AMERICAN UNIVERSITY OF BEIRUT

WELL BALANCED CENTRAL SCHEME FOR THE TWO- DIMENSIONAL SHALLOW WATER EQUATIONS

by
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AN ABSTRACT OF THE THESIS OF

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Major: Computational Sciences

Title: Well Balanced Central Schemes for the Two-Dimensional Shallow Water Equations

We aim to develop a new class of well-balanced non-oscillatory second-order accurate central schemes for the approximating solution of general two-dimensional hyperbolic systems, and in particular to approximate the solution of shallow water equation systems (SWE) on Cartesian grids. The base scheme evolves the numerical solution on a unique Cartesian grid and avoids the resolution of the Riemann problems arising at the cell interfaces thanks to a layer of ghost staggered cells implicitly used while updating the solution.

The system of shallow water equations

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} h \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\frac{\partial b}{\partial x} \\ -gh\frac{\partial b}{\partial y} \end{pmatrix}$$

represents a good mathematical model for the hydrodynamics of coastal oceans, simulation of flows in channels and rivers, study of large-scale waves and vertically averaged regimes in the atmosphere and ocean. Here *h* denotes the water depth, *(u,v)* represents the flow velocity, *g* is the gravitational constant, and *b* is the function that models the water bed topography. *b* vanishes in the case of a flat riverbed and the resulting system becomes a hyperbolic system. Most numerical schemes fail to maintain the steady state constraint of shallow water equation problems and generate numerical (nonphysical) waves and storms. In this project, we shall investigate several approaches that could be coupled with our numerical base scheme in order to ensure, when necessary, the steady state condition of SWE systems.

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1. INTRODUCTION

Numerical solutions of hydrodynamic problems allow us to predict the behavior of water flow and other fluids in real life situations. The shallow water equations model the propagation of disturbances in incompressible fluids. The equations are derived from depth-integrating the Navier-Stokes equations, in the case where the horizontal length scale is much greater than the vertical length scale.

The independent variables are the time t, and the two space coordinates x and y, while the dependent variables are the fluid depth h, and the two-dimensional fluid velocity field $(u,\, v)$. The system of shallow water equations is given by [13]:

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ gh\left(-\frac{\partial z}{\partial x}\right) \\ gh\left(-\frac{\partial z}{\partial y}\right) \end{pmatrix}$$

where $z(x, y)$ is the riverbed function and g is the gravitational constant.

In the recent years, many papers were devoted to treat the numerical solution of the shallow water equations using Riemann solvers [1,2,3,4,9,10,11,13,14,17,18,20,21,22]. The discretization of the source term was the focus point of a considerable number of authors [3,4,5,6,7]. In [17], a parallel algorithm for the solution of shallow water equations, which is based on Arakawa and Lamb's scheme [18], is developed. One of the issues associated with the numerical solution of the Shallow Water Equations is how to treat the non linear advective terms [19,20]. In [21], the two-stage Galerkin method, combined with a high accuracy compact approximation of the first derivative is used. Multi-layer problems are considered in [22], in which the derivation is performed by averaging the governing Euler equations over each layer. Numerical base schemes for general hyperbolic systems were developed in [13,15,23,24], whose multidimensional extensions on Cartesian grids (squares or cubes) or unstructured grids (triangles or tetrahedrons) were intensively and successfully used to solve problems arising in aerodynamics [24,25], astrophysics and magnetohydrodynamics [26,27]. Galerkin methods for the shallow water equations were discussed in [31,32,33]. In [28], well balanced central schemes on staggered grids for the Saint-Venant model is considered. The schemes are called shock capturing schemes since the shocks are identified by the regions with large gradients. Among shock capturing schemes, the most commonly used are finite volume methods, in which the basic unknowns represent the cell average of the unknown field. High order central schemes on staggered grid for conservation laws have been derived [29,30]. In case of the SWE, the crucial balancing between the flux gradient and the source term leads to very accurate and robust numerical schemes [8,9,10]. In [8], the conservation property is directly connected with steady solutions. As in the one-dimensional case derived in [10] and based on the finite volume method, the main advantage of our two-dimensional scheme is simplicity and accuracy by avoiding the time-consuming process of solving Riemann problems arising at the cell interfaces and by evolving piecewise linear numerical solutions. In this thesis, we construct, analyze and implement a new class of unstaggered, nonoscillatory, second-order accurate central schemes for the one and two-dimensional system of Shallow Water Equations. Our two-dimensional scheme is an extension of the one-dimensional method presented in [10]. The main two features of our scheme are the well-balancing of the source term with the flux divergence, and the proper discretization of the water height according to the Surface Gradient method discussed in [11] which is based on an accurate reconstruction of the conservative variables at cell interfaces. The numerical base scheme evolves the numerical solution on a single Cartesian grid, but implicitly uses ghost staggered cells to avoid the resolution of the Riemann problems arising at the cell interfaces.

Based on [10], two one-dimensional central schemes for the shallow water equations will be developed. The first consists of balancing the source term with the flux divergence, and will therefore be called "the Well-Balanced Central Scheme". As an extension of the well balanced scheme presented in [10], it discretizes the source term not only using the MinMod limiter, but also the MC-θ limiter by introducing the central difference for the derivatives. The adaptation of the one-dimensional scheme, called the "Interface-type scheme", goes further by discretizing the water level instead of the water depth, following the Surface Gradient Method introduced in [11].

This work aims to extend the discussed one-dimensional schemes to the case of two-dimensional shallow water equations. The two-dimensional well balanced scheme and the interface-type reformulation that will be developed follow the same strategy used for the one-dimensional case. The resulting scheme maintains the steady state condition by using the Surface Gradient method [11], where the riverbed features discontinuities or challenging irregularities.

The robustness and efficiency of our central scheme are confirmed by solving several two-dimensional Shallow Water problems. Our numerical scheme is capable of maintaining the steady state condition and the numerical results are in good agreement with corresponding ones appearing in the recent literature.

2. ONE-DIMENSIONAL CENTRAL SCHEMES FOR THE SHALLOW WATER EQUATIONS

2.1 Well-Balanced Central Scheme for the SWE

In this section, we consider the one-dimensional hyperbolic balance law system

$$\begin{cases} \partial_t U + \partial_x F(U) = S(U,x), U = U(x,t), x \in \Omega \subset \mathbb{R}, t > 0 \\ U(x,0) = U_0(x) \end{cases} \tag{2.1}$$

In order to ensure well balancing and the steady state condition, an appropriate extension of the central NT scheme should be constructed. Several possible approaches are given in [12], in which central NT schemes are extended to nonhomogenous systems and applied to thermodynamics. The discretization of the source term that depends on the particular balance law ensure the good accuracy of the numerical scheme we develop.

We apply the nonstaggered central NT scheme [11] to the one-dimensional shallow water equations

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hv \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} hv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ gh\left(-\frac{\partial z}{\partial x}\right) \end{pmatrix} \tag{2.2}$$

where $h = h(x,t)$ is the water depth, $v = v(x,t)$ is the water velocity, g is the gravitational constant, and $z = z(x)$ is the bottom topography function.

The crucial property we want to satisfy is the exact C-property [8] in which the steady state of the quiescent flow ($h + z = const$, $v = 0$) is exactly preserved. Since in this case, the balancing will be done between the flux gradient and the source term, we refer to the developed scheme by the "Well-Balanced Scheme".

The computational domain Ω is discretized using the subintervals $C_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ centered at the nodes x_i. The cells C_i are the original cells of the scheme while the ghost staggered dual cells are $D_{i+\frac{1}{2}} = [x_i, x_{i+1}]$ centered at $x_{i+\frac{1}{2}}$, endpoints of the original control volumes. The unstaggered central scheme evolves the numerical solution on a unique grid and avoids the time consuming process of solving the Riemann problems arising at the cell interfaces.

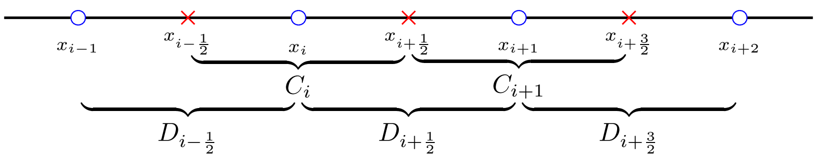

Fig. 2.1: Original control volumes C_i's and staggered control volumes $D_{i+\frac{1}{2}}$'s.

We assume that the solution U_i^n is known at time t^n on the control cells C_i and we integrate equation (2.1) over the rectangle $R^n_{i+\frac{1}{2}} = [x_i, x_{i+1}] \times [t^n, t^{n+1}]$

which results in: $\int\int_{R^n_{i+\frac{1}{2}}} [U_t + F(U)_x] dR = \int\int_{R^n_{i+\frac{1}{2}}} S(U,x) dR$

and applying *Green's formula:* $\int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_{\partial R} (Pdx + Qdy)$

with $\frac{\partial Q}{\partial x} = F(U)_x$ and $\frac{\partial P}{\partial y} = -U_t$, we obtain:

$\oint_{\partial R^n_{i+\frac{1}{2}}} (F(U)dt - Udx) = \int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} S(U,x) dxdt$

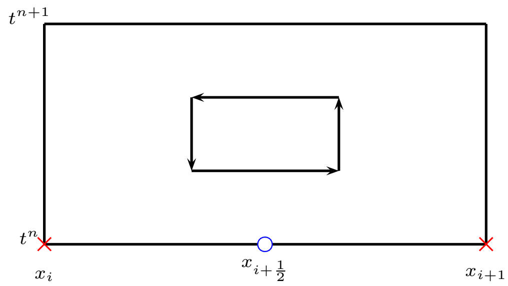

Fig. 2.2: The integration domain $R^n_{i+\frac{1}{2}}$.

Expanding the integral to the left hand side, we obtain

$$\int_{x_i}^{x_{i+1}} \left[F\left(U(x,t^n)\right)dt - U(x,t^n)dx\right] + \int_{t^n}^{t^{n+1}} \left[F\left(U(x_{i+1},t)\right)dt - U(x_{i+1},t)dx\right]$$
$$+ \int_{x_{i+1}}^{x_i} \left[F\left(U(x,t^{n+1})\right)dt - U(x,t^{n+1})dx\right] + \int_{t^{n+1}}^{t^n} \left[F\left(U(x_i,t)\right)dt - U(x_i,t)dx\right]$$
$$= \int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} S(U,x)dxdt$$

Rearranging the integrals, we thus obtain

$$-\int_{x_i}^{x_{i+1}} U(x,t^n)dx + \int_{t^n}^{t^{n+1}} F\left(U(x_{i+1},t)\right)dt + \int_{x_i}^{x_{i+1}} U(x,t^{n+1})dx - \int_{t^n}^{t^{n+1}} F\left(U(x_i,t)\right)dt$$
$$= \int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} S(U,x)dxdt \quad (2.3)$$

Since the solution $U(x,t)$ is assumed piecewise linear defined at the cell centers, the Mean-Value theorem gives $\int_{x_i}^{x_{i+1}} U(x,t^{n+1})dx = \Delta x U^{n+1}_{i+\frac{1}{2}}$ and $\int_{x_i}^{x_{i+1}} U(x,t^n)dx = \Delta x U^n_{i+\frac{1}{2}}$

Equation (2.2) becomes

$$-\Delta x U^n_{i+\frac{1}{2}} + \int_{t^n}^{t^{n+1}} F\left(U(x_{i+1},t)\right) dt + \Delta x U^{n+1}_{i+\frac{1}{2}} - \int_{t^n}^{t^{n+1}} F\left(U(x_i,t)\right) dt = \int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} S(U,x) dx dt$$

The solution at time t^{n+1} on the dual cells $D_{i+\frac{1}{2}}$ can be calculated as follows

$$U^{n+1}_{i+\frac{1}{2}} = U^n_{i+\frac{1}{2}} - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} F\left(U(x_{i+1},t)\right) dt - \int_{t^n}^{t^{n+1}} F\left(U(x_i,t)\right) dt\right] + \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} S(U,x) dx dt \quad (2.4)$$

The flux integrals are approximated using the midpoint rule: $\int_{t^n}^{t^{n+1}} F\left(U(x_i,t)\right) dt \approx F(U_i^{n+\frac{1}{2}}) \cdot \Delta t$ and $\int_{t^n}^{t^{n+1}} F\left(U(x_{i+1},t)\right) dt \approx F(U_{i+1}^{n+\frac{1}{2}}) \cdot \Delta t$, where $U_i^{n+\frac{1}{2}}$ is approximated using Taylor expansion in time and equation (2.1)

$$\begin{aligned} U_i^{n+\frac{1}{2}} &= U_i^n + \frac{\Delta t}{2}\partial_t(U_i^n) \\ &= U_i^n + \frac{\Delta t}{2}\left[-F(U_i)_x + S(U_i,x_i)\right] \end{aligned}$$

The partial flux derivative is calculated using the chain rule to obtain
$U_i^{n+\frac{1}{2}} = U_i^n + \frac{\Delta t}{2\Delta x}\left(-F_i^{'} + S_i^n \cdot \Delta x\right)$ with $F_i^{'} = \frac{\partial F}{\partial U} * \frac{\delta_i^n}{\Delta x}$ and $\left(\frac{\partial U}{\partial x}\right)_{x_i,t^n} \approx \frac{\delta_i^n}{\Delta x}$ with $\frac{\delta_i^n}{\Delta x}$
Equation (2.4) becomes:

$$U^{n+1}_{i+\frac{1}{2}} = U_i^n - \frac{\Delta t}{\Delta x}\left[F(U_{i+1}^{n+\frac{1}{2}}) - F(U_i^{n+\frac{1}{2}})\right] + \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} S\left(U(x,t),x)\right) dx dt$$

The integral of the source term is approximated using a second order quadrature rule as follows:

$$\int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} S\left(U(x,t),x\right) dxdt \simeq \Delta t \cdot \Delta x \cdot S(U_i^{n+\frac{1}{2}}, U_{i+1}^{n+\frac{1}{2}})$$

with

$$S(U_i^{n+\frac{1}{2}}, U_{i+1}^{n+\frac{1}{2}}) = \begin{pmatrix} 0 \\ g\frac{h_{i+1}^{n+\frac{1}{2}} + h_i^{n+\frac{1}{2}}}{2}\left(-\frac{z_{i+1}-z_i}{\Delta x}\right) \end{pmatrix}$$

Thus the numerical scheme summarizes as follows: starting with U_i^n, we first obtain the solution at the intermediate time $t^{n+\frac{1}{2}}$ as follows

$$U_i^{n+\frac{1}{2}} = U_i^n + \frac{\Delta t}{2\Delta x}\left(-F_i^{'} + S_i^n \cdot \Delta x\right) \tag{2.5}$$

next the solution at time t^{n+1} on the staggered dual cells calculated as follows

$$U_{i+\frac{1}{2}}^{n+1} = U_{i+\frac{1}{2}}^n - \frac{\Delta t}{\Delta x}\left[F(U_{i+1}^{n+\frac{1}{2}}) - F(U_i^{n+\frac{1}{2}})\right] + \Delta t S(U_i^{n+\frac{1}{2}}, U_{i+1}^{n+\frac{1}{2}}) \tag{2.6}$$

where the projection of the numerical solution on the original and staggered grids are obtained using Taylor expansions in space as follows:

$$U_{i+\frac{1}{2}}^n = \frac{1}{2}\left(U_i^n + U_{i+1}^n\right) + \frac{\Delta x}{8}\left(U_i^{'} - U_{i+1}^{'}\right) \tag{2.7}$$

$$U_i^{n+1} = \frac{1}{2}\left(U_{i-\frac{1}{2}}^{n+1} + U_{i+\frac{1}{2}}^{n+1}\right) + \frac{\Delta x}{8}\left(U_{i-\frac{1}{2}}^{'} - U_{i+\frac{1}{2}}^{'}\right) \tag{2.8}$$

In contrast with the scheme constructed in [10] where S_i^n was defined using the MinMod limiter, in this work, we propose a new formulation in terms of the MC-θ limiter that leads to sharper capture of discontinuities as follows:

$$S_i^n = S_{i,L}^n + S_{i,R}^n + S_{i,C}^n$$

with,

$$S_{i,L}^n = s_i^2 \frac{1-s_i}{6}(2-s_i) \begin{pmatrix} 0 \\ -gh_i^n \theta \frac{z_i - z_{i-1}}{\Delta x} \end{pmatrix}$$

$$S_{i,R}^n = s_i^2 \frac{1+s_i}{2}(2-s_i) \begin{pmatrix} 0 \\ -gh_i^n \theta \frac{z_{i+1} - z_i}{\Delta x} \end{pmatrix}$$

$$S_{i,C}^n = s_i \frac{(s_i+1)(s_i-1)}{6} \begin{pmatrix} 0 \\ -gh_i^n \frac{z_{i+1} - z_{i-1}}{2\Delta x} \end{pmatrix}$$

As for the parameter s_i in the i^th cell, it is defined by:

$$s_i = \begin{cases} -1 & \text{if } h_i^{'} = \theta \frac{h_i^n - h_{i-1}^n}{\Delta x}, \text{ ie, backward difference}, \\ 1 & \text{if } h_i^{'} = \theta \frac{h_{i+1}^n - h_i^n}{\Delta x}, \text{ ie, forward difference}, \\ 0 & \text{if } h_i^{'} = 0, \\ 2 & \text{if } h_i^{'} = \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x}, \text{ ie, central difference}. \end{cases}$$

and garantees that $z_i^{'}$ and $h_i^{'}$ are discretized in the same way $(1 \leqslant \theta \leqslant 2)$.

Definition: *(Quiescent Flow)*

Applied to the Shallow Water Equations, the quiescent flow case represents the steady case of the water, ie., when the surface of the liquid is initially at rest, corresponding to an initial velocity of zero. In the quiescent flow case, the shallow water equations are such that:

$$U = \begin{pmatrix} h \\ 0 \end{pmatrix}, \; F = \begin{pmatrix} 0 \\ \frac{1}{2}gh^2 \end{pmatrix}, \; S = \begin{pmatrix} 0 \\ gh \cdot \left(-\frac{dz}{dx}\right) \end{pmatrix}$$

Theorem:

For a quiescent flow, equations (2.5) and (2.6) respectively reduce to:

- $$U_i^{n+\frac{1}{2}} = U_i^n \tag{2.9}$$

- $$U_{i+\frac{1}{2}}^{n+1} = U_{i+\frac{1}{2}}^n \tag{2.10}$$

Proof:

- If $h_i^{'}$ is discretized using the backward difference, ie., $s_i = -1$,
$S_{i,L}^n = \begin{pmatrix} 0 \\ -gh_i^n\theta\frac{z_i - z_{i-1}}{\Delta x} \end{pmatrix}$, $S_{i,R}^n = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $S_{i,C}^n = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Therefore,

$$\begin{aligned} S_i^n &= S_{i,L}^n + S_{i,C}^n + S_{i,R}^n \\ &= \begin{pmatrix} 0 \\ -gh_i^n\theta\frac{z_i - z_{i-1}}{\Delta x} \end{pmatrix} \end{aligned}$$

But,

$$U_i^{n+\frac{1}{2}} = U_i^n + \frac{\Delta t}{2\Delta x}(-F_i^{'} + S_i^n \Delta x)$$

and in the quiescent case, $F = \begin{pmatrix} 0 \\ \frac{1}{2}gh^2 \end{pmatrix}$ so $F_i^{'} = \begin{pmatrix} 0 \\ gh_i \cdot h_i^{'} \end{pmatrix}$

Therefore,

$$\begin{aligned}
U_i^{n+\frac{1}{2}} &= U_i^n + \frac{\Delta t}{2\Delta x}\left[\begin{pmatrix} 0 \\ -gh_i^n h_i^{'} \end{pmatrix} + \begin{pmatrix} 0 \\ -gh_i^n \theta \frac{z_i - z_{i-1}}{\Delta x} \end{pmatrix} \Delta x\right] \\
&= U_i^n + \frac{\Delta t}{2\Delta x}\begin{pmatrix} 0 \\ -gh_i^n [h_i^{'} + \theta(z_i - z_{i-1})] \end{pmatrix} \\
&= U_i^n + \frac{\Delta t}{2\Delta x}\begin{pmatrix} 0 \\ -gh_i^n [\theta(h_i + z_i) - \theta(h_{i-1} + z_{i-1})] \end{pmatrix}
\end{aligned}$$

But since $h_i + z_i = H =$ constant for all i then $\theta(h_i + z_i) - \theta(h_{i-1} + z_{i-1}) = 0$ leading to

$$U_i^{n+\frac{1}{2}} = U_i^n$$

In a similar way, we show that $U_i^{n+\frac{1}{2}} = U_i^n$ remains valid if $s_i = 0, 1, 2$.

Now we show that $U_{i+\frac{1}{2}}^{n+1} = U_{i+\frac{1}{2}}^n$

$U_{i+\frac{1}{2}}^{n+1}$ is calculated using equation (2.6) as follows:

$$U_{i+\frac{1}{2}}^{n+1} = U_{i+\frac{1}{2}}^n - \frac{\Delta t}{\Delta x}(F(u_{i+1}^{n+\frac{1}{2}}) - F(u_i^{n+\frac{1}{2}})) + \Delta t S(u_i^{n+\frac{1}{2}}, U_{i+1}^{n+\frac{1}{2}})$$

where the source term is discretized using the formula

$$S(U_i^{n+\frac{1}{2}}, U_{i+1}^{n+\frac{1}{2}}) = \begin{pmatrix} 0 \\ g\frac{h_i^{n+\frac{1}{2}} + h_{i+1}^{n+\frac{1}{2}}}{2}\left(-\frac{z_{i+1} - z_i}{\Delta x}\right) \end{pmatrix}$$

In the case of a quiescent flow we have $U = \begin{pmatrix} h \\ 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 \\ \frac{1}{2}gh^2 \end{pmatrix}$, therefore

$F(U_{i+1}^{n+\frac{1}{2}}) = \begin{pmatrix} 0 \\ \frac{1}{2}g(h_{i+1}^{n+\frac{1}{2}})^2 \end{pmatrix}$, and $F(U_i^{n+\frac{1}{2}}) = \begin{pmatrix} 0 \\ \frac{1}{2}g(h_i^{n+\frac{1}{2}})^2 \end{pmatrix}$

Equation (2.6) becomes:

$$\begin{aligned} U_{i+\frac{1}{2}}^{n+1} &= U_{i+\frac{1}{2}}^{n} - \frac{\Delta t}{\Delta x}\left[\begin{pmatrix} 0 \\ \frac{1}{2}g(h_{i+1}^{n+\frac{1}{2}})^2 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{2}g(h_i^{n+\frac{1}{2}})^2 \end{pmatrix}\right] \\ &+\Delta t \begin{pmatrix} 0 \\ g\frac{h_i^{n+\frac{1}{2}}+h_{i+1}^{n+\frac{1}{2}}}{2}\left(-\frac{z_{i+1}-z_i}{\Delta x}\right) \end{pmatrix} \end{aligned}$$

Basic algebra operations give

$$U_{i+\frac{1}{2}}^{n+1} = U_{i+\frac{1}{2}}^{n} - \frac{\Delta t}{\Delta x}\begin{pmatrix} 0 \\ \frac{1}{2}g\left[(h_{i+1}^{n+\frac{1}{2}} + h_i^{n+\frac{1}{2}})\left[(h_{i+1}^{n+\frac{1}{2}} + z_{i+1}) - (h_i^{n+\frac{1}{2}} + z_i)\right]\right] \end{pmatrix}$$

But since $h_i^n + z_i = H =$ constant for all i then $(h_i^n + z_i) - (h_{i+1}^n + z_{i+1}) = 0$

According to relation (2.9),

$$U_i^{n+\frac{1}{2}} = U_i^n$$

means

$$\begin{pmatrix} h_i^{n+\frac{1}{2}} \\ (hv)_i^{n+\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} h_i^n \\ (hv)_i^n \end{pmatrix}$$

Therefore, $(h_i^{n+\frac{1}{2}} + z_i) - (h_{i+1}^{n+\frac{1}{2}} + z_{i+1}) = (h_i^n + z_i) - (h_{i+1}^n + z_{i+1}) = 0,$

leading to

$$U_{i+\frac{1}{2}}^{n+1} = U_{i+\frac{1}{2}}^{n}$$

which means that if the steady state requirement was satisfied at time t^n, it will remain as such at time t^{n+1}.

2.2 Balanced Central NT Scheme: The Interface Type Reformulation

Due to the fact that the riverbed function and the water level are continuous on the original cells but not on the staggered ones, the steady state is not properly maintained in case of variable or discontinuous riverbeds. A new reformulation of the scheme, will be developed to handle such cases. The Interface Type reformulation [10] is a modification of the Balanced Central NT Scheme derived previously. The bottom topography is defined at the interfaces of each cell, ie., $z_{i+\frac{1}{2}}$ is defined at $x_{i+\frac{1}{2}}$. $z(x)$ is therefore approximated over the whole control cell C_i centered at x_i using linear interpolation:

$$z(x) = z_i + \frac{1}{\Delta x}(z_{i+\frac{1}{2}} - z_{i-\frac{1}{2}})(x - x_i)$$

and, at the cell centers, we define

$$z_i = \frac{z_{i+\frac{1}{2}} + z_{i-\frac{1}{2}}}{2}$$

The interface type reformulation follows the scheme derived previously by using the formulae:

$$U_i^{n+\frac{1}{2}} = U_i^n + \frac{\Delta t}{2\Delta x}\left(-F_i^{'} + S_i^n \Delta x\right)$$

and,

$$U_{i+\frac{1}{2}}^{n+1} = U_{i+\frac{1}{2}}^n - \frac{\Delta t}{\Delta x}\left[F(U_{i+1}^{n+\frac{1}{2}}) - F(U_i^{n+\frac{1}{2}})\right] + \Delta t S(U_i^{n+\frac{1}{2}}, U_{i+1}^{n+\frac{1}{2}})$$

Nevertheless, it differs from the previous scheme in the way it moves the numerical solution from the original control cells to the staggered dual cells and vice versa. In the interface type reformulation, the components of $U_{i+\frac{1}{2}}^n$ and U_i^{n+1} are computed differently, taking into consideration the non-linearity of the water height and the riverbed function over the staggered dual cells.

The hv component is computed by using exactly the Well-Balanced Scheme derived previously (equations (2.5) and (2.6)). The reformulation of the new version of the scheme is based on a special approximation of the water depth h at each step of the algorithm.

$h_i^{n+\frac{1}{2}}$ and $h_{i+\frac{1}{2}}^{n+1}$ are computed using equations (2.5) and (2.6). Nevertheless, in order to approximate

$h^n_{i+\frac{1}{2}}$ and h^{n+1}_i, relations (2.7) and (2.8) need to be updated using the Surface Gradient Method discussed in [11], in which the water level H is first updated, then the water height h is calculated according to the relation $h + z = H$.

Using the unstaggered central scheme, the water height h and the bottom topography z are considered to be linear inside each original control cell C_i.

In this interface-type reformulation, the linearization of the water height is made indirectly by first linearizing the water level $H(x)$, then using $h(x) = H(x) - z(x)$.

$H(x) = H_i + H_i^{'}(x - x_i)$ for all $x \in C_i$ with $H_i^{'}$ computed using a slope limiting procedure on the cell values: $H_i = h_i + z_i$, which will lead to:

$$h_i^{'} = H_i^{'} - z_i^{'}$$

ie., using the cental difference,

$$h_i^{'} = H_i^{'} - \frac{1}{\Delta x}(z_{i+\frac{1}{2}} - z_{i-\frac{1}{2}}) \tag{2.11}$$

Applying relation (2.7) to h results in:

$$h^n_{i+\frac{1}{2}} = \frac{1}{2}(h^n_i + h^n_{i+1}) + \frac{\Delta x}{8}(h_i^{'} - h_{i-1}^{'}) \tag{2.12}$$

Equations (2.10) and (2.11) lead to:

$$h^n_{i+\frac{1}{2}} = \frac{1}{2}\left(h^n_i + h^n_{i+1}\right) + \frac{\Delta x}{8}\left[\left(H_i^{'} - \frac{z_{i+\frac{1}{2}} - z_{i-\frac{1}{2}}}{\Delta x}\right) - \left(H_{i+1}^{'} - \frac{z_{i+\frac{3}{2}} - z_{i+\frac{1}{2}}}{\Delta x}\right)\right] \tag{2.13}$$

which in the quiescent case reduces to:

$$h^n_{i+\frac{1}{2}} = \frac{1}{2}\left(h^n_i + h^n_{i+1}\right) + \frac{1}{2}\left(z_{i+\frac{1}{2}} - \frac{z_i + z_{i+1}}{2}\right) \tag{2.14}$$

Remark:

In the quiescent flow case, H(x) = constant reduces $H^{'}$ to zero.

In addition, $z_{i+\frac{1}{2}} - z_{i-\frac{1}{2}} = 2(z_{i+\frac{1}{2}} - z_i)$ and $z_{i+\frac{3}{2}} - z_{i-\frac{1}{2}} = 2(z_{i+1} - z_{i-\frac{1}{2}})$

As for the target solution h_i^{n+1}, it is approximated as follows:

On the original control volumes, the relation $H_i = h_i + z_i$ applies.

Similarly, over the staggered control volumes:

$$\tilde{\mathrm{H}}_{i+\frac{1}{2}} = h_{i+\frac{1}{2}} + \tilde{\mathrm{z}}_{i+\frac{1}{2}} \tag{2.15}$$

$\tilde{\mathrm{H}}_{i+\frac{1}{2}}$ and $\tilde{\mathrm{z}}_{i+\frac{1}{2}}$ are defined differently due to the fact that the riverbed bottom z(x) is linear inside the original control cells C_i but not inside the staggered dual cells $D_{i+\frac{1}{2}}$.

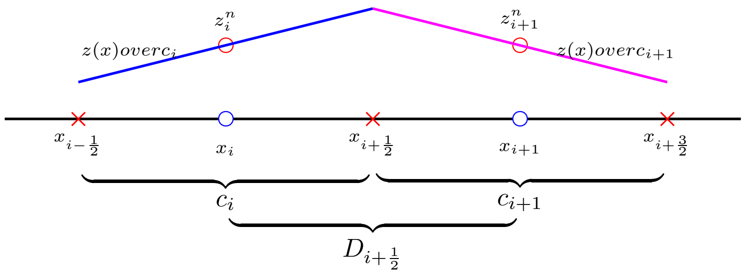

Fig. 2.3: The bottom topography $z(x)$ is linear on each original control volume, but NOT linear on the staggered control volumes.

Therefore, define $\tilde{\mathrm{z}}_{i+\frac{1}{2}}$ as follows:

$$\tilde{\mathrm{z}}_{i+\frac{1}{2}} = z_{i+\frac{1}{2}} - \frac{1}{2}\left(z_{i+\frac{1}{2}} - \frac{z_i + z_{i+1}}{2}\right) \tag{2.16}$$

$\tilde{\mathrm{H}}_{i+\frac{1}{2}}$ will be defined accordingly; leading to:

$$h_{i+\frac{1}{2}} = \tilde{\mathrm{H}}_{i+\frac{1}{2}} - \tilde{\mathrm{z}}_{i+\frac{1}{2}}$$

The numerical dericatives can be calculated now as follows:

$$h^{'}_{i+\frac{1}{2}} = H^{'}_{i+\frac{1}{2}} - \tilde{\mathrm{z}}^{'}_{i+\frac{1}{2}}$$

thus we obtain

$$h^{'}_{i+\frac{1}{2}} = H^{'}_{i+\frac{1}{2}} - \frac{1}{\Delta x}(z_{i+1} - z_i) \tag{2.17}$$

The discrete derivatives $H^{'}_{i+\frac{1}{2}}$ are derived from the staggered values $\tilde{\mathrm{H}}_{i+\frac{1}{2}}$ using a slope limiting procedure.

Using relations (2.14) and (2.16) in (2.15) leads to:

$$\begin{aligned}
\tilde{\mathrm{H}}_{i+\frac{1}{2}} &= h_{i+\frac{1}{2}} + \tilde{\mathrm{z}}_{i+\frac{1}{2}} \\
&= \frac{1}{2}(h_i + h_{i+1}) - \frac{1}{2}\left(z_{i+\frac{1}{2}} - \frac{z_i + z_{i+1}}{2}\right) + z_{i+\frac{1}{2}} - \frac{1}{2}\left(z_{i+\frac{1}{2}} - \frac{z_i + z_{i+1}}{2}\right) \\
&= \frac{1}{2}(h_i + h_{i+1} + z_i + z_{i+1}) \\
&= \frac{1}{2}(H_i + H_{i+1})
\end{aligned}$$

which, in the quiescent flow case, results in $\tilde{\mathrm{H}}_{i+\frac{1}{2}} = \text{constant}$ for all grid points since $H_i = H_{i+1} = \text{constant}$.

Finally, using relation (2.17) in (2.7) for h results in:

$$h_i^{n+1} = \frac{1}{2}\left(h^{n+1}_{i-\frac{1}{2}} + h^{n+1}_{i+\frac{1}{2}}\right) + \frac{\Delta x}{8}\left(h^{'}_{i-\frac{1}{2}} - h^{'}_{i+\frac{1}{2}}\right)$$

and thus

$$\begin{aligned}
h_i^{n+1} &= \frac{1}{2}\left(h^{n+1}_{i-\frac{1}{2}} + h^{n+1}_{i+\frac{1}{2}}\right) \\
&\quad + \frac{\Delta x}{8}\left(H^{'}_{i-\frac{1}{2}} - \frac{z_i - z_{i-1}}{\Delta x} - H^{'}_{i+\frac{1}{2}} + \frac{z_{i+1} - z_i}{\Delta x}\right)
\end{aligned} \tag{2.18}$$

Thus we conclude that the interface type procedure, when applied with the unstaggered central scheme, ensures the quiescent flow requirement.

3. CENTRAL SCHEMES FOR THE TWO-DIMENSIONAL SHALLOW WATER EQUATIONS

3.1 Well-Balanced Central Scheme for the Shallow Water Equations

In this chapter, we extend the one-dimensional Well-Balanced central scheme presented in chapter 2 to the case of two-dimensional hyperbolic balance law system

$$\partial_t U + \partial_x F(U) + \partial_y G(U) = S(U, x, y) \tag{3.1}$$

We apply the unstaggered central scheme discussed in [13] to the two-dimensional system of shallow water equations defined with

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ gh\left(-\frac{\partial z}{\partial x}\right) \\ gh\left(-\frac{\partial z}{\partial y}\right) \end{pmatrix} \tag{3.2}$$

where $h = h(x, y, t)$ is the water depth, $u = u(x, y, t)$ and $v = v(x, y, t)$ are the water velocities in the x and y directions, g is the gravitational constant, and $z = z(x, y)$ is the water bed function.

The C-property of the numerical solution that was proven to be satisfied in the one-dimensional case is also meant to be satisfied in the two-dimensional case.

We discretize the computational domain using the rectangles $C_{i,j} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$

representing the original control volume centered at the node (x_i, y_j), and $D_{i+\frac{1}{2},j+\frac{1}{2}} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ representing the staggered control volume, centered at $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$, corner of the original cell C_i. As in 1D, the developed scheme evolves the numerical solution on a single grid but avoids the time consuming process of solving Riemann problems arising at the cell interfaces by implicitly using the staggered dual cells.

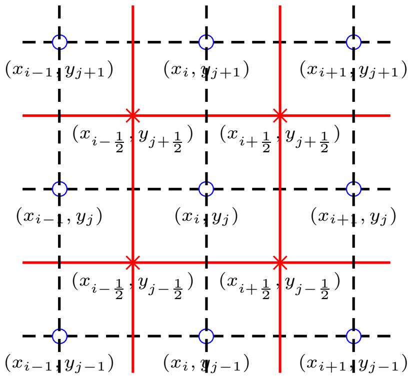

Fig. 3.1: Original control volumes $C_{i,j}$ and staggered control volumes $D_{i+\frac{1}{2},j+\frac{1}{2}}$'s.

We assume that the numerical solution $U^n_{i,j}$ is known at the time t^n as a piecewise linear numerical solution defined at the center (x_i, y_j) of the cells $C_{i,j}$. We integrate equation (3.2) over $R^n_{i+\frac{1}{2},j+\frac{1}{2}} = D_{i+\frac{1}{2},j+\frac{1}{2}} \times [t^n, t^{n+1}]$

$$\int\int\int_{R^n_{i+\frac{1}{2},j+\frac{1}{2}}} [U_t + F(U)_x + G(U)_y] dR = \int\int\int_{R^n_{i+\frac{1}{2},j+\frac{1}{2}}} S(U, x, y) dR$$

and we apply the divergence theorem to the flux integral, thus we obtain

$$\int\int\int_{R^n_{i+\frac{1}{2},j+\frac{1}{2}}} U_t dR + \int\int\int_{R^n_{i+\frac{1}{2},j+\frac{1}{2}}} [F(U)_x + G(U)_y] \, dR = \int\int\int_{R^n_{i+\frac{1}{2},j+\frac{1}{2}}} S(U, x, y) dR$$

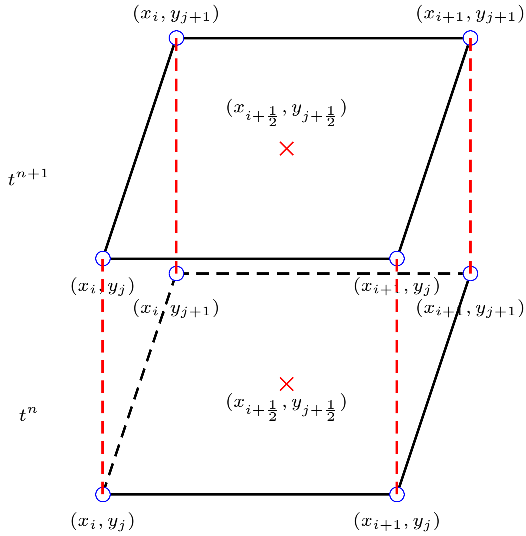

Fig. 3.2: The integration domain $R^n_{i+\frac{1}{2},j+\frac{1}{2}} \times [t^n, t^{n+1}]$.

$$\int_{y_i}^{y_{j+1}} \int_{x_i}^{x_{i+1}} \left(U^{n+1} - U^n\right) dxdy + \int\int\int_{R^n_{i+\frac{1}{2},j+\frac{1}{2}}} \left[F(U)_x + G(U)_y\right] dR$$
$$= \int\int\int_{R^n_{i+\frac{1}{2},j+\frac{1}{2}}} S(U,x,y)dR \tag{3.3}$$

But $\int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} U(x,y,t^{n+1})dxdy = \Delta y \Delta x U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}$
and $\int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} U(x,y,t^n)dxdy = \Delta y \Delta x U^n_{i+\frac{1}{2},j+\frac{1}{2}}$ (Mean-Value Theorem).

Therefore, using the divergence theorem, equation (3.3) becomes:

$$\begin{aligned} \Delta x \Delta y U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} &= \Delta x \Delta y U^{n}_{i+\frac{1}{2},j+\frac{1}{2}} \\ &- \int_{t^n}^{t^{n+1}} \int_{\partial D_{i+\frac{1}{2},j+\frac{1}{2}}} F(U) \cdot n_x dAdt \\ &- \int_{t^n}^{t^{n+1}} \int_{\partial D_{i+\frac{1}{2},j+\frac{1}{2}}} G(U)_y \cdot n_y dAdt \\ &+ \int \int \int_{R^n_{i+\frac{1}{2},j+\frac{1}{2}}} S(U,x,y) dR \end{aligned} \tag{3.4}$$

with $R_{xy} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$

where $n = (n_x, n_y)$ is the unit normal vector to the boundary of $D_{i+\frac{1}{2},j+\frac{1}{2}}$.

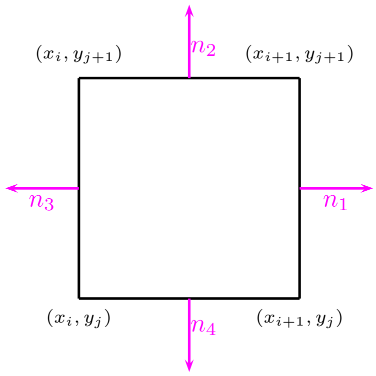

Fig. 3.3: $D_{i+\frac{1}{2},j+\frac{1}{2}}$ *and normal outer vectors* $n_1 =< 1, 0, 0 >$, $n_2 =< 0, 1, 0 >$, $n_3 =< -1, 0, 0 >$ *and* $n_4 =< 0, -1, 0 >$ *to its boundary.*

Equation (3.4) is therefore equivalent to:

$$\begin{aligned} U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} &= U^{n}_{i+\frac{1}{2},j+\frac{1}{2}} - \frac{1}{\Delta x \Delta y} \int_{t^n}^{t^{n+1}} \int_{\partial D_{i+\frac{1}{2},j+\frac{1}{2}}} F(U) \cdot n_x dAdt \\ &- \frac{1}{\Delta x \Delta y} \int_{t^n}^{t^{n+1}} \int_{\partial D_{i+\frac{1}{2},j+\frac{1}{2}}} G(U) \cdot n_y dAdt \\ &+ \frac{1}{\Delta x \Delta y} \int \int \int_{R^n_{i+\frac{1}{2},j+\frac{1}{2}}} S(U,x,y) dR \end{aligned} \tag{3.5}$$

Let's approximate the flux integrals: $A = \int_{t^n}^{t^{n+1}} \int_{\partial R_{xy}} F(U) \cdot n_x dAdt$ and $B = \int_{t^n}^{t^{n+1}} \int_{\partial R_{xy}} G(U) \cdot n_y dxdydt$

A is equivalent to four integrals over the four vertical sides of the cube $R_{i+\frac{1}{2},j+\frac{1}{2}}$:

$$
\begin{aligned}
A &= \int_{t^n}^{t^{n+1}} \int_{\partial R_{xy}} F(U^{n+\frac{1}{2}}) \cdot n_x dxdy \\
&= \int_{t^n}^{t^{n+1}} \int_{y_j}^{y_{j+1}} F(U(x_{i+1}, y, t^{n+\frac{1}{2}})) \cdot (1,0,0)_x dy \\
&\quad + \int_{t^n}^{t^{n+1}} \int_{x_{i+1}}^{x_i} F(U(x, y_{j+1}, t^{n+\frac{1}{2}})) \cdot (0,1,0)_x dx \\
&\quad + \int_{t^n}^{t^{n+1}} \int_{y_{j+1}}^{y_j} F(U(x_i, y, t^{n+\frac{1}{2}})) \cdot (-1,0,0)_x dy \\
&\quad + \int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} F(U(x, y_j, t^{n+\frac{1}{2}})) \cdot (0,-1,0)_x dx
\end{aligned}
$$

Each of these integrals is approximated using the midpoint rule:

$$
\begin{aligned}
A &= \Delta t \Delta y \left[\frac{F(U(x_{i+1}, y_j, t^{n+\frac{1}{2}})) + F(U(x_{i+1}, y_{j+1}, t^{n+\frac{1}{2}}))}{2} \right] \\
&\quad - \Delta t \Delta y \left[\frac{F(U(x_i, y_j, t^{n+\frac{1}{2}})) + F(U(x_i, y_{j+1}, t^{n+\frac{1}{2}}))}{2} \right] \\
&= \Delta t \Delta y \left[\frac{F(U_{i+1,j}^{n+\frac{1}{2}}) + F(U_{i+1,j+1}^{n+\frac{1}{2}}) - F(U_{i,j}^{n+\frac{1}{2}}) - F(U_{i,j+1}^{n+\frac{1}{2}})}{2} \right]
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
B &= \int_{t^n}^{t^{n+1}} \int_{\partial D_{i+\frac{1}{2},j+\frac{1}{2}}} G(U) \cdot n_y dxdydt \\
&= \Delta t \Delta x \left[\frac{G(U_{i,j+1}^{n+\frac{1}{2}}) + G(U_{i+1,j+1}^{n+\frac{1}{2}}) - G(U_{i,j}^{n+\frac{1}{2}}) - G(U_{i+1,j}^{n+\frac{1}{2}})}{2} \right]
\end{aligned}
$$

The update of the numerical solution at time t^{n+1} is now as follows

$$\begin{aligned} U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} &= U^{n}_{i+\frac{1}{2},j+\frac{1}{2}} \\ &-\frac{\Delta t}{2\Delta x}\left[F(U^{n+\frac{1}{2}}_{i+1,j}) + F(U^{n+\frac{1}{2}}_{i+1,j+1}) - F(U^{n+\frac{1}{2}}_{i,j}) - F(U^{n+\frac{1}{2}}_{i,j+1})\right] \\ &-\frac{\Delta t}{2\Delta y}\left[G(U^{n+\frac{1}{2}}_{i,j+1}) + G(U^{n+\frac{1}{2}}_{i+1,j+1}) - G(U^{n+\frac{1}{2}}_{i,j}) - G(U^{n+\frac{1}{2}}_{i+1,j})\right] \\ &+\frac{1}{\Delta x\Delta y}\int\int\int_{R^n_{i+\frac{1}{2},j+\frac{1}{2}}} S(U,x,y)dR \end{aligned} \tag{3.6}$$

Now we discretize the integral of the source term using the midpoint quadrature rule with respect to both time and space:

$$\int\int\int_{R^n_{i+\frac{1}{2},j+\frac{1}{2}}} S(U,x,y)dR \simeq \Delta t\Delta x\Delta y \cdot S\left(U(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^{n+\frac{1}{2}}), x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}\right)$$

with

$$S\left(U^{n+\frac{1}{2}}_{i,j}, U^{n+\frac{1}{2}}_{i+1,j}, U^{n+\frac{1}{2}}_{i,j+1}, U^{n+\frac{1}{2}}_{i+1,j+1}\right) = \begin{pmatrix} 0 \\ g\cdot h^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \cdot (-\frac{\partial z}{\partial x}) \\ g\cdot h^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \cdot \left(-\frac{\partial z}{\partial y}\right) \end{pmatrix}$$

Finally, equation (3.3) becomes:

$$\begin{aligned} U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} &= U^{n}_{i+\frac{1}{2},j+\frac{1}{2}} \\ &-\frac{\Delta t}{2\Delta x}\left[F(U^{n+\frac{1}{2}}_{i+1,j}) + F(U^{n+\frac{1}{2}}_{i+1,j+1}) - F(U^{n+\frac{1}{2}}_{i,j}) - F(U^{n+\frac{1}{2}}_{i,j+1})\right] \\ &-\frac{\Delta t}{2\Delta y}\left[G(U^{n+\frac{1}{2}}_{i,j+1}) + G(U^{n+\frac{1}{2}}_{i+1,j+1}) - G(U^{n+\frac{1}{2}}_{i,j}) - G(U^{n+\frac{1}{2}}_{i+1,j})\right] \\ &+\Delta t \cdot S\left(U^{n+\frac{1}{2}}_{i,j}, U^{n+\frac{1}{2}}_{i+1,j}, U^{n+\frac{1}{2}}_{i,j+1}, U^{n+\frac{1}{2}}_{i+1,j+1}\right) \end{aligned} \tag{3.7}$$

with

$$S\left(U_{i,j}^{n+\frac{1}{2}}, U_{i+1,j}^{n+\frac{1}{2}}, U_{i,j+1}^{n+\frac{1}{2}}, U_{i+1,j+1}^{n+\frac{1}{2}}\right) = \frac{1}{2}\begin{pmatrix} 0 \\ -g\frac{h_{i+1,j}+h_{i,j}}{2}\frac{z_{i+1,j}-z_{i,j}}{\Delta x} \\ -g\frac{h_{i,j+1}+h_{i,j}}{2}\frac{z_{i,j+1}-z_{i,j}}{\Delta y} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ -g\frac{h_{i+1,j+1}+h_{i,j+1}}{2}\frac{z_{i+1,j+1}-z_{i,j+1}}{\Delta x} \\ -g\frac{h_{i+1,j+1}+h_{i+1,j}}{2}\frac{z_{i+1,j+1}-z_{i+1,j}}{\Delta y} \end{pmatrix}$$

where $\dfrac{z_{i+1,j} - z_{i,j}}{\Delta x}$ approximates $-\dfrac{\partial z}{\partial x}$ along the layer $y = y_j$

and $\dfrac{z_{i+1,j+1} - z_{i,j+1}}{\Delta x}$ approximates $-\frac{\partial z}{\partial x}$ along the layer $y = y_{j+1}$

As for $U_{i,j}^{n+\frac{1}{2}}$, it is approximated using Taylor' expansion:

$$U_{i,j}^{n+\frac{1}{2}} = U_{i,j}^{n} + \frac{\Delta t}{2}\partial_t(U_i^n)$$

and the system of equations, $U_t = -F(U)_x - G(U)_y + S(U, x, y)$

leading to,

$$U_{i,j}^{n+\frac{1}{2}} = U_{i,j}^{n} + \frac{\Delta t}{2}\left[-F(U_{i,j})_x - G(U_{i,j})_y + S(U_{i,j}, x_i, y_j)\right]$$

Using the chain rule, $F(U_{i,j}^n)_x = \left(\frac{\partial F}{\partial U}\right)_{i,j}^n \cdot \frac{\partial \delta_{i,j}^n}{\partial x}$ and $G(U_{i,j}^n)_y = \left(\frac{\partial G}{\partial U}\right)_{i,j}^n \cdot \frac{\partial \sigma_{i,j}^n}{\partial y}$ with $\frac{\delta_{i,j}^n}{\Delta x}$ and $\frac{\sigma_{i,j}^n}{\Delta y}$ approximating respectively the partial derivative of U with respect to x and y at time t^n leading to $U_{i,j}^{n+\frac{1}{2}} = U_{i,j}^n + \frac{\Delta t}{2}\left(-\frac{F'_{i,j}}{\Delta x} - \frac{G'_{i,j}}{\Delta y} + S_{i,j}^n\right)$ with $F'_{i,j} = \frac{\partial F}{\partial U} \cdot \frac{\delta_{i,j}}{\Delta x}$ and $G'_{i,j} = \frac{\partial F}{\partial U} \cdot \frac{\sigma_{i,j}}{\Delta y}$

with

$$S_{i,j}^n = \begin{pmatrix} 0 \\ -gh_{i,j}^n \frac{\partial z}{\partial x} \\ -gh_{i,j}^n \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

with $S_1 = 0$,

$$S_2 = S_{2,L} + S_{2,C} + S_{2,R}$$

Define $S_{2,L}$, $S_{2,C}$, and $S_{2,R}$ as follows ($1 \leq \theta \leq 2$):

$$S_{2,L} = s_i^2 \frac{1 - s_i}{6}(2 - s_i)\left(-gh_{i,j}^n \theta \frac{z_{i,j} - z_{i-1,j}}{\Delta x}\right)$$

$$S_{2,C} = s_i \frac{(s_i + 1)(s_i - 1)}{6}(2 - s_i)\left(-gh_{i,j}^n \frac{z_{i+1,j} - z_{i-1,j}}{2\Delta x}\right)$$

$$S_{2,R} = s_i^2 \frac{1 + s_i}{2}(2 - s_i)\left(-gh_{i,j}^n \theta \frac{z_{i+1,j} - z_{i,j}}{\Delta x}\right)$$

with

$$s_i = \begin{cases} -1 & \text{if } h_x = \theta\left(\frac{h_{i,j}^n - h_{i-1,j}^n}{\Delta x}\right), \text{ ie, backward difference}, \\ 0 & \text{if } h_x = 0\,, \\ 1 & \text{if } h_x = \theta\left(\frac{h_{i+1,j}^n - h_{i,j}^n}{\Delta x}\right), \text{ ie, forward difference}, \\ 2 & \text{if } h_x = \frac{h_{i+1,j}^n - h^n i - 1,j}{2\Delta x}\,. \end{cases}$$

Similarly, we define S_3 as follows ($1 \leq \theta \leq 2$):

$$S_3 = S_{3,L} + S_{3,C} + S_{3,R}$$

with

$$S_{3,L} = t_j^2 \frac{1 - t_j}{6}(2 - t_j)\left(-gh_{i,j}^n \theta \frac{z_{i,j} - z_{i,j-1}}{\Delta y}\right)$$

$$S_{3,C} = t_j \frac{(t_j + 1)(t_j - 1)}{6}(2 - t_j)\left(-gh_{i,j}^n \frac{z_{i,j+1} - z_{i,j-1}}{2\Delta y}\right)$$

$$S_{3,R} = t_j^2 \frac{1+t_j}{2}(2-t_j)\left(-gh_{i,j}^n \theta \frac{z_{i,j+1}-z_{i,j}}{\Delta y}\right)$$

with

$$t_j = \begin{cases} -1 & \text{if } h_y = \theta\left(\frac{h_{i,j}^n - h_{i,j-1}^n}{\Delta y}\right), \text{ ie, backward difference}, \\ 0 & \text{if } h_y = 0, \\ 1 & \text{if } h_y = \theta\left(\frac{h_{i,j+1}^n - h_{i,j}^n}{\Delta y}\right), \text{ ie, forward difference}, \\ 2 & \text{if } h_y = \frac{h_{i,j+1}^n - h^n i,j-1}{2\Delta y}. \end{cases}$$

Therefore, we obtain

$$S_2 = \begin{cases} -gh_{i,j}^n \theta \frac{z_{i,j}-z_{i-1,j}}{\Delta x} & \text{if } s_i = -1, \\ 0 & \text{if } s_i = 0, \\ -gh_{i,j}^n \theta \frac{z_{i+1,j}-z_{i,j}}{\Delta x} & \text{if } s_i = 1, \\ -gh_{i,j}^n \frac{z_{i+1,j}-z_{i-1,j}}{2\Delta x} & \text{if } s_i = 2. \end{cases}$$

$$S_3 = \begin{cases} -gh_{i,j}^n \theta \frac{z_{i,j}-z_{i,j-1}}{\Delta y} & \text{if } t_j = -1, \\ 0 & \text{if } t_j = 0, \\ -gh_{i,j}^n \theta \frac{z_{i,j+1}-z_{i,j}}{\Delta y} & \text{if } t_j = 1, \\ -gh_{i,j}^n \frac{z_{i,j+1}-z_{i,j-1}}{2\Delta y} & \text{if } t_j = 2. \end{cases}$$

Thus the well balanced unstaggered central scheme summarizes as follows: starting with $U_{i,j}^n$, we obtain the solution at the intermediate time $t^{n+\frac{1}{2}}$ as follows:

$$U_{i,j}^{n+\frac{1}{2}} = U_{i,j}^n + \frac{\Delta t}{2}\left(-\frac{F_{i,j}^{'}}{\Delta x} - \frac{G_{i,j}^{'}}{\Delta y} + S_{i,j}^n\right) \tag{3.8}$$

next we obtain the solution at the next time t^{n+1} as follows

$$\begin{aligned} U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} &= U^{n}_{i+\frac{1}{2},j+\frac{1}{2}} \\ &-\frac{\Delta t}{2\Delta x}\left[F(U^{n+\frac{1}{2}}_{i+1,j}) + F(U^{n+\frac{1}{2}}_{i+1,j+1}) - F(U^{n+\frac{1}{2}}_{i,j}) - F(U^{n+\frac{1}{2}}_{i,j+1})\right] \\ &-\frac{\Delta t}{2\Delta y}\left[G(U^{n+\frac{1}{2}}_{i,j+1}) + G(U^{n+\frac{1}{2}}_{i+1,j+1}) - G(U^{n+\frac{1}{2}}_{i,j}) - G(U^{n+\frac{1}{2}}_{i+1,j})\right] \\ &+\Delta t \cdot S\left(U^{n+\frac{1}{2}}_{i,j}, U^{n+\frac{1}{2}}_{i+1,j}, U^{n+\frac{1}{2}}_{i,j+1}, U^{n+\frac{1}{2}}_{i+1,j+1}\right) \end{aligned} \tag{3.9}$$

where $U^{n}_{i+\frac{1}{2},j+\frac{1}{2}}$ is calculated using linear interpolants:

$$\begin{aligned} U^{n}_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{4}\left(U^{n}_{i,j} + U^{n}_{i+1,j} + U^{n}_{i,j+1} + U^{n}_{i+1,j+1}\right) \\ &+\frac{1}{16}\left(\delta_{i,j} + \delta_{i,j+1} - \delta_{i+1,j} - \delta_{i+1,j+1}\right) \\ &+\frac{1}{16}\left(\sigma_{i,j} - \sigma_{i,j+1} + \sigma_{i+1,j} - \sigma_{i+1,j+1}\right) \end{aligned} \tag{3.10}$$

and the projection of the solution at time t^{n+1} on the original grid is obtained as follows

$$\begin{aligned} U^{n+1}_{i,j} &= \frac{1}{4}\left(U^{n+1}_{i-\frac{1}{2},j-\frac{1}{2}} + U^{n}_{i+\frac{1}{2},j-\frac{1}{2}} + U^{n}_{i-\frac{1}{2},j+\frac{1}{2}} + U^{n}_{i+\frac{1}{2},j+\frac{1}{2}}\right) \\ &+\frac{1}{16}\left(\delta_{i-\frac{1}{2},j-\frac{1}{2}} + \delta_{i-\frac{1}{2},j+\frac{1}{2}} - \delta_{i+\frac{1}{2},j-\frac{1}{2}} - \delta_{i+\frac{1}{2},j+\frac{1}{2}}\right) \\ &+\frac{1}{16}\left(\sigma_{i-\frac{1}{2},j-\frac{1}{2}} + \sigma_{i-\frac{1}{2},j+\frac{1}{2}} - \sigma_{i+\frac{1}{2},j-\frac{1}{2}} - \sigma_{i+\frac{1}{2},j+\frac{1}{2}}\right) \end{aligned} \tag{3.11}$$

Equation (3.10) is derived as follows:

Linear approximation in 2D:

$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}_{(a,b)} \cdot (x-a) + \frac{\partial f}{\partial y}_{(a,b)} \cdot (y-b)$$

Given $U^n_{i,j}$, we can approximate the solution $U^n(x,y)$ at the cell $C_{i,j}$ as follows:

$$U^n_{C_{i,j}}(x,y) = U^n_{i,j} + (x - x_i)\frac{\delta_{i,j}}{\Delta x} + (y - y_j)\frac{\sigma_{i,j}}{\Delta y}$$

We then compute the solution values $U^n_{i+\frac{1}{2},j+\frac{1}{2}}$ using linear interpolants:

$$\begin{aligned} U^n_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{4}U_{C_{i,j}}(x_i + \alpha\Delta x, y_j + \beta\Delta y, t^n) \\ &+\frac{1}{4}U_{C_{i+1,j}}(x_{i+1} - \alpha\Delta x, y_j + \beta\Delta y, t^n) \\ &+\frac{1}{4}U_{C_{i,j+1}}(x_i + \alpha\Delta x, y_{j+1} - \beta\Delta y, t^n) \\ &+\frac{1}{4}U_{C_{i+1,j+1}}(x_{i+1} - \alpha\Delta x, y_{j+1} - \beta\Delta y, t^n) \end{aligned}$$

with $\alpha \in [0; \frac{1}{2}]$ and $\beta \in [0; \frac{1}{2}]$.

But ,

$$\begin{aligned} U_{C_{i,j}}(x_i + \alpha\Delta x, y_j + \beta\Delta y, t^n) &= U^n_{C_{i,j}}(x_i + \alpha\Delta x, y_j + \beta\Delta y) \\ &= U^n_{i,j} + (x_i + \alpha\Delta x - x_i)\frac{\delta_{i,j}}{\Delta x} \\ &\quad +(y_j + \beta\Delta y - y_j)\frac{\sigma_{i,j}}{\Delta y} \end{aligned}$$

$$U_{C_{i,j}}(x_i + \alpha\Delta x, y_j + \beta\Delta y, t^n) = U^n_{i,j} + \alpha\delta_{i,j} + \beta\sigma_{i,j}$$

Similarly,

$$U_{C_{i+1,j}}(x_{i+1} - \alpha\Delta x, y_j + \beta\Delta y, t^n) = U^n_{i+1,j} - \alpha\delta_{i+1,j} + \beta\sigma_{i+1,j}$$

$$U_{C_{i,j+1}}(x_i + \alpha\Delta x, y_{j+1} - \beta\Delta y, t^n) = U^n_{i,j+1} + \alpha\delta_{i,j+1} - \beta\sigma_{i,j+1}$$

$$U_{C_{i+1,j+1}}(x_{i+1} - \alpha\Delta x, y_{j+1} - \beta\Delta y, t^n) = U^n_{i+1,j+1} - \alpha\delta_{i+1,j+1} - \beta\sigma_{i+1,j+1}$$

Replacing in $U^n_{i+\frac{1}{2},j+\frac{1}{2}}$, we obtain:

$$\begin{aligned} U^n_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{4}\left(U^n_{i,j} + \alpha\delta_{i,j} + \beta\sigma_{i,j}\right) \\ &+\frac{1}{4}\left(U^n_{i+1,j} - \alpha\delta_{i+1,j} + \beta\sigma_{i+1,j}\right) \\ &+\frac{1}{4}\left(U^n_{i,j+1} + \alpha\delta_{i,j+1} - \beta\sigma_{i,j+1}\right) \\ &+\frac{1}{4}\left(U^n_{i+1,j+1} - \alpha\delta_{i+1,j+1} - \beta\sigma_{i+1,j+1}\right) \end{aligned}$$

$$\begin{aligned} U^n_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{4}\left(U^n_{i,j} + U^n_{i+1,j} + U^n_{i,j+1} + U^n_{i+1,j+1}\right) \\ &+\frac{\alpha}{4}\left(\delta_{i,j} + \delta_{i,j+1} - \delta_{i+1,j} - \delta_{i+1,j+1}\right) \\ &+\frac{\beta}{4}\left(\sigma_{i,j} - \sigma_{i,j+1} + \sigma_{i+1,j} - \sigma_{i+1,j+1}\right) \end{aligned}$$

Setting $\alpha = \beta = \frac{1}{4}$ ensures second order accuracy in space [13].

$$\begin{aligned} U^n_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{4}\left(U^n_{i,j} + U^n_{i+1,j} + U^n_{i,j+1} + U^n_{i+1,j+1}\right) \\ &+\frac{1}{16}\left(\delta_{i,j} + \delta_{i,j+1} - \delta_{i+1,j} - \delta_{i+1,j+1}\right) \\ &+\frac{1}{16}\left(\sigma_{i,j} - \sigma_{i,j+1} + \sigma_{i+1,j} - \sigma_{i+1,j+1}\right) \end{aligned}$$

Equation (3.11) is derived using a similar strategy.

The discretization of the source term, along with relations (3.8) and (3.9), will lead to the following relations in the quiescent case:

$$U^{n+\frac{1}{2}}_{i,j} = U^n_{i,j} \tag{3.12}$$

and,

$$U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} = U^n_{i+\frac{1}{2},j+\frac{1}{2}} \tag{3.13}$$

Quiescent Flow as defined in [10]:

In the case of Shallow Water Equations, the quiescent flow represents the steady state of the water, ie., when the surface of the liquid is initially at rest, corresponding to an initial velocity of zero, it remains at rest at any later time. In the quiescent flow case, the unknown U, the flux and the source term vector reduce to:

$$U = \begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix}, F = \begin{pmatrix} 0 \\ \frac{1}{2}gh^2 \\ 0 \end{pmatrix}, G = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}gh^2 \end{pmatrix}, S = \begin{pmatrix} 0 \\ gh \cdot (-\frac{\partial z}{\partial x}) \\ gh \cdot \left(-\frac{\partial z}{\partial y}\right) \end{pmatrix}$$

Let's prove that for all values taken by the parameters s_i and t_j relation (3.12) holds:

According to relation (3.8),

$$U_{i,j}^{n+\frac{1}{2}} = U_{i,j}^n + \frac{\Delta t}{2}\left(-\frac{F'_{i,j}}{\Delta x} - \frac{G'_{i,j}}{\Delta y} + S_{i,j}^n\right)$$

- *For $s_i = -1$ and $t_j = -1$*

$$S_{i,j}^n = \begin{pmatrix} S_1 \\ S_{2,L} \\ S_{3,L} \end{pmatrix} = \begin{pmatrix} 0 \\ -gh_{i,j}^n \theta \frac{z_{i,j} - z_{i-1,j}}{\Delta x} \\ -gh_{i,j}^n \theta \frac{z_{i,j} - z_{i,j-1}}{\Delta y} \end{pmatrix}$$

For a steady state we have:

$$U = \begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix}, F = \begin{pmatrix} 0 \\ \frac{1}{2}gh^2 \\ 0 \end{pmatrix}, G = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}gh^2 \end{pmatrix}$$

with $h_{i,j} + z_{i,j} = H = constant$

Therefore, $F'_{i,j} = \begin{pmatrix} 0 \\ gh^n_{i,j} h_x|^n_{i,j} \\ 0 \end{pmatrix}$ and $G'_{i,j} = \begin{pmatrix} 0 \\ 0 \\ gh^n_{i,j} h_y|^n_{i,j} \end{pmatrix}$

where $h_x|^n_{i,j}$ and $h_y|^n_{i,j}$ denote the derivatives of the water height h with respect to x and y, respectively.

Equation (3.8) becomes:

$$\begin{aligned} U^{n+\frac{1}{2}}_{i,j} &= U^n_{i,j} + \frac{\Delta t}{2} \left[\begin{pmatrix} 0 \\ -gh^n_{i,j} \frac{h_x|^n_{i,j}}{\Delta x} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -gh^n_{i,j} \frac{h_y|^n_{i,j}}{\Delta y} \end{pmatrix} \right] \\ &\quad + \frac{\Delta t}{2} \begin{pmatrix} 0 \\ -gh^n_{i,j} \theta \frac{z_{i,j} - z_{i-1,j}}{\Delta x} \\ -gh^n_{i,j} \theta \frac{z_{i,j} - z_{i,j-1}}{\Delta y} \end{pmatrix} \\ &= U^n_{i,j} + \frac{\Delta t}{2} \begin{pmatrix} 0 \\ -gh^n_{i,j} \left(\frac{h_x|^n_{i,j}}{\Delta x} + \theta \frac{z_{i,j} - z_{i-1,j}}{\Delta x} \right) \\ -gh^n_{i,j} \left(\frac{h_y|^n_{i,j}}{\Delta y} + \theta \frac{z_{i,j} - z_{i,j-1}}{\Delta y} \right) \end{pmatrix} \end{aligned}$$

and since $s_i = -1$ and $t_j = -1$, then,

$$h_x|^n_{i,j} = \theta \left(h^n_{i,j} - h^n_{i-1,j} \right)$$

$$h_y|^n_{i,j} = \theta \left(h^n_{i,j} - h^n_{i,j-1} \right)$$

Replacing $h_x|_{i,j}^n$ and $h_y|_{i,j}^n$ by their values, we obtain:

$$\begin{aligned} U_{i,j}^{n+\frac{1}{2}} &= U_{i,j}^n + \frac{\Delta t}{2}\begin{pmatrix} 0 \\ -gh_{i,j}^n \theta\left(\frac{h_{i,j}-h_{i-1,j}}{\Delta x} + \frac{z_{i,j}-z_{i-1,j}}{\Delta x}\right) \\ -gh_{i,j}^n \theta\left(\frac{h_{i,j}-h_{i,j-1}}{\Delta y} + \frac{z_{i,j}-z_{i,j-1}}{\Delta y}\right) \end{pmatrix} \\ &= U_{i,j}^n + \frac{\Delta t}{2}\begin{pmatrix} 0 \\ -gh_{i,j}^n \frac{\theta}{\Delta x}\left[(h_{i,j}+z_{i,j}) - (h_{i-1,j}+z_{i-1,j})\right] \\ -gh_{i,j}^n \frac{\theta}{\Delta y}\left[(h_{i,j}+z_{i,j}) - (h_{i,j-1}+z_{i,j-1})\right] \end{pmatrix} \end{aligned}$$

But $h_{i,j} + z_{i,j} = H =$ constant for all (x_i, y_j), therefore,

$$U_{i,j}^{n+\frac{1}{2}} = U_{i,j}^n + \frac{\Delta t}{2}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = U_{i,j}^n$$

- *In a similar way, we show that* $U_{i,j}^{n+\frac{1}{2}} = U_{i,j}^n$ *for any value of* s_i *and* t_j*.*

For all values of the parameters s_i and t_j, i.e., whether we are using the backward, forward, central differences, or zero derivative to the case of quiescent flow, the numerical scheme leads to

$$U_{i,j}^{n+\frac{1}{2}} = U_{i,j}^n$$

Proof of relation (3.13): $U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} = U^{n}_{i+\frac{1}{2},j+\frac{1}{2}}$

According to equation (3.9):

$$\begin{aligned} U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} &= U^{n}_{i+\frac{1}{2},j+\frac{1}{2}} \\ &-\frac{\Delta t}{2\Delta x}\left[F(U^{n+\frac{1}{2}}_{i+1,j}) + F(U^{n+\frac{1}{2}}_{i+1,j+1}) - F(U^{n+\frac{1}{2}}_{i,j}) - F(U^{n+\frac{1}{2}}_{i,j+1})\right] \\ &-\frac{\Delta t}{2\Delta y}\left[G(U^{n+\frac{1}{2}}_{i,j+1}) + G(U^{n+\frac{1}{2}}_{i+1,j+1}) - G(U^{n+\frac{1}{2}}_{i,j}) - G(U^{n+\frac{1}{2}}_{i+1,j})\right] \\ &+\Delta t \cdot S\left(U^{n+\frac{1}{2}}_{i,j}, U^{n+\frac{1}{2}}_{i+1,j}, U^{n+\frac{1}{2}}_{i,j+1}, U^{n+\frac{1}{2}}_{i+1,j+1}\right) \end{aligned}$$

with

$$\begin{aligned} S\left(U^{n+\frac{1}{2}}_{i,j}, U^{n+\frac{1}{2}}_{i+1,j}, U^{n+\frac{1}{2}}_{i,j+1}, U^{n+\frac{1}{2}}_{i+1,j+1}\right) &= \frac{1}{2}\begin{pmatrix} 0 \\ -g\frac{h_{i+1,j}+h_{i,j}}{2}\frac{z_{i+1,j}-z_{i,j}}{\Delta x} \\ -g\frac{h_{i,j+1}+h_{i,j}}{2}\frac{z_{i,j+1}-z_{i,j}}{\Delta y} \end{pmatrix} \\ &+\frac{1}{2}\begin{pmatrix} 0 \\ -g\frac{h_{i+1,j+1}+h_{i,j+1}}{2}\frac{z_{i+1,j+1}-z_{i,j+1}}{\Delta x} \\ -g\frac{h_{i+1,j+1}+h_{i+1,j}}{2}\frac{z_{i+1,j+1}-z_{i+1,j}}{\Delta y} \end{pmatrix} \end{aligned}$$

For a quiescent flow, the shallow water system is such that

$$U = \begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix}, F = \begin{pmatrix} 0 \\ \frac{1}{2}gh^2 \\ 0 \end{pmatrix}, G = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}gh^2 \end{pmatrix}$$

We assume that at time $t = t^n$, the steady state requirement is maintained,

i.e., $h_{i,j} + z_{i,j} = H = constant$

- Consider the second component:

$$
\begin{aligned}
U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}|_2} &= U^{n}_{i+\frac{1}{2},j+\frac{1}{2}|_2} \\
&- \frac{\Delta t g}{4\Delta x}\left[\left(h^{n+\frac{1}{2}}_{i+1,j}\right)^2 - \left(h^{n+\frac{1}{2}}_{i,j}\right)^2 + \left(h^{n+\frac{1}{2}}_{i+1,j+1}\right)^2 - \left(h^{n+\frac{1}{2}}_{i,j+1}\right)^2\right] \\
&+ \frac{\Delta t}{2}\left[-g\frac{h^{n+\frac{1}{2}}_{i+1,j} + h^{n+\frac{1}{2}}_{i,j}}{2} * \frac{z_{i+1,j} - z_{i,j}}{\Delta x}\right] \\
&- \frac{\Delta t}{2}\left[g\frac{h^{n+\frac{1}{2}}_{i+1,j+1} + h^{n+\frac{1}{2}}_{i,j+1}}{2} * \frac{z_{i+1,j+1} - z_{i,j+1}}{\Delta x}\right] \\
&= U^{n}_{i+\frac{1}{2},j+\frac{1}{2}|_2} \\
&- \frac{\Delta t g}{4\Delta x}\left[(h^{n+\frac{1}{2}}_{i+1,j} + h^{n+\frac{1}{2}}_{i,j})(h^{n+\frac{1}{2}}_{i+1,j} - h^{n+\frac{1}{2}}_{i,j})\right] \\
&- \frac{\Delta t g}{4\Delta x}\left[(h^{n+\frac{1}{2}}_{i+1,j+1} + h^{n+\frac{1}{2}}_{i,j+1})(h^{n+\frac{1}{2}}_{i+1,j+1} - h^{n+\frac{1}{2}}_{i,j+1})\right] \\
&- \frac{\Delta t g}{4\Delta x}\left[(h^{n+\frac{1}{2}}_{i+1,j} + h^{n+\frac{1}{2}}_{i,j})(z_{i+1,j} - z_{i,j})\right] \\
&- \frac{\Delta t g}{4\Delta x}\left[(h^{n+\frac{1}{2}}_{i+1,j+1} + h^{n+\frac{1}{2}}_{i,j+1})(z_{i+1,j+1} - z_{i,j+1})\right] \\
&= U^{n}_{i+\frac{1}{2},j+\frac{1}{2}|_2} - \frac{\Delta t g}{4\Delta x}\left[(h^{n+\frac{1}{2}}_{i+1,j} + h^{n+\frac{1}{2}}_{i,j})(h^{n+\frac{1}{2}}_{i+1,j} - h^{n+\frac{1}{2}}_{i,j} + z_{i+1,j} - z_{i,j})\right] \\
&- \frac{\Delta t g}{4\Delta x}\left[(h^{n+\frac{1}{2}}_{i+1,j+1} + h^{n+\frac{1}{2}}_{i,j+1})(h^{n+\frac{1}{2}}_{i+1,j+1} - h^{n+\frac{1}{2}}_{i,j+1} + z_{i+1,j+1} - z_{i,j+1})\right] \\
&= U^{n}_{i+\frac{1}{2},j+\frac{1}{2}|_2} \\
&- \frac{\Delta t g}{4\Delta x}\left[(h^{n+\frac{1}{2}}_{i+1,j} + h^{n+\frac{1}{2}}_{i,j})\left[(h^{n+\frac{1}{2}}_{i+1,j} + z_{i+1,j}) - (h^{n+\frac{1}{2}}_{i,j} + z_{i,j})\right]\right] \\
&- \frac{\Delta t g}{4\Delta x}\left[(h^{n+\frac{1}{2}}_{i+1,j+1} + h^{n+\frac{1}{2}}_{i,j+1})\left[(h^{n+\frac{1}{2}}_{i+1,j+1} + z_{i+1,j+1}) - (h^{n+\frac{1}{2}}_{i,j+1} + z_{i,j+1})\right]\right]
\end{aligned}
$$

But $h^n_{i,j} + z_{i,j} = H =$ constant for all i, j, and since $U^{n+\frac{1}{2}}_{i,j} = U^n_{i,j}$ thus

$$U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}|_2} = U^{n}_{i+\frac{1}{2},j+\frac{1}{2}|_2}$$

- And similarly for the third component:

$$\begin{aligned}
U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}|_3} &= U^{n}_{i+\frac{1}{2},j+\frac{1}{2}|_3} \\
&- \frac{\Delta t g}{4\Delta y}\left[\left(h^{n+\frac{1}{2}}_{i,j+1}\right)^2 - \left(h^{n+\frac{1}{2}}_{i,j}\right)^2 + \left(h^{n+\frac{1}{2}}_{i+1,j+1}\right)^2 - \left(h^{n+\frac{1}{2}}_{i+1,j}\right)^2\right] \\
&+ \frac{\Delta t}{2}\left[-g\frac{h^{n+\frac{1}{2}}_{i,j+1} + h^{n+\frac{1}{2}}_{i,j}}{2} * \frac{z_{i,j+1} - z_{i,j}}{\Delta y}\right] \\
&+ \frac{\Delta t}{2}\left[-g\frac{h^{n+\frac{1}{2}}_{i+1,j+1} + h^{n+\frac{1}{2}}_{i+1,j}}{2} * \frac{z_{i+1,j+1} - z_{i+1,j}}{\Delta y}\right] \\
&= U^{n}_{i+\frac{1}{2},j+\frac{1}{2}|_3} \\
&- \frac{\Delta t g}{4\Delta y}\left[(h^{n+\frac{1}{2}}_{i,j+1} + h^{n+\frac{1}{2}}_{i,j})(h^{n+\frac{1}{2}}_{i,j+1} - h^{n+\frac{1}{2}}_{i,j})\right] \\
&- \frac{\Delta t g}{4\Delta y}\left[(h^{n+\frac{1}{2}}_{i+1,j+1} + h^{n+\frac{1}{2}}_{i+1,j})(h^{n+\frac{1}{2}}_{i+1,j+1} - h^{n+\frac{1}{2}}_{i+1,j})\right] \\
&- \frac{\Delta t g}{4\Delta y}\left[(h^{n+\frac{1}{2}}_{i,j+1} + h^{n+\frac{1}{2}}_{i,j})(z_{i,j+1} - z_{i,j})\right] \\
&- \frac{\Delta t g}{4\Delta y}\left[(h^{n+\frac{1}{2}}_{i+1,j+1} + h^{n+\frac{1}{2}}_{i+1,j})(z_{i+1,j+1} - z_{i+1,j})\right] \\
&= U^{n}_{i+\frac{1}{2},j+\frac{1}{2}|_3} - \frac{\Delta t g}{4\Delta y}\left[(h^{n+\frac{1}{2}}_{i,j+1} + h^{n+\frac{1}{2}}_{i,j})(h^{n+\frac{1}{2}}_{i,j+1} - h^{n+\frac{1}{2}}_{i,j} + z_{i,j+1} - z_{i,j})\right] \\
&- \frac{\Delta t g}{4\Delta y}\left[(h^{n+\frac{1}{2}}_{i+1,j+1} + h^{n+\frac{1}{2}}_{i+1,j})(h^{n+\frac{1}{2}}_{i+1,j+1} - h^{n+\frac{1}{2}}_{i+1,j} + z_{i+1,j+1} - z_{i+1,j})\right] \\
&= U^{n}_{i+\frac{1}{2},j+\frac{1}{2}|_3} - \frac{\Delta t g}{4\Delta y}\left[(h^{n+\frac{1}{2}}_{i,j+1} + h^{n+\frac{1}{2}}_{i,j})\left[(h^{n+\frac{1}{2}}_{i,j+1} + z_{i,j+1}) - (h^{n+\frac{1}{2}}_{i,j} + z_{i,j})\right]\right] \\
&- \frac{\Delta t g}{4\Delta y}\left[(h^{n+\frac{1}{2}}_{i+1,j+1} + h^{n+\frac{1}{2}}_{i+1,j})\left[(h^{n+\frac{1}{2}}_{i+1,j+1} + z_{i+1,j+1}) - (h^{n+\frac{1}{2}}_{i+1,j} + z_{i+1,j})\right]\right]
\end{aligned}$$

But $h^n_{i,j} + z_{i,j} = H =$ constant for all i, j, and since $U^{n+\frac{1}{2}}_{i,j} = U^n_{i,j}$

$$U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}|_3} = U^{n}_{i+\frac{1}{2},j+\frac{1}{2}|_3}$$

So we conclude that, in the steady state,

$$U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} = U^{n}_{i+\frac{1}{2},j+\frac{1}{2}}$$

3.2 2D Balanced Central NT Scheme: The 2D Interface Type Reformulation

As in 1D, the Interface Type reformulation is a particular adaptation of the Balanced Central NT Scheme. The bottom topography is defined at the interfaces of each cell $C_{i,j}$, ie., $z_{i+\frac{1}{2},j+\frac{1}{2}}$ is defined at $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$.

At the cell centers, we define

$$z_{i,j} = \frac{z_{i-\frac{1}{2},j-\frac{1}{2}} + z_{i-\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{1}{2}}}{4}$$

The interface type reformulation is a particular adaptation of the well balanced central scheme derived previously. Assuming that for a quiescent flow, the numerical solution $U^n_{i,j}$ satisfies the steady state requirement at time t^n, the numerical solution $U^{n+1}_{i,j}$ is calculated as follows:

First we calculate the solution at the intermediate time $t^{n+\frac{1}{2}}$:

$$U^{n+\frac{1}{2}}_{i,j} = U^n_{i,j} + \frac{\Delta t}{2}\left(-\frac{F'_{i,j}}{\Delta x} - \frac{G'_{i,j}}{\Delta y} + S^n_{i,j}\right)$$

where

$$S^n_{i,j} = \begin{pmatrix} 0 \\ -gh^n_{i,j}\frac{\partial z}{\partial x} \\ -gh^n_{i,j}\frac{\partial z}{\partial y} \end{pmatrix}$$

and, next we calculate the solution at time t^{n+1} on the staggered grid as follows

$$\begin{aligned} U^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} &= U^n_{i+\frac{1}{2},j+\frac{1}{2}} \\ &-\frac{\Delta t}{2\Delta x}\left[F(U^{n+\frac{1}{2}}_{i+1,j}) + F(U^{n+\frac{1}{2}}_{i+1,j+1}) - F(U^{n+\frac{1}{2}}_{i,j}) - F(U^{n+\frac{1}{2}}_{i,j+1})\right] \\ &-\frac{\Delta t}{2\Delta y}\left[G(U^{n+\frac{1}{2}}_{i,j+1}) + G(U^{n+\frac{1}{2}}_{i+1,j+1}) - G(U^{n+\frac{1}{2}}_{i,j}) - G(U^{n+\frac{1}{2}}_{i+1,j})\right] \\ &+\Delta t \cdot S\left(U^{n+\frac{1}{2}}_{i,j}, U^{n+\frac{1}{2}}_{i+1,j}, U^{n+\frac{1}{2}}_{i,j+1}, U^{n+\frac{1}{2}}_{i+1,j+1}\right) \end{aligned}$$

with

$$S\left(U_{i,j}^{n+\frac{1}{2}}, U_{i+1,j}^{n+\frac{1}{2}}, U_{i,j+1}^{n+\frac{1}{2}}, U_{i+1,j+1}^{n+\frac{1}{2}}\right) = \frac{1}{2}\begin{pmatrix} 0 \\ -g\frac{h_{i+1,j}+h_{i,j}}{2}\frac{z_{i+1,j}-z_{i,j}}{\Delta x} \\ -g\frac{h_{i,j+1}+h_{i,j}}{2}\frac{z_{i,j+1}-z_{i,j}}{\Delta y} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ -g\frac{h_{i+1,j+1}+h_{i,j+1}}{2}\frac{z_{i+1,j+1}-z_{i,j+1}}{\Delta x} \\ -g\frac{h_{i+1,j+1}+h_{i+1,j}}{2}\frac{z_{i+1,j+1}-z_{i+1,j}}{\Delta y} \end{pmatrix}$$

The adaptation of the surface gradient method or the interface type reformulation is based on a particular projection of the numerical solution on the original grid and on the staggered one.
The hu and hv components are computed by using exactly the Well-Balanced scheme derived previously. However, a special approximation of the water depth h at each step of the algorithm is required.
$h_{i,j}^{n+\frac{1}{2}}$ and $h_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1}$ are computed using equations (3.7) and (3.8). Nevertheless, in order to approximate $h_{i+\frac{1}{2},j+\frac{1}{2}}^{n}$ and $h_{i,j}^{n+1}$, equations (3.9) and (3.10) need to be updated using the Surface Gradient Method discussed in [10].
Using the two-dimensional unstaggered central scheme, the water height h and the bottom topography z are considered to be linear inside each original control volume $C_{i,j}$.
In this interface-type reformulation, the linearization of the water height is made indirectly by first linearizing the water level $H(x, y)$, then using $h(x, y) = H(x, y) - z(x, y)$.
$H(x, y) = H_{i,j} + H_{x|i,j}(x - x_i) + H_{y|i,j}(y - y_j)$ for all $(x, y) \in C_{i,j}$ with H_x and H_y computed using a slope limiting procedure.

$$H_{i,j}^n = h_{i,j}^n + z_{i,j}$$

This will lead to:

$$h_x|_{i,j}^n = H_x|_{i,j}^n - z_x|_{i,j}$$

and

$$h_y|_{i,j}^n = H_y|_{i,j}^n - z_y|_{i,j}$$

ie., using the cental difference,

$$h_x|_{i,j} = H_x|_{i,j} - \frac{1}{\Delta x}\left(\frac{z_{i,j}+z_{i+1,j}}{2} - \frac{z_{i-1,j}+z_{i,j}}{2}\right) \tag{3.14}$$

$$h_y|_{i,j} = H_y|_{i,j} - \frac{1}{\Delta y}(\left(\frac{z_{i,j}+z_{i,j+1}}{2} - \frac{z_{i,j-1}+z_{i,j}}{2}\right) \tag{3.15}$$

Applying relation (3.10) to h results in:

$$\begin{aligned} h^n_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{4}\left(h^n_{i,j}+h^n_{i+1,j}+h^n_{i,j+1}+h^n_{i+1,j+1}\right) \\ &\quad +\frac{\alpha\Delta x}{4}\left(h_{x|i,j}+h_{x|i,j+1}-h_{x|i+1,j}-h_{x|i+1,j+1}\right) \\ &\quad +\frac{\beta\Delta y}{4}\left(h_{y|i,j}-h_{y|i,j+1}+h_{y|i+1,j}-h_{y|i+1,j+1}\right) \end{aligned} \tag{3.16}$$

Equations (3.13), (3.14) and (3.15) lead to:

$$
\begin{aligned}
h^n_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{4}\left(h^n_{i,j}+h^n_{i+1,j}+h^n_{i,j+1}+h^n_{i+1,j+1}\right)\\
&+\frac{\alpha\Delta x}{4}\left(H^n_{x|i,j}-\frac{\frac{z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}-\frac{z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}}}{2}}{\Delta x}\right)\\
&+\frac{\alpha\Delta x}{4}\left(H^n_{x|i,j+1}-\frac{\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}}{2}-\frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{3}{2}}}{2}}{\Delta x}\right)\\
&-\frac{\alpha\Delta x}{4}\left(H^n_{x|i+1,j}-\frac{\frac{z_{i+\frac{3}{2},j-\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}}{2}-\frac{z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}}{\Delta x}\right)\\
&-\frac{\alpha\Delta x}{4}\left(H^n_{x|i+1,j+1}-\frac{\frac{z_{i+\frac{3}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{3}{2}}}{2}-\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}}{2}}{\Delta x}\right)\\
&+\frac{\beta\Delta y}{4}\left(H_{y|i,j}-\frac{\frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}-\frac{z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}}{2}}{\Delta y}\right)\\
&+\frac{\beta\Delta y}{4}\left(H^n_{y|i+1,j}-\frac{\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}}{2}-\frac{z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{3}{2},j-\frac{1}{2}}}{2}}{\Delta y}\right)\\
&-\frac{\beta\Delta y}{4}\left(H^n_{y|i,j+1}-\frac{\frac{z_{i-\frac{1}{2},j+\frac{3}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}}{2}-\frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}}{\Delta y}\right)\\
&-\frac{\beta\Delta y}{4}\left(H^n_{y|i+1,j+1}-\frac{\frac{z_{i+\frac{1}{2},j+\frac{3}{2}}+z_{i+\frac{3}{2},j+\frac{3}{2}}}{2}-\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}}{2}}{\Delta y}\right)
\end{aligned}
\tag{3.17}
$$

The water depth $h^{n+1}_{i,j}$ at time t^{n+1} on the original grid is approximated as follows:

We first note that on the original control volumes, the relation $H_{i,j} = h_{i,j} + z_{i,j}$ applies, while on the staggered control volumes, a similar relation holds

$$\tilde{\mathrm{H}}_{i+\frac{1}{2},j+\frac{1}{2}} = h_{i+\frac{1}{2},j+\frac{1}{2}} + \tilde{\mathrm{z}}_{i+\frac{1}{2},j+\frac{1}{2}} \tag{3.18}$$

where $\tilde{\mathrm{H}}_{i+\frac{1}{2},j+\frac{1}{2}}$ and $\tilde{\mathrm{z}}_{i+\frac{1}{2},j+\frac{1}{2}}$ are defined differently due to the fact that the riverbed bottom $z(x,y)$ is linear inside the original control volumes $C_{i,j}$ but not inside the staggered control volumes $D_{i+\frac{1}{2},j+\frac{1}{2}}$.

Therefore, define $\tilde{\mathrm{z}}_{i+\frac{1}{2},j+\frac{1}{2}}$ as follows:

$$\tilde{\mathrm{z}}_{i+\frac{1}{2},j+\frac{1}{2}} = z_{i+\frac{1}{2},j+\frac{1}{2}} - \frac{1}{2}\left(z_{i+\frac{1}{2},j+\frac{1}{2}} - \frac{z_{i,j} + z_{i+1,j} + z_{i,j+1} + z_{i+1,j+1}}{4}\right) \tag{3.19}$$

$\tilde{\mathrm{H}}_{i+\frac{1}{2},j+\frac{1}{2}}$ will be defined accordingly; leading to:

$$h^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} = \tilde{\mathrm{H}}^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} - \tilde{\mathrm{z}}_{i+\frac{1}{2},j+\frac{1}{2}}$$

The partial derivative can be now calculated as follows

$$h_{i+\frac{1}{2},j+\frac{1}{2}|x} = H_{i+\frac{1}{2},j+\frac{1}{2}|x} - \frac{\frac{z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2} - \frac{z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}}}{2}}{\Delta x} \tag{3.20}$$

and

$$h_{i+\frac{1}{2},j+\frac{1}{2}|y} = H_{i+\frac{1}{2},j+\frac{1}{2}|y} - \frac{\frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2} - \frac{z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}}{2}}{\Delta y} \tag{3.21}$$

The discrete derivatives $H_{i+\frac{1}{2},j+\frac{1}{2}|x}$ and $H_{i+\frac{1}{2},j+\frac{1}{2}|y}$ are derived from the staggered values $\tilde{\mathrm{H}}_{i+\frac{1}{2},j+\frac{1}{2}}$ using a slope limiting procedure.

Finally, using relations (3.19) and (3.20) in equation (3.11) for h:

$$\begin{aligned}
h_{i,j}^{n+1} &= \frac{1}{4}\left(h_{i-\frac{1}{2},j-\frac{1}{2}}^{n+1} + h_{i+\frac{1}{2},j-\frac{1}{2}}^{n+1} + h_{i-\frac{1}{2},j+\frac{1}{2}}^{n+1} + h_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1}\right) \\
&+ \frac{\alpha\Delta x}{4}\left(H_{x|i-\frac{1}{2},j-\frac{1}{2}}^{n+1} - \frac{\frac{z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}}{2} - \frac{z_{i-\frac{3}{2},j-\frac{1}{2}}+z_{i-\frac{1}{2},j-\frac{1}{2}}}{2}}{\Delta x}\right) \\
&+ \frac{\alpha\Delta x}{4}\left(H_{x|i-\frac{1}{2},j+\frac{1}{2}}^{n+1} - \frac{\frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2} - \frac{z_{i-\frac{3}{2},j+\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}}}{2}}{\Delta x}\right) \\
&- \frac{\alpha\Delta x}{4}\left(H_{x|i+\frac{1}{2},j-\frac{1}{2}}^{n+1} - \frac{\frac{z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{3}{2},j-\frac{1}{2}}}{2} - \frac{z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}}{2}}{\Delta x}\right) \\
&- \frac{\alpha\Delta x}{4}\left(H_{x|i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - \frac{\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}}{2} - \frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}}{\Delta x}\right) \\
&+ \frac{\beta\Delta y}{4}\left(H_{y|i-\frac{1}{2},j-\frac{1}{2}}^{n+1} - \frac{\frac{z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}}}{2} - \frac{z_{i-\frac{1}{2},j-\frac{3}{2}}+z_{i-\frac{1}{2},j-\frac{1}{2}}}{2}}{\Delta y}\right) \\
&+ \frac{\beta\Delta y}{4}\left(H_{y|i+\frac{1}{2},j-\frac{1}{2}}^{n+1} - \frac{\frac{z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2} - \frac{z_{i+\frac{1}{2},j-\frac{3}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}}{2}}{\Delta y}\right) \\
&- \frac{\beta\Delta y}{4}\left(H_{y|i-\frac{1}{2},j+\frac{1}{2}}^{n+1} - \frac{\frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{3}{2}}}{2} - \frac{z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}}}{2}}{\Delta y}\right) \\
&- \frac{\beta\Delta y}{4}\left(H_{y|i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - \frac{\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}}{2} - \frac{z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}}{\Delta y}\right) \qquad (3.22)
\end{aligned}$$

Proposition:

The 2D Interface-Type (equations (3.7), (3.8), (3.17), (3.22)) maintains the Steady State Condition in the way it moves from the original cells to the staggered ones, and vice versa

Proof:

In order to prove that the transformations between the original and the staggered grid maintain the steady state condition, we will use the following two relations (3.17) and (3.22) from the

2D-Interface-Type reformulation.

The main idea consists in replacing relation (3.16) in (3.21) by using the fact that in the steady state, $h^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} = h^{n}_{i+\frac{1}{2},j+\frac{1}{2}}$ as proved in equation (3.12).

- Simplify equation (3.22) in case of a steady state:

  Let A be the term involving the x-derivatives and B the terms involving the y-derivatives in equation (3.22).

  The reduction of A and B uses the fact that in the steady state,

  $H_x = H_y = 0$.

$$\begin{aligned}
A &= \frac{\alpha\Delta x}{2*4\Delta x}\left[-(z_{i-\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j-\frac{1}{2}}) + (z_{i-\frac{3}{2},j-\frac{1}{2}} + z_{i-\frac{1}{2},j-\frac{1}{2}})\right] \\
&\quad + \frac{\alpha\Delta x}{2*4\Delta x}\left[-(z_{i-\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{1}{2}}) + (z_{i-\frac{3}{2},j+\frac{1}{2}} + z_{i-\frac{1}{2},j+\frac{1}{2}})\right] \\
&\quad + \frac{\alpha\Delta x}{2*4\Delta x}\left[(z_{i+\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{3}{2},j-\frac{1}{2}}) - (z_{i-\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j-\frac{1}{2}})\right] \\
&\quad + \frac{\alpha\Delta x}{2*4\Delta x}\left[(z_{i+\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{3}{2},j+\frac{1}{2}}) - (z_{i-\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{1}{2}})\right] \\
&= \frac{\alpha}{8}\left[z_{i-\frac{3}{2},j-\frac{1}{2}} + z_{i-\frac{1}{2},j-\frac{1}{2}} + z_{i-\frac{3}{2},j+\frac{1}{2}} + z_{i-\frac{1}{2},j+\frac{1}{2}}\right] \\
&\quad + \frac{\alpha}{8}\left[z_{i+\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{3}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{3}{2},j+\frac{1}{2}}\right] \\
&\quad - \frac{\alpha}{8}\left[z_{i-\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j-\frac{1}{2}} + z_{i-\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{1}{2}}\right] \\
&\quad - \frac{\alpha}{8}\left[z_{i-\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j-\frac{1}{2}} + z_{i-\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{1}{2}}\right] \\
&= \frac{\alpha}{8}\left(4z_{i-1,j} + 4z_{i+1,j} - 8z_{i,j}\right) \\
&= \frac{\alpha}{2}\left(z_{i-1,j} + z_{i+1,j} - 2z_{i,j}\right)
\end{aligned}$$

$$
\begin{aligned}
B &= \frac{\beta\Delta y}{2*4\Delta y}\left[-(z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}})+(z_{i-\frac{1}{2},j-\frac{3}{2}}+z_{i-\frac{1}{2},j-\frac{1}{2}})\right]\\
&\quad+\frac{\beta\Delta y}{2*4\Delta y}\left[-(z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}})+(z_{i+\frac{1}{2},j-\frac{3}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}})\right]\\
&\quad+\frac{\beta\Delta y}{2*4\Delta y}\left[(z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{3}{2}})-(z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}})\right]\\
&\quad+\frac{\beta\Delta y}{2*4\Delta y}\left[(z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}})-(z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}})\right]\\
&= \frac{\beta}{8}\left[z_{i-\frac{1}{2},j-\frac{3}{2}}+z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{3}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}\right]\\
&\quad+\frac{\beta}{8}\left[z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{3}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}\right]\\
&\quad-\frac{\beta}{8}\left[z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}\right]\\
&\quad-\frac{\beta}{8}\left[z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}\right]\\
&= \frac{\beta}{8}\left(4z_{i,j-1}+4z_{i,j+1}-8z_{i,j}\right)\\
&= \frac{\beta}{2}\left(z_{i,j-1}+z_{i,j+1}-2z_{i,j}\right)
\end{aligned}
$$

For $\alpha=\beta=\frac{1}{4}$, equation (3.22) becomes:

$$
\begin{aligned}
h^{n+1}_{i,j} &= \frac{1}{4}\left(h^{n+1}_{i-\frac{1}{2},j-\frac{1}{2}}+h^{n+1}_{i+\frac{1}{2},j-\frac{1}{2}}+h^{n+1}_{i-\frac{1}{2},j+\frac{1}{2}}+h^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}\right)\\
&\quad+\frac{1}{8}(z_{i-1,j}+z_{i+1,j}+z_{i,j-1}+z_{i,j+1}-4z_{i,j})
\end{aligned}
$$

But according to equation (3.13), $h^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}=h^{n}_{i+\frac{1}{2},j+\frac{1}{2}}$, which leads to

$$
\begin{aligned}
h^{n+1}_{i,j} &= \frac{1}{4}\left(h^{n}_{i-\frac{1}{2},j-\frac{1}{2}}+h^{n}_{i+\frac{1}{2},j-\frac{1}{2}}+h^{n}_{i-\frac{1}{2},j+\frac{1}{2}}+h^{n}_{i+\frac{1}{2},j+\frac{1}{2}}\right)\\
&\quad+\frac{1}{8}(z_{i-1,j}+z_{i+1,j}+z_{i,j-1}+z_{i,j+1}-4z_{i,j})
\end{aligned}
\tag{3.23}
$$

- Simplify equation (3.16) in case of a steady state:

  In case of a steady state, $H_x = Hy = 0$, so we obtain:

$$
\begin{aligned}
h^n_{i+\frac{1}{2},j+\frac{1}{2}} \quad = \quad & \frac{1}{4}\left(h^n_{i,j} + h^n_{i+1,j} + h^n_{i,j+1} + h^n_{i+1,j+1}\right) \\
& + \frac{\alpha\Delta x}{4}\left(-\frac{\frac{z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2} - \frac{z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}}}{2}}{\Delta x}\right) \\
& + \frac{\alpha\Delta x}{4}\left(-\frac{\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}}{2} - \frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{3}{2}}}{2}}{\Delta x}\right) \\
& - \frac{\alpha\Delta x}{4}\left(-\frac{\frac{z_{i+\frac{3}{2},j-\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}}{2} - \frac{z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}}{\Delta x}\right) \\
& - \frac{\alpha\Delta x}{4}\left(-\frac{\frac{z_{i+\frac{3}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{3}{2}}}{2} - \frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}}{2}}{\Delta x}\right) \\
& + \frac{\beta\Delta y}{4}\left(-\frac{\frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2} - \frac{z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}}{2}}{\Delta y}\right) \\
& + \frac{\beta\Delta y}{4}\left(-\frac{\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}}{2} - \frac{z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{3}{2},j-\frac{1}{2}}}{2}}{\Delta y}\right) \\
& - \frac{\beta\Delta y}{4}\left(-\frac{\frac{z_{i-\frac{1}{2},j+\frac{3}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}}{2} - \frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}}{\Delta y}\right) \\
& - \frac{\beta\Delta y}{4}\left(-\frac{\frac{z_{i+\frac{1}{2},j+\frac{3}{2}}+z_{i+\frac{3}{2},j+\frac{3}{2}}}{2} - \frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}}{2}}{\Delta y}\right)
\end{aligned}
$$

  But

$$
\frac{z_{i+\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{1}{2}}}{2} - \frac{z_{i-\frac{1}{2},j-\frac{1}{2}} + z_{i-\frac{1}{2},j+\frac{1}{2}}}{2} = 2\left(\frac{z_{i+\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{1}{2}}}{2} - z_{i,j}\right)
$$

$$
\frac{z_{i+\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{3}{2}}}{2} - \frac{z_{i-\frac{1}{2},j+\frac{1}{2}} + z_{i-\frac{1}{2},j+\frac{3}{2}}}{2} = 2\left(\frac{z_{i+\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{3}{2}}}{2} - z_{i,j+1}\right)
$$

$$
\frac{z_{i+\frac{3}{2},j-\frac{1}{2}} + z_{i+\frac{3}{2},j+\frac{1}{2}}}{2} - \frac{z_{i+\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{1}{2}}}{2} = 2\left(z_{i+1,j} - \frac{z_{i+\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}\right)
$$

$$\frac{z_{i+\frac{3}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{3}{2}}}{2}-\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}}{2}=2\left(z_{i+1,j+1}-\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}}{2}\right)$$

and

$$\frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}-\frac{z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}}{2}=2\left(\frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}-z_{i,j}\right)$$

$$\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}}{2}-\frac{z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{3}{2},j-\frac{1}{2}}}{2}=2\left(\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}}{2}-z_{i+1,j}\right)$$

$$\frac{z_{i-\frac{1}{2},j+\frac{3}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}}{2}-\frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}=2\left(z_{i,j+1}-\frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}\right)$$

$$\frac{z_{i+\frac{1}{2},j+\frac{3}{2}}+z_{i+\frac{3}{2},j+\frac{3}{2}}}{2}-\frac{z_{i+\frac{1}{2},j+\frac{3}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}}{2}=2\left(z_{i+1,j+1}-\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}}{2}\right)$$

Therefore,

$$
\begin{aligned}
h^n_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{4}\left(h^n_{i,j}+h^n_{i+1,j}+h^n_{i,j+1}+h^n_{i+1,j+1}\right) \\
&+\frac{\alpha}{4}\left[-2\left(\frac{z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}-z_{i,j}\right)\right] \\
&+\frac{\alpha}{4}\left[-2\left(\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}}{2}-z_{i,j+1}\right)\right] \\
&+\frac{\alpha}{4}\left[2\left(z_{i+1,j}-\frac{z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}\right)\right] \\
&+\frac{\alpha}{4}\left[2\left(z_{i+1,j+1}-\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}}{2}\right)\right] \\
&+\frac{\beta}{4}\left[-2\left(\frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}-z_{i,j}\right)\right] \\
&+\frac{\beta}{4}\left[-2\left(\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}}{2}-z_{i+1,j}\right)\right] \\
&+\frac{\beta}{4}\left[2\left(z_{i,j+1}-\frac{z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{2}\right)\right] \\
&+\frac{\beta}{4}\left[2\left(z_{i+1,j+1}-\frac{z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}}{2}\right)\right] \\
&= \frac{1}{4}\left(h^n_{i,j}+h^n_{i+1,j}+h^n_{i,j+1}+h^n_{i+1,j+1}\right) \\
&+\frac{\alpha}{4}\left(2z_{i,j}+2z_{i,j+1}+2z_{i+1,j}+2z_{i+1,j+1}\right) \\
&-\frac{\alpha}{4}\left(2z_{i+\frac{1}{2},j-\frac{1}{2}}+4z_{i+\frac{1}{2},j+\frac{1}{2}}+2z_{i+\frac{1}{2},j+\frac{3}{2}}\right) \\
&+\frac{\beta}{4}\left(2z_{i,j}+2z_{i+1,j}+2z_{i,j+1}+2z_{i+1,j+1}\right) \\
&-\frac{\beta}{4}\left(2z_{i-\frac{1}{2},j+\frac{1}{2}}+4z_{i+\frac{1}{2},j+\frac{1}{2}}+2z_{i+\frac{3}{2},j+\frac{1}{2}}\right)
\end{aligned}
$$

For $\alpha = \beta = \frac{1}{4}$,

$$\begin{aligned}
h^n_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{4}\left(h^n_{i,j} + h^n_{i+1,j} + h^n_{i,j+1} + h^n_{i+1,j+1}\right) \\
&\quad + \frac{1}{4}\left(z_{i,j} + z_{i+1,j} + z_{i,j+1} + z_{i+1,j+1}\right) \\
&\quad - \frac{1}{8}\left(4z_{i+\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{3}{2}} + z_{i-\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{3}{2},j+\frac{1}{2}}\right) \\
&= H \\
&\quad - \frac{1}{8}\left(4z_{i+\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{3}{2}} + z_{i-\frac{1}{2},j+\frac{1}{2}} + z_{i+\frac{3}{2},j+\frac{1}{2}}\right) \qquad (3.24)
\end{aligned}$$

with $H = h_{i,j} + z_{i,j}$ for all i, j.

Similarly,

$$h^n_{i-\frac{1}{2},j-\frac{1}{2}} = H - \frac{1}{8}\left(4z_{i-\frac{1}{2},j-\frac{1}{2}} + z_{i-\frac{1}{2},j-\frac{3}{2}} + z_{i-\frac{1}{2},j+\frac{1}{2}} + z_{i-\frac{3}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j-\frac{1}{2}}\right)$$

$$h^n_{i+\frac{1}{2},j-\frac{1}{2}} = H - \frac{1}{8}\left(4z_{i+\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{1}{2},j-\frac{3}{2}} + z_{i+\frac{1}{2},j+\frac{1}{2}} + z_{i-\frac{1}{2},j-\frac{1}{2}} + z_{i+\frac{3}{2},j-\frac{1}{2}}\right)$$

$$h^n_{i-\frac{1}{2},j+\frac{1}{2}} = H - \frac{1}{8}\left(4z_{i-\frac{1}{2},j+\frac{1}{2}} + z_{i-\frac{1}{2},j-\frac{1}{2}} + z_{i-\frac{1}{2},j+\frac{3}{2}} + z_{i-\frac{3}{2},j+\frac{1}{2}} + z_{i+\frac{1}{2},j+\frac{1}{2}}\right)$$

- Replace $h^n_{i-\frac{1}{2},j-\frac{1}{2}}, h^n_{i+\frac{1}{2},j-\frac{1}{2}}, h^n_{i-\frac{1}{2},j+\frac{1}{2}}, h^n_{i+\frac{1}{2},j+\frac{1}{2}}$ by their values in equation (3.22):

$$
\begin{aligned}
h^{n+1}_{i,j} &= \frac{1}{4}(H+H+H+H) \\
&+\frac{1}{4*8}\left(4z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}\right) \\
&+\frac{1}{4*8}\left(4z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i-\frac{1}{2},j-\frac{3}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i-\frac{3}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}\right) \\
&+\frac{1}{4*8}\left(4z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{3}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{3}{2},j-\frac{1}{2}}\right) \\
&+\frac{1}{4*8}\left(4z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{3}{2}}+z_{i-\frac{3}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}\right) \\
&+\frac{1}{8}(z_{i-1,j}+z_{i+1,j}+z_{i,j-1}+z_{i,j+1}-4z_{i,j}) \\
&= H-\frac{1}{8}\left(z_{i-1,j}+z_{i+1,j}+z_{i,j-1}+z_{i,j+1}+4z_{i,j}\right) \\
&+\frac{1}{8}\left(z_{i-1,j}+z_{i+1,j}+z_{i,j-1}+z_{i,j+1}-4z_{i,j}\right) \\
&= H-z_{i,j} \\
&= h^n_{i,j}+z_{i,j}-z_{i,j} \\
&= h^n_{i,j}
\end{aligned}
$$

(using: $z_{i,j} = \frac{z_{i-\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{1}{2}}}{4}$ and $h^n_{i,j}+z_{i,j}=H$)

We finally prove that $\tilde{\text{H}}_{i+\frac{1}{2},j+\frac{1}{2}}$ is maintained constant in the quiescent case:

Using equations (3.19) and (3.24) in equation (3.18) we obtain:

$$
\begin{aligned}
\tilde{\text{H}}_{i+\frac{1}{2},j+\frac{1}{2}} &= h_{i+\frac{1}{2},j+\frac{1}{2}}+\tilde{\text{z}}_{i+\frac{1}{2},j+\frac{1}{2}} \\
&= H-\frac{1}{8}\left(4z_{i+\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{1}{2},j-\frac{1}{2}}+z_{i+\frac{1}{2},j+\frac{3}{2}}+z_{i-\frac{1}{2},j+\frac{1}{2}}+z_{i+\frac{3}{2},j+\frac{1}{2}}\right) \\
&+z_{i+\frac{1}{2},j+\frac{1}{2}}-\frac{1}{2}\left(z_{i+\frac{1}{2},j+\frac{1}{2}}-\frac{z_{i,j}+z_{i+1,j}+z_{i,j+1}+z_{i+1,j+1}}{4}\right) \\
&= H
\end{aligned}
$$

End of Proof

Concluding Remark:

The Stability condition is defined as in [15] by: $\Delta t = min(\Delta t_1, \Delta t_2)$,
with $\Delta t_1 = CFL * \frac{\Delta x}{max(max(LF))}$ and $\Delta t_2 = CFL * \frac{\Delta y}{max(max(LG))}$ where LF and LG are the matrices containing respectively the eigen values of $\frac{\partial F}{\partial U}$ and $\frac{\partial G}{\partial U}$.
In our computations, we considered a CFL number equal to 0.485.

4. NUMERICAL EXPERIMENTS

4.1 *One-dimensional numerical experiments:*

4.1.1 Toro's problem

This first example features a constant riverbed ($z(x) = 0$) with a discontinous initial condition as discussed in [13]. The computational domain [0,40] is discretized using 600 gridpoints and the final solution is calculated at time $t = 2$ using the interface-type reformulation. The initial condition for h is given by

$$h(x,0) = \begin{cases} 2.5 & \text{, if } 17.5 < x < 22.5\,, \\ 0.5 & \text{, otherwise}\,. \end{cases}$$

with initial velocity $v(x,0) = 0$.

The water height is shown in figure 4.1 and compared to the one returned by the numerical base scheme derived in [13]. For this problem, the numerical base scheme is capable to generate the exact profile without additional treatment, since the source term is set to zero.

4.1.2 Dam Break over a rectangular bump

The second example features a rapidly varying flow over a discontinuous bottom as discussed in [10]. The computational domain is [0;1500] discretized using 600 grid points. The computations are performed at $t = 15$s.

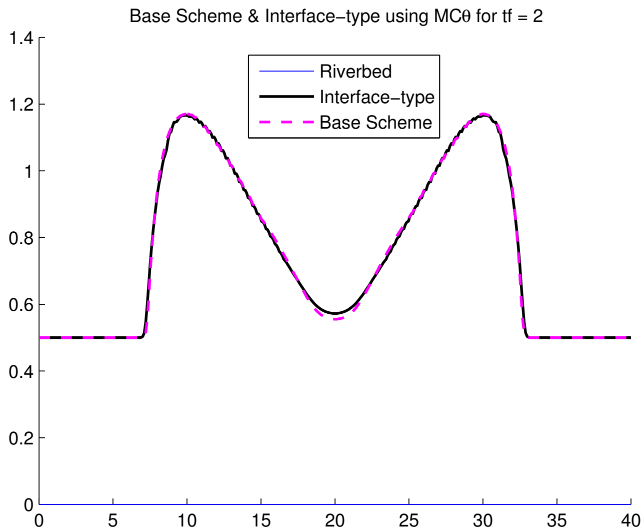

Fig. 4.1: Toro's problem: Water height at $t = 2$ using the MC-θ limiter for the Base scheme and Interface-type reformulation.

The discontinuous riverbed is given with

$$z(x) = \begin{cases} 8 & , \text{if } |x - \frac{1500}{2}| < \frac{1500}{4}\,, \\ 0 & , \text{otherwise}\,. \end{cases}$$

The initial water level is:

$$H(x) = \begin{cases} 20 & , \text{if } x < \frac{1500}{2}\,, \\ 15 & , \text{otherwise}\,. \end{cases}$$

with $v(x, 0) = 0$. The nonphysical oscillations returned by the well-balanced algorithm at the points of discontinuity of the riverbed disappear whenever the interface type reformulation is considered (figures 4.2, 4.3) showing, numerically, that the interface-type reformulation eliminates the non physical oscillations whenever the riverbed is discontinuous.

Figure 4.4 shows a comparison of the numerical results returned by the interface type reformu-

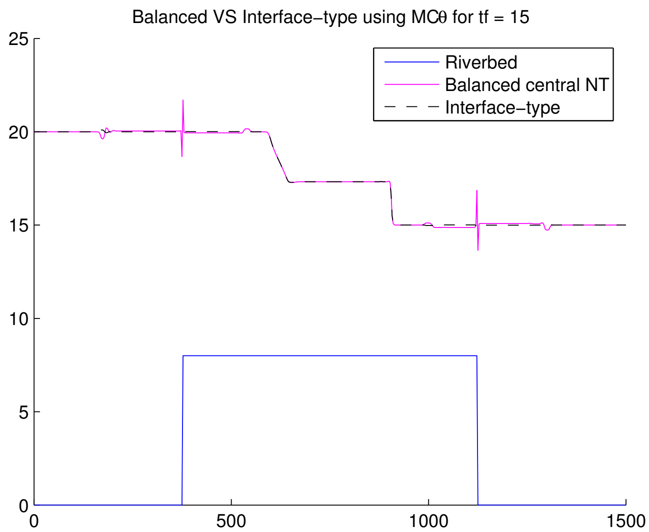

Fig. 4.2: Dam Break problem: Water height at $t = 15$ using the well-balanced algorithm and the Interface-type reformulation

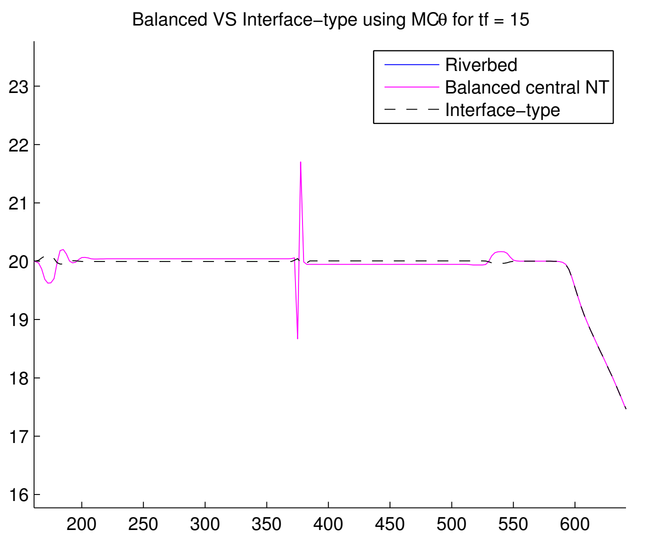

Fig. 4.3: Dam Break problem: Water height using the well-balanced algorithm and the Interface-type reformulation (zoomed)

lation using two different limiting procedures: MinMod and MC-θ ($\theta = 1.5$). As expected, MC-θ returns sharper results, showing that our extension of the discretization of the source term from the MinMod limiter to the MC-θ limiter resulted in less diffusive solutions.

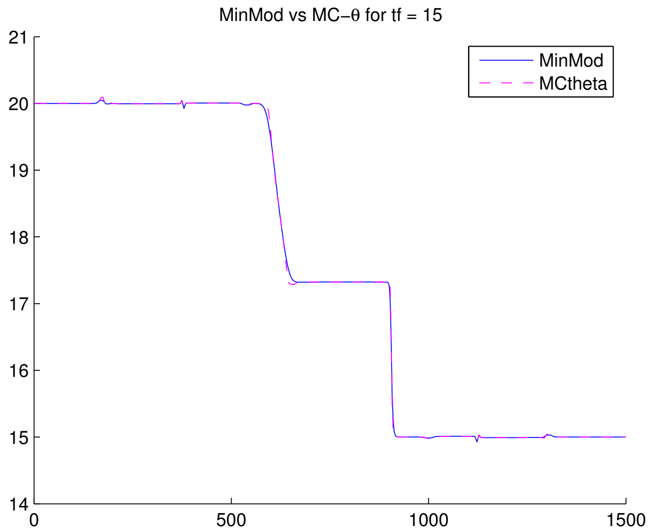

Fig. 4.4: Dam Break problem: comparison of the limiters for $\theta = 1.5$

The validation of the method is made by finding the numerical solutions for different grid spacings: Δx, $\frac{\Delta x}{2}$, and $\frac{\Delta x}{4}$. Considering the solution returned for $\frac{\Delta x}{4}$ to be the most precise solution, compare the two other results to it. Concluding that our numerical solution does not depend on Δx results in the validation of the method used.

4.1.3 Quiescent flow over an irregular riverbed

We consider the case of a gradually varied flow. The riverbed is defined through a set of points in [14] and in section 3.1 of [10]. The computational domain is [0,1500] and the computations are performed with $\Delta x = 7.5$. The water is initially at rest with a height of $12m$. The right boundary condition is $v(1500, t) = 0$.

Figures 4.8 and 4.9 show the numerical results returned by the interface-type reformulation

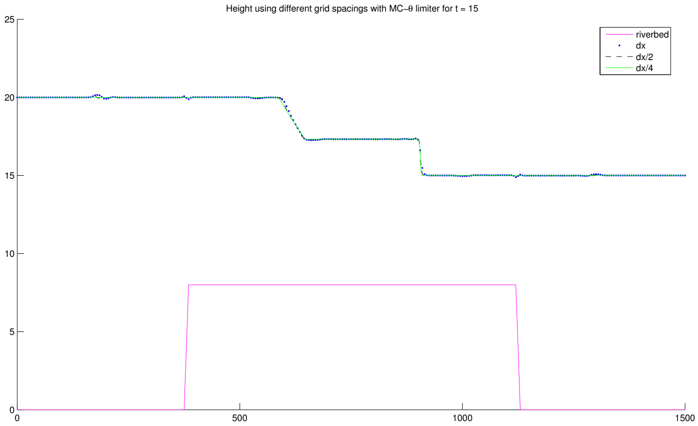

Fig. 4.5: Dam Break problem: validation of the method. Water height for different grid spacings with $\Delta x = 5$

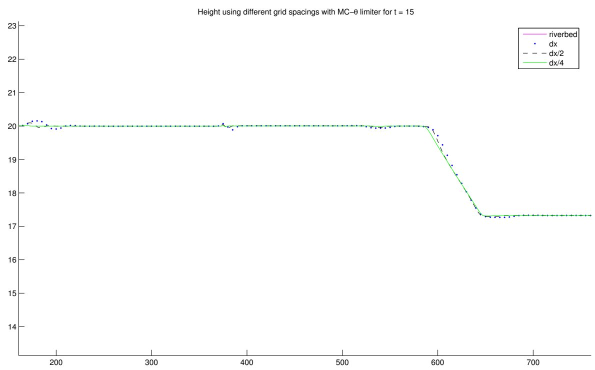

Fig. 4.6: Dam Break problem: validation of the method. Water height for different grid spacings with $\Delta x = 5$ (zoomed)

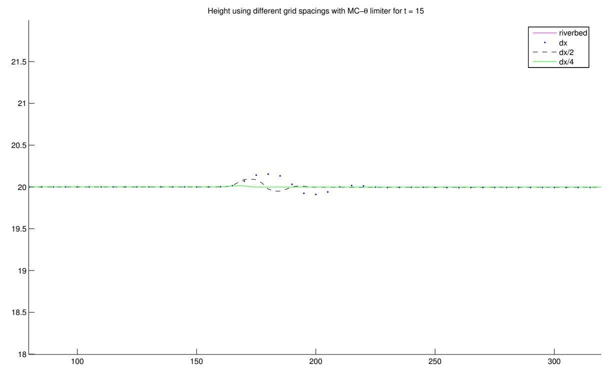

Fig. 4.7: Dam Break problem: validation of the method. Water height for different grid spacings with $\Delta x = 5$ (more zoomed)

compared to those returned by the well-balanced algorithm. The nonphysical oscillations showed by the well-balanced algorithm at the right boundary are treated by the interface type algorithm, proving that the modified scheme maintains the steady state condition, even when the riverbed is irregular.

4.2 Two-dimensional numerical experiments

4.2.1 Toro's problem

This first two-dimensional example features a non variable riverbed ($z(x, y) = 0$) with a discontinous initial condition as discussed in [13]. The computational domain [0,40]×[0,40] is discretized using 100^2 gridpoints. The initial conditions are $u(x, y, 0) = 0$, $v(x, y, 0) = 0$, and

$$h(x,y,0) = \begin{cases} 2.5 & , \text{if } 17.5 < x < 22.5\,, \\ 0.5 & , \text{otherwise}\,. \end{cases}$$

Figure 4.10 shows the profile of the water height at the final time $t = 4.7$s obtained using the Interface-type reformulation; and figure 4.11 shows a plot of the height h along the line $y = 10$

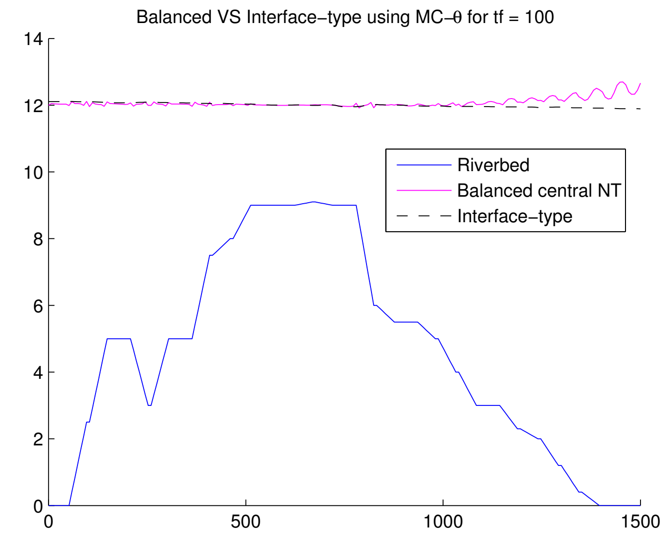

Fig. 4.8: Quiescent flow: comparison of results for $t = 100$s using the MC-θ limiter for the well-balanced algorithm and the interface-type reformulation

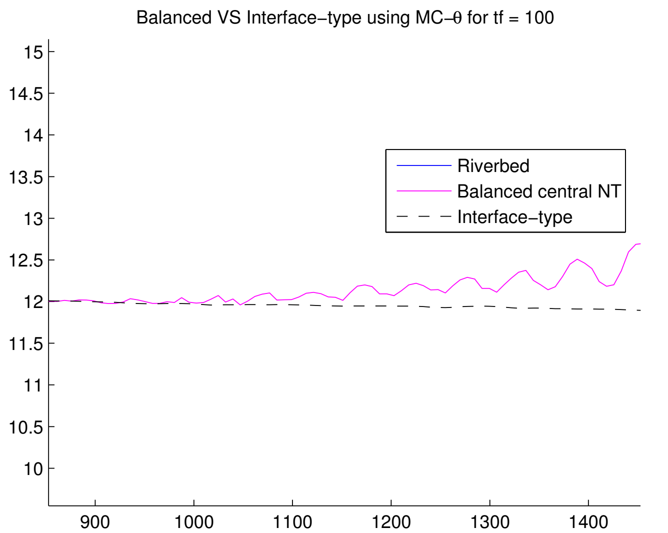

Fig. 4.9: Steady state problem: comparison of results for $t = 100$s using the MC-θ limiter for the well-balanced algorithm and the interface-type reformulation (zoomed)

obtained using the base scheme (solid line) and the well balanced and interface-type scheme (dotted line). The results are in good agreement with the ones shown in [13], in which the two-dimensional shallow water equations are numerically solved using the base scheme.

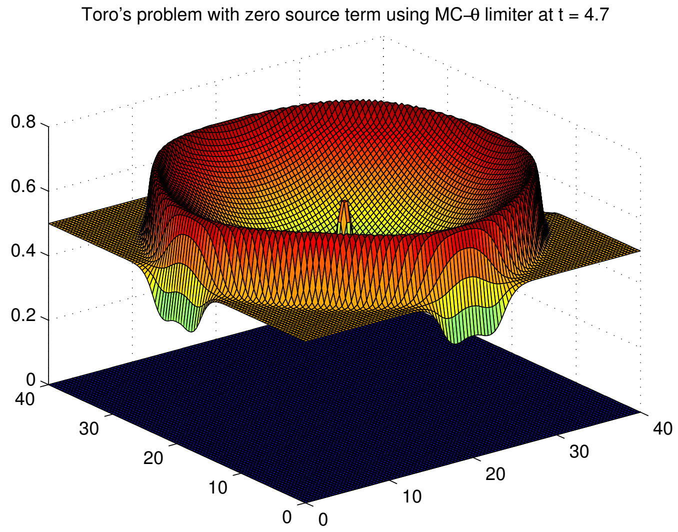

Fig. 4.10: Two-Dimensional Toro's problem at t=4.7s, 100x100 gridpoints, MC-θ

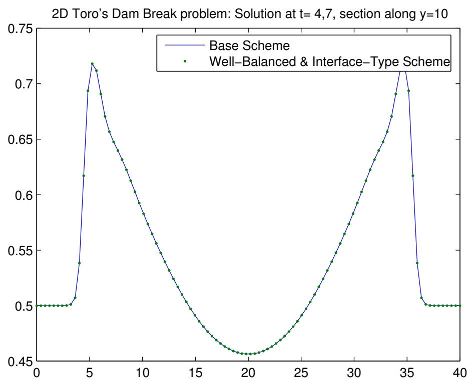

Fig. 4.11: two-dimensional Toro's problem: comparison of the Interface-type scheme with the Base scheme for y = 10, 100×100 gridpoints, MC-θ

4.2.2 Dam Break over a rectangular bump

This second problem is a two-dimensional extension of the rapidly varying flow over a discontinuous bottom discussed in [10].

The computational domain [0;1500]x[0;1500] is discretized using 600×11 gridpoints and the computations are performed at the final time $t = 15$s.
The discontinuous riverbed is defined by

$$z(x,y) = \begin{cases} 8, & \text{if } |x - \frac{1500}{2}| < \frac{1500}{4}\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

The initial water level is defined as follows:

$$H(x,y,0) = \begin{cases} 20, & \text{if } x < \frac{1500}{2}\,, \\ 15, & \text{otherwise}\,. \end{cases}$$

with $u(x,y,0) = v(x,y,0) = 0$.
The nonphysical oscillations returned by the well-balanced algorithm at the points of discontinuity of the riverbed disappear whenever the interface type reformulation is considered (figure 4.13), showing that the derived two-dimensional interface-type scheme reduces the oscillations returned at the discontinuities in case of a discontinuous riverbed.
The method is validated by comparing it to the one-dimensional interface-type reformulation derived in chapter 2, which was also derived in [10]. The cross sectional result of the water height returned by the two-dimensional interface-type scheme in figure 4.14 is in great agreement with the water height obtained from the one-dimensional scheme, validating the two-dimensional extension of the scheme.

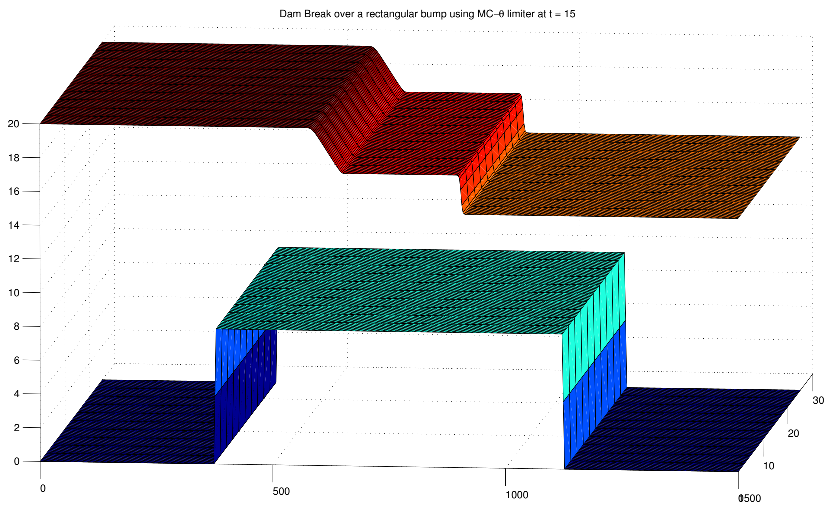

Fig. 4.12: Two-Dimensional Dam Break over a Rectangular Bump at t=15, 600×11 gridpoints, MC-θ

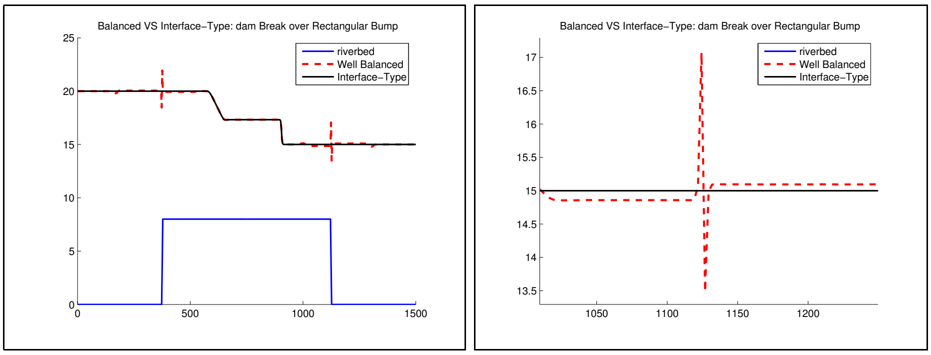

Fig. 4.13: Two-Dimensional Dam Break problem: Well-Balanced VS Interface-type

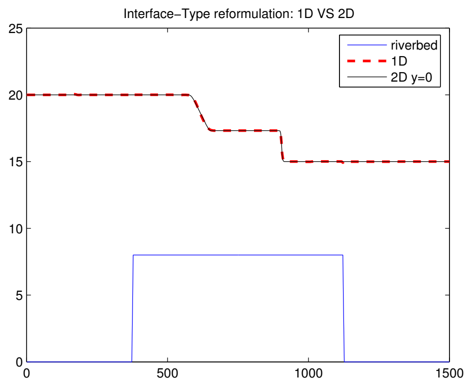

Fig. 4.14: Dam Break problem over a rectangular bump: Interface-Type, 1D VS 2D

4.2.3 Dam Break over a flat bottom

This problem is meant to prove the validity of the two-dimensional interface-type scheme by comparing the numerical solution obatined by our scheme to the reference solution returned by the Riemann solver CLAWPACK. It is of a great benefit to compare our scheme, which avoids Riemann prolems, to a solver that is based on the strong Riemann procedure.

The computational domain [-5, 5]x[-5, 5] is discretized using 600×11 gridpoints and the solution is computed at the final time $t = 2$s. The riverbed function is set to zero and the initial water level is defined by:

$$H(x, y) = \begin{cases} 3 & , \text{if } x < 0\,, \\ 1 & , \text{otherwise}\,. \end{cases}$$

with zero initial velocities. The Interface-type solution is computed using the MC-θ limiter and represented for y = 0 (figure 4.15).

The two solutions agree, with a relatively small relative error (figure 4.16) confirming the efficiency and the potential of the Non-Riemann Interface-type scheme, compared to a Riemann solver.

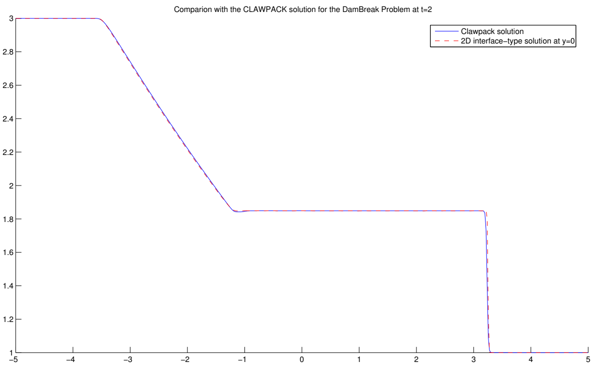

Fig. 4.15: Dam Break problem over a flat bottom: reference solution (solid line) obtained using the CLAWPACK solver and the numerical solution using our scheme (dashed line) through the line y = 0 using the 2D Interface-type scheme at t=2 using the MC-θ limiter

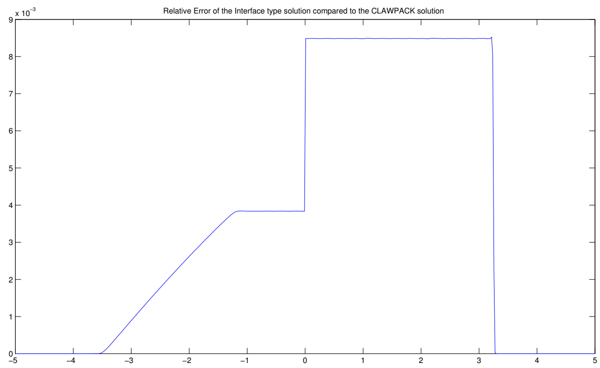

Fig. 4.16: Dam Break problem with zero source term: Absolute error of the interface-type solution compared to the clawpack solution at t=2 using the MC-θ limiter

Grid Size	$\|E\|_2^2$	O_{h_1}(E)
100x100	0.8848	-
200x200	0.3771	1.23
400x400	0.1257	1.6

Time Step	$\|E\|_2^2$	O_{h_1}(E)
$\frac{\Delta t}{3}$	0.0055	-
$\frac{\Delta t}{4}$	0.0038	1.3
$\frac{\Delta t}{5}$	0.00261	1.7

Tab. 4.1: Interface-Type Scheme, Dam Break problem: $L2$ norms of the errors of the numerical solution compared to the reference solution with respect to space (n=800) and time ($\frac{\Delta t}{7}$)

4.2.4 Continuous water level over a flat bottom

In this problem we consider a continuous water level problem over a flat bottom. The computational domain [0;1500]×[0;1500] is discretized using 600×11 gridpoints and the numerical solution is computed at the final time $t = 2$s. The initial velocities are set to zero, and the inital water level is defined by

$$H(x,y,0) = \begin{cases} 20, & \text{if } x < 500\,, \\ -\frac{x}{100} + 25, & \text{if } x < 1000\,, \\ 15, & \text{otherwise}\,. \end{cases}$$

This problem was used to study the numerical accuracy of our scheme in time and space. Taking the numerical solution returned using the finest grid (800 gridpoints) for the Dam Break problem as the exact solution, a simple variation of the number of gridpoints proves the quadratic numerical accuracy of the scheme. Similarly, considering the solution returned for smallest time step Δt to be the exact one, variations of the time step prove the quadratic accuracy of the Interface-type reformulation with respect to time; although this quadratic accuracy is theoretically maintained due to the fractional time step used in the base scheme. The scheme is also proven to be not dependent on the number of gridpoints or the time step used (figures 4.17, 4.18). Figures 4.17 and 4.18 show the superposition of the solutions returned by the Interface-Type scheme for this problem for different number of gridpoints and different time steps, respectively.

According to figures 19 and 20, showing the $L2$ norm of the error with respect to the grid size and the time step on a LOGLOG scale, the numerical accuracy of the scheme is O(1.6) with respect to space and O(1.7) with respect to time.

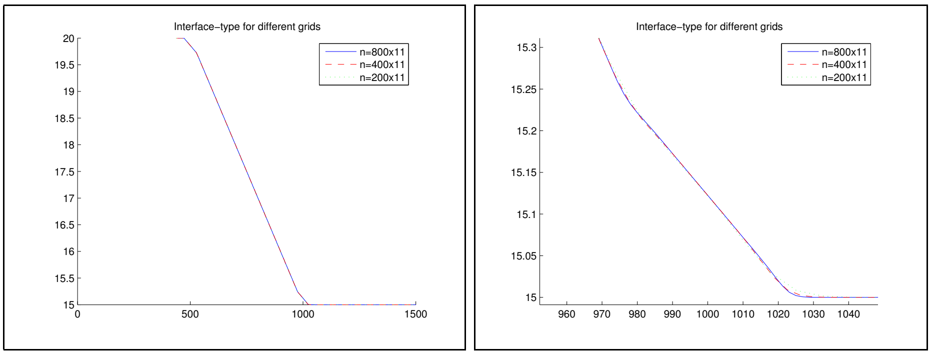

Fig. 4.17: Continuous water level over a flat riverbed: solutions returned for different grid sizes compared to the finest grid using the MC-θ limiter at t=2

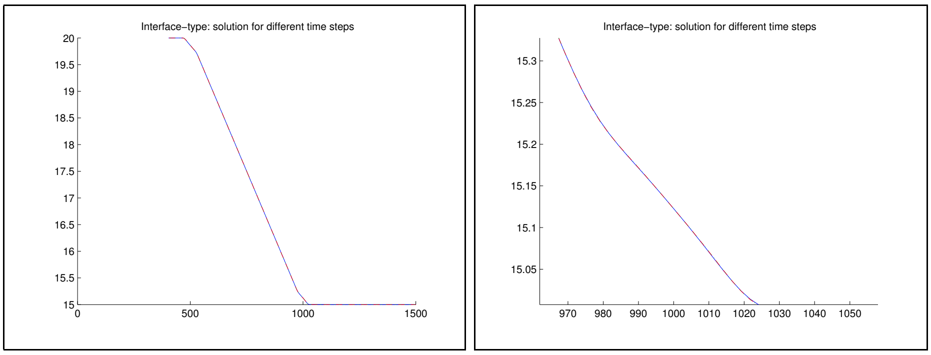

Fig. 4.18: Continuous water level over a flat riverbed: solutions returned for different time steps compared to the smallest time step using the MC-θ limiter at t=2

4.2.5 Dam Break over a discontinuous riverbed

This two-dimensional problem features a rapidly varying flow over a discontinuous bottom. The computational domain [0;1500]x[0;40] is discretized using 600×11 gridpoints and the numerical solution is calculated at time $t = 15$s using the MC-θ.

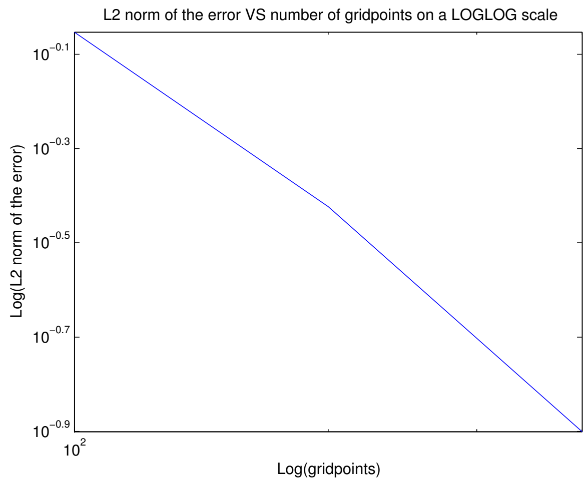

Fig. 4.19: Dam Break problem with continuous riverbed: LOGLOG: L2 norm of the absolute error of the solutions returned for different time steps compared to the smallest time step using the MC-θ limiter

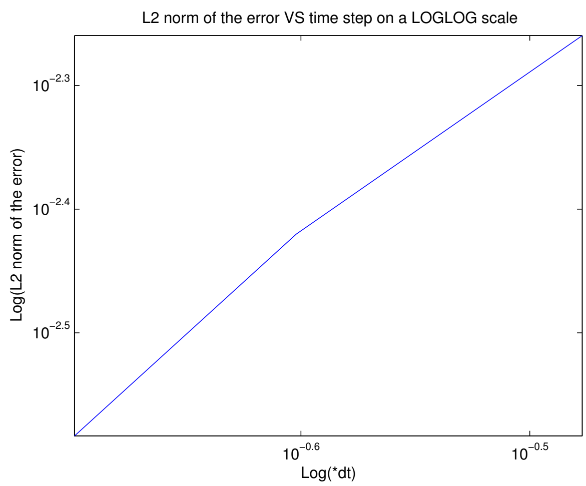

Fig. 4.20: Dam Break problem with continuous riverbed: LOGLOG: L2 norm of the absolute error of the solutions returned for different time steps compared to the smallest time step using the MC-θ limiter

The discontinuous riverbed is defined by

$$z(x,y) = \begin{cases} 8, & \text{if } 700 < x < 800\,, \\ 7, & \text{if } 600 < x < 700, \text{ or } 800 < x < 900\,, \\ 6, & \text{if } 500 < x < 600, \text{ or } 900 < x < 1000\,, \\ 5, & \text{if } 400 < x < 500, \text{ or } 1000 < x < 1100\,, \\ 4, & \text{if } 300 < x < 400, \text{ or } 1100 < x < 1200\,, \\ 3, & \text{if } 200 < x < 300, \text{ or } 1200 < x < 1300\,, \\ 2, & \text{if } 100 < x < 200, \text{ or } 1300 < x < 1400\,, \\ 1, & \text{otherwise}\,. \end{cases}$$

The initial water level is defined by:

$$H(x,y) = \begin{cases} 20, & \text{if } x < \frac{1500}{2}\,, \\ 15, & \text{otherwise}\,. \end{cases}$$

and the initial velocities are set to zero.

The whole purpose of such a problem is to challenge the interface-type reformulation in case of multiple discontinuities. Our scheme showed a great performance in such a case; figure 4.21 represents the water height returned by the new reformulation, not showing any oscillation at the discontinuities.

4.2.6 Quiescent flow over an irregular riverbed

We consider the case of a varying flow over the riverbed shown in figure 4.22. The problem is an extension of the one-dimensional example discussed in section 4.1.3 and in [14]. The computational domain [0,1500]x[0,1500] is discretized using 200^2 grid points and the solution is calculated at the final time $t = 0.5914$. The water is initially at rest with a height of $90m$.

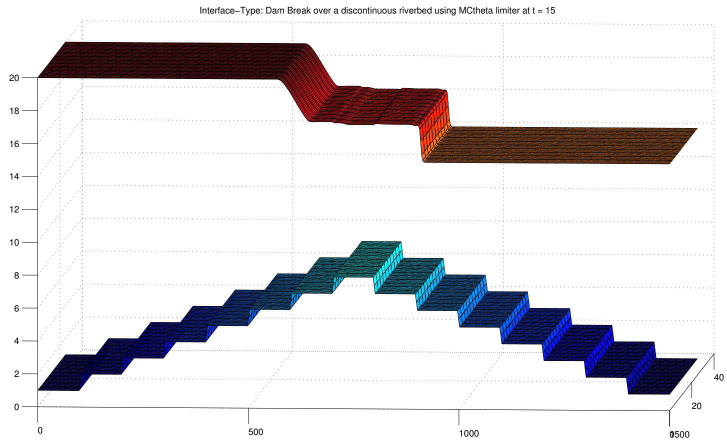

Fig. 4.21: Interface-Type scheme: Dam Break over a discontinuous riverbed at t= 15

Figure 4.22 shows the steady state of the water height at time $t = 0.5914$s. Figures 4.23 and 4.24 show the water height returned by the interface-type reformulation compared to that returned by the well-balanced algorithm. The nonphysical oscillations showed by the well-balanced algorithm at the right boundary are erased by the interface type algorithm, thus confirming the high potential and efficiency of the scheme and its capability to maintain the steady state of the water even when an irregular riverbed is considered.

4.2.7 Two Rarefaction waves over a zero riverbed

We consider the problem of two rarefaction waves propagating in opposite directions as presented in [16], where the initial conditions are given with $H(x, y, 0) = 2$ and

$$u(x,y,0) = v(x,y,0) = \begin{cases} -5.0, x < 25, \\ 5.0, x > 25. \end{cases}$$

with non variable riverbed ($z(x, y) = 0$).

The solution consists of two rarefaction waves, presented at the final time $t = 0.3$s. The profile

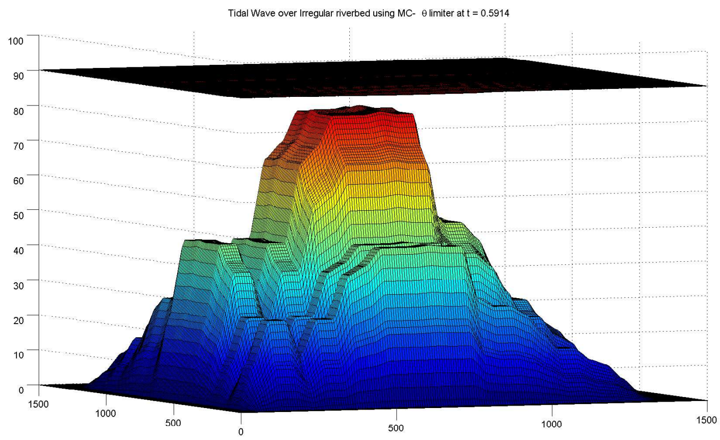

Fig. 4.22: Two-Dimensional quiescent flow problem: Steady State at t=0.5914, 200^2 gridpoints, MC-θ

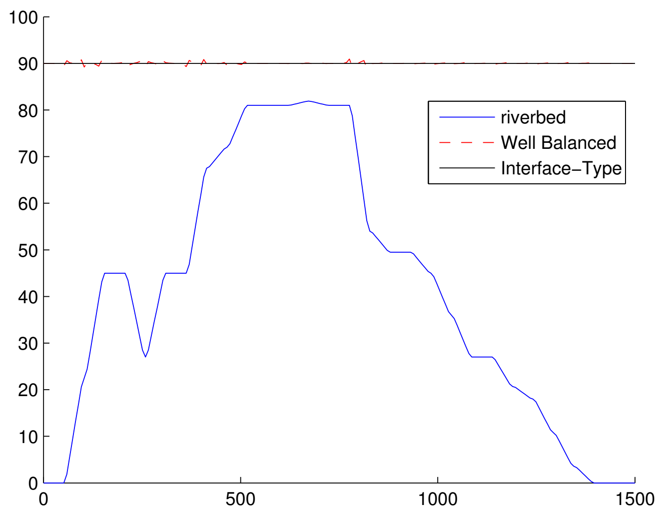

Fig. 4.23: Two-Dimensional quiescent flow: Comparison of results obtained at $t = 0.5914$s using the MC-θ limiter for the well-balanced algorithm and the interface-type reformulation.

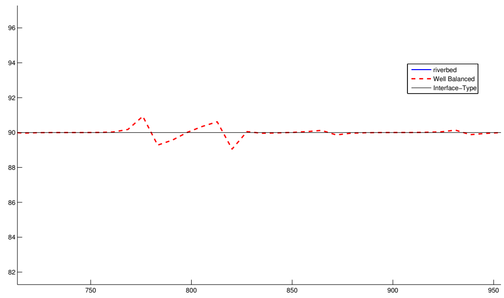

Fig. 4.24: Two-Dimensional quiescent flow: Comparison of results obtained at $t = 0.5914$s using the MC-θ limiter for the well-balanced algorithm and the interface-type reformulation (magnified).

of the water height is shown in figure 4.25 and a cross section along the line $y = 0$ is presented in figure 4.26. The obtained results are in good agreement with the corresponding ones presented in [5], showing the validity of the two-dimensional interface-type reformulation whenever non zero initial velocities are considered.

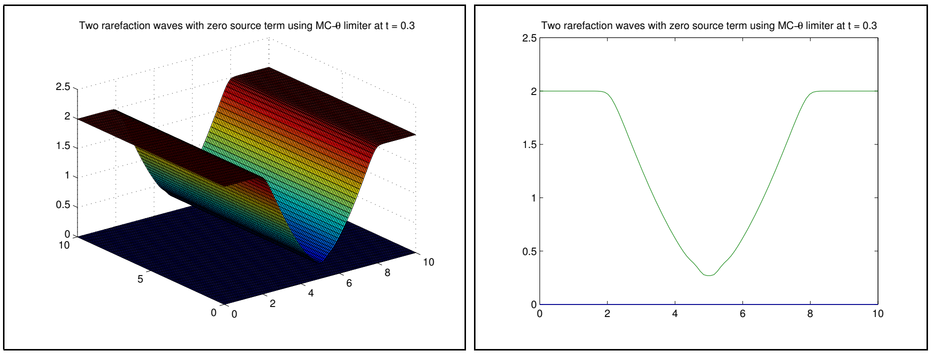

Fig. 4.25: Two-Dimensional Interface-Type Scheme: Rarefaction waves problem at $t = 0.3$ and a cross section for y=0 using the MC-θ limiter

5. CONCLUSION

In this work, we presented a class of one and two-dimensional well-balanced unstaggered central schemes for the approximation of the solution of balanced laws with geometrical source terms. The schemes are first applied to the one-dimensional shallow water equations and then extended to the two-dimensional case. The one-dimensional shallow water equations are first numerically solved using the well-balanced central scheme and an adaptation of the surface gradient method, the interface-type reformulation, both extensions of the work discussed in [10]. They focus on discretizing the source term according to the flux divergence, aiming therefore at balancing the shallow water system in the case of a quiescent flow. The interface-type reformulation, adaptation of the well-balanced scheme, is based on a particular discretization of the water height by first linearizing the water level, in which case the quiescent flow is maintained. The method is then extended to the case of two-dimensional shallow water equations, which is the main objective of this thesis. As in the one-dimensional case, the method guarantees well-balancing by discretizing the source term according to the flux divergence using some parameters based on the MinMod and the MC-θ limiters. Furthermore, the proposed method maintains the steady state by following the Surface Gradient method discussed in[11], according to which the water depth is discretized using the water level. This last feature assures the well performance of the schemes in cases of quiescent flows. The computations performed on several test problems show very good results in both steady and unsteady flow cases, as well as in discontinuous and irregular riverbed cases, and thus confirm the high potential and efficiency of the proposed methods.

As a future work one could investigate the extension of our method to the case of unstructured grids by applying the well balanced scheme and the interface-type reformulation to triangular grids instead of rectangular ones. The treatment of the dry state is also of great importance whenever it comes to shallow water systems. In some problems, the falling of an important water dam might cause the dryness of the riverbed, leading in most cases to negative water height or even to the divergence of the numerical schemes.

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