COMMUNICATIONS SYSTEMS

and

ORTHOGONAL FUNCTIONS

By

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INTRODUCTION

Signals travelling from a transmitter to a receiver, whether over a wire or through the air, are altered by static, electrical disturbances, and other uncontrollable perturbations which lumped together are called "noise". Various methods have been used to "sweep" the noise from a signal, the most recent and probably the best of which is reported in a research paper labled "Orthogonal Coding".[1] The first part of this thesis is an explanation of the principles on which orthogonal coding is based.

The electrical equipment used in the orthogonal coding method is not formidable, but any reduction in such equipment is desirable. The electrical engineers and mathematicians collaborating on the problem were convinced that the circuitry involved could be reduced if a set of orthogonal functions having particular additional properties could be found. The second and third parts of this thesis are concerned with the purely mathematical investigation of such sets of functions.

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[1] "An Orthogonal Coding Technique for Communications", G.A. Franco and G. Lachs, General Dynamics/Electronics Research Division, Rochester, 1960.

PART I

The transmission of messages by telegraph or teletype requires twenty-six different signals, one for each letter of the alphabet, and a few auxiliary signals for punctuation and special codes. Thirty-two different signals are usually sufficient.

A radio wave or telephone signal is more complex. By virtue of a powerful theorem in mathematics, however, from a continuous wave of given bandwidth emitted by electrical equipment it is possible to extract a series of pulses with the property that the complete wave is determined by this series of pulses and can be obtained from it.[2] Furthermore, let each pulse be approximated by a pulse from a predetermined set of exactly n levels. An approximation to the complete wave can then be obtained from the series of approximation pulses. Exhaustive experiment has shown that when n equals thirty-two, the difference between the original wave and its approximation is indistinguishable to the human ear.[3]

The problem of transmitting a radio wave or a telephone signal is therefore, essentially the same as that of sending a message by telegraph or teletype; that is, a series of transmissions of one of a set of thirty-two pulses or signals during consecutive intervals of time

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[2]See Appendix I.

[3]See Appendix II.

(extremely brief intervals, to be sure!). Let us now turn our attention to that simplified problem.

We make another modification. Since one pulse takes only a fraction of its given time interval, then during most of that time interval nothing is transmitted. Hence, let us fill up that interval with one of a set of thirty-two functions which are in a one-to-one correspondence with the original set of thirty-two pulses. Transmission of the functions is equivalent to transmission of the pulses, and at the receiver an incoming function identifies its corresponding pulse.[4]

Consider the transmission of one of a set of thirty-two signals from one station to another where each signal is represented by a real-valued function S_i, $i = 1,2,3,\ldots,32$. If one of the signals (functions) S_i is distorted during transmission by uncontrolled interference - i.e., noise - then how can the distorted S_i be distinguished from the other thirty-one possible signals in S?

System I

To facilitate the identification we impose a condition on S. Let S be a normalized orthogonal set over the interval $(-a, +a)$, then

$$\int_{-a}^{+a} S_i S_j = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases} \quad \text{for } S_i \text{ and } S_j \text{ in } S.$$

If the signal S_i is transmitted and is distorted by noise, then the received signal may be put in the form $S_i + N$, where N is the noise

[4] See Appendix II.

factor. Then let

$$I_i(j) = \int_{-a}^{+a}(S_i+N)S_j = \begin{cases} 0 + \int_{-a}^{+a} N\ S_j & j \neq i \\ 1 + \int_{-a}^{+a} N\ S_j & j = i \end{cases}$$

Hence, if the noise N is not too great; i.e., such that

$$+\tfrac{1}{2} > \int_{-a}^{+a} N\ S_j > -\tfrac{1}{2} \quad \text{for } j = 1,2,3,\ldots,32$$

then

$$I_i(i) = \int_{-a}^{+a} N\ S_i + 1 > -\tfrac{1}{2} + 1 = \tfrac{1}{2} > \int_{-a}^{+a} N\ S_j = I_i(j) \quad \text{for } j \neq i.$$

The largest number among the $I_i(j)$ for $j = 1,2,3,\ldots,32$ would then be $I_i(i)$. Thus, it could be determined that S_i was the originally transmitted signal. The schematic diagram in figure 1 illustrates how electronic components may be used in this method.

Although System I is feasible, the amount of equipment is excessive. A function generator is more complicated than a multiplier or an integrator, and this method uses thirty-two such function generators. In the next section a modification of System I is given which requires only four function generators.

System II

Let $T = \{T_1, T_2, T_3, T_4\}$ be a set of four functions which are normalized and orthogonal over $(-a, +a)$. Let $A = \{a_{i,j}\}$ be a matrix

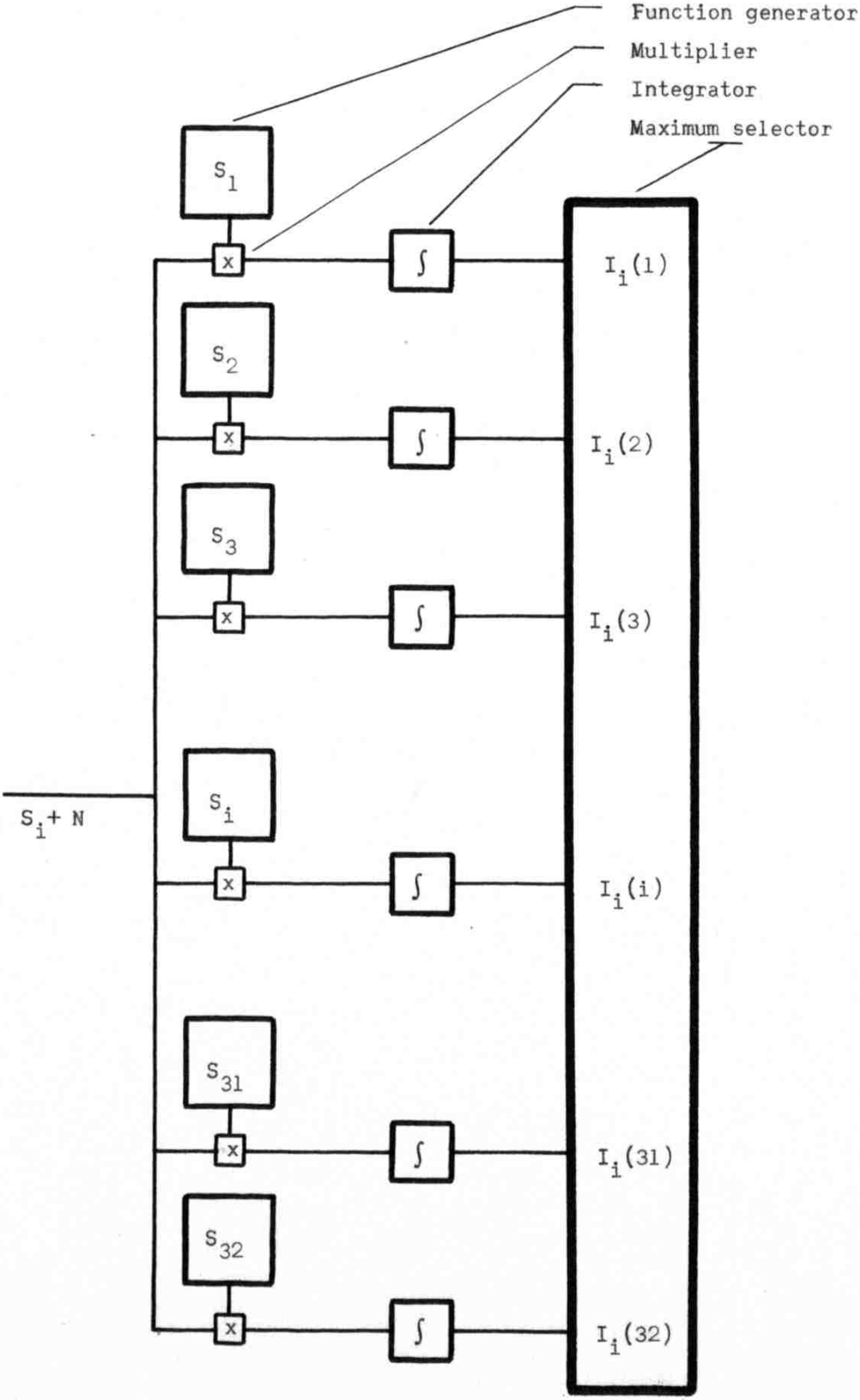
Function generator
Multiplier
Integrator
Maximum selector
S1
S2
S3
Si
S31
S32
x
Si + N
Ii(1)
Ii(2)
Ii(3)
Ii(i)
Ii(31)
Ii(32)

Figure 1

of coefficients having thirty-two rows and four columns, and form a set S of thirty-two functions by setting:

$$
\begin{aligned}
S_1 &= a_{1,1}\,T_1 + a_{1,2}\,T_2 + a_{1,3}\,T_3 + a_{1,4}\,T_4 \\
S_2 &= a_{2,1}\,T_1 + a_{2,2}\,T_2 + a_{2,3}\,T_3 + a_{2,4}\,T_4 \\
&\vdots \\
S_{32} &= a_{32,1}T_1 + a_{32,2}T_2 + a_{32,3}T_3 + a_{32,4}T_4 \,.
\end{aligned}
$$

That is,

$$S = \{S_i | S_i = a_{i,1}T_1 + a_{i,2}T_2 + a_{i,3}T_3 + a_{i,4}T_4 \text{ and } i = 1,2,3,\ldots,32\}.$$

Also, for convenience call

$$A_i = \{a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4}\}.$$

Hence,

$$\int_{-a}^{+a} S_i S_j = \int_{-a}^{+a} (a_{i,1}T_1 + a_{i,2}T_2 + a_{i,3}T_3 + a_{i,4}T_4)(a_{j,1}T_1 + a_{j,2}T_2 + a_{j,3}T_3 + a_{j,4}T_4)$$

$$= \int_{-a}^{+a} a_{i,1}a_{j,1}T_1^2 + a_{i,2}a_{j,2}T_2^2 + a_{i,3}a_{j,3}T_3^2 + a_{i,4}a_{j,4}T_4^2$$

$$= \begin{cases} a_{i,1}a_{j,1} + a_{i,2}a_{j,2} + a_{i,3}a_{j,3} + a_{i,4}a_{j,4} & j \neq i \\ a_{i,1}^2 + a_{i,2}^2 + a_{i,3}^2 + a_{i,4}^2 & j = i \,. \end{cases}$$

The problem is now to choose the matrix A so that as few different numbers $a_{i,j}$ as possible are used and so that

$$\min(a_{i,1}^2+a_{i,2}^2+a_{i,3}^2+a_{i,4}^2) > \max(a_{i,1}a_{j,1}+a_{i,2}a_{j,2}+a_{i,3}a_{j,3}+a_{i,4}a_{j,4}) \quad j\neq i \qquad (1)$$

The following matrix, found by experiment, satisfies these requirements. Let every A_i be such that one of the four elements of A_i is zero and the other three elements of A_i are either -1 or +1. The elements $a_{i,j}$ are chosen in this manner in order that each S_i might require an equal amount of power to be transmitted (see Summary and Conclusion). The number of different A_i is $4(2)^3 = 32$, precisely the number of A_i which are needed. The matrix can be ordered as follows:

	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	$a_{i,4}$
A_1	1	1	1	0
A_2	-1	1	1	0
A_3	1	-1	1	0
A_4	1	1	-1	0
A_5	-1	-1	1	0
A_6	-1	1	-1	0
A_7	1	-1	-1	0
A_8	-1	-1	-1	0
A_9	1	1	0	1
A_{10}	-1	1	0	1
A_{11}	1	-1	0	1
⋮	⋮	⋮	⋮	⋮
A_{31}	0	1	-1	-1
A_{32}	0	-1	-1	-1

It can be seen from the table that

$$\min(a_{i,1}^2 + a_{i,2}^2 + a_{i,3}^2 + a_{i,4}^2) = a_{i,1}^2 + a_{i,2}^2 + a_{i,3}^2 + a_{i,4}^2 = 3$$

$$\text{for all } i = 1,2,3,\ldots,32$$

and that

$$\max(a_{i,1}\, a_{j,1} + a_{i,2}\, a_{j,2} + a_{i,3}\, a_{j,3} + a_{i,4}\, a_{j,4}) = 2$$

since if both $a_{i,1}\, a_{j,1}$ and $a_{i,2}\, a_{j,2}$ are one, then $a_{i,3}\, a_{j,3}$ must be either minus one or zero. If $a_{i,3}\, a_{j,3}$ is zero, then $a_{i,4}\, a_{j,4}$ must be either minus one or zero. Therefore, condition (1) holds and thus implies

$$\int_{-a}^{+a} S_i^2 \geq \int_{-a}^{+a} S_i S_j + 1 \qquad j \neq i .$$

Now, if the signal S_i is transmitted and is distorted by noise, as in the first system call the incoming signal $S_i + N$ where N is the noise factor.

Again let

$$I_i(j) = \int_{-a}^{+a}(S_i+N)S_j = \begin{cases} \int_{-a}^{+a} S_i S_j + \int_{-a}^{+a} N\, S_j \leq 2 + \int_{-a}^{+a} N\, S_j & j \neq i \\ \int_{-a}^{+a} S_i^2 + \int_{-a}^{+a} N\, S_i = 3 + \int_{-a}^{+a} N\, S_i & j = i \end{cases}$$

If the noise N is such that

$$\tfrac{1}{2} > \int_{-a}^{+a} N\, S_j > -\tfrac{1}{2} \quad \text{for} \quad j = 1,2,3,\ldots 32 \qquad (2)$$

then

$$I_i(i) = \int_{-a}^{+a} N\, S_i + 3 > -\tfrac{1}{2} + 3 = 2\tfrac{1}{2} > \int_{-a}^{+a} N\, S_j + 2 \geq I_i(j) \quad \text{for } j \neq i \,.$$

The largest number among the $I_i(j)$ for $j = 1,2,3,\ldots,32$ would then be $I_i(i)$, indicating that S_i was the originally transmitted signal. The schematic diagram in figure 2 shows how $I_i(i)$ could be obtained electronically; note the use of only four multipliers, four integrators and four function generators.

Thus, we have an effective method for determining which signal out of a set of thirty-two possible signals was transmitted through interference.

System III

Let us consider a modification and an improvement of System II. The basic principle underlying System II is that when one of thirty-two signals S_i is transmitted through interference and is distorted into a different signal $S_i + N$, then

$$I_i(i) \text{ will remain larger than every other } I_i(j) \quad \text{for } j \neq i. \qquad (3)$$

Hence, S_i can be electronically identified from the other members of S. Condition (3) depends on two variables, the size of N and the relative differences between $\int_{-a}^{+a} S_i S_j$ for (j and i) = 1,2,3,...,32. If the minimum relative difference d between $\int_{-a}^{+a} S_i S_j$ and $\int_{-a}^{+a} S_i^2$ for $j \neq i$ can be increased, then condition (3) will continue to hold true for bigger N; i.e., greater interference.

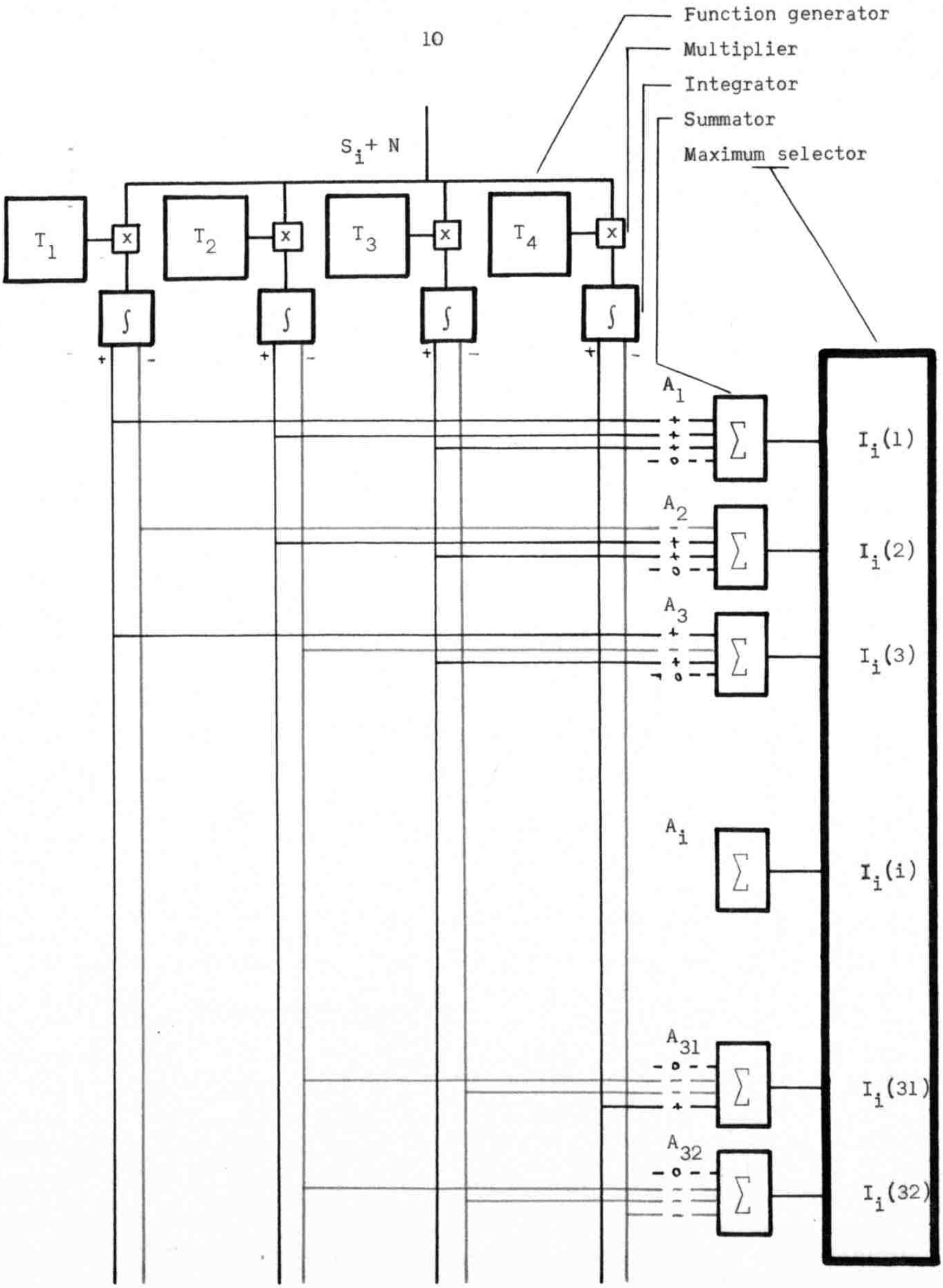
Function generator
Multiplier
Integrator
Summator
Maximum selector
S_i+ N
T_1
T_2
T_3
T_4
A_1
A_2
A_3
A_i
A_{31}
A_{32}
$I_i(1)$
$I_i(2)$
$I_i(3)$
$I_i(i)$
$I_i(31)$
$I_i(32)$

Figure 2

Since, in System II

$$S_i = \sum_{j=1}^{4} a_{i,j}\, T_j \qquad i = 1,2,3,\ldots,32$$

and

$$\int_{-a}^{+a} S_i S_j = \sum_{k=1}^{4} a_{i,k}\, a_{j,k} \qquad j \neq i$$

$$\int_{-a}^{+a} S_i^2 = \sum_{k=1}^{4} a_{i,k}^2$$

one way to allow for more latitude among the $\int_{-a}^{+a} S_i S_j$ is to increase the number of terms in each S_i. Let S_i in general be given by

$$S_i = \sum_{j=1}^{n} a_{i,j}\, T_j \qquad i = 1,2,3,\ldots,32$$

where the $a_{i,j}$ are either plus one, minus one, or zero.[5] Also,

$$d = \min\left[\int_{-a}^{+a} S_i^2 - \int_{-a}^{+a} S_i S_j\right] = \min\left[\sum_{k=1}^{n} a_{i,k}^2 - \sum_{k=1}^{n} a_{i,k} a_{j,k}\right] \quad \text{for } j \neq i.$$

Although no theoretical demonstration was found indicating that under the above conditions d could be maximized, by extensive experiment the following system was found to be most satisfactory: Let each S_i contain eight terms

$$S_i = \sum_{j=1}^{8} a_{i,j}\, T_j \qquad i = 1,2,3,\ldots,32$$

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[5]See Summary and Conclusion at the end of the Thesis.

such that every A_i contains exactly two zero elements. The matrix A was chosen by experiment such that

$$\sum_{k=1}^{8} a_{i,k}^2 = 6 \quad \text{for all } i = 1,2,3,\ldots,32$$

and

$$\sum_{k=1}^{8} a_{i,k}\, a_{j,k} \leq 4 \quad \text{for } j \neq i .$$

Therefore, if

$$1 > \int_{-a}^{+a} N\, S_j > -1 \quad \text{for } j = 1,2,3,\ldots,32 \tag{4}$$

with respect to a distorted signal $S_i + N$, then

$$I_i(i) = \int_{-a}^{+a} S_i^2 + \int_{-a}^{+a} N\, S_i = 6 + \int_{-a}^{+a} N\, S_i > 6-1 = 5$$

and

$$I_i(j) = \int_{-a}^{+a} S_i S_j + \int_{-a}^{+a} N\, S_j \leq 4 + \int_{-a}^{+a} N\, S_j < 4+1=5,$$

and we obtain the desired result

$$I_i(i) > I_i(j) \quad \text{for all } j \neq i.$$

Since the inequality (4) is less restrictive than the inequality (2) upon the amount of noise N through which the system can operate, System III, as we shall call the above system, is much improved over System II. However, the amount of electronic equipment required for System III includes eight function generators and is nearly double the amount required for System II.

PART II

As can be seen from figure 2, the set of orthogonal functions $T = \{T_1, T_2, T_3, T_4\}$ must be generated each time the function $S_i + N$ is processed. Since it is much easier electronically to generate one function and its three derivatives rather than four unrelated functions, let us investigate the possibility of a set of four, five or even eight orthogonal functions such that the set is of the form

$$T = \{T_n | \; T_1 = f, T_2 = f', T_3 = f'', \ldots\} .$$

<u>Definition</u>: A set T of functions which is orthogonal over the interval (a,b) and is of the form $T = \{T_n | \; T_1 = f, T_2 = f', \ldots, T_p = f^{(p-1)}\}$ will be called <u>p-step orthogonal over (a,b)</u>. In short, we say that the function f is p-step orthogonal over (a,b).

The following theorem was soon apparent and forms the basis of most of the work that follows it. One should first note that if a set T is p-step orthogonal over (a,b), then a subset of T is also q-step orthogonal over (a,b) for every $q < p$.

<u>Theorem I</u>: There is no p-step orthogonal set of functions over (a,b) for $p \geq 5$.

<u>Proof</u>:

1. It suffices to show that there is no 5-step orthogonal set over (a,b). Assume the contrary. Thence,

2. $\int_a^b f^{(n)} f^{(n+1)} = \frac{1}{2}(f^{(n)})^2\Big]_a^b = 0$ for $n = 0,1,2,3.$

$$f(b)^2 = f(a)^2 \qquad f''(b)^2 = f''(a)^2$$
$$f'(b)^2 = f'(a)^2 \qquad f'''(b)^2 = f'''(a)^2 \tag{1}$$

Integrating by parts,

3. $\int_a^b f^{(n)} f^{(n+2)} = f^{(n)} f^{(n+1)}\Big]_a^b - \int_a^b (f^{(n+1)})^2 = 0;\ n = 0,1,2.$

$$f(b)f'(b) - f(a)f'(a) = \int_a^b (f')^2$$
$$f'(b)f''(b) - f'(a)f''(a) = \int_a^b (f'')^2 \tag{2}$$
$$f''(b)f'''(b) - f''(a)f'''(a) = \int_a^b (f''')^2$$

4. $\int_a^b f^{(n)} f^{(n+3)} = f^{(n)} f^{(n+2)}\Big]_a^b - \int_a^b f^{(n+1)} f^{(n+2)} = 0;\quad n = 0,1.$

$$f(b)f''(b) = f(a)f''(a)$$
$$f'(b)f'''(b) = f'(a)f'''(a) \tag{3}$$

5. $\int_a^b f \cdot f^{(iv)} = f\, f'''\Big]_a^b - \int_a^b f' f''' = 0$

$$f(b)f'''(b) = f(a)f'''(a). \tag{4}$$

6. By (1) either $f(b) = f(a)$ or $f(b) = -f(a)$. If $f(b) = f(a)$, then by (2) $f'(b) = -f'(a)$; otherwise, $f'(b) = f'(a)$ by (1) and this

implies $\int_a^b (f')^2 = 0$ by (2). Similarly, $f''(b) = f''(a)$ and $f'''(b) = -f'''(a)$. However, by (4) $f(b) = f(a)$ implies that $f'''(b) = f'''(a)$. Hence, we have a contradiction.

7. If $f(b) = -f(a)$, then by reasoning similar to the above $f'(b) = f'(a)$, $f''(b) = -f'''(a)$, and $f'''(b) = f'''(a)$. However, by (4) $f(b) = -f(a)$ implies that $f'''(b) = -f'''(a)$. This contradiction establishes the theorem.

q.e.d.

However, there do exist four-step orthogonal functions. The first such function was found in the following manner: consider the set $\{f, f', f'', f'''\}$. Let us place one condition on f - <u>let f be an even function</u> (a similar argument would follow if f were an odd function). Then f' is an odd function since f is even implies

$$\frac{f(x + h) - f(x)}{h} = - \frac{f(-x -h) - f(-x)}{-h}$$

which implies

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f(-x -h) - f(-x)}{-h} = -f'(-x).$$

And it also follows that f'' will be an even function and f''' an odd function. Therefore,

$$\int_{-a}^{+a} f^{(n)} f^{(n+1)} = 0 \quad \text{for} \quad n = 0,1,2,$$

and

$$\int_{-a}^{+a} f\ f''' = 0 \ .$$

Thus, we have established the following theorem:

<u>Theorem II</u>: If and only if a function f is an even function (or an odd function), and

$$\int_{-a}^{+a} (f^{(n)})^2 \neq 0 \quad \text{for } n = 0,1,2,3 \tag{5}$$

$$\int_{-a}^{+a} f\ f'' = 0 \tag{6}$$

$$\int_{-a}^{+a} f'\ f''' = 0 \tag{7}$$

then will f be a four-step orthogonal function over $(-a, +a)$.

In view of the foregoing theorem, let us make the definition:

<u>Definition</u>: A function which is either an odd or an even function will be called a <u>signed function</u>.

Now, let us examine <u>polynomial</u> functions which are also signed functions. A signed polynomial function f which is four-step orthogonal must contain at least three terms. If f has only two terms and is of the form $f(x) = px^m + qx^n$, then the conditions of equations (6) and (7) imply that $p = q = 0$ in contradiction to the conditions of equations (5). On the other hand, if f contains more than three terms, then there are superfluous terms which unnecessarily complicate calculations (this will soon become clear). The signed polynomial function of three terms and of lowest degree which satisfies equations (5), (6) and (7) is an even function of degree four:

Example 1:

$$f(x) = px^4 + qx^2 + r$$

$$f'(x) = 4px^3 + 2qx$$

$$f''(x) = 12px^2 + 2q$$

$$f'''(x) = 24px$$

where p, q, and r are constants. Solving for p, q, and r; equation (6),

$$\int_{-a}^{+a} f'f''' = 0,$$

implies

$$24r(2)(\tfrac{2}{5}px^5 + \tfrac{1}{3}qx^3\Big]_{-a}^{+a} = 0$$

$$2(\tfrac{2}{5}a^5 p + \tfrac{a^3}{3}q) = 0$$

$$\tfrac{2}{5}a^2 p + \tfrac{1}{3}q = 0$$

and a solution to the equation is

$$p = \frac{5}{a^2}$$

$$q = -6$$

Next, equation (7),

$$\int_{-a}^{+a} f\, f'' = 0,$$

implies

$$\left. \frac{75}{7a^4}x^7 - \frac{105}{5a^2}x^5 + \frac{15r + 18a^2}{3a^2}x^3 - 3rx \right]_{-a}^{+a} = 0$$

$$r = \frac{15}{7}\, a^2.$$

Therefore,

$$f(x) = \frac{5}{a^2}\, x^4 - 6x^2 + \frac{15}{7}\, a^2$$

is a four-step orthogonal function over $(-a, +a)$.

In particular,

$$f(x) = 35x^4 - 42a^2x^2 + 15a^4,$$

and in general,

$$f(x) = k(\frac{5}{a^2}\, x^4 - 6x^2 + \frac{15}{7}\, a^2)$$

for any constant $k \neq 0$ are four-step orthogonal functions over $(-a, +a)$.

Another class of four-step orthogonal functions can be found using the same technique.

<u>Example 2</u>:

$$f(x) = px^{2n} + qx^2 + r$$

where n is an integer such that $n \geq 2$, and p, q and r are parameters depending upon n and a (but not x). As was done above, we can determine from equation (6)

$$p = \frac{4n - 3}{a^{2n-2}}$$

$$q = -2n^2 + n .$$

Hence, equation (7) implies that

$$r = \frac{2n - 1}{3n - 3}\left[5n^2 - 16n + 18 - \frac{9(8n-1)}{(4n - 1)(2n + 1)}\right] a^2 .$$

For example, when

$$n = 2; \quad p = \frac{5}{a^2}, \quad q = -6, \quad r = \frac{15}{7}a^2$$

$$n = 3; \quad p = \frac{9}{a^4}, \quad q = -15, \quad r = \frac{790}{77}a^2$$

$$n = 6; \quad p = \frac{21}{a^{10}}, \quad q = -66, \quad r = 73.76a^2 \quad \text{approximately.}$$

In the foregoing examples, only one of the three terms of f was of degree greater than two. However, to construct a four-step orthogonal function from a signed polynomial where two terms are of degree greater than two is complicated. The most simple of such polynomials is the following example over the interval $(-1, +1)$:

Example 3:

$$f(x) = px^6 + qx^4 + r$$

$$f'(x) = 6px^5 + 4qx^3$$

$$f''(x) = 30px^4 + 12qx^2$$

$$f'''(x) = 120px^3 + 24qx \; .$$

From equation (6)

$$175p^2 + 195pq + 42q^2 = 0 \; .$$

Solving for r in equation (7), we obtain

$$r = \frac{1}{3p+2q}\left(\frac{15}{11}p^2 + \frac{7}{3}pq + \frac{6}{7}q^2\right).$$

Although there are no integral values of p, q and r which satisfy the above equations, approximate solutions are given by $p = .30$, $q = -1.00$ and $r = -.25$.

Similarly, example 3 can be generalized.

Example 4:

$$f(x) = px^{2n} + qx^{2m} + r$$

where $n > m \geq 2$. Any function of this form can be made four-step orthogonal by proper choice of p, q and r. Also, p, q and r will be real-valued.

Notice that in example 3 the solution of equation (6) does not involve the parameter r. This is also true in example 4. The reason behind this fact is that the third term of f in both examples 3 and 4 is a constant term.

The next example of a four-step signed polynomial presents more difficult equations in p, q and r. Notice that the following function has degree five, and it contains no constant term.

Example 5: Over the interval (-1, +1), let

$$f(x) = px^5 + qx^3 + rx$$

$$f'(x) = 5px^4 + 3qx^2 + r$$

$$f''(x) = 20px^3 + 6qx$$

$$f'''(x) = 60px^2 + 6q$$

equations (6) and (7) imply

$$\frac{50}{7} p^2 + 7pq + \frac{10pr + 3q^2}{3} + qr = 0$$

$$\frac{10}{9} p^2 + \frac{13}{7} pq + \frac{20pr + 6q^2}{5} + 2qr = 0 .$$

As evident, it would be difficult to find real values for p, q and r (if they exist) from the above equations.

Generalizing the result of example 5, given any signed function of the form

$$f(x) = px^m + qx^n + rx^k$$

where $m > n > k \geq 2$, it would be quite difficult to find real values for p, q and r such that f is a four-step orthogonal function over $(-1, +1)$.

There are also signed polynomial functions of more than three terms which are four-step orthogonal. However, these functions are unduly complicated.

Functions other than polynomials which are four-step orthogonal exist. For example,

$$f(x) = p \sin x + q \sin 2x + r \qquad \text{over } \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$$

and

$$g(x) = p \cos x + q \cos 2x + r \cos 3x \qquad \text{over } \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$$

are four-step orthogonal functions for the proper choice of the parameters p, q and r. However, the calculation of p, q and r in f and g would involve equations as difficult to solve as those in examples 3 and 5 respectively.

PART III

As seen from the closing remarks of Part I, it would be desirable to have a set T of eight orthogonal functions such that

$$S_i = \sum_{j=1}^{8} a_{i,j} T_j .$$

However, there is no eight-step orthogonal function, and there is not even a five-step orthogonal function as was shown in Theorem I.

If there were a five-step orthogonal function f, such that $f = T_1$, $f' = T_2$, ..., $f^{(iv)} = T_5$, then how useful would be the set of S_i where

$$S_i = \sum_{j=1}^{5} a_{i,j} T_j \ ?$$

A system of interference reduction using such S_i would be an improvement over System II although not as powerful as System III. Compared with System II, the extra term in each S_i would allow more latitude in choosing the matrix A, and it would be possible to incorporate a redundancy check in the system.[6] Thence, a five-step orthogonal function (if it existed) would be quite useful.

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[6] Fazlollah M. Reza's, An Introduction to Information Theory, McGraw-Hill.

On the other hand, an examination of condition (1) of Part I shows that it is not really necessary that the set T be an orthogonal set: i.e.,

$$\int_{-a}^{+a} T_i T_j = \begin{cases} M_i & j = i \\ 0 & j \neq i \end{cases} \quad \text{for } (i \,\&\, j) = 1,2,3,\ldots .$$

An alternate condition would be that the set T is <u>almost orthogonal</u>; i.e.,

$$\int_{-a}^{+a} T_i T_j = \begin{cases} M_i & j = i \\ e_{i,j} & j \neq i \end{cases} \quad \text{for } (i \,\&\, j) = 1,2,3,\ldots ,$$

where $\min(M_i)$ is significantly larger than $\max(e_{i,j})$.

The remainder of the thesis concerns five-step almost orthogonal functions. The basis of the discussion is the following conjecture (the proof of which would conclude the thesis):

<u>Conjecture</u>: There is no five-step almost orthogonal function.

<u>Evidence</u>:

From now on let us assume that all functions f are signed functions. Then

1. $$\int_{-a}^{+a} f^{(n)} f^{(n+1)} = \tfrac{1}{2}\left(f^{(n)}\right)^2\Big]_{-a}^{+a} = 0 \qquad \text{for } n = 0,1,2,3,$$

Hence,

$$e_{0,1} = e_{1,2} = e_{2,3} = e_{3,4} = 0.$$

2. $\int_{-a}^{+a} f^{(n)} f^{(n+2)} = f^{(n)} f^{(n+1)}\Big]_{-a}^{+a} - \int_{-a}^{+a} f^{(n+1)})^2 = e_{n,n+2}$ for n = 0,1,2.

Therefore,

$$2f^{(n)}(a)\ f^{(n+1)}(a) - M_{n+1} = e_{n,n+2} \qquad \text{for } n = 0,1,2.$$

3. $\int_{-a}^{+a} f^{(n)} f^{(n+3)} = f^{(n)} f^{(n+2)}\Big]_{-a}^{+a} - \int_{-a}^{+a} f^{(n+1)} f^{(n+2)} = e_{n,n+3}$ for n=0,1.

Therefore,

$$0 - e_{n+1,n+2} = e_{n,n+3},$$

hence,

$$e_{n,n+3} = 0 \qquad \text{for } n = 0,1.$$

4. Finally,

$$\int_{-a}^{+a} f\ f^{(iv)} = f\ f'''\Big]_{-a}^{+a} - \int_{-a}^{+a} f' f''' = e_{0,4}.$$

Therefore,

$$2f(a)\ f'''(a) - e_{1,3} = e_{0,4}.$$

5. In summary, let f be a signed function. Then over the interval (-a, +a)

$$e_{0,1} = e_{1,2} = e_{2,3} = e_{3,4} = e_{0,3} = e_{1,4} = 0$$

and

$$M_1 + e_{0,2} = 2f(a)\ f'(a) \qquad (1)$$

$$M_2 + e_{1,3} = 2f'(a) f''(a) \qquad (2)$$

$$M_3 + e_{2,4} = 2f''(a)\ f'''(a) \tag{3}$$

$$e_{1,3} + e_{0,4} = 2f(a)\ f'''(a) \tag{4}$$

6. Dividing the product of (1) and (3) by (2), we obtain

$$\frac{(M_1+e_{0,2})(M_3+e_{2,4})}{(M_2 + e_{1,3})} = \frac{2f(a)\ f'(a)\ 2f''(a)\ f'''(a)}{2f'(a)\ f''(a)}$$

$$= 2f(a)\ f'''(a)$$

$$= e_{1,3} + e_{0,4}\ .$$

7. If $e_{0,2}$, $e_{i,3}$ and $e_{2,4}$ were very small compared to M_1, M_2 and M_3, then

$$\frac{(M_1+e_{0,2})(M_3+e_{2,4})}{(M_2 + e_{1,3})} = \frac{M_1M_3+e_{0,2}M_3+e_{2,4}M_1}{M_2(1+\frac{e_{1,3}}{M_2})} + \frac{e_{0,2}\ e_{2,4}}{(M_2+e_{1,3})}$$

$$= \left[\frac{M_1M_3}{M_2} + \frac{e_{0,2}M_3+e_{2,4}M_1}{M_2}\right]\left[1 + \frac{e_{1,3}}{M_2} + \left(\frac{e_{1,3}}{M_2}\right)^2 + \left(\frac{e_{1,3}}{M_2}\right)^3 + \cdots\right] + \frac{e_{0,2}e_{2,4}}{(M_2+e_{1,3})}\ .$$

which is approximately

$$\frac{M_1M_3}{M_2}\ .$$

8. Hence, under the foregoing conditions,

$$\frac{M_1 M_3}{M_2} = e_{1,3} + e_{0,4}, \quad \text{approximately.} \qquad (5)$$

9. If the function f were five-step almost orthogonal, then condition (5) would be implied. Condition (5) can be interpreted as meaning that the size of $M_2 = \int_{-a}^{+a}(f'')^2$ must be of a degree three times as great as the degree of size of

$$(M_1 M_3) = \left[\int_{-a}^{+a}(f')^2\right]\left[\int_{-a}^{+a}(f''')^2\right].$$

The foregoing line of attack was not carried any further.

The following procedure was adapted to check for a counter-example to the conjecture. Let us take a function which is four-step orthogonal and extend it into a five-step almost orthogonal function. For illustration, the function $f(x) = 5x^4 - 6x^2 + \frac{15}{7}$ is four-step orthogonal over $(-1, +1)$.

Case 1:

$$f(x) = 5x^4 - 6x^2 + \frac{15}{7}$$
$$f'(x) = 20x^3 - 12x$$
$$f''(x) = 60x^2 - 12$$
$$f'''(x) = 120x$$

and also,

$$f^{(iv)}(x) = 120 \ .$$

Now

$$\int_{-1}^{+1} f' f^{(iv)} = \int_{-1}^{+1} f''' f^{(iv)} = 0$$

since f' and f''' are odd functions and $f^{(iv)}$ is an even function. Then

$$\int_{-1}^{+1} f\ f^{(iv)} = 120 \int_{-1}^{+1} (5x^4 - 6x^2 + \frac{15}{7}) dx = 2(120)\left[\frac{5}{5} - \frac{6}{3} + \frac{15}{7}\right] = 274\frac{2}{7} = e_{0,4}$$

$$\int_{-1}^{+1} f'' f^{(iv)} = 12(120)\int_{-1}^{+1} (5x^2 - 1) dx = 2(12)(120)\left(\frac{5}{3} - 1\right) = 1920 = e_{2,4} \ .$$

However, by (1)

$$M_1 = 2f(1)f'(1) = 2(5 - 6 + \frac{15}{7})(20 - 12) = 18\frac{2}{7} \ .$$

Obviously, M_1 is not significantly bigger than either $e_{0,4}$ or $e_{2,4}$; thus, the function f is not a counterexample to the conjecture.

There is one other way of extending the function f - by integration. Consider the function g where

$$g(x) = F(x) = x^5 - 2x^3 + \frac{15}{7}x \qquad \text{such that } F' = f$$

$$g'(x) = f(x) = 5x^4 - 6x^2 + \frac{15}{7}$$

$$g''(x) = f'(x) = 20x^3 - 12x$$

$$g'''(x) = f''(x) = 60x^2 - 12$$

$$g^{(iv)}(x) = f'''(x) = 120x \quad .$$

Now

$$\int_{-1}^{+1} g\, g' = \int_{-1}^{+1} g\, g''' = 0 \ .$$

But

$$\int_{-1}^{+1} g\, g^{(iv)} = 120 \int_{-1}^{+1} (x^6 - 2x^4 + \tfrac{15}{7}x)dx = 2(120)\left(\frac{1}{7} - \frac{2}{5} + \frac{5}{7}\right) = 109\tfrac{5}{7} = e_{0,4} \ .$$

However, by (2)

$$M_2 = 2g'(1)\, g''(1) = 2(5-6+\tfrac{15}{7})(20-12) = 18\tfrac{2}{7} \quad .$$

Since M_2 is not significantly bigger than $e_{0,4}$, the function g is not a counterexample to the conjecture. That is, neither the function f nor the function g where $g' = f$ are five-step almost orthogonal functions.

This same extension procedure was tried on other more general four-step orthogonal functions. The following is an example of the results:

Case 2:

$$f(x) = p\,x^{2n} + q\,x^2 + r \quad \text{(where the values of p,q and r are given on page 19.)}$$

over the interval $(-a, +a)$ and such that $n \geq 2$. This is a more general example which includes case 1 as the special case when $n = 2$ and $a = 1$.

First of all, when $a = 1$ and $n \geq 2$, we have

$$M_1 = 2\,f(1)\,f'(1) \quad \text{and} \quad e_{0,4} = 2\,f(1)\,f'''(1)$$

such that $M_1 > e_{0,4}$ implies that either

$$f(1) > 0 \quad \text{and} \quad f'(1) > f'''(1)$$

or

$$f(1) < 0 \quad \text{and} \quad f'(1) < f'''(1)\ .$$

But assuming

$$f'(1) = 2np + 2q > 2n(2n-1)(2n-2)p = f'''(1)$$

implies that

$$n(4n-3) + (-2n^2+n) > n(2n-1)(2n-2)(4n-3)$$

$$1 > (2n-1)(4n-3)$$

or

$$0 > 8n^2 - 10n + 3 = h(n). \qquad \text{for } n \geq 2$$

However,

$$h(2) > 0 \quad \text{and } h'(n) > 0 \qquad \text{for } n \geq 2$$

which is a contradiction.

Hence, $M_1 > e_{0,4}$ implies that $f(1) < 0$ which means

$$p + q + r < 0$$

which can also be shown to lead to a contradiction (in a complicated proof). Therefore, f is not a five-step almost orthogonal function when $a = 1$ and $n \geq 2$.

In the general case when a is arbitrary and $n \geq 2$, we have

$$M_1 = 2\, f(a)\, f'(a), \qquad e_{0,4} = 2\, f(a)\, f'''(a),$$

and

$$f(a) = pa^{2n} + qa^2 + r = a^2 f(1) > 0.$$

Thus, $M_1 > e_{0,4}$ implies that $f'(a) > f'''(a)$, or

$$2npa^{2n-1} + 2qa > 2n(2n - 1)(2n - 2)pa^{2n-3}$$

or

$$a^2 > (2n - 1)(4n - 3). \qquad (6)$$

Also

$$M_4 = \int_{-a}^{+a} [(2n)^2(2n-1)^2(2n-2)^2(2n-3)^2p^2]\ x^{4n-8}dx$$

$$= \frac{2K^2p^2}{4n-7}\ a^{4n-7} \quad \text{where} \quad K = 2n(2n-1)(2n-2)(2n-3)$$

and

$$e_{2,4} = \int_{-a}^{+a} [2n(2n-1)p\ x^{2n-2} + 2q]\ [Kp\ x^{2n-4}]\ dx$$

$$= 2Kp\ [\frac{2n(2n-1)p}{4n-5}\ a^{4n-5} + \frac{2q}{2n-3}\ a^{2n-3}]$$

$$= 2(2n)(2n-1)Kp\ [\frac{4n-3}{4n-5} - \frac{1}{2n-3}]\ a^{2n-3} \quad .$$

Thus, $M_4 > e_{2,4}$ implies that

$$\frac{2K^2p^2}{4n-7}\ a^{4n-7} > 2(2n)(2n-1)Kp\ [\frac{4n-3}{4n-5} - \frac{1}{2n-3}]\ a^{2n-3}$$

$$\frac{Kp}{4n-7}\ a^{2n-4} > 2n(2n-1)\ [\frac{8n^2-22n+14}{(4n-5)(2n-3)}]$$

$$\frac{2n(2n-1)(2n-2)(2n-3)(4n-3)}{a^2(4n-7)} > 2n(2n-1)\ \frac{(4n-7)(2n-2)}{(4n-5)(2n-3)} \ .$$

Simplifying the inequality,

$$(\frac{2n-3}{4n-7})^2(4n-3)(4n-5) > a^2 \ . \qquad (7)$$

Combining conditions (6) and (7) we obtain

$$(\frac{2n-3}{4n-7})^2(4n-3)(4n-5) > (2n-1)(4n-3)$$

$$(\frac{2n-3}{4n-7})^2 > (\frac{2n-1}{4n-5})$$

(note that the inequality fails to hold true for $n = 2$. Using the inequality

$$\frac{2n-1}{4n-5} > \frac{2n-3}{4n-5}$$

we find

$$\frac{2n-3}{(4n-7)^2} > \frac{1}{4n-5}$$

and

$$8n^2 - 22n + 15 > 16n^2 - 56n + 49$$

$$0 > 4n^2 - 17n + 17 = h(n) .$$

But notice that $h(3) > 0$ and that $h'(n) > 0$ for $n \geq 3$. This contradiction establishes that the function f is not a five-step almost orthogonal function.

SUMMARY AND CONCLUSION

Let us see how well the four-step orthogonal functions and the improbable five-step almost orthogonal functions satisfy the requirements of the interference reduction systems. There is one other important condition that now must be imposed on the S_i of Part I. For electronic reasons, each of the functions S_i must take an equal amount of power to be transmitted.

To satisfy this requirement, we impose two conditions on S_i. Since

$$S_i = \sum_{j=1}^{n} a_{i,j} T_j ,$$

one condition is that each $A_i = \{a_{i,1}, a_{i,2}, a_{i,3}, \ldots, a_{i,n}\}$ contain only ±1 or 0, and have the same number of non-zero elements (as indicated in Part I). The second condition is that each T_j be such that

$$\int_{-a}^{+a} T_j^2 = K \qquad \text{for } j = 1,2,\ldots,n$$

where K is a non-zero constant. In practice,

$$T_{2j-1} = \sin(j)$$

$$T_{2j} = \cos(j)$$

over the interval $(-\pi, +\pi)$ worked quite satisfactorily.

The foregoing two conditions imply that

$$\int_{-a}^{+a} S_i^2 = \int_{-a}^{+a} \left(\sum_{j=1}^{n} a_{i,j} T_j \right)^2 = \int_{-a}^{+a} (a_{i,1}^2 T_1^2 + a_{i,2}^2 T_2^2 + \dots + a_{i,n}^2 T_n^2)$$

and thus,

$$\int_{-a}^{+a} S_i^2 = rK \, . \qquad \text{for } i = 1,2,\dots,n$$

Therefore, each S_i takes an equal amount of transmission power.

However, if a function $f = T_1$ were a five-step almost orthogonal function (assuming that one does exist), then f would not satisfy the equal power requirement. Condition (5) of Part III is in direct contradiction to such a requirement.

Finally, let us see which four-step orthogonal functions satisfy the equal power requirement. Take example 1 of Part II

$$f(x) = px^4 + qx^2 + r$$

where

$$M_0 = \int_{-a}^{+a} (px^4 + qx^2 + r)dx = 2\left(\frac{25}{9} - \frac{60}{7} + \frac{36}{5} + \frac{150}{35} - \frac{180}{21} + \frac{225}{49}\right)a^5 = 3\tfrac{1}{5}\, a^5$$

$$M_1 = 2\, f(a)\, f'(a) = 2\left(\tfrac{8}{7}\right)(8)\, a^3 = 18\tfrac{2}{7}\, a^3$$

$$M_2 = 2\, f'(a)\, f''(a) = 2(8)(60-12) = 768\, a$$

$$M_3 = \int_{-a}^{+a} (24px)^2 dx = 2\, \frac{576}{3a^4}\, a^3 = \frac{9600}{a} \; .$$

Evidently, no value of the parameter a will make the M_i close in size.

Example 2 of Part II behaves in a similar manner.

$$f(x) = px^{2n} + qx^2 + r$$

where

$$M_1 = \int_{-a}^{+a} (2npx^{2n-1} + 2qx)^2 dx$$

$$= 8n^2\left[\frac{(4n-3)^2}{4n-1} - 2\,\frac{(4n-3)(2n-1)}{2n+1} + \frac{(2n-1)^2}{3}\right] a^3$$

$$M_2 = 2\ f'(a)\ f''(a)$$

$$= 64n^2(n-1)^2(2n-1)\ a$$

$$M_3 = 2\ f''(a)\ f'''(a) - e_{2,4}$$

$$= 32n^2(n-1)^2(2n-1)^2\ \frac{(4n-3)^2}{4n-5} \cdot \frac{1}{a}\ .$$

We already know that for $n = 2$ the M_i are not close in size for any value of a. By direct substitution, the same is found to be true for $n = 3$ and $n = 4$. For bigger values of n, notice that M_1 has degree 4 in n, M_2 has degree 5, and M_3 has degree 7. Hence, the bigger the value that n takes, the more do the M_i diverge in size.

Furthermore, none of the other examples of four-step orthogonal functions examined in Part II satisfy the equal power requirement. Thus, the usefulness of such functions in system II is of a small degree. As yet, no other applications of the use of four-step orthogonal functions is known to the author.

APPENDIX I

Consider a communications signal in the form of a continuous function G such that G has a finite bandwidth and hence, contains no frequencies greater than some constant W. Then G is completely determined by the set of ordinates of G taken at intervals of length $\frac{1}{2W}$, the set extending throughout the whole time domain.[7] If the time domain is $(-\infty, +\infty)$, then an analytical expression for G in terms of the values of G at the sampling points is

$$G(t) = \sum_{i=-\infty}^{+\infty} G\left(\frac{i}{2W}\right) \frac{\sin(2\pi Wt - i\pi)}{2\pi Wt - i\pi} = \sum_{i=-\infty}^{+\infty} G\left(\frac{i}{2W}\right) Sa(2\pi Wt - i\pi)$$

$$= \ldots + G(0)\frac{\sin(2\pi Wt)}{2\pi Wt} + G\left(\frac{1}{2W}\right) \frac{\sin(2\pi Wt-\pi)}{2\pi Wt-\pi} + G\left(\frac{2}{2W}\right) \frac{\sin(2\pi Wt - 2\pi)}{2\pi Wt - 2\pi} + \ldots$$

where the function $Sa(x) = \frac{\sin x}{x}$ is called the <u>sampling function</u> (see figure <u>3</u>).

As an example of the behavior of the expansion of G, let us examine the three terms in the expansion corresponding to n = 0,1,2 where each of the three terms is represented graphically in figure <u>4</u> .

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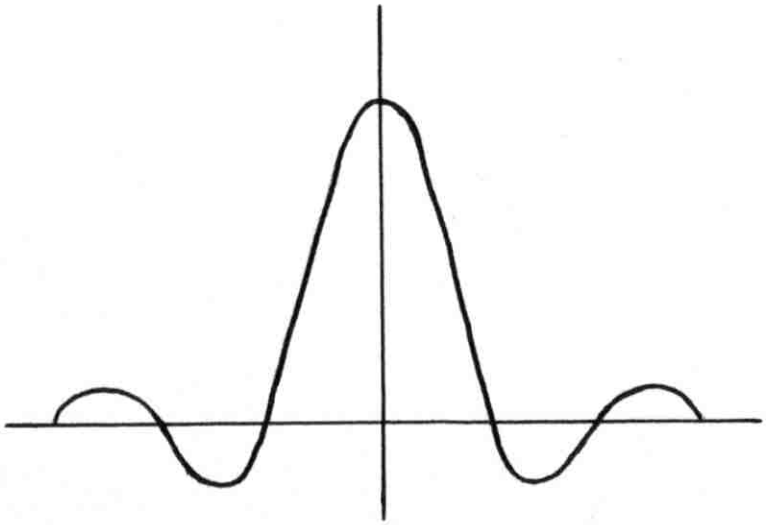
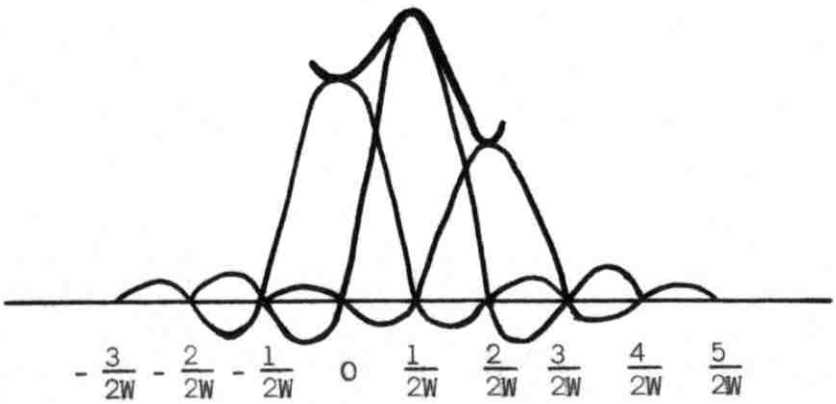
[7]This is a theorem known as the <u>sampling theorem in the time domain</u>. The proof of this theorem along with all the other material presented in Appendix I can be found in Stanford Goldman's Book, <u>Information Theory</u>, Prentice-Hall, Inc., New York, 1953; pp.67-71.



The Sampling Function $Sa(x) = \frac{\sin x}{x}$

Figure 3



$$G(0)\ \frac{\sin(2\pi Wt)}{2\pi Wt}\ , \quad n = 0$$

$$G\left(\frac{1}{2W}\right) \frac{\sin(2\pi Wt - \pi)}{2\pi Wt - \pi}\ , \quad n = 1$$

$$G\left(\frac{2}{2W}\right) \frac{\sin(2\pi Wt - 2\pi)}{2\pi Wt - 2\pi}\ , \quad n = 2$$

Figure 4

As a result of the rapid attenuation of the sampling function, the effects of any term in the expansion of G will be of consequence only within a relatively small number of intervals from that term. This can be seen in figure 4; the term corresponding to $n = 2$ has a diminished effect on the term corresponding to $n = 0$. Therefore, the function G in a neighborhood of the sampling point $(\frac{n}{2W})$ is almost completely determined by the value of G at $(\frac{n}{2W})$ and by a few values at sampling points on either side of $(\frac{n}{2W})$. It is this last property of the expansion of G that makes the sampling theorem so useful.

APPENDIX II

Let a communications signal be considered as a continuous function G. By the preceding appendix, if G contains no frequencies higher than W, then G is completely determined by values of G taken at intervals $\ldots, \frac{-1}{2W}, 0, \frac{1}{2W}, \frac{2}{2W}, \ldots$. Let $p_i = G(\frac{i}{2W})$ for $i = \ldots, -1, 0, 1, 2, \ldots$ (see figure 5). Call each ordinate p_i a pulse.

Let each pulse p_i be approximated in the following manner: If the amplitude of p_i is less than or equal to n units of amplitude and greater than n-1 units of amplitude, then let us say that p_i is approximately equal to a pulse level of $n - \frac{1}{2}$ units of amplitude denoted by P_n. In practice, the range of G is divided into thirty-two units such that the set of all possible P_n contains no more than thirty-two members corresponding to $n = 1,2,\ldots,31, 32$ (see figure 6).

Finally, let there be established a one-to-one correspondence between the set of P_n and the set of thirty-two functions S_n described in the main text such that $P_n \sim S_n$ for $n = 1,2,\ldots,31, 32$. Hence, if the functions S_n identified with the pulses P_n replace the communications signal G, then the total time interval is filled with the S_n (see figure 7).

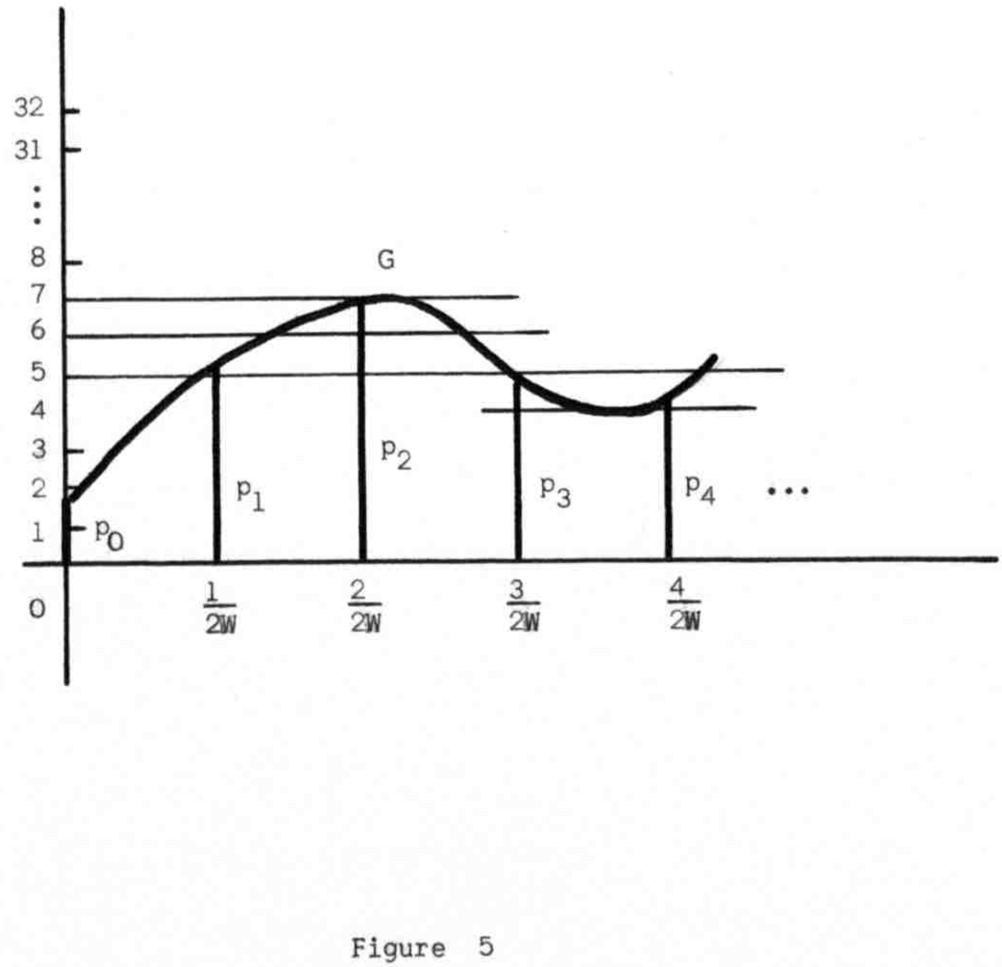
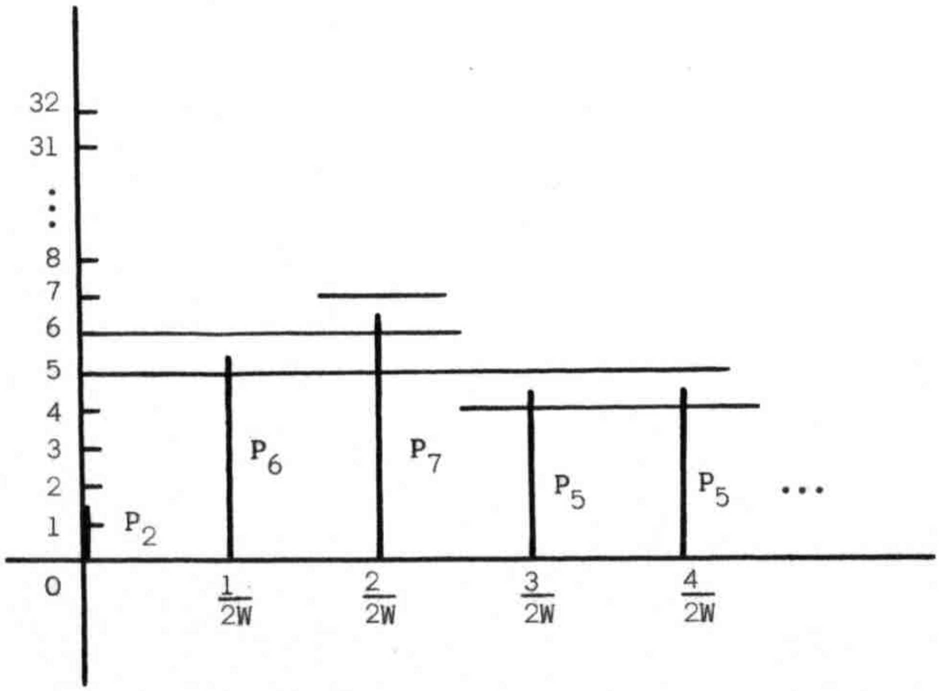


Figure 5

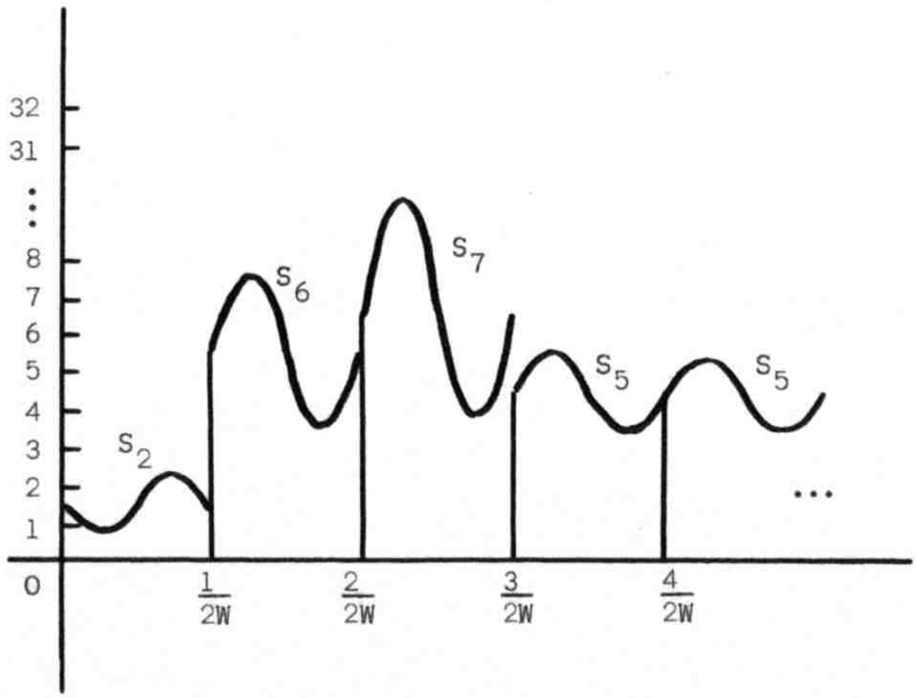


$P_2 \approx p_0$

$P_6 \sim p_1$ $P_5 \sim p_3$

$P_7 \sim p_2$ $P_5 \sim p_4$

Figure 6



$S_2 \sim P_2$

$S_6 \sim P_6$ $S_5 \sim P_5$

$S_7 \sim P_7$ $S_5 \sim P_5$

Figure 7

Consider the reverse problem, the method of obtaining the signal G from the series of functions S_n. Each S_n identifies a pulse level P_n. Form the series

$$G'(t) = \ldots + P_{n_0} \frac{\sin 2\pi Wt}{2\pi Wt} + P_{n_1} \frac{\sin(2\pi Wt-\pi)}{2\pi Wt-\pi} + P_{n_2} \frac{\sin(2\pi Wt-2\pi)}{2\pi Wt-2\pi} + \ldots$$

$$= \sum_{i=-\infty}^{+\infty} P_{n_i} \frac{\sin(2\pi Wt - i\pi)}{2\pi Wt - i\pi}$$

where the pulse level P_{n_i} was obtained from the pulse p_i. By the sampling theorem of Appendix I, G' will be an approximation to the signal G, the degree of accuracy of the approximation depending on the number of values that n takes. As stated before, extensive research has shown that when n takes thirty-two values, the difference between G' and G is inaudible to the human ear.