \ \text{ in } \ \Omega, \quad Nu = \phi \ \text{ on } \ \partial\Omega$$

has a unique $C^{2,\alpha}(\bar{\Omega})$ solution for all $f \in C^\alpha(\bar{\Omega})$ and $\phi \in C^{1,\alpha}(\partial\Omega)$ [9].

2.6 Useful Inequalities

Young's Inequality. [7] *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad (a, b > 0).$$

Young's Inequality with ϵ. [7]

$$ab \leq \epsilon a^p + C(\epsilon) b^q \quad (a, b > 0, \epsilon > 0)$$

for $C(\epsilon) = (\epsilon p)^{-q/p} q^{-1}$.

Gronwall's Inequality (Differential form). [7] *Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies for a.e. t the differential inequality*

$$\eta'(t) \leq \varphi(t)\eta(t) + \psi(t),$$

where $\varphi(t)$ *and* $\psi(t)$ *are nonnegative, summable functions on* $[0, T]$. *Then*

$$\eta(t) \leq e^{\int_0^t \varphi(s)\, ds}[\eta(0) + \int_0^t \psi(s)\, ds]$$

for all $0 \leq t \leq T$.

Chapter 3

Existence and Uniqueness of Solution of the Bidomain Model

The content of this chapter is a replicate of the paper of Bourgault et al. [3], but some proofs are added for convenience.

In this chapter we investigate existence and uniqueness of solutions of the bidomain equations in the framework of the FHN model.

3.1 The bidomain model as an initial value problem

Consider a bounded subset Ω of $\mathbb{R}^d$, $(d = 2, 3)$, representing the myocardium. The model is usually written as two degenerate parabolic PDEs with boundary condititons, coupled to an ODE, and some initial data:

$$\frac{\partial u}{\partial t} + f(u,w) - \nabla \cdot (\sigma_i \nabla u_i) = s_i \text{ in } (0,+\infty) \times \Omega, \tag{3.1}$$

$$\frac{\partial u}{\partial t} + f(u,w) + \nabla \cdot (\sigma_e \nabla u_e) = -s_e \text{ in } (0,+\infty) \times \Omega, \tag{3.2}$$

$$\frac{\partial w}{\partial t} + g(u,w) = 0 \text{ in } (0,+\infty) \times \Omega, \tag{3.3}$$

$$u = u_i - u_e \text{ in } (0,+\infty) \times \Omega, \tag{3.4}$$

$$\sigma_i \nabla u_i \cdot n = 0, \quad \sigma_e \nabla u_e \cdot n = 0, \text{ in } (0,+\infty) \times \partial\Omega, \tag{3.5}$$

$$u(0) = u_0, \quad w(0) = w_0, \text{ in } \Omega. \tag{3.6}$$

The unknowns are the functions $u_i(t,x) \in \mathbb{R}$, $u_e(t,x) \in \mathbb{R}$ and $w(t,x) \in \mathbb{R}$, which are respectively the normalized intra- and extra-cellular potentials and the recovery variable. The variable u denotes the normalized transmembrane potential and n is the outward unit normal to $\partial\Omega$.

The tensors $\sigma_{i,e}(x)$ are conductivity matrices in the global coordinates. These are functions of the space variable $x \in \Omega$ with coefficients in $L^\infty(\Omega)$ and uniformly elliptic. In other words, the assumption is made that there exist constants $0 < m < M$ such that

$$m|\xi|^2 \leq \xi^t \sigma_{i,e} \xi \leq M|\xi|^2, \text{ for each } \xi \in \mathbb{R}^d, \tag{3.7}$$

for a.e. $x \in \Omega$. As stated in Chapter 1, these symmetric matrices have the same eigenbasis, and for $x \in \partial\Omega$ the normal $n(x)$ to $\partial\Omega$ is an eigenvector of both $\sigma_i(x)$ and $\sigma_e(x)$:

$$\sigma_{i,e}(x)n(x) = \lambda_{i,e}^d(x)n(x), \text{ a.e. } x \in \partial\Omega, \tag{3.8}$$

with $\lambda_{i,e}^d(x) \geq m > 0$.

The fact that no current flows out of the myocardium in an isolated heart is represented by the boundary conditions (3.5).

The other data $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are functions representing the ionic activity in the myocardium, and $s_{i,e} : (0, +\infty) \times \Omega \mapsto \mathbb{R}$ are external applied current sources.

3.2 A new strong formulation

In order to overcome the degeneracy in the temporal derivative, we use a reformulation of (3.1) and (3.2) as a parabolic PDE coupled to an elliptic one. So u_i is substituted by $u + u_e$ in (3.1) to get:

$$\frac{\partial u}{\partial t} + f(u, w) - \nabla \cdot (\sigma_i \nabla u) - \nabla \cdot (\sigma_i \nabla u_e) = s_i \text{ in } (0, +\infty) \times \Omega. \tag{3.9}$$

Now substract (3.2) from (3.9):

$$\nabla \cdot (\sigma_i \nabla u + (\sigma_i + \sigma_e) \nabla u_e) = -(s_i + s_e) \text{ in } (0, +\infty) \times \Omega. \tag{3.10}$$

Equation (3.3) remains unchanged:

$$\frac{\partial w}{\partial t} + g(u, w) = 0 \text{ in } (0, +\infty) \times \Omega. \tag{3.11}$$

Also, substitute $u_i = u + u_e$ in the first equation of (3.5) and add them together to get:

$$\sigma_i \nabla u \cdot n + \sigma_i \nabla u_e \cdot n = 0 \text{ in } (0, +\infty) \times \partial\Omega \tag{3.12}$$

$$\sigma_i \nabla u \cdot n + (\sigma_i + \sigma_e) \nabla u_e \cdot n = 0 \text{ in } (0, +\infty) \times \partial\Omega \tag{3.13}$$

and

$$u(0) = u_0, \quad w(0) = w_0 \ \text{ in } \ \Omega. \tag{3.14}$$

The problem consists now of finding u, u_e and w verifying (3.9-3.14). The regular solutions of (3.1-3.6) are naturally the same as those of (3.9-3.14). To further reduce the problem, equations (3.9) and (3.10) are reformulated in a single equation by introducing a new *bidomain operator* A. In this approach, u_e in (3.9) is replaced by the solution of (3.10) with boundary conditions (3.12) and (3.13). So the system consisting of Equations (3.9-3.14) becomes a Cauchy problem for a single parabolic equation with unknowns u and w, which reads

$$\frac{\partial u}{\partial t} + f(u,w) + Au = s, \tag{3.15}$$

$$\frac{\partial w}{\partial t} + g(u,w) = 0, \tag{3.16}$$

$$u(0) = u_0, \quad w(0) = w_0, \tag{3.17}$$

where A is an integro-differential second order elliptic operator accounting for boundary conditions (3.12) and (3.13) and s is a modified source term, both given formally by

$$Au = -\nabla \cdot (\sigma_i \nabla u) + \nabla \cdot \Big(\sigma_i \nabla (\{\nabla \cdot (\sigma_i + \sigma_e)\nabla\}^{-1}(\nabla \cdot \sigma_i \nabla u))\Big),$$

$$s = s_i - \nabla \cdot (\sigma_i \nabla (\{\nabla \cdot (\sigma_i + \sigma_e)\nabla\}^{-1}(s_i + s_e))).$$

Afterwards, u_i and u_e are recovered with

$$u_i = u + u_e, \quad u_e = \{\nabla \cdot (\sigma_i + \sigma_e)\nabla\}^{-1}(-(s_i + s_e) - \nabla \cdot \sigma_i \nabla u). \tag{3.18}$$

The unknowns u_i and u_e are defined up to an additional constant.

Note that $s_i + s_e$ must have a zero mean value, due to the physical fact that there is no current flowing out of the heart as expressed by the boundary conditions (3.12) and (3.13). In other words, integrating equation (3.10) over Ω, and using Green's formula we get:

$$\int_{\Omega} (s_i(x) + s_e(x)) \ \ dx = 0. \tag{3.19}$$

Lemma 1. *Suppose that Ω has a C^1 boundary $\partial\Omega$, $Q(x)$ and $\Lambda_{i,e}(x)$ are $C^0(\bar{\Omega})$. For functions $u, u_e \in H^2(\Omega)$, the conditions (3.5), and the conditions (3.12) and (3.13), and the homogeneous Neumann conditions*

$$\nabla u_i \cdot n = 0, \ \ \nabla u_e \cdot n = 0, \ \ \ in \ \partial\Omega$$

are equivalent.

Proof. Conditions (3.5), (3.12), and (3.13) are linear combination one of the other. Now, $\sigma_{i,e}(x)$ being real and symmetric, we have

$$\sigma_{i,e}^* = \sigma_{i,e}^T = \sigma_{i,e}.$$

So $\nabla u_{i.e} \cdot n = 0$ implies

$$\nabla u_{i,e} \cdot \lambda_{i,e}^d n = 0$$

then

$$\nabla u_{i,e} \cdot \sigma_{i,e} n = 0$$

so

$$\nabla u_{i,e} \cdot \sigma_{i,e}^T n = 0$$

hence

$$\sigma_{i,e} \nabla u_{i,e} \cdot n = 0.$$

Similarly, we get from (3.5) the above Neumann conditions. $\square$

3.3 The bidomain operator

Now we need to study the bidomain operator that was previously introduced. Hence, we study the system

$$-\nabla \cdot (\sigma_i \nabla u) - \nabla \cdot (\sigma_i \nabla u_e) = s_i, \text{ in } \Omega, \tag{3.20}$$

$$\nabla \cdot (\sigma_i \nabla u + (\sigma_i + \sigma_e) \nabla u_e) = -(s_i + s_e), \text{ in } \Omega, \tag{3.21}$$

with the boundary conditions (3.12) and (3.13), given conductivity matrices $\sigma_{i,e}(x)$ and data $s_{i,e}(x)$ such that $\int_\Omega (s_i + s_e) = 0$.

3.3.1 Variational formulation

Equations (3.20) and (3.21) with the homogeneous Neumann boundary conditions (3.12) and (3.13) have solutions (u, u_e). The nonlinear term determines u in $H^1(\Omega)$ but u_e is defined up to an additive constant. Weak solutions will be found in $H^1(\Omega)/\mathbb{R}$, using a bilinear form in $H^1(\Omega)/\mathbb{R} \times H^1(\Omega)/\mathbb{R}$, that is extended to $H^1(\Omega) \times H^1(\Omega)$ in order to address the bidomain equations.

Given a Banach space X of functions integrable over Ω, its subspace $X/\mathbb{R}$ is defined as $X/\mathbb{R} = \{u \in X, \int_\Omega u = 0\} \subset X$ and it is a Banach space with the norm $\|u\|_{X/\mathbb{R}} = \|u\|_X$. Also note that for any $u \in X$, $[u] = u - \frac{1}{|\Omega|} \int_\Omega u \in X/\mathbb{R}$. The spaces involved are $V = H^1(\Omega)$, $H = L^2(\Omega)$ endowed with their usual norms, and $U = V/\mathbb{R}$. So we have the following continuous embeddings

$$U \subset H/\mathbb{R} \equiv (H/\mathbb{R})' \subset U',$$

$$\mathcal{D}(\Omega) \subset V \subset H \equiv H' \subset V'$$

and the injections $V \hookrightarrow H$ and $U \hookrightarrow H/\mathbb{R}$ are compact. We can define on U the norm $|u|_U = \left(\int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}}$, which is equivalent to the norm on V by Poincaré-Wirtinger inequality. An element s in $H/\mathbb{R}$ or H is identified to the element $s \in (H/\mathbb{R})'$ (or H') by $< s, v >= \int_\Omega sv \ \ \forall v \in H/\mathbb{R}$ (or H). An element $s \in V'$ is identified to an element of U' by just restricting the duality product $< s, v >:=< s, v >_{V' \times V}$ to functions v in the subspace U of V. Conversely, an element $s \in U'$ can only be extended to the whole space V independently of the value $v - [v]$ using a special condition like $< s, 1 >= 0$.

Given a regular solution $u \in H^2(\Omega)$ and $u_e \in H^2(\Omega)/\mathbb{R}$, we obtain a variational formulation by multiplying equation (3.20) by a test function $v \in U$, equation (3.21) by $v_e \in U$ and integrating over Ω

$$-\int_\Omega \nabla \cdot (\sigma_i \nabla u) v \ \ dx - \int_\Omega \nabla \cdot (\sigma_i \nabla u_e) v \ \ dx = \int_\Omega s_i v \ \ dx \ \text{ in } \ \Omega$$

$$\int_\Omega \nabla \cdot (\sigma_i \nabla u + (\sigma_i + \sigma_e) \nabla \cdot u_e) v_e \ \ dx = -\int_\Omega (s_i + s_e) v_e \ \ dx \ \text{ in } \ \Omega.$$

Now we integrate by parts the second order terms and the boundary terms disappear due to (3.12) and (3.13). So we get

$$\int_\Omega \sigma_i \nabla u \cdot \nabla v \ \ dx + \int_\Omega \sigma_i \nabla u_e \cdot \nabla v \ \ dx = \int_\Omega s_i v \ \ dx \ \text{ in } \ \Omega$$

$$\int_\Omega [\sigma_i \nabla u + (\sigma_i + \sigma_e) \nabla \cdot u_e] \cdot \nabla v_e \ \ dx = \int_\Omega (s_i + s_e) v_e \ \ dx \ \text{ in } \ \Omega.$$

Adding above equations we obtain

$$\int_\Omega \sigma_i \nabla u \cdot \nabla v \ \ dx + \int_\Omega \sigma_i \nabla u_e \cdot \nabla v \ \ dx + \int_\Omega [\sigma_i \nabla u + (\sigma_i + \sigma_e)\nabla \cdot u_e] \cdot \nabla v_e \ \ dx$$
$$= \int_\Omega s_i v \ \ dx + \int_\Omega (s_i + s_e) v_e \ \ dx.$$

For simplicity, we introduce the bilinear forms $a_{i,e}(\cdot,\cdot)$ on $U \times U$ which are defined as

$$a_i(u,v) = \int_\Omega \sigma_i \nabla u \cdot \nabla v \ \ dx, \ \ a_e(u,v) = \int_\Omega \sigma_e \nabla u \cdot \nabla v \ \ dx, \ \ \forall (u,v) \in U \times U.$$

The resulting variational problem reads: Find $(u, u_e) \in U \times U$ such that

$$a_i(u,v) + a_i(u_e,v) + a_i(u,v_e) + (a_i + a_e)(u_e,v_e) =< s_i, v > + < s_i + s_e, v_e >, \tag{3.22}$$

for all $(v, v_e) \in U \times U$, where s_i, $(s_i + s_e) \in V'$ are given source terms.

Under hypothesis (3.7), the bilinear forms $a_{i,e}(\cdot,\cdot)$ are symmetric, continuous and uniformly elliptic on U. Again, we define on $U \times U$ the bilinear form

$$b((u,u_e),(v,v_e)) = a_i(u,v) + a_i(u_e,v) + a_i(u,v_e) + (a_i + a_e)(u_e,v_e).$$

We have

Lemma 2. *The bilinear form $b(.,.)$ is symmetric, continuous and uniformly elliptic on $(U \times U) \times (U \times U)$ for the norm $|(v,v_e)|_{U\times U} = \max(|v|_U, |v_e|_U)$.*

Proof. (Detailed) $b(.,.)$ can be rewritten as

$$\begin{aligned} b((u,u_e),(v,v_e)) &= a_i(u+u_e,v) + a_i(u,v_e) + a_i(u_e,v_e) + a_e(u_e,v_e) \\ &= a_i(u+u_e,v) + a_i(u+u_e,v_e) + a_e(u_e,v_e) \\ &= a_i(u+u_e,v+v_e) + a_e(u_e,v_e). \end{aligned}$$

Since $a_{i,e}(\cdot,\cdot)$ are bilinear and symmetric, obviously $b(\cdot,\cdot)$ is bilinear and symmetric.

We have the estimate

$$\frac{1}{2}|(u,u_e)|^2_{U\times U} \le |u+u_e|^2_U + |u_e|^2_U \le 5|(u,u_e)|^2_{U\times U}\ . \quad (*)$$

Proof of $(*)$:

$$\begin{aligned}
|u+u_e|^2_U + |u_e|^2_U &\le \textstyle\int |\nabla u|^2 + 2\int \nabla u\cdot\nabla u_e + 2\int |\nabla u_e|^2 \\
&\le \textstyle\int |\nabla u|^2 + 2\sqrt{\int |\nabla u|^2}\sqrt{\int |\nabla u_e|^2} + 2\int |\nabla u_e|^2 \\
&= \Big(|u|_U + |u_e|_U\Big)^2 + |u_e|^2_U \\
&\le 5\ \max(|u|^2_U; |u_e|^2_U) \\
&= 5|(u,u_e)|^2_{U\times U}.
\end{aligned}$$

Analogously, we get

$$\begin{aligned}
|u+u_e|^2_U + |u_e|^2_U &\ge \textstyle\int |\nabla u|^2 - 2\sqrt{\int |\nabla u|^2}\sqrt{\int |\nabla u_e|^2} + 2\int |\nabla u_e|^2 \\
&= (|u|_U - |u_e|_U)^2 + |u_e|^2 \\
&\ge \tfrac{1}{2}\ \max(|u|^2_U; |u_e|^2_U) \\
&= \tfrac{1}{2}|(u,u_e)|^2_{U\times U}.
\end{aligned}$$

Now,

$$\begin{aligned}
|b((u,u_e),(v,v_e))| &= |a_i(u+u_e, v+v_e) + a_e(u_e,v_e)| \\
&\le M\|\nabla(u+u_e)\|_{L^2(\Omega)}\|\nabla(v+v_e)\|_{L^2(\Omega)} + M\|\nabla u_e\|_{L^2(\Omega)}\|\nabla v_e\|_{L^2(\Omega)} \\
&\le M\{(\|\nabla u\|_{L^2(\Omega)} + \|\nabla u_e\|_{L^2(\Omega)})(\|\nabla v\|_{L^2(\Omega)} + \|\nabla v_e\|_{L^2(\Omega)}) \\
&\quad +\|\nabla u_e\|_{L^2(\Omega)}\|\nabla v_e\|_{L^2(\Omega)}\} \\
&= M(\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\|\nabla v_e\|_{L^2(\Omega)} \\
&\quad +\|\nabla u_e\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + 2\|\nabla u_e\|_{L^2(\Omega)}\|\nabla v_e\|_{L^2(\Omega)} \\
&\le 5M\ \max(|u|_U, |u_e|_U)\ \max(|v|_U, |v_e|_U) \\
&= 5M|(u,u_e)|_{U\times U}|(v,v_e)|_{U\times U}.
\end{aligned}$$

Similarly, we get

$$b((u,u_e),(u,u_e)) \geq \frac{m}{2}|(u,u_e)|^2_{U\times U}.$$

Hence, b is continuous and cœrcive. □

Theorem 3. *Let $s_i, s_e \in V'$ and $u \in U$ be given. The variational equations*

$$(a_i + a_e)(\tilde{u_e}, v_e) = -a_i(u, v_e), \quad \forall v_e \in U, \tag{3.23}$$

$$(a_i + a_e)(\bar{u_e}, v_e) =< s_i + s_e, v_e >, \quad \forall v_e \in U, \tag{3.24}$$

have unique solutions $\tilde{u_e}, \bar{u_e} \in U$. For any $u, v \in U$, we can define the mappings

$$\bar{a}(u,v) = b((u,\tilde{u_e}),(v,0)), \quad < s, v >=< s_i, v > -a_i(\bar{u_e}, v). \tag{3.25}$$

The mapping $\bar{a}$ is bilinear, symmetric, continuous and uniformly elliptic on $U \times U$, and the mapping s is linear and continuous on U. Equation (3.22) has a unique solution (u, u_e) where u is also the unique solution of

$$\bar{a}(u,v) =< s, v >, \quad \forall v \in U, \tag{3.26}$$

and $u_e = \tilde{u_e} + \bar{u_e}$, where $\tilde{u}_e, \bar{u}_e$ are the solutions of (3.23) and (3.24).

Proof. Clearly,

$$b((u,u_e),(v,v_e)) =< s_i, v > + < s_i + s_e, v_e >, \;\; \forall (v,v_e) \in U \times U \quad (3.27)$$

$$\Leftrightarrow \;\; b((u,u_e),(v,0)) =< s_i, v >, \;\; \forall v \in U \quad (3.28)$$

$$b((u,u_e),(0,v_e)) =< s_i + s_e, v_e >, \;\; \forall v_e \in U \quad (3.29)$$

$$\Leftrightarrow \;\; b((u,\tilde{u_e}),(v,0)) =< s_i, v > -b((0,\bar{u_e}),(v,0)), \;\; \forall v \in U$$

$$b((u,\tilde{u_e}),(0,v_e)) + b((0,\bar{u_e}),(0,v_e)) =< s_i + s_e, v_e >, \;\; \forall v_e \in U$$

$$u_e = \tilde{u_e} + \bar{u_e}$$

$$\Leftrightarrow \;\; b((u,\tilde{u_e}),(v,0)) =< s_i, v > -b((0,\bar{u_e}),(v,0)), \;\; \forall v \in U \quad (3.30)$$

$$b((u,\tilde{u_e}),(0,v_e)) = 0 \;\; \forall v_e \in U \quad (3.31)$$

$$b((0,\bar{u_e}),(0,v_e)) =< s_i + s_e, v_e >, \;\; \forall v_e \in U \quad (3.32)$$

$$u_e = \tilde{u_e} + \bar{u_e} \quad (3.33)$$

The last equivalence results from the fact that $\tilde{u_e}, \bar{u_e}$ are solutions to (3.23) and (3.24).

Equations (3.31) and (3.32) are exactly equations (3.23) and (3.24), respectively; and equation (3.30) is exactly (3.26) with the notations (3.25).

Note that the mappings

$$U \ni v_e \;\; \mapsto \;\; a_i(u, v_e) \in \mathbb{R}$$

$$U \ni v_e \;\; \mapsto \;\; < s_i + s_e, v_e > \in \mathbb{R}$$

are linear and continuous. Hence knowing that $(a_i + a_e)$ is bilinear, symmetric, bicontinuous and cœrcive,(3.23) and (3.24) have unique solutions by the theorem of Lax-Milgram. We also have

$$m|u|_U^2 \leq |a_i(u,u)| = |(a_i + a_e)(\tilde{u_e}, u)| \leq 2M|\tilde{u_e}|_U|u|_U,$$

$$2m|\tilde{u_e}|_U^2 \leq (a_i + a_e)(\tilde{u_e}, \tilde{u_e}) = |a_i(u, \tilde{u_e})| \leq M|u|_U|\tilde{u_e}|_U,$$

and

$$2m|\bar{u_e}|_U^2 \leq |(a_i+a_e)(\bar{u_e}, \bar{u_e})| = |< s_i+s_e, \bar{u_e} >| \leq \|s_i+s_e\|_{V'}\|\bar{u_e}\|_V \leq C|\bar{u_e}|_U\|s_i+s_e\|_{U'}$$

where C is the constant of Poincaré-Wirtinger inequality. Hence we get

$$\frac{m}{2M}|u|_U \leq |\tilde{u_e}|_U \leq \frac{M}{2m}|u|_U, \;\; |\bar{u_e}|_U \leq \frac{C}{2m}\|s_i + s_e\|_{U'}. \qquad (3.34)$$

Obviously, the mapping s defined in (3.25) is linear and continuous since

$$\forall v \in U, \;\; |< s, v >| \leq \Big(C\|s_i\|_{U'} + \frac{CM}{2m}\|s_i + s_e\|_{U'}\Big)|v|_U.$$

It remains to prove that $\bar{a}$ is bilinear on $U \times U$, continuous, uniformly elliptic (cœrcive) and symmetric. Knowing that

$$\bar{a}(u,v) = b((u, \tilde{u_e}), (v, 0)),$$

and b is bilinear, then $\bar{a}$ is bilinear. Also, considering the function $\tilde{v_e} \in U$ constructed like $\tilde{u_e}$ as the solution of $b((v, \tilde{v_e}), (0, v_e)) = 0 \; \forall v_e \in U$, we get

$$\begin{aligned} \bar{a}(u,v) &= b((u,\tilde{u_e}),(v,0)) &&= b((u,\tilde{u_e}),(v,\tilde{v_e})) \\ & &&= b((v,\tilde{v_e}),(u,\tilde{u_e})) = b((v,\tilde{v_e}),(u,0)) = \bar{a}(v,u). \end{aligned}$$

Moreover, continuity and cœrcivity of $\bar{a}$ follow from the following inequalities

$$|\bar{a}(u,v)| \leq 5M\Big(1+\frac{M}{2m}\Big)|u|_U|v|_U, \quad \bar{a}(u,v) \geq \frac{m}{2}\Big(1+\frac{m}{2M}\Big)|u|_U^2.$$

By straightforward computation, one can show that (3.22) and (3.26) are equivalent given that $\tilde{u_e}$ and $\bar{u_e}$ are the solutions of (3.23) and(3.24). □

Remark 4. Conversely, let u be the solution to (3.26) and $u_e = \tilde{u_e} + \bar{u_e}$ be given by (3.23) and (3.24). The space U does not contain the space $\mathcal{D}(\Omega)$, but for any $v \in \mathcal{D}(\Omega)$, we have $\nabla v = \nabla[v]$ and $[v] \in U$. We must impose the extra condition $< s_i + s_e, 1 >= 0$ to get $< s_i + s_e, v >=< s_i + s_e, [v] >$. In that case $u_e = \tilde{u_e} + \bar{u_e}$ can easily be proved to verify

$$\nabla \cdot (\sigma_i \nabla u + (\sigma_i + \sigma_e)\nabla u_e) = -(s_i + s_e), \text{ in } \mathcal{D}'(\Omega).$$

Additionally, if u, $u_e \in H^2(\Omega)$, then the equation is verified a.e. in Ω and the boundary condition (3.13) holds a.e. in $\partial\Omega$.

In order to state the full bidomain problem the operator $\bar{a}$ is extended to $V \times V$.

Definition 5. The *bidomain bilinear form* a is defined on $V \times V$ by

$$a(u,v) = \bar{a}([u],[v]), \quad \forall (u,v) \in V \times V.$$

Given $s_i, s_e \in V'$ such that $< s_i + s_e, 1 >= 0$, the source term s is extended to a linear form on V, (denoted s), by

$$< s, v >=< s_i, v > -a_i(\bar{u_e}, [v]), \ \ \forall v \in V,$$

where $\bar{u_e}$ is given by (3.24).

Theorem 6. *The bilinear form $a(\cdot,\cdot)$ is symmetric, continuous and positive in V,*

$$\forall u \in V, \quad \alpha\|u\|_V^2 \leq a(u,u) + \alpha\|u\|_H^2, \tag{3.35}$$

$$\forall u, v \in V, \quad |a(u,v)| \leq \mathbf{M}\|u\|_V\|v\|_V, \tag{3.36}$$

where $\alpha = \frac{m}{2}\Big(1 + \frac{m}{2M}\Big)$ and $\mathbf{M} = 5M\Big(1 + \frac{M}{2m}\Big)$. *There exists an increasing sequence* $0 = \lambda_0 < ... \leq \lambda_i \leq ...$ *in* $\mathbb{R}$ *and an orthonormal Hilbert basis of eigenvectors* $(\psi_i)_{i\in\mathbb{N}}$ *such that for all* $i \in \mathbb{N}, \psi_i \in V$ *and*

$$\forall v \in V, \quad a(\psi_i, v) = \lambda_i(\psi_i, v). \tag{3.37}$$

Given $s_i, s_e \in V'$ *such that* $< s_i, 1 >=< s_e, 1 >= 0$, *if* $u \in V$ *is a solution to*

$$a(u,v) =< s, v >, \quad \forall v \in V, \tag{3.38}$$

and $u_e = \tilde{u_e} + \bar{u_e} \in U$ *is given by (3.23) and (3.24), then* (u, u_e) *is a weak solution to (3.19), (3.20) with the boundary conditions (3.12), (3.13). If additionally,* $u, u_e \in H^2(\Omega)$, *then they verify (3.19),(3.20) a.e. in* Ω *and (3.12), (3.13) a.e. in* $\partial\Omega$.

Proof. Clearly, $a(\cdot,\cdot)$ is well-defined and symmetric. Using the proof of theorem 3, $a(\cdot,\cdot)$ verifies (3.35) :

$$\begin{aligned}
|a(u,v)| &\leq 5M\Big(1 + \tfrac{M}{2m}\Big)|[u]|_U|[v]|_U \\
&= 5M\Big(1 + \tfrac{M}{2m}\Big)(\|u\|_V^2 - \|u\|_H^2)^{\frac{1}{2}}(\|v\|_V^2 - \|v\|_H^2)^{\frac{1}{2}} \\
&\leq 5M\Big(1 + \tfrac{M}{2m}\Big)\|u\|_V\|v\|_V,
\end{aligned}$$

and (3.36):

$$\begin{aligned} a(u,u) &\geq \tfrac{m}{2}\Big(1+\tfrac{m}{2M}\Big)\|[u]\|_U^2 \\ &= \alpha(\|u\|_V^2 - \|u\|_H^2). \end{aligned}$$

Since the injection $V \to H$ is compact (Rellich-Kondrachov theorem) and the bilinear form $a(\cdot,\cdot)$ is symmetric and positive, the spectral results become evident by Theorem (17) in the previous chapter. In this case, we have $\lambda_0 = 0$ because a vanishes only for constant functions.

The equivalence with the strong formulation and the boundary conditions is a classical result, stated partly in Remark 4, and easily deduced from Definition 5:

$$a(u,v) =< s,v > \quad \Leftrightarrow \quad \begin{cases} b((u,u_e),([v],0)) =< s_i, v > \quad \forall v \in V, \\ b((u,u_e),(0,v_e)) =< s_i + s_e, v > \quad \forall v_e \in U. \end{cases}$$

The second equation is equivalent to (3.23) and (3.24) to define u_e from $[u] \in U$ and the first equation is obviously the weak form of (3.19). (Note: These have been explicitly done in the proof of Theorem 3.) □

Remark 7. The two conditions $< s_i, 1 >=< s_e, 1 >= 0$ are needed for the solution $u \in U = V/\mathbb{R}$ of (3.38) to be interpreted as a weak solution in V of the PDEs (3.19), (3.20) with the homogeneous Neumann boundary conditions (3.12) and (3.13).

For the full nonlinear bidomain problem, only $< s_i + s_e, 1 >= 0$ will be required with no zero-average condition on s_i and s_e taken individually. Definition 5 requires only that $< s_i + s_e, 1 >= 0$.

3.3.2 Weak operator

The variational process can be handled through weak operators defined from U onto U' or V onto V'. By duality, $A_{i,e}$ and $\bar{A}$ are defined as:

$$< A_{i,e}u, v >= a_{i,e}(u, v), \quad < \bar{A}u, v >= \bar{a}(u, v), \quad \forall (u, v) \in U \times U.$$

They are all one-to-one continuous mappings from U onto U', with continuous inverse (from Lax-Milgram theorem).

Lemma 8. *Given s_i, $s_e \in U'$, the source term $s \in U'$ defined in Theorem 3 is such that*

$$s = s_i - A_i(A_i + A_e)^{-1}(s_i + s_e) = -s_e + A_e(A_i + A_e)^{-1}(s_i + s_e),$$

and we have

$$\bar{A} = A_i(A_i + A_e)^{-1}A_e = (A_i^{-1} + A_e^{-1})^{-1}.$$

Proof. (3.23) and (3.24) can be rewritten as

$$(A_i + A_e)\tilde{u_e} = -A_i u, \quad (A_i + A_e)\bar{u_e} = s_i + s_e.$$

Substituting $\tilde{u_e}$ and $\bar{u_e}$ in (3.25), we get

$$\begin{aligned}
< s, v > &=< s_i, v > -a_i(\bar{u_e}, v) \\
&=< s_i, v > - < A_i\bar{u_e}, v > \\
&=< s_i, v > - < A_i(A_i + A_e)^{-1}(s_i + s_e), v > \\
&=< s_i - A_i(A_i + A_e)^{-1}(s_i + s_e), v >,
\end{aligned}$$

i.e.

$$s = s_i - A_i(A_i + A_e)^{-1}(s_i + s_e).$$

But

$$s - s = s_i + s_e - (A_i + A_e)(A_i + A_e)^{-1}(s_i + s_e),$$

so

$$s = s_i - A_i(A_i + A_e)^{-1}(s_i + s_e) = -s_e + A_e(A_i + A_e)^{-1}(s_i + s_e).$$

Now,

$$\begin{aligned} < \bar{A}u, v > \; = \bar{a}(u, v) \; &= a_i(u, v) + a_i(\tilde{u_e}, v) \\ &=< A_i u, v > + < A_i \tilde{u_e}, v > \\ &=< A_i u, v > + < -A_i(A_i + A_e)^{-1} A_i u, v > \\ &=< (A_i - A_i(A_i + A_e)^{-1} A_i u, v >, \end{aligned}$$

so that

$$\begin{aligned} \bar{A} \; = A_i - A_i(A_i + A_e)^{-1} A_i \; &= A_i(I - (A_i + A_e)^{-1} A_i) \\ &= A_i(A_i + A_e)^{-1}(A_i + A_e - A_i) \\ &= A_i(A_i + A_e)^{-1} A_e. \end{aligned}$$

The second equality defining $\bar{A}$ comes from

$$(A_i^{-1} + A_e^{-1})^{-1} = A_i(A_i + A_e)^{-1} A_e. \quad \square$$

Lemma 9. *Define $J : u \in V \mapsto [u] \in U$ and its transpose $J^T : U' \to V'$. For any s_i, $s_e \in V'$ with $< s_i + s_e, 1 >= 0$, the source term $s \in V'$ and the bilinear operator a given by definition 5 are such that:*

$$s = s_i - J^T A_i(A_i + A_e)^{-1}(s_i + s_e) = -s_e + J^T A_e(A_i + A_e)^{-1}(s_i + s_e),$$

and

$$a(u,v) = < Au, v >, \;\; \forall (u,v) \in V \times V, \;\; with \; A = J^T \bar{A} J : V \to V'.$$

Proof.

$$a(u,v) = \bar{a}([u],[v]) = \bar{a}(Ju, Jv) = < \bar{A}Ju, Jv > = < J^T \bar{A} Ju, v >, \;\; \forall (u,v) \in V \times V.$$

Concerning s we have:

$$\begin{aligned} < s, v > \;\; &= < s_i, v > - a_i(\bar{u_e}, [v]) \;\; = < s_i, v > - < A_i(A_i + A_e)^{-1}(s_i + s_e), Jv > \\ &= < s_i - J^T A_i (A_i + A_e)^{-1}(s_i + s_e), v > . \quad \square \end{aligned}$$

3.3.3 Strong Operator

Now we want to see A_i and A_e as operators in $H/\mathbb{R}$. Hence we suppose that Ω has a C^2 boundary $\partial\Omega$ and that $\sigma_{i,e}$ have $C^1(\bar{\Omega})$ coefficients. Using the regularity results of Chapter 2, for all $f \in H/\mathbb{R}$ we have $u = A_{i,e}^{-1} f \in H^2(\Omega)/\mathbb{R}$, or $u = (A_i + A_e)^{-1} f \in H^2(\Omega)/\mathbb{R}$. As a consequence, we have

$$\begin{aligned} &\nabla \cdot (\sigma_{i,e} \nabla u) = f \;\; a.e. \text{ in } \Omega, && \sigma_{i,e} \nabla u \cdot n = 0 \;\; a.e. \text{ in } \partial\Omega, \\ &\nabla \cdot ((\sigma_i + \sigma_e) \nabla u) = f \;\; a.e. \text{ in } \Omega, && (\sigma_i + \sigma_e) \nabla u \cdot n = 0 \;\; a.e. \text{ in } \partial\Omega. \end{aligned}$$

With Lemma 1 in addition, unbounded operators in $H/\mathbb{R}$, still denoted by A_i, A_e and $A_i + A_e$, are defined on domains $D(A_i) = D(A_e) = D(A_i + A_e) = D(A)/\mathbb{R}$ by

$$A_i u = \nabla \cdot (\sigma_i \nabla u), \quad A_e u = \nabla \cdot (\sigma_e \nabla u), \quad (A_i + A_e) u = \nabla \cdot ((\sigma_i + \sigma_e) \nabla u), \quad (3.39)$$

with

$$D(A) = \Big\{ u \in H^2(\Omega), \; \nabla u \cdot n = 0 \text{ a.e.in } \partial\Omega \Big\} \subset H. \qquad (3.40)$$

Lemma 11. (detailed) *The operators A_i, A_e, $A_i + A_e$ are maximal monotone in $H/\mathbb{R}$ and self-adjoint. They have compact inverses in $H/\mathbb{R}$.*

Proof. The operators A_i, A_e, $A_i + A_e$ verify, for all $(u, v) \in D(A)/\mathbb{R} \times D(A)/\mathbb{R}$,

$$< A_{i,e}u, v >= a_{i,e}(u, v), \quad < (A_i + A_e)u, v >= (a_i + a_e)(u, v).$$

Maximality follows from the theorem of Lax-Milgram applied on the operator $I + A$, and monotonicity and symmetry result from the positivity and symmetry of $a_{i,e}$. The inverses

$$(A_{i,e}^{-1}) : H/\mathbb{R} \to H/\mathbb{R} \quad (A_i + A_e)^{-1} : H/\mathbb{R} \to H/\mathbb{R}$$

are compact operators since their range is $D(A)/\mathbb{R}$ and the injection $D(A)/\mathbb{R} \to H/\mathbb{R}$ is compact. Actually, let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H/\mathbb{R}$ and let

$$v_n = A_{i,e}^{-1} u_n \ \ \forall n.$$

Then (v_n) is a bounded sequence in $D(A)/\mathbb{R}$. Now, by the theorem of Rellich-Kondrachov, the injection $D(A)/\mathbb{R} \to H/\mathbb{R}$ is compact. Hence, there is a convergent subsequence of (v_n) in $H/\mathbb{R}$ and $A_{i,e}^{-1}$ is compact. □

Definition 12. Given $s_{i,e} \in H$ such that $s_i + s_e \in H/\mathbb{R}$, we define the strong bidomain operator $A : D(A) \subset H \to H$ and the source tem $s \in H$ by:

$$Au = A_i(A_i + A_e)^{-1}A_e[u], \quad \forall u \in D(A), \tag{3.41}$$

$$s = s_i - A_i(A_i + A_e)^{-1}(s_i + s_e) = -s_e + A_e(A_i + A_e)^{-1}(s_i + s_e). \tag{3.42}$$

Theorem 13. *Consider s_i, $s_e \in H$ such that $s_i + s_e \in H/\mathbb{R}$. the strong bidomain* *operator A of Definition 12 is self-adjoint and maximal-monotone. We have*

$$\forall (u,v) \in D(A) \times D(A), \quad (Au, v) = a(u,v),$$

and the source term $s \in H$ of Definition 12 can be identified to the source term $s \in V'$ of Definition 5 through the identity $< s, v >= (s,v)$ for all $v \in V \subset H$. The sequence $(\lambda_i)_{i\in\mathbb{N}}$ and the orthonormal Hilbert basis $(\psi_i)_{i\in\mathbb{N}}$ of eigenvectors defined in Theorem 6 are such that $\psi_i \in D(A)$ for all $i \in \mathbb{N}$ and

$$D(A) = \Big\{ u \in H, \ \sum_{i\geq 0} \lambda_i^2 (u,\psi_i)^2 < \infty \Big\}, \quad Au = \sum_{i\geq 0} \lambda_i (u,\psi_i)\psi_i. \tag{3.43}$$

For $u \in H$, we have

$$Au = s \ \text{ and } \ u_e = (A_i + A_e)^{-1}((s_i + s_e) - A_i u) \in D(A)/\mathbb{R} \Leftrightarrow$$

(u, u_e) verify (3.19) and (3.20) a.e. in Ω and the boundary conditions (3.12)and (3.13) a.e. in $\partial\Omega$.

Proof. For $u \in D(A)$, $Au \in H/\mathbb{R} \subset H$ is well defined. Consider the solutions to (3.23) and (3.24) which can be written as $\tilde{u_e} = -(A_i + A_e)^{-1} A_i[u] \in D(A)/\mathbb{R}$ and $\bar{u_e} = (A_i + A_e)^{-1}(s_i + s_e) \in D(A)/\mathbb{R}$. By simple computation we get $Au = A_i[u] - A_i(A_i + A_e)^{-1} A_i[u]$ for all $u \in D(A)$ and then for all $v \in D(A)$,

$$\begin{aligned}
(Au, v) &= (A_i[u], [v]) + (A_i \tilde{u_e}, [v]) &&= a_i([u],[v]) + a_i(\tilde{u_e},[v]) \\
&= b(([u], \tilde{u_e}), (v, 0)) &&= a(u,v), \\
(s, v) &= (s_i, v) - (A_i \bar{u_e}, [v]) &&= (s_i, v) - a_i(\bar{u_e}, [v]) \qquad =< s, v > .
\end{aligned}$$

The remaining results follow from Theorem 6. By positivity and symmetry of a, and by equation (3.35) A is maximal-monotone and self-adjoint in H. The eigenvectors $\psi_i \in V = H^1(\Omega)$ are such that $A\psi_i = \lambda_i \psi_i$ in V' and then $\psi_i \in D(A)$ (regularity result). As a consequence $A\psi_i = \lambda_i \psi_i$ in H and (3.43) is valid. The equivalence is true because if $u \in D(A) \subset H^2(\Omega)$ then $Au = s \;\; \Leftrightarrow \;\; a(u,v) =< s, v > \;\forall v \in V$ and $u_e = (A_i + A_e)^{-1}((s_i + s_e) - A_i u) = \tilde{u_e} + \bar{u_e}$ where $\tilde{u_e}$ and $\bar{u_e}$ are the solutions to (3.23) and (3.24). □

3.4 Strong solutions

The existence and uniqueness of strong solutions for (3.15)-(3.17) is established in the framework of analytical semigroups as presented in Section 2.4 of Chapter 2.

3.4.1 Specific assumptions and notations

In order to apply the definition and lemma from Section 3.3.3, we assume that Ω has a C^2 boundary $\partial\Omega$ and that $\sigma_{i,e}$ have $C^1(\bar{\Omega})$ coefficients. The recovery variable w will be searched in a Banach space $B^m = B \times B ... \times B$ where

- either $B = L^\infty(\Omega)$
- or $B = C^\nu(\Omega)$, $0 < \nu < 1$. This last choice will be needed to establish the regularity of the solutions.

The integer m can be chosen as large as one wishes.

The functions $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ are:

- locally Lipschitz continuous functions on $\mathbb{R} \times \mathbb{R}^m$ when assuming $B = L^\infty(\Omega)$, hence covering a wide range of physiological ionic models,
- $C^2(\mathbb{R} \times \mathbb{R}^m)$ regular functions when assuming that $B = C^\nu(\Omega)$.

Lastly, the functions s_i, s_e are assumed to be locally ν-Hölder continuous in time, $s_{i,e} \in C^\nu_{loc}([0, +\infty), H)$ for some $\nu > 0$:

$$\forall [t_1, t_2] \subset [0, +\infty), \exists C > 0, \ / \ \delta_1, \delta_2 \in [t_1, t_2] \Rightarrow \|s_{i,e}(\delta_1) - s_{i,e}(\delta_2)\|_H \leq C\|\delta_1 - \delta_2\|^\nu. \tag{3.44}$$

Consider $Z = H \times B^m$, with the norm $\|(u, w)\|_Z = \max(\|u\|_H, \|w\|_{B^m})$. It is a Banach space. We introduce the unbounded operator $\mathcal{A}$ in Z defined by

$$\mathcal{A} : D(\mathcal{A}) \subset Z \to Z, \quad \mathcal{A}z = (Au, 0) \in Z, \quad \text{for } \ z = (u, w) \in Z,$$

with $D(\mathcal{A}) = D(A) \times B^m$; and the source term $S : t \in [0, +\infty) \to (s(t), 0) \in Z$ where A and $s(t)$ are given in Definition 12.

Lemma 14. *The unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset H \times B^m \to H \times B^m$ is a sectorial operator.*

Proof. Since the operators $A : D(A) \to H$ and $w \in B^m \mapsto 0 \in B^m$ are self-adjoint and bounded below, they are sectorial. Thus $\mathcal{A}$ is sectorial being the Cartesian product of two sectorial operators. □

Lemma 15. *If $s_{i,e} : [0, +\infty) \to H$ are locally ν-Hölder continuous functions with $s_i(t) + s_e(t) \in H/\mathbb{R}$ for all $t \geq 0$, then $S : [0, +\infty) \to Z$ is locally ν-Hölder* *continuous.*

Proof. Consider $[t_2, t_2] \subset [0, +\infty)$, and the constant $C > 0$ such that (3.44) holds. If $\delta_1,\ \delta_2 \in [t_1, t_2]$, then

$$s(\delta_1) - s(\delta_2) = s_i(\delta_1) - s_i(\delta_2) - A_i(A_i + A_e)^{-1}(s_i(\delta_1) - s_i(\delta_2) + s_e(\delta_1) - s_e(\delta_2)).$$

Since $A_i(A_i + A_e)^{-1} : D(A)/\mathbb{R} \to D(A)/\mathbb{R}$ is bounded, we get

$$\|s(\delta_1) - s(\delta_2)\|_H \leq C'|\delta_1 - \delta_2|^\nu,$$

and the result follows. □

Our next problem is to define the mapping

$$F : (u, w) \in Z \mapsto (f(u, w), g(u, w)) \in Z.$$

To get rid of that difficulty, we introduce the fractional powers $\mathcal{A}^\alpha$ and the interpolation spaces $Z^\alpha = D(\mathcal{A}^\alpha)$. For $\alpha \geq 0$ the unbounded operator $\mathcal{A}^\alpha : D(\mathcal{A}^\alpha) \subset Z \to Z$ is defined by:

$$Z^\alpha = \Big\{u \in H, \sum_{i\geq 0} \lambda_i^{2\alpha}(u, \psi_i)^2 < \infty\Big\}, \quad \mathcal{A}^\alpha(u, w) = \Big(\sum_{i\geq 0} \lambda_i^\alpha(u, \psi_i)\psi_i, 0\Big).$$

The spaces Z^α, with the norm $\|u\|_\alpha = \|u + \mathcal{A}^\alpha u\|_Z$, are Banach spaces. Moreover, for any $0 \leq \alpha \leq \beta$, we have the continuous and dense embedding $Z^\beta \subset Z^\alpha$. These spaces form a sequence of decreasing functional spaces composed of functions whose regularity increases from Z $(\alpha = 0)$ to $D(\mathcal{A}) \subset H^2(\Omega) \times B^m$ $(\alpha = 1)$ ([10], p 29).

Lemma 16. *(Case $B = L^\infty(\Omega)$). For $B = L^\infty(\Omega)$, f, g locally Lipschitz continuous on $\mathbb{R} \times \mathbb{R}^m$, we have*

$$Z^\alpha \subset L^\infty(\Omega) \times B^m \quad \textit{if } \frac{d}{4} < \alpha < 1,$$

and in that case, $F : z \in Z^\alpha \mapsto F(z) \in Z$ is locally Lipschitz continuous.

Lemma 17. *(Case $B = C^\nu(\Omega)$). For $B = C^\nu(\Omega)$, f, g C^2 functions on $\mathbb{R} \times \mathbb{R}^m$, and $\alpha < 1$, we have*

$$Z^\alpha \subset C^\nu(\Omega) \times B^m, \quad \textit{if } 0 < \nu < 2\alpha - d/2,$$

and in that case, $F : z \in Z^\alpha \mapsto F(z) \in Z$ is locally Lipschitz continuous.

Proof. Since the operator $\mathcal{A}$ is sectorial, we use Theorem 19 of Chapter 2 with $m = 2,\ p = 2,\ n = d,\ q = \infty$ and $k = 0$, for lemma 16, and $m = 2,\ p = 2$ and $n = d$ for lemma 17.

A locally Lipschitzian function $f : \mathbb{R} \to \mathbb{R}$ induces a locally Lipschitzian function $f : L^\infty(\Omega) \to L^\infty(\Omega)$, so that F can be extended to a locally Lipschitz continuous function $F : Z^\alpha \to Z$.

A C^2 function $f : \mathbb{R} \to \mathbb{R}$ induces a locally Lipschitz function $f : C^\nu(\Omega) \to C^\nu(\Omega)$ for $(0 < \nu < 1)$. The mapping F can be extended to a locally Lipschitzian function $F : Z^\alpha \to Z$. $\square$

3.4.2 Existence and uniqueness of local in time solution

The strong local in time solution is defined as follows:

Definition 18 (Strong Solution.) Consider $\tau > 0$ and the functions $z : \ t \in [0, \tau) \mapsto z(t) = (u(t), w(t)) \in Z$ and $u_e : \ t \in [0, \tau) \mapsto u_e(t) \in H$. Given $(u_0, w_0) \in Z$, we say that (u, u_e, w) is a strong solutions to (3.9-(3.14) iff,

1. $z : [0, \tau) \to Z$ is continuous and $z(0) = (u_0, w_0)$ in Z (that is (3.14)),
2. $z : (0, \tau) \to Z$ is Fréchet differentiable,
3. $t \in [0, \tau) \mapsto (f(u(t), w(t)), g(u(t), w(t))) \in Z$ is well defined, locally ν-Hölder continuous on $(0, \tau)$ (for $0 < \nu < 1$) and is continuous at $t = 0$,
4. for all $t \in (0, \tau)$, $u(t) \in H^2(\Omega)$, $u_e(t) \in H^2(\Omega)/\mathbb{R}$, and (u, u_e, w) verify (3.9)-(3.11) for all $t \in (0, \tau)$ and for a.e. $x \in \Omega$, and the boundary conditions (3.12) and (3.13) for all $t \in (0, \tau)$ and for a.e. $x \in \partial\Omega$.

Condition (4) in the above definition can be easily replaced by the following characterization.

Lemma 19. *The functions $z = (u, w)$ and u_e are a strong solution to (3.9)-(3.15) iff conditions (1)-(3) of Definition 18 and condition (4') below are satisfied:*

$(4')$ $u(t) \in D(A)$, $\ u_e(t) \in D(A)/\mathbb{R}$, *and* z *verify* $\forall t \in (0, \tau)$

$$\frac{dz}{dt}(t) + \mathcal{A}z(t) + F(z(t)) = S(t) \ \text{ in } \ Z, \tag{3.45}$$

using the previous definitions of $\mathcal{A}$ and F, while u_e is given by

$$u_e(t) = (A_i + A_e)^{-1}(s_i(t) + s_e(t) - A_i[u(t)]) \in D(A)/\mathbb{R}. \tag{3.46}$$

Theorem 20 (Local Existence and Uniqueness). *Consider* $0 < \alpha < 1$ *defined by Lemma 16 (case* $B = L^\infty(\Omega)$*) or Lemma 17 (case* $B = C^\nu$*) such that* $F : Z^\alpha \to Z$ *is well-defined and locally Lipschitzian. Then for any* $(u_0, w_0) \in Z^\alpha$*, there exists* $T > 0$ *such that the problem (3.1)-(3.6) has a unique solution on* $[0, T)$ *in the sense of Definition 18.*

Proof. This theorem is a direct application of the local existence and uniqueness theorem in [10] (or Theorem 20 in Chapter 2) since:

- there always exists $0 \leq \alpha < 1$ such that F extends to a function $F : Z^\alpha \to Z$ locally Lipschitzian, for $d = 1, 2, 3$.
- $\mathcal{A}$ is sectorial (Lemma 14),
- $t \mapsto S(t)$ is locally ν-Hölder continuous for some $\nu > 0$. □

3.4.3 Regularity of the solutions

Let $0 < \nu < 1$. Throughout this subsection, we will assume that $B = C^\nu(\Omega)$ and that the reaction terms f and g have C^2 regularity on $\mathbb{R} \times \mathbb{R}^m$. Also, we will assume that the boundary $\partial\Omega$ has $C^{2+\nu}$ regularity, and that $\sigma_{i,e}$ have their coefficients in $C^{1+\nu}(\bar{\Omega})$.

The operator $\mathcal{A}$ has a smoothing effect on the solutions of (3.45): for an initial data $u_0 \in D(\mathcal{A}^\alpha)$, the solution satisfies $u(t) \in D(A)$ for $t > 0$. This is due to the following elliptic regularity result [9]:

Lemma 21. *Let σ be a uniformly elliptic tensor on Ω whose components belong to $C^{1+\nu}(\bar{\Omega})$ for some $\nu > 0$. We also assume the boundary $\partial\Omega$ to have $C^{2+\nu}$ regularity. If $u \in D(A)$ satisfies $\nabla \cdot (\sigma \nabla u) \in C^{\nu}(\Omega)$, then $u \in C^{2+\nu}(\Omega)$.*

Moreover, some regularity in time takes place ([10] p71):

Lemma 22. *Let $z : t \in (0,T) \mapsto z(t) \in D(\mathcal{A}) = Z^1$ be the solution of the Cauchy problem (3.25) given by Theorem 20. We have $Z^1 \subset Z^{\nu}$ for any $\nu \leq 1$, and the solution moreover satisfies: $t \in (0,T) \mapsto z(t) \in Z^{\nu}$ is continuously Fréchet differentiable for any $\nu < 1$.*

The above two lemma imply that the solutions for (3.45) are classical solutions provided that the initial data w_0 for the second variable w is smooth enough.

Theorem 23 (Regularity of the strong solution). *Consider $d/4 < \alpha < 1$ and $0 < \nu < 2\alpha - d/2$, and assume that $s_{i,e} : [0, +\infty) \to H$ are locally ν-Hölder continuous and such that $s_{i,e}(t) \in C^{\nu}(\Omega)$ for all $t \geq 0$. For $z_0 = (u_0, w_0) \in Z^{\alpha}$ the unique solution z of (3.45) defined on $[0,T)$ for some $T > 0$ satisfies furthermore:*

- *Given any $x \in \bar{\Omega}$, the function $t \in (0,T) \mapsto z(x,t)$ is continuously differentiable in t,*
- *Given any $t \in (0,T)$, the function $x \in \bar{\Omega} \mapsto u(x,t)$ is twice continuously differentiable in x, i.e. $u(\cdot,t) \in C^2(\bar{\Omega})$.*

Proof. Using Lemma 22 ensures that the solution $t \in (0,T) \mapsto z(t) \in C^\nu(\Omega) \times (C^\nu(\Omega))^m$ is continuously (Fréchet) differentiable. This implies that $(t,x) \in (0,T) \times \bar{\Omega} \mapsto z(t,x) = (u(t,x), w(t,x))$ is continuously differentiable in t.

To prove the second assertion, let us show that $Au(t) \in C^\nu(\Omega)$ for $t \in (0,T)$. Indeed, $Au(t) = -\frac{du}{dt}(t) - f(u(t), w(t)) + s(t)$ and $f(u(t), w(t)) \in C^\nu(\Omega)$. Also $du/dt \in C^\nu(\Omega)$ thanks to Lemma 22. Now $s(t) = -s_e(t) + A_e(A_i + A_e)^{-1}(s_i(t) + s_e(t))$ and $(s_i + s_e)(t) \in C^\nu(\Omega)$ by assumption. By Lemma 21, the function $(A_i + A_e)^{-1}(s_i(t) + s_e(t))$ belongs to $C^{2+\nu}(\Omega)$ and then $s(t) \in C^\nu(\Omega)$. Consequently $Au(t) \in C^\nu(\Omega)$ for $t \in (0,T)$.

Observing that $Au(t) = -A_e(A_i + A_e)^{-1}A_i[u(t)]$ and using Lemma 21, we get $A_e^{-1}Au(t) \in C^{2+\nu}(\Omega)$, and therefore $(A_i + A_e)A_e^{-1}Au(t) \in C^\nu(\Omega)$. Lastly, we obtain by the same Lemma $[u(t)] = -A_i^{-1}(A_i + A_e)A_e^{-1}Au(t) \in C^{2+\nu}(\Omega)$. This implies that $x \mapsto u(t,x) \in C^2(\bar{\Omega})$ for $t \in (0,T)$. $\square$

3.5 Global solution based on a variational formulation

The existence of *weak solutions* for (3.9)-(3.15) is established by a Faedo-Galerkin technique.

3.5.1 Specific assumptions and notations

Minimal regularity assumptions are required for the existence of a weak solution: Ω has a Lipschitz boundary $\partial\Omega$, $\sigma_{i,e}$ have $L^\infty(\Omega)$ coefficients, and $s_{i,e}$:

$[0, +\infty) \to V'$ have zero mean value, i.e. $< s_i(t) + s_e(t), 1 >= 0$ for a.e. $t > 0$, in order to use the bilinear form a and the source term $s : t \in [0, +\infty) \mapsto s(t) \in V'$ as in Definition 5. For the sake of simplicity, we assume that $m = 1$, i.e. $w(t, x) \in \mathbb{R}$.

In order to write (3.9) in V' and (3.11) in $H' \equiv H$, we need assumptions on $f, g : \mathbb{R}^2 \to \mathbb{R}$ so that $(u, w) \in V \times H \mapsto (f(u, w), g(u, w)) \in V' \times H'$ is well-defined. We assume that f and g satisfy some structural and growth conditions. More precisely, we suppose:

(H1) the Sobolev embedding $V = H^1(\Omega) \subset L^p(\Omega)$ holds: $p \geq 2$ if $d = 2$; or $2 \leq p \leq 6$ if $d = 3$ [see Section 2.1.1];

(H2) the functions f and g are affine with respect to w:

$$f(u, w) = f_1(u) + f_2(u)w, \quad g(u, w) = g_1(u) + g_2 w, \tag{3.47}$$

where $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : \mathbb{R} \to \mathbb{R}$, $g_1 : \mathbb{R} \to \mathbb{R}$ are continuous functions and $g_2 \in \mathbb{R}$;

(H3) there exist constants $c_i \geq 0 (i = 1...6)$ such that for any $u \in \mathbb{R}$,

$$|f_1(u)| \leq c_1 + c_2|u|^{p-1}, \tag{3.48}$$

$$|f_2(u)| \leq c_3 + c_4|u|^{\frac{p}{2}-1}, \tag{3.49}$$

$$|g_1(u)| \leq c_5 + c_6|u|^{\frac{p}{2}}, \tag{3.50}$$

(H4) there exist constants $a, \lambda > 0, b, c \geq 0$ such that for any $(u, w) \in \mathbb{R}^2$,

$$\lambda u f(u, w) + w g(u, w) \geq a|u|^p - b\Big(\lambda|u|^2 + |w|^2\Big) - c, \tag{3.51}$$

Using hypothesis (H1), we are in the framework

$$V \subset L^p(\Omega) \subset H \equiv H' \subset L^{p'}(\Omega) \subset V'. \tag{3.52}$$

In particular, an element $u \in H'$ or $u \in (L^p(\Omega))'$ is identified to an element $u \in H$ or $u \in L^{p'}(\Omega)$ by $< u, v >= \int_\Omega uv$.

Lastly, we use the classical spaces $L^q(0,T;X)$ $(1 \leq q \leq \infty)$ of measurable vector valued functions $f : t \in (0,T) \mapsto f(t) \in X$ where X is a separable Banach space (X alternatively is U, U', V, V' or H here) [see Section 2.1.2]. The derivative $\partial_t f$ or $\frac{df}{dt}$ of this function is taken in the space of vector valued distributions from $(0,T)$ onto X. A distribution f and a function $\underline{f} \in L^q(0,T;X)$ are identified if

$$< f, \phi >= \int_0^T \underline{f}(t)\phi(t)\, dt \;\; (\text{in } X) \;\; \forall \phi \in \mathcal{D}(0,T).$$

Recall that if $f \in L^1(0,T;X)$ and $\partial_t f \in L^1(0,T;X)$, then f is equal a.e. to a function in $C^0([0,T], X)$ [7].

Lemma 25. *Under hypotheses (H2) and (H3), the mappings* $(u,w) \in L^p(\Omega) \times H \mapsto f(u,w) \in L^{p'}(\Omega)$ *and* $(u,w) \in L^p(\Omega) \times H \mapsto g(u,w) \in H$ *are well-defined. Specifically, for any* $(u,w) \in L^p(\Omega) \times H$, *we have*

$$\|f(u,w)\|_{L^{p'}(\Omega)} \leq A_1|\Omega|^{1/p'} + A_2\|u\|_{L^p(\Omega)}^{p/p'} + A_3\|w\|_H^{2/p'}, \tag{3.53}$$

$$\|g(u,w)\|_H \leq B_1|\Omega|^{1/2} + B_2\|u\|_{L^p(\Omega)}^{p/2} + B_3\|w\|_H, \tag{3.54}$$

where the $A_i \geq 0$ $(i = 1, ..., 3)$ *and* $B_i \geq 0$ $(i = 1, ..., 3)$ *are numerical constants that depend only on the* c_i $(i = 1...6)$ *and on* p.

Proof. For $(u, w) \in \mathbb{R}^2$, we have from (H2) and (H3)

$$\begin{aligned}|f(u,w)| &\leq c_1 + c_2|u|^{p-1} + c_3|w| + c_4|w||u|^{p/2-1},\\ |g(u,w)| &\leq B_1 + B_2|u|^{p/2} + B_3|w|,\end{aligned}$$

with exactly, $B_1 = c_5$, $B_2 = c_6$ and $B_3 = |g_2|$.

If $p \neq 2$, using Young's inequality we get

$$|w||u|^{p/2-1} \leq \frac{|w|^\beta}{\beta} + \frac{|u|^{(p/2-1)\beta'}}{\beta'},$$

where $\beta = \frac{2}{p'} > 1$ and $\frac{1}{\beta} + \frac{1}{\beta'} = 1$.

Since $(\frac{p}{2} - 1)\beta' = (\frac{p}{2} - 1)2\frac{p-1}{p-2} = p - 1$, we have

$$|f(u,w)| \leq c_1 + (c_2 + \frac{c_4}{\beta'})|u|^{p-1} + c_2|w| + \frac{c_4}{\beta}|w|^\beta.$$

But $\beta > 1$ and $|w| \leq \frac{|w|^\beta}{\beta} + \frac{1}{\beta'}$, then positive constants $A_1,\ A_2$ and A_3 can be found such that

$$|f(u,w)| \leq A_1 + A_2|u|^{p-1} + A_3|w|^\beta.$$

If $p = 2$, this inequality is still valid with $A_1 = c_1,\ \ A_2 = c_2\ \ A_3 = c_3 + c_4$.

Now for $(u, v) \in L^p(\Omega) \times H$, we can write

$$\begin{aligned}\|f(u,w)\|_{L^{p'}(\Omega)} &\leq \|A_1 + A_2|u|^{p-1} + A_3|w|^\beta\|_{L^{p'}(\Omega)},\\ &\leq \|A_1\|_{L^{p'}(\Omega)} + \|A_2|u|^{p-1}\|_{L^{p'}(\Omega)} + \|A_3|w|^\beta\|_{L^{p'}(\Omega)},\\ &\leq A_1|\Omega|^{1/p'} + A_2\|u\|^{p/p'}_{L^p(\Omega)} + A_3\|w\|^{2/p'}_H,\end{aligned}$$

because $(p-1)p' = p,\ \beta p' = 2$, and similarly,

$$\begin{aligned}\|g(u,w)\|_H &\leq \|B_1 + B_2|u|^{p/2} + B_3|w|\|_H,\\ &\leq \|B_1\|_H + \|B_2|u|^{p/2}\|_H + \|A_3|w|\|_H,\\ &\leq B_1|\Omega|^{1/2} + B_2\|u\|^{p/2}_{L^p(\Omega)} + B_3\|w\|_H. \ \square\end{aligned}$$

3.5.2 Existence for the initial value problem

Definition 26 (*Weak Solutions*). Consider $\tau > 0$ and the three functions $u : t \in [0, \tau) \mapsto u(t) \in H$, $u_e : t \in [0, \tau) \mapsto u_e(t) \in H$, $w : t \in [0, \tau) \mapsto w(t) \in H$. Given $(u_0, w_0) \in H$, we say that (u, u_e, w) is a weak solution to (3.9)-(3.14) iff, for any $T \in (0, \tau)$,

1. $u : [0, T] \to H$ and $w : [0, T] \to H$ are continuous, and $u(0) = u_0$, $w(0) = w_0$ in H, (that is (3.14);

2. for a.e. $t \in (0, \tau)$, we have $u(t) \in V$, $u_e(t) \in V/\mathbb{R}$, and $u, w \in L^2(0, T; V) \cap L^p(Q_T)$, where $Q_T = (0, T) \times \Omega$;

and (u, u_e, w) verify in $\mathcal{D}'(0, T)$:

$$\tfrac{d}{dt}(u(t), v) + \int_\Omega \sigma_i \nabla(u(t) + u_e(t)) \cdot \nabla v + \int_\Omega f(u(t), w(t))v =< s_i(t), v >,$$

$$\tfrac{d}{dt}(w(t), v) + \int_\Omega g(u(t), w(t))v = 0,$$

respectively for all $v \in V$ and for all $v \in H$, and

$$\int_\Omega \sigma_i \nabla u(t) \cdot \nabla v_e + \int_\Omega (\sigma_i + \sigma_e) \nabla u_e(t) \cdot \nabla v_e =< s_i(t) + s_e(t), v_e >, \quad \forall v_e \in V/\mathbb{R}. \tag{3.55}$$

Remark 27. The weak derivatives of $u : t \in [0, T] \mapsto H$ and $w : t \in [0, T] \mapsto H$ identify to functions $\partial_t u \in L^2(0, T; V') + L^{p'}(Q_T)$ and $\partial_t w \in L^2(0, T; V') + L^{p'}(Q_T)$. Indeed the following equalities are true in $D'(0, T)$:

$$\begin{aligned} < \partial_t u, v > &= \tfrac{d}{dt}(u(t), v) \quad \forall v \in V = V \cap L^p(\Omega), \\ < \partial_t w, v > &= \tfrac{d}{dt}(w(t), v) \quad \forall v \in H. \end{aligned}$$

Naturally we have the following lemma:

Lemma 28. *The functions* (u, u_e, w) *are a weak solution to (3.9)-(3.14) iff conditions (1)-(2) of Definition 26 hold and* (u, w) *verify in* $\mathcal{D}'(0, T)$*:*

$$\begin{aligned} &\tfrac{d}{dt}(u(t), v) + a(u(t), v) + \textstyle\int_\Omega f(u(t), w(t))v =< s(t), v > \quad \forall v \in V, \\ &\tfrac{d}{dt}(w(t), v) + \textstyle\int_\Omega (u(t), w(t))v = 0 \quad \forall v \in H, \end{aligned} \tag{3.56}$$

where $a(\cdot, \cdot)$ *and* $s \in V'$ *are given in Definition 5. The function* u_e *is then recovered from (3.55).*

Lemma 29. *Any strong solution* (u, u_e, w) *on* $[0, \tau)$ *is a weak solution on* $[0, \tau)$*. Conversely, if* $\partial\Omega$ *is* C^1 *regular any weak solution* (u, u_e, w) *on* $[0, \tau)$*, such that* $u(t), \;\; u_e(t) \in H^2(\Omega)$ *for a.e.* $t \in [0, \tau)$*, is a strong solution.*

Proof. This proof is added here because it is important.

Clearly, one can easily get the first assertion of the lemma. Now, assuming that $\partial\Omega$ is C^1 regular and (u, u_e, w) is a weak solution such that $u(t), \;\; u_e(t) \in H^2(\Omega)$ for a.e. $t \in [0, \tau)$, we get $\forall v \in \mathcal{D}(\Omega)$:

$$< \partial_t u, v > + < \sigma_i \nabla(u(t) + u_e(t)), \nabla v > + < f(u(t), (w(t)), v >=< s_i(t), v >,$$

$$\Rightarrow < \partial_t u, v > - < \nabla \cdot (\sigma_i \nabla(u(t) + u_e(t)), v > + < f(u(t), w(t)), v >=< s_i(t), v >,$$

$$\Rightarrow < \partial_t u - \nabla \cdot (\sigma_i \nabla(u(t) + u_e(t)) + f(u(t), w(t)), v >=< s_i(t), v >,$$

$$\Rightarrow \; \partial_t u - \nabla \cdot (\sigma_i \nabla(u(t) + u_e(t)) + f(u(t), w(t)) = s_i(t) \;\text{ in } \mathcal{D}'(\Omega)$$

Since $s_i(t) \in L^2(\Omega)$, the above result is true a.e. in Ω. Hence,

$$\partial_t u - \nabla \cdot (\sigma_i \nabla(u + u_e)) + f(u, w) = s_i \;\text{ a.e. in } (0, \tau) \times \Omega.$$

Similarly, one gets

$$\partial_t w + g(u, w) = 0, \;\text{ a.e.in } (0, \tau) \times \Omega,$$

and

$$\nabla \cdot (\sigma_i \nabla u + (\sigma_i + \sigma_e)\nabla u_e) = -(s_i + s_e) \text{ a.e. in } (0,\tau) \times \Omega.$$

Now, in order to get the Neumann boundary conditions, we multiply by $v \in H^1(\Omega)$ and integrate the following equation:

$$\partial_t u - \nabla \cdot (\sigma_i \nabla (u + u_e)) + f(u,w) = s_i \text{ a.e.in } (0,\tau) \times \Omega,$$

and using Green's theorem we obtain:

$$\int_\Omega \partial_t uv - \int_\Omega \sigma_i \nabla u \cdot \nabla v + \int_\Omega \sigma_i \nabla u_e \cdot \nabla v - \int_{\partial\Omega} (\sigma_i \nabla u \cdot n + \sigma_i \nabla u_e \cdot n) v + \int_\Omega f(u,w) v = \int_\Omega s_i v.$$

Comparing with the original weak formulation, we get:

$$\int_{\partial\Omega} (\sigma_i \nabla u \cdot n + \sigma_i \nabla u_e \cdot n) v \, d\sigma = 0 \ \ \forall v \in H^1(\Omega).$$

But by the trace map $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$, $\gamma(H^1(\Omega))$ is dense in $L^2(\partial\Omega)$. Hence

$$\sigma_i \nabla u \cdot n + \sigma_i \nabla u_e \cdot n = 0 \text{ a.e.in } (0,\tau) \times \partial\Omega.$$

Similarly, one gets the other boundary condition. □

Theorem 30 (Global existence of a weak solution). *Let Ω be a bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, and let $\sigma_{i,e}$ be uniformly elliptic conductivity matrices with coefficients in $L^\infty(\Omega)$. Suppose that hypotheses (H1) to (H4) on f, g hold for some $p \geq 2$. Let there be given $u_0, w_0 \in H$ and $s_i, s_e \in L^2(\mathbb{R}^+; V')$ such that $< s_i(t) + s_e(t), 1 >= 0$ for a.e. $t > 0$. Then the system (3.9)-(3.14) has a weak solution (u, u_e, w) in the sense of Definition 26 with $\tau = +\infty$.*

Proof. Using Lemma 28, it is given in the next subsections, in three parts:

- construction of an approximate solution using the Faedo-Galerkin technique;
- *a priori* estimates on the approximate solution;
- compactness results, and convergence of the approximate solution towards a weak solution. $\square$

Construction of an approximate solution

In the following subsections, we will use the orthonormal Hilbert basis (in H) $(\psi_i)_{i\in\mathbb{N}}$ of eigenvectors defined in Theorem 6. For $m \geq 1$, we define $V_m = \text{span}(\psi_0, ..., \psi_m) \subset V$. We are looking for a couple of functions $t \mapsto (u_m(t), w_m(t))$ with

$$u_m(t) = \sum_{i=0}^{m} u_{im}(t)\psi_i \in V_m, \quad w_m(t) = \sum_{i=0}^{m} w_{im}(t)\psi_i \in V_m,$$

where $(u_{im}(t), w_{im}(t))_{i=0...m}$ are real valued functions solutions of

$$\frac{d}{dt}u_{im}(t) + \lambda_i u_{im}(t) + \int_\Omega f(u_m(t), w_m(t))\psi_i =< s(t), \psi_i >, \tag{3.57}$$

$$\frac{d}{dt}w_{im}(t) + \int_\Omega g(u_m(t), w_m(t))\psi_i = 0, \tag{3.58}$$

for $i = 0...m$, and with initial data

$$u_m(0) = u_{m0}, \quad w_m(0) = w_{m0}. \tag{3.59}$$

Since u_0 and w_0 are in H, we can take u_{m0} and w_{m0} to be the H orthogonal projections of u_0 and w_0 on V_m :

$$\|u_{m0} - u_0\|_H \to 0, \quad \|w_{m0} - w_0\|_H \to 0, \quad \text{as } m \to \infty. \tag{3.60}$$

Equations (3.57) and (3.58) make sense because $u_m(t) \in V \subset L^p(\Omega), w_m(t) \in H$ so that $f(u_m(t), w_m(t)) \in L^{p'}(\Omega) \subset V'$, $g(u_m(t), w_m(t)) \in H$ (from Lemma 25), $\psi_i \in V \subset L^p(\Omega)$ and $\psi_i \in H$. Also, one easily observes that the last three terms in equation (3.57) and the last term in equation (3.58) are continuous functions of u_{im} and w_{im}.

The initial value problem composed of the $2m+2$ differential equations (3.57) and (3.58) with initial data (3.59) has a maximal solution defined for $t \in [0, t_m)$ with u_{im} and w_{im} in C^1 (theorem of Cauchy-Peano) [18]. If (u_m, w_m) is not a global solution (i.e. $t_m < +\infty$) then it is unbounded in $[0, t_m)$. It will be shown in the next subsection that (u_m, w_m) remains bounded for all time, namely $t_m = +\infty$.

A priori estimates

The following lemma establishes uniform bounds for any $T \in (0, t_m)$, on the sequences u_m and w_m in $L^\infty(0, T; H)$, then on the sequences u_m, u'_m respectively in $L^p(Q_T) \cap L^2(0, T; V)$ and its dual $L^{p'}(Q_T) + L^2(0, T; V')$, and on the sequences w_m, w'_m in $L^2(Q_T)$. We use the norm $\|\cdot\|_{L^p(Q_T)\cap L^2(0,T;V)} = \max(\|\cdot\|_{L^p(Q_T)}, \|\cdot\|_{L^2(0,T;V)})$ and the dual norm $\|u\|_{L^{p'}(Q_T)+L^2(0,T;V')} = \inf_{u=u_1+u_2}\Big(\|u_1\|_{L^{p'}(Q_T)} + \|u_2\|_{L^2(0,T;V')}\Big)$.

Lemma 31.*(A priori Estimates). The maximal solution of the Cauchy problem (3.57)-(3.59) is defined for any $t > 0$; and for any $T > 0$, there exist positive constants C_1, C_2, C_3, C_4 such that*

$$\lambda \|u_m(t)\|_H^2 + \|w_m(t)\|_H^2 \le C_1, \quad \forall t \in [0, T], \tag{3.61}$$

$$\|u_m\|_{L^p(Q_T)\cap L^2(0,T;V)} \le C_2, \tag{3.62}$$

$$\|u'_m\|_{L^{p'}(Q_T)+L^2(0,T;V')} \le C_3, \tag{3.63}$$

$$\|w'_m\|_{L^2(Q_T)} \le C_4, \tag{3.64}$$

where $u'_m(t) = \sum_{i=0}^m u'_{im}(t)\psi_i$, $w'_m(t) = \sum_{i=0}^m w'_{im}(t)\psi_i$ *are the derivatives of* $u_m : [0,+\infty) \to V$ *and* $w_m : [0,+\infty) \to H$.

Proof. Multiplying (3.57) by λu_{im} ($\lambda > 0$ defined in hypothesis (H4)), multiplying (3.58) by w_{im}, and summing over $i = 1...m$ yields, for any $t \in [0, t_m)$,

$$\frac{1}{2}\lambda\frac{d}{dt}\|u_m\|_H^2 + \frac{1}{2}\frac{d}{dt}\|w_m\|_H^2 + \lambda a(u_m,u_m) + \int_\Omega(\lambda f(u_m,w_m)u_m + g(u_m,w_m)w_m) = \lambda < s, u_m > .$$

Using the properties of $a(\cdot,\cdot)$ from Theorem 6 (positivity and continuity) and hypothesis (H4), we have for any $t \in [0, t_m)$ and for any $\xi > 0$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\Big) + \alpha\lambda\|u_m(t)\|_V^2 + a\int_\Omega |u_m(t)|^p \\
&\le \frac{1}{2}\frac{d}{dt}\Big(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\Big) + \lambda a(u_m,u_m) + \alpha\lambda\|u_m(t)\|_H^2 \\
&\quad + \int_\Omega(\lambda u_m f(u_m,w_m) + w_m g(u_m,w_m)) + \int_\Omega b(\lambda|u_m(t)|^2 + |w_m(t)|^2) + c|\Omega| \\
&\le \lambda < s, u_m > + b(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2) + c|\Omega| + \alpha\lambda\|u_m(t)\|_H^2 \\
&\le (b+\alpha)\Big(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\Big) + c|\Omega| + \lambda\|s(t)\|_{V'}\|u_m(t)\|_V \\
&\le (b+\alpha)\Big(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\Big) + c|\Omega| + \frac{1}{2\xi}\|s(t)\|_{V'}^2 + \frac{\xi}{2}\|u_m(t)\|_V^2,
\end{aligned}$$

where we have used Young's inequality in the last line. Choosing $\xi = \alpha\lambda$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\Big) + \frac{\alpha\lambda}{2}\|u_m(t)\|_V^2 + a\int_\Omega |u_m(t)|^p \\
&\le \tilde{b}\Big(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\Big) + c|\Omega| + \frac{1}{2\alpha\lambda}\|s(t)\|_{V'}^2,
\end{aligned} \tag{3.65}$$

with $\tilde{b} = b + \alpha$.

We know that $\|u_m(0)\|_H \leq \|u_0\|_H$, $\|w_m(0)\|_H \leq \|w_0\|_H$, Ω is bounded and $\mathcal{S}_t := \int_0^t \|s(\tau)\|_{V'}^2 \, d\tau < +\infty$. From Gronwall's inequality there exists a constant $C_1 > 0$ that depends only on $\sigma_{i,e}, f, g, u_0, w_0, \Omega, s_{i,e}$ and t_m, such that

$$0 \leq t < t_m \quad \Rightarrow \lambda \|u_m(t)\|_H^2 + \|w_m(t)\|_H^2 \leq C_1.$$

Now, for any fixed $T > 0$, we have found a constant $C_1 > 0$ such that (3.61) is valid. Actually, the estimate (3.61) is the bound in $L^\infty(0, T; H)$ for u_m and w_m; and one can easily derive from it the bound for w_m in $L^2(Q_T)$. Consequently, u_m, w_m cannot explode in finite time, and the solution is global.

Coming back with C_1 into (3.65) we immediately have the estimate (3.62) of Lemma 31 with

$$C_2 = \max\Big(\Big(\frac{2C_T}{\alpha\lambda}\Big)^{1/2}, \Big(\frac{C_T}{a}\Big)^{1/p}\Big),$$

where

$$C_T = \frac{1}{2}\Big(\lambda\|u_0\|_H^2 + \|w_0\|_H^2\Big) + \tilde{b}TC_1 + CT|\Omega| + \frac{1}{2\alpha\lambda} S_T.$$

To obtain the remaining estimates on u_m', w_m', we consider the projection: $P_m : V' \to V'$ defined for $u \in V'$ by

$$P_m u = \sum_{i=1}^{m} < u, \psi_i > \psi_i.$$

It is equivalently defined as the unique element of V_m such that $< u, v > = < P_m u, v >$ $\forall\, v \in V_m$. For any $v \in V$ and any $t > 0$, remark that

$$\tfrac{d}{dt}(u_m(t), v) = (u_m'(t), v) = < u_m'(t), v >,$$

$$\textstyle\int_\Omega f(u_m(t), w_m(t))v = < f(u_m(t), w_m(t)), v >,$$

because $u'_m(t) \in V_m \subset V'$ and $f(u_m(t), w_m(t)) \in L^{p'}(\Omega)$ while $v \in V \subset L^p(\Omega)$. And then equation (3.57) reads

$$\forall v \in V_m, \forall t > 0, \;\; < u'_m(t), v >= - < Au_m(t) + f(u_m(t), w_m(t)), v > + < s(t), v >,$$

so that

$$\forall t > 0, \quad u'_m(t) = -P_m(Au_m(t) + f(u_m(t), w_m(t)) - s(t)), \tag{3.66}$$

where A is the weak operator associated to the bilinear form $a(\cdot,\cdot)$ on $V \times V$, as defined in Lemma 9.

For $T > 0$ fixed, we have from (3.62) and the continuity of A,

$$\|Au_m\|_{L^2(0,T;V')} \leq \mathbf{M}\Big(\int_0^T \|u_m(t)\|_V^2\Big)^{1/2} \leq \mathbf{M}C_2$$

and from (3.61), (3.62) and Lemma 25,

$$\begin{aligned}
\|f(u_m, w_m)\|_{L^{p'}(Q_T)} &\leq \Big\| A_1|\Omega|^{1/p'} + A_2\|u_m(t)\|_{L^p(\Omega)}^{p/p'} + A_3\|w_m(t)\|_H^{2/p'} \Big\|_{L^{p'}(0,T)} \\
&\leq A_1(|\Omega|T)^{1/p'} + A_2\|u_m\|_{L^p(Q_T)}^{p/p'} + A_3\|w_m\|_{L^2(Q_T)}^{2/p'} \\
&\leq A_1(|\Omega|T)^{1/p'} + A_2C_2^{p/p'} + A_3(C_1T)^{1/p'}.
\end{aligned}$$

It remains to bound the projection operator P_m. First, remark that the restriction of P_m to V can be viewed as an operator from V onto V (since $P_m(V') \subset V_m \subset V$), given by

$$\forall u \in V, \quad P_m u = \sum_{i=1}^m (u, \psi_i)\psi_i.$$

For $u \in H$, $P_m u$ is the orthogonal projection of u on V_m, and $\|P_m u\|_H \leq \|u\|_H$. The transpose P_m^T of $P_{m|V}$ identifies with $P_m : V' \to V'$ (simple computation), and then we have $\|P_m\|_{\mathcal{L}(V',V')} = \|P_m\|_{\mathcal{L}(V,V)}$. For $u \in V$ we can compute

$$\begin{aligned} a(P_m u, P_m u) &= \sum_{i=0}^{+\infty} \lambda_i (P_m u, \psi_i)(P_m u, \psi_i) \\ &= \sum_{i=0}^{m} \lambda_i (u, \psi_i)(u, \psi_i) \\ &\leq \sum_{i=0}^{+\infty} \lambda_i |(u, \psi_i)|^2 \\ &= a(u, u). \end{aligned}$$

As a consequence, for all $u \in V$,

$$\alpha \|P_m u\|_V^2 \leq a(P_m u, P_m u) + \alpha \|P_m u\|_H^2 \leq \mathbf{M} \|u\|_V^2 + \alpha \|u\|_H^2 \leq (\mathbf{M} + \alpha) \|u\|_V^2.$$

It shows that P_m is uniformly bounded in V': $\|P_m\|_{\mathcal{L}(V',V')} \leq \left(1 + \frac{\mathbf{M}}{\alpha}\right)^{1/2}$, and we have

$$\begin{aligned} &\|P_m(Au_m)\|_{L^2(0,T;V')} \leq \left(1 + \tfrac{\mathbf{M}}{\alpha}\right) \mathbf{M} C_2, \\ &\|P_m(f(u_m, w_m))\|_{L^{p'}(Q_T)} \leq \left(1 + \tfrac{\mathbf{M}}{\alpha}\right) \left(A_1 (|\Omega| T)^{1/p'} + A_2 C_2^{p/p'} + A_3 (C_1 T)^{1/p'}\right), \\ &\|P_m s\|_{L^2(0,T;V')} \leq \left(1 + \tfrac{\mathbf{M}}{\alpha}\right) \|s\|_{L^2(0,T;V')}. \end{aligned}$$

The bound (3.63) is a consequence of these estimates and of (3.66).

In a similar way, equation (3.58) reads

$$\forall v \in V_m \subset H,\ \forall t > 0,\ \ < w_m'(t), v >= - < g(u_m(t), w_m(t)), v >,$$

so that

$$\forall t > 0, \ \ w_m'(t) = -P_m(g(u_m(t), w_m(t))), \tag{3.67}$$

where the operator P_m can be restricted to the orthogonal projection $P_{m|H}$, in particular $\|P_m\|_{\mathcal{L}(H,H)} \leq 1$.

For $T > 0$ fixed, from (3.61), (3.62) and Lemma 25, we have (3.64):

$$\begin{aligned}
\|w'_m\|_{L^2(Q_T)} &\leq \|g(u_m, w_m)\|_{L^2(Q_T)} \\
&\leq \left\| B_1|\Omega|^{1/2} + B_2\|u_m(t)\|^{p/2}_{L^p(\Omega)} + B_3\|w_m(t)\|_H \right\|_{L^2(0,T)} \\
&\leq B_1(|\Omega|T)^{1/2} + B_2\|u_m\|^{p/2}_{L^p(Q_T)} + B_3\|w_m\|_{L^2(Q_T)} \\
&\leq B_1(|\Omega|T)^{1/2} + B_2(C_2)^{p/2} + B_3(C_1T)^{1/2} := C_4. \ \square
\end{aligned}$$

Convergence towards a solution

It is easy to see that $L^{p'}(Q_T) + L^2(0,T;V') \subset L^{p'}(0,T;V')$ since $p' \leq 2$ and $L^{p'}(\Omega) \subset V'$, by the Sobolev embedding theorem. Hence the sequence (u'_m) remains in a bounded set of $L^{p'}(0,T;V')$ while (u_m) remains in a bounded set of $L^2(0,T;V)$. It follows that (u_m) is bounded in $H^1(Q_T) \subset L^2(Q_T)$ (with compact embedding). So it has a subsequence that converges in $L^2(Q_T)$.

As a consequence, we can construct subsequences of u_m and w_m, still denoted by u_m and w_m, such that

1. $u_m \to u$ weakly in $L^p(Q_T) \cap L^2(0,T;V)$ and $u'_m \to \tilde{u}$ weakly in $L^{p'}(Q_T) + L^2(0,T;V')$,

2. $w_m \to w$ weakly in $L^2(Q_T)$ and $w'_m \to \tilde{w}$ weakly in $L^2(Q_T)$, and from the compactness result,

3. $u_m \to u$ strongly in $L^2(Q_T)$, and then a.e. in Q_T, where $u \in L^p(Q_T) \cap L^2(0,T;V)$, $w \in L^2(Q_T)$, and $\tilde{u} \in L^{p'}(Q_T) + L^2(0,T;V')$, $\tilde{w} \in L^2(Q_T)$.

For $i \geq 1$ fixed and $\phi \in \mathcal{D}(0,T)$, we naturally have

$$\begin{array}{rcl} -\int_0^T \int_\Omega u'_m \psi_i \phi & = & \int_0^T \int_\Omega u_m \psi_i \phi' \;\;\rightarrow\;\; \int_0^T \int_\Omega u \psi_i \phi', \\ -\int_0^T \int_\Omega w'_m \psi_i \phi & = & \int_0^T \int_\Omega w_m \psi_i \phi' \;\;\rightarrow\;\; \int_0^T \int_\Omega w \psi_i \phi', \end{array}$$

because $\psi_i \phi' \in L^2(Q_T) \cap L^p(Q_T) \cap L^2(0,T;V)$. As a consequence, we have in the space $\mathcal{D}'(0,T)$ of distribution on (0,T),

$$\frac{d}{dt}(u(t), \psi_i) =< \tilde{u}(t), \psi_i >, \quad \frac{d}{dt}(w(t), \psi_i) =< \tilde{w}(t), \psi_i > \tag{3.68}$$

Since $a(\cdot,\cdot)$ is bilinear and continuous on $V \times V$ and $\psi_i \phi \in L^p(Q_T) \cap L^2(0,T;V)$ for any $\phi \in \mathcal{D}(0,T)$, we have

$$\forall \phi \in \mathcal{D}(0,T), \quad \int_0^T a(u_m(t), \phi(t)\psi_i)\, dt \rightarrow \int_0^T a(u(t), \phi(t)\psi_i)\, dt.$$

Concerning the nonlinear terms, we use hypothesis (H2) to write

$$f(u_m, w_m) = f_1(u_m) + f_2(u_m)w_m = f_1(u_m) + (f_2(u_m) - f_2(u))w_m + f_2(u)w_m,$$

$$g(u_m, w_m) = g_1(u_m) + g_2 w_m.$$

Now, we have $u_m \rightarrow u$ a.e. in Q_T and f_1 is continuous, so that $f_1(u_m) \rightarrow f_1(u)$ a.e. in Q_T; and $f_1(u_m)$ is uniformly bounded in $L^{p'}(Q_T)$,

$$\|f_1(u_m)\|_{L^{p'}(Q_T)} \leq \|c_1 + c_2|u_m|^{p-1}\|_{L^{p'}(Q_T)} \leq c_1(|\Omega|T)^{1/p'} + c_2\|u_m\|^{p/p'}_{L^p(Q_T)}.$$

Knowing that $L^{p'}(Q_T)$ is reflexive ($6/5 \leq p' \leq 2$), there exists a subsequence $f_1(u_m)$ which is weakly convergent. Hence $f_1(u_m) \rightarrow f_1(u)$ weakly in $L^{p'}(Q_T)$:

$$\forall \phi \in \mathcal{D}(0,T), \quad \int_0^T (f_1(u_m(t)), \phi(t)\psi_i)\, dt \rightarrow \int_0^T (f_1(u(t)), \phi(t)\psi_i)\, dt.$$

Similarly, g_1 is continuous and then $g_1(u_m) \rightarrow g_1(u)$ a.e. in Q_T; and $g_1(u_m)$ is bounded in $L^2(Q_T)$,

$$\|g_1(u_m)\|_{L^2(Q_T)} \leq \|c_5 + c_6|u_m|^{p/2}\|_{L^2(Q_T)} \leq c_5(|\Omega|T)^{1/2} + c_6\|u_m\|^{p/2}_{L^p(Q_T)},$$

and then $g_1(u_m) \to g_1(u)$ weakly in $L^2(Q_T)$,

$$\forall \phi \in \mathcal{D}(0,T), \quad \int_0^T (g_1(u_m(t)), \phi(t)\psi_i)\, dt \to \int_0^T (g_1(u(t)), \phi(t)\psi_i)\, dt.$$

Since $w_m \to w$, weakly in $L^2(Q_T)$ we naturally have

$$\forall \phi \in \mathcal{D}(0,T), \quad \int_0^T (g_2 w_m(t), \phi(t)\psi_i)\, dt \to \int_0^T (g_2 w(t), \phi(t)\psi_i)\, dt.$$

As $f_2(u)\phi(t)\psi_i \in L^2(Q_T)$ from hypothesis (H3), the weak convergence of w_m in $L^2(Q_T)$ also implies that

$$\forall \phi \in \mathcal{D}(0,T), \quad \int_0^T (f_2(u) w_m(t), \phi(t)\psi_i)\, dt \to \int_0^T (f_2(u) w(t), \phi(t)\psi_i)\, dt.$$

The remaining term in f is such that

$$\Big| \int_0^T \int_\Omega (f_2(u_m(t)) - f_2(u(t))) w_m(t)\phi(t)\psi_i \, dxdt \Big| \leq \|(f_2(u_m) - f_2(u))\phi\psi_i\|_{L^2(Q_T)} \|w_m\|_{L^2(Q_T)},$$

and

$$\|(f_2(u_m) - f_2(u))\phi\psi_i\|^2_{L^2(Q_T)} = < (f_2(u_m) - f_2(u))^2, (\phi\psi_i)^2 > .$$

the duality product on the right hand side makes sense because $(\phi\psi_i)^2 \in L^{p/2}(Q_T)$ and $f_2(u_m)^2$ and $f_2(u)^2$ are bounded in $L^\beta(Q_T)$ where $\beta > 1$ is given by $\frac{2}{\beta} + \frac{2}{p} = 1$:

$$\|f_2(u_m)^2\|_{L^\beta(Q_T)} \leq \|c_3 + c_4 |u_m|^{p/2-1}\|_{L^\beta(Q_T)} \leq c_3(|\Omega|T)^{1/\beta} + c_4 \|u_m\|^{1/\beta}_{L^p(Q_T)},$$

because $(p/2-1)\beta = p$. Again we have $(f_2(u_m) - f_2(u))^2 \to 0$ weakly in $L^\beta(Q_T)$. Consequently,

$$\forall \phi \in \mathcal{D}(0,T), \quad \|(f_2(u_m) - f_2(u))\phi\psi_i\|_{L^2(Q_T)} \to 0.$$

Since $\|w_m\|_{L^2(Q_T)}$ is bounded, we finally have

$$\forall \phi \in \mathcal{D}(0,T), \quad \int_0^T ((f_2(u_m(t)) - f_2(u(t)))w_m, \phi(t)\psi_i)\, dt \to 0.$$

Gathering all these results, the functions u and w verify, for any $i \geq 1$,

$$\frac{d}{dt}(u(t), \psi_i) + a(u(t), \psi_i) + < f(u(t), w(t)), \psi_i > = < s(t), \psi_i >$$

$$\frac{d}{dt}(w(t), \psi_i) + < g(u(t), w(t)), \psi_i > = 0,$$

in the space of distributions $\mathcal{D}'(0, T)$, for any $i \geq 0$. Since $(\psi_i)_{i \geq 0}$ is dense in V, this is exactly the desired result (Lemma 28).

3.5.3 Continuity

We have $u \in L^2(0, T; V) \cap L^p(Q_T)$, $w \in L^2(Q_T)$ and their weak derivatives $\partial_t u$ and $\partial_t w$ are respectively in $L^2(0, T; V') + L^{p'}(Q_T)$ and $L^2(Q_T)$ (by(3.68)). Therefore, the function $u : t \in [0, T] \mapsto u(t) \in V$ is weakly continuous, and the function $w : t \in [0, T] \mapsto w(t) \in H$ is continuous.

But having the following identity in $\mathcal{D}'(0, T)$:

$$< \partial_t u(t), u(t) > = \frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2,$$

we get:

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 = -a(u(t), u(t)) - < f(u(t), w(t)), u(t) > + < s(t), u(t) >,$$

so that $t \mapsto \|u(t)\|_H^2$ is $H^1(0, T)$, and then it is continuous from $[0, T]$ to $\mathbb{R}$. As a consequence, the function $u : t \in [0, T] \mapsto u(t) \in H$ is continuous. Since $u_m(0) \to u_0$ and $w_m(0) \to w_0$ in H, we easily prove that $u(0) = u_0$ and $w(0) = w_0$.

3.5.4 Uniqueness

Assume that (u_1, u_{e_1}, w_1) and (u_2, u_{e_2}, w_2) are two weak solutions of (3.9)-(3.14) with the same initial data $u_1(0) = u_2(0) = u_0$ and $w_1(0) = w_2(0) = w_0$. For any $u \in L^2(0,T;V) \cap L^p(Q_T)$ and $w \in L^2(Q_T)$ with $\partial_t u \in L^2(0,T;V') + L^{p'}(Q_T)$ and $\partial_t w \in L^2(Q_T)$, we have in $\mathcal{D}'(0,T)$ that

$$< \partial_t u(t), u(t) >= \frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2, \quad < \partial_t w(t), w(t) >= \frac{1}{2}\frac{d}{dt}\|w(t)\|_H^2.$$

As a consequence, one can easily prove that

$$\frac{1}{2}\frac{d}{dt}\|u_1 - u_2\|_H^2 + a(u_1 - u_2, u_1 - u_2) + \int_\Omega (f(u_1, w_1) - f(u_2, w_2))(u_1 - u_2) = 0,$$

and

$$\frac{1}{2}\frac{d}{dt}\|w_1 - w_2\|_H^2 + \int_\Omega (g(u_1, w_1) - g(u_2, w_2))(w_1 - w_2) = 0.$$

With a linear combination of these two equations, we will be able to conclude using Gronwall's inequality if we can bound below

$$\Phi(u_1, w_1, u_2, w_2) = \int_\Omega \Big[\mu(f(u_1, w_1) - f(u_2, w_2))(u_1 - u_2) + (g(u_1, w_1) - g(u_2, w_2))(w_1 - w_2)\Big]\, dx$$

for some $\mu > 0$. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(u, w) = \begin{bmatrix} \mu f(u, w) \\ g(u, w) \end{bmatrix}, \quad \forall (u, w) \in \mathbb{R}^2,$$

and denote by $z \in \mathbb{R}^2$ the vector $z = (u, w)^T \in \mathbb{R}^2$. Then we have

$$\Phi(u_1, w_1, u_2, w_2) = \Phi(z_1, z_2) = \int_\Omega (F(z_1) - F(z_2)) \cdot (z_1 - z_2)\, dx,$$

where $\cdot$ denotes the inner product in $\mathbb{R}^2$. Here F is continuously differentiable, so that Taylor expansion with an integral remainder implies that $\forall z_1, z_2 \in \mathbb{R}^2$

$$F(z_1) - F(z_2) = \int_0^1 [\nabla F(z_\theta)](z_1 - z_2)\, d\theta$$

where $z_\theta = \theta z_1 + (1-\theta)z_2\theta$ and $\nabla F = \begin{pmatrix} \mu\partial_u f & \mu\partial_w f \\ \partial_u g & \partial_w g \end{pmatrix}$. Now, let $Q(z) = \frac{1}{2}(\nabla F(z)^T + \nabla F(z))$ be the symmetric part of ∇F for $z \in \mathbb{R}^2$, and denote by $\lambda_1(z) \leq \lambda_2(z)$ its eigenvalues. We can complete the proof under the hypothesis that

$$\exists C \in \mathbb{R},\ \ C < 0,\ \ \forall z \in \mathbb{R}^2,\ \ \lambda_2(z) \geq \lambda_1(z) \geq C. \tag{3.69}$$

In that case, we have for any $z_1, z_2 \in \mathbb{R}^2$,

$$\begin{aligned} \Phi(z_1, z_2) &= \int_\Omega \int_0^1 (z_1 - z_2)^T [\nabla F(z_\theta)](z_1 - z_2)\, d\theta dx \\ &\geq C \int_\Omega \int_0^1 |z_1 - z_2|^2\, d\theta dx \\ &\geq C\max(1, \mu^{-1})\Big(\mu\|u_1 - u_2\|_H^2 + \|w_1 - w_2\|_H^2\Big). \end{aligned}$$

Taking $Y(t) = \Big(\mu\|u_1(t) - u_2(t)\|_H^2 + \|w_1(t) - w_2(t)\|_H^2\Big)$, we get:

$$\frac{1}{2}Y'(t) \leq -C\max(1, \mu^{-1})Y(t), \tag{3.70}$$

for any $t \in [0, T]$. Now using the lemma of Gronwall, we obtain the result.

Theorem 32. *If the condition (3.69) is satisfied, then the solution obtained in Theorem 30 is unique.*

3.5.5 Stability with respect to the data

This part was not done in the article of Bourgault et al.

In this section, we prove the stability of the solution with respect to the data.

Theorem 33 (Stability with respect to the data). *Assume that there exist numbers* $\mu > 0,\ d_1 > 0$*, such that*

$$\begin{aligned} & \mu\Big(f(u,w) - f(u',w')\Big)(u - u') + \Big(g(u,w) - g(u',w')\Big)(w - w') \\ \geq\ & -d_1(\mu|u - u'|^2 + |w - w'|^2) \quad \forall u,\ u',\ w,\ w' \in \mathbb{R}. \qquad (*) \end{aligned}$$

Assume that $s_{i,e} \in L^2(0,T;L^2(\Omega))$.

Let (u^k, w^k) be solutions of (3.9)-(3.14) with respect to data $s^k_{i,e}$, u^k_0, w^k_0, $(k = 1, 2)$, respectively.

Then there exist numbers $C > 0$ and $D > 0$, which depend only on μ, d_1 and m such that:

$$\begin{aligned} &\mu\|u^1(t) - u^2(t)\|^2_H + \|w^1(t) - w^2(t)\|^2_H \\ \leq\ & (e^{Dt} - 1)C\Big(\|s^1_i(t) - s^2_i(t)\|_H + \|s^1_e(t) - s^2_e(t)\|_H\Big) \\ &+e^{Dt}\Big(\mu\|u^1_0 - u^2_0\|^2_H + \|u^1_0 - u^2_0\|^2_H\Big). \end{aligned} \tag{3.71}$$

Remark. $(*)$ is satisfied for instance if f and g satisfy global Lipschitz conditions.

Proof. Set $U = u^1 - u^2$, $W = w^1 - w^2$, and similarly for U_i and U_e. Using equations (3.9) and (3.10), one can easily prove that:

$$\begin{aligned} &\int_\Omega U_t U + \int_\Omega \Big(f(u^1, w^1) - f(u^2, w^2)\Big)U + \int_\Omega \sigma_i \nabla U_i \cdot \nabla U_i + \int_\Omega \sigma_e \nabla U_e \cdot \nabla U_e \\ =\ & \int_\Omega (s^1_i - s^2_i)U_i + \int_\Omega (s^1_e - s^2_e)U_e. \end{aligned}$$

Since $m|x|^2 \leq x^T\sigma_{i,e}x \leq M|x|^2$, $\quad (x \in \mathbb{R}^d)$, then

$$\begin{aligned} &\textstyle\int_\Omega U_t U + \int_\Omega \Big(f(u^1, w^1) - f(u^2, w^2)\Big)U \\ \leq\ & -m \textstyle\int (|\nabla U_e|^2 + |\nabla U_i|^2) + \int_\Omega (s^1_i - s^2_i)U + \int_\Omega (s^1_i + s^1_e - s^2_i - s^2_e)U_e. \end{aligned}$$

Using Poincaré's inequality, we have $\int |\nabla U_{i,e}|^2 \geq c_0\|U_{i,e}\|^2_V$. This implies:

$$\begin{aligned} &\textstyle\int_\Omega U_t U + \int_\Omega \Big(f(u^1, w^1) - f(u^2, w^2)\Big)U \\ \leq\ & -mc_0(\|U_e\|^2_V + \|U_i\|^2_V) + \|s^1_i - s^2_i\|_H\|U\|_H + (\|s^1_i - s^2_i\|_H + \|s^1_e - s^2_e\|_H)\|U_e\|_H \\ \leq\ & \|s^1_i - s^2_i\|_H\|U\|_H + \tfrac{1}{4mc_0}\Big(\|s^1_i - s^2_i\| + \|s^1_e - s^2_e\|\Big)^2 . (**) \end{aligned}$$

Similarly,

$$\int_\Omega W_t W + \int_\Omega \Big(g(u^1, w^1) - g(u^2, w^2)\Big)W = 0. \qquad (***)$$

Then $(*)$ and $(**)$ yield:

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\Big(\mu\|U\|_H^2 + \|W\|_H^2\Big) \\
\leq\ & \mu\|s_i^1 - s_i^2\|_H\|U\|_H + \tfrac{\mu}{4mc_0}\Big(\|s_i^1 - s_i^2\|_H + \|s_e^1 - s_e^2\|_H\Big)^2 + d_1(\mu\|U\|_H^2 + \|W\|_H^2) \\
\leq\ & \sqrt{\mu}\Big(\|s_i^1 - s_i^2\|_H + \|s_e^1 - s_e^2\|_H\Big)\sqrt{\mu\|U\|_H^2 + \|W\|_H^2} \\
& + \tfrac{\mu}{4mc_0}\Big(\|s_i^1 - s_i^2\|_H + \|s_e^1 - s_e^2\|_H\Big)^2 + d_1(\mu\|U\|_H^2 + \|W\|_H^2) \\
\leq\ & \tfrac{2mc_0+1}{4mc_0}\mu\Big(\|s_i^1 - s_i^2\|_H + \|s_e^1 - s_e^2\|_H\Big)^2 + (d_1 + \tfrac{\mu}{2})(\mu\|U\|_H^2 + \|W\|_H^2).
\end{aligned}
$$

Set $z = \mu\|U\|^2 + \|W\|^2$, then $z' \leq Az + B$, where

$$A = 2d_1 + \mu,$$

$$B = \frac{2mc_0 + 1}{4mc_0}\mu\Big(\|s_i^1 - s_i^2\|_H + \|s_e^1 - s_e^2\|_H\Big)^2, \tag{3.72}$$

and

$$z(0) = \mu\|u_0^1 - u_0^2\|_H^2 + \|w_0^1 - w_0^2\|_H^2 =: z_0. \tag{3.73}$$

Therefore, by Gronwall's inequality we obtain:

$$z(t) \leq \frac{-B}{A} + \Big(z_0 + \frac{B}{A}\Big)e^{At}, \tag{3.74}$$

and the theorem follows. □

3.6 Examples

In this section we check conditions (3.51) (see (H4)), (3.69), and $(*)$ in three typical situations.

3.6.1 FitzHugh-Nagumo

The FitzHugh-Nagumo model reads as

$$f(u,w) = u(u-a)(u-1) + w, \quad g(u,w) = -\epsilon(ku - w),$$

with $0 < a < 1, k, \epsilon > 0$. The functions f and g are obviously of the form (3.47) with f_1, f_2, g_1 continuous and $g_2 = \epsilon$. Using Young's inequality, we have

$$|u|^2 \leq \frac{2}{3}|u|^3 + \frac{1}{3}, \quad |u| \leq \frac{|u|^3}{3} + \frac{2}{3}, \quad |u| \leq \frac{|u|^2}{2} + \frac{1}{2}, \tag{3.75}$$

and then (H3) holds with $p = 4$ (and $c_4 = 0$):

$$\begin{aligned} |f_1(u)| &= |u(u-a)(u-1)| \leq \tfrac{2}{3}a + \tfrac{1}{3}(1+a) + \Big(\tfrac{1}{3}a + \tfrac{2}{3}(1+a) + 1\Big)|u|^3, \\ |f_2(u)| &= 1, \\ |g_1(u)| &= \epsilon k|u| \leq \tfrac{1}{2}\epsilon k + \tfrac{1}{2}\epsilon k|u|^2. \end{aligned}$$

Consider the function $E(u,w) = \epsilon k u f(u,w) + w g(u,w)$ defined in $\mathbb{R}^2$. We have

$$E(u,w) = \epsilon k u^4 - \epsilon k(1+a)u^3 + \epsilon k a u^2 + \epsilon w^2 \geq \epsilon k\Big(|u|^4 - (1+a)|u|^3\Big).$$

with Young's inequality, we find a constant $\gamma > 0$ such that

$$(1+a)|u|^3 \leq \frac{|u|^4}{2} + \gamma.$$

Consequently,

$$E(u,w) + \epsilon k\gamma \geq \frac{\epsilon k}{2}|u|^4,$$

which is exactly (H4) with $\lambda = k\epsilon, a = k\epsilon/2, b = 0$ and $c = k\epsilon\gamma$.

Regarding the uniqueness of the solution, we verify condition (3.69) to apply Theorem 32. One easily calculates

$$\nabla F(z) = \begin{bmatrix} \mu(3u^2 - 2(1+a)u + a)) & \mu \\ -\epsilon k & \epsilon \end{bmatrix}.$$

Taking $\mu = \epsilon k$, we get rid of the antisymmetric part in the quadratic form (i.e. $Q(z) = \begin{bmatrix} \epsilon k(3u^2 - 2(1+a)u + a) & 0 \\ 0 & \epsilon \end{bmatrix}$), and we bound the eigenvalues ($\lambda_1(z) = \epsilon k(3u^2 - 2(1+a)u + a)$, $\lambda_2(z) = \epsilon$) by $C = \epsilon \min(k(a - (1+a)^2/3), 1)$.

Now we show the stability of the model by applying Theorem 33. Hence, we verify $(*)$. We have;

$$\begin{aligned} & u(u-a)(u-1) - u'(u'-a)(u'-1) \\ = \; & f_u(u' + \theta(u-u'))(u-u'), \qquad\qquad (\theta \in (0,1)) \\ = \; & \Big\{(u_\theta - a)(u_\theta - 1) + u_\theta(u_\theta - 1) + u_\theta(u_\theta - a)\Big\}(u-u') \\ = \; & \{3u_\theta^2 + u_\theta(-2a-2) + a\}(u-u'). \end{aligned}$$

Then

$$\begin{aligned} & \mu\Big(f(u,w) - f(u',w')\Big)(u-u') + \Big(g(u,w) - g(u',w')\Big)(w-w') \\ = \; & \mu\{3u_\theta^2 + u_\theta(-2a-2) + a\}(u-u')^2 + \mu(w-w')(u-u') - \epsilon k(u-u')(w-w') \\ & +\epsilon(w-w') \\ \geq \; & -d_1(\mu|u^1 - u^2|^2 + |w^1 - w^2|^2), \end{aligned}$$

where the last line follows because $3u_\theta^2 + u_\theta(-2a-2) + a$ is bounded from below.

3.6.2 Aliev-Panfilov

The Aliev-Panfilov model is

$$f(u,w) = ku(u-a)(u-1) + uw, \quad g(u,w) = \epsilon(ku(u-1-a) + w),$$

with $0 < a < 1, k, \epsilon > 0$. the functions f and g are obviously of the form (3.47) with f_1, f_2, g_1 continuous and $g_2 = \epsilon$. Using the inequalities (3.75), we get (H3) with

$p = 4$ (and $c_4 = 1, c_3 = 0$):

$$\begin{aligned}
|f_1(u)| &= k|u(u-a)(u-1)| \le \tfrac{2}{3}ka + \tfrac{1}{3}k(1+a) + \Big(\tfrac{1}{3}a + \tfrac{2}{3}(1+a) + 1\Big)k|u|^3, \\
|f_2(u)| &= |u|, \\
|g_1(u)| &= \epsilon k|u(u-1-a)| \le \tfrac{1}{2}\epsilon k(1+a) + \Big(\tfrac{1}{2}(1+a) + 1\Big)\epsilon k|u|^2.
\end{aligned}$$

Now, we consider the function $E(u,w) = \lambda u f(u,w) + w g(u,w)$.It is

$$E(u,w) = \lambda k u^4 - \lambda k(1+a)u^3 + \lambda k a u^2 + (\lambda + \epsilon k)u^2 w - \epsilon k(1+a)uw + \epsilon w^2. \quad (3.76)$$

With $\lambda = \epsilon k$, we write

$$|(1+a)u^3| \le \frac{3}{4}\Big(\alpha|u|^3\Big)^{4/3} + \frac{1}{4}\Big(\frac{1+a}{\alpha}\Big)^4, \quad (3.77)$$

$$|u^2 w| \le \frac{1}{2}\Big(\beta|u|^2\Big)^2 + \frac{1}{2}\Big(\frac{|w|}{\beta}\Big)^2, \quad (3.78)$$

$$|uw| \le \frac{1}{2}|u|^2 + \frac{1}{2}|w|^2, \quad (3.79)$$

for any $\alpha > 0$ and $\beta > 0$, and then

$$\begin{aligned}
E(u,w) \ge{}& \Big(\epsilon k^2 - \epsilon k^2 \tfrac{3}{4}\alpha^{4/3} - \epsilon k \beta^2\Big)|u|^4 \\
& -\tfrac{1}{4}\epsilon k^2 \Big(\tfrac{1+a}{\alpha}\Big)^4 - \epsilon k \tfrac{1}{\beta^2}|w|^2 - \epsilon k \tfrac{1+a}{2}|u|^2 - \epsilon k \tfrac{1+a}{2}|w|^2 + \epsilon|w|^2 + \epsilon k^2 a|u|^2.
\end{aligned}$$

Taking $\frac{3}{4}\alpha^{4/3} = \frac{1}{2}$,and $\frac{1}{4}\epsilon k^2 = \epsilon k \beta^2$, we get (3.51) with

$$\begin{aligned}
a &= \tfrac{1}{4}\epsilon k^2, \\
b &= max\Big(\epsilon k\Big(\tfrac{1}{\beta^2} + \tfrac{1+a}{2}\Big) - \epsilon, \tfrac{1+a}{2} - ak\Big), \\
c &= \tfrac{1}{4}\epsilon k^2 \Big(\tfrac{1+a}{\alpha}\Big).
\end{aligned}$$

3.6.3 McCulloch

The model introduced by McCulloch is

$$f(u,w) = bu(u-a)(u-1) + uw, \quad g(u,w) = \epsilon(-ku + w),$$

with $0 < a < 1$, and $b, k, \epsilon > 0$. The functions f and g are obviously of the form (3.47) with f_1, f_2, g_1 continuous and $g_2 = \epsilon$. Using the inequalities (3.75), we get (H3) with $p = 4$ (and $c_4 = 1, c_3 = 0$):

$$
\begin{aligned}
|f_1(u)| &= b|u(u-a)(u-1)| \le \tfrac{2}{3}ba + \tfrac{1}{3}b(1+a) + \Big(\tfrac{1}{3}a + \tfrac{2}{3}(1+a) + 1\Big)b|u|^3, \\
|f_2(u)| &= |u|, \\
|g_1(u)| &= \epsilon k|u| \le \tfrac{1}{2}\epsilon k + \tfrac{1}{2}\epsilon k|u|^2.
\end{aligned}
$$

Using again (3.77)-(3.79), we have this time

$$
\begin{aligned}
E(u,w) &= \lambda b u^4 - \lambda b(1+a)u^3 + \lambda b a u^2 + \lambda u^2 w - \epsilon k u w + \epsilon w^2 \\
&\ge \lambda\Big(b - \tfrac{3}{4}\alpha^{4/3}b - \tfrac{\beta^2}{2}\Big)u^4 - \tfrac{1}{4}\Big(\tfrac{1+a}{\alpha}\Big)^4 \lambda b \\
&\quad - \tfrac{1}{2\beta^2}\lambda|w|^2 - \tfrac{\epsilon k}{2}|u|^2 - \tfrac{\epsilon k}{2}|w|^2 + \epsilon|w|^2 + \lambda b a|u|^2,
\end{aligned}
$$

and (3.51) holds if we take

$$
\frac{3}{4}\alpha^{3/4} = \frac{1}{2}, \quad \text{and} \quad \frac{1}{4}b = \frac{\beta^2}{2}.
$$

Chapter 4

Regularity of the weak solution

In this chapter, we study the regularity of the weak solution of the bidomain model.

By Chapter 3, the weak solution (u, u_e, w) verifies for a.e. $t \in (0, T)$:

$$\frac{d}{dt}(u(t), v) + \int_\Omega \sigma_i \nabla(u(t) + u_e(t)) \cdot \nabla v + \int_\Omega f(u(t), w(t))v = \int_\Omega s_i(t)v \quad \forall v \in H^1(\Omega), \tag{4.1}$$

$$\frac{d}{dt}(w(t), v) + \int_\Omega g(u(t), w(t))v = 0 \quad \forall v \in L^2(\Omega), \tag{4.2}$$

$$\int_\Omega \sigma_i \nabla u(t) \cdot \nabla v_e + \int_\Omega (\sigma_i + \sigma_e)\nabla u_e(t) \cdot \nabla v_e = \int_\Omega (s_i(t) + s_e(t))v_e \quad \forall v_e \in H^1(\Omega)/\mathbb{R}. \tag{4.3}$$

Here we assume $\sigma_{i,e} \in C^1(\bar{\Omega})$, $s_{i,e} \in L^2(0, T; L^2(\Omega))$ and the following conditions (that were obtained using Lemma 25 in Chapter 3 with $p = 2$) on f and g:

$$\|f(u, w)\|_{L^2(\Omega)} \leq A_1|\Omega|^{1/2} + A_2\|u\|_{L^2(\Omega)} + A_3\|w\|_{L^2(\Omega)}, \tag{4.4}$$

$$\|g(u, w)\|_{L^2(\Omega)} \leq B_1|\Omega|^{1/2} + B_2\|u\|_{L^2(\Omega)} + B_3\|w\|_{L^2(\Omega)}. \tag{4.5}$$

Remark. The above inequalities are satisfied for instance if f and g are quadratic polynomials. However, such an assumption is too restrictive for the models discussed in Section (3.6). Equations (4.4) and (4.5) would also follow if we could show that the weak solutions that have been obtained in Chapter 3 belong to $L^\infty(Q_T)$.

The existence of a global weak solution was proved in the previous chapter using Galerkin approximation. Indeed, we used the special orthonormal Hilbert basis (in H) $(\psi_i)_{i\in\mathbb{N}}$ of eigenvectors defined in Theorem 6 of chapter 3. For $m \geq 1$, we note $V_m = \text{span}(\psi_0, ..., \psi_m) \subset H^1(\Omega)$. The approximate solution is the couple of functions $t \to (u_m(t), w_m(t))$ with

$$u_m(t) = \sum_{i=0}^{m} u_{im}(t)\psi_i \ \in V_m, \quad w_m(t) = \sum_{i=0}^{m} w_{im}(t)\psi_i \ \in V_m,$$

where $(u_{im}(t), w_{im}(t))_{i=0,...,m}$ are real valued functions solutions of

$$\frac{d}{dt}u_{im}(t) + \lambda_i u_{im}(t) + \int_\Omega f(u_m(t), w_m(t))\psi_i =< s(t), \psi_i >, \tag{4.6}$$

$$\frac{d}{dt}w_{im}(t) + \int_\Omega g(u_m(t), w_m(t))\psi_i \ = 0, \tag{4.7}$$

for $i = 0, ..., m$, and with initial data

$$u_m(0) = u_{m0}, \quad w_m(0) = w_{m0}, \tag{4.8}$$

where u_{m0} and w_{m0} are the orthogonal projections of u_0 and w_0 on V_m, and $s(t)$ is given by Definition 5 of Chapter 3.

Proposition 1. *There exists a constant $c > 0$, depending on Ω, σ_i, and σ_e, such* *that*

$$\|u_e(t)\|_V^2 \le c\Big(\|s_i(t)+s_e(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)}\Big), \qquad \forall t \in (0,T).$$

Proof. Equation (4.3) reads:

$$\int_\Omega (\sigma_i+\sigma_e)\nabla u_e \nabla v_e = \int_\Omega (s_i+s_e)v_e - \int \sigma_i \nabla u \nabla v_e, \quad \forall\, v_e \in V/\mathbb{R},$$

in particular for $v_e = u_e$ we have:

$$\int_\Omega (\sigma_i+\sigma_e)|\nabla u_e|^2 = \int_\Omega (s_i+s_e)u_e - \int \sigma_i \nabla u \nabla u_e.$$

Now using cœrcivity of σ_i and σ_e, continuity of σ_i and the fact that its coefficients are in $L^\infty(\Omega)$, and Cauchy-Schwarz inequality, we obtain:

$$\alpha\|\nabla u_e(t)\|^2_{L^2(\Omega)} \le \|s_i(t)+s_e(t)\|_{L^2(\Omega)}\|u_e(t)\|_{L^2(\Omega)} + M\|\nabla u(t)\|_{L^2(\Omega)}\|\nabla u_e(t)\|_{L^2(\Omega)},$$

by Young's inequality for some appropriate $\epsilon > 0$ we have:

$$\alpha\|\nabla u_e(t)\|^2_{L^2(\Omega)} \le \frac{1}{\epsilon}\|s_i(t)+s_e(t)\|^2_{L^2(\Omega)} + \epsilon\|u_e(t)\|^2_{L^2(\Omega)} + \frac{M}{\epsilon}\|\nabla u(t)\|^2_{L^2(\Omega)} + M\epsilon\|\nabla u_e(t)\|^2_{L^2(\Omega)}.$$

So,

$$c_1\|\nabla u_e(t)\|^2_{L^2(\Omega)} \le \frac{1}{\epsilon}\|s_i(t)+s_e(t)\|^2_{L^2(\Omega)} + \frac{M}{\epsilon}\|\nabla u(t)\|^2_{L^2(\Omega)}.$$

Now using Poincaré-Wirtinger inequality, we get:

$$c_2\|u_e(t)\|^2_V \le \frac{1}{\epsilon}\|s_i(t)+s_e(t)\|^2_{L^2(\Omega)} + \frac{M}{\epsilon}\|\nabla u(t)\|^2_{L^2(\Omega)},$$

hence

$$\|u_e(t)\|_V^2 \le c\Big(\|s_i(t)+s_e(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)}\Big), \quad \text{a.e. } t. \quad \square$$

The next theorem follows from a simple regularity result, see for instance ([4], Th.IX.26 and Rem 25, p.182).

Theorem 1. *Let $f \in L^2(\Omega)$ such that $\int_\Omega f\,dx = 0$, and σ a matrix in $C^1(\bar{\Omega})$ which is uniformly elliptic. Let u be the unique weak solution of*

$$\int_\Omega \sigma \nabla u \cdot \nabla v dx = \int_\Omega f v\, dx, \quad \forall v \in \Big\{ v \in H^1(\Omega) : \int_\Omega v\, dx = 0 \Big\}.$$

Then $u \in H^2(\Omega)/\mathbb{R}$, $\|u\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}$ and $\sigma \nabla u \cdot n = 0$ on $\partial\Omega$.

Corollary 1.

- *If $u \in H^2(\Omega)$ with $\sigma_i \nabla u \cdot n = 0$ on $\partial\Omega$, then $\tilde{u}_e$ (solution of equation (3.23)) is in $H^2(\Omega)/\mathbb{R}$, $\|\tilde{u}_e\|_{H^2(\Omega)/\mathbb{R}} \leq c\|u\|_{H^2(\Omega)}$ and $(\sigma_i + \sigma_e)\nabla \tilde{u}_e \cdot n = 0$ on $\partial\Omega$.*

- *If $s_i + s_e \in L^2(\Omega)/\mathbb{R}$, then $\bar{u}_e$ (solution of equation (3.24)) is in $H^2(\Omega)/\mathbb{R}$, $\|\bar{u}_e\|_{H^2(\Omega)/\mathbb{R}} \leq c\|s_i + s_e\|_{L^2(\Omega)}$ and $(\sigma_i + \sigma_e)\nabla \bar{u}_e \cdot n = 0$ on $\partial\Omega$.*

- *If $u \in H^2(\Omega)$ and $s_i + s_e \in L^2(\Omega)/\mathbb{R}$, then $u_e = \bar{u}_e + \tilde{u}_e \in H^2(\Omega)/\mathbb{R}$ and $\|u_e\|_{H^2(\Omega)} \leq c(\|u\|_{H^2(\Omega)} + \|s_i + s_e\|_{L^2(\Omega)})$.*

Proposition 2. If $u \in H^2(\Omega)$ such that $\sigma_i \nabla u \cdot n = 0$ on $\partial\Omega$ and $v \in H^1(\Omega)$, then the bilinear form $a(\cdot,\cdot)$ defined in Chapter 3 (Definition 5) is such that

$$|a(u,v)| \leq C\|u\|_{H^2(\Omega)}\|v\|_{L^2(\Omega)}, \quad \text{where} \quad C = C(\sigma_{i,e}, \Omega).$$

Proof.

$$
\begin{aligned}
a(u,v) &= \int_\Omega \sigma_i \nabla[u]\cdot\nabla[v] + \int_\Omega \sigma_e \nabla \tilde{u}_e \cdot \nabla[v] \\
&= \int_\Omega \sigma_i \nabla u \cdot \nabla v + \int_\Omega \sigma_e \nabla \tilde{u}_e \cdot \nabla v \qquad (4.9)\\
&= -\int_\Omega \nabla(\sigma_i \nabla u)\cdot v \; - \int_\Omega \nabla(\sigma_e \nabla \tilde{u}_e)\cdot v.
\end{aligned}
$$

So,

$$
\begin{aligned}
|a(u,v)| &\le \Big(\|\nabla(\sigma_i\nabla u)\|_{L^2(\Omega)} + \|\nabla(\sigma_e \nabla \tilde{u}_e)\|_{L^2(\Omega)}\Big)\|v\|_{L^2(\Omega)} \\
&\le C_1(\|u\|_{H^2} + \|\tilde{u}_e\|_{H^2})\|v\|_{L^2}.
\end{aligned}
$$

Now, $\|\tilde{u}_e\|_{H^2} \le c\|[u]\|_{H^2} = c\|u-\bar{u}\|_{H^2} \le c(\|u\|_{H^2} + \|\bar{u}\|_{H^2})$.

But $\|\bar{u}\|_{H^2} = \|\bar{u}\|_{L^2} = |\bar{u}||\Omega|^{1/2} \le |\Omega|^{1/2}\int_\Omega |u|\,dx \le |\Omega|\|u\|_{L^2}$.

Hence $\|\tilde{u}_e\|_{H^2} \le c'\|u\|_{H^2}$. □

Theorem 2. *Assume $u_0,\ w_0 \in H^1(\Omega)$, $s_{i,e} \in L^2(0,T;L^2(\Omega))$. Suppose also $(u,w) \in L^2(0,T;H^1(\Omega)) \times L^2(Q_T)$ with $(u',w') \in L^2(0,T;(H^1(\Omega))') \times L^2(Q_T)$ is the weak solution of (3.9-3.14). Then*

$$u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \quad u' \in L^2(0,T;L^2(\Omega)).$$

If in addition $u_0 \in H^2(\Omega), s'_{i,e} \in L^2(0,T;L^2(\Omega))$, and

$$\|f_u(u,w)\|_{L^2(\Omega)} \le C_1 + C_2\|u\|_{L^2(\Omega)} + C_3\|w\|_{L^2(\Omega)}, \qquad (4.10)$$

for some positive constants C_1, C_2, and C_3, then

$$u \in L^\infty(0,T;H^2(\Omega)), \quad u' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

Proof.

Step 1. We multiply equation (4.6) by $u'_{im}(t)$ and sum over $i = 0,...,m$:

$$\|u'_m(t)\|^2_{L^2(\Omega)} + a(u_m(t),u'_m(t)) + (f(u_m(t),w_m(t)),u'_m(t)) =< s(t),u'_m(t) > .$$

Now, employing Cauchy-Schwarz and Young's inequality consecutively, we obtain:

$$\begin{aligned} & \|u_m'(t)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\tfrac{d}{dt}a(u_m(t),u_m(t)) \\ \leq\ & \tfrac{1}{\epsilon}\|s(t)\|^2_{L^2(\Omega)} + 2\epsilon\|u_m'(t)\|^2_{L^2(\Omega)} + \tfrac{1}{\epsilon}\|f(u_m(t),w_m(t)\|^2_{L^2(\Omega)}. \end{aligned}$$

After integrating over $(0,t)$ and considering the sup over $0 \leq t \leq T$, we get:

$$\begin{aligned} & \int_0^T \|u_m'(t)\|^2_{L^2(\Omega)}\, dt + \frac{1}{2}\sup_{0\leq t\leq T} a(u_m(t),u_m(t)) \\ \leq\ & \frac{1}{2}a(u_m(0),u_m(0)) \\ & + \int_0^T \Big(\frac{\|s(t)\|^2_{L^2(\Omega)}}{\epsilon} + 2\epsilon\|u_m'(t)\|^2_{L^2(\Omega)} + \frac{\|f(u_m(t),w_m(t))\|^2_{L^2(\Omega)}}{\epsilon}\Big)\, dt. \end{aligned}$$

Taking $\epsilon = \frac{1}{4}$:

$$\begin{aligned} & \int_0^T \|u_m'(t)\|^2_{L^2(\Omega)}dt + \sup_{0\leq t\leq T} a(u_m(t),u_m(t)) \\ \leq\ & a(u_m(0),u_m(0)) + 8\int_0^T \Big(\|s(t)\|^2_{L^2(\Omega)} + \|f(u_m(t),w_m(t))\|^2_{L^2(\Omega)}\Big)dt \\ \leq\ & \mathbf{M}\|u_m(0)\|^2_V + 8\int_0^T \Big(\|s(t)\|^2_{L^2(\Omega)} + \|f(u_m(t),w_m(t))\|^2_{L^2(\Omega)}\Big)dt. \end{aligned}$$

Knowing that $\int_0^T \|s(t)\|^2_{L^2(\Omega)}dt < +\infty$ and using equations (4.3) and (3.61), we can bound the last two terms of the above inequality by a constant C_1. Moreover, we have by equation (3.35):

$$\int_0^T \|u_m'(t)\|^2_{L^2(\Omega)}dt + \alpha \sup_{0\leq t\leq T}\|u_m(t)\|^2_V \leq \mathbf{M}\|u_m(0)\|^2_V + \alpha\sup_{0\leq t\leq T}\|u_m(t)\|^2_{L^2(\Omega)} + C_1,$$

Since $u_m(0) = u_{m0}$ which is the projection of $u_0 \in H^1(\Omega)$ on V_m, we have $\mathbf{M}\|u_m(0)\|^2_{H^1(\Omega)} \leq \mathbf{M}\|u_0\|^2_{H^1(\Omega)} \leq C_2$. And by equation (3.61), we get $\sup_{0\leq t\leq T}\|u_m(t)\|^2_{L^2(\Omega)} \leq C_3$. As a result, we obtain:

$$\int_0^T \|u_m'(t)\|_{L^2(\Omega)}dt + \alpha\sup_{0\leq t\leq T}\|u_m(t)\|^2_{H^1(\Omega)} \leq C.$$

Hence, $u_m \in L^\infty(0,T;H^1(\Omega))$ and $u_m' \in L^2(0,T;L^2(\Omega))$.

Therefore, there exists a subsequence u_m' weakly convergent to some $z \in L^2(0,T;L^2(\Omega))$.

But $u'_m \to u'$ in $L^2(0,T;(H^1(\Omega))') \supset L^2(0,T;L^2(\Omega))$. So $u' = z \in L^2(0,T;L^2(\Omega))$. Similarly, $u \in L^\infty(0,T;H^1(\Omega))$. By Proposition 1, it comes out that $u_e \in L^\infty(0,T;H^1(\Omega))$.

Step 2. The weak solution (u, u_e, w) satisfies

$$\frac{d}{dt}(u(t),v) + \int_\Omega \sigma_i \nabla u_i(t) \cdot \nabla v + \int_\Omega f(u(t),w(t))v = \int_\Omega s_i(t)v \quad \forall v \in H^1(\Omega), \tag{4.11}$$

$$\frac{d}{dt}(u(t),v) - \int_\Omega \sigma_e \nabla u_e(t) \cdot \nabla v + \int_\Omega f(u(t),w(t))v = -\int_\Omega s_e(t)v \quad \forall v \in H^1(\Omega). \tag{4.12}$$

Using $u' \in L^2(0,T;L^2(\Omega))$, Equations (4.11) and (4.12) become

$$\int_\Omega \sigma_i \nabla u_i(t) \cdot \nabla v = \int_\Omega \Big(s_i(t) - f(u(t),w(t)) - u'(t)\Big)v \tag{4.13}$$

$$\int_\Omega \sigma_e \nabla u_e(t) \cdot \nabla v = \int_\Omega \Big(s_e(t) + f(u(t),w(t)) + u'(t)\Big)v \tag{4.14}$$

for all $v \in H^1(\Omega)$ and in particular for all $v \in H^1(\Omega)/\mathbb{R}$. Therefore by Theorem 1, we conclude that u_i and u_e are in $L^2(0,T;H^2(\Omega)/\mathbb{R})$, and we have

$$\|u_i(t)\|_{H^2(\Omega)} \leq \|s_i(t) - f(u(t),w(t)) - u'(t)\|_{L^2(\Omega)} \tag{4.15}$$

$$\|u_e(t)\|_{H^2(\Omega)} \leq \|s_e(t) + f(u(t),w(t)) + u'(t)\|_{L^2(\Omega)}. \tag{4.16}$$

Consequently $u = u_i - u_e \in L^2(0,T;H^2(\Omega))$ and we have for a.e. $t \in (0,T)$

$$\|u(t)\|_{H^2(\Omega)} \leq \|s_i(t)\|_{L^2(\Omega)} + \|s_e(t)\|_{L^2(\Omega)} + 2\|f(u(t),w(t))\|_{L^2(\Omega)} + 2\|u'(t)\|_{L^2(\Omega)}. \tag{4.17}$$

Step 3. If $u_0 \in H^2(\Omega)$, $s'_{i,e} \in L^2(0,T;L^2(\Omega))$ and f_u satisfies equation (4.10), we continue the proof as follows.

Differentiate equation (4.6) with respect to t:

$$u''_{im}(t) + \lambda_i u'_{im}(t) + (\nabla f(u_m(t),w_m(t)) \cdot (u'_m(t),w'_m(t)),\psi_i) =< s'(t),\psi_i >,$$

then multiply by $u'_{im}(t)$ and sum over $i = 0, ..., m$ to get:

$$\begin{aligned} & (u''_m(t), u'_m(t)) + a(u'_m(t), u'_m(t)) \\ = & < s'(t), u'_m(t) > -(\nabla f(u_m(t), w_m(t)) \cdot (u'_m(t), w'_m(t)), u'_m(t)). \end{aligned}$$

Now using equation(3.35), we get:

$$\begin{aligned} & \tfrac{1}{2}\tfrac{d}{dt}\|u'_m(t)\|^2_{L^2(\Omega)} + \alpha\|u'_m(t)\|^2_{H^1(\Omega)} \\ \leq & < s'(t), u'_m(t) > -(\nabla f(u_m(t), w_m(t)) \cdot (u'_m(t), w'_m(t)), u'_m(t)) + \alpha\|u'_m(t)\|^2_{L^2(\Omega)}. \end{aligned}$$

Integrating and using Cauchy-Schwarz then Young's inequalities, we obtain:

$$\begin{aligned} & \frac{1}{2}\sup_{0\leq t\leq T}\|u'_m(t)\|^2_{L^2(\Omega)} + \alpha\int_0^T \|u'_m(t)\|^2_{H^1(\Omega)}dt \\ \leq & \frac{1}{2}\|u'_m(0)\|^2_{L^2(\Omega)} + (1+\alpha)\int_0^T \|u'_m(t)\|^2_{L^2(\Omega)} + 2\int_0^T \|s'(t)\|^2_{L^2(\Omega)} \\ & +2\int_0^T \Big(\|f_u(u_m(t), w_m(t))\ u'_m\|^2_{L^2(\Omega)} + \|f_w(u_m(t), w_m(t))\ w'_m\|^2_{L^2(\Omega)}\Big), \end{aligned}$$

now using the assumptions on $s'_{i,e}$ and f_u, (note that $f_w = f_2(u)$ as in (H2) and by (H3) with $p = 2$ it is uniformly bounded above), and the first part of this theorem, we have:

$$\frac{1}{2}\sup_{0\leq t\leq T}\|u'_m(t)\|^2_{L^2(\Omega)} + \beta\int_0^T \|u'_m(t)\|^2_V dt \leq C_1 + C_2\|u'_m(0)\|^2_{L^2(\Omega)}.$$

Step 4. We have:

$$\|u'_m(t)\|^2_{L^2(\Omega)} + a(u_m(t), u'_m(t)) + (f(u_m(t), w_m(t)), u'_m(t)) =< s(t), u'_m(t) > .$$

Then using proposition 2,

$$\begin{aligned} \|u'_m(t)\|^2_{L^2(\Omega)} = & < s(t), u'_m(t) > -(f(u_m(t), w_m(t)), u'_m(t)) - a(u_m(t), u'_m(t)) \\ \leq & \tfrac{1}{\epsilon}\|s(t)\|^2_{L^2(\Omega)} + 2\epsilon\|u'_m(t)\|^2_{L^2(\Omega)} + \tfrac{1}{\epsilon}\|f(u_m(t), w_m(t))\|^2_{L^2(\Omega)} \\ & +\tfrac{C}{\gamma}\|u_m(t)\|^2_{H^2(\Omega)} + C\gamma\|u'_m(t)\|^2_{L^2(\Omega)}. \end{aligned}$$

Since $s, s' \in L^2(0,T;L^2(\Omega))$, $s \in H^1(0,T;L^2(\Omega))$. So $s \in C([0,T];L^2(\Omega))$ and $\sup\limits_{0\le t\le T} \|s(t)\|_{L^2(\Omega)} \le c_0$ for some $c_0 > 0$. Also, we can easily get
$\sup\limits_{0\le t\le T} \|f(u_m(t), w_m(t))\|_{L^2(\Omega)} \le c_1$ for some $c_1 > 0$. Now, take $\epsilon = C\gamma = \frac{1}{4}$:

$$\|u_m'(t)\|^2_{L^2(\Omega)} \le c_2 + c_3\|u_m(t)\|^2_{H^2(\Omega)}, \quad \forall 0 \le t \le T.$$

In particular for t=0,

$$\|u_m'(0)\|^2_{L^2(\Omega)} \le c_2 + c_3\|u_m(0)\|^2_{H^2(\Omega)}.$$

Also, $\|u_m(0)\|^2_{H^2(\Omega)} \le \|u_0\|^2_{H^2(\Omega)}$ and $\|u_0\|^2_{H^2(\Omega)} \le c_3$ by assumption. As a result, we get:

$$\sup_{0\le t\le T} \|u_m'(t)\|^2_{L^2(\Omega)} + \beta' \int_0^T \|u_m'(t)\|^2_V dt \le C'.$$

Hence, $u_m' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$.

Consequently, by the same argument as in step 1, we get

$u' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$.

By Equation (4.17) we obtain $u \in L^\infty(0,T;H^2(\Omega))$.

This ends the proof of the theorem. $\square$

Chapter 5

Numerical Simulation of Cardiac Electrical Activity

5.1 Finite Difference

5.1.1 Monodomain model

First of all, we start by numerically solving the monodomain equations using finite differences. The continuous equations read as follows:

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla \cdot (\sigma \nabla u) + f(u,w) = 0 & \text{in} \quad \Omega \times (0,T), \\ \frac{\partial w}{\partial t} + g(u,w) = 0 & \text{in} \quad \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x) & \text{in} \quad \Omega, \\ \sigma \nabla u \cdot n = 0 & \text{on} \quad \partial\Omega \times (0,T). \end{cases}$$

One dimensional case

In the one dimensional case, we use a forward difference at time t_n and a second order centered difference for the space derivative at node x_i. Actually, we have:

$$\nabla \cdot (\sigma \nabla u^n)_i \sim \frac{1}{\Delta x^2}[\sigma_i(u^n_{i+1} - u^n_i) - \sigma_{i-1}(u^n_i - u^n_{i-1})],$$

and we get the discrete recurrence equations:

$$\begin{cases} u_i^{n+1} = u_i^n - \Delta t \nabla \cdot (\sigma \nabla u^n)_i - \Delta t\ f(u_i^n, w_i^n), \quad \forall\ 1 \le i \le N, \quad \forall\ n, \\ w_i^{n+1} = w_i^n - \Delta t\ g(u_i^n, w_i^n), \quad \forall\ 1 \le i \le N, \quad \forall\ n, \\ u_2^n = u_1^n, \quad u_{N+1}^n = u_{N-1}^n, \quad \forall\, n \\ u_i^0 = u_0(x_i), \quad w_i^0 = w_0(x_i), \quad \forall\ 1 \le i \le N. \end{cases}$$

Figure 5.1 represents the propagation of the action potential along a fibre of 70

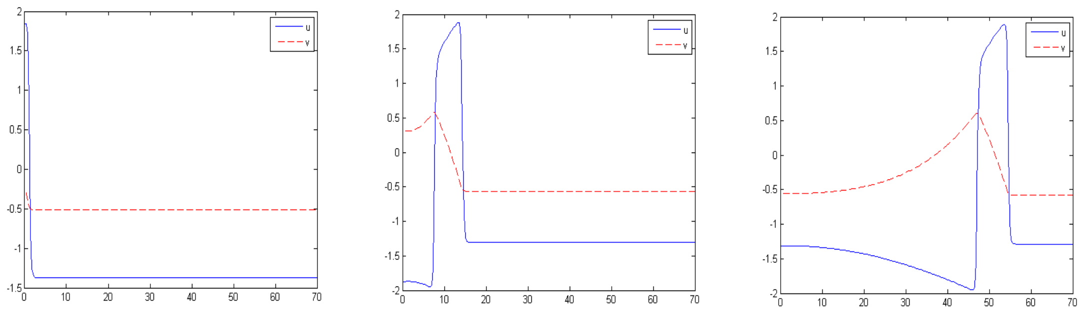

Figure 5.1: The action potential obtained from the 1D monodomain model.

units long. To generate such an action potential, we use the FitzHugh-Nagumo model with parameters:

$$\epsilon = 0.1, \quad \beta = 1, \quad \gamma = 0.5, \quad \sigma = 1.$$

Note that in this simulation we have: $\Delta x = 0.35$, $\Delta t = 0.01$.

Two dimensional case

Similarly, using a forward difference at time t_n and a second order centered difference for the space derivative at node (x_i, y_j) we obtain the discrete system in the two dimensional case:

$$\begin{cases} u_{i,j}^{n+1} = u_{i,j}^n - \Delta t \nabla \cdot (\sigma \nabla u^n)_{i,j} - \Delta t \ f(u_{i,j}^n, w_{i,j}^n), \ \ \forall \ 1 \leq i,j \leq N, \quad \forall \ n, \\ w_{i,j}^{n+1} = w_{i,j}^n - \Delta t \ g(u_{i,j}^n, w_{i,j}^n), \quad \forall \ 1 \leq i,j \leq N, \quad \forall \ n, \\ u_{2,j}^n = u_{1,j}^n, \quad u_{N+1,j}^n = u_{N-1,j}^n, \ \ \forall \, n, \quad \forall \ 1 \leq j \leq N, \\ u_{i,2}^n = u_{i,1}^n, \quad u_{i,N+1}^n = u_{i,N-1}^n, \ \ \forall \, n, \quad \forall \ 1 \leq i \leq N, \\ u_{i,j}^0 = u_0(x_i, y_j), \ \ w_{i,j}^0 = w_0(x_i, y_j), \quad \forall \ 1 \leq i,j \leq N \end{cases}$$

where

$$\begin{aligned} \nabla \cdot (\sigma \nabla u^n)_{i,j} \ \sim \ & \tfrac{1}{\Delta x^2}[\sigma_{x_{i,j}}(u_{i+1,j}^n - u_{i,j}^n) - \sigma_{x_{i-1,j}}(u_{i,j}^n - u_{i-1,j}^n)] \\ & + \tfrac{1}{\Delta y^2}[\sigma_{y_{i,j}}(u_{i,j+1}^n - u_{i,j}^n) - \sigma_{y_{i,j-1}}(u_{i,j}^n - u_{i,j-1}^n)], \end{aligned}$$

and $\sigma = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{pmatrix}$. Figure 5.2 represents the propagation of the action potential in a square [-30,30]×[-30,30] with $\Delta x = \Delta y = 0.6$ and $\Delta t = 0.05$. To generate such an action potential, we use the FitzHugh-Nagumo model with parameters:

$$\epsilon = 0.2, \quad \beta = 1, \quad \gamma = 0.5, \quad \sigma_x = 1, \quad \sigma_y = 0.5.$$

In order to initiate a spiral wave, we reset one half of the mesh to the minimal value of the action potential after the plane wave has propagated some distance. The remaining half of the plane wave then curls up and forms a spiral wave.

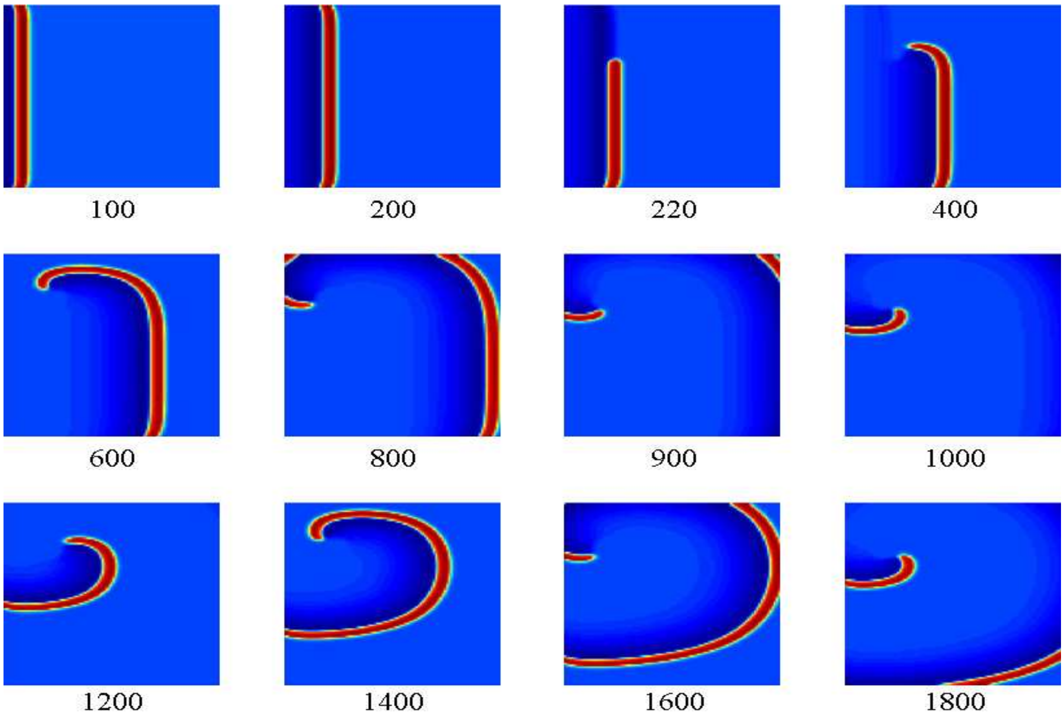

Figure 5.2: Spiral wave generated from the monodomain model by resetting half of the mesh to the minimal value of the action potential.

5.1.2 Bidomain model

Now, we will approximate the solution of the bidomain equations. One more equation is involved and we need to find another unknown which is u_e, by discretizing an elliptic equation. The system to be solved is:

$$
\begin{cases}
\frac{\partial u}{\partial t} - \nabla \cdot (\sigma_{int} \nabla u) - \nabla \cdot (\sigma_{int} \nabla u_e) + f(u,w) = 0 & \text{in} \quad \Omega \times (0,T), \\
\nabla \cdot (\sigma_{int} \nabla u + (\sigma_{int} + \sigma_{ext}) \nabla u_e) = 0, & \text{in} \quad \Omega \times (0,T), \\
\frac{\partial w}{\partial t} + g(u,w) = 0 & \text{in} \quad \Omega \times (0,T), \\
u(x,0) = u_0(x), \quad w(x,0) = w_0(x) & \text{in} \quad \Omega, \\
\sigma_{int} \nabla u \cdot n + \sigma_{int} \nabla u_e \cdot n = 0, & \text{on} \quad \partial\Omega \times (0,T), \\
\sigma_{int} \nabla u \cdot n + (\sigma_{int} + \sigma_{ext}) \nabla u_e \cdot n = 0, & \text{on} \quad \partial\Omega \times (0,T),
\end{cases}
$$

where σ_{int} and σ_{ext} denote the intracellular and extracellular conductivity matrices respectively.

One dimensional case

The one dimensional problem is treated in the same way as in the case of the monodomain problem. We obtain the discrete system:

$$\begin{cases} u_i^{n+1} = u_i^n - \Delta t \nabla \cdot (\sigma_{int} \nabla u^n)_i - \Delta t \nabla \cdot (\sigma_{int} \nabla u_e^n)_i - \Delta t \; f(u_i^n, w_i^n), \\ \qquad\qquad \forall \;\; 1 \leq i \leq N, \; \forall \;\; n, \\ w_i^{n+1} = w_i^n - \Delta t \; g(u_i^n, w_i^n), \quad \forall \;\; 1 \leq i \leq N, \quad \forall \;\; n, \\ u_e^{n+1} = A^{-1} F, \\ u_2^n = u_1^n, \quad u_{N+1}^n = u_{N-1}^n, \;\; \forall \, n \\ u_{e_2}^n = u_{e_1}^n, \quad u_{e_{N+1}}^n = u_{e_{N-1}}^n, \;\; \forall \, n \\ u_i^0 = u_0(x_i), \;\; w_i^0 = w_0(x_i), \quad \forall \;\; 1 \leq i \leq N, \end{cases}$$

where A is a tridiagonal $N \times N$ matrix given by:

$$A_{i,i-1} = \sigma_{int_{i-1}} + \sigma_{ext_{i-1}},$$

$$A_{i,i+1} = \sigma_{int_i} + \sigma_{ext_i},$$

$$A_{i,i} = -[(\sigma_{int_i} + \sigma_{ext_i}) + (\sigma_{int_{i-1}} + \sigma_{ext_{i-1}})],$$

and F is the vector given by:

$$F_i = -h^2 \nabla \cdot (\sigma_{int} \nabla u^{n+1})_i, \quad \text{where} \quad h = \Delta x.$$

Similar to the one-dimensional monodomain simulation, we used for the one-dimensional bidomain model the FitzHugh-Nagumo ionic model with parameters:

$$\epsilon = 0.1, \quad \beta = 1, \quad \gamma = 0.5, \quad \sigma_i = \sigma_e = 1.$$

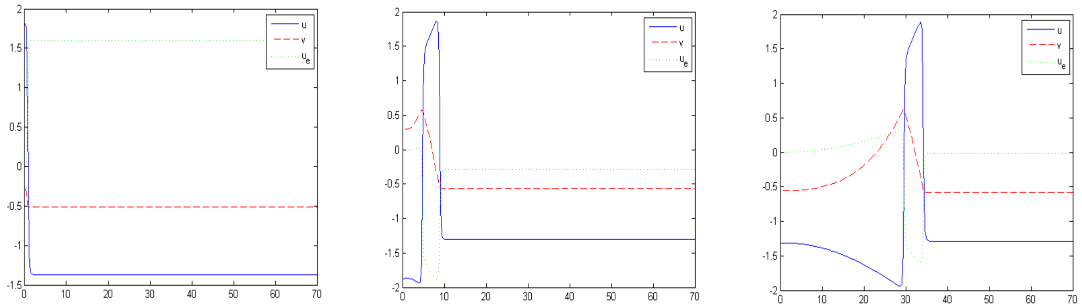

Figure 5.3: The action potential obtained from the 1D bidomain model.

Figure 5.3 represents three snapshots of the propagation of the action potential along a fibre of 70 units long with $\Delta x = 0.35$, $\Delta t = 0.01$.

Two dimensional case

Similarly, the two dimensional case results in the following discrete system:

$$\begin{cases} u_{i,j}^{n+1} = u_{i,j}^{n} - \Delta t \nabla \cdot (\sigma_{int} \nabla u^n)_{i,j} - \Delta t \nabla \cdot (\sigma_{int} \nabla u_e^n)_{i,j} - \Delta t \, f(u_{i,j}^n, w_{i,j}^n), \\ \qquad \forall \;\; 1 \leq i,j \leq N, \quad \forall \;\; n, \\ w_{i,j}^{n+1} = w_{i,j}^{n} - \Delta t \, g(u_{i,j}^n, w_{i,j}^n), \quad \forall \;\; 1 \leq i,j \leq N, \quad \forall \;\; n, \\ u_e^{n+1} = A^{-1} F, \\ u_{2,j}^n = u_{1,j}^n, \quad u_{N+1,j}^n = u_{N-1,j}^n, \quad \forall \, n, \quad \forall \;\; 1 \leq j \leq N, \\ u_{i,2}^n = u_{i,1}^n, \quad u_{i,N+1}^n = u_{i,N-1}^n, \quad \forall \, n, \quad \forall \;\; 1 \leq i \leq N, \\ u_{e_{2,j}}^n = u_{e_{1,j}}^n, \quad u_{e_{N+1,j}}^n = u_{e_{N-1,j}}^n, \quad \forall \, n, \quad \forall \;\; 1 \leq j \leq N, \\ u_{e_{i,2}}^n = u_{e_{i,1}}^n, \quad u_{e_{i,N+1}}^n = u_{e_{i,N-1}}^n, \quad \forall \, n, \quad \forall \;\; 1 \leq i \leq N, \\ u_{i,j}^0 = u_0(x_i, y_j), \quad w_{i,j}^0 = w_0(x_i, y_j), \quad \forall \;\; 1 \leq i,j \leq N. \end{cases}$$

In this case, when solving for u_e , we had to re-index by considering $k = (j-1)n + i$, i.e. we considered the points column by column. As such, the matrix A is a pentadiagonal $N^2 \times N^2$ matrix given by:

$$A_{k,k} = -[(\sigma_{int_x} + \sigma_{ext_x})_{i,j} + (\sigma_{int_x} + \sigma_{ext_x})_{i-1,j} + (\sigma_{int_y} + \sigma_{ext_y})_{i,j} + (\sigma_{int_y} + \sigma_{ext_y})_{i,j-1}],$$

$$A_{k,k-1} = (\sigma_{int_x} + \sigma_{ext_x})_{i-1,j},$$

$$A_{k,k+1} = (\sigma_{int_x} + \sigma_{ext_x})_{i,j},$$

$$A_{k,k-N} = (\sigma_{int_y} + \sigma_{ext_y})_{i,j-1},$$

$$A_{k,k+N} = (\sigma_{int_y} + \sigma_{ext_y})_{i,j},$$

and F is the N^2 vector obtained using the relation:

$$F(k) = -h^2 \nabla \cdot (\sigma_{int} u^{n+1})_{i,j}, \quad \text{where} \quad h = \Delta x = \Delta y.$$

Again, we used the FitzHugh-Nagumo ionic model in the two-dimensional bidomain model in order to simulate the propagation of the action potential in the square [-30,30]×[-30,30] with $\Delta x = \Delta y = 0.6$ and $\Delta t = 0.05$. The following values for the parameters have been used:

$$\epsilon = 0.2, \quad \beta = 1, \quad \gamma = 0.5, \quad \sigma_{int_x} = 1, \quad \sigma_{int_y} = 0.5, \quad \sigma_{ext_x} = 1, \quad \sigma_{ext_y} = 1.$$

A spiral wave has been initiated by resetting one half of the mesh to the minimal value of the action potential after the plane wave has propagated some distance. The remaining half of the plane wave then curls up and forms a spiral wave similar to the one obtained with the monodomain model. However, we can see that the plane wave generated with the bidomain model is slower than the one generated by the monodomain model. We noticed this behavior also in the one-dimensional models. Figure 5.4 illustrates a sequence of snapshots of the plane wave before and after initiation of the spiral wave.

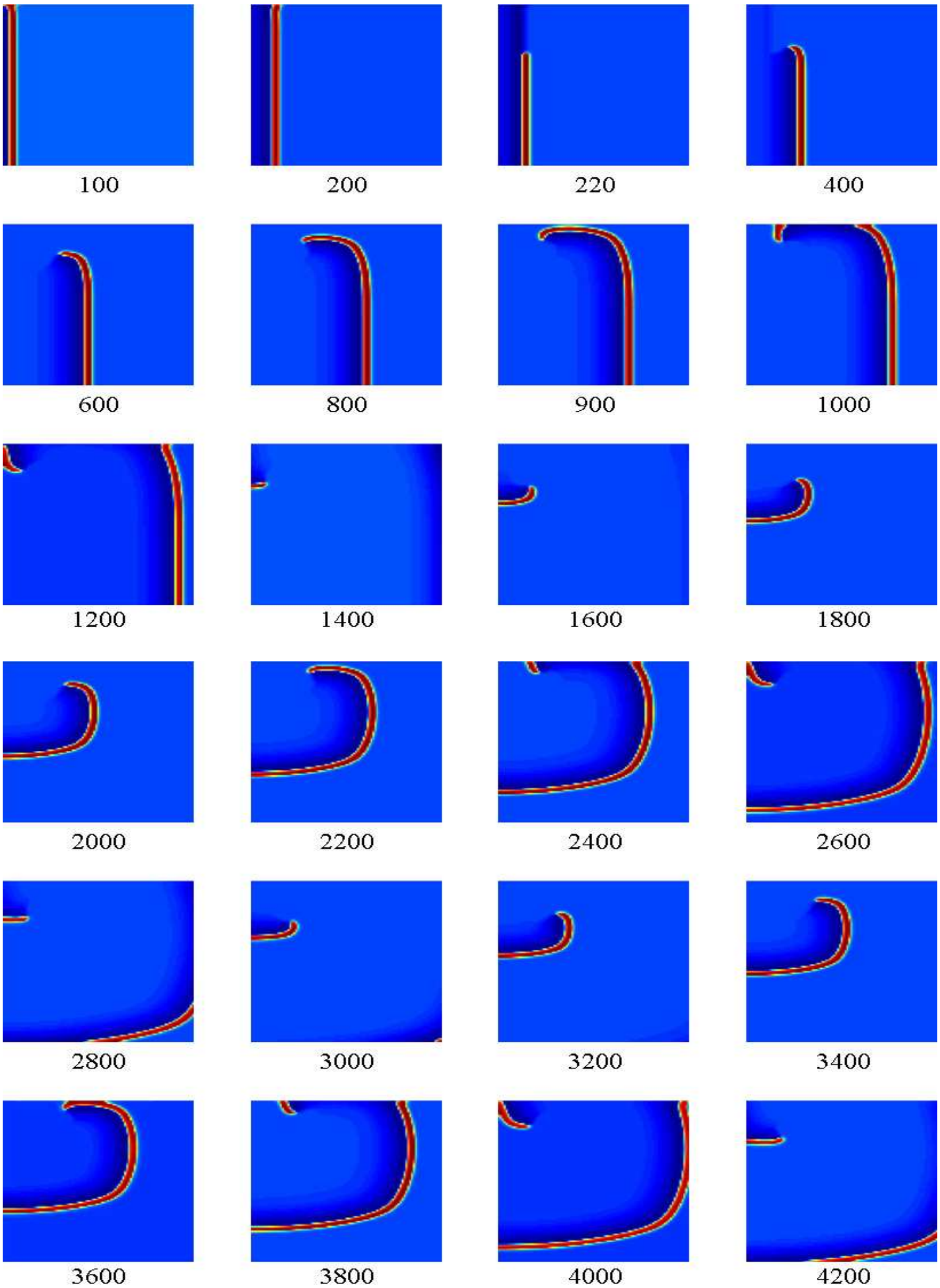

Figure 5.4: Spiral wave generated from the bidomain model.

5.2 Discretization by the finite element method

In this section, we numerically solve the two-dimensional monodomain and bidomain models by finite elements using the software Freefem++ [11]. We initiate spiral waves by the same technique as in the previous section. The domain $\Omega = [-30, 30] \times [-30, 30]$ has been meshed using Delauney triangulation, see Figure 5.5.

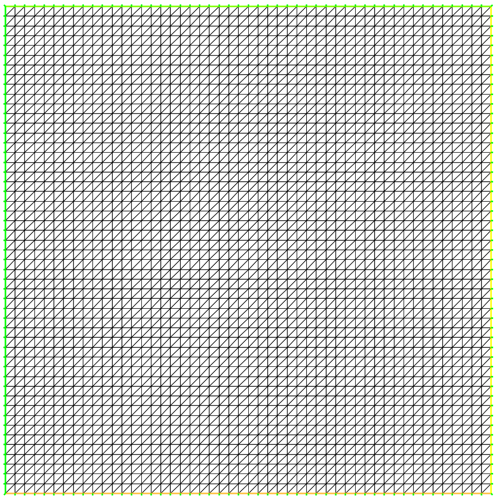

Figure 5.5: The triangular 2D mesh.

5.2.1 The 2D monodomain model

We semi-discretize in time the monodomain model given by the continuous equations of section 5.1.1. The nonlinear terms are considered in explicit form, we get the following system:

$$\begin{aligned} &-h\nabla \cdot (\sigma \nabla u^{n+1}) + u^{n+1} = u^n - hf(u^n, w^n), \\ &w^{n+1} = w^n - hg(u^n, w^n), \end{aligned} \tag{5.1}$$

where $h = \Delta t$. The weak formulation corresponding to system (5.1) reads as follows:

$$h \int_\Omega \sigma \nabla u^{n+1} \cdot \nabla v \, dx + \int_\Omega u^{n+1} v \, dx = \int_\Omega \Big(u^n - h f(u^n, w^n) \Big) v \, dx, \tag{5.2}$$

$$w^{n+1} = w^n - h g(u^n, w^n). \tag{5.3}$$

We proceed in two steps. First, we solve equation (5.2) for u^{n+1} using finite elements with elements of type P_2. Second, we compute w^{n+1} from equation (5.3).

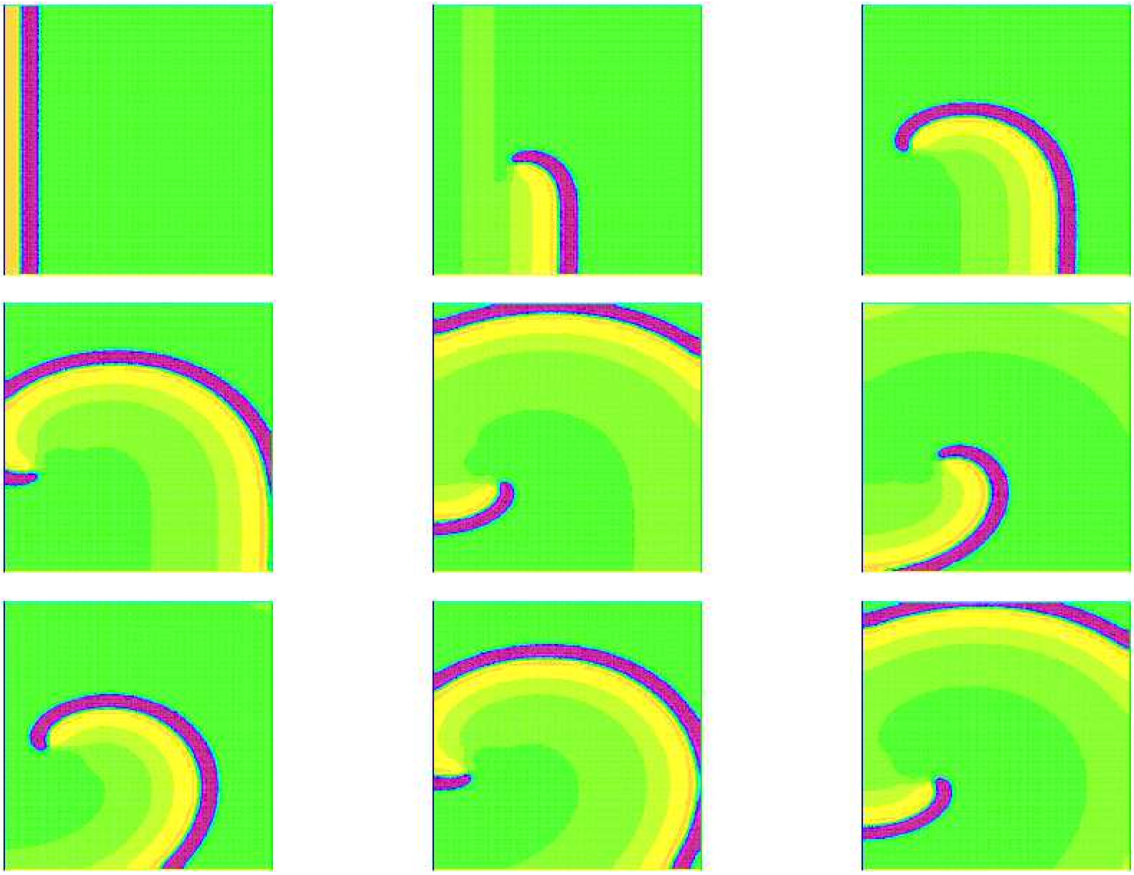

Figure 5.6: Spiral wave with the monodomain model using the finite element method. The snapshots correspond to the following iteration numbers: 100, 400, 600, 800, 1000, 1200, 1400, 1600, 1800.

Figure 5.6 illustrates snapshots of the propagation of a spiral wave using the FitzHugh-Nagumo ionic model with the same parameters as in section 5.1.1 (the 2D case).

5.2.2 The 2D bidomain model

Analogously to the monodomain case, we semi-discretize in time the bidomain model given by the equations of section 5.1.2. The nonlinear terms are considered in explicit form, we get the following system:

$$\begin{aligned}
&-h\nabla\cdot(\sigma_i\nabla u^{n+1}) + u^{n+1} = u^n - hf(u^n,w^n) + h\nabla\cdot(\sigma_i\nabla u_e^n),\\
&-\nabla\cdot((\sigma_i+\sigma_e)\nabla u_e^{n+1}) = \nabla\cdot(\sigma_i\nabla u^{n+1}),\\
&w^{n+1} = w^n - hg(u^n,w^n).
\end{aligned} \tag{5.4}$$

The weak formulation corresponding to system (5.4) reads as follows:

$$\int_\Omega (h\sigma_i\nabla u^{n+1}\cdot\nabla v + u^{n+1}v)\,dx = \int_\Omega \Big(u^n v - hf(u^n,w^n)v + h\sigma_i\nabla u_e^n\cdot\nabla v\Big)\,dx, \tag{5.5}$$

$$\int_\Omega (\sigma_i+\sigma_e)\nabla u_e^{n+1}\cdot\nabla v\,dx = \int_\Omega \sigma_i\nabla u^{n+1}\cdot\nabla v\,dx, \tag{5.6}$$

$$w^{n+1} = w^n - hg(u^n,w^n). \tag{5.7}$$

We proceed in three steps. First, we solve equation (5.5) for u^{n+1} and second we solve equation (5.6) for u_e^{n+1} using finite elements with elements of type P_2. Finally, we compute w^{n+1} from equation (5.7).

Figure 5.7 illustrates snapshots of the propagation of a spiral wave using the FitzHugh-Nagumo ionic model with the same parameters as in section 5.1.2 (the 2D case).

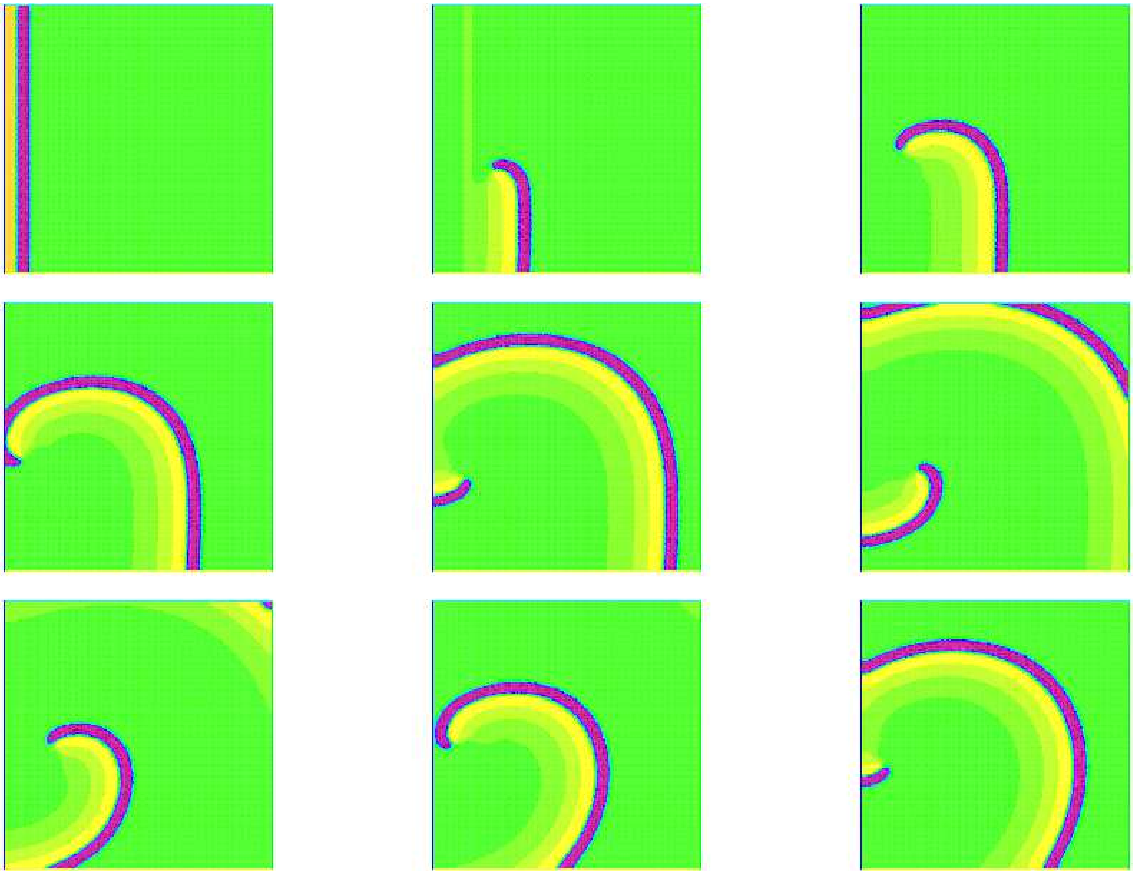

Figure 5.7: Spiral wave with the bidomain model using the finite element method. The snapshots correspond to the following iteration numbers: 100, 400, 600, 800, 1000, 1200, 1400, 1600, 1800.

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