ON SINGULAR FUNCTIONS
OF BOUNDED VARIATION

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The present thesis is the outcome of my endeavour to make a detailed study of the class of functions known as Singular Functions of Bounded Variation.

While taking a course in set theory and topology, I was introduced to functions which are continuous and have zero differential coefficient almost everywhere, without being constant, and later to a function (Koepeke's function) which is continuous, everywhere oscillating and has finite differential coefficient everywhere. This aroused my curiosity to know more about the nature of everywhere oscillating functions. A systematic study of these functions requires a study of non-monotonic functions, which in turn needs a comprehensive study of singular functions of bounded variation.

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ABSTRACT

A detailed study of a class of functions known as 'singular functions of bounded variation' is the theme of this thesis. This class of functions for the purpose of this study has been divided into four subclasses as follows:

1. Step Functions.
2. Jump Functions.
3. Strictly Increasing Singular Functions.
4. Non Monotonic Singular Functions.

To each of the above subclasses, one chapter has been devoted. The thesis therefore comprises of five chapters, Chapter I being an introduction to the subject.

The introduction chapter, namely Chapter I, deals with three known definitions and some interesting properties of a singular function of bounded variation. It also includes an article on various decompositions of a function of bounded variation. At the end of the chapter, an arithmetic method of defining a function is discussed.

In Chapter II, continuous step functions have been studied. This chapter can be taken as a summary of all the work done on this topic from the time of G. Cantor (1884) till today. Works of Cantor [1], Hille & Tamarkin [1], T. Carleman [1], R.E. Gilman [1], H. Kober [2] and [4], have been specially studied in this chapter.

Chapter III is concerned with another class of singular functions, namely Jump Functions. These are inverse functions of step functions, and therefore the inverse of the functions mentioned in Chapter II have been studied.

Chapter IV deals with that subclass of singular functions which are strictly increasing functions defined by A. Denjoy [1], W. Sierpinski [1], and S. Saks [1] have been studied. Lastly in Chapter V, existence of non monotonic singular functions have been established and some interesting properties of such functions have been studied.

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CHAPTER I

INTRODUCTION

The class of functions of bounded variation[1] has played a very important role in the development of the theory of functions of a real variable. The classical Lebesgue decomposition (also known as Jordan's decomposition) of a function of bounded variation into two monotonic functions has proved to be a powerful tool in investigating a large number of interesting and important properties of a function of a real variable. With the development of Lebesgue's theory of measure, the importance of a function of bounded variation increased all the more in the literature of real functions.

Results like

> "A function of bounded variation has finite differential coefficient almost everywhere (p.p)"

are basic in Lebesgue's theory of integration.

In this thesis we have studied in some detail a subclass of functions of bounded variation, namely the class of singular functions

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(1) A function $f(x)$ is said to be of bounded variation in (a,b), if for any mode of division of (a,b)

$$a = x_0 < x_1 < \dots < x_n = b$$

the sum $\sum_{v=0}^{n-1} |f(x_{v+1})-f(x_v)|$ is bounded [see Titchmarsh [1], p.355].

of bounded variation. The earliest definition of a singular function of bounded variation is the following:

A function $f(x)$ $(a \leq x \leq b)$ of bounded variation is said to be a singular function is the differential coefficient is zero everywhere except at a set of measure 0.

Two other definitions of a singular function were given subsequently. We shall deal with these in the following section.

Singular functions have been known for a long time. G. Cantor [1] in 1884 gave an example of such a function. Later many examples of singular functions were given and many properties were proved.

G. Cantor [1], L. Scheeffer [1], H. Lebesgue [1], W. Sierpinski [1], H. Hahn [1], T. Carleman [1], E. Hille & J.D. Tamarkin [1], O.D. Kellog [1], G. Vitali [1], R.E. Gilman [1], H. Kober [1], [2], [3], [4] have all made some investigations concerning singular functions.

In this thesis, we have classified the class of singular functions into the following subclasses:

1. Step functions
2. Jump functions
3. Strictly increasing (decreasing)singular functions
4. Non monotonic singular functions.

A singular step function can be defined to be a single valued function which is constant on intervals of an open set whose measure is equal to the length of the fundamental interval. A jump function is defined to be the inverse of step function. Strictly increasing singular functions are monotone non decreasing singular functions which are nowhere constant. Non monotonic singular functions are singular functions, which are not monotone in any subinterval, no matter how small. In this connection it may be mentioned, that if a non monotonic function is continuous, then it is an everywhere oscillating function[2].

In this chapter we discuss some general properties of singular functions, and in the succeeding chapters we discuss in detail each of the subclasses mentioned above.

§1 On Singular Functions

The definition of a singular function given above assumes that the differential coefficient exists almost everywhere. This limits the applicability of the concept of a singular function. This led S. Saks [1] to give the following definition of a singular function which coincides with the original definitions whenever the differential coefficient exists almost everywhere.

I. A function $f(t)$ of $V_{0,a}$ [3] is said to be singular if given $\varepsilon > 0$, there exists non overlapping interval (t_k, t_k') $(k = 1,\dots n)$ in $(0,a)$ such that,

$$\sum_{k=1}^{n} (t_k' - t_k) < \varepsilon \quad \text{and} \quad \Sigma|f(t_k') - f(t_k)| > V_{0,a} f - \varepsilon .$$ [4]

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(2) A function $f(x)$ is said to be everywhere oscillating in (a,b), if it has an everywhere dense set of maxima and minima. See Hobson [1], pp. 374

(3) $f(t)$ of $V_{0,a}$ means that $f(t)$ is of bounded variation in the interval $(0,a)$.

(4) $V_{0,a} f$ denotes the total variation of $f(t)$ in $(0,a)$.

Later H. Kober [1] gave the following equivalent definition:

II. $f(t)$ of $V_{0,a}$ is singular, if the length of the curve $y = f(t)$ joining the points $(0,f(0))$, $(a,f(a))$ is

$$L_{0,a}f(t) = a + V_{0,a}f .$$

We shall state some well known results of the theory of functions of a real variable, from which we can deduce some properties of a singular function.

<u>Theorem 1</u>: If for a function $f(x)$ $(a \leq x \leq b)$, $f'(x) = 0$ p.p., and if $f(x)$ is absolutely continuous in (a,b), then it follows that $f(x)$ is constant in (a,b).

<u>Theorem 2</u>: If $f(x)$ is continuous and $f'(x) = 0$ everywhere except at an enumerable set, then $f(x)$ is constant in the fundamental interval.

From Theorem 1, it follows that any singular function $f(x)$ cannot be absolutely continuous (A.C.) except in the degenerate case where $f(x)$ is a constant. Theorem 2 tells us something about the set ε where $f'(x) \neq 0$. This set, if $f(x)$ is continuous, is unenumerable. Therefore, it is impossible to construct continuous singular functions with $f'(x) = 0$ everywhere except at an enumerable set. The following theorem tells us more about this set, in the case where $f(x)$ is a non decreasing singular function:

Theorem 3[5]: If (i) $y = f(t)$ $(0 \leq t \leq a)$ is a non decreasing singular function not reducing to a constant and if (ii) the sum of its jumps when there are any is smaller than $V_{0,a} f = f(a) - f(0)$, then there exists a non enumerable set E in $(0,a)$ such that $f'(t) = \infty$ for $t \in E$.

The following theorem[6] is an important result.

Theorem 4: If $y = f(t)$ is a non decreasing singular function not reducing to a constant, then the inverse function $t = g(y)$ is also a singular function.

These step functions which are continuous and are constant on the intervals of an open set having measure equal to the length of the fundamental interval are an important subclass of singular functions and these are said to be basic functions. The following theorem gives us a necessary and sufficient condition for a function to be a singular function.

Theorem 5: A function $w(t)$ is a basic function if and only if it is inverse of a jump function.

A jump function can be defined to be a function having points of discontinuities of the first kind at an everywhere dense set.

§2 Decomposition of a Function of Bounded Variation and Contravariation Functions

2.1 Decomposition of a function of bounded variation into an A.C. function and a singular function.

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(5) See H. Kober [1]

(6) See H. Kober [1].

Let $f(x)$ be a function of bounded variation, then its derivative $f'(x)$ exists almost everywhere. Let

$$Q(x) = \int_a^x f'(x)\,dx .$$

Because $Q(x)$ is an integral $Q(x)$ is an absolutely continuous function.

Let

$$r(x) = f(x) - Q(x) .$$

By the so-called fundamental theorem of integral calculus

$$\frac{d}{dx} Q(x) = \frac{d}{dx} \int_a^x f'(x)\,dx = f'(x) \qquad \text{p.p.}$$

It follows that $r'(x) = f'(x) - Q'(x) = 0$ p.p., and therefore $r(x)$ is a singular function. Hence we get the decomposition

$$f(x) = Q(x) + r(x)$$

of a function of bounded variation into the sum of an A.C. function and a singular function.

Here it is relevant to give the definition of Lusin condition, (N), with which a necessary and sufficient condition is given for a continuous function to be A.C.

A function $f(x)$ is said to satisfy Lusin condition (N) on a set E if $|f[H]|$[1] $= 0$ for every set H E of measure zero.

With the help of this the following theorem can be proved.

[1] By $|f[H]|$, is meant the measure of the image of H under f.

Theorem 6: In order that a function $f(x)$ which is continuous and of bounded variation on a bounded closed set be A.C. on E, it is necessary and sufficient that $f(x)$ fulfil the condition (N) on this set.

Let us show that the decomposition above is unique. Suppose

$$f(x) = Q_1(x) + r_1(x) = Q_2(x) + r_2(x)$$

where

$$Q_1(x),\ Q_2(x) \quad \text{are A.C.}$$

and

$$r_1(x),\ r_2(x) \quad \text{are singular,}$$

then

$$Q_1(x) - Q_2(x) = r_2(x) - r_1(x)\ .$$

Let

$$Q(x) = Q_1(x) - Q_2(x),$$

we have $Q(x)$ is an A.C. function and $Q'(x) = 0$ p.p.

Hence it follows that $Q(x)$ is a constant.

This proves that the decomposition of a function of bounded variation into an A.C. function and a singular function is unique up to a constant.

By adding to a function, a singular function, the differential coefficient is the same almost everywhere, while the behaviour of the function might be completely changed (for example different types of singularities can be added to the function).

2.2 A function of bounded variation as the difference of two monotone functions.

It was mentioned, if $f(x)$ is of B.V., then

$$f(x) = Q_1(x) - Q_2(x)$$

where $Q_1(x)$, $Q_2(x)$ are both monotone of the same nature.

This decomposition is very important and useful. However it has one disadvantage, that it is not unique. This led H. Kober [3], to put the following condition on $Q_1(x)$, $Q_2(x)$,

$$V_{(0,x)}f(x) = V_{(0,x)}Q_2(x) + V_{0,x)}Q_1(x) = Q_1(x) + Q_2(x) .$$

With this condition $Q_1(x)$, $Q_2(x)$ are unique, and he calls them contravariation functions (CAV's). We shall now give the formal definitions:

"Two functions $g(t)$ and $h(t)$ are said to be the contravariation functions of $f(t)$ if

(i) $g(t)$ and $h(t)$ are non decreasing

(ii) $f(t) = g(t) - h(t)$, and

(iii) $V_{0,t}f(t) = V_{0,t}g(t) + V_{0,t}h(t) = g(t) + h(t)$".

He gave the following criterion for two functions to be contravariation functions.

"The functions $g(t)$ and $h(t)$ $(0 \leq t \leq 1)$ are (CAV's) if and only if

(i) $g(t)$, $h(t)$ are non decreasing, $g(0) = h(0) = 0$.

(ii) given $\varepsilon > 0$, there exists a finite set of disjoint closed intervals $< t_k, T_k >$ such that

$$\Sigma \; g(T_k) - g(t_k) \quad > g(1) - \varepsilon ,$$

$$\Sigma \; h(T_k) - h(t_k) \quad < \varepsilon .$$

A decomposition of a singular function is given by the following theorem due to Kober [1].

Theorem 6: If $f(t) \varepsilon V_{0,a}$ is singular, then there exists a jump function $j(t) \varepsilon V_{0,a}$, and given $\varepsilon > 0$, a basic function $w(t) \varepsilon V_{0,a}$ and a continuous function $h(t)$; $V(h) < \varepsilon$ and

$$f(t) = j(t) + w(t) + h(t) .$$

If $f(t)$ is non decreasing, then so are $j(t)$, $w(t)$ and $h(t)$.

§ 3 Arithmetic Representations of Numbers, Sets, and Functions

Arithmetic representation of numbers, sets and functions has played a very important part in the definition of functions with certain peculiarities such as singularities and certain specified properties of the function or its derivative at points of a certain set. Arithmetic representation has the advantage over other representation (geometric, by series) that itis comparatively easy to study the behaviour of the function at particular points, while it is extremely difficult to find the behaviour of the function at particular points when it is defined geometrically or as a series of functions.

Radix representation of numbers. Let α be a +ve integer greater than 2. Then we can represent a point in $(0,1)$ as follows:

$$(1) \qquad x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \dots + \frac{a_n}{\alpha^n} + \dots . \qquad a_i = 0,1,\dots(\alpha-1)$$

A special case of this, is when $\alpha = 10$, and in that case we get the usual decimal system.

We can group the points in $(0,1)$ into two classes:

Class (i) consists of all points in which from and after some place all a's are zero or all are $(\alpha-1)$.

Class (ii) consists of all points in $(0,1)$ in which not all a's are zero or all a's are $(\alpha-1)$, after a certain place.

Points belonging to class (i) are called <u>primary</u> points and points representable only as in class (ii) are called <u>secondary</u> points.

Primary points have double representation. Let x be a primary point, then

$$x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \dots + \frac{a_n}{\alpha^n} + \frac{\alpha-1}{\alpha^{n+1}} + \frac{\alpha-1}{\alpha^{n+2}} + \dots .$$

This is equal to

$$= \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \dots + \frac{a_n+1}{\alpha^n} .$$

Primary points are enumerable and form an everywhere dense set in $(0,1)$, and hence form a set of the first category. From this it follows that the set of secondary points is a residual set.

Because the points of class (i) are everywhere dense, then we can define a function $f(x)$ at all the primary points. Then by applying the principle of continuity we can extend the definition to all points in $(0,1)$. However, this can be done only when the resulting function is continuous. This method of constructing continuous functions has been used by various mathematicians (e.g. see Broden [1]).

Radix representation of numbers can be generalized as follows:

Let $\{k_n\}$ be an increasing sequence of integers, then we can represent all points in $(0,1)$ as follows:

$$x = \frac{a_1}{k_1} + \frac{a_2}{k_1 . k_2} + \ldots + \frac{a_n}{k_1 k_2 \ldots k_n} + \ldots$$

where $a_n = 0,1,\ldots\, k_n - 1 \quad (n = 1,2,\ldots)$.

This representation corresponds to a division of the interval $(0,1)$ first into k_1 parts, then each of these parts into k_2 parts and so on. This method of representation of numbers has been used by A.N. Singh [1] to give a general method to construct perfect sets of positive measure.

Let $k_1, k_2, \ldots k_n, \ldots$ be a sequence of odd integers such that the series $\Sigma \frac{1}{k_n}$ is convergent, then the set of all points x where

$$x = \frac{1}{2}\left(c_0 + \frac{c_1}{k_1} + \frac{c_2}{k_1 k_2} + \ldots + \frac{c_n}{k_1 k_2 \ldots k_n} + \ldots\right)$$

$$c_0 = 0 \text{ or } 1 \qquad c_n \neq \frac{k_n - 1}{2} \qquad 0 \leq c_n \leq k_n - 1 \qquad (n = 1,2,\ldots)$$

can be shown to be a non dense perfect set of positive measure equal to

$$\pi \left(1 - \frac{1}{k_n}\right) .$$

We now give a representation of numbers which is powerful to construct functions with some kind of singularities at an everywhere dense set of the first category which is of +ve measure, and even of measure equal to the length of the fundamental interval.

This method works by using any non dense perfect set, but for simplification, we shall use Cantor's non dense perfect set in $(0,1)$.

The set of points x, where

$$x = \frac{c_{11}}{3} + \frac{c_{12}}{3^2} + \dots + \frac{c_{1n}}{3^n} + \dots \qquad \text{where} \quad c_{1n} = 0 \text{ or } 2$$

is Cantor's set in $(0,1)$.

Let the contiguous intervals be given by (ξ', ξ'') where

$$\xi' = \frac{c_{11}}{3} + \frac{c_{12}}{3^2} + \dots + \frac{0}{3^n} + \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \dots$$

$$= \frac{c_{11}}{3} + \frac{c_{12}}{3^2} + \dots + \frac{1}{3^n}$$

$$\xi'' = \frac{c_{11}}{3} + \frac{c_{12}}{3^2} + \dots + \frac{a}{3^n} .$$

If in all such contiguous intervals we again set up Cantor's perfect set, and again in the contiguous intervals obtained, we set up Cantor's perfect set. If this process is carried on indefinitely many times, we get a set, whose points admit of the following representation:

$$x = (\frac{c_{11}}{3} + \frac{c_{12}}{3^2} + \dots + \frac{1}{3^{n1}} + \frac{1}{3^{n1}}(\frac{c_{21}}{3} + \frac{c_{22}}{3^2} + \dots + \frac{1}{3^{n2}} + \frac{1}{3^{n2}} (\dots$$

$$(\dots + \frac{1}{3^{n_{r-1}}} + \frac{1}{3^{n_{r-1}}} (\frac{c_{r1}}{3} + \frac{c_{r2}}{3^2} + \dots + \dots (\dots(\dots$$

where the number of brackets may be finite or infinite. In case the number of brackets is finite we get the primary points, and where the number of brackets is infinite we get the secondary points.

A point x representing a primary point can have double representation.

$$x = (\frac{c_{11}}{3} + \frac{c_{12}}{3^2} + \dots + \frac{1}{3^{n_1}} + \frac{1}{3^{n_1}} (\frac{c_{21}}{3} + \frac{c_{22}}{3^2} + \dots + \frac{1}{3^{n_2}}(\dots$$

$$(\dots + \frac{1}{3^{n_{r-1}}} + \frac{1}{3^{n_{r-1}}} (\frac{c_{r1}}{3} + \frac{c_{r2}}{3^2} + \dots + \frac{2}{3^m}) \dots)$$

x can also be represented as follows:

$$x = (\frac{c_{11}}{3} + \frac{c_{12}}{3^2} + \dots + \frac{1}{3^{n_1}} + \frac{1}{3^{n_1}} (\frac{c_{21}}{3} + \frac{c_{12}}{3^2} + \dots + \frac{1}{3^{n_2}} + \frac{1}{3^{n_2}}(\dots$$

$$(\dots + \frac{1}{3^{n_{r-1}}} + \frac{1}{3^{n_{r-1}}} (\frac{c_{r1}}{3} + \frac{c_{r2}}{3^2} + \dots + \frac{1}{3^m} + \frac{1}{3^m}(\frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \dots)$$

This representation of numbers has been used to define functions with certain singularities at a set of positive measure. U.K. Shukla [1] has used this representation of numbers to define a singular function which is non monotonic in every subinterval.

CHAPTER II

ON STEP FUNCTIONS

In this chapter we shall discuss a class of functions known as step functions. A single valued real function $f(x)$ is said to be a step function, if in every interval of the fundamental interval, there exists a subinterval in which the function takes a constant value. Intervals in which a step function is constant are called lines of invariability. Clearly at every point of a line of invariability of the function, the differential coefficient is zero. In this chapter we will discuss step functions which are continuous and have zero differential coefficient almost everywhere.

Cantor [1] (1884) defined a general class of functions, which for a special case reduces to the classical example of a step function, known as Cantor's step function. Cantor's step function was studied by Scheeffer [1] (1884), and again studied in detail by Hille & Tamarkin [1] (1929). Cantor's step function was generalized by Carleman [1] (1923), then was further generalized by Gilman [1] (1931). All these functions, however are special cases of a class of functions defined by Kober [3] (1947).

In § 1 we study the works of G. Cantor on step functions. In §2 we give an analysis of step functions that was given by E. Hille and

J. Tamarkin. § 3 gives Carleman's generalization of Cantor's step function and § 4 gives E. Gilman's generalization of Cantor's step function. Finally § 5 gives a study of step function from a different angle and that is the one given by Kober.

§ 1 Works of G. Cantor

Cantor defines a class of step functions as follows:

Let P be any non dense perfect set defined in (0,1). Using the fact that C(P) is made up of open intervals (contiguous to P) which are at most enumerable, arrange the contiguous intervals of P (call them u_n) in order of magnitude of their lengths. In case, some intervals are of equal length, then their order is immaterial, since there can be only a finite number of them.

Let this arrangement be

$$u_1, u_2, \dots, u_n \dots$$

where $u_n = (a_n, b_n)$, $a_n, b_n \varepsilon P$ and $(a_n, b_n) \in C(P)$.

The end points a_n, b_n not only belong to P, but determine P completely, since P consists of such points plus their limiting points (call them g).

We classify the points of P in the following three classes:

Class 1 consists of points which are left end points of contiguous intervals . We denote the set of all such points by P^-.

Class 2 consists of points which are right end points of contiguous intervals. We denote the set of all such points by P^+.

<u>Class 3</u> consists of points which are limiting points of P^- or of P^+ but are not points of P^- or of P^+. We denote the set of all such points by P^0.

To define the function let $\{\varphi_n\}$ be any sequence of points everywhere dense in (0,1), such that $\{\varphi_n\}$ does not contain the point 0 or the point 1. We shall set up a 1-1 correspondence between the contiguous intervals $\{u_n\}$ and the points $\{\varphi_n\}$ in the following way:

To u_1, let there correspond any point φ_{n_1}. Put $\varphi_{n_1} = \psi_1$. To u_2 let there correspond a point φ_{n_2}, such that n_2 is the smallest integer for which the following is true:

if u_2 is to the left of u_1, then $\varphi_{n_2} < \varphi_{n_1}$

if u_2 is to the right of u_1, then $\varphi_{n_2} > \varphi_{n_1}$

Put $\varphi_{n_2} = \psi_2$.

To u_3, let there correspond a φ_{n_3}, such that n_3 is the smallest index for which

if u_3 is to the left of u_2 or u_1, then $\varphi_{n_3} < \varphi_{n_2}$ or $\varphi_{n_3} < \varphi_{n_1}$

and

if u_3 is to the right of u_2 or u_1, then $\varphi_{n_3} > \varphi_{n_2}$ or $\varphi_{n_3} > \varphi_{n_1}$

i.e. φ_{n_3} is that point in the sequence $\{\varphi_n\}$ with the smallest index, such that φ_{n_3} has the same position in (0,1) relative to $\{\varphi_{n_2}$ and φ_{n_1}, as u_3 has relative to u_2 and u_1.

Put $\varphi_{n_2} = \psi_3$.

In general, to u_3, let there correspond a point $\varphi_{n_s} = \psi_3$ such that n_s is the smallest integer for which φ_{n_s} has the same position relative to $\psi_1, \dots \psi_{s-1}$, as the position of u_s relative to $u_1, \dots u_{s-1}$.

We can see that such a correspondence is possible since both the set of contiguous intervals as well as the set of points $\{\varphi_n\}$ are everywhere dense in (0,1).

Now we shall extend this correspondence to all points of the perfect set P as well as to all points in (0,1) in the following way:

Let $g \varepsilon P^0$, then there will exist a sequence of points of P^+ or of P^- denote it by $\{g_n\}$ such that $\lim_n g_n = g$. Form the sequence $\{h_n\}$ where h_n is that point of the sequence $\{\varphi_n\}$ which is the correspondent of g_n as has been defined above. Define $h = \lim_n h_n$ to be the correspondent of g. We know that this limit will exist and will be unique, since this sequence has the same relative position in (0,1) as the sequence $\{g_n\}$, and because the sequence $\{g_n\}$ converges, it must necessarily follow that $\{h_n\}$ converges.

Since the derived set of $\{a_n\}$, $\{b_n\}$ is the set P, and the derived set of $\{\varphi_n\}$ is the set of all points in (0,1), then this correspondence will have an extension to all points of P and all points in (0,1).

Now we shall define $f(x)$ in the following way:

If u_n is a contiguous interval of P with end points a_n, b_n then

$$\text{if} \quad x \in u_n \quad \text{i.e.} \quad x \in (a_n, b_n)$$

then $$f(x) = f(a_n) = f(b_n) = \psi_n$$

and for $$x \in P^0 \quad x = \lim g_n$$

where h_n is the correspondent of g_n.

This defines a single valued function, which is continuous and constant in all contiguous intervals of P. This function f(x) is singular only if P is of measure 0.

1.1 <u>Cantor's step function</u>. As a special case of the class of functions defined in §1, Cantor considers the following example:

Let P be Cantor's ternary set. Then P will be the set of all points x having the following representation:

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \dots + \frac{a_n}{3^n} + \dots \qquad a_i = 0 \text{ or } 2 .$$

The set P^+ will consist of points which have the following representation:

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \dots + \frac{2}{3^n} .$$

The set P^- will consist of points x which have the following representation:

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \dots + \frac{0}{3^n} + \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \dots$$

$$= \frac{a_1}{3} + \frac{a_2}{3^2} + \dots + \frac{1}{3^n}$$

and the set P^0 will consist of points x which have a representation in which there exists an infinite number of a_n's equal to 2 and an infinite number equal to 0, i.e.

$$x = \sum_{i=1} \frac{a_{\lambda_i}}{3^{\lambda i}} + \sum_{j=1} \frac{a_{\mu_j}}{3^{\mu j}} \qquad \lambda_i \neq \mu_i \quad \text{for all } i,j$$

$$\text{and} \quad a_{\lambda_i} = 0 \quad a_{\mu j} = 2.$$

A contiguous interval $u_n = (\alpha_n, \beta_n)$ will be of length $\frac{1}{3^n}$, and its end points will be equal to

$$\alpha_n = \frac{a_1}{3} + \frac{a_2}{3^2} + \dots + \frac{0}{3^n} + \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \dots$$

$$\beta_n = \frac{a_1}{3} + \frac{a_2}{3^2} + \dots + \frac{2}{3^n} \quad .$$

This is a non dense perfect set. Now define $f(x)$ in the following way:

$$\text{if } x \in P, \text{ i.e. } x = \frac{a_1}{3} + \frac{a_2}{3^2} + \dots + \frac{a_n}{3^n} + \dots \qquad a_i = 0 \text{ or } 2$$

$$\text{then} \qquad f(x) = \frac{b_1}{2} + \frac{b_2}{2^2} + \dots + \frac{b_n}{2^n} + \dots \qquad b_i = \frac{a_i}{2} \quad .$$

If $x \notin P$, $x \in (\alpha_n, \beta_n)$, then define

$$f(x) = f(\alpha_n) = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{0}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots \qquad b_i = \frac{a_i}{2}$$

$$= \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{1}{2^n} = f(\beta_n) \; .$$

Thus we have defined a single valued function in (0,1) constant in all intervals contiguous to Cantor's ternary set.

§2 Works of E. Hille & J. Tamarkin

Hille & Tamarkin have analysed Cantor's step function; they have proved the following properties:

(i) $f(x)$ is monotone, and increases from 0 to 1 as x goes from 0 to 1, the intervals of deletion being intervals of constancy.

[Note: We shall denote the intervals of deletion at the p^{th} stage by δ_{p_h}, and the remaining intervals by n_{p_h}. The length of δ_{p_h} will be $\frac{1}{3^p}$, there being 2^{p-1} of them, the length of n_{p_h} will be $\frac{1}{3^p}$, there being 2^p of them.]

(ii) $f(x)$ is continuous but not absolutely continuous continuity at interior points of deleted intervals is apparent, since the function of such points takes on a constant value. Therefore we have to show continuity at points of P. Let $x \varepsilon P^-$ then

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \ldots + \frac{0}{3^n} + \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \ldots$$

Continuity from right is obvious.

To prove continuity from left, let ξ be a point to the left of x, then

$$\xi = \frac{a_1}{3} + \frac{a_2}{3^2} + \ldots + \frac{0}{3^n} + \ldots + \frac{0}{3^v} + \ldots$$

where some terms equal to 2 have been replaced by 0, a_v being such a one. We have

$$f(x) = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{0}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots + \frac{1}{2^v} + \ldots$$

$$f(\xi) = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{0}{2^n} + \ldots + \frac{0}{2^v} + \ldots$$

$f(\xi) - f(x) = \frac{1}{2^v} + \ldots$ terms of order greater 2^{-v}

as $\xi \to x$, i.e. $v \to \infty$ $f(\xi) \to f(x)$

therefore continuity from right at the point x.

Similarly, we can prove continuity at all points of P. To show that $f(x)$ is not absolutely continuous, let us find the variation over the remaining intervals μ_{p_h}, after the p^{th} stage.

$$V_{n_{p_h}} = \sum_{h=1}^{2^P} 2^{-P} = 1$$

while the length of the remaining intervals is

$$= \sum_{=1}^{2^P} 3^{-P} = (\tfrac{2}{3})^P .$$

By taking P arbitrarily large, the length of the remaining intervals can be made arbitrarily small, while the variation over them is always 1.

This proves that $f(x)$ is not absolutely continuous.

(iii) $f(x)$ is a singular function.

Since $f'(x) = 0$ at all points of contiguous intervals of P, and $m(P) = 0$ it follows that $f'(x) = 0$ p p. Hence $f(x)$ is a singular function.

(iv) The function $f(x)$ satisfies a Lipschitz condition of order $\alpha = \log 2/\log 3$, i.e.

$$|\omega(x + h) - \omega(x)| \leq \Lambda|h|^\alpha .$$

(v) Define

$$\varphi_h(x) = f(x + h) - f(x) .$$

Let $T(h)$ be the variation of $\varphi_n(x)$ over $(0,1)$ and let

$$z = \max T(h) \qquad 0 \leq h \leq z ,$$

then z is constant and is equal to 2.

The importance of this function lies in the fact that a necessary and sufficient condition for a function $f(x)$ to be absolutely continuous is that $T(h) \to 0$ as $h \to 0$. This has been proved by A. Plessner.

§ 3 Works of T. Carleman

Carleman defined a class of step functions which includes Cantor's step function as a special case.

Let α be any positive integer. Let P be the set of all points $x \varepsilon (0,1)$ having the following representation:

$$x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \ldots + \frac{a_n}{\alpha^n} + \ldots \qquad \text{where } a_i = 0 \text{ or } \alpha-1 .$$

Then P is a non dense perfect set. A contiguous interval pf P has end points ξ', ξ'', where

$$\xi' = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \ldots + \frac{0}{\alpha^n} + \frac{\alpha-1}{\alpha^{n+1}} + \frac{\alpha-1}{\alpha^{n+2}} + \ldots$$

and

$$\xi'' = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \ldots + \frac{\alpha-1}{\alpha^n} .$$

Define $f(x)$ as follows:

$$\text{if } x \varepsilon P, \text{ i.e. } x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \ldots + \frac{a_n}{\alpha^n} + \ldots \qquad a_i = 0 \text{ or } \alpha-1$$

then

$$f(x) = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{b_n}{2^n} + \ldots \qquad \text{where } b_i = \frac{a_i}{\alpha-1}$$

for

$$x \not\varepsilon P, \quad x \varepsilon (\xi', \xi'')$$

then

$$f(x) = f(\xi') = f(\xi'') .$$

This defines the function $f(x)$ for all points in $(0,1)$. Then setting $\alpha = 3$, $f(x)$ reduces to Cantor's step function.

§ 4 Works of E. Gilman

The technique Gilman follows in constructing a class of step functions, is that he first constructs a class of non dense perfect sets of zero measure in the interval $(0,1)$. Then using the fact that all perfect sets have the power of the continuum and that the contiguous intervals are enumerable, he sets up a 1-1 correspondence between the contiguous intervals and the rationals in $(0,1)$, and between the set of limiting points and the set of irrationals in $(0,1)$.

Therefore we shall first turn to a discussion of the class of perfect sets:

§ 4.1 A class of perfect sets. Let α be any positive integer, represent all points in $(0,1)$ in radix representation to the base α. for all $x \in (0,1)$ let

$$x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \dots + \frac{a_n}{\alpha^n} + \dots \quad . \tag{1}$$

Let

$$\alpha - 1 = q(\beta - 1) \; . \qquad q \geq 2, \;\; \beta \geq 2$$

Now let $\alpha - 1 = q(\beta - 1)$, $\qquad q \geq 2, \;\; \beta \geq 2$

and hold β, q fixed, then define $P_{\alpha,\beta}$ to be the set of points in $(0,1)$ having the above representation with the a_n's being multiples of q,

i.e. $P_{\alpha,\beta}$ consists of all points x;

$$x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \dots + \frac{a_n}{\alpha^n} + \dots \qquad \text{where} \;\; a_i = qb_i \; .$$

We will show that P is a non dense perfect set of measure 0. $C(P)$ is an everywhere dense set of intervals, a typical interval of $C(P)$ will have left and right end points ξ', ξ'', where

$$\xi' = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \ldots + \frac{a_n}{\alpha^n} + \frac{\alpha-1}{\alpha^{n+1}} + \frac{\alpha-1}{\alpha^{n+2}} + \ldots$$

$$= \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \ldots + \frac{a_n+1}{\alpha^n} \qquad \text{where all } a_i = qb_i \text{ and in particular } a_n = qb_n$$

and

$$\xi'' = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \ldots + \frac{a_n+q}{\alpha^n} .$$

These end points of contiguous intervals belong to P and P will consist of such points and their limiting points. Because of this property, P is closed and because every point is a limiting point, P is perfect.

The classification of points of P into sets P^+, P^-, P^0 will be

P^- consists of points ξ' having the following representation

$$\xi' = \frac{a_1}{a} + \frac{a_2}{a^2} + \ldots + \frac{a_n}{\alpha^n} + \frac{\alpha-1}{\alpha^{n+1}} + \frac{\alpha-1}{\alpha^{n+2}} + \ldots \qquad a_i = qb_i \quad (i=1,..n)$$

P^+ consists of points ξ'' having the following representation

$$\xi'' = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \ldots + \frac{a_n+q}{\alpha^n} \qquad a_i = qb_i \quad (i=1,..n).$$

P^- consists of points having the following representation

$$x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \ldots + \frac{a_n}{\alpha^n} + \ldots \quad \text{where } \exists \text{ an infinity of } a_n\text{'s different from } \alpha-1 \text{ and } 0.$$

To show that P is of measure 0, first the length of a contiguous interval at the n^{th} stage is $\frac{\alpha-1}{\alpha^n}$. At first stage we have $(\beta-1)$ contiguous intervals, at second stage $\beta(\beta-1)$ contiguous intervals and in general at the n^{th} stage $\beta^{n-1}(\beta-1)$ contiguous interval, hence total length of contiguous intervals is

$$(\beta-1)\frac{\alpha-1}{\alpha} + \beta(\beta-1)\frac{(\alpha(-1))}{\alpha^2} + \beta^2(\beta-1)\frac{(\alpha-1)}{\alpha^3} + \ldots$$

$$+ \beta^{n-1}(\beta-1)\frac{(\alpha-1)}{\alpha^n} + \ldots$$

$$= \sum_{n=1} \beta^{n-1}(\beta-1)\frac{(\alpha-1)}{\alpha^n} = \frac{(\beta-1)(\alpha-1)}{\alpha} \sum_{n=0} \left(\frac{\beta}{\alpha}\right)^n$$

$$= \frac{(\beta-1)(\alpha-1)}{\alpha} \frac{\alpha}{(\alpha-\beta)} = \frac{(\beta-1)(\alpha-1)}{(\alpha-\beta)} = \frac{\alpha-1-\beta+1}{\alpha-\beta} = 1 .$$

Hence $m(p) = 0$.

Now we will define the class of step function.

§ 4.2 <u>A class of step function</u>. Define $w_{\alpha,\beta}(x)$ in the following way. When

$$x \in P_{\alpha,\beta}$$

i.e.

$$x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \ldots + \frac{a_n}{\alpha^n} + \ldots \qquad \text{where } a_i = qb_i$$

then

$$w_{\alpha,\beta}(x) = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \dots + \frac{b_n}{\beta^n} + \dots \quad .$$

Hence if ξ', ξ'' are end points of some contiguous interval

$$\xi' = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \dots + \frac{a_n}{\alpha^n} + \frac{\alpha-1}{\alpha^{n+1}} + \frac{\alpha-1}{\alpha^{n+2}} + \dots \qquad a_i = qb_i$$

$$\xi'' = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \dots + \frac{a_n+q}{\alpha^n} \qquad (\alpha-1) = q(\beta-1)$$

then

$$w_{\alpha,\beta}(\xi') = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \dots + \frac{b_n}{\beta^n} + \frac{\beta-1}{\beta^{n+1}} + \frac{\beta-1}{\beta^{n+2}} + \frac{(\beta-1)}{\beta^{n+3}} + \dots$$

$$= \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \dots + \frac{b_{n+1}}{\beta^n} = w_{\alpha,\beta}(\xi'') \quad .$$

For $x \notin P$, $x \in$ a contiguous intervals (ξ', ξ'') then we shall define

$$w_{\alpha,\beta}(x) = w_{\alpha,\beta}(\xi') = w_{\alpha,\beta}(\xi'') \ .$$

We can state the definition in a somewhat different manner. Let all points in $(0,1)$ have representation (1). If

$$x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \dots + \frac{a_n}{\alpha^n} + \dots \qquad a_i = qb_i$$

then

$$w_{\alpha,\beta}(x) = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \dots + \frac{b_n}{\beta^n} + \dots$$

and if $a_n \neq qb_n$ (where a_n is the first such term) then

$$w_{\alpha,\beta}(x) = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \dots + \frac{b_n}{\beta^n} . \quad \text{where } b_n = [\frac{a_n}{q}] + 1$$

The following properties are easy to prove:

(i) The function $w_{\alpha,\beta}(x)$ is monotone increasing, with the contiguous intervals of P being intervals of invariability.

(ii) $w_{\alpha,\beta}(x)$ is continuous but not absolutely continuous continuity at all points of $C(P)$ is apparent, we shall prove continuity at only right end points of P, since for other points of P the proof is exactly the same.

Let $x \varepsilon P^+$

$$x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \dots + \frac{a_n}{\alpha^n} \qquad a_i = qb_i$$

continuity from left is obvious since the function there takes a constant value. Hence that a point $x' > x$ such that $x' \varepsilon P$

$$x' = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \dots + \frac{a_n}{\alpha^n} + 0 \dots + \frac{a_v}{\alpha^v} \quad \text{where } a_v \text{ is first non zero digit}$$

$$w_{\alpha,\beta}(x') - w_{\alpha,\beta}(x) = \frac{b_v}{\beta^v} + \dots$$

and

$$\lim \text{as } x' \to x \text{ of } w_{\alpha,\beta}(x') - w_{\alpha,\beta}(x) \text{ is } \lim_{v} \frac{b_v}{\beta^v} = 0 .$$

To show non absolute continuity, $w'_{\alpha,\beta}(x) = 0$ almost everywhere but the function is not a constant, hence by a well known theorem $w_{\alpha,\beta}(x)$ is not absolutely continuous.

(iii) The function $w_{\alpha,\beta}(x)$ satisfies a Lipschitz condition of order

$$u = \frac{\log \beta}{\log \alpha}, \quad \beta = \alpha^{\mu}$$

the Lipschitz coefficient, being not greater than

$$c = \beta(q-1)^{-\mu}$$

and the best value of μ possible, i.e. the inequality

$$|w_{\alpha,\beta}(x') - w_{\alpha,\beta}(x)| \leq c_0 \; |x-x'|^{\mu_0}$$

will not hold good for $\mu_0 \geq \mu$ even though $c_0 < c$.

<u>Proof</u>: Let $x > x'$ and first let $x, x' \varepsilon P$

$$x = \sum_{i=1} a_i \alpha^{-i} \qquad x' = \sum_{i=1} a_i' \alpha^{-i} .$$

Let a_k be the first term which is different from a_h', then

$$w_{\alpha,\beta}(x') - w_{\alpha,\beta}(x) = b_h - b_h'(\beta^{-h} + \sum_{i=h+1} (b_i - b_i')\beta^{-i} \leq (\beta-1)\beta^{-h} + \beta^{-h}$$
$$\leq \beta^{-h+1} = \beta\alpha^{-h\mu}$$

and

$$x' - x = (a_h - a_h')\alpha^{-h} + \sum_{i=h+1} (a_i - a_i')\alpha^{-i} \geq (q-1)\alpha^{-h} .$$

Hence

$$|\alpha' - x|^{\mu} \geq (q-1)\alpha^{-h\mu}$$

and

$$\frac{w_{\alpha,\beta}(x') - w_{\alpha,\beta}(x)}{|x' - x|^{\mu}} \leq \frac{\beta}{q-1}$$

i.e.

$$|w_{\alpha,\beta}(x') - w_{\alpha,\beta}(x)| \le \frac{\beta}{q-1} |x' - x|^{\mu} .$$

Now if x or $x' \varepsilon C(P)$, let ξ_2 be the left end point of interval containing x and ξ_1 right end point of interval containing x', then

$$w_{\alpha,\beta}(x) - w_{\alpha,\beta}(x') = w_{\alpha,\beta}(\xi_2) - w_{\alpha,\beta}(\xi_1) \le c(\xi_2 - \xi_1)^{\mu} \le c(x-x')^{\mu} .$$

Now we shall turn to a study of the derivatives of $w(x)$. We first shall prove the following theorems which deal with derivatives at points of P^+, P^- and $C(P)$.

<u>Theorem 1</u>: For (i) $x \varepsilon C(P)$ $w'(x)$ <u>exists and</u> $w'(x) = 0$

(ii) $x \varepsilon P^+$ $D^+w(x) = +\infty$ $D^-w(x) = 0$

(iii) $x \varepsilon P^-$ $D^+w(x) = 0$ $D^-w(x) = +\infty$

<u>Proof</u>: The first statement is obvious. We shall prove only the second statement since the third one is derivable from the second and by the transformation $x' = 1 - x$.

That $D^-w(x) = 0$ is clear since the function to the left of a point P^+ is a constant.

Let $x \varepsilon P^+$

then

$$x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \dots + \frac{a_n}{\alpha^n} \qquad a_i = qb_i (i=1,\dots n).$$

Take a point x' to the right of x, i.e. $x' > x$

$$x' = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \dots + \frac{a_n}{\alpha^n} + \frac{0}{\cdot\cdot} + \frac{a_v}{\alpha^v} + \dots$$

then

$$w(x') - w(x) = \frac{b_v}{\beta^v} + \dots \text{ terms of order greater } \beta^{-v} \geq \frac{b_v}{\beta^v}$$

and

$$x' - x = \frac{a_v}{\alpha^v} + \dots \quad \leq \frac{a_{v+1}}{\alpha^v} .$$

Hence

$$\frac{w(x') - w(x)}{x' - x} \geq \frac{\dfrac{b_v}{\beta^v}}{\dfrac{a_{v+1}}{\alpha^v}} = \frac{b_v}{a_{v+1}} \left(\frac{\alpha_v}{\beta}\right)^v$$

and

$$\lim_{v} \frac{b_v}{a_{v+1}} \left(\frac{\alpha}{\beta}\right)^v = +\infty.$$

Hence

$$D^+ w(x) = +\infty .$$

We shall now study the behaviour of the derivatives of $w(x)$ at points of P^0. We shall need to use the following notation:

Let

$$x \in P^0 \qquad x = \sum_{h=1} a_{v_h} \alpha^{-v_h} \qquad a_{v_h} = qb_{v_h}$$

let

$$\alpha_b = a_{v_h} \neq 0 \qquad\qquad \beta_h \equiv b_{v_h} \neq 0$$

(we assume $a_{v_h} \neq \alpha - 1$, and $v_{h+1} - v_a \neq 1$),

let x' designate the variable point going into x, and

$$Q(x') = \frac{w(x') - w(x)}{x' - x} ,$$

let

$$x_n = \sum_{h=1}^{n} \alpha_h \alpha^{-v_h}$$ a sequence of right end points approaching x,

and

$$y_n = \sum_{h=1}^{n-1} \alpha_h \alpha^{-v_h} + \frac{\alpha_n - q + 1}{\alpha^{v_n}}$$ a sequence of left end points going into x,

and let

$$\boldsymbol{\varepsilon}_n = [x_{n-1}, x_n]$$

and let s_n denote the number of non zero digits between a_{v_n} and $a_{v_{n+1}}$ i.e.

$$S_n = v_{n+1} - v_n \quad - 1$$

and let

$$r_n = \frac{S_v}{v_n}$$

and let ε_n, M_n denote two sequences of +ve numbers;

$$\varepsilon_n \quad O \quad M_n \quad O \quad \varepsilon_n M_n \qquad v_n M_n < M$$ where M is a +ve constant,

and let

$$\lambda = \frac{1}{\mu} - 1 = \frac{\log \alpha}{\log \beta} - 1 \, .$$

We now prove the following lemmas:

<u>Lemma 1</u>: When $x' \rightarrow x$ on the sequence x_n , then

$$Q(x_n) > \frac{\beta_{n+1}}{\alpha_{n+1} + 1} \left(\frac{\alpha}{\beta}\right)^{v_{n+1}}$$

<u>Proof</u>: $$Q(x_n) = \frac{w(x) - w(x')}{x - x'}$$

$$w(x) - w(x') = \sum_{h=n+1} \beta_h \beta^{-v_h} > \beta_{n+1} \cdot \beta^{-v_{h+1}}$$

and

$$x - x' = \sum_{h=n+1} \alpha_h \alpha^{-v_h} < (\alpha_{n+1}+1)\alpha^{-v_{h+1}} .$$

Hence

$$\frac{w(x) - w(x')}{x - x'} > \frac{\beta_{n+1}}{(\alpha_{n+1}+1)} \left(\frac{\alpha}{\beta}\right)^{v_h+1} .$$

<u>Lemma 2</u>: For x' approaching x on the sequence y_n

$$Q(y_n) < \frac{\beta_{n+1}+1}{q-1} \alpha^{v_n} \beta^{-v_n+1}$$

$$Q(y_n) = \frac{w(y_n) - w(x_n)}{y_n - x}$$

$$w(y_n) - w(x) = \sum_{h=n} \beta_h \beta^{-v_h} - \frac{\alpha_n - q+1}{q} + 1$$

$$= \sum_{h=n+1} \beta_h \beta^{-v_h} < (\beta_{n+1}+1)\beta^{-v_{n+1}}$$

and

$$y_n - x = \sum_{h=n} a_v \alpha^{-v_h} - \frac{\alpha_n - q+1}{\alpha^{v_n}} = \sum_{h=n+1} a_h \alpha^{-v_h} + \frac{q-1}{\alpha^{v_n}} \geq \frac{q-1}{\alpha^{v_n}} .$$

Hence

$$Q(y_n) < \frac{(\beta_{n+1}+1)}{q-1} \alpha^{v_n} \beta^{-v_n+1} .$$

Lemma 3: When $x' \varepsilon \delta_n = [x_{n-1}, x_n]$

then

$$Q(x') > \frac{\beta_{n+1}}{a_n+1} \alpha^{v_n} \beta^{-v_{n+1}} > \frac{1}{\alpha\beta} Q(y_n).$$

Proof: Since $x_{n-1} < x' < x_n$

then

$$w(x_{n-1}) \leq w(x') \leq w(x_n)$$

therefore

$$w(x) - w(x') \geq w(x) - w(x_n) > \beta_{n+1}\beta^{-v_{n+1}} \qquad \text{(Lemma 1)}$$

while

$$x - x' \leq x - x_{n-1} < (\alpha_{n+1}) \alpha^{-v_n}.$$

Hence applying Lemma 2, we have

$$\frac{Q(x')}{Q(y_n)} > \frac{\beta_{n+1}(q-1)}{(\alpha_n+1)(\beta_{n+1}+1)} > \frac{1}{\alpha\beta} .$$

We will give now a necessary and sufficient coefficient for the existence of a derivative in the left of a point belonging to P^0. This is contained in the following theorem.

Theorem 2: A necessary and sufficient condition that at a point $x \varepsilon P^0$, $w(x)$ have a derivative in the left is that

$$Q(y_n) \to \infty \quad \text{as} \quad n \to \infty .$$

The derivative is equal to $+\infty$ if it exists at all.

Proof: The necessity of the condition follows from the fact that $Q(x_n) \to \infty$, hence if the derivative exists then $\lim Q(y_n) = \lim Q(x_n)$. Hence if the derivative exists at all $Q(y_n) \to \infty$. The sufficiency of the condition follows from Lemma 3. For if $Q(y_n) \to \infty$, then from Lemma 3 follows that $Q(x') \to \infty$ and from Lemma 1 $Q(x_n) \to \infty$. Therefore

$$\lim Q(x_n) = \lim Q(y_n) = \lim Q(x') = +\infty .$$

Hence the derivative will be equal to $+\infty$.

By using the transformation $x' = 1-x$ we get similar information about the derivative from the left, since $\xi > x \quad \xi' < x'$, where $\xi' = 1 - \xi$.

We shall give now more definite results which are contained in this theorem.

Theorem 3: If $x \in P^0$ and $r_n \leq \lambda - \varepsilon_n$ for $n > N$ that is for all values greater than a certain N, then the function $w(x)$ has a derivative on the left equal to $+\infty$. If however, we have $r_n > \lambda - \epsilon_n$ for infinitely many values of n then $w(x)$ has no derivative on the left.

(Same results for derivative on right).

Now we can classify the point in $(0,1)$ according to the behaviour of the derivative of the function $w(x)$

(i) $D^+w(x) = D^-w(x) = 0$ for $x \varepsilon C(P)$

(ii) $D^+w(x) = +\infty$ $D^-w(x) = 0$ for $x \varepsilon P^+$

(iii) $D^+w(x) = 0$ $D^-w(x) = +\infty$ $x \varepsilon P^-$

(iv) $D^+w(x) = +\infty$ $D^-w(x)$ not defined)

(v) $D^+w(x)$ does not exist $D^-w(x) = +\infty$) $x \varepsilon P^0$

(vi) $D^+w(x) = D^-w(x) = +\infty$)

(vii) $D^+w(x)$ does not exist

$D^-w(x)$ does not exist.

§ 5 Works of Kober

Kober defined a class of continuous step functions $w(r',\alpha,\beta)$, β an integer and $\alpha > \beta \geq 2$, which includes all the step functions already mentioned as special cases. For $\alpha = 3$, $\beta = 2$ we obtain Cantor's step function, $\beta = 2$, and α any positive integer gives us the class of functions defined by Carleman, and for α, β any positive integers, we obtain the class of functions that was defined by Gilman.

Kober deals with this class of function from two points of view as follows:

1) $w(r',\alpha,\beta)$ is defined as the inverse function of the function $r = G(y)$ defined as follows:

$$r = G(y) = q \sum_{\substack{m>-\infty \\ n\beta^{-m}<y}} \sum_{n=1}' \alpha^{-m}$$

$$= q \sum_{m>-\infty} a_{m,y} \alpha^{-m}$$

(where $q = \frac{\alpha-\beta}{\beta-1}$ $\quad 0 \leq y <$; $m > -\log \alpha/_{\log \beta}$) ,

where the dash indicates that the summation should not include n's which are multiples of β, and $a_{m,y}$ is the number of positive integers n such that $n \nmid \beta$, and given β, $n < y\,\beta^m$.

As the function defined by (1) is a jump function with discontinuities at all rational points, the study of $w(t',\alpha,\beta)$ as the inverse of $G(y)$ will be taken up in the chapter on jump functions.

2) The function $w(t',\alpha,\beta)$ can be defined uniquely by the following equations:

$$w(0) = 0 \qquad w(1) = 1$$

(A) $$w(ty_\alpha) = \frac{1}{\beta}\, w(t)$$

$$w\left(t + \frac{\alpha+1}{\alpha}\right) = w(t) + \frac{1}{\beta} \ . \qquad \left(0 \leq t \leq 1 - \frac{\alpha+1}{\alpha} = 1 - \frac{\alpha-1}{\alpha(\beta-1)}\right)$$

Kober has proved that the equations (A) determine $w(t)$ completely and this is a consequence of the following theorem.

<u>Theorem 1</u>: <u>If a function</u> (t',α,β) (i) <u>does not decrease in any interval</u> $0 \leq t \leq \delta$ <u>and</u> (ii) <u>satisfies the equations (A) then it is identical with</u> $w(t',\alpha,\beta)$ <u>defined</u> in (1).

This follows from

<u>Theorem 2</u>: <u>The Fourier-Stieltjes transform</u> $F(x) = \int_0 e^{ixt}\, d\ (t)$ <u>of a function</u> (t) <u>satisfying the conditions in Theorem (1) is given by</u>

$$F(x) = e^{i\,x/2} \prod_{n=0} \frac{\sin(\beta\, x\, \alpha^{-n}\, \Delta/2)}{\beta\,\sin(x\,\alpha^{-n}\,\Delta/2)} \quad . \qquad \text{where} \quad \Delta = \frac{1-\alpha^{-1}}{\beta-1}$$

That is the conditions of Theorem (1), determine the Fourier-transform of the function uniquely, hence by a known uniqueness theorem, the function defined by the conditions of Theorem (1), is unique.

CHAPTER III

ON JUMP FUNCTIONS

In this chapter we discuss the class of singular functions known as jump functions. It has been proved in Chapter I that if $y = f(t)$ is a non decreasing singular function, then the inverse function $t = g(y)$ is singular. In case $f(t)$ is a step function then the function $g(y)$ is a jump function.

In §1, we study the jump function which is the inverse of Cantor's step function. § 2 and 3 deal with the jump functions which are inverse functions of Carleman's and Gilman's step functions respectively. In § 4 we study Kober's works on jump functions.

§ 1 Inverse of Cantor's Step Function

Let $f(x)$ denote Cantor's step function defined in $(0,1)$. Then

$$f(x) = \frac{b_1}{2} + \frac{b_2}{2^2} + \dots + \frac{b_n}{2^n} + \dots$$

when

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \dots + \frac{a_n}{3^n} + \dots \qquad a_i = 0 \text{ or } 2 \text{ and } b_i = \frac{a_i}{2}.$$

and

$$f(x) = \frac{b_1}{2} + \frac{b_2}{2^2} + \dots + \frac{b_n}{2^n}$$

when

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \ldots + \frac{a_n}{3^n} + \ldots \quad . \qquad a_n = 1 \text{ and } a_i(i<n) = 0 \text{ or } 2$$

$$\text{and } b_i = \frac{a_i}{2} \; (i<n)$$

$$b_n = \frac{a_n+1}{2}$$

The inverse function $x = f^{-1}(y) = g(y)$ will be given by:

if

$$y = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{b_n}{2^n}$$

i.e. whenever y has finite radix representation to the base 2,

then

$$x = g(y) = 2(\frac{b_1}{3} + \frac{b_2}{3^2} + \ldots + \frac{b_n}{3^n}) \; .$$

If y has infinite representation only i.e.

$$y = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{b_n}{2^n} + \ldots$$

then

$$g(y) = 2(\frac{b_1}{3} + \frac{b_2}{3^2} + \ldots + \frac{b_n}{3^n} + \ldots) \; .$$

Since $y = f(x)$ has an everywhere dense set of intervals where the function is constant, then $f^{-1}(y)$ will have an everywhere dense set of points of discontinuities. In fact $f^{-1}(y)$ is discontinuous at all points y having finite radix representation to the base 2. Let y be such a point, then

$$y = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{1}{2^n} \; .$$

Let $\quad y' < y$

then

$$y' = \frac{b_1}{2} + \frac{b_2}{2^2} + \dots + \frac{0}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots + \frac{1}{2^v}$$

we have

$$g(y) = 2(\frac{b_1}{3} + \frac{b_2}{3^2} + \dots + \frac{1}{3^n})$$

and

$$g(y') = 2(\frac{b_1}{3} + \frac{b_2}{3^2} + \dots + \frac{0}{3^n} + \frac{1}{3^{n+1}} + \dots + \frac{1}{3^v})$$

$$\lim_{y' \to y} g(y) - g(y') = \lim_{v} [\frac{2}{3^n} - (\frac{1}{3^{n+1}} + \dots + \frac{1}{3^v})] = \frac{1}{3^n} .$$

Hence $g(y)$ is discontinuous from left at all points with finite representation, the measure of the discontinuity being $\frac{1}{3^n}$. At such points however $g(y)$ is continuous from the right.

Let

$$y' > y$$

$$y' = \frac{b_1}{2} + \frac{b_2}{2^2} + \dots + \frac{1}{2^n} + \frac{0}{2^{n+1}} + \dots + \frac{0}{2^{v-1}} + \frac{1}{2^v}$$

$$g(y') - g(y) = \frac{2}{3^v}$$

$$\lim_{y' \to y} g(y') - g(y) = \lim_{v} \frac{2}{3^v} = 0.$$

At all other points in $(0,1)$, i.e. at all points y that have infinite radix representation only to the base 2, the function is continuous. Let y be such a point, then

$$y = \frac{b_1}{2} + \frac{b_2}{2^2} + \dots + \frac{b_n}{2^n} + \dots$$

infinite number of b_n's =1

infinite number of b_n's =0

let

$y' > y$, then

$$y' = \frac{b_1}{2} + \frac{b_2}{2^2} + \dots + \frac{b'_v}{2^v} + \dots$$

where $b_v = 0$ and

$b'v = 1$

i.e. to get y' we replace some bv's which are zeros by 1's. Then

$$g(y') - g(y) = 2(\frac{b'v}{3^v} + \dots \text{ terms of higher order})$$

$$\lim_{y' \to y} g(y') - g(y) = \lim_{v \to \infty} 2(\frac{b'v}{3^v} + \dots)$$

$$= 0 .$$

Similarly at such points $g(y)$ is continuous from the left. In fact $g(y)$ is an example of a function discontinuous at an enumerable dense set, and continuous elsewhere. Since $g(y)$ is monotone then its differential coefficients exist almost everywhere, and almost everywhere it will be equal to zero.

Let y be a point that has infinite representation only, then

$$y = \frac{b_1}{2} + \frac{b_2}{2^2} + \dots + \frac{b_n}{2^n} + \dots$$

infinite number of b_n's=1

Infinite number of b_n's=0

$$= \sum_i \frac{b_{\lambda i}}{2^{\lambda_i}} + \sum_j \frac{b_{\mu j}}{2^{\mu j}} .$$

$\lambda_i \neq \mu_j$ for all i,j

$b_{\lambda i} = 1$ $b_{\mu j} = 0$

To find progressive derivative, let $y' > y$, then

$$y' = \Sigma \frac{b_{\lambda i}}{2^{\lambda i}} + \Sigma \frac{b'_{\mu j}}{3^{\mu j}} . \qquad b'_{\mu j} = 1$$

Hence

$$g(y') - g(y) = 2\Sigma \frac{b'_{\mu j}}{3^{\mu j}}$$

and

$$\frac{g(y') - g(y)}{y' - y} = \frac{2 \sum\limits_{j=v} \frac{b'_{\mu j}}{3^{\mu j}}}{\sum\limits_{j=v} \frac{b'_{\mu j}}{2^{\mu j}}}$$

$$= \frac{2 \sum\limits_{j=v} \frac{1}{3^{\mu j}}}{\sum\limits_{j=v} \frac{1}{2^{uj}}} \leq \frac{2 \cdot \frac{1}{3^{uv}}}{\frac{2}{2^{uv}}}$$

$$\leq \frac{2}{3^{\mu v}} \cdot \frac{2^{\mu v}}{2}$$

$$D^{+}g(y) = \lim_{y' \to y} \frac{g(y') - g(y)}{y' - y} \leq \lim_{v \to \infty} \frac{2}{3^{\mu v}} \cdot \frac{2^{\mu}}{2} = 0 .$$

Because $g(y)$ is monotone $D^{+}g(y) \geq 0$. Hence

$$D^{+}g(y) = 0 .$$

To find $D^{-}g(y)$, let $y' < y$, then

$$y' = \sum_{i=1}^{v} \frac{b_{\lambda i}}{2^{\lambda i}} + \sum_{i=v+1} \frac{b'_{\lambda i}}{2^{\lambda i}} + \Sigma \frac{b_{\mu j}}{2^{\mu j}} . \qquad b'_{\lambda i} = 0$$

Then

$$\frac{g(y) - g(y')}{y - y'} = \frac{2 \sum\limits_{i=v+1} \frac{b_{\lambda i}}{3^{\lambda i}}}{\sum\limits_{i=v+1} \frac{b_{\lambda i}}{2^{\lambda i}}} = \frac{2 \sum\limits_{i=v+1} \frac{1}{3^{\lambda i}}}{\sum\limits_{i=v+1} \frac{1}{2^{\lambda i}}}$$

$$\leq \frac{2 \frac{1}{3^{\lambda v+1}}}{\frac{2}{2^{\lambda v+1}}} .$$

$$D^{-}g(y) = \lim_{y' \to y} \frac{g(y)-g(y')}{y-y'} \leq \lim_{v \to \infty} 2 \frac{1}{3^{\lambda v+1}} \cdot \frac{2^{\lambda v+1}}{2} = 0$$

$$D^{+}g(y) = D^{-}g(y) = 0 .$$

Because the points which have finite representation are enumerable, the differential coefficient exists and is equal to zero almost everywhere, namely everywhere except at points having finite representation.

Let us now study the behaviour of the derivatives at points having finite representation. Since $g(y)$ is discontinuous from the left at such points, the regressive derivative will be infinite. To compute the progressive derivative, let y be a point having finite radix representation to the base 2, then

$$y = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{1}{2^n}$$

let

$$y' > y,$$

$$y' = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{1}{2^n} + \frac{0}{2^{n+1}} + \frac{0}{2^{n+2}} + \ldots + \frac{0}{2^{v-1}} + \frac{1}{2^v}$$

and

$$g(y') - g(y) = 2\left(\frac{1}{2^v} + \ldots \text{ terms of higher order}\right)$$

and

$$\frac{g(y') - g(y)}{y' - y} = \frac{2\,\frac{1}{3^v} + \ldots)}{\frac{1}{2^v} + \ldots)} \leq \frac{\frac{2}{3^v}}{\frac{2}{2^v}} = \left(\frac{2}{3}\right)^v$$

$$D^+g(y) = \lim_{y' \to y} \frac{g(y')-g(y)}{y'-y} \leq \lim_{v \to \infty} \left(\frac{2}{3}\right)^v \leq 0$$

$$D^+g(y) = 0.$$

These properties of the derivative are true of the class of functions which are inverse functions of Carleman's, Gilman's, Kober's functions.

§ 2 Inverse Function of Carleman's Functions

Carleman's class of functions which include Cantor's function as a special case was defined in Chapter II as follows:

Let α be any positive integer, then represent all points in $(0,1)$ in radix representation to the base α, then

$$y = f(x) = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{b_n}{2^n} + \ldots \qquad b_i = \frac{a_i}{\alpha-1}$$

when

$$x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \ldots + \frac{a_n}{\alpha^n} + \ldots \qquad \alpha_i = 0 \text{ or } \alpha - 1$$

and

$$f(x) = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{1}{2^n}$$

when

$$x = \frac{a_1}{a} + \frac{a_2}{a^2} + \ldots + \frac{a_n}{\alpha^n} + \ldots \qquad a_i \ (1 < n) = 0 \text{ or } \alpha-1$$

$$a_n \neq 0 \text{ or } \alpha-1$$

$$x = g(y) = f^{-1}(y)$$

will be given as follows:

For a point having finite representation,

$$y = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{1}{2^n}$$

then

$$g(y) = (\alpha-1)\left[\frac{b_1}{\alpha} + \frac{b_2}{\alpha^2} + \ldots + \frac{1}{\alpha^n}\right] .$$

For points that have infinite representation only

$$y = \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{b_n}{2^n} + \ldots$$

then

$$g(y) = (\alpha-1)\left[\frac{b_1}{\alpha} + \frac{b_2}{\alpha^2} + \ldots + \frac{b_n}{\alpha^n} + \ldots\right] .$$

As was shown in § 1, similarly it can be shown that $g(y)$ defines a strictly increasing function, discontinuous at all points having finite radix representation to the base 2, continuous at remaining points, at such points the differential coefficient exists and is equal to zero. For $\alpha = 3$, this reduces to the jump function discussed in § 1, namely the inverse function of Cantor's step function. A more general class of functions having these properties will be discussed in the following section.

§3 Inverse Function of Gilman's Function

Gilman's functions were defined as follows:

Representing all points in $(0,1)$ in radix representation to the base α and letting $\alpha-1 = q(\beta-1)$, then

$$f(x) = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \ldots + \frac{b_n}{\beta^n} + \ldots$$

whenever

$$x = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \ldots + \frac{a_n}{\alpha^n} + \ldots \qquad a_i = qb_i$$

and if a_n is the first term which is not a multiple of q, then

$$f(x) = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \ldots + \frac{[\frac{a_n}{q}] + 1}{\beta^n}$$

where $[\frac{a_n}{q}]$ denotes the greatest integer in $\frac{a_n}{q}$.

The inverse function will be given by

$$x = g(y) = f^{-1}(y) = q(\frac{b_1}{\alpha} + \frac{b_2}{\alpha^2} + \ldots + \frac{b_n}{\alpha^n})$$

whenever y has finite radix representation to the base α. i.e. whenever

$$y = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \ldots + \frac{b_n}{\beta^n}$$

and

$$x = g(y) = q(\frac{b_1}{\alpha} + \frac{b_2}{\alpha^2} + \ldots + \frac{b_n}{\alpha^n} + \ldots)$$

whenever y has infinite representation only.

We will prove that the properties mentioned above are satisfied by this class of functions. We first discuss the continuity of $x = g(y)$.

Let y be a point having finite representation

$$y = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \dots + \frac{b_n}{\beta^n}$$

let $y' < y$, then

$$y' = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \dots + \frac{b_n - 1}{\beta^n} + \frac{\beta-1}{\beta^{n+1}} + \dots + \frac{\beta-1}{\beta^v}$$

then

$$g(y) - g(y') = q\,[\frac{b_n}{\alpha^n} - \frac{b_n - 1}{\alpha^n} - (\frac{\beta-1}{\alpha^{n+1}} + \dots + \frac{\beta-1}{\alpha^v})$$

$$= q\,[\frac{1}{\alpha^n} - (\frac{\beta-1}{\alpha^{n+1}} + \dots + \frac{\beta-1}{\alpha^v})]$$

$$\lim_{y' \to y} g(y) - g(y') = \lim_{v \to \infty} q[\frac{1}{\alpha^n} - (\frac{\beta-1}{\alpha^{n+1}} + \dots + \frac{\beta-1}{\alpha^v})]$$

$$= \frac{1}{\alpha^n} - \frac{(\beta-1)(\alpha-1)}{\alpha^n} = \frac{\alpha-\beta}{(\alpha-1)\,\alpha^n}\,.$$

At such points, the function is discontinuous from the left, however, at such points it is continuous from the right.

Let $y' > y$, then

$$y' = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \dots + \frac{b_n}{\beta^n} + \frac{0}{\beta^{n+1}} + \dots + \frac{b_v}{\beta^v} + \dots$$

then

$$g(y) - g(y') = q(\frac{b_v}{\alpha^v} + \dots) \underset{v \to \infty}{\longrightarrow} 0\,.$$

Similarly we can show that at points having infinite representation only $g(y)$ is continuous. Moreover we will show that at such points the differential coefficient exists and is equal to zero.

Let y be a point having infinite representation only and let $y' > y$ then

$$y = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \dots + \frac{b_n}{\beta^n} + \dots$$

$$y' = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \dots + \frac{b_n}{\beta^n} + \dots + \frac{b'v}{\beta^v} + \dots \qquad b'v > b_v$$

then

$$\frac{g(y') - g(y)}{y' - y} = q \frac{[\frac{b'v-b_v}{\alpha^v} + \dots]}{[\frac{b'v-bv}{\beta^v} + \dots]}$$

$$\leq \frac{q\,\frac{b'v-bv}{\alpha^v}}{\frac{b'v-bv_{+1}}{\beta^v}} = q\,\frac{b'v-bv}{b'v-b_{v+1}}\left(\frac{\beta}{\alpha}\right)^v \underset{v\to\infty}{\longrightarrow} 0 .$$

Hence $D^+g = 0$.

To find D^-g let $y' < y$, then

$$y' = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \dots + \frac{b_n}{\beta^n} + \dots + \frac{b'v}{\beta^v} + \dots \qquad b'v < b_v$$

then

$$\frac{g(y)-g(y')}{y-y'} = \frac{q[\frac{b'v}{\alpha^v} + \dots]}{[\frac{b'v}{\beta^v} + \dots]} \leq q\,\frac{b'_v - b_v}{\alpha^v} \Big/ \frac{b'_v + 1 - b_v}{\beta^v} \underset{v\to\infty}{\to} 0 .$$

Therefore $D^-g = 0$.

Hence the differential coefficient exists and is equal to zero almost everywhere. This proves that $g(y)$ is a singular function.

§ 4 Kober's Works on Jump Functions

Kober in 1948 defined a class of jump functions $t = G(y)$ which includes all the functions previously mentioned as special cases. $t = G(y)$ is defined as follows

$$(1) \qquad t = G(y) = q \sum_{\substack{m > -\infty \\ n\beta^{-m} < y}} \sum_{n=1}' \alpha^{-m} = q \sum_{m > -\infty} a_{m,y}\, \alpha^{-m}$$

$$(0 \le y < \infty; \quad m > -\log y / \log \beta)$$

where β is an integer, $\alpha > \beta \ge 2$, $q = (\alpha-\beta)(\beta-1)^{-1}$, and where the dash means that summation does not include n's which are multiples of β; and $a_{m,y}$ is the number of +ve integers n which are not divisible by β such that given m, $n < y\,\beta^m$. From this we obtain the inverse of Cantor's step function by setting $\alpha = 3$, $\beta = 2$, the inverse of Carleman's function for $\beta = 2$, α any integer, and the inverse of Gilman's function by putting $y \le 1$ and α integral.

The relation (1) $t = G(y)$ defines a function with discontinuities at all points of the form $y_{m,n} = n\,\beta^{-m}$ $(n \nmid \beta)$ for, letting

$$G(y_{m,n-}) = G(y_{m,n}) = t_{m,n} = q \sum_{m > -\infty} a_{m,y}\, \alpha^{-m}$$

then

$$G(y_{m,n+}) = q \sum_{m \ge -\infty} a'_{m,y}\, \alpha^{-m}$$

where $a'_{m,y}$ is the number of +ve integers n's ; $n \nmid \beta$ and $n \leq y\beta^m$, and therefore $a'_{m,y} - a_{m,y} = 1$. In other words

$$G(y_{m,n+}) = G(y_{m,n}) + q/_{\alpha^{-m}} = t_{m,n} + q/_{\alpha^{-m}} .$$

From the definition of $G(y)$ we can deduce the following properties:

$$\text{(i)} \quad G(y/_\beta) = \frac{1}{\alpha} G(y)$$

I $$\text{(ii)} \quad G(0) = 0 \quad G(1) = 1$$

$$\text{(iii)} \quad G(y + \tfrac{1}{\beta}) = G(y) + \frac{q+1}{\alpha} \qquad 0 < y < 1 - \frac{1}{\beta} .$$

Proof of (i)

Let $$G(y/\beta) = q \sum_{m > -\infty} a_{m,y} \, \alpha^{-m}$$

and

$$G(y) = q \sum_{m > -\infty} a'_{m,y} \, \alpha^{-m} .$$

We shall find the relation between $a_{m,y}$ and $a'_{m,y}$. $a_{m,y}$ denotes the number of integers n; $n \nmid \beta$ and given m, $n < y\,\beta^{m-1}$ and $a'_{m,y}$ denotes the number of integers n; $n \nmid \beta$ and given m, $n < y\,\beta^m$.

Proof of (ii)

$G(0) = 0$ This is apparent.

$$G(1) = q \sum_{m > -\infty} a_{m,y} \, \alpha^{-m}$$

where $a_{m,y}$ is the number of +ve integers n; $n \nmid \beta$ and $n < \beta^m$

$$a_{m,y} = \beta^m - \beta^{m-1} = \beta^{m-1}(\beta - 1)$$

$$G(1) = q\left[\frac{\beta-1}{\alpha} + \frac{\beta(\beta-1)}{\alpha^2} + \dots + \frac{\beta^{m-1}(\beta-1)}{\alpha^m} + \dots\right]$$

$$= q\,\frac{\beta-1}{\alpha}\left[1 + \left(\frac{\beta}{\alpha}\right) + \left(\frac{\beta}{\alpha}\right)^2 + \dots + \left(\frac{\beta}{\alpha}\right)^n + \dots\right]$$

$$= q\,\frac{\beta-1}{\alpha}\,\frac{1}{1-\frac{\beta}{\alpha}} = q\,\frac{\beta-1}{\alpha}\,\frac{\alpha}{\alpha-\beta} = \frac{q(\beta-1)}{\alpha-\beta} = 1.$$

Proof of(iii)

We want to show that

$$G\left(y + \frac{1}{\beta}\right) - G(y) = \frac{q+1}{\alpha}\,.$$

$$G\left(y + \frac{1}{\beta}\right) - G(y) = q \sum_{n\beta^{-m} < y+\frac{1}{\beta}}' \alpha^{-m} - q \sum_{m\beta^{-m} < y}'$$

$$= q \sum_{y < n\beta^{-m} < y+\frac{1}{\beta}}' \alpha^{-m}\,.$$

If we consider $0 < y < 1-\frac{1}{\beta}$, then $m > 0$ since $y + \frac{1}{\beta} \leq 1$. The number of n's not divisible by β such that

$$y\beta^m < n < y\beta^m + \beta^{m-1} \quad \text{is} \quad \beta^{m-1}-\beta^{m-2} = \beta^{m-2}(\beta-1).$$

Hence

$$G\left(y+ \frac{1}{\beta}\right) - G(y) = q\left(\frac{1}{\alpha} + \frac{\beta-1}{\alpha^2} + \frac{\beta(\beta-1)}{\alpha^3} + \dots + \frac{\beta^{m-2}(\beta-1)}{\alpha^m} + \dots\right)$$

$$= \frac{q}{\alpha} + \frac{q(\beta-1)}{\alpha^2}\left(1 + \frac{\beta}{\alpha} + \left(\frac{\beta}{\alpha}\right)^2 + \dots\right)$$

$$= \frac{q}{\alpha} + \frac{q(\beta-2)}{\alpha^2}\cdot\frac{\alpha}{\alpha-\beta} = \frac{q}{\alpha} + \frac{1}{\alpha} = \frac{q+1}{\alpha}\,.$$

Hence

$$G\left(y + \frac{1}{\beta}\right) - G(y) = \frac{q+1}{\alpha}\,.$$

To see that $G(y)$ is actually a jump function, it is necessary to show that the inverse function is a step function. Denoting the inverse function by $y = w(t) = G\ (t)$, then since the points $y_{m,n}$ of discontinuities of $G(y)$ are everywhere dense, then $w(t)$ will have an everywhere dense set of intervals of constancy. In fact if

$$G(y_{m,n}) = t_{m,n}$$

then for all points t;

$$t_{m,n} \leq t \leq t_{m,n} + q\ \alpha^{-m}$$

$w(t)$ is constant and takes the value $y_{m,n} = n\beta^{-m}$. Such intervals would be called of order m, there being $\beta^m - \beta^{m-1}$ of them. Therefore total length of intervals on which $w(t)$ is constant is

$$\sum_{m=1}^{\infty} q\ \alpha^{-m}(\beta^m - \beta^{m-1}) = q(\beta-1) \sum_{m=1}^{\infty} \frac{\beta^{m-1}}{\alpha^m}$$

$$= q\ \frac{(\beta-1)}{\alpha}\ [1 + \frac{\beta}{\alpha} + (\frac{\beta}{\alpha})^2 + \ldots + (\frac{\beta}{\alpha})^m + \ldots]$$

$$= q\frac{(\beta-1)}{\alpha} \cdot \frac{\alpha}{\alpha-\beta} = 1.$$

Therefore the function $w(t)$ is a step function, the set of its intervals of constant has measure equal to the length of the fundamental interval, and therefore $w(t)$ defines a singular function, from which we can conclude that $G(y)$ the inverse function is a jump function which is singular.

From the relations mentioned above of $G(y)$, we see that $w(t)$ satisfies the following relations:

$$\text{II} \qquad \begin{aligned} &\text{(i)} \quad w(0) = 0 \qquad w(1) = 1 \\ &\text{(ii)} \quad w(t/\alpha) = \frac{1}{\beta} w(t) \\ &\text{(iii)} \quad w\left(t + \frac{\alpha+1}{\alpha}\right) = w(t) + \frac{1}{\beta} . \end{aligned}$$

As was seen in Chapter II, any non decreasing function $w(t)$ satisfying the above relations, is the function $w(t)$ defined in (I). That is these relations II, with the fact that the function is non decreasing define the function uniquely.

Kober has proved the following inequalities concerning the function $w(t)$.

$$\text{III} \qquad \begin{aligned} &\text{(i)} \quad w(t + \tau) \leq w(t) + w(\tau) \\ &\text{(ii)} \quad w(t) \leq t^{\lambda} \text{ and } w(t) \geq \left(\frac{\beta-1}{\alpha-1}\, t\right)^{\lambda} \qquad \left(\lambda = \frac{\log \beta}{\log \alpha};\ 0 \leq t < \infty\right) \\ &\text{(iii)} \quad 0 \leq w(t + h) - w(t) \leq h^{\lambda} . \qquad (0 \leq t < \infty;\ 0 \leq h < \infty) \end{aligned}$$

It is easy to see that the inequality (III) follows immediately from (I) and (II).

$$w(t + h) - w(t) \leq w(h) \leq h^{\lambda}$$

and this shows that the function $w(t)$ satisfies a Lipschitz condition.

Due to the complicated form of $w(t)$, the proofs tend to be long and rather difficult. However, in a later paper Kober has improved on

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See I, p. 37-38

this by generalizing the function $w(t)$ and putting it in a simple and more compact form.

Let β be any real number greater than one, and $0 \leq x < \infty$. Define

$$\gamma(x) = \gamma(x,\alpha,\beta) = \sum_{m=-\infty}^{\infty} \alpha^{-m} [\beta^m x] = \sum_{m \geq \log x / \log \beta} \alpha^{-m} [\beta^m x]$$

and

$$c_{\alpha,\beta} = \lim_{0 > \varepsilon \to 0} \frac{1}{\gamma}(1-\varepsilon)$$

where $[u]$ denotes the greatest integer in u.

Set

$$\text{(i)} \quad t = g(x) = c_{\alpha,\beta}\gamma(x)$$

$$\text{(ii)} \quad \Lambda(t) = \Lambda(t;\alpha,\beta) = x = g^{-1}(t) . \qquad 0 \leq t < \infty$$

In the case that β is an integer, then $c_{\alpha,\beta}$ will reduce to $\frac{(\alpha-\beta)(\alpha-1)}{\alpha(\beta-1)}$, since in that case

$$c_{\alpha,\beta} = \lim_{\varepsilon \to 0} \frac{1}{\gamma(1-\varepsilon)} = \sum_{m=0}^{\infty} \alpha^{-m}(\beta^m - 1)$$

$$= \sum_{m=0}^{\infty} \frac{(\beta^m - 1)}{\alpha^m} = \frac{\alpha}{\alpha-\beta} - \frac{\alpha}{\alpha-1} = \frac{\alpha(\beta-1)}{(\alpha-1)(\beta-1)} .$$

It is easy to see that in that case, $t = c_{\alpha,\beta}\gamma(x)$ reduces to the function $w(t)$ mentioned above, and $g(x)$ generalizes the function $G(y)$. $g(x)$ is a jump function and is discontinuous at all points of the form $x_{\mu,v} = \mu\beta^{-v}$ ($\mu = 1,2,\ldots,$ $v = 0, \pm 1, \pm 2,\ldots$) which are everywhere dense in $(0,\infty)$.

Let

$$g(x_{\mu,v}) = t_{\mu,v}$$

then

$$g(x_{\mu,v} -) = g(x_{\mu,v} - \varepsilon) = g(\mu\beta^{-v} - \varepsilon)$$

$$= \Sigma\, \alpha^{-m} [\mu\beta^{m-v} - \beta^{m} - \varepsilon]$$

$$= t_{\mu,v} - \frac{1}{\alpha-1} \ .$$

Therefore the function $\Lambda(t) = g^{-1}(t)$ is constant in all intervals $(t_{\mu,v} - \frac{1}{\alpha-1}, t_{\mu,v})$ which are everywhere dense, and therefore is a step function. Many of the properties of $G(x)$ and $w(t)$ hold for the general case. For example, it is easy to see that the following relations are satisfied:

$$g(0) = \Lambda(0) = 0 \qquad g(1) = 1 = \Lambda(1)$$

$$g(x) = \alpha\, g(\tfrac{x}{\beta}) \qquad \Lambda(t) = \beta\, \Lambda(\tfrac{t}{\alpha}) \ .$$

CHAPTER IV

ON STRICTLY INCREASING SINGULAR FUNCTIONS

A. Denjoy [1] has given a method to construct continuous singular functions which are strictly increasing. Subsequently many examples of such functions were given. In § 1 we shall discuss Denjoy's method. In § 2 we apply this method to obtain a continuous strictly increasing singular function. § 3 deals with a function given by W. Sierpinski [1]. Lastly § 4 deals with a function given by S. Saks [1] to illustrate a certain property of functions which are absolutely continuous generalized (A.C.G.).[1] This function happens to be a singular function.

§1 Denjoy's Works

To define the function Denjoy sets up a 1-1 correspondence between two sets which are everywhere dense, and by the principle of continuity extends this to all points in the interval.

Consider two intervals (a,b) and (α,β). Define in (a,b) a

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[1] A function $f(x)$ is said to be A.C.G. on E, if f is continuous on E, and if E is the sum of a finite or enumerable sequence of sets E_n on each of which $f(x)$ is A.C. See Saks [1], p.223.

non dense perfect set P_1 of points x, having positive measure $m(P_1) = \Theta > 0$, and a non dense perfect set π_1 of points ξ, such that $m(\pi_1) = 0$.

On (a,b) define a continuous function $f(x)$ which is constant in all intervals contiguous to P_1, increasing otherwise, such that the range of $f(x)$ is the interval $(0,1)$. Similarly on (α,β) define a continuous function[2] $g(\xi)$ which is constant in all intervals contiguous to π_1, increasing otherwise, and having as range the interval $(0,1)$. [In fact, let the values of $f(x)$ and $g(\xi)$ at end points of contiguous intervals of P_1 and π_1 respectively, be points of a preassigned sequence $\{k_n\}$ which is everywhere dense in $(0,1)$, but includes neither of the points 0 and 1].

Denote the contiguous intervals of P_1 by u_1^P $(P = 1,2,\ldots)$, since these are enumerable, arrange them in a certain order

$$u_1^1,\ u_1^2, \ldots u_1^P, \ldots .$$

Let the contiguous intervals of π_1 be w_1^P $(P = 1,2,\ldots)$. We establish a 1-1 correspondence between u_1^P and w_1^P in the following way:

If

$$u_1^P = (a_1^P,\ b_1^P) \quad \text{and} \quad w_1^P = (\alpha_1^P,\ \beta_1^P)$$

then

$$u_1^P \text{ corresponds to } w_1^P \text{ if and only if}$$

---

(2) Clearly $f(x)$ and $g(\xi)$ are continuous step functions.

$$f(a_1^P) = g(\alpha_1^P) \quad \text{and} \quad f(b_1^P) = g(\beta_1^P).$$

Similarly we establish a 1-1 correspondence between the points of P_1 and those of π_1 as follows:

$$x_1 \in P_1 \text{ corresponds to } \xi_1 \in \pi_1 \text{ if and only if } f(x_1) = g(\xi_1).$$

Moreover, left end points of contiguous intervals have to go into left end points, and right end points have to go into right end points. This defines the correspondence uniquely.

Since both $f(x)$ and $g(\xi)$ are monotone non decreasing functions, it follows that

if $x_1 < x_2$ and $x_1 \leftrightarrow \xi_1$ [3]

$x_2 \leftrightarrow \xi_2$

then $\xi_1 < \xi_2$.

Consider two intervals, u_1^n, w_1^n, contiguous to P_1 and to π_1 respectively, such that they correspond to each other under the correspondence defined above.

Let

$$u_1^n = (a_1^n, b_1^n)$$

$$w_1^n = (\alpha_1^n, \beta_1^n).$$

Place a non dense perfect set P_2^n in u_1^n such that $m(P_2^n) = \theta(b_1^n - a_1^n)$ and a non dense perfect set π_2^n in w_1^n such that $m(\pi_2^n) = 0$. In u_1^n

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[3] By $x_1 \leftrightarrow \xi_1$ we mean ξ_1 is the correspondent of x_1.

Recalling the equivalent definition of a singular function given by Kober [1]

"A function $f(x)$ is singular in (a,b), if given $\varepsilon > 0$, $\exists$ non-overlapping intervals (x_k, x_k') $(k = 1,2,\ldots n)$ in (a,b) such that

$$\sum_{k=1}^{n} (x_k' - x_k) < \varepsilon \qquad \sum_{k=1}^{n} |f(x_k') - f(x_k)| > V_a^b f - \varepsilon$$

where $V_a^b f$ denotes the total variation of $f(x)$ in (a,b)".

It immediately follows that $f(x)$ defined above is a singular function.

§ 2 An Application of Denjoy's Method

Here we apply the method given in § 1, to obtain an example of a strictly increasing singular function.

Consider the interval $(0,1)$. Construct in it the following perfect sets, π_1 of zero measure and P_1 of positive measure, defined in the following way:

π_1 (4) is the set of all points $\xi \in (0,1)$ such that

$$\xi = \frac{\alpha_1}{3} + \frac{\alpha_2}{3^2} + \ldots + \frac{\alpha_n}{3^n} + \ldots \qquad \text{where } \alpha_i = 0 \text{ or } 2$$

and P_1 is the set of all points $x \in (0,1)$, such that

$$x = \frac{1}{2}\left(\frac{a_1}{4} + \frac{a_2}{4^2} + \ldots + \frac{a_n}{4^n} + \ldots\right) \qquad \text{where } a_i = 2^i + 3 \text{ or } 0 .$$

---

(4) π_1 is Cantor's ternary set.

define a step function $f_2^n(x)$ constant in all intervals contiguous to P_2^n, increasing otherwise, and a function $g_2^n(x)$, such that both functions have the same range.

As before, we set up a 1-1 correspondence between the points of P_2^n and those of π_2^n. This will be as follows:

$x_2 \varepsilon P_2^n$ corresponds to $\xi_2 \varepsilon \pi_2^n$ if $f_2^n(x) = g_2^n(x)$, with the condition that left end points of contiguous intervals of P_2^n, go into left end points of contiguous intervals of π_2^n.

Denote the contiguous intervals of P_2^n by

$$u_2^{n1}, u_2^{n2}, u_2^{n3}, \ldots , u_2^{np} \ldots$$

and those of π_2^n by

$$w_2^{n1}, w_2^{n2}, \ldots , w_2^{np} \ldots$$

where

$$u_2^{np} = (a_2^{nP}, b_2^{nP})$$

$$w_2^{nP} = (\alpha_2^{nP}, \beta_2^{nP}) .$$

Let u_2^{nP} correspond to w_2^{nP} if $f_2^n(a_2^{nP}) = g_2^n(\alpha_2^{nP})$

$$f_2^n(b_2^{nP}) = g_2^n(\beta_2^{nP}).$$

Set

$$P_2 = P_1 + u_n P_2$$

$$\pi_2 = \pi_1 + u_n \pi_2 .$$

Thus we have set up a 1-1 correspondence between points of P_2 and those of π_2, and between contiguous intervals of P_2 and those of π_2. Repeating this process infinitely many times, we get sets

$$P_1, P_2, \dots P_n, \dots$$

$$\pi_1, \pi_2, \dots \pi_n, \dots$$

where

$$P_1 \subset P_2 \subset P_3 \subset \dots \subset P_n \subset \dots$$

$$\pi_1 \subset \pi_2 \subset \pi_3 \subset \dots \subset \pi_n \subset \dots .$$

Moreover a 1-1 correspondence has been defined between the points of P_n and those of π_n.

Let P & π be the outer limiting sets, then

$$P = UP_i$$

$$\pi = U\pi_i .$$

P & π are everywhere dense and are sets of the first category. Moreover, there exists a 1-1 correspondence between them which we shall state as follows:

If $x \varepsilon P \quad \xi \varepsilon \pi$, then $\xi = f(x)$.

Since $f(x)$ is a strictly increasing function and the sets P and π are everywhere dense in (a,b) and (α,β) respectively, then by the principle of continuity, the correspondence can be extended to all points in (a,b) and (α,β).

Thus we have defined a strictly increasing function $f(x)$ in (a,b) having (α,β) as its range. To see that $f(x)$ is a singular function, we note that $f(x)$ transforms a set P of full measure (equal to the length of the fundamental interval) into a set π of zero measure.

We have seen in Chapter I that π_1 has zero measure. To see that P_1 has measure equal to $\frac{1}{2}$, denote contiguous intervals of P_1 by $u_n = (x', x'')$, then

$$x' = \frac{1}{2}\left(\frac{a_1}{4} + \frac{a_2}{4^2} + \dots + \frac{0}{4^n} + \frac{2^{n+1}+3}{4^{n+1}} + \frac{2^{n+2}+3}{4^{n+2}} + \dots\right)$$
$$= \frac{1}{2}\left(\frac{a_1}{4} + \frac{a_2}{4^2} + \dots + \frac{2^n+1}{4^n}\right)$$

and

$$x'' = \frac{1}{2}\left(\frac{a_1}{4} + \frac{a_2}{4^2} + \dots + \frac{2^n+3}{4^n}\right).$$

Therefore length of u_n is $\frac{1}{4^n}$, there being $2^{(n-1)}$ such intervals. Hence the measure of $C(P_1)$ is equal to

$$\sum_{n=1} 2^{n-1} \frac{1}{4^n} = \frac{1}{2}$$

from which it follows that $m(P_1) = 0$.

To apply Denjoy's method, we now have to place perfect sets P_1^n, π_1^n, of the same nature in each of the contiguous intervals u_n and w_n of P_1 and π respectively. Letting $P_2 = P_1 + uP_1^n$, $\pi_2 = \pi_1 + u\pi_1^n$, then P_2 and π_2 are perfect non dense sets with $m(P_2) = \frac{1}{2} + \frac{1}{2^n}$ and $m(\pi_2) = 0$.

Also if $x \in P_2$, then

$$x = \frac{1}{2}\left(\frac{a_{11}}{4} + \frac{a_{12}}{4^2} + \dots + \frac{2^{n_1}+1}{4^{n_1}} + \frac{1}{4^{n_1}}\left(\frac{a_{21}}{4} + \frac{a_{22}}{4^2} + \dots + \frac{a_{2n}}{4^n} + \dots\right.\right.$$

where $a_{ij} = 2^j+3$ or 0

and if $\xi \varepsilon \pi_2$, then

$$\xi = \frac{\alpha_{11}}{3} + \frac{\alpha_{12}}{3^2} + \dots + \frac{1}{3^{n1}} + \frac{1}{3^{n2}}(\frac{\alpha_{21}}{3} + \frac{\alpha_{22}}{3^2} + \dots + \frac{\alpha_{2n}}{3^n} + \dots)$$

where $\alpha_{ij} = 0$ or 2.

Again, we have to place perfect sets of the same nature in each of the contiguous intervals of P_2 and π_2 respectively, and if this process is repeated infinitely many times, then we will get two sets P and π each of the first category such that

$$m(P) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = 1 \qquad \text{and} \quad m(\pi) = 0.$$

If x is a point of P, then x admits of the following representation:

$$x = \frac{1}{2}(\frac{a_{11}}{4} + \frac{a_{12}}{4^2} + \dots + \frac{2^n+1}{4^{n1}} + \frac{1}{4^{n1}}(\frac{a_{21}}{4} + \frac{a_{22}}{4^2} + \dots + \frac{2^{n2}+1}{4^{n2}}$$

$$+ \frac{1}{4^{n2}}(\frac{a_{31}}{4} + \dots (\dots (\dots$$

where $a_{ij} = 0$ or 2^j+3

and the number of brackets is finite. According to such representation, we can classify the points of $(0,1)$ into three classes.

<u>Class 1</u> consists of all points x, whose representation has finite number of brackets, last series being finite, i.e.

$$x = \frac{1}{2}\left(\frac{a_{11}}{4} + \frac{a_{12}}{4^2} + \ldots + \frac{2^{n1}+1}{4^{n1}} + \frac{1}{4^{n1}} \left(\ldots\left(\ldots + \frac{2^{nr-1}+1}{4^{nr-1}} + \frac{1}{4^{nr-1}}\left(\frac{a_{r1}}{4}\right.\right.\right.\right.$$

$$\left.\left.\left.\left. + \frac{a_{r2}}{4^2} + \ldots + \frac{a_{rs}}{4^s}\right) \ldots\right)\right)\right)$$

where $a_{ij} = 0$ or 2^j+3

$(i = 1,\ldots n)$

$(j = 1,\ldots n'_{2+1})$.

<u>Class 2</u> consists of all points x, whose representation has finite number of brackets, but last series is infinite, i.e.

$$x = \frac{1}{2}\left(\frac{a_{11}}{4} + \frac{a_{12}}{4^2} + \ldots + \frac{2^{n1}+1}{4^{n1}} + \frac{1}{4^{n1}} \left(\frac{a_{21}}{4} + \ldots\left(\ldots + \frac{2^{nr-1}+1}{4^{nr-1}}\right.\right.\right.$$

$$\left.\left.\left. + \frac{1}{4^{nr-1}} \left(\frac{a_{r1}}{4} + \frac{a_{r2}}{4^2} + \ldots + \frac{a_{rs}}{4^{rs}} + \ldots\right)\right)\right)\right)$$

Class 1 and class 2 constitute all the points of P. The set of points of class 1 is an enumerable set, while the set of points of class 2 has positive measure equal to 1.

<u>Class 3</u> consists of all points x, whose representation can have infinite number of brackets only,

$$x = \frac{1}{2}\left(\frac{a_{11}}{4} + \frac{a_{12}}{4^2} + \ldots + \frac{2^{n1}+1}{4^{n1}} + \frac{1}{4^{n1}} \left(\ldots \left(\ldots + \ldots + \frac{2^{nr-1}+1}{4^{nr-1}}\right.\right.\right.$$

$$+ \frac{1}{4^{nr-1}} \left(\frac{a_{r1}}{4} + \ldots \left(\ldots \left(\ldots \left(\ldots \right.\right.\right.\right. .$$

Such points constitute the points of $C(P)$ which is a residual set having measure 0.

Representing now the points of $(0,1)$ in radix representation to the base 3, again we can classify the points in $(0,1)$ into 3 classes:

Class 1 consists of all points x in $(0,1)$ which can have the following representation:

$$x = \frac{a_{11}}{3} + \frac{a_{12}}{3^2} + \dots + \frac{1}{3^{n1}} + \frac{1}{3^{n1}} \left(\frac{a_{21}}{3} + \dots + \frac{1}{3^{n2}} + \frac{1}{3^{n2}} \left(\frac{a_{31}}{3} + \dots + \frac{1}{3^{nr-1}} \left(\frac{a_{r1}}{3} + \frac{a_{r2}}{3^2} + \dots + \frac{a_{rs}}{3^s}\right) \dots \right) .$$

The representation of such points has finite number of brackets, last series being also finite.

Class 2 consists of all points x, whose representation has finite number of brackets, last series being infinite,

$$x = \frac{a_{11}}{3} + \frac{a_{12}}{3^2} + \dots + \frac{1}{3^{n1}} + \frac{1}{3^{n1}} \left(\frac{a_{21}}{3} + \dots (\dots (\dots + \frac{1}{3^{nr-1}} \left(\frac{a_{r1}}{3} + \frac{a_{r2}}{3^2} + \dots + \frac{a_{rs}}{3} + \dots \right) \dots \right) \qquad \text{where } a_{ij} = 0 \text{ or } 2.$$

Class 3 consists of all points x, whose representation can have infinite number of brackets only,

$$x = \frac{a_{11}}{3} + \frac{a_{12}}{3^2} + \dots + \frac{1}{3^{n1}} + \frac{1}{3^{n1}} \left(\frac{a_{21}}{3} + \dots (\dots (\frac{1}{3^{nr-1}} \left(\frac{a_{r1}}{3} + \dots (\dots (\dots (\dots \right.\right.$$

$$a_{ij} = 0 \text{ or } 2.$$

The points of class 1 and class 2 constitute the points of the set π which is everywhere dense having zero measure, and the points of class 3 constitute the points of $C(\pi)$ which is a residual set and has measure equal to 1.

To define the function, take a point x in $(0,1)$ such that $x \in P$ or $x \in C(P)$, then

$$x = \frac{1}{2}\left(\frac{a_{11}}{4} + \frac{a_{12}}{4^2} + \dots + \frac{2^{n1}+1}{4^{n1}} + \frac{1}{4^{n1}}\left(\frac{a_{12}}{4} + \dots \left(\dots\left(\dots + \frac{1}{4^{nr-1}}\left(\frac{a_{r1}}{4} + \frac{a_{r2}}{4^2} + \dots\right.\right.\right.\right.\right.$$

$$a_{ij} = 0 \text{ or } 2^j+3$$

where the number of brackets may be finite or infinite.

Then

$$f(x) = \frac{b_{11}}{3} + \frac{b_{12}}{3^2} + \dots + \frac{1}{3^{n1}} + \frac{1}{3^{n1}}\left(\frac{b_{21}}{3} + \frac{b_{22}}{3^2} + \dots + \frac{1}{3^{n2}} + \frac{1}{3^{n2}}\left(\frac{b_{31}}{3} + \dots\right.\right.$$

$$+ \left(\dots\left(\dots + \frac{1}{3^{nr-1}}\left(\frac{b_{r1}}{3} + \frac{b_{r2}}{3^2} + \dots\right.\right.\right.$$

where $b_{ij} = 0$ if $a_{ij} = 0$

$b_{ij} = 2$ when $a_{ij} = 2^j+3$.

This defines a single valued continuous function $f(x)$ which transforms a set P having measure equal to 1, into a set π having zero measure, and conversely a set of zero measure $C(P)$, into a set $C(\pi)$ having measure equal to 1. Also because $f(x)$ is a monotone (strictly increasing) function, $f(x)$ is of bounded variation and its differential coefficient exists almost everywhere. To show that $f(x)$ is a singular function, it is sufficient to show that one of the derivatives is zero almost everywhere.

To do this, let

$$x = \frac{1}{2}\left(\frac{a_{11}}{4} + \frac{a_{12}}{4^2} + \dots + \frac{2^{n1}+1}{4^{n1}} + \frac{1}{4^{n1}}\left(\dots\left(\dots + \dots + \frac{2^{nr}-1}{4^{nr-1}} + \right.\right.\right.$$

$$+ \frac{1}{4^{nr-1}}\left(\frac{a_{r1}}{4} + \frac{a_{r2}}{4^2} + \dots + \frac{a_{rs}}{4^s} + \dots\right)$$

where $a_{rs} = 0$ $\quad a_{ij} = 0$ or 2^j+3.

Such points form a set of measure equal to 1. Let us compute one of the right derivatives at such points.

Let

$$\xi = \frac{1}{2}\left(\frac{a_{11}}{4} + \frac{a_{12}}{4^2} + \dots + \frac{2^{n1}+1}{4^{n1}} + \frac{1}{4^{n1}}\left(\dots\left(\dots + \frac{2^{nr-1}+1}{4^{nr-1}} + \frac{1}{4^{nr-1}}\left(\frac{a_{r1}}{4} + \frac{a_{r2}}{4^2} + \right.\right.\right.\right.$$

$$\dots + \frac{a'_{rs}}{4^s} + \dots\Big)$$

where $a'_{rs} = 2^s+3$

then

$$f(\xi)-f(x) = \frac{1}{3^{nr-1}}\left(\frac{2}{3^s} + \dots \text{ terms of higher order}\right)$$

and

$$\frac{f(\xi)-f(x)}{\xi-x} = \frac{\frac{1}{3^{nr-1}}\left(\frac{2}{3^s} + \text{terms of higher order}\right)}{\frac{1}{2}\left[\frac{1}{4^{nr-1}}\left(\frac{2^s+3}{4^s} + \text{terms of higher order}\right)\right.}$$

$$= 2.\left(\frac{4}{3}\right)^{nr-1} \frac{\left(\frac{1}{3^s} + \dots\right)}{\left(\frac{2^s+3}{4^s} + \dots\right)}$$

$$\leq 2\left(\frac{4}{3}\right)^{nr-1} \frac{1}{3^{s-1}} \qquad \leq 6.\left(\frac{4}{3}\right)^{nr-1}\left(\frac{4}{3}\right)^3 \frac{1}{2^s}$$

$$\leq 6.\left(\frac{4}{3}\right)^{nr-1}\left(\frac{2}{3}\right)^s \to 0$$

as $s \to \infty$

Hence $$\lim_{\xi \to x} \frac{f(\xi)-f(x)}{\xi-x} = 0 .$$

Therefore we have proved that one of the derivatives is zero almost everywhere, which shows that the differential coefficient is zero almost everywhere. That is, $f(x)$ is a singular function.

§ 3 Sierpinski's Works

W. Sierpinski [1] defined a strictly increasing singular function using different techniques. By giving two recurrence formulas, he defines $f(x)$ for all points x in $(0,1)$ of the form $\frac{l}{3^n}$ $(l = 0,1,\ldots 3^n)$, and by the principle of continuity, he extends the definition to all points in $(0,1)$.

Let

(1) $$f(0) = 0 \qquad f(1) = 1 .$$

Assume for a given integer n, $f(x)$ is defined for all numbers of the form $\frac{h}{3^{n-1}}$ where $h = 0,1,2,\ldots 3^{n-1}$, then let

(2)
$$f\left(\frac{3h+1}{3^n}\right) = \frac{1}{2}\left[f\left(\frac{h}{3^{n-1}}\right) + f\left(\frac{h+1}{3^{n-1}}\right) - \frac{1}{3^{5n}}\right]$$
$$f\left(\frac{3h+2}{3^n}\right) = \frac{1}{2}\left[f\left(\frac{h}{3^{n-1}}\right) + f\left(\frac{h+1}{3^{n-1}}\right) + \frac{1}{3^{5n}}\right] .$$

Since $f(x)$ is defined for $n = 1$, by induction the function will be defined for all numbers of the form $\frac{k}{3^n}$ $(n = 0,1,2,\ldots,\ k = 0,1,\ldots 3^n)$.

Before extending the definition to all points in $(0,1)$, we have to show that it is possible to apply the principle of continuity. To do this we have to give some relations. The following are apparent.

$$f(\frac{3h+1}{3^n}) - f(\frac{3h}{3^n}) = \frac{1}{2}\left[f(\frac{h+1}{3^{n-1}}) - f(\frac{h}{3^{n-1}}) - \frac{1}{3^{5n}}\right]$$

(3)
$$f(\frac{3h+2}{3^n}) - f(\frac{3h+1}{3^n}) = \frac{1}{3^{5n}}$$

$$f(\frac{3h+3}{3^n}) - f(\frac{3h+2}{3^n}) = \frac{1}{2}\left[f(\frac{h+1}{3^{n-1}}) - f(\frac{h}{3^{n-1}}) - \frac{1}{3^{5n}}\right] .$$

By induction on n we can prove the following inequality:

(4)
$$\frac{}{3^{5m}} \leq f(\frac{h+1}{3^m}) - f(\frac{h}{3^m}) \leq \frac{1}{2^m} \qquad m = 0,1,2,\ldots \quad h = 0,1,\ldots\ 3^m-1.$$

<u>Proof:</u> For $m = 0$ the inequality is true.

Assume it is true for $m = n-1$, then

$$\frac{1}{3^{5n-s}} \leq f(\frac{h+1}{3^{n-1}}) - f(\frac{h}{3^{n-1}}) \leq \frac{1}{2^{n-1}} \qquad l = 0,1,\ldots\ 3^{n-1}-1.$$

We will show it is true for $m = n$

$$\frac{1}{3^{5n}} < \frac{1}{2}\left[\frac{1}{3^{5n-5}} - \frac{1}{3^{5n}}\right] \leq f(\frac{3h+1}{3^n}) - f(\frac{h}{3^n}) < \frac{1}{2^n}$$

But

$$\frac{1}{3^{5n}} = f(\frac{3h+2}{3^n}) - f(\frac{3h+1}{3^n}) < \frac{1}{2^n}$$

$$\frac{1}{3^{5n}} < \frac{1}{2}\left[\frac{1}{3^{5n-5}} - \frac{1}{3^{5n}}\right] \leq f(\frac{3h+3}{3^n}) - f(\frac{3h+2}{3^n}) < \frac{1}{2^n} .$$

This proves

$$\frac{1}{3^{5n}} \leq f(\frac{h+1}{3^n}) - f(\frac{h}{3^n}) \leq \frac{1}{2^n} .$$

Now we will show that $f(x)$ as defined for the points of the form $\frac{k}{3^n}$ is a strictly increasing function. That is,

$$\text{if} \quad \frac{k}{3^n} < \frac{h}{3^m} \quad \text{then} \quad f(\frac{k}{3^n}) < f(\frac{h}{3^m}) .$$

Denote by r max (m,n), and write

$$\frac{k}{3^n} = \frac{k'}{3^r} \quad \text{and} \quad \frac{h}{3^m} = \frac{h'}{3^r} \quad \text{where} \quad h' > k' .$$

We have, on applying (4)

$$f(\frac{k'}{3^r}) < f(\frac{k'+1}{3^r}) < f(\frac{k'+2}{3^r}) < \ldots < f(\frac{h'}{3^r})$$

which gives

$$f(\frac{k}{3^n}) < f(\frac{h}{3^m}) .$$

Since the points $\frac{k}{3^n}$ $(n = 0,1,\ldots;\ h = 1,\ldots 3^n)$ are everywhere dense in the interval $(0,1)$, and $f(x)$ is strictly increasing on these points, we can apply the principle of continuity to define $f(x)$ for the remaining points in $(0,1)$. Such points cannot be represented in the form $\frac{h}{3^n}$, but can be expressed in infinite radix representation to the base 3.

Let x be such a point, then

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \ldots + \frac{a_n}{3^n} + \ldots .$$

Define $f(x)$ as follows:

$$f(x) = \lim_{n \to \infty} f(a_1, a_2, \ldots a_n) .$$

That this limit exists and is unique follows from the fact that $f(. a_1 \, a_2 \ldots a_n)$ is a monotone bounded sequence.

Thus we have defined $f(x)$ for all points in $(0,1)$. This function is continuous and strictly increasing in $(0,1)$ and can be drawn to be a singular function.

§ 4 Saks Works

Saks gave an example of a continuous function $f(x)$ increasing in $J_0 = [0,1]$, which has its lower right hand derivative at every point of a set E, without being A.C.G. on E. This function happens to be a strictly increasing singular function.

To define the function, let us first denote by $H(x)$, a function defined in an interval $I = [a,b]$, such that $H(x)$ is continuous and monotone non decreasing, and it satisfies the following conditions.

(i) $H(x)$ is constant in intervals of a sequence I_h such that the total length of these intervals is equal to the length of the interval I, i.e. $\Sigma|I_h| = |I|$[5] and for all k $|I_h| < |I|/_2$.

(ii) $H(x) - H(a) \leq x-a \quad H(b) - H(x) \leq b-x \quad$ for all $x \varepsilon I$.

Now we define a sequence of functions $f_n(x)$ in the following way:

Given $f(x)$ defined in J_0, let $I_h^{(n)} = [a_h^{(n)}, b_h^{(h)}]$ be the sequence of intervals of $f_n(x)$ for each $I_h^{(h)}$, we define a function $H_h^{(n)}(x)$ satisfying the above condition and is defined in $I_h^{(n)}$.

Define $f_{n+1}(x)$ as follows:

$$f_{n+1}(x) = \sum_i^{(x)}[H_i^{(n)}(b_i^{(n)}) - H_i^{(n)}(a_i^{(n)})] \qquad \text{for } x \varepsilon J_0 - \sum_k I_h^{(n)}$$

$$H_h^{(n)}(x) - H_h^{(n)}(a_h^{(n)}) + \sum_i^{(x)}[H_i^{(n)}(b_i^{(n)}) - H_i^{(n)}(a_i^{(n)})] \qquad \text{for } x \varepsilon I_h^{(n)}.$$

The sum $\Sigma_i^{(x)}$ being extended over all $i;\ b_i^{(n)} \leq x$.

Now define $f(x)$ as follows:

$$f(x) = \sum_n f_n(x)/_{2^n}.$$

Saks proved that the function $f(x)$ defined as above is a continuous, strictly increasing singular function.

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[5] By $|I|$ we mean the measure of I (in this case the length of I).

CHAPTER V

NON MONOTONIC SINGULAR FUNCTIONS

This chapter deals with a study of singular functions which are non monotonic in every subinterval of the fundamental interval.

"A function is said to be non monotonic everywhere in (a,b) if there exists no subinterval (α,β) in which the function is monotone".

The existence of such functions has been established by U.K. Shukla [1], by actually defining such a function. In § 1, we give the definition of the function. § 2 deals with some properties of the CAV's of a non monotonic singular function. In § 3 we use the results proved in § 2 to show that the function defined in § 1 is a non monotonic singular function.

§ 1 A Non Monotonic Singular Function

We first give the geometrical representation of two functions $\psi_1(x), \psi_2(x)$, where these functions are the CAV's of the non monotonic singular function $f(x)$. Therefore we have

$$f(x) = \psi_1(x) - \psi_2(x).$$

The definition of the functions $\Psi_1(x), \Psi_2(x)$ is based on the following geometrical operations:

We shall first make the following notations:

Let

$$P_n = \frac{1}{2}\left[\prod_{r=n}\left(1 + \frac{1}{8.2^r}\right) + 1\right]$$

$$N_n = \frac{1}{2}\left[\prod_{r=n}\left(1 + \frac{1}{8.2^r}\right) - 1\right]$$

and

$$\Sigma_1^n = (1 + 2 + 3 + \dots + n). \qquad \Sigma_1^0 = 0$$

Given a segment AB of +ve gradient such that the vertical distance between A and B is

$$\frac{\alpha_n P_{n+r}}{8^{n+r} . 2^{\Sigma_1^{n+r}}} . \qquad \alpha_n = 1 \text{ or } 8.2^n - 1$$

Then we shall transform the segment AB, into the enclosed polygon AB,CA,B as follows:

Let ab be the projection of AB on the x-axis. Let a_1, b_1, c_1 be points on ab such that c_1 is the midpoint of ab and of a_1b_1 and $|a_1b_1| = \frac{|ab|}{4^n}$[1] .

Now draw the points $B_1C_1A_1$ such that $b_1c_1a_1$ are their projections on the axis respectively, and their vertical position is given as follows.

---

[1] By $|ab|$ we mean the length of ab.

Vertical distance between A and B_1 is $\dfrac{\alpha_n P_{n+r+1}}{8^{n+r}.\ 2^{\Sigma_1^{n+r+1}}}$

Vertical distance between A and C_1 is $\dfrac{\alpha_n(8.2^{n+r+1}.P_{n+r+1})}{8^{n+r+1}.2^{\Sigma_1^{n+r+1}}}$

Vertical distance between A and A_1 is $\dfrac{\alpha_n(8.2^{n+r+1}P_{n+r+1}+N_{n+r+1})}{8^{n+r+1}.\ 2^{\Sigma_1^{n+r+1}}}$

Starting with $A = (0,0)$ $B = (1,1)$, and repeating this process infinitely many times for AB_1, B_1C, CA_1, A_1B, we will get a graphical representation of $\Psi_1(x)$ (see figure I).

Similarly we can get a graphical representation for $\Psi_2(x)$, if the P_n's and the C_n's are interchanged in the above operations.

From this graphical representation we can work out step by step the arithmetic definition of $\Psi_1(x)$, $\Psi_2(x)$. We get the following results.

Let m_λ $(\lambda = 0,1,\ldots)$ denote +ve integers and $m_0 = 0$ and

$$M_r = \sum_{\lambda=0}^{r} m_r . \qquad\qquad M_0 = m_0 = 0$$

Define

X_{M_r} , $X_{M_{r,\infty}}$ as follows:

$$X_{M_{r,\infty}} = \frac{P_{M_r+1}}{4^{M_{r+1}}} + \frac{P_{M_r+2}}{4^{M_{r+2}}} + \ldots + \frac{P_{M_r+h}}{4^{M_{r+h}}} + \ldots$$

where

$$P_{M_r+h} = 0 \quad \text{or} \quad 2^h + 3 \qquad (h = 1,2,\ldots)$$

and

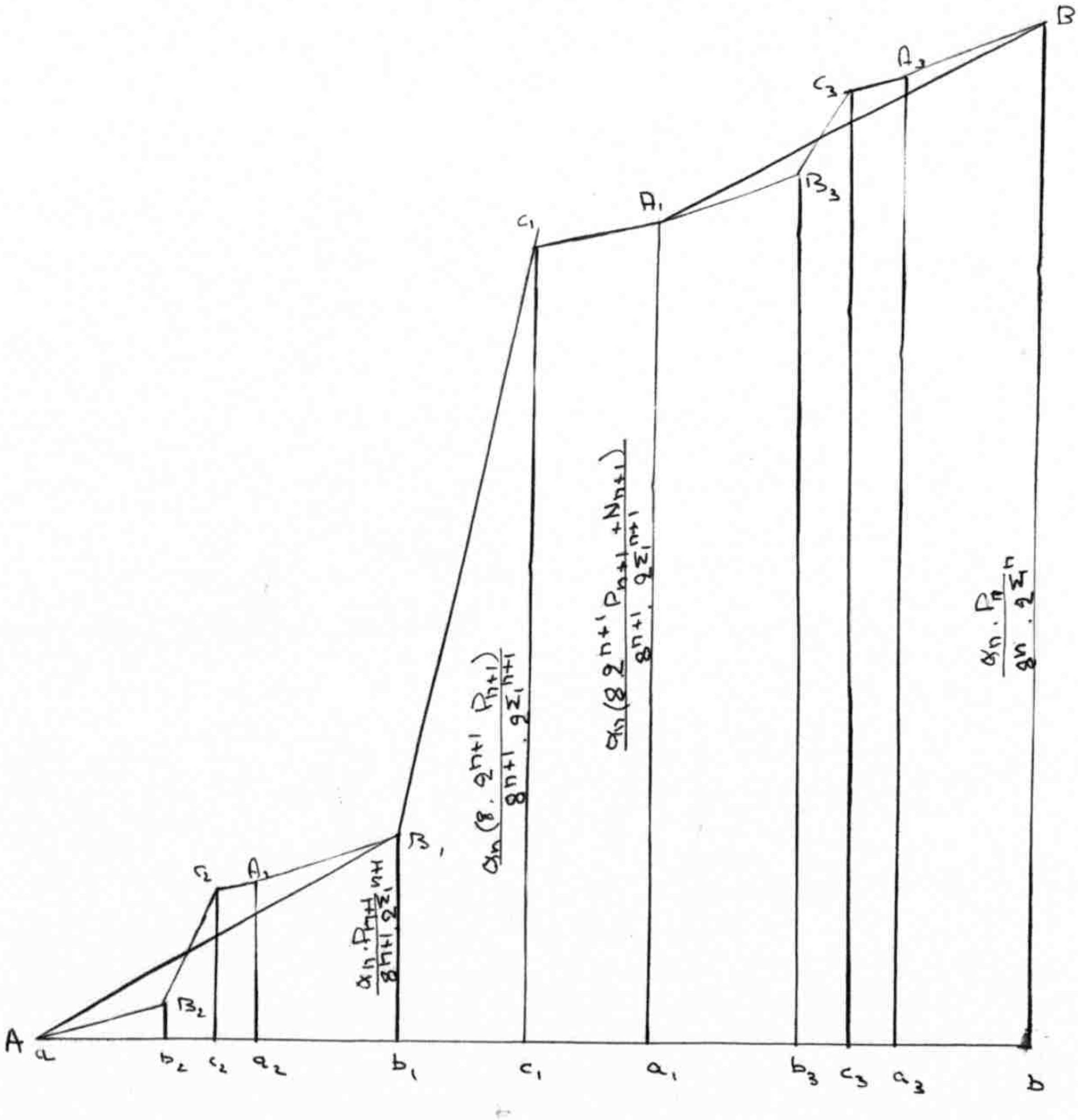

FIG I

GRAPHICAL REPRESENTATION OF $\psi_1(x)$ (not to scale)

$$X_{M_r} = \frac{P_{M_{r-1}+1}}{4^{M_{r-1}+1}} + \frac{P_{M_{r-2}+2}}{4^{M_{r-1}+2}} + \dots + \frac{P_{M_r}}{4^{M_r}}$$

where

$$P_{M_{r-1}+k} = 0 \quad \text{or} \quad 2^k + 3$$

and

$$P_{M_r} = 2^r+1 \quad \text{or} \quad 2^r + 2 \; .$$

Using the above notations, we can represent all points in $(0,1)$ as follows:

$$x = \tfrac{1}{2}(X_{M_1} + \tfrac{1}{2}(X_{M_2} + \dots(\dots(\dots + \tfrac{1}{2}(X_{M_r} + \dots(\dots(\dots$$

where the number of brackets may be finite or infinite. Corresponding to such an x, we define $\psi_1(x)$ and $\psi_2(x)$.

$$\psi_1(x) = (U_{M_1} + \alpha_{M_1}(U_{M_2} + \dots(\dots(\dots + \alpha_{M_{r-1}}(U_{M_r} + \dots(\dots(\dots$$

and

$$\psi_2(x) = (V_{M_1} + \alpha_{M_1}(V_{M_2} + \dots(\dots(\dots + \alpha_{M_{r-1}}(V_{M_r} + \dots(\dots(\dots$$

where

$$\alpha_{M_r},\ U_{M_r},\ V_{M_r} \quad \text{corresponding to} \quad X_{M_r} = \frac{P_{M_{r-1}+1}}{4^{M_{r-1}+1}} + \frac{P_{M_{r-1}+2}}{4^{M_{r-1}+2}} + \dots + \frac{P_{M_r}}{4^{M_r}}$$

are given by

$$\alpha_{M_r} = 8.2^{M_r} - 1 \quad \text{if} \quad P_{M_r} = 2^{m_r} + 1$$

$$= 1 \qquad \text{if} \quad P_{M_r} = 2^{m_r} + 2$$

and

$$U_{M_r} = \frac{s_{M_{r-1}+1}}{8^{M_{r-1}+1} \cdot 2^{\Sigma_1^{M_{r-1}+1}}} + \frac{s_{M_{r-1}+2}}{8^{M_{r-1}+2} \cdot 2^{\Sigma_1^{M_{r-1}+2}}} + \dots + \frac{s_{M_r}}{8^{M_r} \cdot 2^{\Sigma_1^{M_r}}}$$

and

$$V_{M_r} = \frac{t_{M_{r-1}+1}}{8^{M_{r-1}+1} \cdot 2^{\Sigma_1^{M_{r-1}+1}}} + \frac{t_{M_{r-1}+2}}{8^{M_{r-1}+1} \cdot 2^{\Sigma_1^{M_{r-1}+2}}} + \dots + \frac{t_{M_r}}{8^{M_r} \cdot 2^{\Sigma_1^{M_r}}}$$

where

$$\Sigma_1^{M_{r-1}+h} \quad \text{denotes} \quad \sum_{n=1}^{n=M_{r-1}+h} n \; .$$

Similarly to $X_{M_{r-1},\infty}$, there will correspond $U_{M_{r-1},\infty}$ and $V_{M_{r-1},\infty}$. The values $s_{M_{r-1}+h}$ and $t_{M_{r-1}+h}$, S_{M_r}, T_{M_r} depend on $P_{M_{r-1}}$ and on whether

$$[\sum_{\lambda=1}^{r-1} P_{M_\lambda} + (r-1)];$$

is even or odd. For simplicity, they will be given by the following table:

TABLE I

$[\sum_{\lambda=1}^{r-1} P_{M_\lambda} + (r-1)]$	$P_{M_{r-1}+k}$	$S_{M_{r-1}+k}$	$t_{M_{r-1}+k}$
Even	0 2^k+3	0 $8.2^{M_{r-1}+k} . P_{M_{r-1}+k}$ $+ N_{M_{r-1}+k}$	0 $8.2^{M_{r-1}+k} . N_{M_{r-1}+k}$ $+ P_{M_{r-1}+k}$
Odd	0 2^h+3	0 $8.2^{M_{r-1}+k} . N_{M_{r-1}+k}$ $+ P_{M_{r-1}+k}$	0 $8.2^{M_{r-1}+k} . P_{M_{r-1}+k}$ $+ N_{M_{r-1}+k}$
$[\sum_{\lambda=1}^{r-1} P_{M_\lambda} + (r-1)]$	P_{M_r}	S_{M_r}	t_{M_r}
Even	$2^{m_r}+1$ $2^{m_r}+2$	P_{M_r} $8.2^{M_r} . P_{M_r}$	N_{M_r} $8.2^{M_r} . N_{M_r}$
Odd	$2^{m_r}+1$ $2^{m_r}+2$	N_{M_r} $8.2^{M_r} . N_{M_r}$	P_{M_r} $8.2^{M_r} . P_{M_r}$

Now $f(x)$ will be given by

$$f(x) = \psi_1(x) - \psi_2(x) .$$

In the following sections we show that $\psi_1(x), \psi_2(x)$ are the contravariation functions, and using this fact with some results given in the following section we shall prove that $f(x)$ is a non monotonic singular function.

§ 2 Contravariation Functions

In accordance with the definition given by Kober [1][2] two functions $g(x)$ and $h(x)$ $(a \leq x \leq b)$ are said to be the contravariation functions (CAV's) of $f(x)$ if

$$f(x) = g(x) - h(x)$$

and

$$V_a^b f(x) = V_a^b g(x) + V_a^b h(x) = g(b) + h(b)$$

$g(a), h(a)$ are assumed to be zero.

The problem of determining whether two functions $g(t)$ and $h(t)$ are contravariation functions, or not has been solved partly by Kober [3]. He has given the following criterion:

The function $g(t)$ and $h(t)$ are CAV's in $(0 \leq t \leq 1)$ if and only if

---

[2] See Chapter 1, p.

(i) $g(t)$ and $h(t)$ are monotone non decreasing and $g(0) = h(0) = 0$.

(ii) Given $\varepsilon > 0$, $\exists$ a finite set of disjoint closed intervals $\langle t_K, T_K \rangle$ such that

$$\Sigma \; g(T_K) - g(t_K) \quad > g(1) - \varepsilon$$

and

$$\Sigma \; h(T_K) - h(t_K) \quad < \varepsilon.$$

It may be noted here that since $g(t)$ and $h(t)$ are monotone, then $g(1)$, $h(1)$ are the total variation of $g(t)$ and $h(t)$ respectively in $(0,1)$.

The CAV's of a non monotonic singular function have interesting properties. The following results have been proved by Shukla .

Theorem 1: A necessary and sufficient condition that a function of bounded variation be a singular function is that its contravariation functions are themselves singular.

Theorem 2: A necessary and sufficient condition that a function of bounded variation be non monotonic in every subinterval is that its both contravariation functions are strictly increasing functions.

From these two theorems, it follows that to have $F(x)$ a non monotonic singular function, it is necessary and sufficient that its contravariation functions are strictly increasing singular functions.

§3 f(x) is a Non Monotonic Singular Function

It can be shown that $\Psi_1(x), \Psi_2(x)$ are the contravariation functions of $f(x)$, by actually finding disjoint intervals (t_K, T_K) which satisfy the criterion given by Kober. Using this fact, we now prove that $f(x)$ is a non monotonic singular function by showing that $\Psi_1(x), \Psi_2(x)$ are strictly increasing singular functions.

$\Psi_1(x), \Psi_2(x)$ are strictly increasing

Let x have the representation

$$x = \frac{1}{2}\,(X_{M_1} + \frac{1}{2}\,(X_{M_2} + \ldots(\ldots + \frac{1}{2}\,(X_{M_r} + \ldots(\ldots(\ldots$$

where

$$X_{M_r} = \frac{P_{M_{r-1}+1}}{4^{M_{r-1}+1}} + \frac{P_{M_{r-1}+2}}{4^{M_{r-1}+2}} + \ldots + \frac{P_{M_{r-1}+\lambda}}{4^{M_{r-1}+\lambda}} + \ldots$$

Let $x' > x$, then x' is given by

$$x' = \frac{1}{2}(X_{M_1} + \frac{1}{2}(X_{M_2} + \ldots(\ldots + \frac{1}{2}(X'_{M_r} + \ldots(\ldots(\ldots$$

where

$$X'_{M_r} = \frac{P_{M_{r-1}+1}}{4^{M_{r-1}+1}} + \ldots + \frac{P'_{M_{r-1}+\lambda}}{4^{M_{r-1}+\lambda}} + \ldots$$

where

$$P'_{M_{r-1}+\lambda} > P_{M_{r-1}+\lambda} .$$

We can easily see from the table that

$$S'_{M_{r-1}+\lambda} > S_{M_{r-1}+\lambda}$$

which proves that

$$\Psi_1(x') > \Psi_1(x).$$

Similarly we can show for $x' > x$

$$\Psi_2(x') > \Psi_2(x).$$

From this it follows that $f(x) = \Psi_1(x) - \Psi_2(x)$ is non monotonic, and is of bounded variation. To show that $f(x)$ is singular it is sufficient to show that one of its derivatives is zero almost everywhere. For this purpose, let x have the following representation.

$$x = \frac{1}{2}(X_{M_1} + \frac{1}{2}(X_{M_2} + \ldots(\ldots + \frac{1}{2}(X_{M_{r-1},\infty})\ldots)$$

where

$$X_{M_{r-1},\infty} = \frac{P_{M_{r-1}+1}}{4^{M_{r-1}+1}} + \frac{P_{M_{r-1}+2}}{4^{M_{r-1}+2}} + \ldots + \frac{P_{M_{r-1}+\lambda}}{4^{M_{r-1}+\lambda}} + \ldots$$

where $P_{M_{r-1}+\lambda} = 0$ or $2^{\lambda}+3$.

Such points form a set whose measure is 1.

We will show that one of the left derivatives is zero at such points. Let $x' < x$ then

$$x' = \frac{1}{2}(X_{M_1} + \frac{1}{2}(X_{M_2} + \ldots + (\ldots + \frac{1}{2}(X'_{M_{r-1},\infty}) \ldots)$$

where

$$X'_{M_{r-1},\infty} = \frac{P_{M_{r-1}+1}}{4^{M_{r-1}+1}} + \frac{P_{M_{r-1}+2}}{4^{M_{r-1}+2}} + \ldots + \frac{P'_{M_{r-1}+\lambda}}{4^{M_{r-1}+\lambda}} + \ldots$$

where $P'_{M_{r-1}+\lambda} = 0$ whereas $P_{M_{r-1}+1} = 2^{\lambda}+3$.

We have

$$\frac{f(x)-f(x')}{x-x'} = (a_{M_1}\cdots a_{M_{r-1}})\left[\frac{q'_{M_{r-1}+\lambda}}{8^{M_{r-1}+\lambda}\,2^{\Sigma_1^{M_{r-1}+\lambda}}} - \frac{q_{M_{r-1}+\lambda}}{8^{M_{r-1}+\lambda}\,2^{\Sigma_1^{M_{r-1}+\lambda}}}\right] - \frac{1}{2}\left[\frac{2^{\lambda}+3}{4^{M_{r-1}+\lambda}}\right]$$

where $q_{M_{r-1}+k} = S_{M_{r-1}+h} - t_{M_{r-1}+k}$

$$\text{or } \frac{f(x)-f(x')}{x-x'} = \frac{2^{r}.4^{M_{r-1}+\lambda}\,(a_{M_1}\cdots a_{M_{r-1}})[\pm\,(8.2^{M_{r-1}+\lambda}-1)]}{8^{M_{r-1}+\lambda}.2^{\Sigma_1^{M_{r-1}+\lambda}}[-(2^{\lambda}+3)]}$$

$$\lim_{x'\to x}\frac{f(x)-f(x')}{x-x'} = \lim_{\lambda\to\infty}\frac{f(x)-f(x')}{x-x'} = 0 .$$

This shows that $f'(x) = 0$ everywhere except possibly on a set of zero measure. Hence we have shown that $f(x)$ is a non monotonic singular function.

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