15 pt. 2

Reinforced concrete

THESIS

SECOND PART

DESIGN OF A REINFORCED CONCRETE ARCH

by

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Introduction

The present design deals with the analysis of a hingeless arch supported on abutments that may be considered as rigid. This is the most common problem encountered in a reinforced concrete arch design. In ordinary massonry and concrete arches, tensile stresses are not permissible. The arch ring must therefore be designed so that the line of pressure will not pass outside the middle third. In reinforced concrete arches this limitation does not exist. The arch ring is a beam, and if properly reinforced may carry heavy bending moments involving tensile stresses in the steel.

Theoretically the gain in economy by the use of steel in a concrete arch is not great. If the pressure line does not depart from the middle third, the steel reinforces only in compression and in this respect is not as economical as concrete. If the line of pressure deviates farther from the center, resulting in tensile stresses on the section, the conditions are such that those stresses must be provided far by the use of the steel at very low working values. Thatis to say, the direct compression in the arch is so large a factor that the limiting stresses in the concrete will always result in relatively small unit tensile stresses in the steel where any tension exists atall.

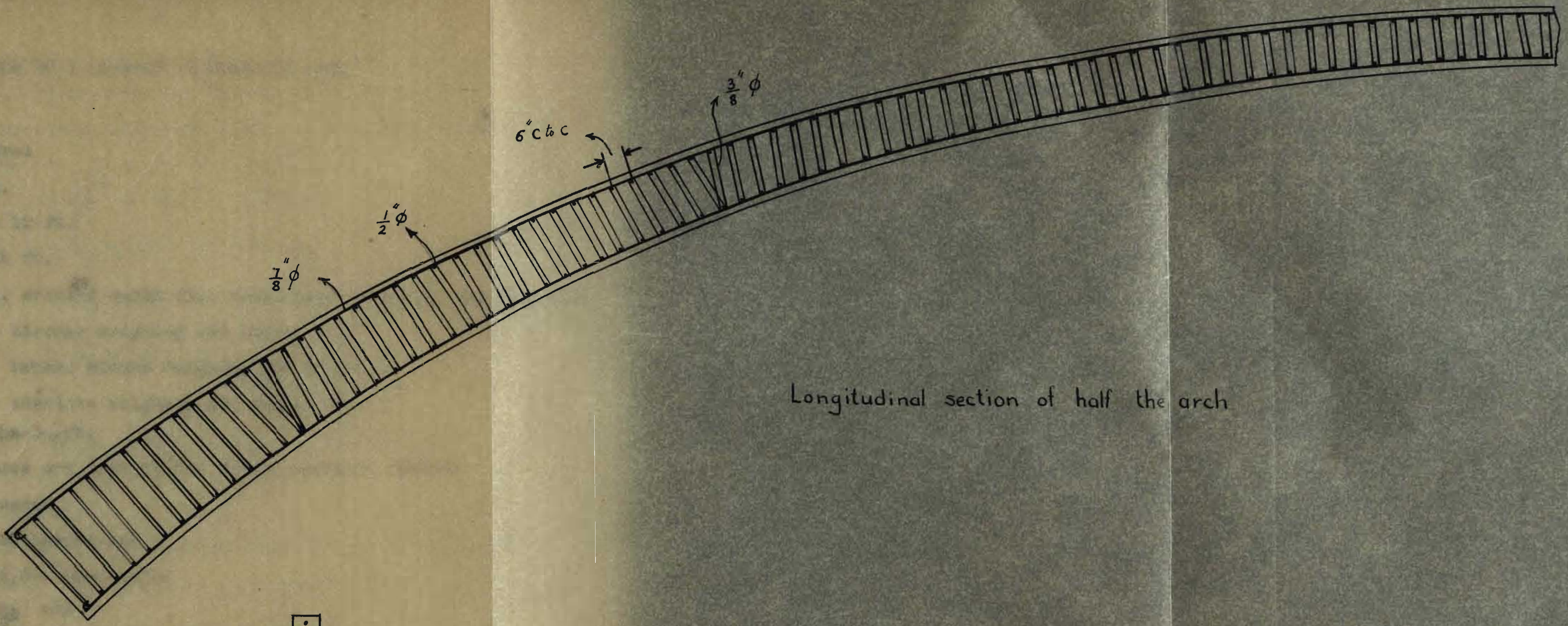
Practically the value of reinforcement is very considerable. It renders an arch a much more secure and reliable structure, it greatly aids in preventing cracks due to any slight settlement, and by furnishing a form of construction of greater reliability makes possible the use of working stresses in the concrete considerably higher than are usual in plain massonry. Consequently in long span arches where the dead load constitutes by far the larger part of the load, the possible increase in average working stress counts greatly toward economy. It affects not only the arch but also the abutments and foundations.

The most common type of reinforcement consists of longitudinal bars placed near extrados and intrados with a few transverse bars to prevent longitudinal cracks and to equalize the load on the arch. The outer and inner bars are connected by ties as in a "tied column". The total amount of longitudinal steel at the crown section is usually about 1%, and a common practice is to use the same number and spacing of bars throughout, giving a reduced steel ratio at the pringing section. This arrangement requires, for equal stresses at crown and springing, a relatively large ratio of depth at springing to depth at crown, usually from 2 to 2.5. If a smaller ratio is desired, the reinforcement may be increased towards the springing, a practice more commonly used for long spans.

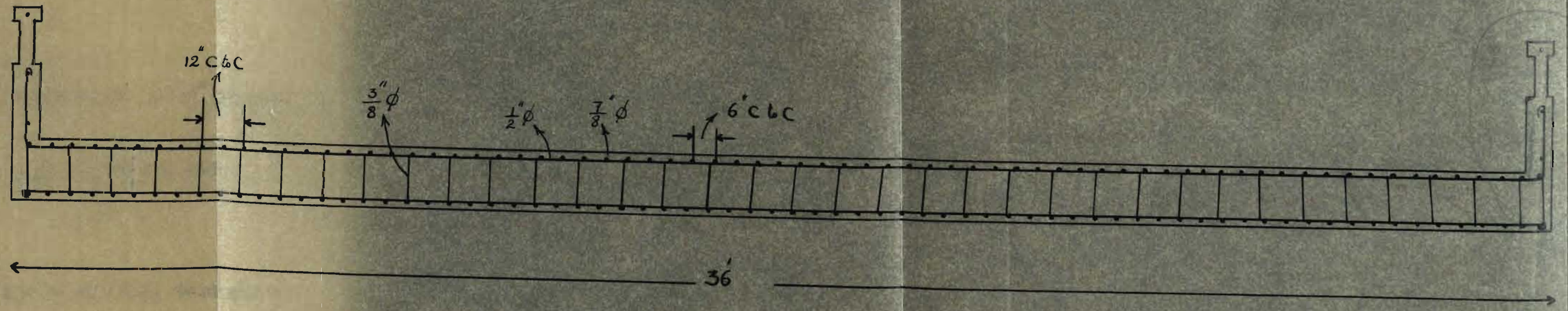
A hingless arch is indeterminate to the third degree, there being six unknown elements (three at each reaction : H, V, and M) In order to analyze the stresses it is necessary to find three equations in addition to the three of equilibrium for a non-concurrent co-planar force system. This may be done in various ways by consideration of the relations between the elastic deformations of the arch and the internal and external stresses. The method of analysis presented here is that of Least Work which is probably a more convenient method than the others. There are two great advantages to this method: first, its ease and accuracy of application, and second, the fact that it is the application of a simple and familiar theorem and so involves the minimum of new detail for the learner.

In order to apply this or any other method of analysis based on the elastic properties of the arch it is first necessary to make certain assumptions. The labor of an exact analysis is so great that it is desirable to have available some simple method for arriving at a trial section that will require little if any change upon closer study. Mr. Charles S. Whitney has prepared an adaptation of the elastic theory, simple and rapid of use, which even exceeds the longer methods in accuracy when applied to arches proportioned in accordance with his fundamental equations.

Reference: Reinforced Concrete Design by Sutherland and Clifford.



Longitudinal section of half the arch



Cross section at the crown

Scale 1" = 2.5'

Steel Design

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DESIGN OF A REINFORCED CONCRETE ARCH

DATA

Type - Filled spandrel

Clear span = 70 ft.

Rise of intrados = 12 ft.

Width of arch = 36 ft.

Dead load = 12 in. saturated earth fill above crown weighing 120 Lbs/cu.ft.

= 6 in. blocage weighing 150 Lbs/cu.ft.

= 4 in. broken stones weighing 150 Lbs/cu.ft.

= 2 in. idéalite weighing 150 Lbs/cu.ft.

Live load = 120 Lbs/sq.ft.

Impact allowances are omitted for filled spandrel arches.

Allowable stresses:

f_c = 650 Lbs/sq.in.

f_s = 16,000 Lbs/sq.in.

$n = \frac{E_s}{E_c} = 15$

E_c = 2,000,000 Lbs/sq.in.

The arch is to be designed for a fall in temperature of 50° Fahrenheit and a rise of 25°.

Design is done by considering a strip of arch of one ft. wide.

Calculations:

Crown thickness:

The following empirical formula devised by Mr. F.F. Weld gives the crown thickness:

$$d_c = \sqrt{L_1} + \frac{L_1}{10} + \frac{W_L}{200} + \frac{W_c}{400}$$

where d_c = crown thickness in inches.

L_1 = clear span in feet = 70

W_L = live load in Lbs per square foot = 120

W_c = dead load at crown in Lbs per square foot.

In order to calculate W_c, assume a crown thickness of 17 inches:

weight of arch	$= \frac{17}{12} \times 150$	=	213 Lbs
earth fill	$= \frac{12}{12} \times 120$	=	120 Lbs
blocage	$= \frac{6}{12} \times 150$	=	75 Lbs
broken stones	$= \frac{4}{12} \times 150$	=	50 Lbs.
idéalite	$= \frac{2}{12} \times 150$	=	25 Lbs
W_c	=		483 Lbs/sq.ft.

Therefore:

$$d_c = \sqrt{70} + \frac{70}{10} + \frac{120}{200} + \frac{483}{400} = 17.17 \text{ inches}$$

Consider crown thickness = 17 in.

Springing line thickness:

The thickness of the arch at the springing is usually about twice or more that at the crown. In this case, it is taken 35 in.

Preliminary Analysis

Trial Computations based on Whitney's method

In order to locate the axis of the arch and find its rib thickness, the value of $\frac{W_s}{W_c}$ should be known; W_s being the dead load per linear foot at springing and W_c being the dead load per linear foot at crown.

The ratio $\frac{W_s}{W_c}$ will be found by trial in the following steps.

Data required for analysis:

Reinforcement = 1% at crown = 0.01 x 17 x 12 = 2.04 sq.in.

Use $\frac{7}{8}$ in. ϕ bars 6 in. ∅ to ∅ in each face.

where A'_s = 1.2 sq.in./foot width in each face.

The same rods are used throughout the length of the barrel and so the steel ratio at the springing is considerably less than at the crown.

These rods are placed 2 in. from the rib end on both sides.

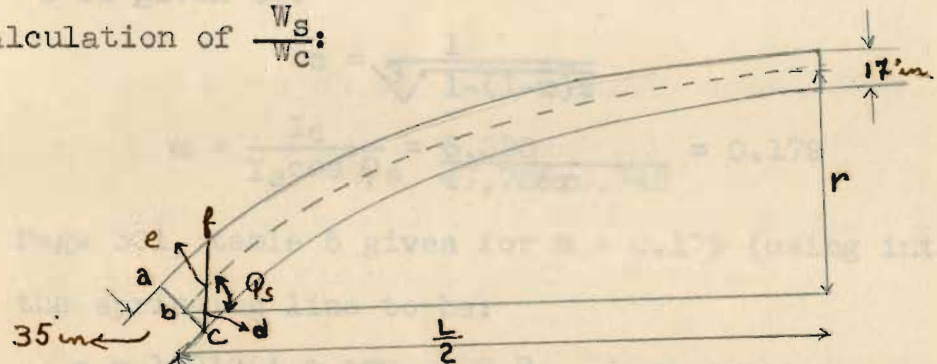
Moment of inertia at crown $I_c = \frac{12x\overline{17}^3}{12} + (15-1)1.2x2x(\frac{17}{2}-2)^2 = 6,330 \text{ in}^4$.

Moment of inertia at springing $I_s = \frac{12x\overline{35}^3}{12} + (15-1)1.2 \times 2 (\frac{35}{2}-2)^2 =$
$= 47,725 \text{ in}^4$

Crown section Area A_c = 12 x 17 + (15-1)1.2 x2 = 237.6 sq.in.

Springing section Area A_s = 12 x 35 + (15-1)1.2 x 2 = 443.6 sq.in.

Calculation of $\frac{W_s}{W_c}$:



Assuming $\varphi_s = 42^o$, a f as a straight line and fc = ac =35 in.

Hence ce = $\frac{1}{2}$ fc =$\frac{1}{2}$ x 35 = 17.5 in.

L = Clear span + 2(17.5 cosφ_s sinφ_s); cos 42° = 0.743 sin 42° = 0.669

= 70 x 12 + 2 x 17.5 x 0.743 x 0.669 = 857.5 in ; tan 42° = 0.90

r = Rise of intrados + $\frac{17}{2}$ - 17.5 $\cos^2 \varphi_s$

= 12 x 12 -8.5 - 17.5 x $(0.743)^2$ = 142.8 in = 11.9 ft.

$\frac{L^2}{r^2} \tan^2 \varphi_s = (\frac{857.5 \times 0.90}{142.8})^2 =$ 29.20 at the springing

Page 300, table 4 at Z = 1.0 (springing line) gives the nearest value to 29.20 to be 29.441 and a corresponding value of $\frac{W_s}{W_c}$ to be 4.324. By interpolation, the corresponding value of $\frac{W_s}{W_c}$ to 29.20 is given by:

$$\frac{W_s}{W_c} = 3.5 + 0.824 \times \frac{2.524}{2.765} = 3.5 + 0.753 = 4.253$$

The arch being symmetrical, only one half of it will be considered.

The coordinates of its axis are given from table 2, page 298 for $\frac{W_s}{W_c} = 4.253$

Calculation of rib thickness:

Using the following equation:

$$d_x = d_c \, c \sqrt[6]{1 + \tan^2 \varphi}$$

where d_x = rib thickness at any point.

d_c = crown thickness.

φ = angle formed by the tangent to the arch axis and a horizontal line

c is given by:

$$c = \frac{1}{\sqrt[3]{1-(1-m)z}}$$

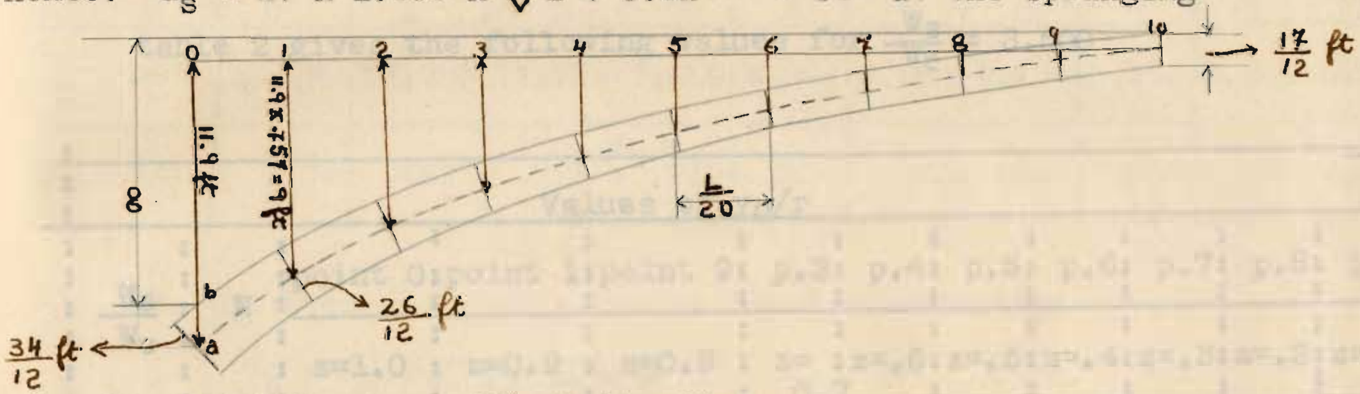
$$m = \frac{I_c}{I_s \cos \varphi_s} = \frac{6,330}{47,725 \times 0.743} = 0.179$$

Page 301, table 5 gives for m = 0.179 (using interpolation) a value of c at the springing line to be:

$$c = 1.710 + 0.172 \times \frac{2.8}{5} = 1.710 + 0.096 = 1.806$$

$$\tan^2 \varphi_s = \frac{2}{0.9} = 0.81$$

Hence: $d_s = 17 \times 1.806 \times \sqrt[6]{1 + 0.81} = 34"$ at the springing



For point 1, c = 1.431 (table 5)

Table 4 gives for $\frac{W_s}{W_c} = 4.253$ at point 1, a corresponding value of $\frac{L^2}{r^2} \tan^2 \varphi_1 = 18.79$

Hence:

$$\tan^2 \varphi_1 = \frac{18.79}{\left(\frac{857.5}{142.8}\right)^2} = 0.52$$

$$d_{x_1} = 17 \times 1.431 \times \sqrt[6]{1 + 0.52} = 26 \text{ in.}$$

Verification of $\frac{W_s}{W_c}$ graphically:

The earth above the arch ring weighs less than the concrete and so the depth g will be represented by $g \times \frac{12}{15}$; W_s will be represented graphically by $a\,b + \frac{12}{15} g + 1$ and W_c by $\frac{17}{12} + 1 \times \frac{12}{15} + 1$ (1 being the layer above the fill)

The value of $\frac{W_s}{W_c}$ comes to be graphically 3.5 instead of 4.253 and φ_s to be 40°30' instead of 42°

No need to find the coordinates and rib thickness for points 2,3,4, etc. since the calculations will be repeated on the basis of φ_s = 40°30'

$L = 70 \times 12 + 2(17.5 \cos \varphi_s \sin \varphi_s)$

cos 40°30' = 0.76
sin " " = 0.649
tan " " = 0.854

$L = 840 + 2 \times 17.5 \times 0.76 \times 0.649 = 857.25$ in. 71.5 ft.

$r = 12 \times 12 + \frac{17}{2} - 17.5 \cos^2 \varphi_s$

$= 144 + 8.5 - 17.5 \times (0.76)^2 = 142.4$ in. $= 11.9$ ft.

$\frac{L^2}{r^2} \tan^2 \varphi_s = (\frac{71.5 \times 0.854}{11.9})^2 = 26.5$

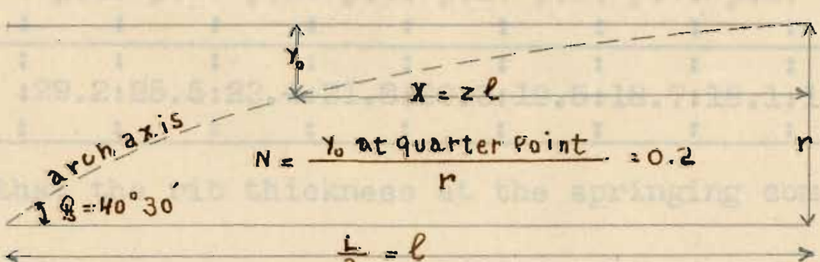
Table 4, point 0 (springing line) gives for 26.5 a corresponding value of $\frac{W_s}{W_c} = 3.500$

Location of arch axis:

table 2 gives the following values for $\frac{W_s}{W_c} = 3.500$

Values of y_o/r

$\frac{W_s}{W_c}$	N	point 0	point 1	point 2	p.3	p.4	p.5	p.6	p.7	p.8	p.9	p.10
		z=1.0	z=0.9	z=0.8	z= 0.7	z=.6	z=.5	z=.4	z=.3	z=.2	z=.1	z=.0
3.5	.2	1.0	.7662	.5757	.4215	.2978	.2000	.1245	.0686	.0300	.0074	.0000
$y_o = \frac{y_o}{r} \times r$		11.9	9.12	6.85	5.01	3.54	2.38	1.48	.817	.358	.088	0



Determination of rib thickness:

$m = \frac{I_c}{I_s \cos \varphi_s} = \frac{6,330}{47,725 \times 0.76} = 0.175$

Table 5 gives the corresponding values of c(using interpolation)

Value of $c = \frac{1}{\sqrt[3]{1-(1-m)z}}$

Point	point 0	point 1	point 2	point 3	point 4	point 5	point 6	point 7	point 8	point 9	point 10
m	z=1.0	z=.9	z=.8	z=.7	z=.6	z=.5	z=.4	z=.3	z=.2	z=.1	z=.0
0.175	1.796	1.625	1.433	1.333	1.256	1.194	1.143	1.099	1.062	1.029	1.000

Table 4 gives for $\frac{W_s}{W_c} = 3.5$, corresponding values of $\frac{L^2}{r^2}\tan^2\varphi$ and $\tan^2\varphi = \frac{\text{values given by table 4}}{(\frac{L}{r})^2} = \frac{\text{Values given by table 4}}{(\frac{71.5}{11.9})^2}$

	Point 0	point 1	point 2	point 3	point 4	point 5	point 6	point 7	point 8	point 9	point 10
$\frac{L^2}{r^2}\tan^2\varphi$	26.676	17.780	11.734	7;629	4.844	2.964	1.706	0.883	0.369	0.089	0
$\tan^2\varphi$	0.74	.492	.325	.213	.134	.082	.047	.024	.01	.0025	0

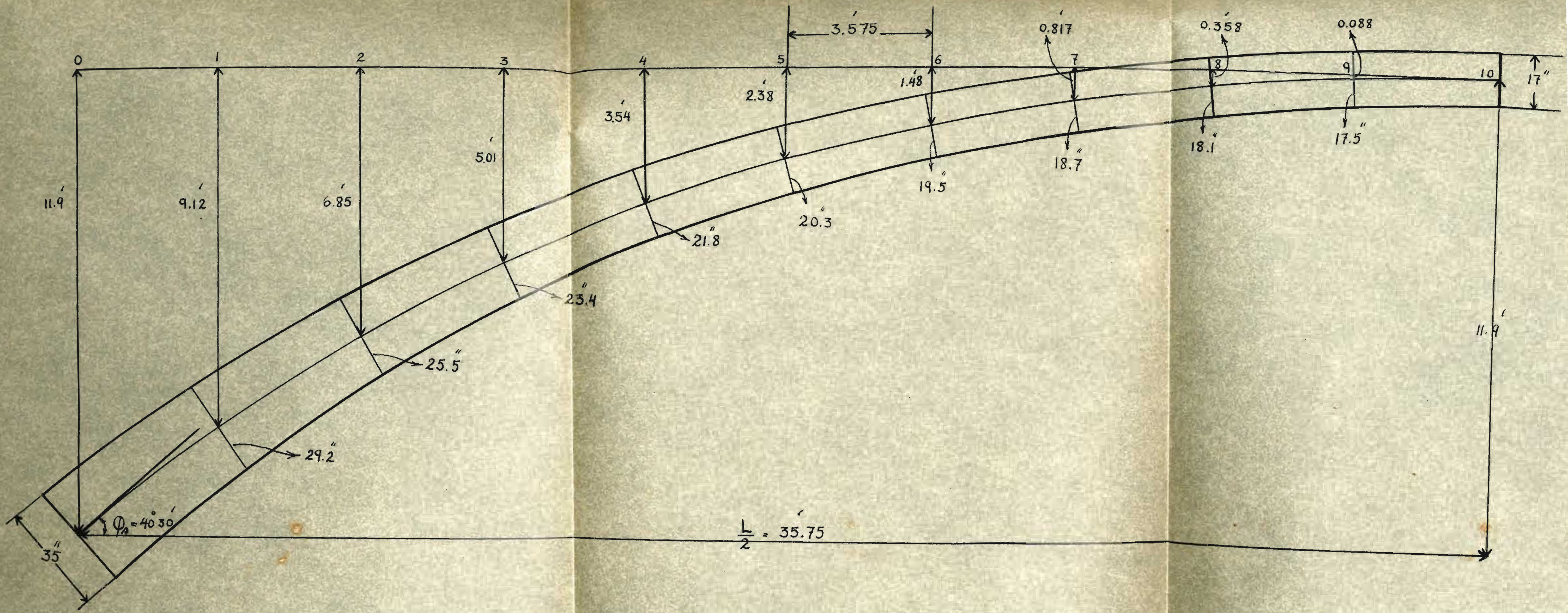
Rib thickness at each point is found by applying:

$$d_x = d_c \, c \sqrt[6]{1 + \tan^2\varphi}$$

since d_c, c and $\tan^2\varphi$ are known. The following table gives the thickness at each point in inches.

	Point 0 springing	p.1	p.2	p.3	p.4	p.5	p.6	p.7	p.8	p.9	p.10 crown
d_x	34	29.2	25.5	23.4	21.8	20.3	19.5	18.7	18.1	17.5	17

It is seen that the rib thickness at the springing comes to be 34 inches



Location of arch axis and determination of rib thickness

FIG. I

Scale 1" = 2.5'

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while the assumed one is 35 inches and hence a difference of one inch at the springing; this difference will be much less towards the crown. These dimensions will be considered except that at the springing which remains as before 35 inches.

$\frac{W_s}{W_c}$ and φ_s are measured graphically from figure I and their values agree with those given in the last trial i.e. $\frac{W_s}{W_c} = 3.5$ and $\varphi_s = 40°30'$.

Dead load equilibrium polygon:

The moment about the springing of all the dead loads on one half of the arch divided by the rise of the axis gives the crown thrust. So the arch axis is divided into ten equal parts as shown by the vertical dotted lines. The earth fill above the arch ring weighs less per cubic foot than the concrete and so the line showing the depth of earth is multiplied by $\frac{12}{15} = 0.8$, and then drawn showing the depth at which this material would stand where it compressed to the same density as the concrete. To the height of the arch and the reduced height of the earth is added the height of one foot representing the weight of materials above the earth (blockage, broken stones, idéalite) weighing the same as the concrete. On fig. II, the equilibrium polygon is drawn to scale. These areas are calculated graphically and their values are written on the same figure. The line of action of each load division is found graphically by the intersection of two lines, in each area, formed by joining the end of the basis to a point located at $\frac{1}{3}$ the opposite line from the corresponding end as shown in fig. II.

The moment about any panel point (about the dotted lines) of the dead loads between it and the crown is calculated in the following manner in the table , on the basis of the general moment equation which is reduced in this case to:

$$M = M_o + V_o x + P_a$$

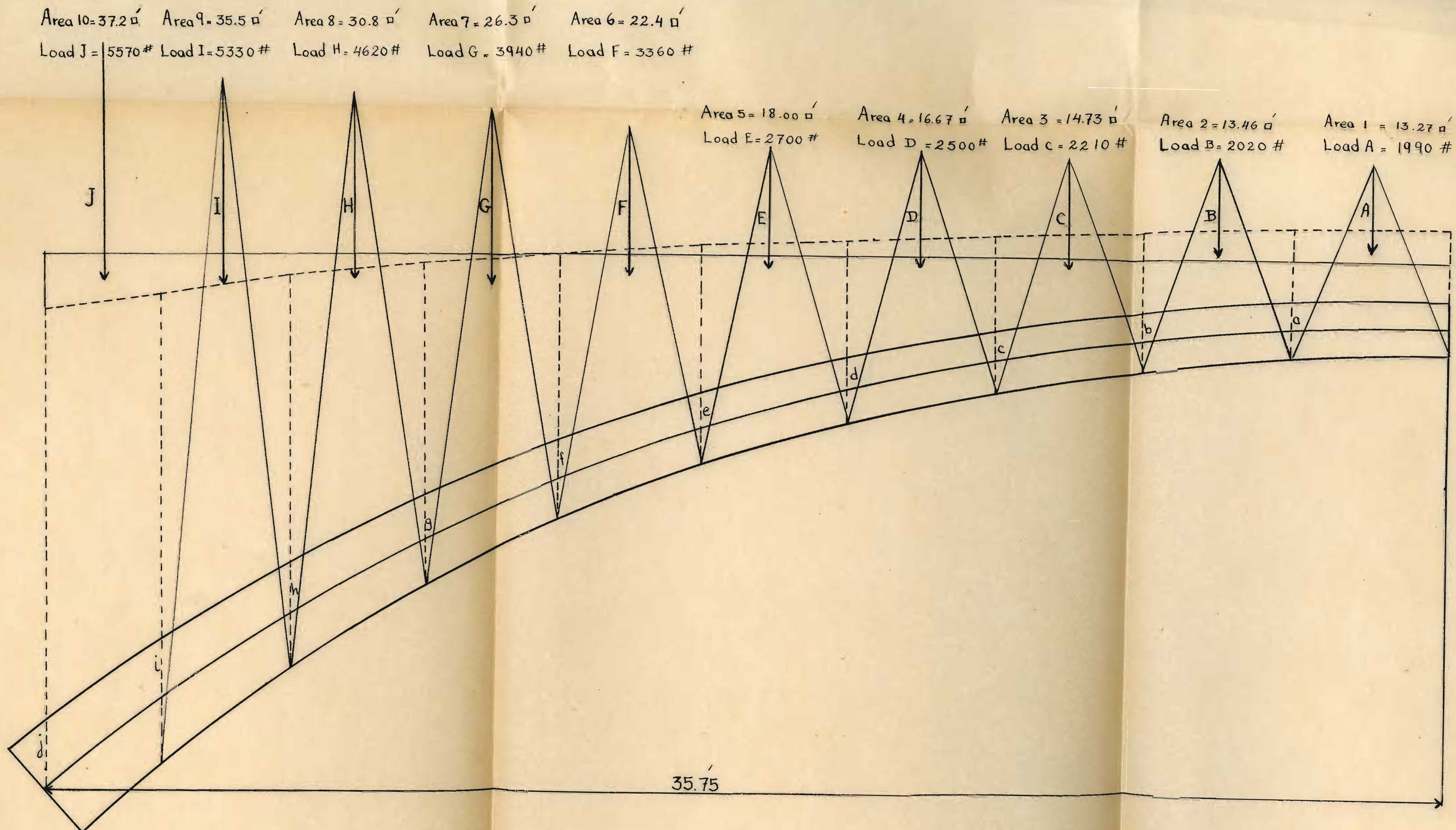

FIG II

Scale 1" = 2.5'

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Origin of moment	Load in Lbs.	Arm in ft.	Moment in ft. Lbs.
a	A = 1,990	2.00	3,980
b	A = 1,990	3.82	7,600
	B = 2,020	1.90	3,840
			15,420
c	4,010	3.80	15,200
	C = 2,210	1.85	4,080
			34,700
d	6,220	3.75	23,300
	D = 2,500	1.88	4,700
			62,700
e	8,720	3.70	32,300
	E = 2,700	1.75	4,730
			99,730
f	11,420	3.62	41,350
	F = 3,360	1.75	5,900
			146,980
g	14,780	3.50	51,800
	G = 3,940	1.70	6,700
			205,480
h	18,720	3.38	63,200
	H = 4,620	1.63	7,520
			276,200
i	23,340	3.25	75,800
	I = 5,330	1.55	8,300
			360,300
j	28,670	3.00	86,000
	J = 5,570	1.50	8,400
	34,240		454,700

Dead Thrust at crown $= \frac{\text{Dead moment}}{\text{Rise}} = \frac{454700}{11.9} = 38,200$ Lbs.

Vertical dead load at springing = 34,240 Lbs.

Live loads:

Values of $W_L L$; $W_L L^2$, $\frac{W_L L^2}{r}$

$W_L = 120$ Lbs. per ft. ; $L = 71.5$ ft. ; $r = 11.9$ ft.

Therefore:

$$W_L L = 120 \times 71.5 = 8,572.5 \text{ Lbs.}$$

$$W_L L^2 = 120 \times \overline{71.5}^2 = 612,000 \text{ Ft. Lbs.}$$

$$\frac{W_L L^2}{r} = \frac{120 \times \overline{71.5}^2}{11.9} = 51,500 \text{ Lbs.}$$

With the aid of influence lines which show clearly the position of the loading for maximum stress at the critical sections (crown, springing) special plates are prepared by Mr Whitney, which give coefficients for obtaining the maximum values of live load moment at the crown, quaterpoint and springing, with the corresponding horizontal component of crown thrust. In this preliminary analysis, critical stresses due to maximum positive moment at crown and maximum negative moment at springing will be considered only.

Live load thrust for maximum positive moment at crown is given by:

$$H_l = C_l \frac{W_L L^2}{r}$$

For N = 0.20 and m = 0.175 , fig. 104, page 303, gives a value of $C_l = 0.0723$

Hence :

$$H_l = 0.0723 \times 51,500 = 3720 \text{ Lbs.}$$

Maximum positive live load moment at crown is given by :

$$M_l = K_l W_L L^2$$

For N =0.20 and m = 0.175 , Fig. 104 gives a value of $K_l = 0.0052$

Hence :

$$M_l = 0.0052 \times 612,000 = 3180 \text{ ft. Lbs.}$$

Rib-shortening:

Any load, live or dead, causes compression in an arch rib and ~~consequen~~

consequently a general shortening of the fibers. This shortening sets up a stress of exactly the same sort as a fall in temperature and therefore it must be combined with thelive and dead load stresses to give the true stress. This effect is called rib-shortening. The amount of horizontal pull on the abutments thus induced may be expressed as:

$$H_{R.S.} = - H u'$$

where H is the thrust produced by the live and dead loading and

$$u' = \frac{I_c}{A_c\ C'_m\ Cr^2}$$

Here, at the crown $I_c = 6330\ in^4$; $A_c = 237.6$ sq. in.; $r = 11.9 \times 12$ in.

C and C_m' are coefficients given by table 7 , page 302 and fig. 111. page 307 respectively. In this case, for t. $\frac{11.9}{71.5} = 0.166$ and $m = 0.175$, fig. 111 gives a value of $C_m' = 1.175$. Table 7, gives a value of ~~C = 0.030~~ C = 0.0304.

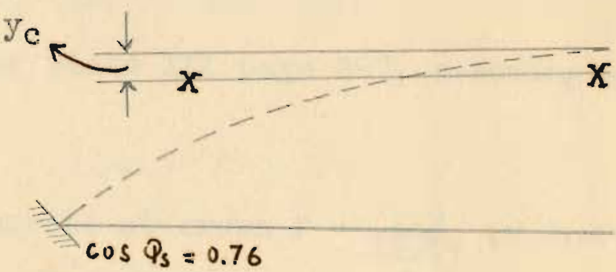
Therefore:

$$u' = \frac{6330}{237.6 \times 1.175 \times 0.0304 \times (11.9)^2 \times (12)^2} = 0.0366$$

$$H_{R.S.} = - (38,200 + 3720) \times 0.0366 = -1520 \text{ Lbs.}$$

Temperature stresses (Whitney):

It is necessary to locate the reference axis. Table 6, page 302 prepared by Mr Whitney, gives for a certain value of N and m, the ratio $\frac{y_c}{r} = 0.1865$, where ~~the~~ y_c is the distance between the reference axis and the tangent line drawn to the axis of the arch at the crown as shown in Diag. For m = 0.175 and N =0.20



$$\frac{y_c}{r} = 0.1865$$

$$y_c = 0.1865 \times 11.9 = 2.22 \text{ ft.}$$

Temperature thrust is given by

$$H_t = C_t \frac{t\ I_c}{r^2}$$

where $C_t = \frac{\alpha E}{C}$ (α being the coefficient of expansion = 0.000006)

For N = 0.20 and m = 0.175, fig. 110 page 306, gives a value of C_t = 2.73

t = temperature change in Degrees Fahrenheit and I_c as before, r = rise in ft.

For 50° fall

$$H_t = -2.73 \times \frac{50 \times 6,330}{(11.9)^2} = -6120 \text{ Lbs.}$$

$$M_c = 6,120 \times 2.22 = +13,600 \text{ ft. Lbs.}$$

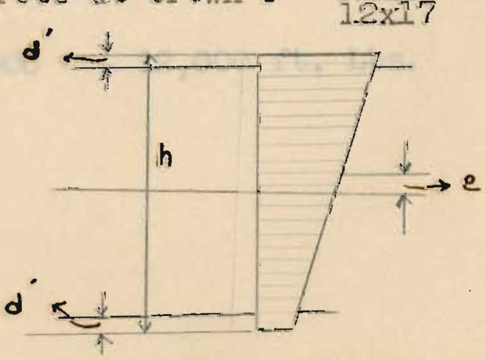
The following table, done for maximum moment at crown, gives the combination of thrust and moment respectively, due to dead and live loads, rib-shortening and temperature.

Maximum positive moment at Crown (Whitney)

	Thrust = H_o	Moment = M_c
Dead	38,200	—
Live	3,720	3,180
Rib-shortening	- 1,520	1520x2.22 =3,380
	40,400	6,560
Dead & Live	41,920	3,180
Temperature	- 6,120	13,600
Rib-shortening	- 1,520	3,380
	34,300	20,160

Approximate stress analysis: Using plate XII page 397, assuming always $\frac{d'}{h} = \frac{1}{10}$

Actual percentage of steel to concrete at crown $P = \frac{1.2}{12 \times 17} = 0.0059$ in each face.



For dead + live + rib-shortening:

Eccentricity $e = \frac{M_c}{H_o}$

$$e = \frac{M_c}{H_o} = \frac{6560 \times 12}{40,400} = 1.95 \text{ in}$$

$$\frac{e}{h} = \frac{1.95}{17} = 0.115$$

For P = 0.0059 and $\frac{e}{h} = 0.115$, plate XII gives $\frac{M}{b\,h^2\,f_c} = 0.0835$

Therefore:

$$\text{Maximum } f_c = \frac{6560 \times 12}{12 \times (17)^2 \times 0.0835} = \underline{272} \quad \text{Lbs/sq.in.}$$

Dead and live plus temperature plus rib-shortening (plateXII)

$$e = \frac{20,160 \times 12}{34,300} = 7.05 \text{ in.}$$

$$\frac{e}{h} = \frac{7.05}{17} = 0.415$$

$$\text{Maximum } f_c = \frac{20160 \times 12}{12 \times (17)^2 \times 0.137} = 510 \text{ Lbs/sq.in.}$$

<u>Maximum negative</u> moment at springing:

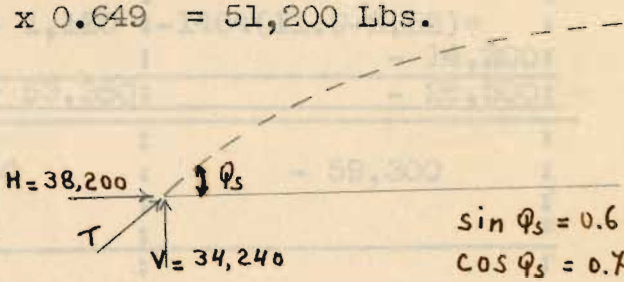
Dead load thrust at springing:

$T = H \cos \varphi_s + V \sin \varphi_s$

$= 38,200 \times 0.76 + 34,240 \times 0.649 = 51,200$ Lbs.

Live load thrust:

$$H_l = C_l \frac{W_L L^2}{r}$$



$\sin \varphi_s = 0.649$
$\cos \varphi_s = 0.76$

Fig 107 gives a value of $C_l = 0.035$

$H_l = 0.035 \times 51,500 = 1,800$ Lbs.

Fig 102 page 296, gives:

$V_l = 0.34\, W_L\, L = 0.34 \times 8,5725 = 2920$ Lbs.

Therefore

$T_l = H_l \cos \varphi_s + V_l \sin \varphi_s$

$= 1,800 \times 0.76 + 2,900 \times 0.649 = 3270$ Lbs.

Fig 107 gives $K_l = 0.0196$

$M_l = -0.0196 \times 612,000 = -12,000$ ft. Lbs.

Rib-shortening:

$H_{R.S.} = -H u'$

H = Dead thrust + live thrust

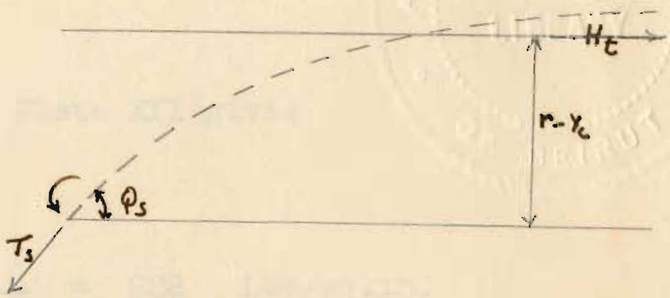
= 38,200 + 1,800 = 40,000

u' = 0.0366 (as before)

Therefore:

$H_{R.S.}$ = -40,000 x 0.0366

= - 1,464 Lbs.



Temperature stresses :

$M_s = - H_t (r-y_c) = - 6,120 \times (11.9 - 2.22)$

= -59,300 ft.Lbs.

$T_s = - H_t \cos \phi_s = -6,120 \times 0.76 = - 4,650$ Lbs.

Maximum negative moment at springing

	T_s	M_s
Dead	51,200	———
Live	3,270	- 12,000
Rib-shortening	-1464x.76 = - 1,120	-1464(11.9-2.22)= - 14,300
	+ 53,350	- 26,300
Temperature	- 4,650	- 59,300
	+ 48,700	-85,600

Stresses at springing:

Approximate stress analysis (using plate XII)

Dead + live + rib-shortening :

Percentage of steel at springing $P = \frac{1.2}{12 \times 35} = 0.00286$ in each face.

$$e = \frac{M_s}{T_s} = \frac{26,300 \times 12}{53,350} = 5.92 \text{ in.}$$

$$\frac{e}{h} = \frac{5.92}{35} = 0.169$$

For $P = 0.00286$ and $\frac{e}{h} = 0.169$, Plate XII gives

$$\frac{M}{b\,h^2\,f_c} = 0.094$$

$$\text{maximum } f_c = \frac{26,300 \times 12}{12 \times (35)^2 \times 0.094} = \underline{228} \text{ Lbs/sq.in.}$$

Dead + live + temperature + rib-shortening (using plate XIII)

$$e = \frac{85,600 \times 12}{48,700} = 21.1 \text{ in.}$$

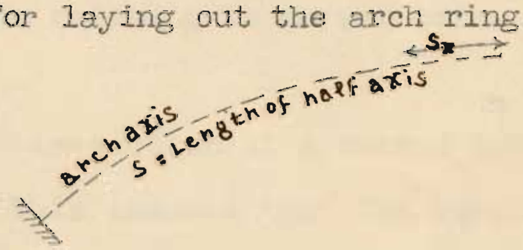
$$\frac{e}{h} = \frac{21.1}{35} = 0.604$$

Therefore:

$$\text{maximum } f_c = \frac{85,600 \times 12}{12 \times (35)^2 \times 0.108} = 650 \text{ Lbs/sq.in.}$$

Checking arch ring by Cochrane's method:

Table 8, page 318, is given by Cochrane for laying out the arch ring. It is reproduced here for comparison with Whitney's section and also because of its ease of application. Anyhow, the thickness computed by Whitney's method is thicker than that recommanded by Cochrane and more safer specially for long spans. For $\frac{d_s}{d_c} = \frac{35}{17} = 2.06$ table 8, gives:



S_x/S	d_x/d_c	d_x(inches)	S_x/S	d_x/d_c	d_x(inches)
0	1.0000	17.00	0.55	1.090	18.53
0.05	1.0047	17.08	0.65	1.180	20.06
0.15	1.0140	17.24	0.75	1.335	22.70
0.25	1.0240	17.40	0.85	1.587	26.89
0.35	1.0330	17.60	0.95	1.893	32.18
0.45	1.048	17.82	1.00	2.060	35.02

The section given with respect to table 8, is approximately the same as that used.

Computations for hingeless arch by method of Least Work:

By making certain reasonable assumptions, the application of the Theorem of Least Work to arch analysis is greatly simplified. The equations thus derived, give results that are closely the same as those obtained by the better known method based on the principles relating to curved bars. The simplifying assumptions are:

a/ The distribution of stress over the cross-section of a curved bar is the same as thogh the bar were straight. This assumes that the familiar expression, $f = \frac{P}{A} \pm \frac{My}{I}$, applies instead of the exact formula which involves the radius of curvature.

b/ The total direct or axial thrust at all sections of the arch is the same and equal to the horizontal component of crown thrust.

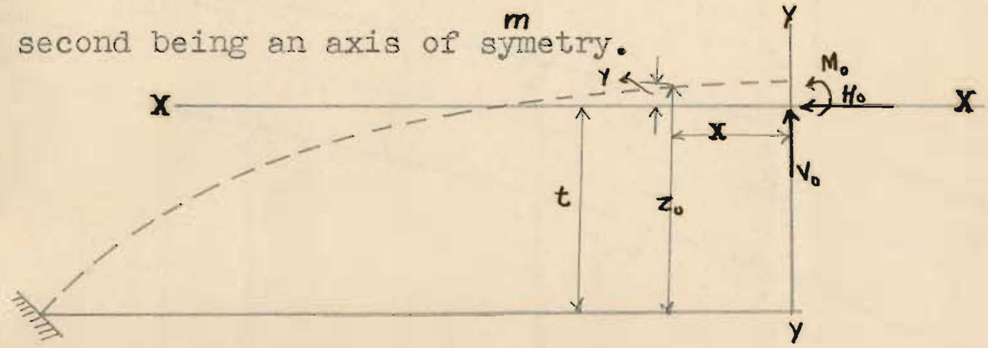
c/ The work due to shear may be neglected as is done universally in the study of indeterminate structures.

So the analysis of a hingeless arch based on the method of Least Work is a proper one since the magnitude of the error was found to be a fraction of one per cent in bigger arches.

In addition to the notation already given, the following will be use

X X and Y Y are two reference axes, the first-located so that $\int \frac{Y\,dS}{E\,I} = 0$

and the second being an axis of symetry.



M_o, H_o, V_o, being the moment, horizontal component of thrust and the shear acting at the level of the reference axis X X.

m_L = moment (ft.Lbs.) at any point (x,y) on the left half axis of tl loads between that point and the crown, the left half of the arch being considered as a cantilever beam fixed at the abutment with the right half replaced by its equivalent M_o, H_o, and V_o.

m_R = same for right half of arch

M_c = actual bending moment at crown (ft.Lbs.) = $M_o - H_o\ (r-t)$

I = moment of inertia (ft^4) at any section normal to axis

A = area (sq.ft.) at any section normal to axis

E. = modulus of elasticity (Lbs. per sq.ft.)

The following working formulas obtained by dividing half the arch int equal parts its axis are:

$$M_o = \frac{\sum \frac{(m_L + m_R)}{I}}{2 \sum \frac{1}{I}}$$

$$H_o = \frac{-\sum \frac{(m_L + m_R) y}{I}}{2 \sum \frac{y^2}{I} + 2 \sum \frac{1}{A}}$$

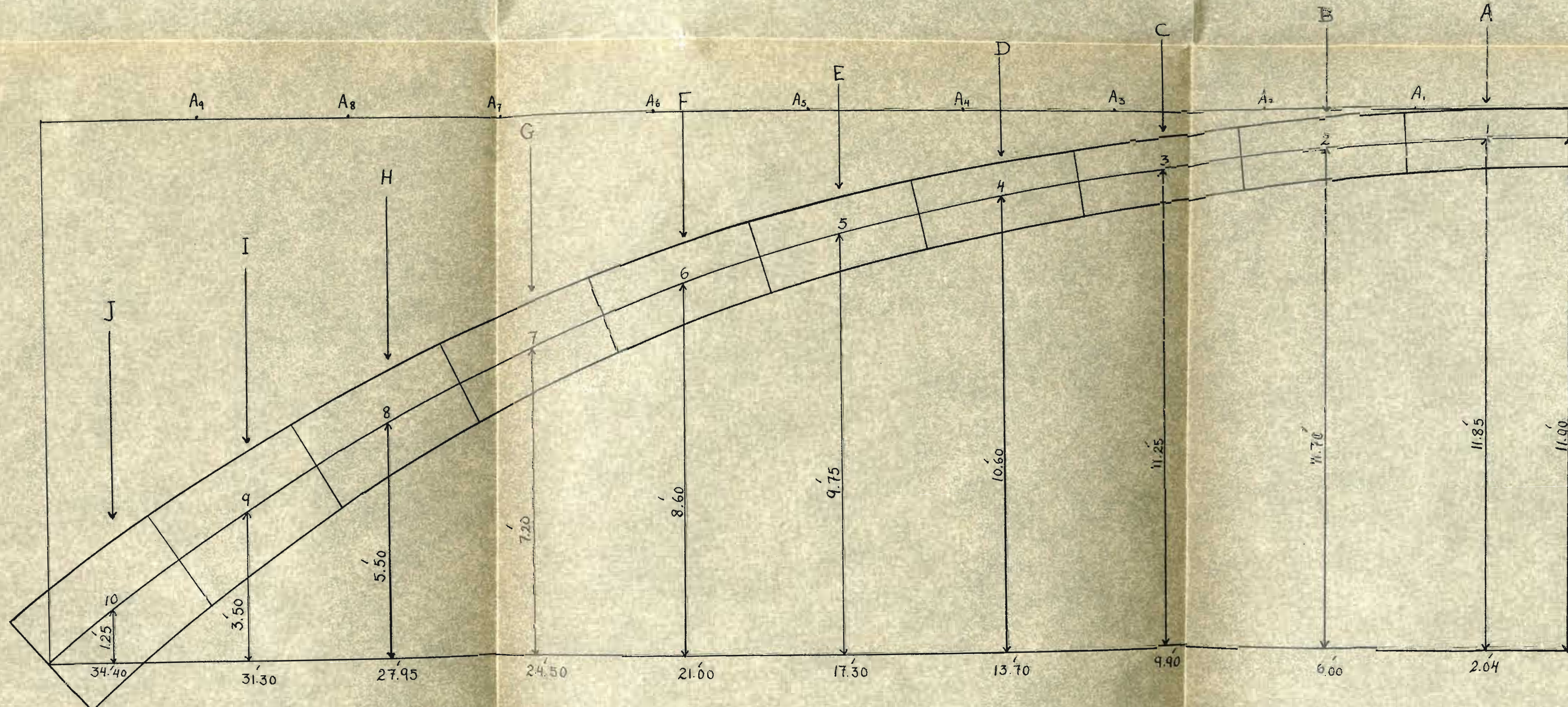


FIG. III

Scale 1" = 2.5'

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$$V_o = \frac{\sum \frac{(m_L - m_R)x}{I}}{2 \sum \frac{x^2}{I}}$$

$$t = \frac{\sum \frac{Z_o}{I}}{\sum \frac{1}{I}}$$

Temperature thrust $H_o = \frac{\alpha \, t \, n \, E}{\sum \frac{y^2}{I} + \sum \frac{1}{A}}$

where α = coefficient of expansion = 0.000006

t = temperature change in degrees Fahrenheit

n = number of divisions = 10

Z_o, x, y, d_x and consequently I are measured graphically from fig 3.

Computations:

In order to set a table to give the reference axis, it is necessary to have, with the aid of Fig. 3 :

Area of any section in ft. $A = d_x \times 1 + \frac{(15-1) \times 2 \times 1.2}{144} = d_x + 0.233$ sq.ft.

$I = \frac{d_x^3 \times 1}{12} + A_s d^2$ (d being the distance from the neutral axis to the steel)

$= \frac{d_x^3}{12} + 0.233 \left(\frac{d_x}{2} - \frac{2}{12}\right)^2 = \frac{d_x^3}{12} + 0.058 (d_x - 0.33)^2$

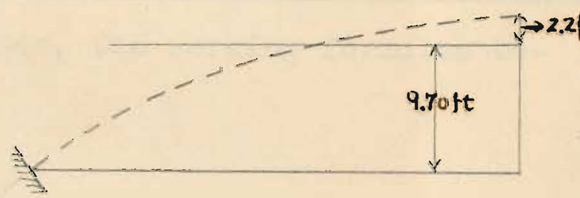
Analysis by Method of Least Work : Reference axis

Sect	Z_o	d_x	$d_x^3/12$	$.058(d_x-.33)^2$	I	Z_o/I	$\frac{1}{I}$
1	11.85	1.43	0.244	0.0702	0.3142	37.70	3.18
2	11.70	1.48	0.270	0.0768	0.3468	33.80	2.88
3	11.25	1.55	0.310	0.0864	0.3964	28.40	2.53
Total forwarded						99.90	8.59

continued on next page

Total copied from previous page						99.90	8.59
4	10.60	1.61	0.348	0.0936	0.4416	24.00	2.27
5	9.75	1.67	0.388	0.1040	0.4920	19.80	2.03
6	8.60	1.80	0.485	0.1253	0.6103	14.10	1.64
7	7.20	1.90	0.572	0.1430	0.7150	10.10	1.40
8	5.50	2.09	0.760	0.1800	0.9400	5.85	1.06
9	3.50	2.40	1.155	0.2484	1.4034	2.50	0.72
10	1.25	2.75	1.733	0.3395	2.0725	0.60	0.48
						176.85	18.19

$\Sigma \frac{Z_o}{I} = 176.85$; $\Sigma \frac{1}{I} = 18.19$



Therefore:

$$t = \frac{\Sigma \frac{Z_o}{I}}{\Sigma \frac{1}{I}} = \frac{176.85}{18.19} = 9.70 \text{ ft.}$$

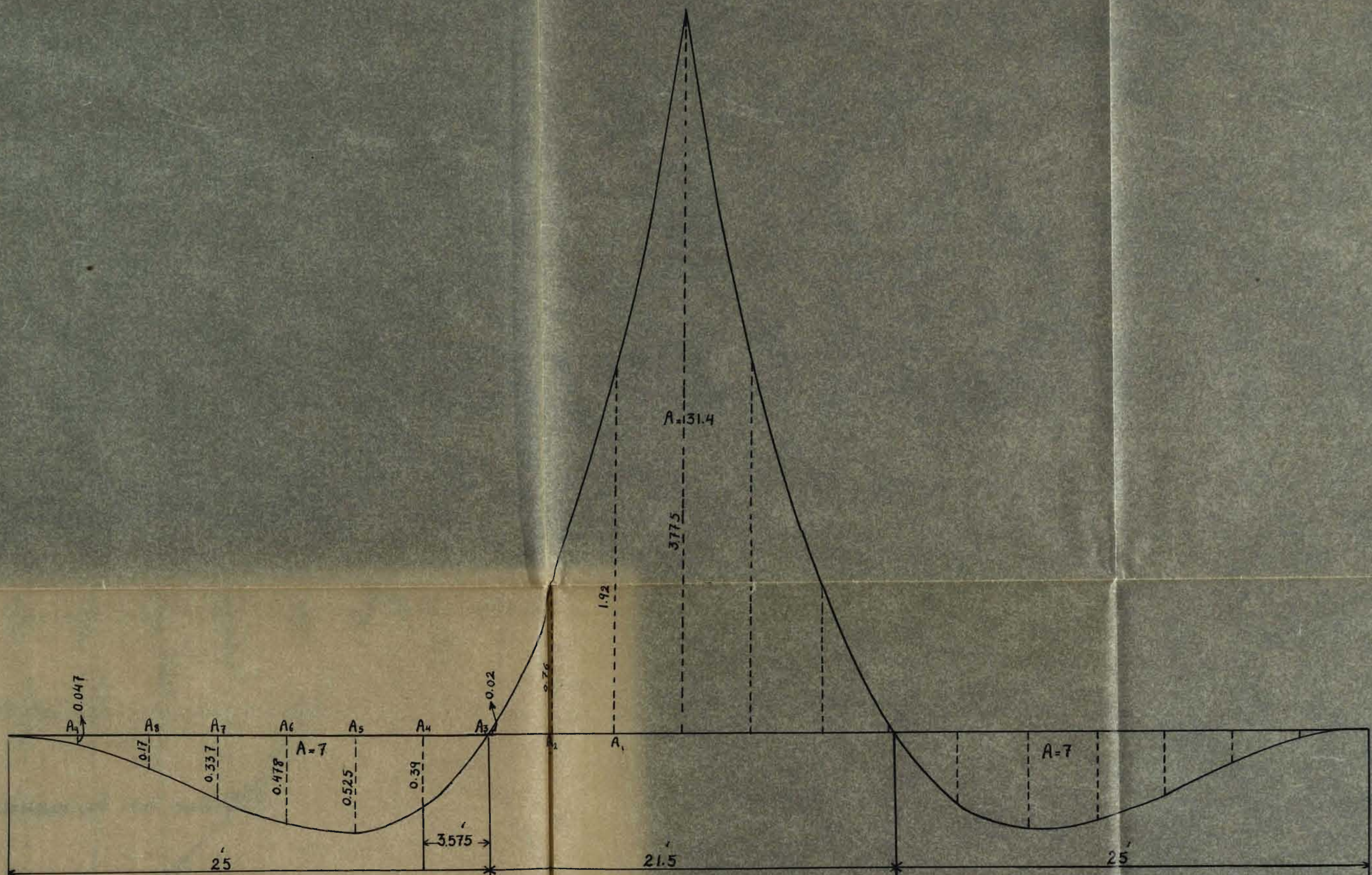
$$y_c = r - t = 11.90 - 9.70 = 2.20 \text{ ft.}$$

The value of y_c in the preliminary analysis being 2.22 ft., the difference is 0.02 ft.

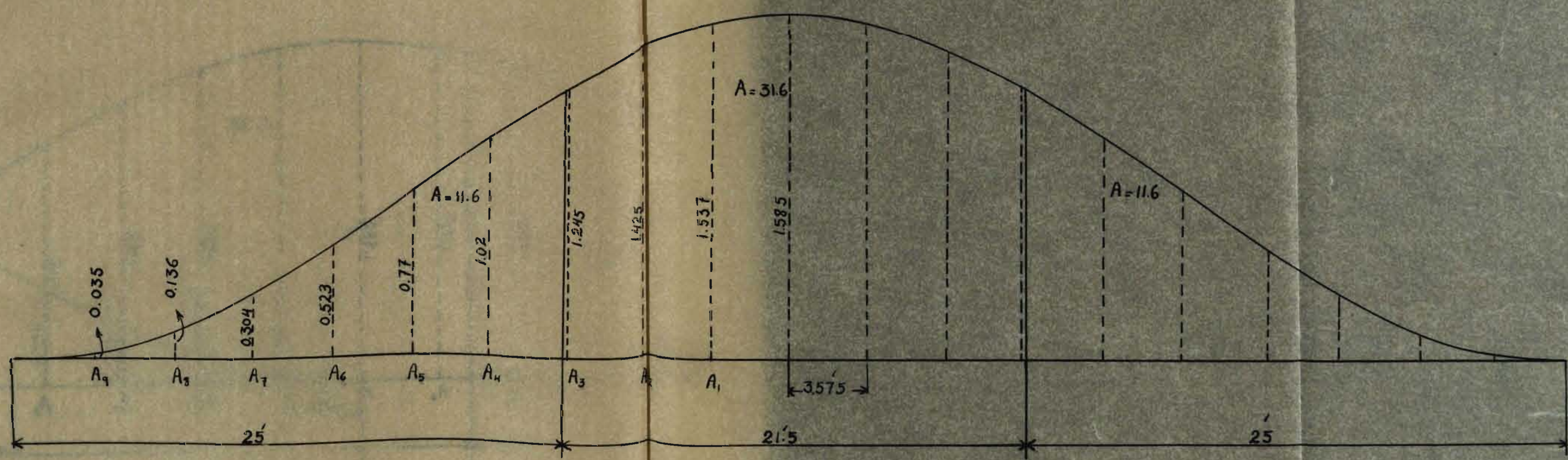
The following table gives the arch constants which figure in the denominator of the working formulas mentioned before, $\Sigma \frac{1}{I}$ was given in the previous one:

Arch Constants

Sect.	x	$y = Z_o - t$	$A = d_x + 0.233$	x^2/I	y^2/I	$\frac{1}{A}$
1	2.04	2.15	1.663	13.25	14.70	0.602
2	6.00	2.00	1.713	103.70	11.50	0.584
Total forwarded				116.95	26.20	1.186



Influence line for moment at crown section



Influence line for thrust at crown section

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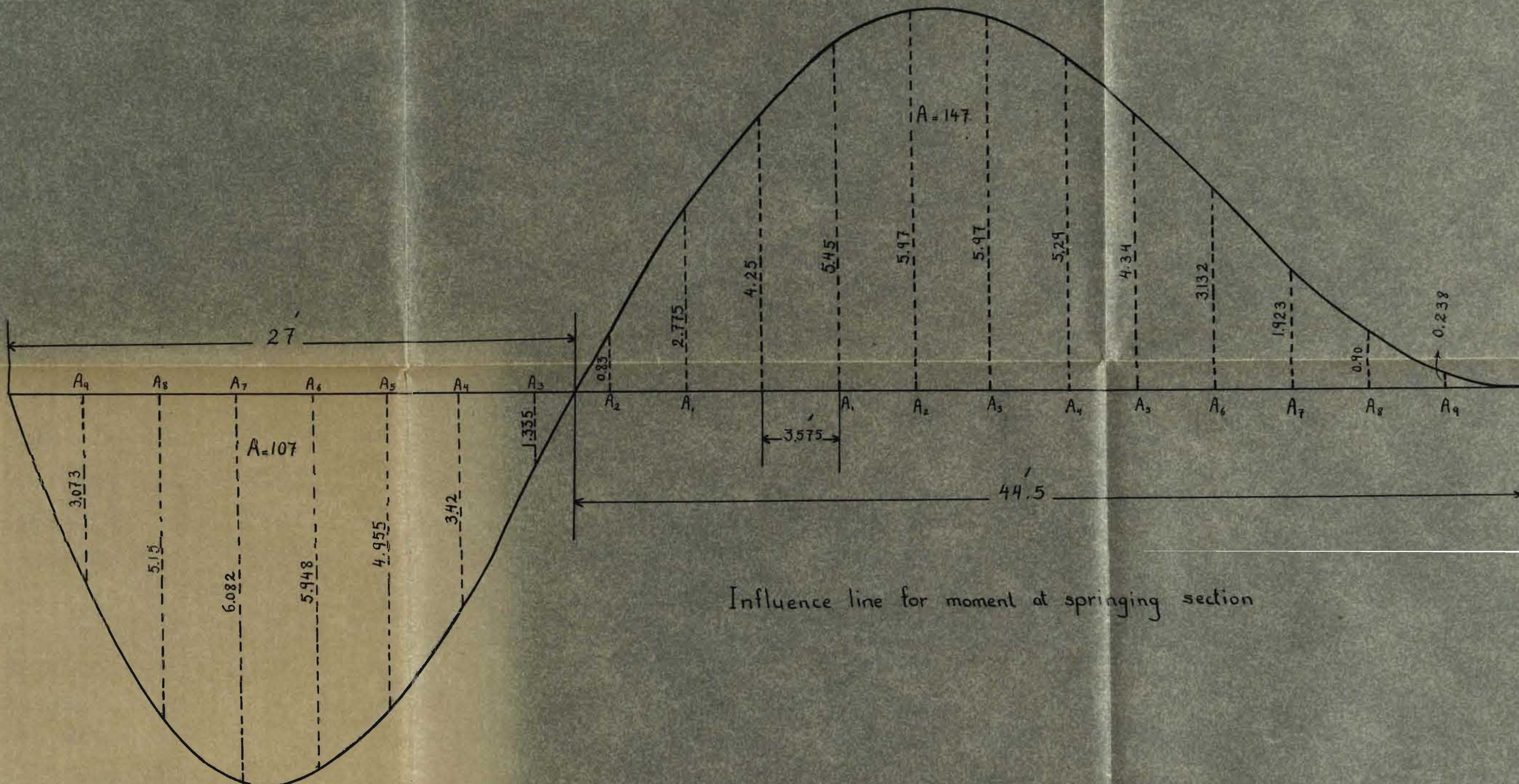
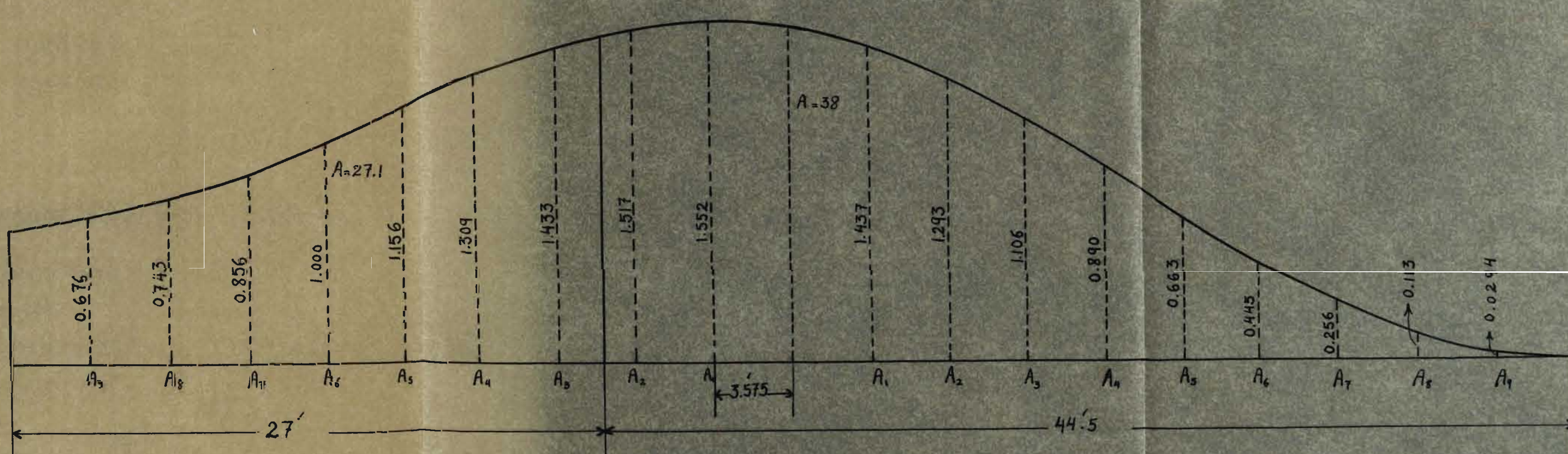
FIG. IV



Influence line for moment at springing section



Influence line for thrust at springing section

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FIG. V

Total copied from previous page					116.95	26.20	1.186
3	9.90	1.55	1.783	247.00		6.07	0.560
4	13.70	0.90	1.843	425.00		1.83	0.543
5	17.30	0.05	1.903	608.00		0.00	0.526
6	21.00	- 1.10	2.033	723.00		1.98	0.500
7	24.50	- 2.50	2.133	840.00		8.75	0.468
8	27.95	- 4.20	2.323	832.00		18.75	0.431
9	31.30	- 6.20	2.633	700.00		27.40	0.380
10	34.40	- 8.45	2.983	572.00		34.50	0.335
				5064.00		125.48	4.929

Substituting arch constants by their values, the working formulas become:

$$M_o = \frac{\sum \frac{m_L + m_R}{I}}{2\sum \frac{1}{I} = 36.38}$$

$$V_o = \frac{\sum \frac{(m_L - m_R)\,x}{I}}{2\sum \frac{x^2}{I} = 10,128}$$

$$H_o = \frac{-\sum \frac{(m_L + m_R)\,y}{I}}{2\sum \frac{y^2}{I} + 2\sum \frac{1}{A} = 260.82}$$

Temperature stress $H_o = \dfrac{\alpha\ t\ n\ E}{\sum \frac{y^2}{I} + \sum \frac{1}{A} = 130.41}$

Calculation of m_L for dead loads:

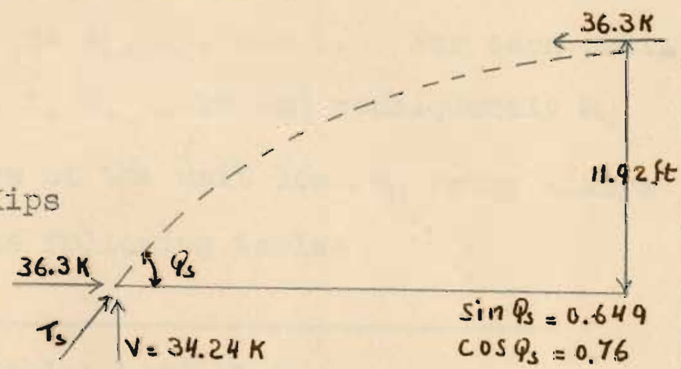
Referring to Fig 3, moments are taken with respect to points 1, 2, 3, etc... Dead loads have been previously obtained. Moment arms are measur graphically from Fig 3. These calculations are shown in the following table:

m_L for dead loads

Origin of moment	Load in Kips	Arm in ft.	Moment in Kips
1	1.990	0	0
2	1.990	3.96	7.88
	2.020		7.88
3	4.010	3.90	15.60
	2.210		23.48
4	6.220	3.80	23.67
	2.500		47.15
5	8.720	3.60	31.40
	2.700		78.55
6	11.420	3.70	42.30
	3.360		120.85
7	14.780	3.50	51.75
	3.940		172.60
8	18.720	3.45	64.60
	4.620		237.20
9	23.340	3.35	78.20
	5.330		315.40
10	28.670	3.10	89.10
	5.570		404.50
	34.240	1.35	46.25
Springing			450.75

Dead Stresses

Section	$m_L = m_R$	$\frac{m_L + m_R}{I}$	$\frac{(m_L + m_R)y}{I}$
1	0	0	0
2	7.88	45.5	+ 91.00
3	23.48	118.3	+183.36
4	47.15	213.3	+191.97
5	78.55	319.0	+ 15.95 (+482.28)
6	120.85	396.5	- 436.15
7	172.60	482.0	-1,205.00
8	237.20	503.0	-2,112.60
9	315.40	448.0	-2,777.60
10	404.50	403.0	-3,405.35 (-9,936.70)
		2,928	-9,455



Hence for dead loads:

For crown:

$$M_o = \frac{2,928}{36.38} = 80.5 \text{ ft.Kips}$$

$$H_o = \frac{-(-9455)}{260.82} = 36.30 \text{ Kips}$$

$$M_c = 80.5 - 36.30 \times 2.2 = 700 \text{ ft. Lbs.}$$

Eccentricity $e = \frac{M_c}{H_o} = \frac{700 \times 12}{36,300} = 0.2$ in. $= 0.02$ ft. above axis

For springing :

$$M_s = -450.75 + 36.30\,(11.9 + 0.02) = -17.50 \text{ ft. Kips}$$

$$T_s = 36.30 \times 0.76 + 34.24 \times 0.649$$

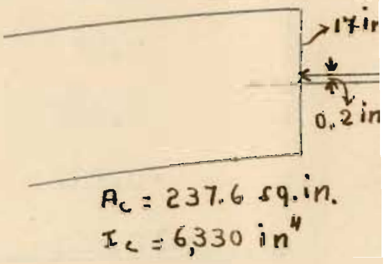
$$= 49.80 \text{ Kips}$$

$$e = \frac{M_s}{T_s}$$

$$= \frac{-17{,}500 \times 12}{49{,}800} = -4.22 \text{ in.} = 0.35 \text{ ft below axis}$$

Stresses - Crown:

$$f_c = \frac{H_o}{A_c} \pm \frac{M_c y}{I_c} = \frac{36{,}300}{237.6} \pm \frac{700 \times 12 \times 8.5}{6{,}330}$$



$= 153 \pm 11 = 164$ Lbs/sq. in. top fiber

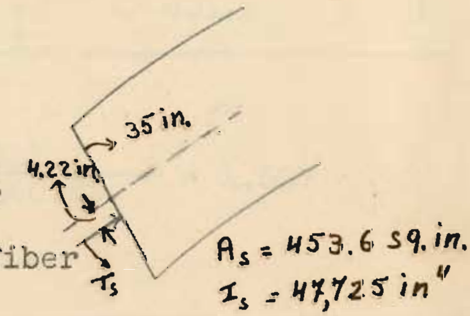
$= 142$ Lbs/sq.in. bottom fiber

Springing:

$$f_c = \frac{49{,}800}{453.6} \pm \frac{17{,}500 \times 12 \times 17.5}{47{,}725}$$

$= 110 \mp 77 = 33$ Lbs/sq.in. top fiber

$= 187$ Lbs/sq.in. bottom fiber



Live loads:

Referring also to fig3, half the span is divided into ten equal parts and a unit load is put respecively at A_1, A_2, etc... For each position, moments are taken around points 1, 2, 3, .. 10 and consequently M_o V_o, and H_o are computed for each position of the unit load. m_R being always zero. The computations are found in the following tables

Least Work: Influence Table: Load at A_1

Section	$m_L = m_L + m_R = m_L - m_R$	$(m_L+m_R)\frac{1}{I}$	$(m_L-m_R)\frac{x}{I}$	$(m_L-m_R)\frac{y}{I}$
2	2.43	7.00	42	+ 14.0
3	6.33	16.00	158	+ 24.8
4	10.13	22.90	314	+ 20.6
Total forwarded		45.90	514	+ 59.4

Total copied			45.90	514	+ 59.4
5		13.73	27.90	482	+ 1.4 (+ 60.8)
6		17.43	28.60	600	- 31.5
7		20.93	29.30	718	- 73.2
8		24.38	26.00	727	- 109.2
9		27.73	19.70	616	- 122.0
10		30.83	14.90	512	- 126.0 (-461.9)
			192.30	4,169	- 401.1

$$M_o = \frac{192.30}{36.38} = 5.30$$

$$V_o = \frac{4,169}{10,128} = 0.412$$

$$H_o = \frac{401.1}{260.82} = 1.537$$

Load at A_2

Section	$m_L = m_L + m_R$ $= m_L - m_R$	$(m_L + m_R)\frac{1}{I}$	$(m_L - m_R)\frac{x}{I}$	$(m_L - m_R)\frac{y}{I}$
3	2.75	6.95	69	+ 10.8
4	6.55	14.80	203	+ 13.3
5	10.15	20.60	356	+ 1.0 (+ 25.1)
6	13.85	22.70	477	- 25.0
7	17.35	24.25	594	- 60.6
8	20.80	22.10	641	- 93.0
9	24.15	17.25	540	- 107.0
10	27.25	13.15	453	- 111.5 (-397.1)
		141.80	3,333	- 372

$M_o = \frac{-141.80}{36.38} = 3.9$ $\qquad H_o = \frac{372}{260.82} = 1.425$

$V_o = \frac{3,333}{10,128} = 0.328$

Load at A_3

Section	$m_L = m_L + m_R$ $= m_L - m_R$	$(m_L - m_R)\frac{1}{I}$	$(m_L - m_R)\frac{X}{I}$	$(m_L - m_R)\frac{Y}{I}$
4	2.98	6.74	92	+ 6.05
5	6.58	13.36	231	" 0.65 (+6.70)
6	10.28	16.85	354	- 18.55
7	13.78	19.30	473	- 48.25
8	17.23	18.35	513	- 77.00
9	20.58	14.65	459	- 91.00
10	23.68	11.40	392	- 96.50 (-331.3)
		100.65	2514	-324.60

$M_o = \frac{100.65}{36.38} = 2.76$ $\qquad H_o = \frac{324.60}{260.82} = 1.245$

$V_o = \frac{2,514}{10,128} = 0.248$

Load at A_4

5	3.00	6.10	106	+ 0.3 (+ 0.3)
6	6.70	10.95	230	- 12.05
7	10.20	14.25	349	- 35.60
8	13.65	14.55	407	- 61.20
9	17.00	12.10	379	- 75.00
10	20.10	9.75	335	- 82.25 (- 266.1)
		67.7	1806	- 265.8

$M_o = \frac{67.7}{36.38} = 1.86$; $V_o = \frac{1,806}{10,128} = 0.178$; $H_o = \frac{265.8}{260.82} = 1.02$

Load at A_5

6	3.13	5.13	108	- 5.65
7	6.63	9.28	228	- 23.20
8	10.08	10.73	300	- 45.10
9	13.43	9.58	300	- 59.50
10	16.53	7.98	275	- 67.50
		42.70	1,211	-200.95

$M_o = \frac{42.70}{36.38} = 1.17$; $V_o = \frac{1,211}{10,128} = 0.12$; $H_o = \frac{200.95}{260.82} = 0.77$

Load at A_6

7	3.05	4.27	105	- 10.65
8	6.50	6.92	193	- 29.10
9	9.85	7.02	220	- 43.50
10	12.95	6.26	215	- 53.00
		24.47	733	- 136.25

$M_o = \frac{24.47}{36.38} = 0.672$; $V_o = \frac{733}{10,128} = 0.073$; $H_o = \frac{136.25}{260.82} = .523$

Load at A_7

8	2.93	3.12	87	- 13.1
9	6.28	4.48	140	- 27.8
10	9.38	4.53	156	- 38.3
		12.13	383	- 79.2

$M_o = \frac{12.13}{36.38} = 0.333$; $V_o = \frac{383}{10.128} = 0.038$; $H_o = \frac{79.2}{260.82} = 0.304$

Load at A_8

9	2.70	1.92	60	- 11.9
10	5.80	2.80	96	- 23.6
		4.72	156	- 35.5

$M_o = \frac{4.72}{36.38} = 0.13$; $V_o = \frac{156}{10,128} = 0.0154$; $H_o = \frac{35.5}{260.82} = 0.136$

Load at A_9

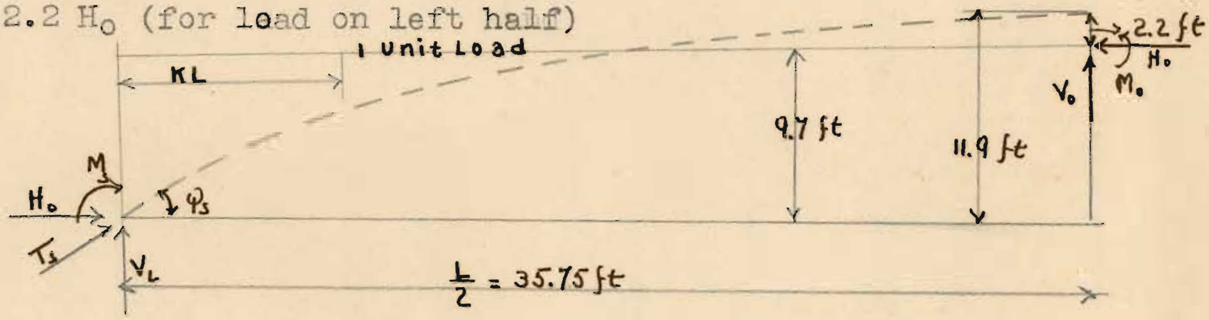
10	2.23	1.08	37	- 9.15

$M_o = \frac{1.08}{36.38} = 0.03$

$H_o = \frac{9.15}{260.82} = 0.035$

$V_o = \frac{37}{10,128} = 0.0037$

$M_C = M_o - 2.2\ H_o$ (for load on left half)



Left $M_s = M_o + 9.7\ H_o + 35.75\ V_o - K\ L$

Right $M_s = M_o + 9.7\ H_o - 35.75\ V_o$

$T_s = V \sin \varphi_s + H \cos \varphi_s = 0.649V + 0.76\ H$

Based on these relations and the known values of M_o, H_o and V_o for live load, the following table is set:

Least Work : Influence Table

Least Work : Influence Table

Unit load at

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
M_o	5.30	3.90	2.76	1.86	1.17	0.672	0.333	0.13	0.03
H_o	1.537	1.425	1.245	1.02	0.77	0.523	0.304	0.136	0.035
$V_o = V_R$	0.412	0.328	0.248	0.178	0.12	0.073	0.038	0.0154	0.0037
V_L	0.588	0.672	0.752	0.822	0.880	0.927	0.962	0.9846	0.9963
$- 2.2\ H_o$	- 3.38	- 3.14	- 2.74	- 2.25	- 1.695	- 1.15	- 0.67	- 0.30	- 0.077
M_c Crown	+ 1.92	+ 0.76	+ 0.02	- 0.39	- 0.525	- 0.478	- 0.337	- 0.17	- 0.047
$(11.9 - 2.2)H_o = 9.7\ H_o$	14.90	13.80	12.07	9.80	7.46	5.07	2.95	1.32	0.34
$35.75\ V_o$	14.75	11.73	8.86	6.37	4.29	2.61	1.36	0.55	0.132
$M_o + 9.7H_o + 35.75\ V_o$	34.950	29.43	23.690	18.03	12.920	8.352	4.643	2.00	0.502
K L	-32.175	- 28.60	- 25.025	- 21.45	- 17.875	- 14.300	- 10.725	- 7.15	- 3.575
M_s left	+ 2.775	+ 0.83	- 1.335	- 3.42	+ 4.955	- 5.948	- 6.082	- 5.15	- 3.073
M_s right	+ 5.45	+ 5.97	+ 5.97	+ 5.29	+ 4.34	+ 3.132	+ 1.923	+ 0.90	+ 0.238
$H_o \cos \varphi_s$	1.170	1.080	0.945	0.775	0.585	0.398	0.231	0.103	0.027
$V_L \sin \varphi_s$	0.382	0.437	0.488	0.534	0.571	0.602	0.625	0.640	0.649
T_s left	1.552	1.517	1.433	1.309	1.156	1.000	0.856	0.743	0.676
$H_o \cos \varphi_s$	1.170	1.080	0.945	0.775	0.585	0.398	0.231	0.103	0.027
$V_R \sin \varphi_s$	0.267	0.213	0.161	0.115	0.078	0.047	0.025	0.010	0.0024
T_s right	1.437	1.293	1.106	0.890	0.663	0.445	0.256	0.113	0.0294

The last table contains all the values of H_o, M_c, T_s, M_s for unit loads as described. Figs. 4 and 5 show the influence lines draw from these values The totals of the H_o, T_s, $+M_c$, $-M_c$, $+M_s$ and $-M_s$ areas are indicated on the thrust and moment influence lines. Influence lines for the quarter point are not calculated nor drawn. Also the influence lines for shear are not drawn because the shearing stresses are so small that they may be neglected. The product of the areas thus found by $W_L = 120$ gives the following:

Positive moment at crown = 3770 ft.Lbs.

Negative moment at crown = 1700 ft. Lbs.

Thrust with positive moment at crown = 3800 Lbs.

Thrust with negative moment at crown = 2750 Lbs.

Positive moment at springing = 17,650 ft.Lbs.

Negative moment at springing = 12,850 ft.Lbs.

Thrust with positive moment = 4,600 Lbs.

Thrust with negative moment = 3,250 Lbs.

Temperature Stress:

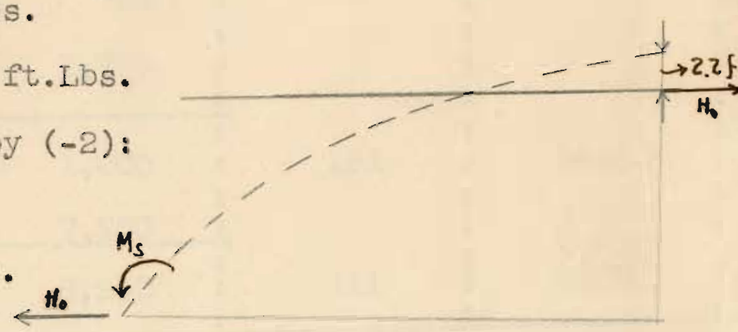
For 50° fall:

$$M_o = -\frac{0.000006 \times 50 \times 10 \times 2{,}000{,}000 \times 144}{130.41} = -6{,}600 \text{ Lbs.}$$

$M_c = 6{,}600 \times 2.2 = +14{,}500$ ft.Lbs.

$T_s = 6{,}600 \times 0.76 = -5050$ Lbs.

$M_s = 6{,}600 \times 9.70 = -64{,}000$ ft.Lbs.



For 25° rise, divide above values by (-2):

$H_o = \frac{6{,}600}{2} = +3{,}300$ Lbs.

$M_c = \frac{-14{,}500}{2} = -7{,}250$ ft. Lbs.

$T_s = \frac{5050}{2} = +2525$ Lbs.

$M_s = \frac{64{,}000}{2} = +32{,}000$ ft.Lbs.

A summary of all thrusts and moments at crown and springing by Least Work are shown in the following tables. f_c is calculated by :

$$\text{Max. } f_c = \frac{H_o}{A_c} + \frac{M_c y}{I_c}$$

y = the distance from the center of the section to the extre fiber

$$\text{Max. } f_s = n\left(\frac{H_o}{A_c} + \frac{M_c\, y_1}{I_c}\right)$$

y_1 = The distance from the center of the section to the steel

The same method is followed for f_c and f_s at the springing

Stress summary and unit stresses

$A_c = 237.6$; $y = 8.5$; $I_c = 6330$; $y_1 = 6.5$; $n = 15$

Crown		Thrust= H_o Lbs.	Moment= M_c ft.Lbs	f_c Lbs/sq.in.	f_s Lbs/sq.in.
Max	Dead	+ 36,300	+ 700		
	Live	+ 3,800	+ 3,770		
		+ 40,100	+ 4,470	241	3360
+ M	Tem. fall	- 6,600	+ 14,500		
		+ 33,500	+ 18,970	447	5620
	Dead	+ 36,300	+ 700		
	Live	+ 2,750	- 1,700		
Max	P				
		+ 39,050	- 1,000	181	2650
-M	Tem. rise	+ 3,300	- 7,250		
		+ 42,350	- 8,250	311	4200

Stress summary and unit stresses

$A_s = 443.6$; $y = 17.5$; $I_s = 47,725$; $y_1 = 15.5$; $n = 15$

Springing		Thrust = T_s	Moment = M_s	f_c	f_s
Max $+ M_s$	Dead	+ 49,800	− 17,500		
	Live	+ 4,600	+ 17,650		
		+ 54,400	+ 150	124	1850
	Tem. rise	+ 2,500	+ 32,000		
		+ 56,900	+ 32,150	270	3800
Max $- M_s$	Dead	+ 49,800	− 17,500		
	Live	+ 3,250	− 12,850		
		+ 53,050	− 30,350	262	3680
	Tem. fall	− 5,050	− 64,000		
		+ 48,000	− 94,350	523	7150

f_c and f_s calculated by Least Work Method are more correct than those found in the preliminary analysis. Anyhow the difference is small. The stresses are within the limits and the arch is safe.