AMERICAN UNIVERSITY OF BEIRUT

THE ECONOMIC LOT SCHEDULING PROBLEM: A REVIEW

by
AMER FAYEZ KAKISH

A project
submitted in partial fulfillment of the requirements
for the degree of Master of Engineering Management
to the Engineering Management Program
of the Faculty of Engineering and Architecture
at the American University of Beirut

Beirut, Lebanon
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AMERICAN UNIVERSITY OF BEIRUT

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AN ABSTRACT OF THE PROJECT OF

Amer Kakish for Master of Science

Major: Engineering Management

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The Economic Lot Scheduling Problem (ELSP) has drawn much attention during the past century. It stems from the fact that when the rate of production exceeds the demand rate, production of a given product has to stop after its inventory has reached a predetermined level. Demand will then be satisfied from inventory, while the production facility can be used to produce other types of products. The problem is then to determine the production quantity of each product as well as the cycle time needed.

Numerous papers have appeared in the literature which dealt with the problem of ELSP. A number of these papers were cited in this project, indicating their working environment, the advantages of each, their limitations and how they compare to other models. In addition, we will present a new two-product single-facility problem that allows for shortages. We will prove that adopting a shortage policy for the two product case is never optimal.

ACKNOWLEDGMENTS

I would like to express my sincere thanks to Dr. Nadim Abboud whom without his precious help, supervision, guidance and advice, this work would not have been accomplished.

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TO MY PARENTS

for the wisdom they give

CHAPTER ONE

INTRODUCTION

The Economic Lot Scheduling problem (ELSP) is encountered in many industries, and has drawn much attention during the past half century. It stems from the fact that when the rate of production exceeds the demand rate, production of a certain product is going to be stopped after its inventory has reached a prescribed level. The production facility is then used to produce other products in order to maximize the utilization of the facility.

When two or more items compete for using the same facility, a well known phenomenon of interference will occur. That is, eventually, the facility will be required to produce more than one item at the same time, which is physically impossible. Thus, the main issue in lot size scheduling is the simultaneous determination of the batch sizes of the different items produced by the facility, and the order in which the lot sizes will be produced. Batch sizing arises due to the setup time and cost incurred when the machine switches from one product to another. The cost may be due to scrap losses or cleaning while adjusting the machine to produce the next product. On the other hand setup times imply a down time during which the machine cannot produce, and consequently implies a need to carry more inventory.

Numerous approaches have been developed that try to solve the ELSP. In general, heuristic approaches are used and they sometimes provide very good solutions. Analytical approaches are also used to solve a restricted version of the ELSP, where restrictions are imposed on the cycle times, the number of times the product is produced in each cycle, batch sizes and others. The main difficulty in obtaining optimal solutions to the general ELSP is in the determination of feasible schedules of the production runs. Researchers have shown that feasibility testing is an NP-Hard problem. We note that feasibility is achieved when no interference occurs and the schedule meets the demand requirements of all products.

Since the feasibility of a schedule is of prime concern, most researchers have focused on the analysis of some policies which facilitate the search for a solution for this outstanding scheduling problem. Special attention has been given to cyclic schedules where batches of the different items are produced in the same sequence indefinitely.

The approaches commonly discussed in the literature can be classified into three categories : (1) the Common Cycle (CC) which is also referred to as Rotational Cycle approach, (2) the Basic Period (BP) approach, and (3) the Dobson's approach.

Define the product cycle time to be the time between successive production batches of a given product. The common cycle approach restricts all the product's cycle time to be equal, and finds the optimal common products cycle time. CC has the advantages of guaranteeing a feasible solution, and the procedure itself is simple.

The BP approach allows different cycle times for different products. However, it restricts each product's cycle time to be an integer multiple of a period of time called a basic period or a fundamental cycle. Furthermore, lots of each item, within a cycle, should be of the same size. In general, this approach is considered to give better solutions than the CC approach, but at a risk of producing infeasible schedules.

Dobson's approach is quite different than the two previously mentioned approaches in the sense that it does not use a basic period of time, and it allows lot sizes to vary over a cyclic schedule.

This project starts with a brief and rather broad review of the ELSP for the case of deterministic systems and infinite time horizon. The discussion views the same problem from the perspective of different research papers that appeared in the literature since 1964. An analysis of a special version of the ELSP problem is then presented, where only two products are involved with allowed shortages.

The chapters of the project are organized as follows. Chapter 2 presents a problem definition along with a description of all the terminology used throughout the project. Detailed discussion of the assumptions under which the

ELSP operates is also given. Chapters 3 and 4 present a "comprehensive" literature review of the ELSP problem since the pioneering work of Maxwell in 1964. In Chapter 4, the review focuses more on the two product scenario and the possible extensions. Chapter 5 introduces a new solution approach for the case of two products with allowed shortages, and we conclude the project in Chapter 6.

CHAPTER TWO

WORKING ENVIRONMENT AND BASIC RESULTS

2.1 Problem definition

There are N products that need to be produced on the same facility. The N items may have different but known production rates, demand rates, setup costs, and inventory carrying costs. Given these parameters and the sequence-dependent setup times, the problem can be stated as finding a feasible 'optimal if possible' solution for scheduling these N products. Therefore, the problem is one of lot size determination and scheduling for the N products.

Before we state the different assumptions used by different researchers we have limited the scope of the problem by the following assumptions. First, the production capacity is sufficient enough to meet the demand of all products. Second, no more than one product can be produced at the same time. Third, the time horizon is infinite, and fourth, there are no uncertainties in the problem.

2.2 Assumptions Used by Researchers

Several assumptions have been imposed by researchers to help find a solution for the outstanding problem ELSP. These assumptions relate to different aspects of the problem, such as, the setup time and cost, inventory accumulation, cycle time used, the objective function used, and whether backordering is allowed or not.

2.2.1 Setup time

Setup time is the time needed to switch from producing one product to another. Some researchers assumed this time to be sequence dependent which created a new problem in addition to the ELSP, that is, determining the best sequence of switch. This problem is known as the traveling salesman problem.

In an attempt to make the problem easier, and due to the fact that some of the modern production systems used are flexible (e.g., FMS systems), the setup time is considered sequence independent. In some cases it is even considered negligible.

2.2.2 Inventory

Due to the setup time and cost encountered in switching production and the fact that no backordering is allowed, a sufficient level of inventory of each product is needed. The question regarding inventories is, what should the initial level of each product be in order to avoid shortages? Some researchers addressed this question while others neglected it.

2.2.3 Cycle time

As defined earlier, the cycle time is the time between two consecutive starts of a production run for a given product. Researchers developed three different approaches in order to deal with cycle times.

1. The Common Cycle approach, which was first developed by Hanssman in 1962, is a long enough cycle to accommodate the production of each item exactly once. Using this cycle time, the solution is guaranteed to be feasible but not necessarily optimal.

2. The Basic Period (BP) approach admits different cycles for the different items, but restricts the cycle to be an integer multiple of a fundamental cycle or a basic period. Moreover, this cycle should be long enough to accommodate the production of all the items. Researchers made further assumptions regarding the multipliers; some assumed the multipliers to be any positive number while others restricted them to be in the order of 2.

3. The third approach is similar to the second with the additional property that items are loaded on two BP's simultaneously. In this case, the BP can assume any value that is large enough to accommodate such simultaneous loading.

2.2.4 Objective Function

Most of the researchers who dealt with the ELSP problem used the criterion of minimizing the sum of the setup costs and holding costs for an objective function. There are special cases, however, where the objective function is different. For example, the criterion could be to minimize warehouse capacity, material procurement, or maximum inventory level among others.

2.2.5 Backorders or Shortages

Backorder cost is the cost incurred when a demand occurs for a certain product and there is not enough items in inventory in order to satisfy this demand. As a result, two different scenarios may occur: either the customer will be lost, or he will willingly wait (backorder) till his demand is satisfied.

Most researchers avoided dealing with backorders, by restricting the inventory level to be greater than or equal to zero. This was a simplification assumption since backordering would render the problem more difficult.

2.3 Notation

N : total number of products to be produced

d_i : demand rate in units per unit time for product i

p_i : production rate in units per unit time for product i

s_{ij} : setup time in units of time per production lot to switch from product i to product j. Subscript i will be deleted if the setup time is sequence independent.

ρ_i : ratio of demand rate to production rate of product i ($\rho_i = \frac{d_i}{p_i}$)

ρ : total utilization factor ($\rho = \sum_{i=1}^{N} \rho_i \leq 1$)

T_i : production cycle for product i

τ_i : the processing time per lot of product i ($\tau_i = \rho_i T_i$)

δ_i : the total production time per lot i, this includes setup time and processing time ($\delta_i = s_i + \tau_i$ for the sequence independent case)

n_i : multiplier for cycle T_i

η_i : frequency of production for product i ($\eta_i = \frac{1}{T_i}$)

T_{cc} : Common Cycle

ω : fundamental cycle

A_i : setup cost of product i per production lot

h_i : holding cost per unit of product i per unit time

μ_i : the cost to produce one unit of product i

C_i : the total cost function per unit time of product i

C : the total cost function per unit time which includes setup and holding costs of all N products ($C = \sum_{i=1}^{N} C_i$)

2.4 Upper and Lower Bound Solution

For the case of sequence independent setup times, it is possible to establish upper and lower bound solutions on the optimal solution.

2.4.1 Upper Bound

If we assume a cycle is long enough to include the production of each item exactly once, a solution providing an upper bound for the objective function can be obtained. The solution is considered an upper bound, since each product is required to maintain a large inventory that is sufficient to satisfy the demand during the production of other products. This approach is referred to as the common cycle CC approach.

It is clear that the total cost per unit time due to product i is given by

$$C_i = \frac{A_i}{T_i} + \frac{T_i}{2} h_i d_i (1 - \rho_i). \tag{2.1}$$

Therefore,

$$C = \frac{1}{T}\sum_{i=1}^{N} A_i + \frac{T}{2}\sum_{i=1}^{N} h_i d_i (1-\rho_i) \ . \tag{2.2}$$

Differentiating (2.2) with respect to T, and setting the derivative equal to zero, we get,

$$T^* = \sqrt{\frac{2\sum_{i=1}^{N} A_i}{\sum_{i=1}^{N} h_i d_i (1-\rho_i)}}, \tag{2.3}$$

which results in a total cost of

$$C^* = \sqrt{2\sum_{i=1}^{N} A_i h_i d_i (1-\rho_i)}. \tag{2.4}$$

The above approach can be extended by the use of Lagrange multipliers to take into account various restrictions such as: a limit on the number of setup hours expressed as a function of the cycle time T_i, a limit on the storage capacity, a limit on the total average inventory, etc. However, these variations are beyond the scope of this project. The interested reader is referred to Parsons (1966).

2.4.2 Lower Bound Solution

The lower bound is obtained by assuming that each product can be manufactured independently of other products. That is, we assume that there exist as many production facilities as the number of products. This approach is referred to as the Independent Solution (IS) approach.

The average cost per unit time when item i is produced in cycles of length T_i is given by (2.1). By setting the derivative of (2.1) with respect to T_i equal to zero, and solving for T_i, we get

$$T_i^* = \sqrt{\frac{2A_i}{h_i d_i (1-\rho_i)}}, \tag{2.5}$$

with a corresponding cost of

$$C_i^* = \sqrt{2A_i h_i d_i (1-\rho_i)}. \tag{2.6}$$

Summing (2.6) over all i should provide a lower bound on C, which is the total cost per unit time.

If the IS solution is a feasible solution, then it must be optimal since it acts as a lower bound. But, because we treat products independently, there is no guarantee that the problem of interference will be avoided, and hence no guarantee on optimality. We note that feasibility testing of IS solutions has been shown to be NP-hard by Hsu (1983). While testing solution feasibility is a difficult problem, it was found that interference is most likely to occur when:

1. The load on the facility exceeds capacity. Therefore overlap in the schedule will appear to be necessary in order to satisfy output requirements.

2. Initiating a production run for say product i, prior to the completion of the preceding production run may be necessary in order to avoid a stockout of product i.

We should also mention that the IS approach provides a very useful method to test the performance of heuristic procedures. That is, the criterion of selecting a good heuristic algorithm could be based on its deviation from the IS solution.

2.5 Testing the Validity of the Common Cycle approach

Unlike the Independent Solution, the Common Cycle schedule is always feasible. The Common Cycle approach schedules only one lot of each product in a time interval called "Rotational Cycle" or T_{cc}. This cycle repeats itself every T_{cc} time units, and can be computed as follows:

$$T_{cc} = \text{Max}\ \{T^{*}, T_{min}\} \tag{2.7}$$

where T^* is as given in (2.3), and

$$T_{min} = \frac{\sum_{i=1}^{N} \tau_i}{1 - \sum_{i=1}^{N} \rho_i}. \tag{2.8}$$

Note that the concept behind using T_{min} is given by Maxwell (1964) and will be discussed later.

Although the CC schedule is always feasible, it is not always optimal. Inman and Jones (1989) have developed conditions under which the CC approach can be used as a good approximation of the optimal cycle. We now briefly discuss how these conditions have been developed.

Let C_{IS} and C_{cc} denote the average total costs when IS and CC approaches are used respectively, and let

$$T_i^{IS} = \text{Max}\{T_i^*, T_i^{Min}\} \tag{2.9}$$

where T_i^* is as given in (2.5), and

$$T_i^{Min} = \frac{s_i}{1 - \rho_i}. \tag{2.10}$$

Then,

$$C_{IS} = \sum_{i=1}^{N} \left(\frac{A_i}{T_i^{IS}} + \frac{T_i^{IS} H_i}{2} \right), \tag{2.11}$$

where $H_i = h_i d_i (1 - \rho_i)$, and

$$C_{cc} = \sum_{i=1}^{N} \left(\frac{A_i}{T_{cc}} + \frac{T_{cc} H_i}{2} \right), \tag{2.12}$$

where T_{cc} is obtained from (2.7). When $T_{min} \leq T^*$ and $T_i^{Min} \leq T_i^*$, (2.11) and (2.12) can be expressed as

$$C_{IS} = \sum_{i=1}^{N} \sqrt{2 A_i H_i} \tag{2.13}$$

and

$$C_{cc} = \sqrt{2 \sum_{i=1}^{N} A_i \sum_{i=1}^{N} H_i} \ , \tag{2.14}$$

respectively. It is worthwhile to mention that in many real world situations the CC schedules are nearly optimal. See Inman and Jones (1989) for more details.

2.5.1 When is the CC Schedule Optimal?

Inman and Jones (1989) have proposed the following lemma which provides sufficient conditions for optimality when the CC approach is used. The lemma is applicable only when the setup times are "relatively small".

Lemma : If the following two conditions hold

1. $\frac{A_1}{H_1} = \frac{A_2}{H_2} = \ldots = \frac{A_n}{H_n}$

2. $T_{min} \leq T^*$

then the CC schedule is identical to the IS schedule and hence is optimal.

Fortunately, the conditions above are frequently satisfied in industry.

2.5.2 Percentage Deviation from Optimal Solution

Given that the A/H ratio's for different products are almost the same, we are interested in knowing how close is the CC solution to the optimal. To simplify the problem, Inman and Jones examined only the case when setup times are relatively small in order to satisfy the equalities: $T_i^{IS} = T_i^*$ and $T_{cc} = T^*$. Let,

$$\frac{A_i}{H_i} = \lambda_i \frac{A_1}{H_1} \text{ for } i = 1,2,\ldots,N \qquad (2.15)$$

where $0 \leq \lambda_i \leq 1$ and A_1/H_1 is the largest ratio among the N ratios calculated.

Since the optimum solution is not available, Inman and Jones studied how the difference $C_{cc} - C_{IS}$ changes as a function of the λ's. They have developed the following results, which will be presented without proofs. For more details, the interested reader is referred to the original paper.

<u>Theorem:</u> *For the N product one machine ELSP with $T_i^{IS} = T_i^*$ for all i; $T_{cc} = T^*$*

$$\frac{C_{cc} - C_{IS}}{C_{IS}} = \sqrt{\frac{\sum_{i=1}^{N} \lambda_i H_i \sum_{i=1}^{N} H_i}{\sum_{i=1}^{N} \sqrt{\lambda_i H_i^2}}}. \qquad (2.16)$$

Based on the above theorem, the following results can be stated, regarding the same conditions.

$$\frac{C_{cc} - C_{IS}}{C_{IS}} = \sqrt{1 + \frac{(1-\lambda^{1/2})^2}{2\lambda}} - 1\,, \tag{2.17}$$

where $\lambda = \text{Min}\{\lambda_i\}$. Furthermore, for the two product scenario (2.17) reduces to

$$\frac{C_{cc} - C_{IS}}{C_{IS}} \leq \sqrt{\frac{\lambda+1}{2\lambda^{1/2}}} - 1\,. \tag{2.18}$$

If we further assume that $H_1/H_2 \geq 1$ or $A_1/A_2 \leq 1$ then (2.18) reduces to

$$\frac{C_{cc} - C_{IS}}{C_{IS}} = \sqrt{1 + (\lambda^{1/2} - 1)^{1/2}} - 1\,. \tag{2.19}$$

Moreover, for the two product case when $H_1/H_2 \geq 1$, the CC schedule is always within 41.4% of the optimum value regardless of the value of λ.

2.5.3 Extension to the Class of Easy ELSP

Through the use of a tighter lower bound that explicitly considers machine capacity, Gallego (1990) extended the conditions under which the rotational schedule is optimal. He was motivated by the results given by Inman and Jones, and he further generalized their conclusions to include the case when the machine capacity is binding. Moreover, he demonstrated how these results can be easily extended to the case where backorders are allowed at a linear time weighted costs.

Gallego introduced a new sufficient condition for the CC to be optimal. His proposition is stated as follows:

A sufficient condition for a CC to be optimal is that the ratios $\frac{A_i}{H_i}$ *and* $\frac{s_i}{H_i}$ *being independent of i.*

2.5.4 Conclusion

The CC schedule has been shown to be near optimal for a wide range of realistic situations. Simple bounds were obtained that guarantee the CC solution to be within a specific percentage of optimality. We can further improve the CC solution through the use of Group Technology (GT), which groups various parts and products with similar design and/or production processes in order to increase the efficiency of production.

CHAPTER THREE

LITERATURE REVIEW FOR THE ELSP

3.1 Introduction

An operational model extensively used in the analysis of inventory problems and production systems, assumes that the demand is constant and known with certainty, and the machine when operating produces at a constant and known rates as well. Note that, these assumptions are the same assumptions used in the derivation of the classical and well known Economic Order Quantity (EOQ), which served as a basic model for analyzing the multiproduct case by a number of researchers like Maxwell, Goyal, Gallego, Hsu and others.

As previously indicated in Chapter 1, the approaches proposed are categorized into two categories. First, analytical approaches, which achieve an optimal solution for a more restricted version of the ELSP problem. Second, heuristic approaches, which achieve a near optimal solution for the ELSP. In this chapter, we will examine the different procedures and approaches developed, indicate the working environment of each, and try to compare between the results available. In addition, we mention some of the different extensions available in literature.

3.2 Maxwell (1964)

One of the earliest papers that analyzed the one machine, multiproduct case, is that of Maxwell (1964). Maxwell's article established the foundation on which other researchers built their algorithms and results.

Maxwell started his analysis by discussing the basic Economic Order Quantity model (EOQ) developed for a single product. He then generalized the model by including setup times in the formulation. He indicated that the frequency of start ups (setups), may impose time requirements which may exceed the time available (i.e., the inventory depletion time). In any arbitrary time period T, the time available for setup is

$$t_s = T(1-\rho). \tag{3.1}$$

With no shortages allowed, the setup time required cannot exceed the time available, t_s. Hence, a lot size q must satisfy the inequality

$$q \geq \frac{ds}{1-\rho}, \tag{3.2}$$

and so, the complete solution for the single product case takes the form

$$q^* = \text{Max}\{\sqrt{\frac{2Ads}{h(1-\rho)}}, \frac{ds}{1-\rho}\}. \tag{3.3}$$

Note, that the setup cost A, was taken as a function of the setup time. Later on, we will see that this assumption has been dropped out by most of the researchers.

Unfortunately, q^* is not applicable for the multiproduct case because of the implicit assumption in the formulation of the EMQ, which states that the entire time of the machine is available for the production of that single product. This statement implies that there exists an independence relation among the N products needed to be produced, which is not the case.

Maxwell tried to test the feasibility of the q^* obtained by the IS. He was faced with the problem of how to avoid scheduling conflicts. That is, can the lot sizes obtained be sequenced with no interference? The answer in general is negative due to the reasons previously discussed.

Using the Common Cycle (CC) approach, the N-dimensional ELSP problem will be reduced to a 1-dimensional problem. This is achieved by modeling the lot sizes as a function of the time interval between production cycles. Assume that this time interval is fixed for all the N products in the rotational cycle. The total cost of production can be easily determined, as in section 2.5, to be

$$C = \sum_{i=1}^{N} \frac{A_i s_i d_i}{q_i} + \sum_{i=1}^{N} \frac{h_i q_i (1-\rho_i)}{2}, \tag{3.4}$$

where, q_i can be expressed as $q_i = d_i T$. Thus, (3.4) can be rewritten as

$$C = \frac{1}{T} \sum_{i=1}^{N} A_i s_i + \frac{T}{2} \sum_{i=1}^{N} h_i d_i (1-\rho_i). \tag{3.5}$$

By the use of calculus the optimal cycle can be found to be

$$T = \sqrt{\frac{2\sum_{i=1}^{N} A_i s_i}{\sum_{i=1}^{N} h_i d_i (1-\rho_i)}}. \tag{3.6}$$

Again as in the single product case, the T^* should be adjusted to include the restriction imposed by product saturation, in addition to the changeover frequency saturation. Hence, the complete solution takes the form

$$T^* = \max\{ \sqrt{\frac{2\sum_{i=1}^{N} A_i s_i}{\sum_{i=1}^{N} h_i d_i (1-\rho_i)}}, \frac{\sum_{i=1}^{N} s_i}{1-\rho} \}. \tag{3.7}$$

The importance of this result, as previously indicated in section 2.5, lies in the ability of the CC approach in developing feasible solutions. It should be noted here that the setup times were assumed to be sequence independent. However, if the setup time is sequence dependent, we must first determine the sequence in which the products are to be produced, after which we determine the optimal production quantities. The first part of the problem is solved as a traveling salesman problem in order to minimize the sum of the setup times, or the changeover cost. After determining the sequence, T^* can be easily found from (3.7).

Producing only one lot of each product in a rotational cycle, may not lead to optimality for the following reasons. First, the imbalance in demand rates and production rates that lead to different individual machine utilization. Second, the setup time and cost are sequence dependent. Third, if most of the demand arises from a small proportion of the products, one might suspect that more frequent production of the highly demanded products, will decrease the inventory of the products at the expense of increasing the inventory level of the other slow moving products. Motivated by these reasons, Maxwell was able to introduce a general formulation for the ELSP that included most of the possible variations that may be encountered.

3.2.1 Problem Definition

Maxwell assumed that setup times are sequence dependent, and it need not be true that $s_{ij} = s_{ji}$. Given the demand rates, production rates, changeover matrix and the cost coefficients, Maxwell stated the problem to be:

In what sequence should the products be produced and for what length of time should the machine continuously produce the product, in order to minimize the sum of inventory carrying costs and changeover costs?

This problem statement served as a guideline for most of the succeeding works. It is the first problem definition recorded in the literature that describe the ELSP as an integration between a sequencing problem and a lot sizing problem.

In addition to the previous notation described in section 2.4, Maxwell adopted the following notation.

q_{ij} : the lot size of product i when it is produced for the j^{th} time

τ_{ij} : the production time of product i when it is produced for the j^{th} time

I_{ij} : the starting inventory of product i when it is produced for the j^{th} time

e_{ij} : idle time after the production of product i for the j^{th} time

$i(k)$: product i is the k^{th} product produced in the sequence

M_i : the average inventory of product i

$\Re_{ij}$: the setup time of product i when it is produced for the j^{th} time

κ_i : the number of production runs of product i in a cycle of length T

κ : the total number of runs in a cycle of length T

3.2.2 Model Development

Using the above notation, Maxwell was able to specify an operating discipline by $i(k)$ and either $(q_{ij}, \Re_{ij})$, $(\Re_{ij}, e_{ij})$ or $(e_{ij}, \tau_{ij}, I_{ij})$. Among these, the triplet $(e_{ij}, \tau_{ij}, I_{ij})$, was the most convenient operating discipline. The model presented by Maxwell (1964) is given below.

The model :

$$Min\ C = \frac{A}{T}\sum_{k=1}^{\eta} s_{i(k),i(k+1)} + \sum_{i=1}^{N} h_i M_i \tag{3.8}$$

Subject to :

1. Non-negativity constraints to ensure that no backorders are permitted, and that idle time is restricted to be positive,

$$I_{ij} \geq 0, \text{ and } e_{ij} \geq 0 . \tag{3.9}$$

2. Sufficient production quantities to ensure that all the demand is satisfied during depletion time,

$$\sum_{j=1}^{\kappa_i} \tau_{ij} = \rho_i T. \tag{3.10}$$

3. Cycle time equation

$$\sum_{i=1}^{N}\sum_{j=1}^{\kappa_i}(\tau_{ij} + e_{ij}) + \sum_{k=1}^{\kappa} s_{i(k),i(k+1)} = T. \tag{3.11}$$

The relation between successive inventory levels of the same product i can be expressed as,

$$I_{i,j+1} = I_{ij} + (p_i - d_i)\tau_{ij} - d_i(S_{ij} + \tau'_{ij} + E_{ij}), \tag{3.12}$$

where :

S_{ij} : the sum of the changeover times between the j^{th} production and the $(j+1)^{st}$ production run of product i.

E_{ij} : the sum of idle times between the j^{th} production and the $(j+1)^{st}$ production run of product i.

τ'_{ij} : the sum of the production times for other products between the j^{th} and $(j+1)^{st}$ production run of product i.

To get the remaining expression for the average inventory level M_i, we examine the general plot of inventory which is given in Figure 3.1. Close examination of Figure 3.1 will reveal that

$$M_i = \frac{1}{2d_i(1-\rho_i)T}\sum_{j=1}^{\eta_i}\left[(I_{ij} + (p_i - d_i)\tau_{ij})^2 - I_{ij}^2\right] \tag{3.13}$$

We note that the above model, not only introduces sequence dependent setup times, but also expresses the setup cost as a function of time. Furthermore, the model allows for unequal production times.

Despite all the generalizations, the solution of Maxwell's model remained complex for several reasons. First, there exist an infinite number of possible combinations of the triplets (e_{ij}, τ_{ij}, I_{ij}), for each possible order. And if we further assume that the order is specified, the problem of picking the best values of the variables, subject to this order is still difficult and time consuming. Second, the objective function is not necessarily convex in the region bounded by the feasibility constraints, especially for the case of a given sequence with zero idle times. Third, the use of idle times as parameters will complicate the model further.

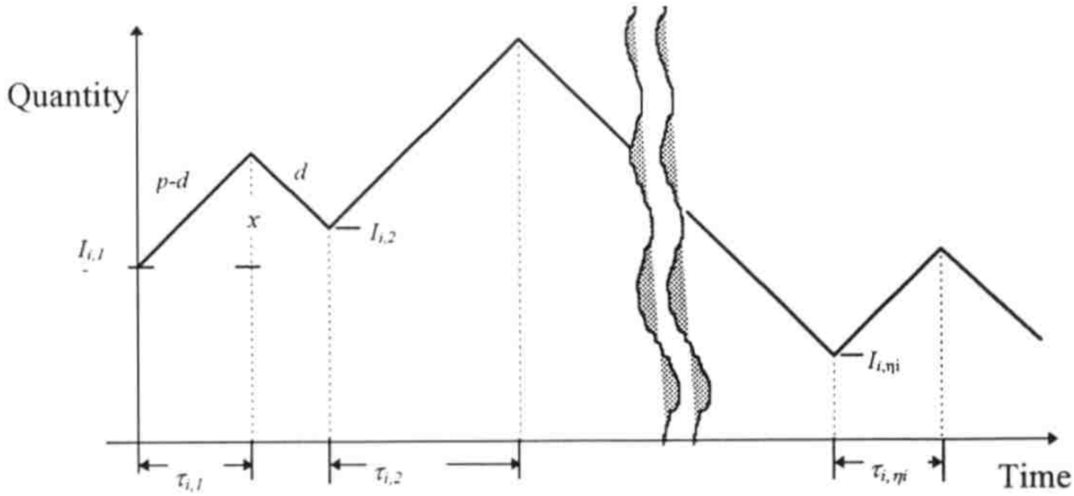

Figure 3.1: The Inventory on Hand

3.2.3 Maxwell's Solution

To Facilitate the obtaining of a solution, Maxwell introduced different assumptions in addition to a number of implicit rules, each of which generates a new version of the ELSP problem. These assumptions in addition to the possible deviations, served as a guideline for most of the researchers in this field.

For a particular product, given the number of times it will be produced per cycle, the best inventory picture would be one in which all lot sizes are equal and production takes place at equidistant points in time. The inventory plot will take the form of a series of identical triangular shapes. Any deviation in this shape will result in higher inventory level, consequently a higher inventory cost. But we should keep in mind that higher inventory cost translates to a decrease in the number of setups required, and eventually to a reduction in the setup cost.

Maxwell suggested a number of possible deviations from the standard triangular shape.

1. *Zero Switch rule*, which states that the production of a certain product cannot start until its inventory level reaches zero. This criterion will be discussed in the next section.

2. *Equal lot sizes*, this rule restrict the lot size of each product *i* to remain constant throughout the planning horizon.

3. *Equal usage interval*, this rule sets the starting inventory levels and lot sizes so that for any particular product, the time intervals during which the product is not produced are of equal lengths.

4. *Maximum inventory rule*, this rule states that, a change in the production from product *i* to product *j* is only possible, if and only if, the inventory level of *i* reaches a prescribed level.

5. *Equal production interval rule*, this rule restricts the time between the production runs for product *i* to be constant.

3.2.4 The Optimality of the Zero Switch Rule

One of the most important implicit rule Maxwell introduced, was the Zero Switch Rule (ZSR), which restricts the production of a certain item to start only when its inventory level hits zero. Matthews (1988) examined the conditions under which ZSR is a necessary condition for optimality given a production capacity equal to the aggregate demand. Matthews' work can be regarded as an extension of the earlier work of Cook et al. (1980). Cook et al.'s approach was based on the concept of the fundamental cycle, in which each item is manufactured at least once during a cycle. Matthews, proved that within the context of the fundamental cycle, the ZSR is indeed a necessary condition for optimality, given approximately equal inventory-holding costs for all items.

To prove the optimality of ZSR, Matthews formulated the problem as a linear programming problem. The assumptions he used are the same as before, with the added restrictions that the process capacity is to equal the aggregate sales rate, and back to back production runs of the same item is not allowed. As a result, the cycle cannot start and end with the same product.

Define

P_{ij} : the duration of the j^{th} run of item i ($P_{ij} > 0$ for all i,j).

S_i : the daily sales rates of item i (expressed as a fraction of the daily capacity of the process).

Y_{i0} : the time duration from the start of the cycle until commencement of the first run of item i.

$I_i(t)$: the inventory level of item i at time t.

Y_{in} : the time starting from the end of the j^{th} run of item i until either the start of its $(j+1)^{st}$ run, or the end of the cycle if Y_{ij} is the last run of the item i.

Then,

$$I_i(0) \geq S_i Y_{i0} \qquad i = 1,\ldots,N, \tag{3.14}$$

and for $r = 1,\ldots, \eta_{i-1}$ and $i = 1,\ldots,N$,

$$I_i(0) + \sum_{j=1}^{r} P_{ij} \geq S_i Y_{i0} + S_i \sum_{j=1}^{r} (P_{ij} + Y_{ij}). \tag{3.15}$$

The inequality (3.14) indicates that the item inventory must be adequate to meet demand until the first production run of the item begins, while (3.15) indicates, that the beginning item inventory plus production through the $r < \eta_i$ must satisfy the sales until the $(r+1)^{st}$ run.

Matthews proved that, when optimizing the model, (3.14) and (3.15) will be binding. Hence, item inventories will always be zero at the beginning of production.

3.3 Bomberger (1966)

Bomberger (1966) published an excellent description for the ELSP problem. His ten product one machine problem became a bench mark for testing the various proposed solutions for the ELSP. Bomberger's model is a dynamic formulation in which restrictions on the cycle time of each product are imposed. It was the first time, the cycle times were restricted to be integer multiples of a fundamental cycle ω. This is the approach that is referred to nowadays as the *Basic Cycle* approach (BP).

Bomberger used two assumptions to assure achieving a feasible solution. First, $T_i = n_i \omega$, where n_i is an integer. Second, the sum of the times required to setup and produce a lot of each item is less than the fundamental cycle ω. The first condition is not restrictive if considered separately from the second. Within any degree of accuracy, cycle times can be made integer multiples of some unit cycle. However, forcing this cycle to be long enough to include the production and setup times of all products is definitely restrictive. This assumption can be stated as

$$\sum_{i=1}^{N} (s_i + \rho_i T_i) \leq \omega , \tag{3.16}$$

where we have used the identity $\tau_i = \rho_i T_i$.

Considering the two restrictions together, we find that the largest common divisor of $T_1, T_2, \ldots, T_N$, must satisfy (3.16). This may inflate the cycle time unduly and cause cost to escalate. Also, machine interference will be avoided, and the problem will be amenable to dynamic programming formulation. Finally, the setup times were assumed to be sequence independent. This assumption was adopted to again facilitate the use of dynamic programming, and to ease the search for a valid solution.

3.3.1 Dynamic Programming Formulation

For some cycle ω, suppose that product $i=1,2,...k-1$ have already been scheduled, and their schedule required X production time units. Let $C(\omega\text{-}X, \omega)$, denote the minimum cost of producing items k to N in the remaining time $(\omega\text{-}X)$. Hence, the cost of ω can be expressed as

$$C(\omega)=\sum_{i=1}^{N}\left[\frac{S_i}{T_i}+\frac{h_i}{2}d_i(1-\rho_i)\omega\right], \tag{3.17}$$

while the cost of product i is given by

$$C_i(n_i,\omega)=\frac{S_i}{n_i\omega}+\tfrac{1}{2}h_i(1-\rho_i)d_i n_i\omega\,. \tag{3.18}$$

Moreover ω must satisfy the inequality

$$\sum_{i=k}^{N}(n_i\rho_i+\frac{S_i}{\omega})\leq\frac{\omega-X}{\omega}\,. \tag{3.19}$$

Using the principle of DP, we can rewrite (3.17), (3.18) and (3.19) as

$$C_k(\omega-X,\omega)=\underset{n_k}{\mathrm{Min}}\left\{C_k(n_k,\omega)+C_k(\omega-T-n_k\rho_k\omega-\tau_k;\omega)\right\} \tag{3.20}$$

where,

$$0 \le n_k \le \frac{\omega - T - \tau_k}{\rho_k \omega},$$

and

$$C_{N+1}(w - T, w) = 0.$$

n_k is integer .

A backward approach using trial values was used to solve this set of equations. These trial values were chosen at any stage by interpolating (extrapolating) between the two prior estimates of ω corresponding to the two lowest costs found.

Given a utilization factor of 90%, the Bomberger solution was found to offer considerable savings over Hassmann (1962). Bomberger also conjectured that the percentage difference between the DP solution and the optimal solution will increase exponentially as the utilization factor increases.

3.4 Analytical Methods

Most often analytical approaches produce an optimal solution for a more restricted ELSP problem. The most trivial solutions for the ELSP problem is that of the Independent Solution and the Common Cycle. The IS approach is rather crude in nature, since it ignores the most elementary feasibility restriction, which states that the total setup and processing times should not exceed the available time. That is,

$$\sum_{i=1}^{N} \frac{\sigma_i}{T_i} \le 1. \tag{3.21}$$

Therefore, the most elementary approach to the ELSP is to guarantee feasibility of the problem by imposing restrictions on the cycle times. Then the best solution is achieved by optimizing the individual cycle durations, subject to the imposed constraints.

Three procedures were cited in the literature with this approach in mind. First, to assume a cycle long enough to accommodate the production of each product exactly once (the common cycle approach). Second, to admit different cycles for the different items, but restrict each T_i to be an integer multiple of a basic period (fundamental cycle), which is long enough to accommodate the production of all items. Third, Similar to the second approach, but "load" the items on two BP's simultaneously. Doing this we can relax the condition that the fundamental cycle is long enough to accommodate such simultaneous loading. Elmaghraby (1978) referred to this approach as the *Extended Basic Period* (EBP) approach.

Researchers were unable to obtain an optimal solution analytically using the Basic Period approach. However, this approach was extensively used in the development of heuristics discussed in the next section. Furthermore, since the CC approach does not provide an optimal solution to the original ELSP problem, in addition to the analytical complexity encountered in the BP approach, researchers started to impose restrictions on the sequence, number of products, etc., in an attempt to make the problem easier. A number of these restrictions will be examined next.

3.4.1 The Basic Period Approach

In this approach one permits varying cycle times $T_i = n_i\omega$, but impose the feasibility constraint that,

$$\sum_{i=1}^{N}\left[s_i + \rho_i n_i \omega\right] \leq \omega . \tag{3.22}$$

This immediately facilitates the use of a dynamic programming formulation. The DP approach was developed by Bomberger (1966), and was discussed in section 3.3.

3.4.2 The Extended Basic Period Approach

This approach is similar to that of Bomberger, except for the method used in loading the items. If items $1,2,\ldots,k-1$ have already been loaded they would occupy ω_1 and ω_2 units of time in the two BP's, ω_1, $\omega_2 \geq 0$ and $\omega_1 + \omega_2 = \sum_{i=1}^{k-1}\sigma_i$. This leaves a residual capacity of $\omega_1 = T - \omega_1$ and $\omega_2 = T - \omega_2$ in the two BP's respectively. Let $F_k(\omega_1, \omega_2)$ denote the minimum cost of producing items $k, k+1, \ldots, N$, when the residual capacities are ω_1 and ω_2. Then

$$F_k(\omega_1,\omega_2) = \min_{n_k} \left| \begin{array}{ll} C_k(n_k,T) + F_{k+1}(\omega_1 - \sigma_k, \omega_2), & \\ C_k(n_k,T) + F_{k+1}(\omega_1, \omega_2 - \sigma_k) & \text{if } n_k \text{ even,} \\ C_k(n_k,T) + F_{k+1}(\omega_1 - \sigma_k, \omega_2 - \sigma_k) & \text{if } n_k \text{ odd,} \end{array} \right. \tag{3.23}$$

where, the first expression in the right-hand side of (3.23) is used when

$$1 \le n_k \le \frac{\omega_1 - s_k}{\rho_k T},$$

the second expression is used when,

$$1 \le n_k \le \frac{\omega_2 - s_k}{\rho_k T},$$

and the last expression is used when

$$1 \le n_k \le \min\{\frac{\omega_1 - s_k}{\rho_k T}; \frac{\omega_2 - s_k}{\rho_k T}\}.$$

$C_k(n_k,T)$ is given as in (3.18); n_k is an integer and $F_{N+1}(*,*) = 0$. If n_k is even, then item k can be produced in only one of the BP's, so we have the choice where to load it. However, if n_k is odd, then it must be loaded in both BP's.

3.4.3 Linear Programming Formulation

Hodgson and Nuttle (1986) where able to model the ELSP problem using linear programming. They focused there attention on determining the run lengths for the N products given a known sequence. Furthermore, they were able to relax the two commonly used assumptions stating that, each product is to be produced on a regular invariant cycle, and that the inventory level at the outset of each production run is zero.

Hodgson and Nuttle proved that if all runs for a given product are equal, while production may start before inventory is exhausted, then the optimal cycle and run lengths may be determined using linear programming methods.

Furthermore, if a sequence generating procedure was used in conjunction with the above result, an optimal solution for this version of the ELSP is easily found. The model described corresponds to that of Maxwell (1964), where equal production runs are used and idle time between the production runs is permitted.

Let , X_j :denote the product run start time $0 \leq X_j \leq T$, $j=1,2,\ldots,L$,

α_i : denote the number of production runs of product i per cycle ,

i_j : denote the product to be processed during run j, $j=1,2,\ldots,L$

J : denote the given sequence $\{ i_1,\ldots,i_j,\ldots,i_L\}$, $i_j \in \{1,2,\ldots,N\}$,

L : denote the number of sequencing positions ($L = \sum_{i=1}^{N} \alpha_i$) ,

e_j : denote the idle time following production run j.

For a given sequence J, all production runs for product i are required to yield $d_i T/\alpha_i$ units. This implies a run length of $d_i T/p_i \alpha_i$ time units.

The objective function can be expressed as minimizing the total cost (which consist of setup costs and inventory costs), subject to, meeting all demand constraints. The relation between start times, idle times and cycle length is expressed as,

$$X_j + \frac{d_{i_j} T}{p_{i_j} \alpha_{i_j}} + e_j + s_{i_{j+1}} = X_{j+1} \quad j = 1,2,\ldots,L-1 , \tag{3.24}$$

and

$$X_L + \frac{d_{i_L} T}{p_{i_L} \alpha_{i_L}} + e_L = T . \tag{3.25}$$

where $X_1 = s_{i1}$.

Let Z_j be the index of the next production run of product i_j after production j, and I_j the inventory level of i_j at the beginning of run j. Then, for $j=1,2,\ldots,L$,

$$I_j + \frac{d_{i_j}T}{\alpha_{i_j}} - d_{i_j}(X_{z_j} - X_j) = I_{z_j} \tag{3.26}$$

Since no backorders are allowed, I_j should be greater than or equal to zero ($I_j \geq 0$). If the I_j's are allowed to be positive, this implies that production of a product may begin before the current supply is exhausted. Also, if the I_j's are restricted to be zero, this will imply the use of the zero switch rule.

Since the production schedule is repeated every T time units, the increase in carrying cost for product i per unit time may be calculated as $h\mu_i \sum_{i=1}^{\alpha_i}\left[I_{i_j}/\alpha_i\right]$. Note that this increase is linearly dependent on the minimum inventory levels. Therefore, the total cost function can be expressed as,

$$\sum_{i=1}^{N} \frac{\alpha_i A_i}{T} + \text{Min}\{h[\sum_{i=1}^{N} c_i(1-\rho_i)\frac{d_i T}{2\alpha_i} + \sum_{j=1}^{L} \frac{c_{i_j} I_j}{\alpha_{i_j}}]\}. \tag{3.27}$$

The LP problem then, is to minimize (3.27), subject to the constraints (3.24), (3.25), (3.26) and the non-negativity constraints.

From an implementation point of view, the following should be noted. First, the production quantities associated with the optimal value of T may be impractical and a need for rounding or specifying a workable number of days may be required. Second, the problem is solved only if the sequence of production is predetermined.

3.5 Heuristic Procedures

Given the difficulty of the ELSP problem, it is only natural that a large number of heuristic procedures will be developed. While many heuristics yield near optimal solutions, the common drawback in these procedures is that they lack a systematic way for testing the feasibility of the solutions they provide. In addition, they do not usually provide a direction to move from an infeasible solution to a feasible one.

In this section we will examine some of the heuristic procedure developed. A number of these heuristics, due to their importance, will be examined in full detail, while others will only be briefly summarized.

3.5.1 Madigan (1968)

Madigan's procedure would first compute the optimal production quantities using the IS approach. He then uses the CC solution to determine the common cycle, and to evaluate the corresponding individual item costs C_i. Then using C_i^* of the IS, Madigan selects the product for which the difference $C_i - C_i^* \geq 0$ is "rather significant." Modification for these lot sizes is performed in integer multiples or integer fractions, in an attempt to bring cost "more in line" with its optimal value C_i^*. A check on feasibility is made at each selection of multipliers. If the multipliers are infeasible, they are modified until feasibility is achieved. Given a set of multipliers $\{n_i\}$, the corresponding BP can be found to be

$$\omega^{*} = \sqrt{\frac{\frac{\sum_{i=1}^{N} A_i}{n_i}}{\frac{1}{2}\sum_{i=1}^{N} h_i d_i n_i (1-\rho_i)}} \, . \tag{3.28}$$

Madigan's procedure suffers from several drawbacks. First, there is no guide to the selection of the items whose cycles are to be modified, and the lack of a systematic way for modifying them. Second, no guide was given to the amount of change of multipliers in each iteration. Third, the objective function used by Madigan is only applicable under the assumption of having cyclic production. Finally, modification is only performed on the multipliers, while the basic period (found using the CC approach), remains constant throughout the procedure.

3.5.2 Stankard and Gupta (1969)

Stankard and Gupta (1969) divided the set of products in groups $G, G^1, G^2, \ldots, G^k$ in which G has cycle time T, and the rest of the groups have cycle time nT. Then they checked for the feasibility of the proposed grouping. The importance of Stankard and Gupta's work lies in demonstrating the restriction of Bomberger's condition on the BP, i.e., that the sum of the times required to setup and produce a lot of each item is less than the BP. However, the heuristic proposed was criticized by a lot of researchers due to the restriction on the multipliers, which are either specified in group *1* or *2*. Second, the grouping technique used is arbitrary, so a numerous amount of grouping exists.

3.5.3 Doll and Whybark (1973)

Doll and Whybark (1973) made use of an iterative procedure that determines the individual multiplier together with the BP. Thus eliminating the restriction imposed on the multipliers by Stankard and Gupta (1973). Their procedure consists of five steps.

Step 1: Determine the IS cycle T_i^*, and set ω = Min $\{T_i^*\}$

Step 2: For all i, find $\frac{T_i^*}{\omega}$, rounding the quotient up to yield n_i^+ and down to yield n_i^- .

Step 3: Evaluate the corresponding cost associated with n_i^+ and n_i^-, and set n_i equal to the multiplier that yields the minimum cost, using

$$C_i(n) = \frac{A_i}{n\omega} + \frac{n\omega}{2} h_i d_i (1-\rho_i)$$

Step 4: Using the set of multipliers found in step 3, compute

$$\omega = \sqrt{\frac{\sum_{i=1}^{N} \frac{A_i}{n_i}}{\frac{1}{2}\sum_{i=1}^{N} h_i d_i (1-\rho_i) n_i}} .$$

Step 5: With the new set $(\omega{:}n_1, n_2, \ldots, n_N)$ return to step 2. Stop when the set is repeated.

Note, that the equation used in step 4 is the same used by Madigan (1968). The procedure was considered as a distinct improvement over Stankard and

Gupta (1969), since it eliminated the restriction imposed on the multipliers. Yet, this heuristic has two serious drawbacks. First, if one decided, rightly or wrongly, on ω and the set of multipliers to be infeasible, the procedure does not give a guide for escaping from infeasibility. Second, if one succeeded in escaping from infeasibility, the author's procedure may lead to oscillatory behavior where convergence is never achieved.

3.5.4 Goyal (1975)

Goyal (1975) was able to develop a new approach based on the results by Maxwell (1964) and Bomberger (1968). Goyal reported that an optimal schedule is possible without altering the optimal manufacturing frequencies, or cycle times for individual products. This is only achieved under the following conditions: (1) low machine utilization (<25%), (2) optimal frequencies for individual products that do not cause interference and, (3) the total production time is free of overlap (i.e., no part of the production time of a product overlaps with any part of the production time of the other products). Unfortunately, in practice such conditions are rarely met. Furthermore, the complexity of the problem increases if machine utilization is high (>70%), and if the ratio of optimal frequency of one product to another is not an integer or inverse of an integer.

Goyal examined the complicating factors and found out that in the first scenario, the complexity increases by the decrease of available time for conducting desirable changes. While for the second case, it was noted that the chance of interference greatly increases if there are some products manufactured 4 times and 6 times in a cycle , 8 and 9 times in a cycle and so on. To overcome this problem, Goyal imposed a restriction that a product can be manufactured 2^k times in a cycle where $k = 0,1,2,3,\ldots$, or $k = -1, -2, -3,\ldots$. This assumption was also used by Bomberger (1968) and Maxwell (1964). This assumption simplified the problem of obtaining a feasible schedule and the method is very likely to yield the optimal schedule when, (1) machine utilization is high (> 80%), (2) number of products is more than eight, and (3) the ratio between maximum and minimum value of the optimal manufacturing frequency is less than eight.

In addition to the assumptions stated in Chapter Two, Goyal assumed that each product can be produced in every manufacturing cycle, twice, four times, eight times, and so on, or once in every two, four, eight,... , manufacturing cycles depending on the its holding costs. Furthermore, he assumed that setup costs are sequence independent.

Let h denote the holding cost per year per unit of investment in stock. Hence, if we have m equally spaced manufacturing cycles per year, then the total variable cost per year for the i^{th} product will be

$$C_i(m,k_i) = s_i m 2^{k_i} + \tfrac{1}{2} h d_i c_i (1-\rho_i) \frac{1}{m 2^{k_i}} . \qquad (3.29)$$

Goyal verified that the above function is convex in k_i, so a local minimum exists at say $k_i = k_i(m)$ provided that,

$$C_i(m,k_i(m)) \leq \mathrm{Min}\{C_i(m,k_i(m)+1), C_i(m,k_i(m)-1)\}. \qquad (3.30)$$

We can reduce (3.30) to

$$2^{2k_i(m)-1} < \frac{R_i^2}{m^2} < 2^{2k_i(m)+1} , \tag{3.31}$$

where

$$R_i^2 = hc_i d_i \frac{(1-\rho_i)}{2s_i} .$$

Hence, using this result (3.29) can be expressed as,

$$C(m) = m\sum_{i=1}^{N} A_i 2^{k_i(m)} + \frac{h}{2m}\sum_{i=1}^{N} \mu_i d_i (1-\rho_i) 2^{-k_i(m)} . \tag{3.32}$$

Differentiating (3.32) with respect to m, and equating the answer to zero we obtain the minimum value of m, say m_o where,

$$m_o = \sqrt{\frac{\frac{h}{2}\sum_{i=1}^{N} d_i \mu_i (1-\rho_i) 2^{-k(m_o)}}{\sum_{i=1}^{N} A_i 2^{k(m_o)}}} . \tag{3.33}$$

Unfortunately, m_o cannot be determined unless $k_i(m_o)$ is known for each product, which in turn cannot be determined unless m_o is known. To overcome this obstacle, Goyal developed a procedure for determining m_o.

Notes on Goyal's method:

* Unlike the method proposed by Maxwell (1964), Bomberger (1968), and others, Goyal assumed that setup times are sequence independent.

* Goyal did not take into account the initial on hand inventory, and as noted by Elmaghraby (1978), the initial conditions are of great importance in analyzing a system of m periods.

* Schweitzer and Silver (1983) have indicated that no restriction was imposed on T to prevent it from becoming too small. Such a constraint arises in real problems due to the presence of setup times.

The above comments demonstrate the importance of a good and clear definition in constructing and stating the constraints, which translate into a well-posed statements of the mathematical problem involved. Goyal's procedure may be criticized from at least three points. As before, there is no systematic way to escape from infeasibility, and the procedure may "cycle" under a given heuristic. Also, we may end up with different answers if we start with a trial value of ω instead of a set of multipliers.

Effect of using the Powers-of-Two

Maxwell and Singh (1983) tried to examine how well the commonly used power-of-two, introduced by Goyal (1975), perform. They, showed that from an economical view point, this restricted the solution space to be very close to an optimal solution in a certain sense.

In their model, both the batch size for each product, and the time interval between the start of successive batches were limited to a single value.

Consequently, the inventory plot will be in the form of a sawtooth. The cost can be expressed in terms of T_i as,

$$C_i(T_i) = \frac{A_i}{T_i} + \tfrac{1}{2} h_i d_i (1-\rho_i) T_i \ . \tag{3.34}$$

As discussed in Chapter Two, a least cost (lower bound) is obtained by choosing optimal values of T_i^* for each product using,

$$T_i^* = \sqrt{\frac{2A_i}{h_i d_i (1-\rho_i)}} \ . \tag{3.35}$$

We can relate the cost incurred if T_i^* is used to the optimal cost by,

$$\frac{C_i(T_i)}{C_i^*} = \frac{1}{2}\left(\frac{T_i^*}{T_i} + \frac{T_i}{T_i^*}\right) . \tag{3.36}$$

Unfortunately, using T_i^* is not always possible. Some experiments revealed the existence of an empirical pattern of the form $T_i = n_i \omega$. This pattern was extensively used by many authors, as a necessary condition in the scheduling problem.

Goyal (1975), Elmaghraby (1978) and Haessler (1979) among others, have proposed that the minimum used cycle time is some value ω, the maximum used cycle time is some value $2^n \omega$, and the other cycle times used are always $2^k \omega$, for $k=1,2,\ldots$. Maxwell and Singh (1983), proved that the results obtained by using the powers-of-two, correspond to an increase of 6.1% relative to the lower bound.

Maxwell and Singh also proposed an easy way to measure the cycle times for the N products needed to be manufactured, through the use of a predetermined set of the form $\omega, \omega k^2, \omega k^3, \ldots$, justified from a cost point of view. Using this set they eliminated the need for precise values of demand rates, production rates, setup costs, etc. Also, the product classification scheme is easy to understand.

3.5.5 Elmaghraby (1978)

Elmaghraby reviewed most of the articles published prior to 1978. He found out, that the analytical and heuristic approaches proposed for the ELSP clearly underscores the nub of the ELSP, which is the question of feasibility and all its ramifications. Elmaghraby's contributions can be summarized by, first, introducing a test procedure which identifies the solution to be either feasible or infeasible. Second, develop an approach for escaping from infeasibility once infeasibility is encountered.

The test procedure proposed takes the form of integer programming, in which a set of multipliers $n_1, n_2, \ldots, n_N$ is tested to see whether it satisfies the developed feasibility equation or not. The approach proposed for escaping from infeasibility is akin to Doll and Whybark (1973), but avoids the pitfalls of cycling and vagueness in the manipulation of the multipliers, that plague the other heuristic procedures previously discussed.

3.5.6 Park and Yun (1984)

Park and Yun developed a new heuristic approach to test the feasibility of the solution obtained. The heuristic relaxes the sufficient feasibility constraints used in the earlier heuristic, by considering all the basic periods in the total cycle simultaneously.

Park and Yun used a stepwise partial enumeration algorithm. This algorithm determines the repeatable total cycle time and the frequency (number of setups of each product in a total cycle time), instead of determining the basic period and multipliers. In addition to the assumption stated in Chapter Two, Park and Yun added a new restriction on the quantities produced to be equal, and considering only repeatable schedules.

The total cost per unit time when each product is produced in cycles of length T_i is given by summing (3.34) over all N. The lower bound is then found as in section 2.5. Unfortunately, it is not always possible to produce each product according to its optimal frequency due to machine interference. To keep the schedule simple, we define a total cycle T and modify the product cycles to be integer fractions $(1/\eta_i)$ of the total cycle T, and

$$T_i = \frac{T}{\eta_i}. \tag{3.37}$$

The problem is then reduced to finding the values of T and η_i such that the cost is minimized. To do so, let L denote the least common multiple of the set $\{\eta_i\}$, and let the basic period $\omega = T/L$. Next, we can express each cycle in terms of the basic period as,

$$T_i = n_i \omega \ , \tag{3.38}$$

where

$$n_i = \frac{1}{\eta_i} \ . \tag{3.39}$$

The cyclical pattern of production over a repeating total cycle can be determined by finding the first production period of each product. Note that the first period t_i must not be later than n_i, since each product must recur exactly η_i times during the total cycle.

The test for feasibility of a given set of scheduling parameters (T,n) requires that the scheduling scheme attempt to equalize the load of each basic period, consequently, minimizing the peak load. In order to ensure a feasible schedule, Park and Yun relied on the theorem that addressed the necessary and sufficient condition for feasibility by,

$$\omega \geq \sum_{k=1}^{K} L_k^* \ , \tag{3.40}$$

where k denotes the number of partitions made for the product set, L_k^* denotes the minimum peak load of the subset k made for the product set, given that only the products belonging to that subset are scheduled independently of the other product subsets.

Calculating L_k^* is difficult, and so, Park and Yun introduced a search procedure that generates L_k^*. The test they suggested, is much closer to the original problem than any other analytical approach. This stems from the fact that no restrictions were imposed on the multipliers, and that all the basic periods in the total cycle were considered simultaneously, instead of considering only two consecutive periods at a time.

Park and Yun proposed an enumeration procedure that generates a workable schedule for the N products needed to be produced. In every iteration step three types of schedule parameter sets were generated, starting from the local schedule parameter set $[T, n, T(n)]$ obtained in the previous step. The procedure can be summarized as follows:

Step 1 : Determine T_i^* for each product using,

$$T_i^* = \sqrt{\frac{2A_i}{h_i d_i (1-\rho_i)}} .$$

Start schedule parameter set $[T^o, \eta(T^o)]$, with the CC solution.

Set $k = 1$.

Step 2 : Change schedule parameters set to $[2T^{k-1}, 2\eta(T^{k-1})]$ (this is the upper bound schedule parameter set of this solution).

Step 3 : Determine lower bound schedule parameter set $[2T^{k-1}, \eta^*(T^{k-1})]$ using the following:

i. for each product divide $2T^{k-1}$ by T_i^*.

ii. round the quantity down to yield the integer $[\eta_i]$ and up to yield $[\eta_i] + 1$.

iii. determine which multiplier yields the least cost.

Step 4 : Enumerate η_i^k within the two bounds $[\eta_i^*(2T^{k-1}),\ \eta_i(T^{k-1})]$

in order to generate new sets of schedule parameters. For each of the schedule parameters, determine the local minimum schedule parameter set. This is accomplished by the use of

$$C^k = \sqrt{2(\sum_{i=1}^{N} A_i \eta_i^k)(\sum_{i=1}^{N} h_i d_i (1-\rho_i)\frac{1}{\eta_i^k})}.$$

Compute the local optimal total cycle time minimizing

$$C(T,n(T)) = \sum_{i=1}^{N}\left[\frac{A_i}{\eta_i T} + \frac{h_i d_i (1-\rho_i)T}{2\eta_i}\right],$$

then, calculate T^k using

$$T^k = \sqrt{\frac{2\sum_{i=1}^{N} A_i \eta_i^k}{\sum_{i=1}^{N} h_i d_i (1-\rho_i)/\eta_i^k}}.$$

Check for the feasibility of the parameters obtained.

Step 5 : Evaluate C^k. If $C^{k-1} - C^k \leq \varepsilon$; $\varepsilon > 0$, terminate the iteration; otherwise, set $k=k+1$ and go to step 2.

3.5.7 Singh and Foster (1987)

One major disadvantage which most ELSP heuristics suffer from, is that the cycle generating procedure is performed separately and without considering the management planning horizon. Singh and Foster (1987) developed a heuristic with two important features. First, the user is allowed to specify a planning horizon and has the guarantee that the schedule will repeat after the end of that horizon. Second, the heuristic will always achieve a feasible solution, given that the solution exists, regardless of the utilization factor.

Several tests were used to validate the heuristic. It was found that the solution obtained will not vary more than 5% from the best value found by Fujita (1978) and Haessler (1979). Moreover, this procedure has the advantage that when there is "ample" production capacity, the solution will tend to have equal lot sizes, and as the capacity decreases, the procedure will move away from symmetrical solutions in order to reach feasibility.

The problem Singh and Foster solved, was to determine a schedule for the N products which minimizes the sum of the inventory costs and setup cost, given that the schedule is repeated after H periods. Singh and Foster ignored initial inventories and backorders, because the model was mainly used for planning purposes. Furthermore, the solution was regarded feasible, as long as the total setup and production time required does not exceed the available time on the machine.

The setup times and costs were assumed to be sequence dependent. However, this assumption did not affect the feasibility of the solution due to three major reasons. First, the setup times consist of less than 10% of the total machine time, and usually machines are scheduled for a utilization factor not exceeding 90%. Second, the sequence dependent component of a product's setup is small enough to be neglected. Finally, there are significant amounts of time allocated for each machine (about 5%) for testing and experimental purposes. So, any small deviations in setup times can be adjusted within these limits.

The heuristic developed breaks the problem into three different stages as follows:

Stage 1: Computes the number of setups required for each product.

i. Using the common EOQ, compute the minimum inventory and setup costs.

ii. Compute the minimum changeover cost for each product *i*.

iii. Using the traveling salesman problem (TSP), determine the production sequence in which the *N* products are to be scheduled.

The heuristic iterates between the EOQ and the TSP until it converges. Convergence is obtained if EOQ setup costs and TSP average setup costs become approximately the same or the number of iterations reaches a user defined maximum.

Stage 2 : Allocates the setups by forming partial sequences, which is similar to Haessler's (1979) method. These partial sequences are then recorded to decrease change over costs.

Stage 3 : Adjusts the run length of the setup so that production runs only start when the inventory of each product reaches exactly zero. Similar to the Zero Switch Rule (ZSR) developed by Maxwell (1964).

The heuristic is structured so that a feasible solution is always obtained, if there is enough production capacity to setup and run every product at least once in the horizon. This procedure was found to work well in practical situations, and provides a feasible solution if one exists. Yet, the solution is not necessarily optimal.

3.5.8 Larraneta and Onieva (1988)

Based on the results obtained by Goyal (1975), Bomberger (1968) and Maxwell (1964), Larraneta and Onieva (1988) were able to develop a new heuristic rule to solve the ELSP. The assumptions adopted were essentially similar to Goyal's (1975).

The policy used regarding the fundamental period, is the same one used in Bomberger (1968). So the cycle time was restricted to be an integer multiple of some basic cycle, i.e., $T_i = n_i\omega$. Moreover, the heuristic rule developed relies heavily on the concept of economic time TE_i, where

$$TE_i = \sqrt{\frac{2A_i}{h_i n_i}}, \tag{3.41}$$

is calculated for each item.

Unlike the way Goyal expressed the total cost, Larraneta and Onieva expressed it per unit time of a single policy of the fundamental period ω and multipliers n_i, so,

$$C(\omega,n_i) = \frac{1}{\omega}\sum_{i=1}^{N}\frac{A_i}{n_i} + \frac{\omega}{2}\sum_{i=1}^{N} n_i h_i d_i \ . \tag{3.42}$$

The values of n_i used, are identical to the ones obtained by Goyal (1975) and Doll and Whybark (1973). The multipliers n_i were selected so that the interval $n_i\omega$ between two successive orders for item i is close to its economic TE_i. That is

$$n_i(n_i - 1) \le \frac{2A_i}{h_i d_i \omega^2} \le n_i(n_i + 1), \tag{3.43}$$

and hence,

$$\omega = \sqrt{\frac{2\sum_{i=1}^{N}\frac{A_i}{n_i}}{\sum_{i=1}^{N} n_i h_i d_i}} \ . \tag{3.44}$$

Note that, the multipliers n_i are chosen to be the closest integers to the fraction values, that represent the relation between the economic time TE_i of the item and the fundamental period ω. Feasibility conditions are expressed as,

$$\sum_{i=1}^{N} \rho_i \le 1, \tag{3.45}$$

and

$$\sum_{i=1}^{N}\left[\frac{s_i + \rho_i T_i}{T_i}\right] \le 1 \ . \tag{3.46}$$

The second condition is not sufficient since the setup times are not divisible.

Elmaghraby (1978) and Hsu (1983) indicated that there is no analytical expression that provides a necessary and sufficient condition for feasibility. Bomberger (1968), expressed this condition as discussed earlier in section 3.3 as

$$\sum_{i=1}^{N}(s_i + n_i\rho_i\omega_i) \leq \omega \,, \tag{3.47}$$

indicating that the fundamental cycle is long enough to include the production of all the N items. The overall problem is then stated as

$$\text{Min} \quad \frac{1}{\omega}\sum_{i=1}^{N}\frac{A_i}{n_i} + \frac{\omega}{2}\sum_{i=1}^{N} n_i h_i d_i$$

$$\text{s.t.} \quad \sum_{i=1}^{N}(s_i + n_i\rho_i\omega_i) \leq \omega$$

$$n_i \geq 1,$$

n_i is integer.

Relaxing the integrality constraints, and by the use of Kuhn-Tucker conditions, the following observations are evident. First, the feasibility constraints hold, if there exists a fundamental cycle ω, smaller than the shortest of the economic times TE_i, and the items are manufactured according to their natural production lots, hence,

$$n_i = \frac{TE_i}{\omega} \,. \tag{3.48}$$

Second, for $\omega > TE_i$, the solution becomes complex. Third, we can distinguish two cases: (1) $n_i > 1$ indicating that product i is replenished in intervals $n_i\omega < TE_i$, and (2) $n_i = 1$ indicating that the products belonging to this set may depend on the values of ρ_i.

In order to guarantee feasibility, the procedure proposed is based on the idea of identifying the smaller group of items manufactured every fundamental cycle,

$$T^*(m) = \sqrt{\frac{2\sum_{i=1}^{m} A_i}{\sum_{i=1}^{m} h_i d_i}}\,, \tag{3.49}$$

where $n_i = 1$ for $i = 1,\ldots,m$, and $n_i = TE_i/T^*(m)$ for $i = m+1,\ldots,N$.

The first m items, whose multipliers are fixed to be one, are manufactured every cycle and define the fundamental period $T^*(m)$. While the rest are manufactured in their natural production lots found using the IS approach. The sequence is then solved for increasing m until $T^*(m)$ satisfies the feasibility restriction. The aim is to select the smallest subgroup of items to be included in every manufacturing order, so that the resulting fundamental period T is feasible.

Several notes can be stated regarding the results obtained. First, the solution is similar to that obtained by Bomberger (1968). Second, the solution seems sound, since it satisfies some reasonable properties. Third, as TE_i decreases, item i will be manufactured more often. Finally, the fundamental period is approximately equal the economic time of the subgroup of items manufactured every time period.

3.5.9 Geng and Vickson (1988a)

Geng and Vickson (1988a) presented two heuristics which guarantee a feasible schedule given that the problem itself is feasible. Both algorithms use the same intuitive procedures in the search for a feasible schedule. Whenever the procedures fail in finding a feasible schedule, both of the approaches adjust the values of ω and n_i according to the same rules. However, the main difference between the two algorithms lies in the use of two different initial starting values of ω and n_i, which consequently implies achieving different sets of answers.

The first algorithm is the ELSPHU which was developed by Geng, and the second algorithm is the ELSPDW, which was developed by Vickson. Note that the, ELSPDW is a modified version of Doll and Whybark (1973) algorithm.

Both the ELSPHU and the ELSPDW make use of the upper (UB) and lower (LB) bounds discussed in Chapter Two, in forming a new performance index ξ,

$$\xi = \frac{HU - LB}{UB - LB}, \tag{3.50}$$

where HU is the average cost obtained using the heuristic procedure. This measure yields more insight into the performance of the heuristic, than does the more common measure,

$$e = \frac{HU - LB}{LB}. \tag{3.51}$$

The reason behind this is that ξ is invariant under rescaling of the input data, while e is not. Also, ξ pinpoints and measures the performance compared to the two bounds UB and LB, while e measures it with respect to only the lower bound (LB).

The ELSPHU Algorithm

The algorithm is composed of seven steps (with some steps repeated). At each step the algorithm alters the values of ω and n_i to the smallest extent necessary, while moving toward feasibility. The multiplier n_i is restricted to powers of 2. This will: (1) guarantee that the new cost will not deviate from the LB by more than 6% (see Goyal (1975)), and (2) increase the likelihood of obtaining a feasible schedule when all multipliers are even.

Step 1 : Determination of the initial multipliers.

i. Find $\text{Min}\{T_i^*\}$ by solving the IS problem

ii. Adjust n_i^* to be integer-valued powers of two,

1. for each i find j such that $2^j \leq n_i^* \leq 2^{j+1}$

2. If $2^{j+1} - n_i^* > n_i^* - 2^j$, let n_i=2^j , else n_i=2^{j+1}

Step 2 : Determine the number of BP's in a complete cycle.

$G = \underset{i}{\text{Max}} \ \{n_i\}$ is the number of BP's in a complete cycle. If G=*1* go to step 6.

Step 3 : Find the best BP by solving,

$$\omega = \sqrt{\frac{X}{Y}},$$

where,

$$X = \sum_{i=1}^{N} \frac{A_i}{k_i} ,$$

and

$$Y = \sum_{i=1}^{N} \rho_i n_i .$$

Step 4: Form a production sequence. The main idea in the procedure is to assign a product to a period such that the time currently assigned to this period is minimal.

Step 5: Check the feasibility of ω and $\{n_i\}$.

Step 6 : Compute the optimal CC solution, compute T using,

$$T^* = \mathrm{Max}\left\{T_{cc}, \sum_{i=1}^{N} \frac{s_i}{1-\rho}\right\}.$$

Step 7 : Compute the average cost of BP solution using,

$$C = \frac{X}{\omega} + Y\omega.$$

ELSPDW Algorithm :

The heuristic outlined by Doll and Whybark (1973) is an iterative procedure. It starts with the same ω and n_i as in the ELSPHU, and then iterates back and forth between new values of ω and $\{n_i\}$, until no further improvement can be found. However, due to the iterative nature of the procedure, different starting values of ω and $\{n_i\}$ can lead to different final values as indicated by Elmaghraby (1978). The contribution of Geng and Vickson lied in adding the calculation of the total cost before and after each iteration, in an attempt to prevent possible cycling.

3.5.10 Dobson (1987)

In an attempt to convert every production sequence obtained into a feasible schedule and to obtain a more uniform utilization of the factory, Dobson (1987) introduced a new formulation for the ELSP. The model Dobson developed, provides feasible schedules by allowing the lot sizes as well as the cycle times of each product to vary over time, taking into account setup times.

Dobson's approach is quite different than most of the ones previously described. First, the idea of a basic period was not used. Second, he allowed lot sizes to vary over the cycle. Third, Dobson succeeded in handling the difficulties caused by accounting for setup times. The approach is similar to that of Maxwell (1964) and to Delporte and Thomas (1977), in the sense of handling setup costs. The solution obtained is sensible in the short run.

The model Dobson developed can be viewed as deciding on a cycle T, and a production sequence $f_1, f_2, \ldots f_m$, (m denotes the number of production runs of all N products in a cycle). The production sequence is established in the chosen cycle length, the cycle can be repeated indefinitely, demand is met, and inventory and setup costs per unit time are minimized.

Dobson used the superscript to refer to the data related to the part produced at the j^{th} position in the sequence, e.g., p^j, d^j, h^j, A^j, s^j. Let Φ be the set of all possible finite sequence of parts, u^j is the idle time for position j and τ^j is the production time of the j^{th} position. Then the number of parts produced is $p\tau^j$, all of which are demanded in the time interval $[o,v]$. Hence, $v = p^j\tau^j/d^j$. Therefore, the highest inventory level is $(p^j - d^j)\tau^j$, and the total inventory cost for the production of part f_j is

$$C^j = \tfrac{1}{2}h^j(p^j - d^j)\rho^j(\tau^j)^2 . \tag{3.52}$$

Let J_i be the positions in a given sequence where part i is produced, and L_k be the position in a given sequence from k (where f_k is produced), up to but not including the position in the sequence where part f_k is produced again. Then the overall model can be written as,

$$\inf_{j\in\Phi} \min_{\substack{\tau\geq 0\\ u\geq o\\ T\geq o}} \{\frac{1}{T}(\sum_{j=1}^{m} \tfrac{1}{2}h^j(p^j - d^j)\rho^j(\tau^j)^2 + \sum_{j=1}^{m} A^j), \tag{3.53}$$

$$\text{s.t.} \quad \sum_{j\in J_i} p_i\tau^j = d_iT \qquad i = 1,\ldots,N, \tag{3.54}$$

$$\sum_{j\in J_i}(\tau^j + s^j + u^j) = r^k\tau^k \quad k = 1,\ldots,m, \tag{3.55}$$

$$\sum_{j=1}^{m}(\tau^j + s^j + u^j) = T . \tag{3.56}$$

The interpretation of the constraints are as follows. Constraint (3.54) means that enough time for each product i is allocated to meet its demand d_iT over the cycle. Constraint (3.55) means that enough stock of product i must be produced each time in order to last until the next period of production. Moreover, the assumption that inventory is zero at the beginning is still enforced. It is clear that the above formulation contains two stages. In the first stage the sequence f is determined, while in the second stage, we determine the production time and idle time.

Dobson stated the feasibility constraint used was stated in the form of a theorem.

Theorem: Let f be a given sequence, fix the idle time $u \geq 0$. Then

there exists a feasible set of production times τ if and only if

$$\sum_{i=1}^{N} \rho_i < 1 .$$

The heuristic proposed by Dobson consisted of two major steps. First, adjusting the frequencies of production of the products in order to be spaced symmetrically. Second, use the new frequencies obtained to produce an actual sequence.

Dobson's contribution lies in the ability of solving problems with setup costs equal to zero. The usual heuristics are either not applicable under this condition, or they have extreme difficulty in finding feasible solutions. However, Dobson's heuristic failed in solving the eternality problem that large batches impose on others. Eternality is defined as imposing a constraint that other parts must be produced in large quantities in order to build a sufficient inventory during the production of the troublesome part.

3.6 Extensions

Several extensions for the ELSP problem were cited. These extensions were used either to generalize the common ELSP, such as assuming the setup times are sequence dependent, or to include a new variation in the problem, such as warehouse capacity. In this section, some of the major extensions will be discussed.

3.6.1 The Finite Planning Horizon

Elmaghraby (1968), presented two models whose formulation springs from the cardinal assumption that the length of the planning horizon is finite. He also addressed the importance of the initial state (initial inventory level), which was neglected by most of the researchers prior to 1968.

The importance of the inventory level is prevailed by considering two cases. Case one, if no inventory is available for any item at the beginning of the planning horizon, then no matter what is done some items will be short for a period of time. So, the condition of satisfying all future demand is violated at the start of the problem. Case two, if an appropriate level of inventory for each item is guaranteed at the start of the planning horizon, then the cost of such a state should be taken into account, else the cost of any solution does not reflect the real optimal cost. Some investigators neglected the initial condition, or they assumed that its effect will disappear. However, this is incorrect, since unlike stochastic systems, deterministic systems are heavily influenced by the initial conditions, whose effect never dies out.

According to Elmaghraby, limiting the horizon to a finite length is more practical and may actually be better if it leads to answering our objective. That is, to determine an optimal workable sequence for scheduling the N products on one machine. Elmaghraby (1968), presented two different types of approaches, which modeled the ELSP under the assumption of finite planning horizon. These approaches are, the Linear Programming modeling technique and the Dynamic Programming technique.

Linear Programming

The linear programming model was used to satisfy two objectives. First, it is always helpful to construct a formal model of the problem on hand. Second, integer LP's where at that time under intensive research, consequently, increasing the possibility of obtaining a solution. The model can be viewed as, minimizing the sum of the setup and holding costs, subject to non-negativity and non-interference constraints.

Needless to say that at that time, the LP formulation is by no means modest in size. For example, given 5 different products, a planning horizon of 20 time periods, and that no more than 7 lots of each product is produced, the resulting LP model has around 5630 constraints. However, with the advances made in linear programming since then, it is now much easier to solve such problems.

A Dynamic Programming Model

Elmaghraby modeled the ELSP using the DP approach. He assumed that each product will be produced at most once. Furthermore, he introduced a fictitious product ($N+1$) which denotes the *do nothing* for one time unit. So, if $N+1$ is not produced at all, the time horizon of length H ($H = \sum_{i=1}^{N} \delta_i$) permits the production of one lot of each product. This can be thought of as introducing a delay in the cycle, which decreases the levels of inventory needed. This concept was also used by Maxwell (1964). Using this concept, the problem is similar to that, of sequencing N jobs on a single facility, treated by Elmaghraby (1968b).

Consider product i with initial inventory I_{i0}, and suppose that its first appearance in the sequence occurred at position k. Then the elapsed time E, can be expressed as

$$E = \sum_{j=1}^{k-1} \delta_{(j)} . \tag{3.57}$$

The continuous demand causes the inventory at time $t = E_{k-1} + s_i$, given a demand rate of d_i, to be I_{i0} - d_{it}, (which is unrestricted in sign). The costs up to time t can be thought of as storage cost, a penalty cost for backordering if any, in addition to setup cost. We denote these costs as γ_{i1}. Considering the time span, between t and $t + \tau_i$, we encounter two types of costs, the cost of production, and the cost of holding the products in inventory. We will denote these costs by γ_{i2}. By definition, a sequence has i produced only once, the interval between $t + \tau_i$ and H witness the depletion of inventory at the rate of d_i, with unrestricted inventory levels. Two types of costs are encountered, the inventory carrying cost and backordering cost. These costs are denoted by γ_{i3} . Let j denote the product in position j, and N' is the number of sequencing position, where $N' \geq N$ due to the insertion of $N+1$ in several locations of the planning horizon. Then the overall model can be viewed as,

$$\text{Min } \sum_{j=1}^{N'} C_j \text{ ,} \tag{3.58}$$

$$\text{s.t.} \qquad \sum_{j=1}^{N'} g_{[j]} \leq H, \tag{3.59}$$

where

$$C_{[j]} = \sum_{j=1}^{3} \gamma_{ji} \text{ ,}$$

and

$$g_{[j]} = \begin{vmatrix} \delta_i \text{ if product i is in position } j \\ 0 \text{ otherwise} \end{vmatrix}$$

The above formulation is almost a knapsack problem, except that the cost C_j is dependent on all other products in sequence [1] through [j-1]. Unfortunately, due to the computational complexity inherent in the two models presented, an optimal solution could not be reached.

3.6.2 Effect of Reducing Production Rates

Management may be interested in reducing the production rates in order to avoid the rapid accumulation of inventories, i.e., to reduce the holding cost or the warehouse capacity needed at peak production times. Moreover, if we allow production rates to be reduced from the current or nominal production rates, a better solution can be obtained.

Several researchers have studied this procedure. Their work can be divided into two different categories. Category I (flexible case), allows the production rate to be changed within the production run. Category II (rigid case), fixes the production rate during the run, but the production rate is adjusted at the start of the run and remains constant throughout the run.

The flexible case holds when setups are only incurred when switching from one item to another, but not after idling and resuming the production of the same item. On the other hand, the rigid case arises in situations where the materials handling system feeds the raw materials at rates that can be adjusted only at the beginning of the run, but not during the production run itself.

Sheldon (1987)

Sheldon studied the problem of inserting arbitrary idle times to reduce the effective production rate in facilities where setups are time consuming. He considered the flexible case and suggested an idle time insertion strategy, that divides the production runs into two parts. The first part consists of inserting idle time uniformly to produce at the demand rate, then resuming at the nominal rate with no further idle time insertion.

Silver (1990)

Silver studied the rigid case by imposing a Common Cycle interval for all items. He showed that at most one item should have its production rate reduced. Silver obtained a closed form solution for the optimal cycle length in addition to the optimal production rates

Inman and Jones (1989)

Inman and Jones were motivated by an actual industrial problem where the flexible case applies. Drawing upon Silver's (1990) results for the rigid case, Inman and Jones determined new production rates by solving a non-linear program. The resulting schedule was not based on the CC approach, but instead items have their production rates determined by inserting idle times uniformly throughout the production run.

Moon, Gallego and Simchilevi (1991)

Moon et al. (1991) studied the flexible case. The CC approach was used to ensure a flexible solution. The problem discussed is a generalized form of Silver's (1990) problem. Their approach can be viewed as follows:

1. Redefine the units h_i, d_i, p_i such that demand rates are 1, (this procedure will ease manipulation).

$$d_i' = \frac{d_i}{d_i}; \quad p_i' = \frac{p_i}{d_i}; \quad h_i' = h_i d_i; \quad \rho_i = \frac{1}{p_i} = \frac{d_i}{p_i}.$$

2. Define, $k = 1 - e^T \rho$, where e denotes a vector of ones.

Then, using the results from Dobson (1987), the authors were able to develop three different propositions which facilitated the search for the solution. But before discussing these propositions, it is of great importance to examine the assumptions used.

First, a cycle is made through the entire family every T years. Second, production rates can be adjusted at the beginning of the production run and during the production run. Third,

$$h_1 \le \sum_{i=1}^{N} \frac{h_i(1-\rho_i)}{2k}, \tag{3.60}$$

the last assumption was used in the proof of the existence of a global minimum.

Proposition 1 : *The optimal schedule that allows the production rates to be changed during the production runs (schedule 1) has costs at most as high as the optimal schedule that allows the reduction only at the beginning of the production run (schedule 2).*

Referring to Figure 3.2 and by the use of proposition 1, we can formulate the problem as follows:

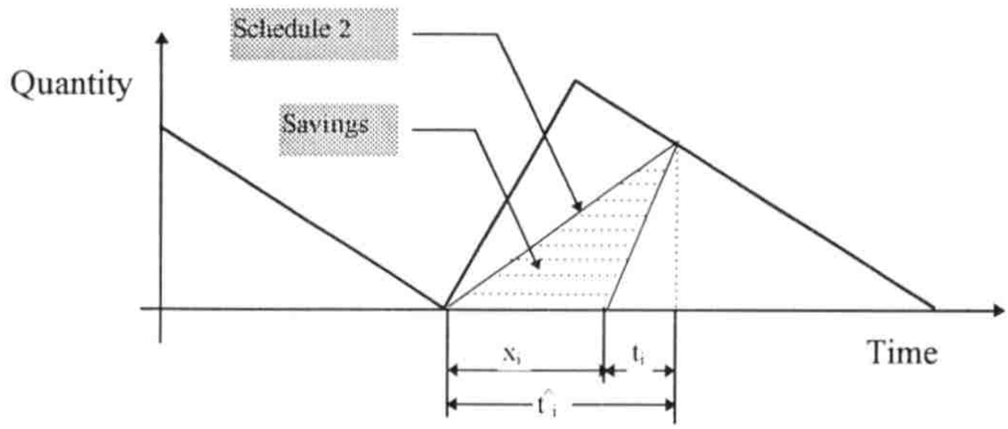

Figure 3.2: Time Varying Production

$$\underset{x,T}{Min} \quad \frac{1}{T}\sum_{i=1}^{N}\left(A_i + \tfrac{1}{2}h_i\overline{\rho}_i(T-x_i)\right)^2, \tag{3.61}$$

s.t.

$$\sum_{i=1}^{N}\frac{s_i}{T} + \sum_{i=1}^{N}\frac{\overline{\rho}_i x_i}{T} \le k. \tag{3.62}$$

Proposition 2 *: The optimal solution of (3.61) satisfies (3.62) as an equality.*

Proposition 3 *: The objective function of (3.61) is strictly convex in (x,T).*

Even though proposition 3 applies, a finite solution may not be attained. Furthermore, Moon et al. proved the existence of a unique global minimum for the objective function. An algorithm was established for solving the above model, in addition to an extension that allows backorders to be included in the formulation.

Moon (1991)

Moon, used the CC approach in analyzing the flexible case. He proved that the insertion strategy suggested by Sheldon (1987) is optimal, and reported savings almost twice as large relative to the strategy that inserts idle times uniformly throughout the production runs.

Gallego (1993)

Gallego studied both the flexible and rigid cases. He provided an efficient test that identifies the items, typically the slowest, with potential for production rate reduction, and decides whether a reduction of the production rate is profitable. Furthermore, Gallego also offered a solution for the flexible case, where the production runs consist of two periods; a period of non-negative length in which production takes place at the demand rate, and a period of positive length, in which the production takes place at maximum rate. These results generalize those of Silver (1990).

3.6.3 Allowing Backorders

Most of the researchers who examined the ELSP problem usually assume that no backorders are allowed. As far as we know, the only paper available prior to 1990 that analyzes backorder situations is that of Krone (1964). The motivation behind studying such a problem were due to two limitations the ELSP models suffer from. The economical restrictions, and the lack of scheduling approaches applicable in real time, i.e., when demands are random or the facility is subject to sudden failure.

Two major contributions were recorded in the literature that analyze the backorder case. One of Gupta (1992) and the other of Gallego and Roundy (1992).

Gupta (1992)

Gupta examined the ELSP by allowing backorders and suggested a closed form solution for it, using the CC approach. The CC approach was used since it always provides a feasible workable schedule. In some cases discussed by Inman and Jones (1989), the total cost per unit time using the CC lot sizes is not significantly higher than that of the IS approach (lower bound).

Gupta presented a complete solution to the ELSP given that two types of shortage costs are incurred. Namely, occurrence dependent and occurrence and duration dependent shortage costs. That is, Gupta allowed for backorders at a cost of π_i per unit short, and π^*_i per unit short per unit time.

<u>**Solution limits**</u>

Lower Bound:

Using the IS Gupta obtains the following results:

Case I : If $2A_i h_i > \pi_i^2 d_i(1-\rho_i)$ then,

$$T_i^{\circ} = \sqrt{\frac{h_i + \overline{\pi}_i}{\overline{\pi}}\left[\frac{2A}{h_i d_i (1-\rho_i)} - \frac{\pi_i^2}{h_i (h_i + \overline{\pi}_i)}\right]}, \tag{3.63}$$

and

$$b_i^{\circ} = (1-\rho_i)\frac{-\pi_i d_i + \sqrt{\frac{2A_i d_i h_i (h_i + \overline{\pi}_i)}{\overline{\pi}_i (1-\rho_i)} - \frac{d_i h_i \pi_i^2}{\overline{\pi}_i}}}{(h_i + \overline{\pi}_i)}, \tag{3.64}$$

where b_i° is the backordered quantity.

Case II : If $2A_i h_i \leq \pi_i^2 d_i (1-\rho_i)$ then,

$$T_i^{\circ} = \sqrt{\frac{2A_i}{h_i d_i (1-\rho_i)}}, \tag{3.65}$$

and $b_i^{\circ} = 0$. Hence, the lower bound cost C_L equals

$$C_L = \sum_{i=1}^{N} C_i (T_i^{\circ}, b_i^{\circ}), \tag{3.66}$$

where,

$$C_i (T_i^{\circ}, b_i^{\circ}) = \frac{A_i}{T_i} + \frac{h_i \left[T_i d_i (1-\rho_i) - b_i^{\circ}\right]^2 + \overline{\pi}_i (b_i^{\circ})^2}{2T_i^{\circ} d_i (1-\rho_i)} + \frac{\pi_i b_i^{\circ}}{T_i^{\circ}}, \tag{3.67}$$

and $q_i^{\circ} = T_i^{\circ} d_i$.

Upper Bound :

The CC solution can be considered to be the Upper bound solution. The problem can be written as

$$\text{Min } C(T,b) = \sum_{i=1}^{N}\left[\frac{A_i}{T} + \frac{h_i (T d_i (1-\rho_i) - b_i^2)^2 + \overline{\pi}_i b_i^2}{2T d_i (1-\rho_i)} + \frac{\pi_i b_i}{T}\right], \tag{3.68}$$

$$\text{s.t.} \qquad \frac{T - \sum_{i=1}^{N} t_i}{1 - \sum_{i=1}^{N} \rho_i} \geq 0 \tag{3.69}$$

$$b_i \geq 0 \qquad \forall i = 1,2,\ldots,N. \tag{3.70}$$

The constraint (3.69) ensures feasibility under the CC approach else, if we want to deviate from the CC approach, we have to add *N-1* constraints similar to (3.69). Note that these constraints are necessary but not sufficient for feasibility.

Moreover, the function $C(T,b)$ is not necessarily convex. However, Gupta proved that $C(T,b)$ is pseudoconvex. He also showed that, for $T > T_{min}$ then $b_i > 0$ and $T > \frac{\pi_i}{h_i}$, where

$$b_i = \frac{d_i(1-\rho_i)(Th_i - \pi_i)}{h_i + \overline{\pi}_i}, \tag{3.71}$$

$$T = \sqrt{\frac{\sum_{i=1}^{N} 2A_i + 2\pi_i b_i + \frac{b_i^2(\overline{\pi}_i + h_i)}{d_i(1-\rho_i)}}{\sum_{i=1}^{N} h_i d_i (1-\rho_i)}}, \tag{3.72}$$

and

$$T_{min} = \frac{T - \sum_{i=1}^{N} t_i}{1 - \sum_{i=1}^{N} \rho_i}. \tag{3.73}$$

Else, if $T \leq \frac{\pi_i}{h_i}$ then $b_i = 0$.

In order to obtain a fast solution and to ease the procedure manipulation, Gupta (1992) developed an iterative procedure. However, this approach is beyond the scope of our project, and for more details, refer to Gupta (1992).

Gallego and Roundy (1992)

Gallego and Roundy (1992) analyzed the situation were customers do not need immediate delivery, and considerable savings can be achieved by backordering. In many manufacturing systems where the items are used to feed production lines, the cost of a temporary stockout is considered to be finite. Lost production at the line is often recovered either by working faster or longer.

In order to measure the amount of savings including backorder costs, Gallego and Roundy (1992) assumed that backorder cost rate is K times larger than the holding cost rate. Given this, it is possible to find feasible schedules with an average cost not larger than $\sqrt{K(K+1)} * 100\%$, compared to an average cost of a schedule with infinite backorder cost rate.

The ELSP literature so far assumes that cyclic schedules can be repeated. This is only possible under the conditions that, (1) the initial inventories are in agreement with the schedule proposed, (2) the facilities are perfectly reliable, (3) setup consumes a constant amount of time, and (4) the demand rates are constant. Any significant amount of randomness in the system will eventually force deviation from the cyclic schedule and make stockouts inevitable. Gallego and Roundy's work can be categorized as follows:

1. Computing an optimal or near optimal cyclic schedule for the case of backorders.

2. Solving an optimal control problem to adjust the length of the production runs of the target schedule in response to disruption in expected inventories.

3. Determining appropriate levels of safety stocks, to achieve the serviceability level.

The results of Gallego and Roundy represent a key for the development of a real-time scheduling tool, capable of managing a multiproduct one machine system under more realistic assumptions.

3.6.4 Multiprocessor Case

Carreno (1990), was the first investigator who analyzed the multiprocessor case. He referred to this problem as the multiprocessor ELSP, since he regarded the existence of several identical parallel processors. The problem goes beyond the scope of finding the lot sizes of each product, to determining a feasible schedule for the production of these lots on each production line. Both functions have to be integrated, so that when one determines the lot sizes, scheduling considerations are also taken into account. Carreno's approach, decomposes the problem into two different sets of problems: an allocation problem, and a lot sizing problem. Carreno's approach, solves the two problems in an alternating fashion, where improvement is performed on one part at a time while the other is held constant.

In 1993, another attempt to solve the multiprocessor scenario was recorded by Najdawi and Kleindorfer (1993). In their research, they provided an optimizing framework of the Common Cycle scheduling problem for the multistage, multiproduct, flow-shop environment under deterministic and stationary conditions. Najdawi and Kleindorfer assumed that a fixed sequence is maintained across all processing stages. The framework considers the costs of work-in-process inventory and determines a jointly optimal common cycle time and production schedule for this type of problems.

It is important to note that in the multiprocessor case, the problem has the advantage of allocating the products to different processors (Grouping Technology). So, one can make use of grouping the products according to a pre defined criterion such as, equal processing time, equal setup time, similar processing steps and others. This grouping was found to offer better results than random allocation to the machines.

3.6.5 Warehouse Restriction

Most of the ELSP problems examined use the cost as an objective function. Geng and Vickson (1988b) altered this, by using the storage space required as an objective function and optimized on the space available. In other words, to minimize the maximum storage required by the machine's output. This problem is referred to as the "Warehouse Restricted Lot Scheduling Problem," or WRLSP.

The WRLSP is similar to the ELSP in the sense that we need to determine a sequence for producing the N products on the same facility. Geng and Vickson used in their analysis the concept of the common cycle in order to obtain a feasible solution more easily. The problem can be stated as follows:

Determine the sequence in which the N products are to be produced in addition to the lot sizes used (using the CC approach), given constant production and demand rates, sequence independent setup times, in addition to the warehouse restriction, namely, all products share the same storage space.

The WRLSP is strongly NP-hard, hence, an exact solution cannot be found analytically. Geng and Vickson (1988b) approached this problem by making use of the potentially explosive combinatorial approach, such as branch and bound procedures. We should point out that the solution procedure was applied to a specific problem in an automobile factory in China.

3.6.6 Setup Reduction and Quality Improvement

Classical ELSP models assume that setup costs, setup times and quality levels are fixed and uncontrollable at the optimal level. However, this is not always true. The success of the JIT manufacturing systems, is partly based on the beliefs that setups and quality are controllable factors, and can be improved through various efforts. Shorter setups result in smaller lot sizes and consequently low inventory holding costs. Which implies a higher production capacity, and a high quality. High quality reduces rework, repair and loss of goodwill.

To improve quality and reduce setups, sustainable efforts have to be made. Moreover, capital investments are needed for the setup reduction and quality improvements. When a single facility produces multiple products, setup times become more important. Setup times determine the production capacity of the system. When setup times are short, more time can be devoted to actually producing products.

Hwang, Kim and Kim (1993) used the multiproduct capacity EOQ Model of Spence and Proteus (1987). They included additional features to the problem by introducing the concept of setup reduction for the case in which, setup costs are proportional to the setup time and overtime is allowed. Furthermore, they extended their model by allowing defective items to be produced in addition to the use of power form investment functions.

The relevant problem is to determine the optimal lot sizes, setup cost, setup time and quality level simultaneously with an objective of minimizing the total cost. That is, the sum of setup costs, inventory holding costs, production costs, costs related to defective items and finally the investment cost for setup reduction and quality improvement. Finally, a closed form solution for the WRLSP was obtained, using the common cycle approach in parallel with a geometric programming formulation.

3.7 Testing Several Heuristic Approaches

A large number of heuristic procedures were proposed and recorded in the literature. Comparing the performance of these procedures is difficult due to several reasons. First, there is not enough empirical evidence in the form of detailed case-studies to allow comparison of like-with-like. Not denying the fact that an optimal solution is not available as well, and the only available data is the lower bound obtained by the Independent Solution approach (IS). Second, it is too much to hope that a given plant could be subjected to alternative control systems in sequence to provide a consistent framework for measurement purposes. If such a plant is available, the introduction of a new package is accomplished by a reorganization of the plant and a complete change in managerial attitudes and style. Hence, it is impossible to determine what proportion of the dramatic improvement is due to the algorithmic process and success of the package itself, and how much is due to the corresponding managerial attitudes and reorientation.

Consequently, the only available tool for testing, is to use a set of a predetermined problems as a base for comparison between the various algorithms available. Perhaps the most quoted test problem in the literature for the ELSP is that of Bomberger dating back to 1966. Although, earlier test problems existed, such as the test of Eilon (1957) and Rogers (1958), the ten-product one-machine problem of Bomberger remained as the most used test in the field of ELSP.

Eilon (1985), addressed a serious pitfall in the Bomberger problem. Namely, that the total production time (excluding setup) required to produce the specified product range amounts to only 22% of the demand time. Consequently this creates a lot of space for maneuvering. Moreover, some of the product, with low-setup costs, may be scheduled frequently in small batches, with little regard to the capacity availability. In fact, with such a low level of capacity utilization, additional products (other than the ten products Bomberger used) can be manufactured.

Despite all the disadvantages and faults in Bomberger's test problem, his problem remains as a bench mark in the test of any heuristic procedure proposed. Finally, most of the heuristic models cited lack a systematic way to test the feasibility of a given set of parameters, and do not provide a procedure for escaping from infeasibility once it is encountered. These two criticisms were outlined in detail by Elmaghraby (1978).

Yazitzoglou (1987), set out to compare the results of nine heuristic procedures, being those of Bomberger (1966), Doll and whybark (1973), Eilon (1962, and 1985), Goyal (1973), Haessler (1971), Hanssman (1962), Hodgson (1970), Madigan (1968) and Stankard and Gupta (1969). Details of these heuristics were discussed in the previous sections.

Virtually, all nine heuristics (with the exception of Bomberger's method) start either with computing the optimal batch sizes for the individual products, or with finding an overall production cycle time and then proceed to finding the batch sizes. Moreover, most of these heuristics take advantage explicitly or implicitly of the fact that the cost function for each product is quite flat around the optimum solution. Consequently, deviations from the individual optimum solutions can be tolerated with only moderate cost penalties.

We are going to discuss only the results based on Bomberger's problem due to its importance. Bomberger's problem consists of ten products with very low inventory holding costs, and a very low utilization factor of only 22%. The costs and other parameters for the ten products, are tabulated in Table 3.1. Note that for three out of the ten products the rate of production exceeds the demand rate by a factor of at least 300, and half the products required very low setup costs (between $5 and $50), and rather low set-up times (from 1 to 8 time units). The high level of over capacity provides a great deal of flexibility for scheduling purposes,

Table 3.1 : Bomberger's Problem

Part Number	Setup Cost	Production Cost	Production Rate	Demand Rate	Setup Time
1	15	0.0065	30000	100	1
2	20	0.1175	8000	100	1
3	30	0.1275	9500	200	2
4	10	0.1000	7500	400	1
5	110	2.7850	2000	20	4
6	50	0.2675	6000	20	2
7	310	1.5000	2400	6	8
8	130	5.9000	1300	85	4
9	200	0.9000	2000	85	6
10	5	0.0400	15000	100	1

so that products can be produced frequently in small batches within a given production cycle, and this means that many low-cost solutions can be easily determined.

In order to increase the number of test problems, Yazitzoglou proceeded by scaling the demand rates. Thus as stated earlier, the original problem involves a utilization factor of 22%. So, doubling the demand rates for the products while leaving the other data intact, the utilization factor increases to 44%. Using this procedure, Yazitzoglou was able to produce five sets of the problem. The results for these five test problems are tabulated in Table 3.2. Furthermore, these results are summarized in Table 3.3.

Table 3.2 : Solutions for Bomberger's Problem

Author(s)	ρ=22%		ρ=44%		ρ=66%		ρ=88%	
	Cost	Deviation	Cost	Deviation	Cost	Deviation	Cost	Deviation
Lower bound	16.87		23.33		27.91		31.42	
Bomberger	17.01	0.80	24.42	4.70	29.90	7.10	36.60	16.50
Doll & Whybark	17.18	1.80	23.71	1.60	28.32	1.50	31.85	1.40
Eilon	17.17	1.80	23.82	2.10	28.32	1.30	31.89	1.50
Goyal	17.16	1.70	23.86	2.30	28.33	1.50	31.96	1.70
Haessler	17.19	1.90	23.74	1.80	36.68	1.50	33.17	5.60
Hanssman	22.50	33.40	30.90	32.40	28.25	31.40	40.96	30.40
Hodgson	17.18	1.80	23.71	1.60	28.56	1.20	33.86	7.80
Madigan	17.29	2.50	23.90	2.40	30.48	2.30	33.73	7.40
Stankard & Gupta	17.47	3.60	25.51	9.30	27.91	9.20	36.29	15.50

Examining Table 3.3, we can see that four of the nine heuristic methods do not seem to perform very well. All the other five, namely those by Doll and

Whybark, Eilon, Goyal, Haesslar and Hodgson, came within 2% of the lower bound.

Table 3.3 : Frequency of achieving good results

Author(s)	<1%	1–2%	<2%
Bomberger	1	1	1
Doll & Whybark	0	4	4
Eilon	0	3	4
Goyal	0	3	4
Haessler	0	3	3
Hanssman	0	0	0
Hodgson	0	3	3
Madigan	0	0	0
Stankard & Gupta	0	0	0

According to Eilon (1978), comparing the results of heuristics for the Bomberger problem led him "*to the realization that we have come to the end of the road, as far as multi-product batch production is concerned, and any further theoretical refinements of what we already have can only result in very marginal improvement.*" This conclusion is confirmed by the results obtained in Table 3.3.

CHAPTER FOUR

THE TWO PRODUCT VERSION OF THE ELSP

4.1 Introduction

In this chapter we will examine the various approaches proposed in the literature, that dealt with the two-product version of the ELSP. One of the first attempts to solve the two-product scenario can be found by Deloporte and Thomas (1976). Deloporte and Thomas suggested a mixture of heuristics and mathematical programming methods. The formulation permits the use of unequal lot sizes for the more frequently manufactured products, consequently increasing the possibility of getting a lower total cost.

Vemuganti (1978) derived a necessary and sufficient condition for feasibility which facilitated the search for an optimal answer. Saipe (1977) presented the optimal solution for the two-product case under the assumption that; the number of setups of a given product is restricted to be an integer multiple of the other. Another approach which also provides an optimal solution is that of Boctor (1982). Boctor assumed that the number of setups over a certain time period for both products are positive integers.

All the above approaches inherently assume that the two products are produced cyclically. Under certain conditions, this assumption may cause significant errors in the solutions obtained. Recently, two more approaches were cited in the literature, one of Goyal (1984) and the other of Lee and Danusaputro (1989). In both approaches an assumption on the number of setups was made. Namely, the number of setups is an integer multiple of the other. Yet both allowed for unequal cycle times for the most frequently manufactured product.

Before we discuss in detail the various proposed approaches to the two-product ELSP problem, we first state the most common assumptions used. These are: (a) only one product can be manufactured at a time; (b) demand rates and production rates are deterministic; (c) stockouts are not allowed; (d) we have an infinite time horizon; and (e) the criterion of optimality is cost minimization. Unless otherwise stated, these assumptions apply for the discussion below.

4.2 Discussion of Approaches Proposed

In this section we will discuss the approaches of Saipe, Boctor, Goyal and Lee and Danusaputro. The discussion will include, the approach proposed, validity of answers obtained, and the working environment under which the approach provides the best solutions.

4.2.1 Saip (1977)

In 1977, Saipe presented an optimal solution for the two product scenario. His model served as a basis for the development of a general heuristic which solves the multiproduct case. In his analysis, Saipe adopted the assumption that setup times are negligible, and that the number of production runs of product one is an integer multiple of the number of runs of product two.

To discuss Saip's model, we first introduce the notation to be used. Let, X denote the number of production days, D_i denote the annual demand for product i, s_i denote the fixed cost of a run of product i, and n_i denote the number of equal-sized runs of product i in one year (not necessarily integer). There are two necessary conditions that should be met in order to guarantee a feasible schedule. These are:

1. Let products be numbered such that $H_1 \leq H_2$, where $H_i = \frac{h_i}{s_i} d_i (1 - \rho_i)$. Assuming there are n_i runs of product i in one year, then $n_1 \leq n_2$. So, in order to obtain a feasible solution it is necessary that

$$n_2 = kn_1 \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad . \tag{4.1}$$

If this condition is violated then we have machine interference.

2. The depletion time of product two's inventory should be at least as long as the production time in a production depletion cycle of product one. That is,

$$\frac{D_2(1-\rho_2)}{n_2 d_2} \geq \frac{D_1}{n_1 p_1}, \tag{4.2}$$

which is equivalent to

$$\frac{X}{n_2}(1-\rho_2) \geq \frac{X}{n_1}\rho_1 . \tag{4.3}$$

Hence, we can express (4.3) as

$$n_1(1-\rho_2) \geq n_2\rho_1 . \tag{4.4}$$

Then by combining (4.2) and (4.4), we obtain

$$k \leq \frac{1}{\rho_1}(1-\rho_2) . \tag{4.5}$$

One might think that another equation is needed to impose the condition for the opposite case, i.e., restricting the depletion time of product one to be as large as the production time of product two. However, this condition reduces to $k \geq \frac{\rho_2}{1-\rho_1}$. But $\frac{\rho_2}{1-\rho_1}$ is always guaranteed to be less than or equal to one, and so, this assumption is redundant.

The cost function can be expressed as

$$C(n_1,n_2)=n_1s_1+n_2s_2+\frac{s_1H_1}{2n_1}+\frac{s_2H_2}{2n_2}\ . \tag{4.6}$$

Therefore, the overall problem can be stated as

$$Min\ C(n_1,n_2)$$

$$\text{s.t.}\quad k\leq\frac{1}{\rho_2}(1-\rho_1) \tag{4.7}$$

$$k=\frac{n_2}{n_1}$$

Using the second constraint, we can reformulate (4.7) as follows:

$$Min\ C(n_1,n_1k)=n(c_1+c_2k)+\frac{1}{2n_1}(c_1H_1+\frac{c_2H_2}{k})$$

$$\text{s.t.}\quad k\leq\frac{1}{\rho_1}(1-\rho_2)\quad \text{for } k=1,2,3... \tag{4.8}$$

$$n_1>0$$

which is a two-dimensional optimization problem. For a fixed value of ***k***, the optimal value of n_1 is,

$$n_1^*(k)=\sqrt{\frac{1}{2}(c_1H_1+\frac{c_2H_2}{k})(c_1+c_2k)}, \tag{4.9}$$

and a corresponding cost of

$$C(n_1^*,k)=\sqrt{2(c_1+c_2k)(c_1H_1+\frac{c_2H_2}{k})}\ . \qquad (4.10)$$

Next, we find the optimal value of k, which is determined by solving the problem ,

$$Min\ C(n_1^*,k)$$

$$\text{s.t.} \qquad k\le\frac{1}{\rho_1}(1-\rho_2) \quad \text{for } k=1,2,...$$

The unconstrained minimization of $C(n_1^*,k)$ over k may be accomplished by rewriting $C(n_1^*,k)$ as

$$C(n_1^*,k)=\sqrt{ah(k)+b} \qquad (4.11)$$

where, $h(k)=H_1k+\dfrac{H_2}{k}$, $a=2s_1s_2$, and b = $s_1^2H_1+s_2^2H_2$. This is equivalent to minimizing $h(k)$, since

$$h(k)-h(k-1)=H_1-\frac{H_2}{k(k-1)}\ . \qquad (4.12)$$

Hence, k^* is the largest integer such that,

$$k(k-1)<\frac{H_2}{H_1}\ . \qquad (4.13)$$

This justifies the numbering conventions previously adopted. Thus the solution for k^* is the largest integer, k , such that (4.5) and (4.13) are satisfied.

The following procedure summarizes the solution method used in Saipe (1977):

Step 1: Calculate the value of k using equation (4.5)

Step 2: Calculate the value of n_1^* *using equation (4.9)*

Step 3: Calculate the value of n_2^* *using equation (4.1)*

Step 4: Calculate the value of $C(n_1^*, k^*)$ *using equation (4.10)*

We note that the solution outlined in Saipe (1977) does not allow the lot sizes to vary over time. Furthermore, if $\rho_1 + \rho_2 = 1$, then k should be less than or equal to one. This implies that the common cycle will produce an optimal solution.

4.2.2 Boctor (1982)

Boctor relied heavily on the assumption that the two products are produced cyclically. He further assumed that $T_1 = n_1\omega$ and $T_2 = n_2\omega$ for some basic cycle $\omega > 0$ and with g.c.d. of $(n_1, n_2) = 1$. n_1 and n_2 are the multipliers of the basic cycle ω. To ensure feasibility, Boctor imposed that the condition $\rho_1 + \rho_2 \leq 1$.

Given n_1, n_2 and ω the cost function can be written as,

$$C(n_1,n_2,\omega)=\frac{\dfrac{A_1}{n_1}+\dfrac{A_2}{n_2}}{\omega+n_1H_1+n_2H_2}, \tag{4.14}$$

where $H_i=\frac{1}{2}d_ih_i(1-\rho_i)$ for $i=1,2$. Differentiating (4.14) with respect to ω and setting the derivative to zero, yields

$$T(n_1,n_2)=\sqrt{\frac{(\dfrac{A_1}{n_1}+\dfrac{A_2}{n_2})}{n_1H_1+n_2H_2}}, \tag{4.15}$$

at a cost of

$$C(n_1,n_2)=2\sqrt{\left[\frac{A_1}{n_1}+\frac{A_2}{n_2}\right]\left[n_1H_1+n_2H_2\right]} \tag{4.16}$$

A necessary condition addressed is to guarantee that,

$$\omega\geq s_1+s_2+\rho_1(n_1T_1)+\rho_2(n_2T_2)\ . \tag{4.17}$$

That is, ω should be long enough to incorporate the production and setup times for the two products. Equivalently,

$$\omega\geq\frac{(s_1+s_1)}{1-n_1\rho_1-n_2\rho_2}\ . \tag{4.18}$$

Since s_1+s_2 is positive, (4.16) reduces to

$$1-n_1\rho_1-n_2\rho_2>0\ . \tag{4.19}$$

Based on the above, a finite combination of n_1 and n_2 that provide a feasible ω_1 and ω_2 exist. For each feasible pair satisfying (4.19), an optimal cycle time and cost can be obtained by

$$\omega^*(n_1, n_2) = \text{Max}\{\omega(n_1, n_2), \frac{s_1+s_2}{1-n_1\rho_1-n_2\rho_2}\}, \tag{4.20}$$

and

$$C^*(n_1, n_2) = \frac{(\frac{A_1}{n_1} + \frac{A_2}{n_2})}{\omega^* + (n_1 H_1 + n_2 H_2)\omega^*}, \tag{4.21}$$

respectively. To obtain the optimal multipliers, we enumerate all possible combinations (n_1, n_2), and choose the combination that yields the minimum cost.

Boctor's algorithm can be summarized as follows:

Step 1 : Enumerate all possible combinations (n_1, n_2) such that

$n_1 \in \{1, 2, \ldots, [1 - \frac{\rho_1}{\rho_2}]\}$ and $n_2 \in \{\frac{1}{\rho_2}[1 - n_1\rho_1]\}$, where [x] is the nearest integer to x. For each pair of multipliers, if the g.c.d. $(n_1, n_2) > 1$, increment n_2; otherwise, compute $\omega^*(n_1, n_2)$ and $C^*(n_1, n_2)$ using (4.20) and (4.21) respectively.

Step 2 : Compare the solutions obtained in step 1, and find the optimal set of multipliers that minimizes cost.

We note that Boctor's solution is optimal whenever the number of setups over a certain period for both products are positive integers.

4.2.3 Goyal (1984)

Goyal (1984) implemented a search procedure in which unequal batch quantities for the more frequently manufactured product is permitted. Assuming that there are X repetitive cycles per unit time, the total cost for the two products can be written as

$$C = \sum_{i=1}^{2} C_i(X) \tag{4.22}$$

where,

$$C_i(X) = XA_i + \frac{d_i h_i (1-\rho_i)}{2X} .$$

Using the basic EMQ model, the economic frequency of manufacturing setups can be found to be

$$\eta_i = \sqrt{\frac{H_i}{A_i}} . \tag{4.23}$$

However, this is only possible under the independence assumption (IS) between the two products, which is not the case. To overcome this obstacle, Goyal used the results of Doll and Whybark (1973) which states; the nearly optimal production quantities can be obtained if the product for which H_i/A_i is largest, and is manufactured the maximum number of times in a repetitive cycle.

Goyal denoted the product with the highest H_i/A_i to be the "First product" and is manufactured the maximum number of times. While the other product is referred to as the "Second product" and is manufactured only once per repetitive cycle. That is, the "Second product" is manufactured X times with a repetitive cycle of $1/X$. While the "First product" is manufactured nX times in the same period. This translates into an interval of $1/nX$ between successive setups.

Suppose there is no delay between the end of the production cycle of the first product and the start of second product setup, and that the first production run of the "First product" starts at time zero. Then, total time required to produce one batch of the First and Second products is $T_1 + T_2 + [\rho_1/n_1 + \rho_2]/X$. To ensure feasibility of the production schedule, Goyal restricted the time between two successive setups of the "First product" to be long enough for setting up and producing both products. Hence,

$$n \leq \frac{1-\rho_1}{X(T_1 + T_2 + \frac{\rho_2}{X})}. \tag{4.24}$$

Arguing that setup times occupy relatively small span of time relative to the production times, Goyal neglected setup times. Under this assumption, the feasibility restriction will be reduced to $n \le \rho_2(1-\rho_1) = \zeta$. In order to ensure a feasible solution, we restrict the value of ζ to be equal or greater than one. If $n \le \zeta$ is not satisfied for any value of *n*, then the holding cost of the first product can be minimized by processing the first setup at the beginning of the manufacturing cycle, and the second setup after an interval of $1/X\zeta$ time units. Hence, setup covers a span of $\frac{1}{XMin(n,\zeta)}$, and the remaining *(n-1)* setups will be

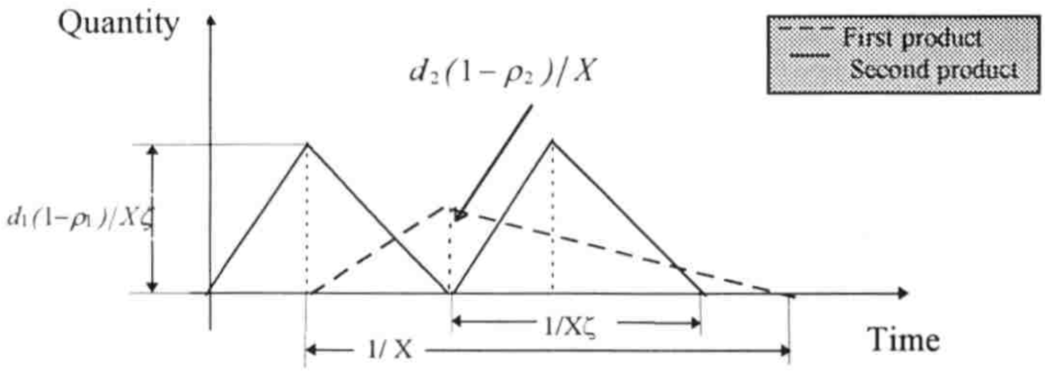

Figure 4.1: $n \le \zeta$ **, plot for** $n=2$

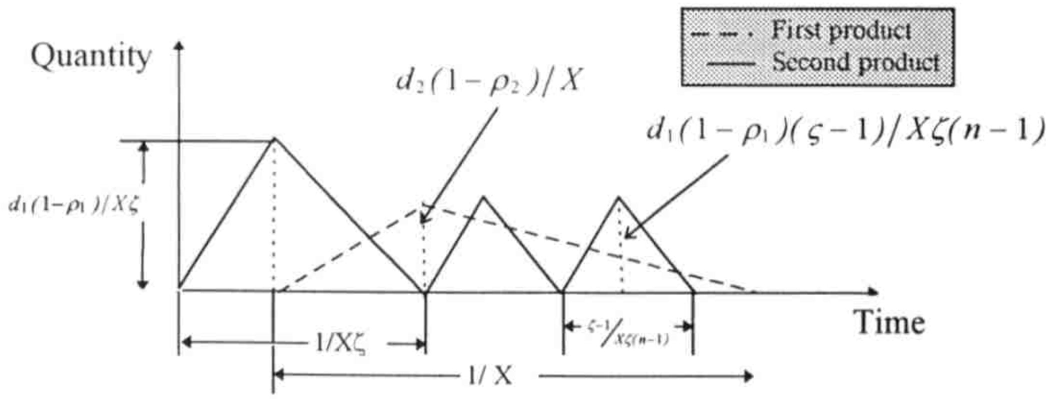

Figure 4.2: $n > \zeta$ **, plot for** $n=3$

undertaken at equal time intervals, covering a time span of $\frac{\{Min(n,\zeta)-1\}}{\{XMin(n,\zeta)\}}$.

Figures 4.1 and 4.2, show possible realizations of such problems.

Let $\alpha_i = X\text{Min}(n,\zeta)$, $\alpha_2 = \text{Min}(n-\zeta)-1$, and $H_i = \frac{1}{2}d_i h_i(1-\rho_i)$. Then the variable cost of the first product can be expressed as,

$$C_1(Xn) = X\left[ns_1 + H_1\left(\frac{1}{\alpha_1}\right)^2 + H_2\left(\frac{1}{n-1}\right)\left(\frac{\alpha_2}{\alpha_1}\right)^2\right]. \qquad (4.25)$$

Therefore, the total variable cost $C(X,\zeta)$, given that product 1 is manufactured Xn times and the second product is manufactured X times per unit time, can be expressed as

$$C(X,\zeta) = X(ns_1 + s_2) + \frac{1}{X}\left[\frac{H_1(1+\{\text{Min}(n,\zeta)-1\}^2)}{(n-1)\{\text{Min}(n,\zeta)\}^2} + H_2\right]. \qquad (4.26)$$

Due to the multidimensionality of (4.26), Goyal developed the following search routine for determining the economic policy for scheduling the two products.

Search procedure:

Step 1: Find A_i/s_i ratio, and arrange them in descending order. Recall that the second product is manufactured once in every repetitive cycle.

Step 2: (i) Determine the integer value of the multiplier $n = a$ such that

$$a(a-1) \le \frac{A_1 s_2}{A_2 s_1} \le a(a+1)$$

(ii) Determine the integer value of $n = b$ such that

$$b \le \rho_2(1-\rho_1) \le b+1$$

(iii) Let $Y = \mathrm{Min}(a,b)$

Step 3: Find $C(n)$ for $Y \le n \le a$ using equation (4.26). The search is terminated when $C(n+1) \ge C(n)$, where $n = n^*$.

Step 4: The economic policy is determined by

$$X(n^*) = \sqrt{\frac{A_2 + \dfrac{A_1}{\{Min(n^*,\zeta)\}^2} + \dfrac{1+\{Min(n^*,\zeta)-1\}^2}{n^*-1}}{s_1 + s_1 n^*}} .$$

The economic production quantity for the second product is given by $d_2 / X(n^*)$, while the first run for the first product is given by

$\dfrac{d_1}{X(n^*)Min(n^*,\zeta)}$, and the rest of the n^*-1 runs are

$\dfrac{d_1[Min(n^*,\zeta)-1]}{(n^*-1)X(n^*)Min(n^*,\zeta)}$.

4.2.4 Lee and Danusaputro (1989)

Lee and Danusaputro have extended the problem that Goyal (1984) solved by including the setup times. The algorithm developed can be regarded as an extension to Boctor's algorithm.

To test the validity of the proposed algorithm, Lee and Danusaputro first calculated the lower bound solution (IS). This is done by setting the setup times to zero. Consequently, the optimal cycle time and cost can be expressed as $T_i = \sqrt{A_i/H_i}$, and $C_i = 2\sqrt{A_i H_i}$ respectively. Without loss of generality it is assumed that $T_2 \geq T_1$. In order to demonstrate how the algorithm functions, we first present the used notation. Let, X denote the largest integer less than or equal to T_2/T_1 , and K is the largest integer such that

$$\sqrt{K(K-1)} \leq \frac{T_2}{T_1} \leq \sqrt{K(K+1)} \ . \tag{4.27}$$

Lee and Danusaputro were particularly interested in analyzing the pair of multipliers $(1,K)$; i.e., $n_1 = 1$, $n_2 = K$. Then, using Boctor's approach we can express the cycle time and the feasibility equations as,

$$T(1,K) = \sqrt{\frac{A_1 + \frac{A_2}{n}}{H_1 + nH_2}} \ , \tag{4.28}$$

$$1 - \rho_1 - n\rho_2 > 0 \ , \tag{4.29}$$

and

$$T(1,K) \geq \frac{(s_1 + s_2)}{1 - \rho_1 - n\rho_2}. \tag{4.30}$$

The authors presented two theorems which can be used to reduce the complexity of obtaining a solution.

Theorem 1: (i) If both (4.29) and (4.30) are satisfied then, $C(n_1,n_2) \geq C(1,K)$ for any feasible n_1 and n_2, such that $\frac{n_2}{n_1} \geq X$ or $\frac{n_2}{n_1} \leq X+1$.

(ii) Let (n_1^*, n_2^*) be the optimal multipliers, then $n_1^* \leq n_2^*$.

(iii) $n_1^* = 1$ and $n_2^* \leq K$ if at least one of the feasibility equations were not satisfied.

Theorem 2 If (4.29) and (4.30) are satisfied then, $C(1,K) \leq 1.015(C_1 + C_2)$. That is, the increase in cost encountered is by no means greater than 1.5% of the IS (lower bound).

Using Goyal's results, product 2 will be produced once per cycle and product 1 will be produced more frequently with unequal cycle times inside product 2's cycles. The concept behind the Lee and Danusaputro algorithm, can be discussed as follows:

i. Given the multiplier n and neglecting the feasibility conditions, both products will be produced cyclically. If we assume a cycle T for product one, then the total cost can be expressed as,

$$C(n,T) = \frac{nA_1 + A_2}{T} + (\frac{s_1}{n} + s_2)T. \tag{4.31}$$

The optimal cycle for products 1 and 2 can then be written as,

$$T_1(n) = n\sqrt{(A_1 + \frac{A_2}{n})(H_1 + nH_2)}\ , \tag{4.32}$$

and

$$T_2(n) = n\sqrt{(nA_1 + A_2)(\frac{H_1}{n} + H_2)}\ , \tag{4.33}$$

respectively. Substituting (4.32) and (4.33) in (4.31), yields the corresponding minimum cost, i.e.,

$$C(n,T(n)) = 2\sqrt{(nA_1 + A_2)(\frac{H_1}{n} + H_2)}\ . \tag{4.34}$$

ii. Based on the results in part one, we can respectively rewrite (4.29) and (4.30) as

$$1 - \rho_1 - n\rho_2 > 0\ , \tag{4.35}$$

and

$$\frac{T(n)}{n} \geq \frac{s_1 + s_2}{1 - \rho_1 - n\rho_2}\ . \tag{4.36}$$

Then, given the multipliers *(1, K)*, the optimal feasible cycle times for products 1 and 2 are *T(n)/n* and *T(n)* respectively. Note that, in this case both products are produced cyclically, with a corresponding cost given by (4.34).

iii. If equation (4.35) is not satisfied, the problem is solved by defining a new function *y(T)* such that

$$y(T)=\frac{s_1+s_2+\rho_2 T}{1-\rho_1}, \tag{4.37}$$

and

$$1-\rho_1<n\rho_2 \ . \tag{4.38}$$

Under this condition, it is impossible for product 1 to be produced cyclically *n* times within a time interval of *T*. Therefore, for product 1 we use a cycle of length *y(T)* only once, while the remaining *T - y(T)* time units are divided into *n-1* cycles. This approach is similar to Goyal's procedure. For more information refer to Figure 4.3. The total cost in this case is given by

$$C'(n,T)=\frac{(nA_1+A_2)}{T}+H_1\lambda+H_2T , \tag{4.39}$$

where

$$\lambda=\frac{[(\frac{T-y(T)}{n-1})^2(n-1)+y(T)^2]}{T} .$$

The optimal *T* is found by differentiating (4.39) with respect to *T* and equating to zero. Hence,

$$T'(n)=\sqrt{\frac{\alpha}{\beta}}, \tag{4.40}$$

where

$$\alpha = H_1 n(s_1 + s_2)^2 + (nA_1 + A_2)(1-\rho_1)^2(n-1),$$

and

$$\beta = H_1\left[(1-\rho_1-\rho_2)^2 + \rho_2^2(n-1)\right] + H_2(1-\rho_1)^2(n-1).$$

The same scenario applies if (4.36) is not satisfied, with only small changes in the cycle time and cost. That is,

$$T''(n) = \frac{ns_1 + s_2}{1-\rho_1-\rho_2}, \tag{4.41}$$

and

$$C''(n) = \frac{nA_1 + A_2}{T''(n)} + H_1\left(\frac{s_1}{1-\rho_1}\right)^2(n-1) + \frac{y(T''(n))^2}{T''(n)} + H_2 T''(n). \tag{4.42}$$

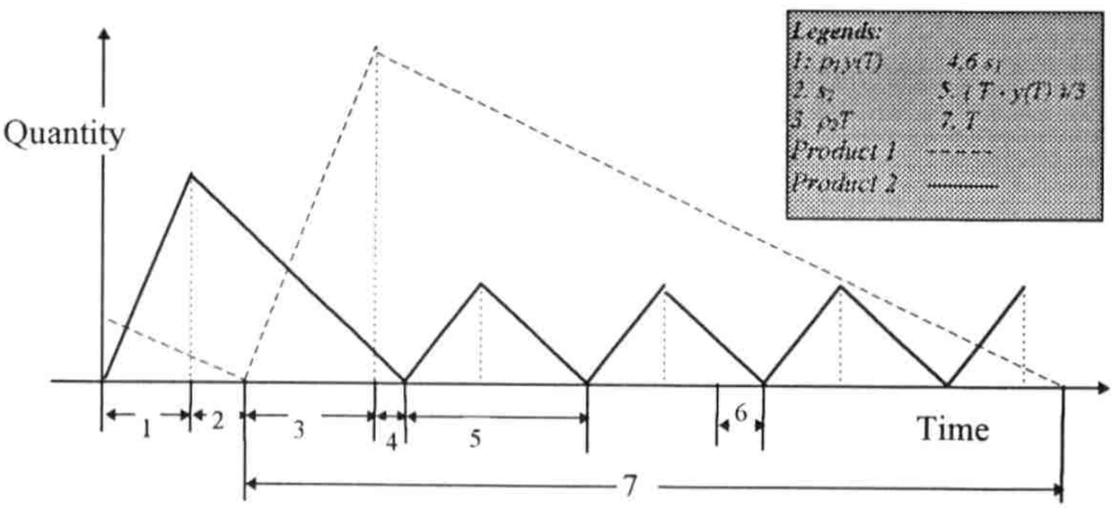

Figure 4.3: Plot of the Time Varying Lot Sizes

More formally, the algorithm can be written as follows.

Step 1: Find K using (4.29)

Step 2: Find $T(n)$ using (4.33). If (4.35) and (4.36) are satisfied then $T^*(n)=T(n)$ and $C^*(n)=C(n)$, and go to step 4. Otherwise, go to step 3.

Step 3: Find $T'(n)$ using (4.40). If $T'(n) > (ns_1+s_2)/(1-\rho_1-\rho_2)$ then $T^*(n)=T'(n)$ and $C^*(n)=C'(n)$ using (4.39) and (4.40) respectively. Otherwise, set $T^*(n)=T''(n)$ and $C^*(n)=C''(n)$ using (4.41) and (4.42) respectively. Go to step 4.

Step 4: If $n = 1$ or both (4.35) and (4.36) are satisfied, then find the optimal solution among all solutions that have been found. That is, find $C^* = Min\{C(i); n \le i \le K\}$ and find the corresponding T^*.

Else set $n = n - 1$ and go to step 2.

We should note, that the above algorithm has a tendency to produce better results than that of Boctor's.

4.3 Extensions in the Two Product Scenario

Several attempts were cited in the literature that deviate from the traditional two product ELSP. Two different attempts will be summarized in this section.

4.3.1 Hwang and Moon (1991)

Most of the models explicitly assume that the raw material required to manufacture the products had been procured in the most economical manner. No extra cost in the objective function (total cost function) is added to account for the cost of ordering materials. However, it is impossible to procure the raw materials in an optimal manner without knowing the production batch sizes of the product under discussion.

Hwang and Moon (1991) developed a model that integrates the production planning problem of the two product case with the inventory replenishment model. It is an iterative heuristic model that provides a near optimal production and raw material policies to minimize total system cost. In their study, they assumed that both the production sequence as well as the replenishment time is known and constant throughout the planning horizon. No assurance for providing an optimal solution is given, but the heuristic was found to have a tendency of providing a feasible solution most of the time.

4.3.2 Ibrahim and Thomas (1991)

Ibrahim and Thomas implemented the super production cycle approach on the two product one machine problem with limited storage capacity. The problem they solved can be viewed as a traditional two product scenario of the ELSP in addition to a new constraint on the amount of items that can be stored.

The super production approach assumes, that there is a super cycle, within which there are exactly n_1 production runs of product one, and n_2 production runs for product two. The problem then is to find a minimum cost schedule for producing the two types of products, given that there is a limit on the total amount of finished products that can be stored prior to delivery. The algorithm outlined gives the optimal schedule among all possible production schedules. Moreover, it addresses a possible extension for solving the more general case (multiproduct case).

CHAPTER FIVE

THE TWO PRODUCT CASE WITH ALLOWED SHORTAGES

5.1 Introduction

In the previous chapter, we examined the two product scenario of the ELSP. It was assumed throughout the discussion that the demand should be satisfied during the planning horizon, with the exception of few papers that allow for backorders. However, no contributions were cited that examine the shortage scenario. In this chapter, we will discuss the case where shortages are allowed, in an attempt to examine their effects on both the annual cost and the quantity produced.

Researchers often referred to shortages as lost sales case. Hadley and Whitin (1963) defined shortages as the demand which occurs when the system is out of stock and this demand is lost forever. However, we will adopt a different definition, and define shortages as the demand which occurs when the system is out of stock and which has to be satisfied at a higher cost.

5.2 Why Do Shortages Occur?

Adopting the second definition of shortages, we can identify a number of situations under which shortages may occur. First, the case were demand has to be met, while the capacity of the firm is not sufficient. For example, if a manufacturer has a monopoly on the market, or he has an agreement for supplying a certain amount of items while his firm capacity is not sufficient for meeting this demand, so he may choose to order some items, from an outside vender or supplier at a higher cost. Second, if backordering cost is too high which makes it impractical for the firm to adopt backordering strategy. Third, when holding cost is high relative to shortage cost, consequently the firm may adopt a strategy that allows shortages, in an attempt to decrease the holding cost. Fourth, when the firm has a preventive maintenance policy which has to be implemented. Finally, if setup times occupy a long period of time, and this time is not negligible relative to the production time, then no matter what level of inventory is maintained some items will be short.

5.3 The One Product Case

Before we discuss the two product case, it is of great importance to identify how shortages affect the solution of the one product case. If demand is occurring when the system is out of stock is supplied from an outside supplier, it is no longer true that the annual revenues received will be independent of the operating policy. On the contrary, they will depend on the length of time for which the system is out of stock, and hence on the operating policy used. Thus, we cannot immediately conclude that maximizing the average annual profit, will yield the same operating policy as that minimizing the average annual cost. However, with a proper definition of the shortage (stockout) cost, minimizing the average annual cost will yield the same result as maximizing the average annual profit.

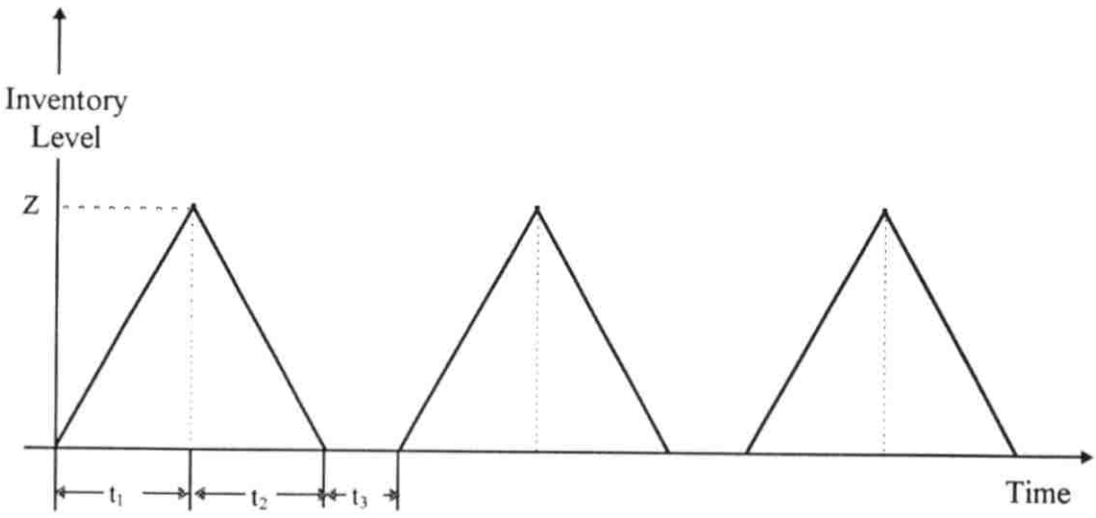

Figure 5.1: The One Product Case

Let X_s be the unit selling price of the item, π_o the additional cost incurred when purchasing the item from an outside supplier (excluding lost profit), X_o the selling price of the outside supplier ($\pi_o = X_s - X_o$). Then, if f_o is the fraction of time during which the system is out of stock, we can express the average annual profit as

$$\begin{aligned} \text{Profit} &= d(X_s - \mu)(1 - f_o) - \pi_o d f_o \\ &\quad - \text{ordering and carrying cost} . \end{aligned} \tag{5.1}$$

We can rewrite (5.1) as,

$$\begin{aligned} \text{Profit} &= d(X_s - \mu) - (\pi_o + X_s - \mu) d f_o \\ &\quad - \text{ordering and carrying cost} . \end{aligned} \tag{5.2}$$

To obtain the optimal lot size, we differentiate (5.5), with respect to t_3 and q, equate the derivations to zero, and solve the two equations simultaneously. That is,

$$\frac{\partial C}{\partial q} = -dA - \pi d^2 t_3 + hd(1-\rho)qt_3 + \tfrac{1}{2}h(1-\rho)q^2 = 0, \tag{5.6}$$

and

$$\frac{\partial C}{\partial t_3} = -d^2 A - \tfrac{1}{2}hq^2(1-\rho)d + \pi d^2 q = 0. \tag{5.7}$$

Solving (5.7) in terms of q yields,

$$q = \frac{\pi d}{h(1-\rho)} \pm \sqrt{\left(\frac{\pi d}{h(1-\rho)}\right)^2 - \frac{2dA}{h(1-\rho)}}. \tag{5.8}$$

If $\pi^2 d < 2Ah(1-\rho)$, then no valid solution exists for q that satisfies (5.8), while if $\pi^2 d = 2Ah(1-\rho)$, then there exists a unique positive value of q that satisfies (5.8). Finally if $\pi^2 d > 2Ah(1-\rho)$, there are two positive values of q which satisfy (5.12), since in this case,

$$\frac{\pi d}{h(1-\rho)} > \sqrt{\left(\frac{\pi d}{h(1-\rho)}\right)^2 - \frac{2dA}{h(1-\rho)}}. \tag{5.9}$$

In the event that there is no real q, there is no t_3, such that $0 < t_3 < \infty$, which will yield a minimum cost; hence the optimal value of t_3 must be zero or infinity. However, $\pi^2 d < 2Ah(1-\rho)$ implies that incurring a shortage cost all the time is cheaper than producing. Consequently, the optimal value of t_3 should be infinity.

On the other hand, for the case where either one or two positive q satisfy (5.8), it was found by substituting (5.8) in (5.6) that the optimal t_3 does not lie in the interval $0 < t_3 < \infty$. In this case, the optimal value of t_3 is zero, since $\pi^2 d \geq 2Ah(1-\rho)$, i.e., the cost of running the system and producing all the time is cheaper than running the system with allowed shortages.

In conclusion, for the one product case, it is never optimal to incur any shortages.

5.4 The Two Product Case

In analyzing the two product case, we will assume that setup times are negligible. That is, no allowance is added to compensate for the setup time. Furthermore, we will assume that no idle times are allowed, the machine is either producing product 1 or product 2. The average annual cost can then be expressed in terms of q_1, q_2, t_3^1, and t_3^2, as

$$C = \sum_{i=1}^{2} \left[\frac{d_i A_i}{q_i + d_i t_3^i} + \frac{h_i' q_i^2}{2(q_i + d_i t_3^i)} + \frac{\pi_i d_i^2 t_3^i}{q_i + d_i t_3^i} \right], \qquad (5.10)$$

where, $h_i' = h_i(1-\rho_i)$ and t_3^i is the shortage time of product i. See Figure 5.2 for more details.

According to the results obtained in the previous section, the lower bound solution can be found using the IS approach presented in Chapter Two. Using this approach we can express the cycle time of each item and the corresponding cost as,

$$T_i^{IS} = \sqrt{\frac{2A_i}{d_i h_i'}}, \tag{5.11}$$

and

$$C^{IS} = \sum_{i=1}^{2} [\frac{A_i}{T_i^{IS}} + \tfrac{1}{2} h_i' d_i T_i^{IS}], \tag{5.12}$$

respectively. While assuming no shortages the upper bound can be established using the Common Cycle procedure, where the cycle time and the corresponding cost are,

$$T^{cc} = \sqrt{\frac{2\sum_{i=1}^{2} A_i}{\sum_{i=1}^{2} d_i h_i'}}, \tag{5.13}$$

and

$$C^{cc} = \sum_{i=1}^{2} [\frac{A_i}{T^{cc}} + \tfrac{1}{2} h_i' d_i T^{cc}]. \tag{5.14}$$

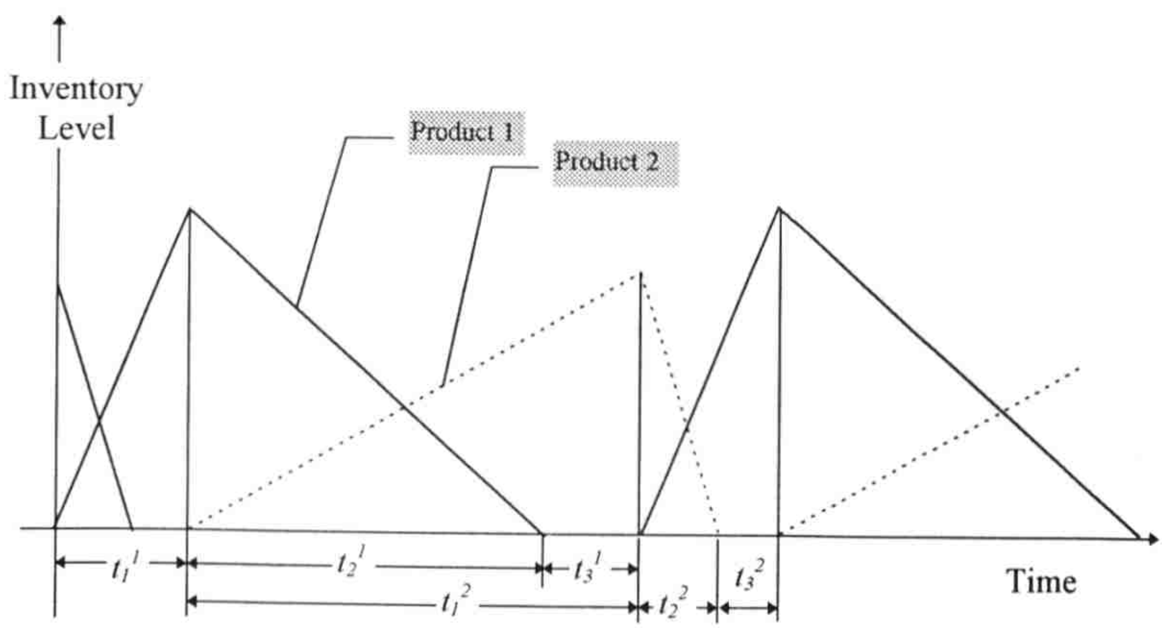

Figure 5.2 : The Two Product Case, with Allowed Shortages

5.4.1 The Common Cycle Approach

Assuming a Common Cycle, we can express q_2 in terms of q_1 as,

$$q_2 = \frac{q_1}{\beta} + d_2(t_3^1 - t_3^2), \tag{5.15}$$

where $\beta = d_1/d_2$. The interpretation of (5.15) means that the production period of product 1 is long enough to incorporate the consumption period of product 2, in addition to a possible shortage period for that product. This is shown in Figure 5.2. Substituting (5.15) in (5.10) yields,

$$C = \frac{1}{q_1 + d_1 t_1}\{d_1 A' + \pi_1 d_1^2 t_1 + \alpha\pi_2 t_2 + \frac{h_1' q_1^2}{2} + \frac{h_2' q_1^2}{2\beta} + h_2' d_2 q_1 t_1 - h_2' d_2 q_1 t_2 \\ + \frac{\alpha h_2' t_1^2}{2} - \alpha h_2' t_1 t_2 + \frac{\alpha h_2' t_2^2}{2}\} \tag{5.16}$$

where, $\alpha = d_1 d_2$, $A' = A_1 + A_2$, and $t_i = t_3^i$.

To find the optimal operating policy, we differentiate (5.16) with respect to q_1, t_1, and t_2 and equate the derivatives to zero. The results obtained are as follows,

$$\frac{\partial C}{\partial t_2} = \alpha\pi_2 - d_2 h_2' q_1 - \alpha h_2' t_1 + \alpha h_2' t_2 = 0, \tag{5.17}$$

and

$$\frac{\partial C}{\partial t_1} = -d_1^2 A' + \pi_1 d_1^2 q_1 - \alpha\pi_2 d_1 t_2 - \frac{h_2' d_1 q_1^2}{2} + \frac{h_2' d_2 q_1^2}{2} + \frac{\alpha h_2' d_1 t_1^2}{2} \\ + \alpha h_2' t_1 q_1 - \frac{\alpha h_2' d_1 t_2^2}{2} = 0, \tag{5.18}$$

and finally,

$$\frac{\partial C}{\partial q_1} = -d_1 A' - \pi_1 d_1^2 t_1 - \alpha \pi_2 t_2 + \frac{h_1' q_1^2}{2} + h_1' d_1 q_1 t_1 + \frac{h_2' q_1^2}{2\beta} + d_2 h_2' q_1 t_1$$
$$+ \alpha h_2' t_1^2 - \frac{\alpha h_2' t_1^2}{2} - \frac{\alpha h_2' t_2^2}{2} = 0 \; . \tag{5.19}$$

These equations can be solved, by substituting (5.17) in both (5.18) and (5.19), and after several steps we obtain,

$$q_1 = \frac{\pi_1 d_1}{h_1'} \pm \sqrt{(\frac{\pi_1 d_1}{h_1'})^2 - \frac{2 d_1 A'}{h_1'} + \frac{\alpha \pi_2^2}{h_1' h_2'}}, \tag{5.20}$$

and

$$t_1 = \frac{q_1^2 h_1' h_2' + \alpha \pi_2^2 - 2 d_1 A' h_2'}{2 h_2' (\pi_1 d_1^2 - q_1 h_1' d_1)} \; . \tag{5.21}$$

By symmetry, we find similar expressions for q_2 and t_2.

Examining (5.20), we notice that a feasible solution is obtained if and only if,

$$\pi_1^2 d_1 + \frac{d_2 h_1' \pi_2^2}{h_2'} > 2 A' h_1', \tag{5.22}$$

is satisfied. However if (5.22) is not satisfied, there is no t_1, $0 < t_1 < \infty$, which yields a minimum cost; hence the optimal value of t_1 must be zero or infinity. The optimal value is infinity since $\pi_1^2 d_1 + \frac{d_2 h_1' \pi_2^2}{h_2'} < 2 A' h_1'$ implies that incurring the shortage cost all the time is cheaper than producing the item in the facility. Therefore, no inventory system exists for product 1.

In the event that (5.22) is satisfied as an equality, there exists a unique positive solution for q_1 that satisfies (5.20). However, this solution is not valid, since the corresponding value of t_1 associated with it is infinity, i.e., the cost of shortage is less than the cost of production. Finally, for the case where (5.22) is satisfied, there exists two values of q_1 that satisfy (5.22). But for the solution to be feasible, the corresponding value of t_1 should lie within the interval $0 < t_1 < \infty$, which translates to satisfying the inequality

$$\pi_1 d_1 > q_1 h_1'. \tag{5.23}$$

Else, if (5.23) is not satisfied, the corresponding value of t_1 will be infinity. Also, it can be shown that for $\pi_2 = 0$, equation (5.20) will reduce to (5.8), meaning that product 2 will not be produced, while product 1 is produced according to (5.11).

Close examination of (5.23) together with (5.20), yields the following. First, (5.23) implies that $q_1 < \dfrac{\pi_1 d_1}{h_1'}$, so in order to obtain a feasible q_1, only the negative sign in (5.20) must be used. Second, given $q_1 < \dfrac{\pi_1 d_1}{h_1'}$, a feasible q_1 exists if and only if,

$$\pi_2 < \sqrt{\frac{2h_2' A'}{d_2}}. \tag{5.24}$$

Third, given the fact that $\dfrac{2d_1 A'}{h_1'} > \dfrac{\alpha \pi_2^2}{h_1' h_2'}$, then it can be shown that,

$$\left(\frac{\pi_1 d_1}{h_1'}\right)^2 > \frac{2d_1 A'}{h_1'} - \frac{\alpha \pi_2^2}{h_1' h_2'}. \tag{5.25}$$

Using (5.25), we can obtain a range of feasible values for π_1. Two extreme cases will be examined, $\pi_2 = 0$ and $\pi_2 = \sqrt{\frac{2h_2'A'}{d_2}}$. For the first case, it follows that

$$\pi_1 > \sqrt{\frac{2h_1'A'}{d_1}}. \tag{5.26}$$

However, as discussed in section 5.3, it is never optimal to allow for shortages when the machine is idle. For the second case, i.e, $\pi_2 = \sqrt{\frac{2h_2'A'}{d_2}}$, we conclude that $\pi_1 > 0$, which is redundant in the presence of (5.26). The same argument can be made if we examine q_2 along with t_2, i.e, for a valid solution to exist

$$\pi_1 > \sqrt{\frac{2h_1'A'}{d_1}} \text{ and } \pi_2 < \sqrt{\frac{2h_2'A'}{d_2}}, \tag{5.27}$$

and

$$\pi_1 < \sqrt{\frac{2h_1'A'}{d_1}} \text{ and } \pi_2 > \sqrt{\frac{2h_2'A'}{d_2}}, \tag{5.28}$$

which is impossible. That is, neither product one nor product two should be allowed to have shortages.

In order to demonstrate these two arguments, we developed a set of problems by varying the values of π_1 and π_2. The break points were set to equal $\sqrt{\frac{2h_i'A'}{d_i}}$. The resulting five cases were examined and the results are tabulated in

Table 5.1. Examining Table 5.1, we notice that for the five different cases, a valid solution was never obtained. This emphasizes the arguments previously discussed.

The overall solution then is, to produce product 1 alone according to (5.11), produce product 2 alone according to (5.11), or to produce products 1 and 2 together but without allowing for shortages. The decision is based on the economical benefits of each strategy. The following procedure was suggested .

Table 5.2: The Effect of Varying the Value of π

	Case 1		Case 2		Case 3		Case 4		case 5	
	Product 1	Product 2	Product 1	Product 2	Product 1	Product 2	Product 1	Product 2	Product 1	Product 2
A_i	10	15	10	15	10	15	10	15	10	15
d_i	200	300	200	300	200	300	200	300	200	300
p_i	500	600	500	600	500	600	500	600	500	600
h_i	0.005	0.002	0.005	0.002	0.005	0.002	0.005	0.002	0.005	0.002
π_i	0.035	0.018	0.005	0.020	0.040	0.005	0.005	0.001	0.040	0.020
ρ_i	0.400	0.500	0.400	0.500	0.400	0.500	0.400	0.500	0.400	0.500
h_i'	0.002	0.001	0.002	0.001	0.002	0.001	0.002	0.001	0.002	0.001
$\pi_i d_i/h_i'$	1414.2	2738.6	200	3000	1600	750	200	150	1600	3000
q_i	0.0324	0.0263	-463.32	1715.5	757.39	-881.72	N/A	N/A	-120.47	-331.67
t_i	3.5353	6.8465	18.723	-4.3984	-2.8344	23.637	N/A	N/A	4.2559	8.7155

Procedure

Step 1: Compute C_1^{IS} and C_2^{IS} using (5.12)

Step 2 : If $C_1^{IS} > \pi_1 d_1$ and $C_2^{IS} > \pi_2 d_2$, then do not produce products 1 and 2. While If $C_1^{IS} < \pi_1 d_1$ and $C_2^{IS} > \pi_2 d_2$, then produce product 1 only, and vice versa. Finally if $C_1^{IS} < \pi_1 d_1$ and $C_2^{IS} < \pi_2 d_2$, then produce the two products using (5.13) or any other alternative method examined in Chapter Four.

5.4.2 The Two Product Case with Non-Negligible Setup Times

In many industries such as the glass, ceramic and chemical industries, setup times occupy a high proportion of time relative to production times. So, neglecting the setup times will affect the quality of the solution obtained.

In this section we will examine the case where setup times are not negligible. We will further assume that shortages are going to occur during the setup times.

Referring to Figure 5.3, the cycle time T can be written as $T = t_1^1 + t_2^1 + t_3^1 + s_1$. Given T the total cost can then be expressed as,

$$C = \sum_{i=1}^{2} [\frac{A_i}{T} + \frac{h_i' q_i^2}{2Td_i} + \frac{\pi_i d_i x_i}{T}], \tag{5.29}$$

where $x_i = t_3^i + s_i$. The overall problem can then be written as,

$$\text{Min } C \tag{5.30}$$

$$\text{s.t.} \quad x_i \geq s_i \, . \qquad \text{for } i=1,2$$

However, based on the results obtained in the previous section, we conclude that if product 1 and product 2 are to be produced, their corresponding values of x_i will tend to converge to s_i, i.e., no extra shortage will be needed.

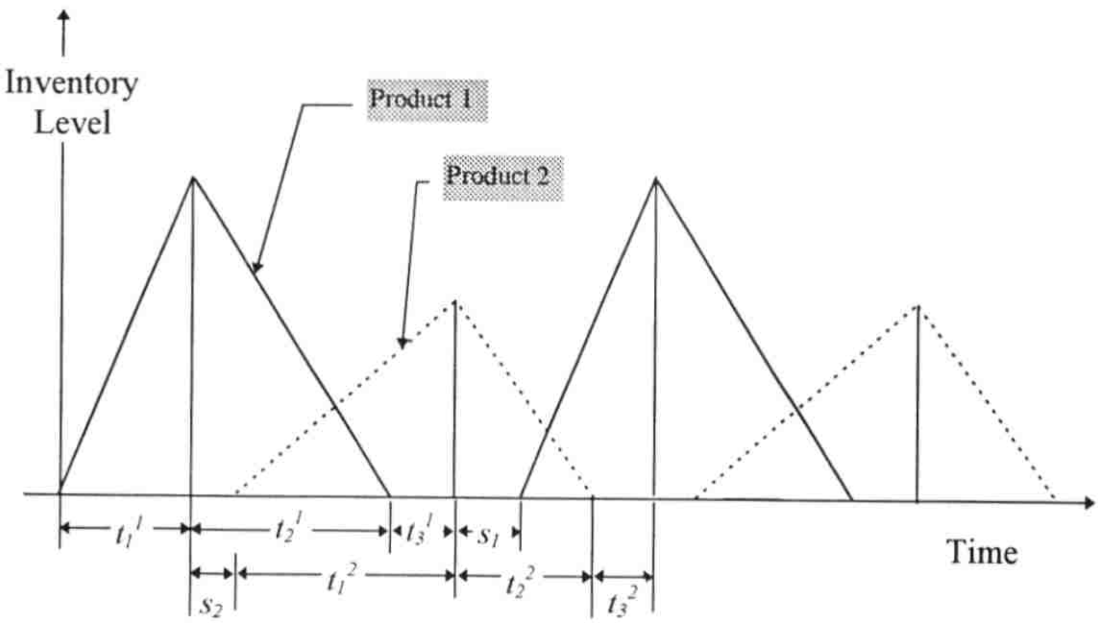

Figure 5.3 : Two Product Case with Non-negligible Setup Times

CHAPTER SIX

CONCLUSION

Numerous approaches were developed that try to solve the outstanding ELSP problem. In the previous chapters we examined a number of these approaches and pointed out their working environment, limitations and advantages. In general, heuristic approaches were used and they sometimes provide very good solutions. Analytical approaches are also used to solve restricted versions of the ELSP.

The main difficulty encountered in solving the problem lies in testing the feasibility of the proposed schedule. Researchers have shown that feasibility testing is an NP-hard problem. Moreover, we were able to classify the techniques used in solving the problem into three major categories, (1) Common Cycle Approach, (2) Basic Period, and (3) the Dobson's approaches.

The Upper and lower bound solutions were established. And it was noticed that in the worst case scenario, the upper bound exceeds the lower bound by 41.2%. However, in many realistic cases this percentage decreases dramatically if the ratios of setup cost to holding cost are approximately equal.

In the last chapter we proved that allowing shortages in the two product case is never optimal. This implies, that the facility either produces the parts or purchase them. That is, for the two product case, the mix between these two extreme strategies is never optimal. A possible extension was addressed that examine the non negligible set up times.

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