STUDY OF THE INTEGRAL EQUATION OF $K^{\pm} \rightarrow 3\pi$ DECAY

By

Fuad T. Abbud

Submitted in Partial Fulfillment for the Requirements
of the Degree Master of Science
in the Physics Department of the
American University of Beirut
Beirut, Lebanon,
1962

$K^{\pm} \rightarrow 3\pi$ DECAY

Fuad T. Abbud

ACKNOWLEDGEMENTS

I would like to express my gratitude to Prof. N. N. Khuri for supervising my thesis. His advice and support at difficult times were a constant source of inspiration and fortitude for me. To him I owe all the enjoyment that I have derived out of this work.

I am also grateful to Prof. A. Zahlan for his wise and philosophic counsel from time to time. My thanks are also due to Miss Mona Jabbour for typing the manuscript.

ABSTRACT

The purpose of this thesis is to solve the singular integral equation for the decay amplitude of the $K^{\pm} \rightarrow 3\pi$ decay. This is done by reducing the equation to a Hilbert type problem.

In chapter I, we discuss and solve a boundary value problem of the theory of analytic functions known as the Hilbert problem. In chapter II, a class of singular integral equations with singular kernels of the Cauchy type are presented. This type, known as the Omnes type, is important in scattering theory. The method by which the Omnes type equation may be reduced to a Hilbert problem is given.

Chapter III is devoted to a discussion of the kinematics of the $K^{\pm} \rightarrow 3\pi$ decay and to the presentation of the equation for the decay amplitude. This equation is solved in chapter IV after performing necessary approximations. Also, a comparison of the solution with some of the available experimental data is presented.

TABLE OF CONTENTS

CHAPTER I

THE HILBERT PROBLEM (1)

1. Introduction

In this chapter, a boundary value problem of the theory of analytic functions will be defined and solved. The problem, known as the Hilbert problem, is to find a analytic function satisfying certain boundary conditions along a contour or arc. Such problems have important applications in the theory of elasticity and the theory of singular integral equations.

The problem considered below was first studied by Hilbert (1) who solved it by reducing it to a Fredholm integral equation. The method used by Hilbert was complicated. However, it is possible to obtain the solution in a straightforward manner by the use of Cauchy integrals. This method was first used by Plemelj (1) in his study of the Hilbert problem. His treatment was less general than the treatment given below. The general method is due to Gakhov (1), and it is this method that is presented here.

The solution of singular integral equations and the Hilbert problem are closely related. In fact, the

Hilbert problem provides an easy and elegant method of solving such equations. The method of solving these equations by use of the Hilbert problem will be given in chapter II.

2. Definition of the Hilbert Problem

i) The Homogeneous Hilbert problem:

The problem is to find an analytic function $\phi(z)$, of finite degree at infinity, such that,

$$\phi^{+}(x) = G(x)\, \phi^{-}(x), \; x \in L , \tag{1.1}$$

where

$$\phi^{+}(x) = \lim_{\varepsilon \to 0} \phi(x + i\varepsilon), \; x \in L , \tag{1.2}$$

and

$$\phi^{-}(x) = \lim_{\varepsilon \to 0} \phi(x - i\varepsilon), \; x \in L . \tag{1.3}$$

L is the x-axis from 1 to infinity.* Thus $\phi^{+}(x)$ and $\phi^{-}(x)$ are the values of $\phi(z)$ as z approaches L from above and below L respectively; G(x) is a function that vanishes nowhere on L and satisfies the following condition on L

---

* The branch line L need not be the x-axis. In general, it may be any smooth contour or arc or any number of non-intersecting smooth contours or arcs.

$$|G(x_1) - G(x_2)| \leq A|x_1 - x_2|^{\mu}, \tag{1.4}$$

where A is a positive constant and $0 < \mu < 1$.

From (1.1), it follows that the function $\phi(z)$ has a discontinuity all along L since the limit of $\phi(z)$ as the x-axis $(1 < x < \infty)$ is approached from above is different from the limit as it is approached from below. It is said that $\phi(z)$ has a branch line or a cut along L.

ii) The non-homogeneous Hilbert problem:

Here, the problem is to find an analytic function $\phi(z)$ of finite degree at infinity satisfying the boundary condition

$$\phi^+(x) = G(x)\,\phi^-(x) + g(x), \quad x \in L, \tag{1.5}$$

where G(x) and g(x) are given functions that vanish at infinity, G(x) being nowhere zero on L. $\phi^+(x)$ and $\phi^-(x)$ have the same meaning as above.

3. The Plemelj Formula*

Let the function F(z) be defined by

$$F(z) = \frac{1}{2\pi i} \int_1^{\infty} \frac{f(x')dx'}{x' - z}, \tag{1.6}$$

---

*Strictly speaking, the relation given here is not the plemelj formula, but a relation that follows from it.

where f(x) is a function that vanishes at infinity. The function F(z) defined by (1.6) will be an analytic function regular in the whole z-plane excluding the branch cut on the x-axis extending from $x = 1$ to $x = \infty$. Also we have $F(z) \to 0$ as $|z| \to \infty$.

We defined the boundary values $F^+(x)$ and $F^-(x)$ of F(z) by

$$F^+(x) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_L \frac{f(x')dx'}{x' - x - i\varepsilon}, \qquad (1.7)$$

and

$$F^-(x) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_1^\infty \frac{f(x')dx'}{x' - x + i\varepsilon}. \qquad (1.8)$$

$F^+(x)$ is the value F(z) approaches as z approaches the x-axis (for $1 < x < \infty$) from above, and $F^-(x)$ as z approaches from below.

The Plemelj formula states that

$$F^+(x) - F^-(x) = f(x), \quad x > 1. \qquad (1.9)$$

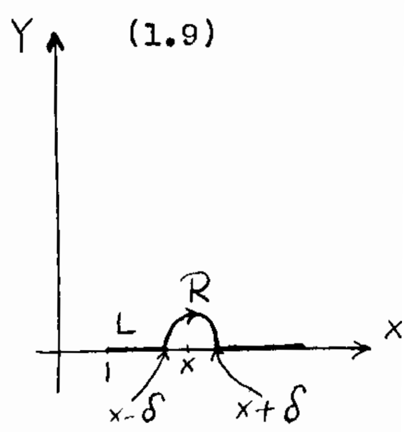

Fig. 1

To show this, the countour is broken up into 3 parts: $1 \le x' < x - \delta$, $x + \delta < x' < \infty$ and a circle R connecting $x - \delta \le x' \le x + \delta$ and enclosing the point of singularity $x + i\varepsilon$, which is thus avoided. Then

$$2\pi i\ F^{+}(x) = \lim_{\substack{\varepsilon\to o\\ \delta\to o}} \left[\int_{o}^{x-\delta} \frac{f(x')dx'}{x' - x - i\varepsilon} + \int_{x+\delta}^{\infty} \frac{f(x')dx'}{x'-x - i\varepsilon} + \int_{R} \frac{f(z')dz'}{z'-x - i\varepsilon} \right. \tag{1.10}$$

$$= \lim_{\delta\to o} \left[\int_{o}^{x-\delta} \frac{f(x')dx'}{x' - x} + \int_{x+\delta}^{\infty} \frac{f(x')dx'}{x' - x} + \int_{R} \frac{f(z')dz'}{z' - x} \right] . \tag{1.11}$$

The first two integrals form the principal value of the integral at x; this is denoted by $P \int \frac{f(x')dx'}{x' - x}$. The third integral may be evaluated by use of the Cauchy Residue Theorem, thus in the limit as $R \to 0$

$$\int_{R} \frac{f(z')dz'}{z' - x} = i\pi\ f(x) , \tag{1.12}$$

and therefore

$$2\pi i\ F^{+}(x) = P \int \frac{f(x')dx'}{x' - x} + i\pi\ f(x) . \tag{1.13}$$

By a similar procedure, it can be shown that

$$2\pi i\ F^{-}(x) = P \int \frac{f(x')dx'}{x' - x} - i\pi\ f(x) . \tag{1.14}$$

Subtracting (1.14) from (1.13) yields the result

$$2\pi i\ [\ F^{+}(x) - F^{-}(x)] = 2\pi i\ f(x) , \tag{1.15}$$

or

$$F^{+}(x) - F^{-}(x) = f(x), \tag{1.16}$$

which is the Plemelj result.

4. Solution of the Hilbert Problem

The Plemelj formula (1.9) can be used to solve both Hilbert problems (1.1) and (1.4).

The Homogeneous problem:

Taking the logarithm of both sides of (1.1) we obtain

$$\log \phi^{+}(x) = \log \phi^{-}(x) + \log G(x) , \tag{1.17}$$

or

$$\log \phi^{+}(x) - \log \phi^{-}(x) = \log G(x). \tag{1.18}$$

Using the Plemelj formula, we get

$$\log \phi(z) = \frac{1}{2\pi i} \int_L \frac{\log G(t)dt}{t - z} = \Gamma(z), \tag{1.19}$$

and therefore

$$\phi(z) = e^{\Gamma(z)} . \tag{1.20}$$

Since $\Gamma(z)$ goes to zero at infinity, $\phi(z)$ approaches 1 as $|z| \to \infty$.

To find a solution of degree k at infinity, it is sufficient to take any polynomial of degree k multiplied by ϕ, this is so since a polynomial function has no branch cuts. Thus

$$\phi_k(z) = P_k(z).\ \phi(z). \tag{1.21}$$

where $P_k(z)$ is a polynomial of degree k, is the required solution of degree k at infinity.

Note that the analyticity of ϕ is guaranteed by the existence of the derivative as ϕ is being defined in terms of an integral.

The non-homogeneous Hilbert problem:

Let $\chi(z)$ be the solution of the homogeneous problem corresponding to (1.4); that is

$$\chi^+(x) = G(x)\chi^-(x), \tag{1.22}$$

or

$$G(x) = \frac{\chi^+(x)}{\chi^-(x)}. \tag{1.23}$$

By substituting (1.23) in (1.4), one gets

$$\phi^+(x) = \frac{\chi^+(x)}{\chi^-(x)}\phi^-(x) + g(x), \tag{1.24}$$

or

$$\frac{\phi^+(x)}{\chi^+(x)} - \frac{\phi^-(x)}{\chi^-(x)} = \frac{g(x)}{\chi^+(x)}. \tag{1.25}$$

From (1.25) and by the Plemelj formula, the function $\frac{\Phi(z)}{X(z)}$ is given by

$$\frac{\Phi(z)}{X(z)} = \frac{1}{2\pi i} \int_L \frac{g(t)dt}{X^+(t)(t.z)} + P_k(z), \qquad (1.26)$$

where $P_k(z)$ is any polynomial function of degree k. This makes the solution of degree k at infinity.

Thus, the solution of the non-homogeneous problem is

$$\Phi(z) = \frac{X(z)}{2\pi i} \int_L \frac{g(t)dt}{X^+(t)(t-z)} + P_k(z) X(z), \qquad (1.27)$$

where

$$X(z) = e^{\Gamma(z)}, \quad \Gamma(z) = \frac{1}{2\pi i} \int_L \frac{\log G(t)dt}{t - z}. \qquad (1.28)$$

Note that (1.27) is made up of two terms, the first being a solution of the non-homogeneous problem and the second being the general solution of the homogeneous problem.

The analyticity of the solution (1.27) is again guaranteed by the existence of the derivative.

CHAPTER II

THE SINGULAR INTEGRAL EQUATION OF THE OMNES TYPE (1)

1. Introduction

In this chapter, we shall study a certain class of singular integral equations. This class has kernels of the Cauchy type and hence possesses singularities of the first order. In general these equations take the form

$$A(x)\,\phi(x) + \frac{1}{\pi}\int_L \frac{K(x,x')\phi(x')}{x' - x}\,dx' = f(x). \qquad (2.1)$$

Where A(x), F(x) and K(x,x') are given functions that are bounded and analytic. L is a smooth contour (or arc), or a number of smooth non-intersecting contours (or arcs) and x is taken to approach L from either side.

The application of dispersion relation techniques to problems of both Weak and Strong Interactions leads in many cases to an integral equation which is a special case of (2.1), namely,

$$\phi(x) = f(x) + \frac{1}{\pi}\int_1^\infty \frac{h\,(x')\,\phi(x')}{x' - x - i\epsilon}\,dx'\,. \qquad (2.2)$$

Here $f(x)$ is a given bounded function and $h(x) = e^{i\delta(x)} \sin \delta(x)$ where $\delta(x)$ is a phase shift for some scattering process, and $\phi(x)$ is a scattering amplitude, decay amplitude or form factor which one desires to find knowing $\delta(x)$ and $f(x)$. The equation (2.2) was first studied in detail by Omnes (1).

Equation (2.1) is quite similar in appearance to the classical Fredholm integral equation. However, because of the singularity of the kernel of (2.1), the methods usually used in solving the Fredholm equation are not applicable here. Instead, the equation will be solved by reducing it to a Hilbert problem.

In section 2 the solution of (2.2) will be given and in section 3 the assymptotic behaviour of the solution will be discussed.

2. Solution of the Omnes Equation

We define the analytic function $F(z)$ by,

$$F(z) = \frac{1}{2\pi i} \int_L \frac{h^*(x')\, \phi(x')}{x' - z}\, dx' , \qquad (2.3)$$

Where L denotes the cut $(1 < x < \infty)$. $F(z)$ is regular in the whole z-plane except for the cut.

As before, we denote the limit from above or below by $F^+(x)$ or $F^-(x)$. We have

$$F^{\pm}(x) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_L \frac{h^*(x')\,\phi(x')}{x' - x \mp i\epsilon}\,dx' . \qquad (2.4)$$

Using the Plemelj formula (1.9) we get

$$F^{+}(x) - F^{-}(x) = h^*(x)\,\phi(x) , \qquad (2.5)$$

or

$$\phi(x) = \frac{1}{h^*(x)}\,[F^{+}(x) - F^{-}(x)] . \qquad (2.6)$$

If we substitute this expression for $\phi(x)$ in (2.2), we get

$$\frac{1}{h^*(x)}\,[F^{+}(x) - F^{-}(x)] = f(x) + 2i\,F^{+}(x) , \qquad (2.7)$$

or

$$[1 - 2ih^*(x)]\,F^{+}(x) - F^{-}(x) = h^*(x)\,f(x) . \qquad (2.8)$$

This is of the same form as (1.4), the non-homogeneous Hilbert problem.

The coefficient of $F^{+}(x)$ in (2.8) can be written as

$$1 - 2ih^*(x) = 1 - 2ie^{-i\delta}\sin\delta = e^{-2i\delta} . \qquad (2.9)$$

The Hilbert problem then becomes,

$$e^{-2i\delta(x)}F^{+}(x) - F^{-}(x) = e^{-i\delta(x)}\sin\delta(x)\,f(x). \qquad (2.10)$$

Using the results of the previous chapter, the function $F(z)$ can be expressed as

$$F(z) = \frac{\chi(z)}{2\pi i} \int_L \frac{f(x')e^{-i\delta(x')} \sin \delta(x')}{\chi^+(x')(x' - z)} dx'$$

$$+ P_k(z)\chi(z) , \qquad (2.11)$$

where the function $\chi(z)$ is the solution of the homogeneous problem corresponding to (2.10), i.e. $\chi(z)$ satisfies the equation

$$e^{-2i\delta} \chi^+(x) - \chi^-(x) = 0 , \qquad (2.12)$$

and is therefore given by [see (1.20)],

$$\chi(z) = e^{\Gamma(z)}, \Gamma(z) = \frac{1}{\pi} \int_L \frac{\delta(x')dx'}{x' - z} . \qquad (2.13)$$

$P_k(z)$ in (2.11) is an arbitrary polynomial of degree k. The second term in (2.11), $P_k(z)\chi(z)$, is the general solution of the homogeneous equation. Thus, again the solution is made up of two terms, the first being a solution of the non-homogeneous equation and the second a general solution of the homogeneous one.

Finally, the solution of equation (2.2) may be obtained from F(z) by use of the relation

$$\phi(x) = f(x) + 2i\ F^+(x) \qquad (2.14)$$

$$= f(x) + \frac{\chi^+(x)}{\pi} \int_1^\infty \frac{f(x')e^{i\delta(x')}\sin \delta(x')dx'}{\chi^+(x')(x' - x - i\epsilon)} .$$

3. Assymptotic Behaviour

In section 2 we have shown that solving the Omnes equation involves solving the homogeneous Hilbert problem

$$\chi^+(x) - e^{2i\delta_0(x)}\chi^-(x) = 0 \ , \qquad (2.12)$$

whose solution is

$$\chi(z) = e^{\Gamma(z)}, \Gamma(z) = \frac{1}{\pi} \int_L \frac{\delta_0(t)dt}{t - z} \ . \qquad (2.13)$$

The real part of $\Gamma(z)$ is given by

$$R_e\Gamma^{\pm}(x) = I(x) = P\,\frac{1}{\pi} \int_a^\infty \frac{\delta_0(t)dt}{t - x} \ , \qquad (2.15)$$

where a = 4.

Thus the question of the convergence of integrals of the form (2.15) becomes important. In this section we shall study briefly the conditions under which (2.15) will converge.

For I(x) to be finite for values of x > a, it is necessary for the function $\delta_0(t)$ to satisfy the following condition, usually known as the Lipschitz condition

$$|\delta_0(t_2) - \delta_0(t_1)| \leq A|t_2 - t_1|^\mu \ , \qquad (2.16)$$

where A and u are positive constants and $0 < \mu < 1$

Let $\delta_0(t)$ be a function which approaches zero at infinity as $\frac{1}{t^\epsilon}$, $\epsilon > 0$. Then it is an elementary matter to show that the integral (2.15) is convergent for all values of x, $x \neq a$. At $x = a$, $\delta_o(t)$ must approach zero as t^{ε}, $\varepsilon > 0$ to ensure the convergence of $I(x)$.

Frequently it happens that the function δ_o does not approach zero at infinity. In such a case, $I(x)$ is divergent. However, it is still possible to find a solution of problem (2.12) as we shall show now.

Let $\delta_o(t)$ be of order $t^{1-\varepsilon}$ as $t \to \infty$, $\varepsilon > 0$. Then

$$\chi'(z) = e^{\Gamma'(z)} , \tag{2.17}$$

where

$$\Gamma'(z) = \frac{z - x_o}{\pi} \int_a^{\infty} \frac{\delta_o(t)dt}{(t-x_o+i\varepsilon)(t-z)} , \quad x_o \in L , \tag{2.18}$$

is a convergent solution of (2.12). This can be shown as follows:

$$\Gamma'^{+}(x) = \lim_{\varepsilon \to o} \frac{x + i\varepsilon - x_o}{\pi} \int_a^{\infty} \frac{\delta_o(t)dt}{(t-x_o+i\varepsilon)(t-x-i\varepsilon)} \tag{2.19}$$

$$\Gamma'^{-}(x) = \lim_{\varepsilon \to o} \frac{x - i\varepsilon - x_o}{\pi} \int_a^{\infty} \frac{\delta_o(t)dt}{(t-x_o+i\varepsilon)t-x+i\varepsilon)} , \tag{2.20}$$

and therefore

$$\Gamma'^{+}(x) - \Gamma'^{-}(x) = 2i\delta_o(x) , \tag{2.21}$$

or

$$\Gamma'^{+}(x) - e^{2i\delta_o(x)} \Gamma'^{-}(x) = 0 , \tag{2.22}$$

i.e. $\chi'(z)$ is a solution of (2.12).

Now, since $\delta_0(t) \to t^{1-\varepsilon}$ as $t \to \infty$, it is easy to show that the integral (2.18) converges. Solution (2.18) is known as the subtracted form of the solution of the Hilbert problem.

It is obvious that if $\delta_0(t)$ approaches infinity faster than $t^{1-\varepsilon}$, then a higher order subtraction can be performed. As long as $\delta_0(t)$ grows as a polynomial at infinity it is possible, by doing a subtraction of a suitable order, to find a solution of (2.12).

CHAPTER III

INTEGRAL EQUATION FOR $K^{\pm} \to 3\pi$ DECAY

1. Introduction

Khuri and Treiman (1) recently used the dispersion relation methods to study the effects of final state pion-pion interactions on the spectrum of $K^{\pm} \to 3\pi$ decay. The approximations adopted in their paper lead to a set of linear singular integral equations for the amplitudes of $K^{\pm} \to 3\pi$ decay. The kernels in these equations depend on the pion-pion S-wave scattering amplitudes. The equations obtained are similar in structure to the ones discussed in the previous chapter, but are unfortunately much more complicated to solve. Before writing down these integral equations, we shall, in the next section, discuss briefly the kinematics of $K^{\pm} \to 3\pi$ decay.

2. Kinematics

Let us denote the three emerging pions by the letters a,b,c, and let k_a, k_b, k_c be their respective 4-momenta, with,

$$k_a^2 = k_b^2 = k_c^2 = -\mu^2, \tag{3.1}$$

where μ is the pion mass*. Similarly, we denote by the 4-momentum of the K-meson and $K^2 = - m^2$.

We have three amplitudes A, B, and C corresponding to the three possible T = 1 isotopic spin final states. The amplitudes A, B and C are functions of the scalar variables.

$$\begin{aligned} S_a &= - (K - k_a)^2 \\ S_b &= - (K - k_b)^2 \\ S_c &= - (K - k_c)^2 . \end{aligned} \qquad (3.2)$$

Only two of these variables are independent since energy - momentum conservation, $K = k_a + k_b + k_c$, implies that

$$S_a + S_b + S_c = m^2 + 3\,\mu^2 . \qquad (3.3)$$

It is however convenient to use all three variables and write $A = A(S_a, S_b, S_c)$, and similarly for B and C.

The variables S_a, S_b, S_c are simply related to the energies of the a, b and c pions in the K-meson rest system,

$$\begin{aligned} S_a &= m^2 + 3\mu^2 - 2mw_a \\ S_b &= m^2 + 3\mu^2 - 2mw_b \\ S_c &= m^2 + 3\mu^2 - 2mw_c \ , \end{aligned} \qquad (3.4)$$

where w_i is the energy of the ith pion.

The functions A, B, and C are related by the symme-

---

*We use the units where $\hbar = c = 1$.

trics imposed by the Bose-Einstein statistics of the three pion system,

$$S_b \leftrightarrow S_c \; ; \; S_a \leftrightarrow S_a \; : \; B \leftrightarrow C, \; A \leftrightarrow A$$

$$S_a \leftrightarrow S_c \; ; \; S_b \leftrightarrow S_b \; : \; A \leftrightarrow C, \; B \leftrightarrow B \qquad (3.5)$$

$$S_a \leftrightarrow S_b \; ; \; S_c \leftrightarrow S_c \; : \; A \leftrightarrow B, \; C \leftrightarrow C.$$

Finally, we shall write down the relation between the amplitudes A, B, and C and the squared matrix element for the decay. There are two modes of decay of the K^+ meson,

τ- mode $\qquad K^+ \rightarrow \pi^+ + \pi^+ + \pi^-$,

τ'- mode $\qquad K^+ \rightarrow \pi^+ + \pi^0 + \pi^0$.

The squared matrix element for the τ- mode is given by

$$|M_\tau|^2 = |A(S_1, S_2, S_3) + B(S_1, S_2, S_3)|^2, \qquad (3.6)$$

Where S_1 and S_2 refer to the two π^+ mesons and S_3 to the π^- meson. Similarly for the τ'- mode

$$|M_{\tau'}|^2 = |C(S_1, S_2, S_3)|^2, \qquad (3.7)$$

where again S_3 refers to the unlike (π^+) pion.

In the usual manner integration of $|M_\tau|^2$ or $|M_{\tau'}|^2$, after multiplication with the phase space factors, will lead to the energy spectrum.

It can be easily shown that in the K-meson rest system the maximum pion kinetic energy is given by

$$T = \frac{(m - \mu)^2 - 4\mu^2}{2m} = 50 \text{ Mev} . \tag{3.8}$$

This corresponds to a maximum energy of

$$W = T + \mu = \frac{m^2 - 3\mu^2}{2m} . \tag{3.9}$$

Using (3.4) we have

$$W_a = \frac{m^2 + \mu^2 - S_a}{2m} . \tag{3.10}$$

The minimum value of S_a is $4\mu^2$ and this immediately leads to (3.9).

In the calculations related to our integral equation it turns out to be sometimes convenient to work in the reference frame where $\vec{k}_b + \vec{k}_c = 0$. Relative to that frame of reference we express A, B, and C as functions of S_a and $\cos \theta_{bc}$, where θ_{bc} is the angle between $\vec{k}_a$ and $\vec{k}_b = - \vec{k}_c$. In that case one needs an expression for $\cos \theta_{bc}$ in terms of S_a, S_b, and S_c. We shall obtain such a relation below and we shall also obtain a relation between S_a and the magnitude k of the vectors $\vec{k}_b$ and $\vec{k}_c$ We have

$$S_a = - (K - k_a)^2 = -(k_b + k_c)^2$$

$$= +(k^\circ{}_b + k^\circ{}_c)^2 . \tag{3.11}$$

The last equality follows from the fact that we are using the $\vec{k}_b + \vec{k}_c = 0$ reference frame. We also have that

$$k^o{}_b = k^o{}_c \ , \tag{3.12}$$

and

$$k^o{}_b{}^2 = + \mu^2 + k^2 \ , \tag{3.13}$$

where

$$k = |\vec{k}_b| = |\vec{k}_c| .$$

Substituting (3.12) and (3.13) in (3.11), we get

$$S_a = 4\mu^2 + 4k^2 \ ,$$

or

$$k = \sqrt{\frac{S_a}{4} - \mu^2} \ , \tag{3.14}$$

which is the desired relation for k.

The relation for $\cos \theta_{bc}$ in terms of S_a, S_b and S_c may be obtained in the following manner

$$S_b = - (K - k_b)^2$$

$$= m^2 + \mu^2 - 2K^o \ k^o{}_b - 2|\vec{K}| \ k \cos \theta_{bc} \ , \tag{3.15}$$

$$S_c = - (K - k_c)^2$$

$$= m^2 + \mu^2 - 2K^o \ k^o{}_b + 2|\vec{K}| \ k \cos \theta_{bc} \ . \tag{3.16}$$

Adding (3.15) to (3.16), we get

$$S_b + S_c = 2(m^2 + \mu^2) - 4K^\circ \, k^\circ{}_b \, . \qquad (3.17)$$

But

$$K^{\circ 2} = + m^2 + |\vec{K}|^2 \, , \qquad (3.18)$$

and

$$S_b + S_c = m^2 + 3\mu^2 - S_a \, . \qquad (3.19)$$

Using relation (3.13), (3.18) and (3.19) in (3.17), we get, after rearranging terms

$$|\vec{K}| = \frac{\sqrt{S_a - (\mu - m)^2} \sqrt{S_a - (\mu + m)^2}}{2\sqrt{S_a}} \, . \qquad (3.20)$$

By subtracting (3.15) from (3.16), we get

$$S_c - S_b = 4|\vec{K}| k \cos \theta_{bc} \, . \qquad (3.21)$$

The final relation for $\cos \theta_{bc}$ may be obtained from (3.21) by substituting (3.20) for $|\vec{K}|$; the relation is

$$\cos \theta_{bc} = \frac{S_c - S_b}{\phi(S_a)} \, , \qquad (3.22)$$

where

$$\phi(S) = \frac{\sqrt{S - 4\mu^2} \, . \, \sqrt{S - (\mu - m)^2} \, . \, \sqrt{S - (\mu + m)^2}}{\sqrt{S}} \, . \qquad (3.23)$$

3. The Integral Equations for $K^{\pm} \to 3\pi$ Decay

Khuri and Treiman (1) derived a dispersion representation for the amplitutes A, B, and C. In the approximation where one retains only the lowest mass intermediate states, the absorptive parts in these representations involve products of the decay amplitudes themselves with the amplitudes for pion-pion scattering. Since the maximum kinetic energy of an outgoing pion in this decay is only about 50 Mev. One can further make the approximation that only the S-wave pion-pion scattering amplitude should be taken into account.

The dispersion representations used are of the subtracted variety and this makes both of the above approximations better. This is because what is neglected in that case is the contribution of the higher mass states and the higher partial waves to the shape of the spectrum rather than their contribution to the absolute value of the decay rate.

With these approximations the dispersion representations become integral equations. The kernels in these equations depend on the S-wave pion-pion scattering amplitudes. The integral equations are

$$\begin{aligned} A(S_a, S_b, S_c) &= D_o + U(S_a) + V(S_b) + V(S_c) \\ B(S_a, S_b, S_c) &= D_o + V(S_a) + U(S_b) + V(S_c) \\ C(S_a, S_b, S_c) &= D_o + V(S_a) + V(S_b) + U(S_c) \; . \end{aligned} \tag{3.24}$$

D_0 is the value of A at the symmetric point S_0, given by

$$S_a = S_b = S_c = S_0 = \frac{m^2 + 3\mu^2}{3} . \qquad (3.25)$$

At this point, we have

$$A = B = C = D_0 . \qquad (3.26)$$

The functions U and V in (3.24) are given by

$$U(S) = \frac{S-S_0}{\pi}\int_{4\mu^2}^{\infty} \frac{\widetilde{A}(S')f_0(S') + \frac{1}{3}[\widetilde{B}(S') + \widetilde{C}(S')][f_0(S')-f_2(S')]ds'}{(S' - S_0 + i\epsilon)(S' - S + i\epsilon)} ,$$

$$(3.27)$$

and

$$V(S) = \frac{(S-S_0)}{\pi}\int_{4\mu^2}^{\infty} \frac{\frac{1}{2}[\widetilde{B}(S') + \widetilde{C}(S')]\, f_2(S')\, ds'}{(S' - S_0 + i\epsilon)(S' - S + i\epsilon)} ,$$

where

$$\widetilde{A}(S_a) = \frac{1}{4\pi}\int A(S_a, \cos\theta_{bc})d\Omega_{bc}$$

$$\widetilde{B}(S_a) = \frac{1}{4\pi}\int B(S_a, \cos\theta_{bc})d\Omega_{bc} \qquad (3.28)$$

$$\widetilde{C}(S_a) = \frac{1}{4\pi}\int C(S_a, \cos\theta_{bc})d\Omega_{bc} .$$

In (3.27) the functions f_0 and f_2 are the T = 0 and T = 2 S-wave pion-pion scattering amplitudes and are related the scattering phase shifts δ_0 and δ_2 by

$$f_j = e^{i\delta_j}\sin\delta_j , \; j = 0 , 2. \qquad (3.29)$$

Equations (3.24) form a system of coupled linear singular integral equations for the amplitudes A, B and C. These equations cannot be solved exactly even if simple expressions for f_o and f_2 are used. Thus, approximate methods are necessary.

In order to obtain an indication of the effects involved, Khuri and Treiman treated the pion-pion effects as a small perturbation and obtained a first order iterative solution of (3.24); i.e. they set $A = B = C = D_o$ in (3.27). They used a scattering length approximation for the phase shifts and set

$$\frac{k}{w} \cot \delta_j = \frac{1}{a_j} \ , \ j = 0, 2, \qquad (3.30)$$

where k and w are the centre of mass momentum and energy, and a_j is the dimensionless scattering length.

The result they obtained for $|M_\tau|^2$ and $|M_{\tau'}|^2$ is

$$\frac{1}{4} |M_\tau|^2 = 1 + \frac{5}{3\pi} \frac{\rho^2}{(1+\frac{1}{2}\rho^2)^{1/2}} (a_2 - a_o)(2t_3 - 1),$$

and

$$|M_{\tau'}|^2 = 1 + \frac{10}{3\pi} \frac{\rho^2}{(1+ \frac{1}{2} \rho^2)^{1/2}} (a_o - a_2)(2t_3 - 1), \qquad (3.31)$$

where $\rho^2 \approx 0.64$ and t_3 refers to the kinetic energy of the unlike pion and is related to k, the magnitude of the momenta of the two like pions in their C.M. system, by

$$k^2 = \rho^2(1 - t_3). \tag{3.32}$$

From comparison with the experimental data on τ-decay, Khuri and Treiman concluded that if both a_2 and a_0 are small then

$$a_2 - a_0 \approx 0.7. \tag{3.33}$$

Obviously, the iterative solution is not very reliable if the phase shifts are large. In the past two years there have been some theoretical arguments which lead to the conclusion that at least one of the pion-pion phase shifts is large at low energies. For example, Desai (1), obtains that $a_0 > 0$ and large and $a_2 \approx 0$. This is in obvious contradiction with (3.33). The disagreement with Desai may be due to the unreliability of the iterative solution used to obtain (3.33). Therefore we shall, in chapter IV, obtain a solution of the equation for the scattering amplitude using a non-iterative method. We shall use the values of δ_0 and δ_2 given by Desai, and we will compare the result with the experimental data of Ferro - Luzzi et al. (1) and of Prowse (1). From the data they obtain the following relation for the decay amplitude

$$|A(S_a)|^2 \approx D_0^2\ [1 + 0.370(S_a - S_0)], \tag{3.34}$$

where S_a is limited to the physical region.

One advantageous feature of the phase shifts proposed by Desai is the fact that $\delta_2 \approx 0$ while δ_0 is relatively large. This allows us to achieve an appreciable simplification of equations (3.24). We set $f_2 \equiv 0$ in (3.24) and (3.27) and obtain

$$\begin{aligned} A(S_a) &= D_0 + U(S_a) \\ B(S_a,\alpha) &= D_0 + U(S_b) = D_0 + U(S_a,\alpha) \\ C(S_a,\alpha) &= D_0 + U(S_c) = D_0 + U(S_a,\alpha) , \end{aligned} \qquad (3.35)$$

and

$$U(S) = \frac{S - S_0}{\pi} \int_4^\infty \frac{\widetilde{A}(S') + \frac{1}{3}[\widetilde{B}(S') + \widetilde{C}(S')]\, f_0(S')\,ds'}{(S' - S_0 + i\epsilon)(S' - S + i\epsilon)}, \qquad (3.36)$$

and

$$V(S) = 0 . \qquad (3.37)$$

Equations (3.35) are the required equations, and we will attempt to solve them in the next chapter.

CHAPTER IV

SOLUTION OF THE INTEGRAL EQUATIONS

1. Introduction

In this chapter we shall obtain a numerical solution of the integral equations for the decay amplitudes, (3.15). These equations, as was mentioned in the previous chapter, are not of the Omnes type and cannot be solved exactly. However, we shall reduce the equations to an Omnes equation by making use of two simplifying approximations.

In section 2, will present the equation and its solution. In section 3, we will give tables of the solution and some functions related to the solution. We will also compare the results we obtain with experimental data.

2. Solution

The equations obtained in chapter III are

$$A(S_a) = D_o + U(S_a),$$

$$B(S_a,\alpha) = D_o + U(S_b) = D_o + U(S_a,\alpha), \quad (3.35)$$

$$B(S_a,\alpha) = D_o + U(S_c) = D_o + U(S_a,\alpha),$$

where*

$$U(S) = \frac{S - S_o}{\pi} \int_4^\infty \frac{\widetilde{A}(S') + \frac{1}{3}[\widetilde{B}(S') + \widetilde{C}(S')]\, f_o(S')\,dS'}{(S' - S_o + i\epsilon)(S' - S + i\epsilon)}, \quad (3.36)$$

and $= \cos \theta_{bc}$.

The integration over the angles in the expressions for $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ may be performed. The result is

$$\widetilde{A}(S) = A(S) = D_o + U(S),$$
$$\widetilde{B}(S) = \frac{1}{2}\int_{-1}^{1} B(S_b)\, d\alpha = D_o + \frac{1}{2}\int_{-1}^{1} U(S_b)\,d\alpha, \quad (4.1)$$
$$\widetilde{C}(S) = \frac{1}{2}\int_{-1}^{1} C(S_c)\, d\alpha = D_o + \frac{1}{2}\int_{-1}^{1} U(S_c)\,d\alpha,$$

Both integrations in the expression for $\widetilde{A}$ can be carried out, since $A(S)$ is not a function of either angular variable. The same is not true for $\widetilde{B}$ and $\widetilde{C}$; they are functions of through S_b and S_c.

We shall now show that

$$\widetilde{B} = \widetilde{C},$$

that is

$$\int_{-1}^{1} U(S_b)\, d\alpha = \int_{-1}^{1} U(S_c)\, d\alpha. \quad (4.2)$$

This can be easily shown by changing variables

---

*We shall work in units of $\mu^2 = 1$.

$$\int_{-1}^{1} U(S_b)d\alpha = -\frac{2}{\phi(S)} \qquad U(\beta)d\beta = \frac{2}{\phi(S)} \int_{\beta -}^{\beta +} U(\beta)d\beta \;, \qquad (4.3)$$

and

$$\int_{-1}^{1} U(S_c)d\alpha = \frac{2}{\phi(S)} \int_{\beta -}^{\beta +} U(\beta)d\beta \;, \qquad (4.4)$$

where

$$\beta(S^{\pm}) = \frac{1}{2}[b \pm \phi(S) - S], \; b = \frac{m^2 + 3}{15.25} \;, \qquad (4.5)$$

and $\phi(S)$ is as defined by (3.23)

By making use of this, we can rewrite equation (3.) for U(S) in the form

$$U(S) = \frac{S-S_o}{\pi} \int_{4}^{\infty} \frac{\frac{5}{3}D_o + U(S') + \frac{2}{3\phi(S')} \int_{\beta -}^{\beta +} U(\beta)d\beta f_o(S')ds'}{(S'- S_o + i\varepsilon)(S' - S + i\varepsilon)} \qquad (4.6)$$

This last equation is a linear integral equation for U(S). The presence of the definite integral in the integrand makes it impossible to solve.

To simplify matters, we shall adopt the following approximation,

$$\frac{2}{\rho(S)} \int_{\beta -}^{\beta +} U(\beta) \; d\beta = 2U(\frac{b-S'}{2})$$

$$= 2U(\frac{b-S'}{2}). \qquad (4.7)$$

This essentially means that the integral is being approximated by the value of the function at the mean point times the interval. It should be noted that

$$\frac{b - S}{2} = \frac{1}{2} [\beta^{+} + \beta^{-}] . \qquad (4.8)$$

The approximation made in (4.7) is exact at $S = 4$, $S = (m^2-1)^2$ and $S = (m+1)^2$. This follows immediately from (4.7) since $\phi(S)$ at those points is zero and $\beta^{+} = \beta^{-}$.

The limits of integration in (4.7) are $\beta^{-} = \frac{b - S'}{2} - \frac{\phi(S')}{2}$ and $\beta^{+} = \frac{b - S'}{2} + \frac{\phi(S')}{2}$. In the physical region $(4 < S < (m-1)^2)$, the approximation will be good since ϕ is small and β^{+} β^{-}. We are therefore justified in saying that (4.7) is a good approximation.

By substituting (4.7) into equation (4.6), we get

$$U(S) = \frac{S - S_o}{\pi} \int_4^{\infty} \frac{\frac{5}{3}D_o + U(S') + \frac{2}{3} U(\frac{b-S'}{2}) \, f_o(S')ds'}{(S' - S_o + i\epsilon)(S' - S + i\epsilon)},$$

or

$$A(S) = D_o + U(S)$$

$$= D_o + \frac{S - S_o}{\pi} \int_4^{\infty} \frac{A(S') + \frac{2}{3} A(\frac{b-S'}{2}) \, f_o(S')ds'}{(S'-S_o+i\epsilon)(S'-S+i\epsilon)}. \qquad (4.9)$$

Equation (4.9) is very similar in appearance to the Omnes equation. However, it turns out to be impossible to solve it by using the same method. The problem cannot be reduced to a Hilbert problem because of the presence of the second term in the numerator of the integrand. To reduce it, we will have to make use of another simplifying approximation. This we will do now.

We define the function F(S) by

$$G(S) = A(S) + \frac{2}{3} A\left(\frac{b-S}{2}\right). \tag{4.10}$$

G(Z) will be analytic function throughout the z-plane except on the real axis. It will have a right branch cut ($4 < z < \infty$) coming from the term $A(z)$ and a left branch cut ($-\infty < z < b - 8$) coming from the term $A(\frac{b-z}{2})$. Thus $G(z)$ may be written as

$$G(z) = G_L(z).\ G_R(z), \tag{4.11}$$

where $G_L(z)$ is a function that has a left branch cut only, and $G_R(z)$ has a right branch cut only. These branch cuts are given by

Right branch cut: $4 < x < \infty$,

Left branch cut : $-\infty < x < b-8$.

By definition, we have $b = m^2 + 3$; this gives b 15.25 when one substitutes for m^2 the physical value for the mass of the K-meson. However, for this value of m the two branch cuts will overlap. To avoid this difficulty we assume that $m^2 < 9$ and therefore $b < 12$; this will make the two cuts distinct. After obtaining the solution for $m^2 < 9$ we analytically continue it in m^2 to values greater than 9.

The boundary condition on G(z)may now be written as two conditions one on the left and one on the right. The right boundary condition is

$$G^+(S) - G^-(S) = G_L(S)(G^+_R(S) - G^-_R(S)), \quad 4 < S < \infty, \quad (4.12)$$

since G_L does not have a right branch cut. But using (4.10) we have

$$G_L(G^+_R(S) - G^-_R(S)) = A^+(S) - A^-(S)$$

$$= 2i(A^-(S) + \frac{2}{3} A^-(\frac{b-s}{2}))f_o(S)$$

$$= 2i\ G_L\ G^-_R\ f_o(S). \quad (4.13)$$

Thus the right boundary condition becomes

$$G^+_R - G^-_R = 2i\ G^-_R\ f_o(S), \quad (4.14)$$

which is a Hilbert type problem and may be solved exactly. The solution (see chapter I, eqs. (1.19) and (1.20)), is

$$G_R(z) = \frac{5}{3} D_o\ e^{\Gamma_1(z)}\ , \quad \Gamma_1(z) = \frac{z-S_o}{\pi} \int_4^\infty \frac{\delta_o(S')ds'}{(S'-z)(S'-S_o+i\epsilon)} \ . \quad (4.15)$$

The left boundary condition is

$$G^+(S) - G^-(S) = G_R(G^+_L - G^-_L), \quad -\infty < S < b-8\ , \quad (4.16)$$

since G_R does not have a branch cut on the left. By (4.10), we have

$$G_R(S)(G^+_L(S) - G^-_L(S) = \frac{2}{3}[A^+(\frac{b-S}{2}) - A^-(\frac{b-S}{2})]. \qquad (4.17)$$

Equation (4.9) may be used to obtain expressions for $A^+(\frac{b-S}{2})$ and $A^-(\frac{b-S}{2})$. These are

$$A^{\pm}(\frac{b-S}{2}) = D_o + \int_4^{\frac{b-S}{2} - S_o} \frac{1}{\pi} \frac{A(S') + \frac{2}{3} A(\frac{b-S'}{L})}{(S'-S_o+i\epsilon)(S'-\frac{b-S}{2} + i\epsilon)} f_o(S')ds' \qquad (4.18)$$

Using (4.18), we can rewrite (4.17) in the form

$$G_R(S)(G^+_L(S) - G^-_L(S)) = 2i\ G_R(S)\ G^-_L(S)\ f_o(\frac{b-S}{2})$$

$$+ 2i[\frac{4}{9}\ A^-(\frac{b+S}{4}) - A^-(S)]\ f_o(\frac{b-S}{2}). \qquad (4.19)$$

Thus, the left boundary condition is

$$G^+_L(S) - G^-_L(S) = 2i\ G^-_L(S)\ f_o(\frac{b-S}{2}) + g(S)f_o(\frac{b-S}{2}) \qquad (4.20)$$

where

$$g(S) = \frac{2i}{G_R(S)}\ [\frac{4}{9}\ A^-(\frac{b+S}{4}) - A^-(S)]. \qquad (4.21)$$

Equation (4.20) is not a Hilbert problem and cannot be solved exactly. To reduce it to a Hilbert problem we suppose that, in a first approximation, f is negigible in comparison with G^-_L. Later on, in the next section, we shall test this approximation by comparing the Cauchy integrals of each of the two functions. More precisely, we will compare the following two integrals

$$\int_{-\infty}^{b-8} \frac{G^{-}_{L}(S')f_{o}(\frac{b-S'}{2})ds'}{S' - S}, \quad \int_{-\infty}^{b-8} \frac{g(S')f_{o}(\frac{b-S'}{2})\, ds'}{S' - S}. \qquad (4.22)$$

If the second integral turns out to be small compared to the first, then that would imply that the effect of g(S) on the solution is small, and hence that the approximation is good.

With this approximation, the left boundary condition becomes

$$G^{+}_{L}(S) - G^{-}_{L}(S) = 2i\, G^{-}_{L}(S)\, f_{o}(\frac{b-S}{2}), \qquad (4.23)$$

which is a Hilbert problem. The solution is

$$G_{L}(z) = e^{\Gamma_2(z)}, \quad \Gamma_2(z) = \frac{z-S_o}{\pi} \int_{-\infty}^{b-8} \frac{\delta_o(\frac{b-S'}{2})ds'}{(S'-S_o+ i\epsilon)(S'-z)}. \qquad (4.24)$$

Now, we can write down an expression for G(z). This is

$$G(z) = G_{L}(z).\, G_{R}(z),$$

$$= \frac{5}{3} D_o\, e^{\Gamma_1(z)}.\, e^{\Gamma_2(z)}. \qquad (4.25)$$

The scattering amplitude A(S) may be obtained by substituting (4.25) into (4.9),

$$A(S) = D_o + \frac{5}{3\pi} D_o(S-S_o) \int_{4}^{\infty} \frac{e^{\Gamma_2(S')}\, e^{\Gamma_1^{-}(S')}}{(S'-S_o+i\epsilon)(S'-S+i\epsilon)} f_o(S')ds'. \qquad (4.26)$$

This is the required solution.

3. Numerical Results

The following is a table of the values of the phase shift δ_0 as obtained by Desai (1).

S	$\delta_0(S)$
4.0	.00
4.5	.46
5.0	.65
5.5	.70
6.0	.74
8.0	.84
20.0	.90
40.0	.89

We have used the above values of $\delta_0(S)$ in our computations.

We are interested in the boundary value of $G_R(S)$ and $G_L(S)$ as we approach from below the cut. The real part of $G_R^-(S)$ is given by

$$R_e\, G^-_R(S) = \frac{5}{3} D_0 \cos\,(I_m\, \Gamma_1^-(S)).\ e^{Re\, \Gamma_1^-(S)}, \qquad (4.27)$$

where

$$R_e\, \Gamma_1^-(S) = \frac{S-S_0}{\pi}\, P \int_4^\infty \frac{\delta_0(S')ds'}{(S'-S_0)(S'-S)}, \qquad (4.28)$$

and

$$I_m \sqrt{1^-}(S) = \delta_o(S_o) - \delta_o(S) \ , \ S > 4$$

$$= \delta_o(S_o), \ S \leq 4. \qquad (4.29)$$

The imaginary part of $G^-_R(S)$ is given by

$$I_m \ G^-_R(S) = \frac{5}{3} D_o \sin I_m \sqrt{1^-}(S) \ e^{R_e \sqrt{1^-}(S)}, \qquad (4.30)$$

where $I_m \sqrt{1^-}(S)$ and $R_e \sqrt{1^-}(S)$ are as given by (4.28) and (4.29).

The real part of $G^-_L(S)$ is given by

$$R_e \ G^-_L(S) = \cos \ (I_m \sqrt{2^-}(S) \ e^{R_e \sqrt{2^-}(S)}, \qquad (4.31)$$

where

$$R_e \sqrt{2^-}(S) = \frac{S-S_o}{\pi} \ P \int_{-\infty}^{b-8} \frac{\delta_o(b-S')ds'}{(S'-S_o)(S'-S)} \ , \qquad (4.32)$$

and

$$I_m \sqrt{2^-}(S) = \delta_o(S_o), \ S \geq 7.25$$

$$= \delta_o(S_o) - \delta_o(\frac{b-S}{2}), \ S < 7.25. \qquad (4.33)$$

The imaginary part of $G^-_L(S)$ is

$$I_m \ G^-_L(S) = \sin \ (I_m \sqrt{2^-}(S)) \ e^{R_e \sqrt{2^-}(S)}. \qquad (4.34)$$

To evaluate $R_e\,\Gamma_2^-(S)$, we used the following relation

$$R_e\,\Gamma_2^-(S) = -\,R_e\,\Gamma_1^-\left(\frac{b-S}{2}\right). \qquad (4.35)$$

This saves us the trouble of evaluating a new integral. It is elementary to prove (4.34). By changing the integration variable to $t = \frac{b-S'}{2}$ in equation (4.32), we get

$$R_e\,\Gamma_2^-(S) = \frac{S-S_o}{\pi}\int_{\infty}^{4}\frac{\delta_o(t)\,dt}{\left[t-\left(\frac{b-S}{2}\right)\right]\left[b-2t-S_o\right]} . \qquad (4.36)$$

Using $b = 3S_o$ in (4.35) and reversing the limits of integration we obtain

$$R_e\,\Gamma_2^-(S) = -\,\frac{\frac{b-S}{2} - S_o}{\pi}\int_{4}^{\infty}\frac{\delta_o(t)\,dt}{(t-S_o)\left[t-\left(\frac{b-S}{2}\right)\right]}$$

$$= -\,R_e\,\Gamma_1^-\left(\frac{b-S}{2}\right),$$

which is the desired equality.

We shall now give a table of the values of $R_e\,\Gamma_1^-(S)$, $R_e\,G^-{}_R(S)$, $I_m\,G^-{}_R(S)$, $R_e\,G^-{}_L(S)$ and $I_m\,G^-{}_L(S)$.

TABLE II. REAL AND IMAGINARY PARTS OF THE SOLUTION ON THE RIGHT (G_R) AND THE SOLUTION ON THE LEFT (G_L)

S	$R_e\sqrt{1^-}(S)$	$R_eG^-{}_R(S)$	$I_mG^-{}_R(S)$	$R_eG^-{}_L(S)$	$I_mG^-{}_L(S)$
4.0	0.66	1.72	1.32	1.09	-0.15
4.5	0.17	1.82	0.65	1.06	-0.02
5.0	0.04	1.73	0.02	1.01	0.00
5.5	-0.08	1.53	-0.05	0.92	0.08
6.0	-0.16	1.42	-0.12	0.95	0.25
8.0	-0.37	1.13	-0.20	0.85	0.65
20.0	-0.84	0.70	-0.17	1.50	1.16
50.0	-1.15	0.52	-0.13	2.02	1.56

The following is a table of the values of $\frac{R_e A(S)}{Do}$, $I_m \frac{A(S)}{Do}$ and $\left|\frac{A(S)}{Do}\right|^2$, while in fig. 2, we give a plot of $\left|\frac{A(S)}{Do}\right|^2$ as obtained from the solution of the equation and from the fit (4.) of Ferra - Luzzi *et al.* and Prowse (1).

TABLE III. REAL AND IMAGINARY PART OF THE DECAY AMPLITUDE $\frac{A}{D_o}$

S	$R_e \frac{A(S)}{D_o}$	$I_m \frac{A(S)}{D_o}$	$\left\lvert\frac{A(S)}{D_o}\right\rvert^2$
4.0	0.79	1.13	1.91
4.5	1.25	0.61	1.93
5.0	1.14	0.00	1.30
5.5	0.74	0.03	0.55
6.0	0.66	0.18	0.47
8.0	0.89	0.30	0.88
20.0	1.06	-0.22	1.17
50.0	0.98	-0.66	1.40

Before discussing this result we will say a few words about the second approximation. To obtain an idea of the extent of this approximation, we have calculated $R_e G^-{}_L(S)$, $I_m\ G^-{}_L(S)$, $R_e\ g(S)$ and $I_m\ g(s)$. We find that in the region of interest the neglected function g(S), in both its real and imaginary parts is, on the average, less than 20°/o of the function $G^-{}_L(S)$. This approximation is good enough for the purpose of this paper, as we are interested in the slope of the curve rather than its absolute value.

The approximation may be made better, by using the solution to calculate the function g(S), and solving equation (4.20) with this known function. This would lead to a non-homogeneous Hilbert problem for G_L which may be easily solved. By successive approximations of this sort, one may obtain as good an approximation as one desires, provided the series obtained is convergent. To decide this, we have emputed the real and imaginary parts of the integrals

$$\int_{-12.75}^{7.25} \frac{g(S')\, f_o(\frac{b-S'}{2})ds'}{S'-S} \quad \text{and} \quad \int_{-11.75}^{7.25} \frac{G^-_L(S')\, f_o(\frac{b-S'}{2})ds'}{S'-S}, \quad (4.36)$$

at S = 4 and S = 6. We find that the second integral is always much larger than the first. This indicates that a solution by successive approximation would enverge quickly.

Note that the integrals (4.36) are the same as the integrals (4.22) except for the lower limit of integration. The integrals from $-\infty$ to -12 will both be small and will not contribute appreciably to the slope.

4. Conclusion

Our result for $\left|\frac{A(S)}{D_o}\right|^2$ is in rough agreement with physical intuition. For the Desai δ_o is a strongly attractive phase shift and this means that in τ' - decay,

$$K^+ \rightarrow \pi^+ + \pi^o + \pi^o,$$

the two π^o mesons are attracted to each other and tend to come out together and in the same direction. This means that in order to conserve momentum the π^+ will tend to come out opposite to the two π^{o}'s. From this one concludes that more π^+'s should come out with high energy than with low energy. Now from (3.7)

$$|M_{\tau'}|^2 = |C(S_e)|^2,$$

and since $S_e = m^2 + \mu^2 - 2mw_e$. we would expect $C(S_e)$ to be a decreasing function of S_e, i.e. an increasing function of w_e, the energy of the π^+ meson in the rest system of the K-meson. This rough intuitive argument is in agreement with our results for A, B, and C.

However there is obviously a strong disagreement between our result for $|A|^2$ and the one that gives rough agreement with experimental data (see eq. (3.34)); the latter increases with S_a. Up to last summer, this would have led us to the conclusion that either

i) the Desai phase shifts are wrong, or

ii) the higher intermediate states or the pion-pion p-wave effects or both have to be taken into account in the dispersion representation of Khuri and Treiman.

During the past year several new strongly unstable particles have been discovered experimentally. The presence of these particles puts the problem of the pion S-wave phase shifts in a different light and provides several new factors which come out together and in the same direction. This means that in order to conserve momentum the π^+ will tend to come out opposite to the two π°'s. From this one concludes that more π^+'s should come out with high energy than with low energy. Now from (3.7)

$$|M_{\tau'}|^2 = |C(S_e)|^2,$$

and since $S_e = m^2 + \mu^2 - 2mw_e$. we would expect $C(S_e)$ to be a decreasing function of S_e, i.e. an increasing function of w_e, the energy of the π^+ meson in the rest system of the K-meson. This rough intuitive argument is in agreement with our results for A, B, and C.

However there is obviously a strong disagreement between our result for $|A|^2$ and the one that gives rough agreement with experimental data (see eq. (3.34)); the latter increases with S_a. Up to last summer, this would have led us to the conclusion that either

i) the Desai phase shifts are wrong, or

ii) the higher intermediate states or the pion-pion p-wave effects or both have to be taken into account in the dispersion representation of Khuri and Treiman.

During the past year several new strongly unstable particles have been discovered experimentally. The presence of these particles puts the problem of the pion S-wave phase shifts in a different light and provides several new factors which might be contributing appreciably to the absorptive parts in the dispersion representation.

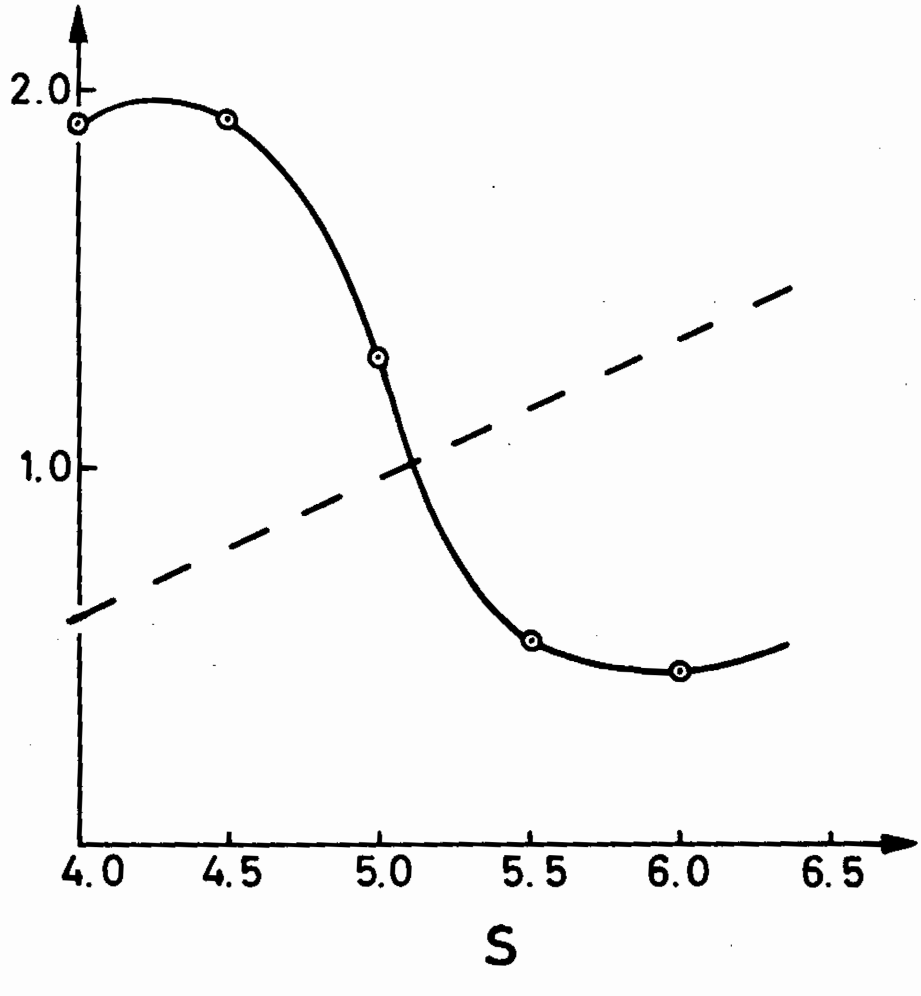

Fig. 2 Plot of $\left|\frac{A(s)}{D_0}\right|^2$

BIBLIOGRAPHY

Desai, B.R. [1] - Low-energy pion-pion S-wave phase shifts, Phys. Rev. Letters, Vol.6, No.9 (1961). Pp. 497-500.

Ferro-Luzzi <u>et al.</u>,[1] - Nuovo Cimento 22,(1961). Pp.1087.

Gakhov,F.D. [1] - Linear boundary problems of the complex function theory, Izv. Phiziko-matem. Obshch. i natushno-issled., 3 ser. Vol. X(1938). Pp. 39-79.

Hilbert, D. [1] - Grundzuge einer allgemeinen Theorie der linearem Integralgleichumgen. Leipzig-Berlin (1912).

Khuri,N.N., [1] - and Treiman S.B., Pion-Pion scattering and $K \to 3\pi$ decay, Phys. Rev., Vol. 119, No. 3 (1960) Pp. 1115-1121.

Muskhelishvili,N.I. [1]-Singular Integral Equations, P. Noordhoff N.V., 1953. Ch. 5.

Omnes, R., [1] - Nuovo Cimento, Vol. VII, No. 2(1958).

Plemelj, J. [1] - Ein Enganzungssatz sur Cauchyschen Integraldarstellung analytischer Funktionen, Randwerte betreffend, Monatshefte fur Math.u. Phys., XIX Jahrgang.

Prowse, D.J. [1]- and Smith, L.T., Un. of Cal. preprint (1961) (unpublished).