ON LINE SEGMENTS TERMINATED BY THE SIDES OF THE TRIANGLE

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ABSTRACT

The reader is assumed to be acquainted with the use of trilinear homogeneous coordinates.

Chapter I contains the definition of trilinear coordinates, representation of linear and quadratic equations, and a necessary and sufficient condition for six points to determine a conic.

In Chapter II six points are selected on the sides of the triangle of reference in such a way that two points on two sides form a segment parallel to the third side. The selected six points are found to lie on a conic whose equation is obtained. Expressions representing the center of the conic are given.

Chapter III is a continuation of Chapter II, conditions on the line segments are imposed and results are derived. Equations of circumscribed and inscribed conics are obtained as well as coordinates of some points. An equation representing the circumscribed Steiner's ellipse is obtained.

In Chapter IV, the six points are selected so that the line segments are equal in length. A discussion parallel to that in Chapters II and III is followed. The equation of Steiner's inscribed ellipse is obtained.

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CHAPTER I

INTRODUCTION

1. Introducing Trilinear Coordinates.

Let three non-concurrent straight lines meet in pairs to form a triangle. Denote the triangle by $A_1A_2A_3$. The coordinates of any point P with respect to $A_1A_2A_3$ are the perpendicular distances from P to the sides of the triangle $A_1A_2A_3$ or any other quantities respectively proportional to these distances. The i-th coordinate is the distance of P from a_i $(i = 1,2,3)$.

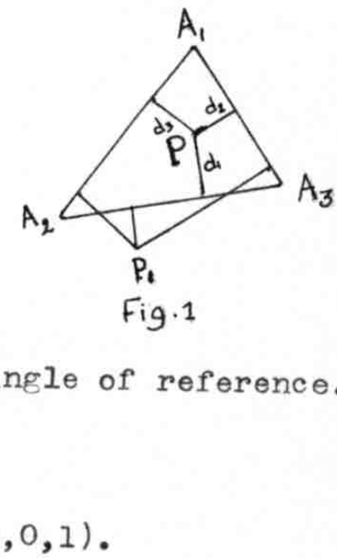

Fig.1

Thus in the adjacent figure, the trilinear coordinates of P are (d_1, d_2, d_3) or (Kd_1, Kd_2, Kd_3) where $K \neq 0$.

2. Notation.

Triangle $A_1A_2A_3$ is called the triangle of reference. The coordinates of A_i are:

$$A_1 = (1,0,0,) \; ; \quad A_2 = (0,1,0) \; ; \quad A_3 = (0,0,1).$$

If P is inside $\Delta\, A_1A_2A_3$, the distances d_i of P are all positive. In general d_i is positive if P is on the same side as A_i from a_i, otherwise negative.

Thus the coordinates of P_1 are $(-d_1, d_2, d_3)$ where each d_i is positive. The first coordinate of P_1 is negative since P_1 and A_1 are on opposite sides from a_1.

Not more than one coordinate of any point need be negative. If two are negative, multiplication by -1 makes them positive.

The coordinates of any point P are connected by the relation:

$$d_1 a_1 + d_2 a_2 + d_3 a_3 = 2\Delta,$$

where d_i are the distances of P and Δ the area of $A_1A_2A_3$. This formula is valid for any position of P, if the distances are given correct signs.

In trilinear coordinates an equation of the first degree takes the form of

$$\Sigma c_i x_i = 0 .$$

An equation of the second degree takes the form

$$\Sigma a_{ij} x_i x_j = 0 , \quad a_{ij} = a_{ji}$$

i.e.

$$a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + 2a_{12} x_1 x_2 + 2a_{23} x_2 x_3 + 2a_{31} x_3x_1=0.$$

The line at infinity is represented by:

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0.$$

In this paper when the subscripts are given as actual integers, the summation symbol indicates the sum of the three expressions obtained by cyclic permutation of the subscripts modulo 3. For example, $\Sigma\, a_1\, b_2$ means

$$a_1\, b_2 + a_2\, b_3 + a_3\, b_1.$$

But when they are variable subscripts, the variable takes the values 1 to 3. For example: $\Sigma\, a_i$ means $a_1 + a_2 + a_3$:

3. <u>Necessary and sufficient condition for six points, no three of which are collinear, to determine aconic:</u>

<u>Statement:</u> A necessary and sufficient condition for the six points: $P_i = (V_{1i}, V_{2i}, V_{3i})$, $Q_i = (W_{1i}, W_{2i}, W_{3i})$, $i = 1,2,3$ to determine a conic is that the determinant:

$$\begin{vmatrix}
V_{11}^2 & V_{21}^2 & V_{31}^2 & V_{11}\, V_{21} & V_{21}\, V_{31} & V_{31}\, V_{11} \\
W_{11}^2 & W_{21}^2 & W_{31}^2 & W_{11}\, W_{21} & W_{21}\, W_{31} & W_{31}\, W_{11} \\
V_{12}^2 & V_{22}^2 & V_{32}^2 & V_{12}\, V_{22} & V_{22}\, V_{32} & V_{32}\, V_{12} \\
W_{12}^2 & W_{22}^2 & W_{32}^2 & W_{12}\, W_{22} & W_{22}\, W_{32} & W_{32}\, W_{12} \\
V_{13}^2 & V_{23}^2 & V_{33}^2 & V_{13}\, V_{23} & V_{23}\, V_{33} & V_{13}\, V_{33} \\
W_{13}^2 & W_{23}^2 & W_{33}^2 & W_{13}\, W_{23} & W_{23}\, W_{33} & W_{13}\, W_{33}
\end{vmatrix}$$

vanishes.

Proof:- (i) The condition is necessary.

Let the six points P_i, Q_i lie on a conic K given by K:

$$A\, x_1^2 + B\, x_2^2 + C\, x_3^2 + D\, x_1\, x_2 + E\, x_2\, x_3 + F\, x_3\, x_1 = 0.$$

Substituting the coordinates of P_i, Q_i in K gives:

$$P_i: AV_{1i}^2 + BV_{2i}^2 + CV_{3i}^2 + DV_{1i}V_{2i} + EV_{2i}V_{3i} + FV_{3i}V_{1i} = 0$$

$$Q_i: AW_{1i}^2 + BW_{2i}^2 + CW_{3i}^2 + DW_{1i}W_{2i} + EW_{2i}W_{3i} + FW_{3i}W_{1i} = 0$$

$i = 1, 2, 3.$

The above six homogeneous equations in the six unknowns, A,B,C,D,E, and F have a non-trivial solution - i.e. at least one of the unknowns is not zero - if and only if the determinant of their coefficients is equal to zero. Hence the condition is necessary. [1, p.47]

(ii) The condition is sufficient:

If the determinant is zero, then the above six equations have a non-trivial solution, and hence the six points determine a conic.

CHAPTER II

UNEQUAL PARALLEL SEGMENTS

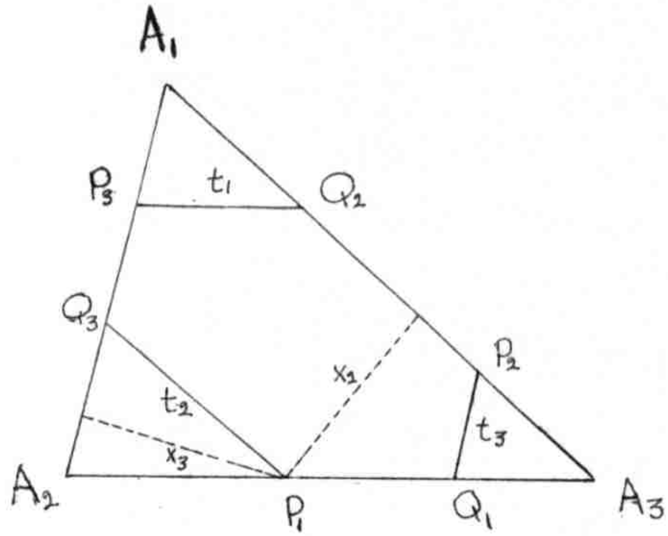

Fig. 2

1. Let P_3 on A_1A_2 and Q_2 on A_1A_3 be selected so that P_3Q_2 is parallel to A_2A_3, and denote the length of P_3Q_2 by t_1. If A_1 lies between P_3Q_2 and A_2A_3, t_1 is considered to be negative. Similarly, we define $P_1Q_3 = t_2$, $P_2Q_1 = t_3$.

2. Coordinates of P_i, Q_i.

$$\sin \delta_1 = \frac{x_3}{t_2}, \quad x_3 = t_2 \sin \delta_1,$$

where x_3 is the perpendicular distance from P_1 to a_3, δ_i being the interior angle at $A_i (i = 1,2,3)$.

$$A_2P_1 = \frac{x_3}{\sin \delta_2} = \frac{t_2 \sin \delta_1}{\sin \delta_2}$$

$$P_1A_3 = a_1 - \frac{t_2 \sin \delta_1}{\sin \delta_2} = \frac{a_1 \sin \delta_2 - t_2 \sin \delta_1}{\sin \delta_2}$$

$$X_2 = P_1A_3 \sin \delta_3 = \sin \delta_3 \frac{a_1 \sin \delta_2 - t_2 \sin \delta_1}{\sin \delta_2},$$

X_2 is the perpendicular distance from P_1 to a_2.

Coordinates of P_1:

$$(X_1, X_2, X_3) = (0, \sin \delta_3 \frac{a_1 \sin \delta_2 - t_2 \sin \delta_1}{\sin \delta_2}, t_2 \sin \delta_1)$$

$$\sim (0, \frac{a_3\ a_1\ (a_2 - t_2)}{a_2}, a_1\ t_2)$$

$$\sim (0, a_3\ (a_2 - t_2), a_2\ t_2).$$

The second result is obtained from the first by using the law of sines:

$$\frac{\sin \delta_1}{\sin \delta_2} = \frac{a_1}{a_2}.$$

In a similar manner or by cyclic permutation of subscripts, the other points can be shown to be as follows:

$P_2\ (a_3\ t_3,\ 0,\ a_1(a_3 - t_3))$

$P_3\ (a_2(a_1 - t_1),\ a_1\ t_1,\ 0)$

$Q_1\ (0,\ a_3\ t_3,\ a_2(a_3 - t_3))$

$Q_2\ (a_3(a_1 - t_1),\ 0,\ a_1\ t_1)$

$Q_3\ (a_2\ t_2,\ a_1(a_2 - t_2),\ 0).$

3. Conic of P_1, Q_1, P_2, Q_2, P_3 and Q_3.

Henceforward this conic will be known as the c-conic of the six points.

a. Existence of the c-conic.

It is sufficient to show that the determinant formed by the P_i and Q_i equal zero. This follows from Statement 1:

The determinant is:

$$\begin{vmatrix} 0 & a_3^2(a_2-t_2)^2 & a_2^2\ t_2^2 & 0 & a_2a_3t_2(a_2-t_2) & 0 \\ 0 & a_3^2\ t_3^2 & a_2^2(a_3-t_3)^2 & 0 & a_2a_3t_3(a_3-t_3) & 0 \\ a_3^2\ t_3^2 & 0 & a_1^2(a_3-t_3)^2 & 0 & 0 & a_1a_3t_3(a_3t_3) \\ a_3^2(a_1-t_1)^2 & 0 & a_1^2\ t_1^2 & 0 & 0 & a_1a_3t_1(a_1-t_1 \\ a_2^2(a_1-t_1)^2 & a_1^2\ t_1^2 & 0 & a_1a_2t_1(a_1-t_1) & 0 & 0 \\ a_2^2\ t_2^2 & a_1^2(a_2-t_2)^2 & 0 & a_1a_2t_2(a_2-t_2) & 0 & 0 \end{vmatrix}$$

Expanding the determinant shows that it is equal zero.

b. The c-conic: Representation

Equation of P_1Q_1 is $x_1 = 0$.

Equation of P_2Q_2 is $x_2 = 0$.

Equation of P_1P_2 is:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 0 & a_3(a_2-t_2) & a_2\ t_2 \\ a_3\ t_3 & 0 & a_1(a_3-t_3) \end{vmatrix} = 0$$

i.e.

$$a_1(a_3 - t_3)\,x_1 + \frac{a_2\,t_2\,t_3}{a_2-t_2}\,x_2 - a_3\,t_3\,x_3 = 0.$$

Equation of Q_1Q_2 is:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 0 & a_3\,t_3 & a_2(a_3-t_3) \\ a_3(a_1-t_1) & 0 & a_1\,t_1 \end{vmatrix} = 0$$

i.e.

$$\frac{a_1\,t_1\,t_3}{a_1-t_1}\,x_1 + a_2(a_3 - t_3)x_2 - a_3\,t_3\,x_3 = 0.$$

Conics through $P_1P_2Q_1Q_2$ are:

$$\left[\frac{a_1\,t_1\,t_3}{a_1-t_1}\;x_1 + a_2(a_3-t_3)x_2 - a_3\,t_3\,x_3\right]\left[a_1(a_3-t_3)x_1 + \frac{a_2t_2t_3}{a_2-t_2}x_2 - a_3t_3x_3\right]$$

$$+ \lambda\,x_1\,x_2 = 0 \quad , \quad \lambda \neq 0 \quad \ldots\ldots\ldots\ldots(1)$$

If P_3 is a point for which $x_1x_2 \neq 0$, there is a unique conic in this family which passes through P_3. If $x_1x_2 = 0$, a degenerate conic is obtained. This will be considered later. To find the value of λ, substitute the coordinates of P_3 in (1) and solve for λ.

Thus:

$$\left[\frac{a_1t_1t_3}{a_1-t_1}.a_2(a_1-t_1)+a_2(a_3-t_3).a_1t_1\right]\left[a_1(a_3-t_3).a_2(a_1-t_1)+\frac{a_2t_2t_3}{a_2-t_2}.a_1t_1\right]$$

$$+ \lambda\,a_1\,t_1.\,a_2(a_1-t_1) = 0 \quad \ldots\ldots\ldots\ldots\ldots(2)$$

from (2)

$$\lambda = \frac{-a_3[a_1a_2(a_1-t_1)(a_2-t_2)(a_3-t_3) + a_1a_2t_1t_2t_3]}{(a_1-t_1)(a_2-t_2)} .$$

Thus the conic: $P_1Q_1P_2Q_2P_3$ is given by

$$\left[\frac{a_1t_1t_3}{a_1-t_1}x_1+a_2(a_3-t_3)x_2-a_3t_3x_3\right]\left[a_1(a_3-t_3)x_1+\frac{a_2t_2t_3}{a_2-t_2}x_2-a_3t_3x_3\right] - \frac{a_1a_2a_3[(a_1-t_1)(a_2-t_2)(a_3-t_3)+t_1t_2t_3]}{(a_1-t_1)(a_2-t_2)}x_1x_2 = 0 \ldots . (3)$$

After removing of the brackets and collecting like terms Equation (3) may be put in the form:

$$\sum \frac{(a_ix_i)^2\ t_i}{a_i-t_i} - \sum a_1a_2\left[1+\frac{t_1\ t_2}{(a_1-t_1)(a_2-t_2)}\right]x_1x_2 = 0 \ \ldots\ldots (4)$$

Where the subscripts are integers, the meaning of Σ is that explained in Chapter I.

The symmetry of this result shows that the equation is also satisfied by the coordinates of Q_3 and therefore represents the c-conic $P_1Q_1P_2Q_2P_3Q_3$.

4. <u>The Center of the C-Conic:</u>

Let the center be (x_1', x_2', x_3').

The polar of the center of a conic c is given by:

$$x_1' \frac{\partial c}{\partial x_1} + x_2' \frac{\partial c}{\partial x_2} + x_3' \frac{\partial c}{\partial x_3} = 0 \qquad [4,p.75].$$

Let c represent let c represent the left-hand side of (4). Then:

$$2\sum_{i=1}^{3}\frac{a_i^2\ t_i}{a_i-t_i}\ x_i'x_i\ -\ \sum_{i<j}a_i\ a_j\left[1+\frac{t_it_j}{(a_i-t_i)(a_j-t_j)}\right]x_j'x_i$$

$$-\ \sum_{i<j}a_i\ a_j\left[1+\frac{t_it_j}{(a_i-t_i)(a_j-t_j)}\right]x_i'x_j\ =\ 0$$

Rewriting the above equation:

$$\sum_{\substack{i=1\\ i\neq j}}^{3}\left(\frac{2a_i^2\ t_i}{a_i-t_i}\ x_i'\ -\ a_ia_j\ \left[1+\ \frac{t_it_j}{(a_i-t_i)(a_j-t_j)}\right]\ x_j'\right)\ x_i=0$$

The polar of the center is the line at infinity i.e. the line

$$a_1\ x_1\ +\ a_2\ x_2\ +\ a_3\ x_3\ =\ 0\ \ldots\ldots\ldots(6)\qquad [4,p.66].$$

Therefore the coefficients of x_1, x_2, x_3 in (5) and (6) are proportional, i.e.

$$\frac{2a_1^2\ t_1}{a_1-t_1}\ x_1'\ -\ a_1a_2\left[1+\frac{t_1t_2}{(a_1-t_1)(a_2-t_2)}\right]\ x_2'\ -\ a_1a_3\ \left[1+\frac{t_1t_3}{(a_1-t_1)(a_3-t_3)}\right]x$$

$$-\ a_1\left[1+\frac{t_1t_2}{(a_1-t_1)(a_2-t_2)}\right]x_1'\ +\ \frac{2a_2t_2}{a_2-t_2}\ x_2'\ -\ a_3\ \left[1+\frac{t_2t_3}{(a_2-t_2)(a_3-t_3)}\right]x_3'$$

$$-\ a_1\left[1+\ \frac{t_1t_3}{(a_1-t_1)(a_3-t_3)}\right]x_1'\ -\ a_2\left[1+\frac{t_2t_3}{(a_2-t_2)(a_3-t_3)}\right]x_2'\ +\ \frac{2a_3t_3}{a_3-t_3}\ x_3'=$$

Consider the determinant of the above equations:

$$\Delta = a_1a_2a_3 \begin{vmatrix} \frac{2t_1}{a_1-t_1} & -\left[1+\frac{t_1t_2}{(a_1-t_1)(a_2-t_2)}\right] & -\left[1+\frac{t_1t_3}{(a_1-t_1)(a_3-t_3)}\right] \\ -\left[1+\frac{t_1t_2}{(a_1-t_1(a_2-t_2)}\right] & \frac{2t_2}{a_2-t_2} & -\left[1+\frac{t_2t_3}{(a_2-t_2)(a_3-t_3)}\right] \\ -\left[1+\frac{t_1t_3}{(a_1-t_1)(a_3-t_3)}\right] & -\left[1+\frac{t_2t_3}{(a_2-t_2)(a_3-t_3)}\right] & \frac{2t_3}{a_3-t_3} \end{vmatrix}$$

If $\Delta \neq 0$, then

$$a_2a_3 x_1' = \frac{\begin{vmatrix} 1 & -\left[1+\frac{t_1t_2}{(a_1-t_1)(a_2-t_2)}\right] & -\left[1+\frac{t_1t_3}{(a_1-t_1)(a_3-t_3)}\right] \\ 1 & \frac{2t_2}{a_2-t_2} & -\left[1+\frac{t_2t_3}{(a_2-t_2)(a_3-t_3)}\right] \\ 1 & -\left[1+\frac{t_2t_3}{(a_2-t_2)(a_3-t_3)}\right] & \frac{2t_3}{a_3-t_3} \end{vmatrix}}{\Delta}$$

x_2' and x_3' are also found by similar expressions.
Simplifying these expressions and rewriting:

$$x_1' \sim \frac{a_1-t_1}{a_1}[a_1a_2^2a_3^2 - a_2^2a_3^2t_1 - (a_1a_2^2t_3^2 + a_1a_3^2t_2^2 - a_2^2a_3t_1t_3 - a_2a_3^2t_1t_2)]$$

advancing the subscripts we get x_2', x_3'. Thus (x_i') give equations that give the center of the c-conic.

CHAPTER III

CONDITIONS AND RESULTS

1. Condition for concurrence of Q_1P_2, Q_2P_3, Q_3P_1.

Equation of Q_1P_2:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 0 & a_3t_3 & a_2(a_3-t_3) \\ a_3t_3 & 0 & a_1(a_3-t_3) \end{vmatrix} = 0$$

i.e.

$$a_1(a_3-t_3)x_1 + a_2(a_3-t_3)x_2 - a_3t_3x_3 = 0 \quad \ldots\ldots\ldots\ldots(1)$$

Equation of Q_2P_3:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_3(a_1-t_1) & 0 & a_1t_1 \\ a_2(a_1-t_1) & a_1t_1 & 0 \end{vmatrix} = 0$$

i.e.

$$-a_1t_1x_1 + a_2(a_1-t_1)x_2 + a_3(a_1-t_1)x_3 = 0 \quad \ldots\ldots\ldots\ldots(2)$$

Equation of Q_3P_1:

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_2t_2 & a_1(a_2-t_2) & 0 \\ 0 & a_3(a_2-t_2) & a_2t_2 \end{vmatrix} = 0$$

i.e.

$$a_1(a_2-t_2)x_1 - a_2t_2x_2 + a_3(a_2-t_2)x_3 = 0 \quad \ldots\ldots\ldots\ldots(3)$$

Q_1P_2, Q_2P_3, Q_3P_1 are concurrent if and only if the determinant of the coefficients in (1), (2), (3) is = 0.

i.e.

$$\begin{vmatrix} a_1(a_3-t_3) & a_2(a_3-t_3) & -a_3t_3 \\ -a_1t_1 & a_2(a_1-t_1) & a_3(a_1-t_1) \\ a_1(a_2-t_2) & -a_2t_2 & a_3(a_2-t_2) \end{vmatrix} = 0$$

Expansion and simplification of the result gives:

$$\underline{a_1a_2t_3 + a_2a_3t_1 + a_1a_3t_2 = 2\,a_1a_2a_3}$$

as the condition required. The point of concurrence is

$$\left(\frac{a_i - t_i}{a_i^2}\right) \quad i = 1,\ 2,\ 3,$$

2. <u>Condition for $P_1Q_1 = P_3Q_2$</u>

$$A_2P_1 = \frac{t_2 \sin \delta_1}{\sin \delta_2}$$

$$Q_1A_3 = \frac{t_3 \sin \delta_1}{\sin \delta_3}$$

$$P_1Q_1 = a_1 - \frac{t_2 \sin \delta_1}{\sin \delta_2} - \frac{t_3 \sin \delta_1}{\sin \delta_3}$$

$$P_3Q_2 = t_1$$

If

$$P_1Q_1 = P_3Q_2,$$

then

$$a_1 - \frac{t_2 \sin \delta_1}{\sin \delta_2} - \frac{t_3 \sin \delta_1}{\sin \delta_3} = t_1 \;\ldots\ldots\ldots\ldots\ldots(1)$$

must hold.

Simplify (1):

$$a_1 \sin \delta_1 \sin \delta_2 - t_2 \sin \delta_1 \sin \delta_3 - t_3 \sin \delta_1 \sin \delta_2 = t_1 \sin \delta_2 \sin \delta$$

$$\frac{a_1}{\sin \delta_3} - \frac{t_2}{\sin \delta_2} - \frac{t_3}{\sin \delta_3} = \frac{t_1}{\sin \delta_1}$$

this gives:

$$\underline{a_2 a_3 t_1 + a_1 a_3 t_2 + a_1 a_2 t_3 = a_1 a_2 a_3}$$

when further simplified.

Thus

$$P_1Q_1 = P_3Q_2$$

if and only if:

$$a_2 a_3 t_1 + a_1 a_3 t_2 + a_1 a_2 t_3 = a_1 a_2 a_3 \;\ldots\ldots\ldots\ldots(2)$$

The symmetry of this condition shows that simultaneously

$$Q_1P_2 = P_3Q_3 \quad \text{and} \quad P_2Q_2 = P_1Q_3.$$

Therefore, (2) is the condition for the hexagon $P_1Q_1P_2Q_2P_3Q_3$ to have each pair of opposite sides equal and parallel. Hence each pair of diagonals P_1Q_2, P_2Q_3, P_3Q_1 are the diagonals of a parallelogram, and these segments bisect each other at the center of the hexagon. This point is found by intersecting P_1Q_2 and P_2Q_3, for example. Their equations are, respectively:

$$a_1t_1(a_2-t_2)x_1 + a_2t_2(a_1-t_1)x_2 - a_3(a_1-t_1)(a_2-t_2)x_3 = 0 \ \ldots(1)$$

$$-a_1(a_2-t_2)(a_3-t_3)x_1 + a_2t_2(a_3-t_3)x_2 + a_3t_3(a_2-t_2)x_3 = 0 \ \ldots(2)$$

These are solved simultaneously to give the center:

$$\left(\frac{a_i-t_i}{a_i^2}\right) i = 1,\ 2,\ 3.$$

3. Condition for the hexagon $P_1Q_1P_2Q_2P_3Q_3$ to circumscribe a conic.

Equation of Q_1P_3 is:

$$a_1t_1(a_3-t_3)x_1 - a_2(a_1-t_1)(a_3-t_3)x_2 + a_3t_3(a_1-t_1)x_3 = 0$$

Equation of P_1Q_2 is:

$$a_1t_1(a_2-t_2)x_1 + a_2t_2(a_1-t_1)x_2 - a_3(a_1-t_1)(a_2-t_2)x_3 = 0$$

Equation of P_2Q_3 is:

$$-a_1(a_2-t_2)(a_3-t_3)x_1 + a_2t_2(a_3-t_3)x_2 + a_3t_3(a_2-t_2)x_3 = 0.$$

If P_1Q_2, P_2Q_3, Q_1P_3 are concurrent, then the hexagon $P_1Q_1P_2Q_2P_3Q_3$ circumscribes a conic [2, p.110]. P_1Q_2, P_2Q_3, Q_1P_3 are concurrent if and only if the determinant of the coefficients of the above equations is 0.

i.e.

$$a_1a_2a_3\begin{vmatrix} t_1(a_3-t_3) & -(a_1-t_1)(a_3-t_3) & t_3(a_1-t_1) \\ t_1(a_2-t_2) & t_2(a_1-t_1) & -a_3(a_1-t_1)(a_2-t_2) \\ -(a_2-t_2)(a_3-t_3) & t_2(a_3-t_3) & t_3(a_2-t_2) \end{vmatrix} = 0$$

Expansion and simplification gives:

$$a_1a_2a_3(a_3-t_3)(a_1-t_1)(a_2-t_2)[a_3t_1t_2-(a_2-t_2)(a_1a_3-a_1t_3-a_3t_1)+a_1t_2t_3$$

Simplifying further:

$$a_1a_2a_3(a_3-t_3)(a_1-t_1)(a_2-t_2)(a_1a_2t_3+a_2a_3t_1+a_1a_3t_2-a_1a_2a_3) = 0$$

i.e.

$$a_1a_2t_3 + a_2a_3t_1 + a_1a_3t_2 = a_1a_2a_3$$

as the condition for the hexagon to circumscribe a conic. Also

$$t_3 = a_3, \; a_1 = t_1 \quad \text{or} \quad a_2 = t_2$$

make the diagonals of the hexagon $P_1Q_1P_2Q_2P_3Q_3$ concur at the center. In this case the hexagon is a degenerate case consisting of two straight lines, one of which contains four of the vertices of the hexagon. Figure 3 illustrates the case

$$t_1 = a_1 \; , \; t_3 \neq a_3 \; , \; t_2 \neq a_2.$$

Let 0 be the center, then:

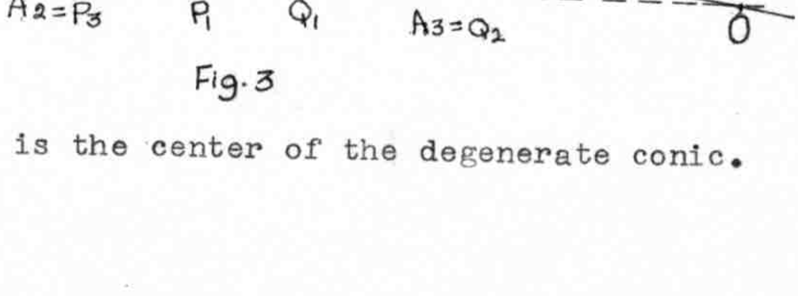

Fig. 3

$0\left(0, -\frac{a_2-t_2}{a_2t_2} \; , \; \frac{a_3-t_3}{a_3t_3}\right)$ is the center of the degenerate conic.

4. Conic Inscribed in the Hexagon.

(This conic will be denoted by I-conic)

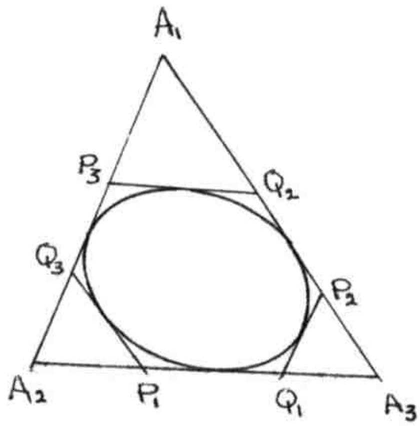

Fig.4

Let (u_1, u_2, u_3) be any point in the plane of $\Delta\ A_1A_2A_3$ but not on one of its sides. M, N, K projections of (u_i) from A_i on a_i $(i = 1, 2, 3)$ respectively.

Statement 1:

The equation of the conic that touches the sides of the triangle $A_1A_2A_3$ at M, N and K is given by:

$$\sum_{i=1}^{3} \left(\frac{x_i}{u_i}\right)^2 - 2 \sum_{i<j} \frac{x_i x_j}{u_i u_j} = 0 \quad \text{...............(1)}$$

Proof:- solve equation (1) simultaneously with $x_1 = 0$: then

$$\left(\frac{x_2}{u_2}\right)^2 + \left(\frac{x_3}{u_3}\right)^2 - 2\left(\frac{x_2}{u_2}\right)\left(\frac{x_3}{u_3}\right) = 0,$$

and hence

$$\left(\frac{x_2}{u_2} - \frac{x_3}{u_3}\right)^2 = 0$$

Hence $x_2 : x_3 = u_2 : u_3$ is a double root.

Therefore, A_2A_3 is tangent to the conic at $(0, u_2, u_3)$. Similarly, A_1A_2 is tangent to the conic at $(u_1, u_2, 0)$ and A_1A_3 is tangent at $(u_1, 0, u_3)$.

<u>Statement 2:</u>

If the center of the above conic is (v_i), then:

$$u_1 \sim \frac{1}{a_1(-a_1v_1 + a_2v_2 + a_3v_3)}$$

$$u_2 \sim \frac{1}{a_2(a_1v_1 - a_2v_2 + a_3v_3)}$$

$$u_3 \sim \frac{1}{a_3(a_1v_1 + a_2v_2 - a_3v_3)}$$

Also if (u_i) is given, the center is given by:

$$\left(\frac{a_2}{u_3} + \frac{a_3}{u_2}\ ,\ \frac{a_3}{u_1} + \frac{a_1}{u_3}\ ,\ \frac{a_1}{u_2} + \frac{a_2}{u_1}\right).$$

or

$$(a_2u_2 + a_3u_3,\ a_3u_3 + a_1u_1,\ a_1u_1 + a_2u_2)$$

Proof:- The line at infinity is given by:

$$a_1x_1 + a_2x_2 + a_3x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$F = \frac{x_1^2}{u_1^2} + \frac{x_2^2}{u_2^2} + \frac{x_3^2}{u_3^2} - 2\,\frac{x_1x_2}{u_1u_2} - 2\,\frac{x_2x_3}{u_2u_3} - 2\,\frac{x_1x_3}{u_1u_3} = 0$$

is the equation of the conic that touches the sides of the triangle of reference.

Let the center of the conic be (v_i), then the polar of (v_i) is of the form:

$$v_1 \frac{\partial F}{\partial x_1} + v_2 \frac{\partial F}{\partial x_2} + v_3 \frac{\partial F}{\partial x_3} = 0 \qquad [4, p.75]$$

i.e.

$$\left(\frac{2x_1}{u_1^2} - \frac{2x_2}{u_1u_2} - \frac{2x_3}{u_1u_3}\right)v_1 + \left(\frac{2x_2}{u_2^2} - \frac{2x_1}{u_1u_2} - \frac{2x_3}{u_2u_3}\right)v_2$$

$$+\left(\frac{2x_3}{u_3^2} - \frac{2x_2}{u_2u_3} - \frac{2x_1}{u_1u_3}\right)v_3 = 0 \text{ ....(2)}$$

rewriting (2):

$$\left(\frac{v_1}{u_1^2} - \frac{v_2}{u_1u_2} - \frac{v_3}{u_1u_3}\right)x_1 + \left(\frac{-v_1}{u_1u_2} + \frac{v_2}{u_2^2} - \frac{v_3}{u_2u_3}\right)x_2$$

$$+\left(\frac{-v_1}{u_1u_3} - \frac{v_2}{u_2u_3} + \frac{v_3}{u_3^2}\right)x_3 = 0 \text{ ....(3)}$$

But the polar of the center is (1) [4, p. 66]

Therefore:

$$\frac{v_1}{u_1^2} - \frac{v_2}{u_1u_2} - \frac{v_3}{u_1u_3} = \lambda a_1$$

$$\frac{-v_1}{u_1u_2} + \frac{v_2}{u_2^2} - \frac{v_3}{u_2u_3} = \lambda a_2$$

$$-\frac{v_1}{u_1u_3} - \frac{v_2}{u_2u_3} + \frac{v_3}{u_3^2} = \lambda a_3 \quad , \lambda \neq 0.$$

$$\Delta = \frac{-4}{u_1^2\ u_2^2\ u_3^2}$$

$$v_1 = \frac{\lambda \begin{vmatrix} a_1 & -\dfrac{1}{u_1u_2} & -\dfrac{1}{u_1u_3} \\ a_2 & \dfrac{1}{u_2^2} & -\dfrac{1}{u_2u_3} \\ a_3 & -\dfrac{1}{u_2u_3} & \dfrac{1}{u_3^2} \end{vmatrix}}{\Delta}$$

$$= \frac{\dfrac{2a_2}{u_1u_2u_3^2} + \dfrac{2\,a_3}{u_1u_2^2u_3}}{\Delta}$$

$$= \frac{2\ \dfrac{a_3u_3 + a_2u_2}{u_1u_2^2u_3^2}}{\Delta}$$

Similarly

$$v_2 = \frac{2\ \dfrac{a_1u_1 + a_3u_3}{u_1^2u_2u_3^2}}{\Delta}$$

$$v_3 = \frac{2\ \dfrac{a_2u_2 + a_1u_1}{u_1^2u_2^2u_3}}{\Delta}$$

Simplify further:

$$v_1 \sim \frac{a_3}{u_2} + \frac{a_2}{u_3}\ , \quad v_2 \sim \frac{a_3}{u_1} + \frac{a_1}{u_3}$$

$$v_3 \sim \frac{a_1}{u_2} + \frac{a_2}{u_1}\ .$$

This proves the second part of statement 2.

To prove the first part of the statement, set:

$$v_1 = \lambda\left(\frac{a_3}{u_2} + \frac{a_2}{u_3}\right)$$

$$v_2 = \lambda\left(\frac{a_3}{u_1} + \frac{a_1}{u_3}\right) \qquad \lambda \neq 0$$

$$v_3 = \lambda\left(\frac{a_2}{u_1} + \frac{a_1}{u_2}\right)$$

$$\Delta = \lambda^3 \begin{vmatrix} 0 & a_3 & a_2 \\ a_3 & 0 & a_1 \\ a_2 & a_1 & 0 \end{vmatrix} = 2a_1a_2a_3\lambda^3$$

$$\frac{1}{u_1} = \frac{\lambda^2 \begin{vmatrix} v_1 & a_3 & a_2 \\ v_2 & 0 & a_1 \\ v_3 & a_1 & 0 \end{vmatrix}}{\Delta}$$

$$= \frac{\lambda^2(-a_1a_1v_1 + a_3a_1v_3 + a_1a_2v_2)}{\Delta}$$

$$= \frac{\lambda^2 a_1(-a_1v_1 + a_3v_3 + a_2v_2)}{\Delta}$$

Similarly

$$\frac{1}{u_2} = \frac{\lambda^2 a_2(a_1v_1 - a_2v_2 + a_3v_3)}{\Delta}$$

$$\frac{1}{u_3} = \frac{\lambda^2 a_3(a_1v_1 + a_2v_2 - a_3v_3)}{\Delta}$$

The result follows immediately.

Under the condition:

$$a_1a_2t_3 + a_2a_3t_1 + a_3a_1t_2 = a_1a_2a_3$$

the center of the I-conic inscribed in the hexagon $P_1Q_1P_2Q_2P_3Q_3$ is the intersection of Q_1P_3 and P_1Q_2.

Q_1P_3: is represented as:

$$a_1t_1(a_3-t_3)x_1 - a_2(a_1-t_1)(a_3-t_3)x_2 + a_3t_3(a_1-t_1)x_3 = 0$$

P_1Q_2: as:

$$a_1t_1(a_2-t_2)x_1 + a_2t_2(a_1-t_1)x_2 - a_3(a_1-t_1)(a_2-t_2)x_3 = 0.$$

Solving for x_1, x_2 and x_3

$$x_1 \sim a_2a_3(a_1-t_1)(a_2a_3-a_3t_2-a_2t_3)$$

$$x_2 \sim a_1a_3^2t_1(a_2-t_2)$$

$$x_3 \sim a_1a_2^2t_1(a_3-t_3)$$

Simplifying:

$$x_1 \sim \frac{a_2a_3}{a_1}(a_1-t_1)(a_1a_2a_3 - a_1a_3t_2 - a_1a_2t_3)$$

But

$$a_1a_2a_3 - a_1a_3t_2 - a_1a_2t_3 = a_2a_3t_1$$

Therefore

$$x_1 \sim \frac{a_2^2a_3^2}{a_1} t_1(a_1-t_1).$$

Multiply the x_i by $\frac{a_1}{t_1}$ then:

$$(x_1,x_2,x_3) \sim (a_2^2a_3^2(a_1-t_1),\ a_1^2a_3^2(a_2-t_2),\ a_1^2a_2^2(a_3 - t_3))$$

$$\sim \left(\frac{a_1-t_1}{a_1^2},\ \frac{a_2-t_2}{a_2^2},\ \frac{a_3-t_3}{a_3^2} \right)$$

as the coordinates of the center. From statement 2,

$$u_1 = \frac{1}{a_1(-\frac{a_1-t_1}{a_1} + \frac{a_2-t_2}{a_2} + \frac{a_3-t_3}{a_3}}$$

$$= \frac{1}{\frac{a_1(-a_2a_3(a_1-t_1)+a_1a_3(a_2-t_2)+a_1a_2(a_3-t_3))}{a_1a_2a_3}}$$

$$u_2 = \frac{1}{\frac{a_2(a_2a_3(a_1-t_1)-a_1a_3(a_2-t_2)+a_1a_2(a_3-t_3))}{a_1a_2a_3}}$$

$$u_3 = \frac{1}{\frac{a_3(a_2a_3(a_1-t_1)+a_1a_3(a_2-t_2)-a_1a_2(a_3-t_3))}{a_1a_2a_3}}$$

Substitution of u_1, u_2, u_3 in

$$\sum_{i=1}^{3} \frac{x_i}{u_1}^2 - 2 \sum_{i<j} \frac{x_i x_j}{u_i u_j} = 0$$

and simplification gives:

$$t_1^2x_1^2 + t_2^2x_2^2 + t_3^2x_3^2 - 2t_1t_2x_1x_2 - 2t_2t_3x_2x_3 - 2t_1t_3x_1x_3 = 0.$$

This may be written as:

$$\sum(t_ix_i)^2 - 2 \sum_{i<j} t_it_jx_ix_j = 0 \text{ ....(3)}$$

(3) represents the I-conic inscribed in the Hexagon. It follows that the points of tangency on A_2A_3, A_3A_1, A_1A_2 are respectively $(0, \frac{1}{t_2}, \frac{1}{t_3})$, $(\frac{1}{t_1}, 0, \frac{1}{t_3})$, $(\frac{1}{t_1}, \frac{1}{t_2}, 0)$.

5. <u>The H-conic determined by $A_1, B_3, A_2, B_1, A_3, B_2$:</u>

B_i intersection of t_jt_k

$i = 1,2,3$, $j = 1,2,3$, $k = 1,2,3$

when $i \neq j \neq k \neq i$

Coordinates of B_i:

$t_3 = Q_1P_2$ has the equation:

$$a_1(a_3-t_3)x_1 + a_2(a_3-t_3)x_2 - a_3t_3x_3 = 0 \text{ ....(1)}$$

$t_2 = P_1Q_3$ has the equation:

$$a_1(a_2-t_2)x_1 - a_2t_2x_2 + a_3(a_2-t_2)x_3 = 0 \text{ ....(2)}$$

$t_1 = P_3Q_2$ has the equation:

$$-a_1t_1x_1 + a_2(a_1-t_1)x_2 + a_3(a_1-t_1)x_3 = 0 \text{ ....(3)}$$

Intersection of t_2 and t_3 gives B_1,

from (1) and (2)

$$x_1 \sim -a_2a_3(a_2a_3 - a_2t_3 - a_3t_2)$$

$$x_2 \sim a_1a_3(a_2a_3 - a_3t_2)$$

$$x_3 \sim a_1a_2(a_2a_3 - a_2t_3)$$

Therefore

$$B_1 \sim \left(-a_2a_3(a_2a_3 - a_2t_3 - a_3t_2),\ a_1a_3^2(a_2-t_2),\ a_1a_2^2(a_3-t_3)\right).$$

Similarly

$$B_2 \sim \left(a_2a_3^2(a_1-t_1),\ -a_1a_3(a_1a_3-a_3t_1-a_1t_3),\ a_1^2a_2(a_3-t_3)\right)$$

$$B_3 \sim \left(a_2^2a_3(a_1-t_1),\ a_1^2a_3(a_2-t_2),\ -a_1a_2(a_1a_2-a_1t_2-a_2t_1)\right).$$

a. The form of a conic that passes through $A_1A_2A_3$:

Let

$$Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_1x_2 + Ex_2x_3 + Fx_1x_3 = 0 \text{ ....(1)}$$

represent the general conic.

Substitution of the coordinates of $A_i (i = 1, 2, 3)$ shows that

$$A = B = C = 0.$$

Therefore,

$$Dx_1x_2 + Ex_2x_3 + Fx_1x_3 = 0 \text{ ....(2)}$$

is the equation of the family of conics that passes through

$$A_i,\ i = 1, 2, 3.$$

To get a symmetric form, we take

$$E = u_1, \quad F = u_2, \quad D = u_3.$$

Then (2) takes the form:

$$u_1x_2x_3 + u_2x_1x_3 + u_3x_1x_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2')$$

or

$$\sum_{i=1}^{3} \frac{u_i}{x_i} = 0 \qquad \text{except when } x_1x_2x_3 = 0$$

b. The conic that passes through A_1, A_2, A_3, B_1 and B_3:

Substituting the coordinates of B_1, B_2 in (2') gives:

$$u_1[(a_2-t_2)(a_3-t_3)a_1^2a_2^2a_3^2] + u_2[-a_2a_3(a_2a_3-a_2t_3-a_3t_2)(a_3-t_3)a_1a_2^2]$$
$$+u_3[-a_2a_3(a_2a_3-a_2t_3-a_3t_2)(a_2-t_2)a_1a_3^2]=0 \quad \ldots (3)$$

$$u_1[-a_1^3a_2a_3(a_1a_3-a_3t_1-a_1t_3)(a_3-t_3)] + u_2[a_1^2a_2^2a_3^2(a_3-t_3)(a_1-t_1)]$$
$$+u_3[-a_2a_3^2(a_1-t_1)(a_1a_3-a_3t_1-a_1t_3)a_1a_3]=0 \quad \ldots (4)$$

Solving equations (3) and (4) for u_1, u_2, u_3:

$$u_1 \sim a_1^2a_2^3a_3^4(a_1-t_1)(a_3-t_3)(a_2a_3-a_2t_3-a_3t_2)x$$
$$[a_2a_3(2a_1-t_1)-a_1(a_3t_2+a_2t_3)]$$

$$u_2 \sim a_1^3a_2^2a_3^4(a_2-t_2)(a_3-t_3)(a_1a_3-a_3t_1-a_1t_3)x$$
$$[a_2a_3(2a_1-t_1)-a_1(a_3t_2+a_2t_3)]$$

$$u_3 \sim a_1^4a_2^4a_3^2(a_3-t_3)^2\, t_3\, [a_2a_3(2a_1-t_1) - a_1(a_3t_2+a_2t_3)]$$

Simplifying further:

$$(u_1,u_2,u_3)\sim\Big(a_2a_3^2(a_1-t_1)(a_2a_3-a_2t_3-a_3t_2),a_1a_3^2(a_2-t_2)(a_1a_3-a_3t_1-a_1t$$
$$a_1^2a_2^2t_3(a_3-t_3)\Big).$$

Substituting for u_1,u_2,u_3 in (2') gives:

$$a_1^2a_2^2t_3(a_3-t_3)x_1x_2 + a_2a_3^2(a_1-t_1)(a_2a_3-a_2t_3-a_3t_2)x_2x_3$$
$$+ a_1a_3^2(a_2-t_2)(a_1a_3-a_3t_1-a_1t_3)x_3x_1 = 0.$$

Divide each term in the above equation by:

$$a_1^2a_2^2a_3^2(a_1-t_1)(a_2-t_2)(a_3-t_3):$$

$$\frac{t_3}{a_3^2(a_1-t_1)(a_2-t_2)}x_1x_2 + \frac{a_2a_3-a_2t_3-a_3t_2}{a_1^2a_2(a_2-t_2)(a_3-t_3)}x_2x_3$$
$$+\frac{(a_1a_3-a_3t_1-a_1t_3)}{a_1a_2^2(a_1-t_1)(a_3-t_3)}x_1x_3 = 0 \;\dots\dots\dots\dots\dots(6)$$

(6) represents the conic that passes through A_1,A_2,A_3,B_1 and B_2.

c. When does (6) pass through B_3?

Substitute the coordinates of B_3 in (6) and simplify. This gives:

$$a_1^2a_2^2t_3(a_3-t_3) - a_3(a_1a_2-a_1t_2-a_2t_1)[2a_1a_2(a_3-t_3)-a_3(a_1t_2+a_2t_1)]=0.$$

Expansion and simplification gives:

$$2 = 3\sum_{i=1}^{3}\frac{t_i}{a_i} - \sum_{i=1}^{3}\frac{t_i}{a_i}^{2} - 2\sum_{i<j}\frac{t_it_j}{a_ia_j} \;\dots\dots\dots\dots(7)$$
$$= 3\sum_{i=1}^{3}\frac{t_i}{a_i} - \left(\sum_{i=1}^{3}\frac{t_i}{a_i}\right)^2.$$

Therefore

$$\sum_{i=1}^{3} \frac{t_i}{a_i}^{2} - 3 \sum_{i=1}^{3} \frac{t_i}{a_i} + 2 = 0.$$

Factoring:

$$\left(\sum_{i=1}^{3} \frac{t_i}{a_i} - 1\right)\left(\sum_{i=1}^{3} \frac{t_i}{a_i} - 2\right) = 0.$$

Therefore

$$\sum_{i=1}^{3} \frac{t_i}{a_i} = 1 \quad \ldots\ldots\ldots(8)$$

or

$$\sum_{i=1}^{3} \frac{t_i}{a_i} = 2 \quad \ldots\ldots\ldots(9)$$

i.e.

$$a_2a_3t_1+a_1a_3t_2+a_1a_2t_3 = a_1a_2a_3 \quad \text{or} \quad 2a_1a_2a_3.$$

But (8) is the condition that the hexagon $P_1Q_1P_2Q_2P_3Q_3$ to have each pair of opposite sides equal, and also to circumscribe a conic where as (9) is the condition for the lines P_1Q_3, P_3Q_2, P_2Q_1 to concur.

Therefore (9) gives a trivial case, since under this condition B_i, i=1, 2, 3 go to one point only, namely $\left(\frac{a_i - t_i}{a_i^2}\right)$ i=1, 2, 3, the point of concurrence of P_1Q_3, P_3Q_2, and P_2Q_1.

The above discussion leads us to state:

$A_1, A_2, A_3, B_1, B_2, B_3$ determine a conic if and only if the hexagon $P_1Q_1P_2Q_2P_3Q_3$ circumscribes a conic.

The condition for $P_1Q_1 = P_3Q_2$, i.e. $\sum_{i=1}^{3} \frac{t_i}{a_i} = 1$, gives:

$$(u_1,u_2,u_3) \sim \left(\frac{a_2^2a_3^3t_1(a_1-t_1)}{a_1} , \frac{a_1^2a_3^3t_2(a_2-t_2)}{a_2} , a_1^2a_2^2t_3(a_3-t_3) \right)$$

$$\sim \left(\frac{t_i(a_i-t_i)}{a_i^3} \right) \quad i = 1,\ 2,\ 3$$

Then the conic that passes through A_1, A_2, A_3, B_1, B_2 takes the form:

$$\frac{t_1(a_1-t_1)}{a_1^3} x_2x_3 + \frac{t_2(a_2-t_2)}{a_2^3} x_1x_3 + \frac{t_3(a_3-t_3)}{a_3^3} x_1x_2 = 0$$

i.e.

$$\sum_{i=1}^{3} \frac{t_i(a_i-t_i)}{a_i^3 x_i} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

It follows from the symmetry of equation (7) that B_3 is also on the conic.

Equation (7) represents the simplest form of the H-conic determined by A_1, A_2, A_3, B_1, B_2 and B_3.

6. Concurrence of A_iB_i.

Equation of A_1B_1 =

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 1 & 0 & 0 \\ -a_2a_3(a_2a_3-a_2t_3-a_3t_2) & a_1a_3^2(a_2-t_2) & a_1a_2^2(a_3-t_3) \end{vmatrix} = 0$$

i.e.

$$0.\ x_1 - a_2^2(a_3-t_3)x_2 + a_3^2(a_2-t_2)x_3 = 0$$

Similarly

Equation of A_2B_2:

$$a_1^2(a_3-t_3)x_1 + 0.x_2 - a_3^2(a_1-t_1)x_3 = 0$$

$$A_3B_3: -a_1^2(a_2-t_2)x_1 + a_2^2(a_1-t_1)x_2 + 0.x_3 = 0.$$

Determinant of the equations of A_iB_i is :

$$\begin{vmatrix} 0 & -a_2^2(a_3-t_3) & a_3^2(a_2-t_2) \\ a_1^2(a_3-t_3) & 0 & -a_3^2(a_1-t_1) \\ -a_1^2(a_2-t_2) & a_2^2(a_1-t_1) & 0 \end{vmatrix}$$

Expanding gives:

$$-a_1^2a_2^2a_3^2(a_1-t_1)(a_2-t_2)(a_3-t_3) + a_1^2a_2^2a_3^2(a_1-t_1)(a_2-t_2)(a_3-t_3)$$

which is 0.

Therefore the straight lines A_iB_i are concurrent irrespective of any condition on the t_i.

Concurrence point of A_iB_i:

$$A_1B_1: \ -a_2^2(a_3-t_3)x_2 + a_3^2(a_2-t_2)x_3 = 0 \ \ldots\ldots\ldots\ldots (1)$$

$$A_2B_2: \ a_1^2(a_3-t_3)x_1 - a_3^2(a_1-t_1)x_3 = 0 \ \ldots\ldots\ldots\ldots (2)$$

$$\frac{x_1}{x_3} = \frac{a_3^2(a_1-t_1)}{a_1^2(a_3-t_3)} \qquad \text{from (2)}$$

$$\frac{x_2}{x_3} = \frac{a_3^2(a_2-t_2)}{a_2^2(a_3-t_3)} \qquad \text{from (1)}$$

Therefore the intersection or concurrence of A_1B_1, A_2B_2 is

$$(x_1, x_2, x_3) \sim \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_3}{x_3} \right)$$

$$= \left(\frac{a_3^2(a_1-t_1)}{a_1^2(a_3-t_3)}, \frac{a_3^2(a_2-t_2)}{a_2^2(a_3-t_3)}, 1 \right)$$

$$\sim \left(\frac{a_1-t_1}{a_1^2}, \frac{a_2-t_2}{a_2^2}, \frac{a_3-t_3}{a_3^2} \right).$$

7. Concurrence of A_iB_i at the centroid.

As a special case, consider the concurrence of A_iB_i at the centroid:

The centroid is $(\frac{1}{a_i})$ $i = 1, 2, 3$.

If A_iB_i concur at the centroid, then

$$\frac{a_i-t_i}{a_i^2} = \frac{\lambda}{a_i} \qquad \lambda \neq 0, \ i = 1, 2, 3.$$

The above relations give:

$$t_i = a_i(1 - \lambda) \quad i = 1, 2, 3$$

or

$$\frac{t_1}{a_1} = \frac{t_2}{a_2} = \frac{t_3}{a_3},$$

which implies that the segments t_i cut equal areas from triangle $A_1A_2A_3$.

Conversely, assume that t_1, t_2, t_3 cut equal areas from triangle $A_1A_2A_3$. We are interested to see whether A_iB_i (B_i defined as before) concur at the centroid or not.

It is easy to see that $t_i = ua_i$, $i = 1,2,3$, $u \neq 0$.

Then

$$B_1 \sim \left(\frac{2u-1}{a_1}, \frac{1-u}{a_2}, \frac{1-u}{a_3} \right)$$

$$B_2 \sim \left(\frac{1-u}{a_1}, \frac{2u-1}{a_2}, \frac{1-u}{a_3} \right)$$

$$B_3 \sim \left(\frac{1-u}{a_1}, \frac{1-u}{a_2}, \frac{2u-1}{a_3} \right)$$

Equation of A_1B_1 takes the form:

$$0.\ x_1 - \frac{1-u}{a_3}x_2 + \frac{1-u}{a_2}x_3 = 0$$

i.e.

$$a_2x_2 - a_3x_3 = 0 \quad \dots\dots\dots\dots\dots\dots(1)$$

But (1) is the equation of the median from A_1 to a_1.

By symmetry A_2B_2 and A_3B_3 are also medians. Therefore A_iB_i are concurrent at the centroid. Thus we have the following result:

A_iB_i concur at the centroid if and only if the t_i cut equal areas from $A_1A_2A_3$.

Also we show that the centroid of $A_1A_2A_3$ is the centroid of $B_1B_2B_3$ and the sides of $A_1A_2A_3$ cut equal areas from $B_1B_2B_3$ when t_i cut equal areas from $A_1A_2A_3$.

Let

$$A_1B_1 .\ A_2A_3 = M, \quad A_1B_1 .\ B_2B_3 = N$$

$$\Delta\ A_1A_2M \sim \Delta\ B_1B_2N$$

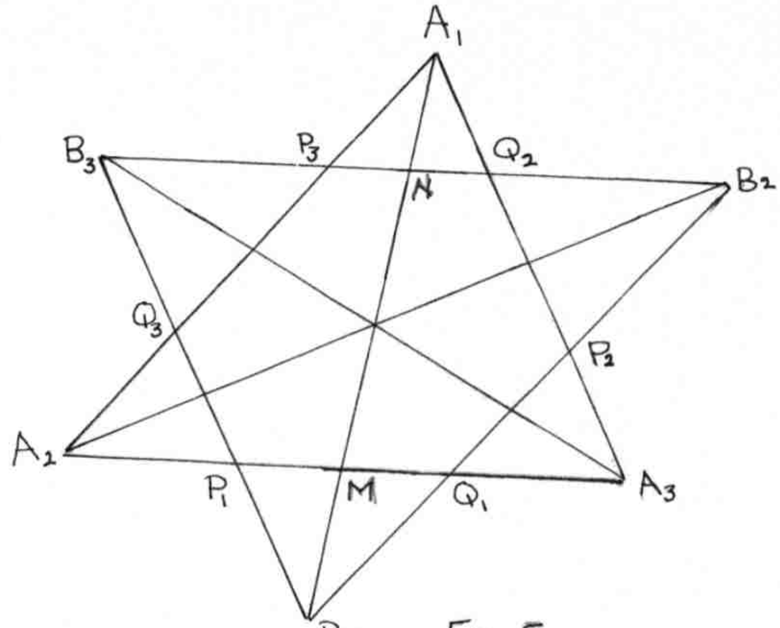

Fig. 5

Therefore

$$\frac{A_1A_2}{B_1B_2} = \frac{A_2M}{B_2N}$$

But

$$\frac{A_1A_2}{B_1B_2} = \frac{A_2A_3}{B_2B_3}\ , \ \Delta\ A_1A_2A_3 \sim \Delta\ B_1B_2B_3$$

Hence

$$\frac{A_2M}{B_2N} = \frac{A_2A_3}{B_2B_3}\ .$$

Therefore N is the midpoint of B_2B_3 since M is the midpoint of A_2A_3.

That is A_1B_1 coincides with the medians of $A_1A_2A_3$ and $B_1B_2B_3$ from A_1 and B_1.

Similarly A_2B_2, A_3B_3 are the medians from A_2, A_3, B_2 B_3 of triangles $A_1A_2A_3$ and $B_1B_2B_3$ respectively. But N is the midpoint of P_3Q_2.

Therefore

$$P_3B_3 = Q_2B_2$$

Hence

$$\Delta\ P_3B_3Q_3 \;=\; \Delta\ Q_2B_2P_2$$

Similarly

$$\Delta\ P_3B_3Q_3 \;=\; \Delta\ P_1B_1Q_1.$$

8. The Conic, Center and Envelope when A_iB_i concur at the Centroid.

The conic:

When A_iB_i concur at the centroid, $t_2 = a_2 \frac{t_1}{a_1}$, $t_3 = a_3 \frac{t_1}{a_1}$. Substituting these values in the equation of the conics that pass through $P_1Q_1P_2Q_2P_3Q_3$ gives:

$$\sum_{i=1}^{3} (a_ix_i)^2 - \left[\frac{a_1-t_1}{t_1} + \frac{t_1}{a_1-t_1}\right] \sum_{i<j} a_ia_jx_ix_j = 0$$

as the equation of the conic when A_iB_i concur at the centroid.

The Center:

In chapter 2, the coordinates of the center were found. Replacing t_2, t_3 by $a_2 \frac{t_1}{a_1}$, $a_3 \frac{t_1}{a_1}$ respectively gives:

$$x_1 \sim a_2^2a_3^2(a_1-t_1)$$

$$x_2 \sim a_1a_2a_3^2(a_1-t_1)$$

$$x_3 \sim a_1a_2^2a_3(a_1-t_1)$$

Therefore $(x_i) \sim (a_2^2a_3^2,\ a_1a_2a_3^2,\ a_1\ a_2^2a_3) \sim (\frac{1}{a_i})$,

which is the centroid.

The Envelope:

Consider the conic given on page 34:

$$\sum_{i=1}^{3}(a_i x_i)^2 - \left[\frac{a_1-t_1}{t_1} + \frac{t_1}{a_1-t_1}\right] \sum_{i<j} a_i a_j x_i x_j = 0 \quad .....(2)$$

Differentiating with respect to t_1 gives:

$$-\left[\frac{-a_1}{t_1^2} + \frac{a_1}{(a_1-t_1)^2}\right] \sum_{i<j} a_i a_j x_i x_j = 0$$

i.e.

$$\left[\frac{a_1}{t_1^2} - \frac{a_1}{(a_1-t_1)^2}\right] \text{ or } \sum_{i<j} a_i a_j x_i x_j = 0.$$

Thus $t_1 = \frac{a_1}{2}$, and hence $t_2 = \frac{a_2}{2}$, $t_3 = \frac{a_3}{2}$. That is the t_i are the midlines of triangle $A_1A_2A_3$. Replacing t_1 by $\frac{a_1}{2}$ in equation (2) gives the envelope as:

$$\sum_{i=1}^{3}(a_i x_i)^2 - 2\sum_{i<j} a_i a_j x_i x_j = 0 .$$

This conic is tangent to the sides of $A_1A_2A_3$ at their midpoints and is known as the inscribed Steiner ellipse. The points of tangency of conic (2) to the envelope are always imaginary except in the case $t_1 = \frac{a_1}{2}$, when the conic coincides with the envelope.

CHAPTER IV

EQUAL AND PARALLEL SEGMENTS

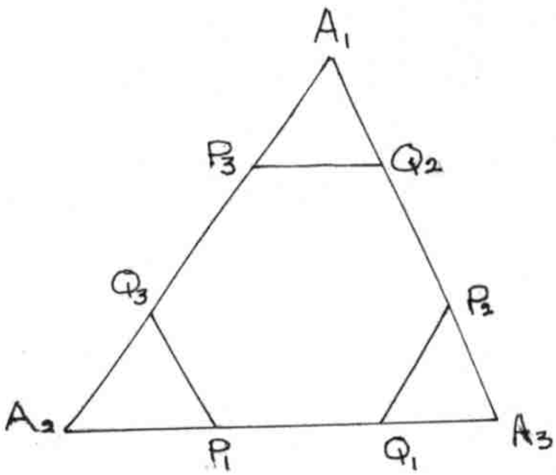

Fig. 6

1. Let P_3 on A_1A_2 and Q_2 on A_1A_3 be selected so that P_3Q_2 is parallel to A_2A_3 and denote the length of P_3Q_2 by t. If A_1 lies between P_3Q_2 and A_2A_3, t is considered to be negative. Similarly we define $P_1Q_3 = t$, $P_2Q_1 = t$.

2. Coordinates of P_i, Q_i.

The coordinates are found as in chapter III, by replacing t_i, $i = 1, 2, 3$, by t. Thus the coordinates are:

$P_1 \quad (0, a_3(a_2-t), a_2t)$

$P_2 \quad (a_3t, 0, a_1(a_3-t))$

$P_3 \quad (a_2(a_1-t), a_1t, 0)$

Q_1 $(0,\ a_3t,\ a_2(a_3-t)$

Q_2 $(a_3(a_1-t),\ 0,\ a_1t)$

Q_3 $(a_2t,\ a_1(a_2-t),\ 0)$.

3. Conic of $P_1, Q_1, P_2, Q_2, P_3, Q_3$. (Hence forward, this conic will be denoted by t-conic).

Replacing t_i, $i = 1, 2, 3$, by t in the equation of the c-conic in chapter III, we get the t-conic:

$$t\sum_{i=1}^{3}\frac{(a_i x_i)^2}{a_i-t} - \sum_{i<j} a_i a_j \left[\frac{t^2}{(a_i-t)(a_j-t)} + 1\right] x_i x_j = 0.$$

Center of the t-conic:

Denote by (x_1', x_2', x_3') the center of the t-conic. The same procedure followed in chapter III for finding the center gives:

$$x_1' \sim \frac{a_1-t}{a_1}\,[(a_2a_3^2+a_2^2a_3-a_1a_3^2-a_1a_2^2)t^2 - a_2^2a_3^2t + a_1a_2^2a_3^2].$$

The other two coordinates of the center are obtained by advanding the subscripts in the expression of x_1'.

4. The Locus of the Center of the t-conic.

The locus is given by parametric representation of the coordinates:

$$x_1' \sim \frac{t-a_1}{a_1}\,[(a_1a_2^2+a_3^2a_1-a_2a_3^2-a_2^2a_3)t^2 + a_2^2a_3^2t - a_1a_2^2a_3^2],$$

x'_2, x'_3 being obtained by advancing the subscripts. The locus of (x'_i) is a cubic curve, but elimination of t is too tedious to carry out here.

Intersection of the Locus and $\Sigma\, c_i x_i = 0$:

Replacing x_i by the expressions of x'_i, $(i=1,2,3)$, in $\Sigma\, c_i x_i = 0$ and arranging the result we get:

$$\sum c_1[a_2^2+a_3^2-\frac{a_2a_3}{a_1}(a_2+a_3)]t^3+\sum c_1[a_2a_3^2+a_2^2a_3-a_1a_3^2-a_1a_2^2+\frac{a_2^2a_3^2}{a_1}]t^2$$
$$-\,2a_1^2a_2^2a_3^2\sum\frac{c_i}{a_i^2}\,t$$
$$+\,a_1^2a_2^2a_3^2\sum\frac{c_i}{a_i} = 0 \quad \ldots\ldots\ldots\ldots(1)$$

where the subscripts are given as actual integers and the meaning of Σ given in Chapter I is applied here. Equation (1) gives $t = 0$ if and only if

$$a_1^2a_2^2a_3^2\sum\frac{c_i}{a_i} = 0$$

i.e.

$$\sum\frac{c_i}{a_i} = 0.$$

Hence the centroid $(\frac{1}{a_i})$ is on the straight line. Also equation (1) gives $t = 0$ twice if and only if

$$\sum\frac{c_i}{a_i} = 0 \quad\text{and}\quad \sum\frac{c_i}{a_i^2} = 0$$

i.e.

$$(c_i)\sim[a_1^2(a_3-a_2),\ a_2^2(a_1-a_3),\ a_3^2(a_2-a_1)].$$

Therefore in this case $\Sigma\, c_i x_i = 0$ is represented as:

$$a_1^2(a_3-a_2)x_1 + a_2^2(a_1-a_3)x_2 + a_3^2(a_2-a_1)x_3 = 0 \quad \ldots\ldots\ldots\ldots (2)$$

(2) Is tangent to the locus when $t = 0$, that is at the centroid $(\frac{1}{a_i})$. Besides the double root $t = 0$, this tangent cuts the curve at the point $t = \frac{a_1a_2a_3}{a_1a_2+a_2a_3+a_3a_1}$.

When $t \to 0$, then the t-conic approaches the Steinerellipse circumscribed about $A_1A_2A_3$ and the center $(x_i')\sim(\frac{1}{a_i})$, the centroid. But if $t = a_i$, then $x_i' = 0$. In particular if $t = a_1$, then $x_1' = 0$, and $x_2' \sim \frac{1}{a_2(a_3-a_1)}$, $x_3'\sim\frac{1}{a_3(a_1-a_2)}$.

That is

$$(x_i')\sim(0,\ \frac{1}{a_2(a_3-a_1)},\ \frac{1}{a_3(a_1-a_2)})$$

which is a point on A_2A_3. Similar results are obtained when $t = a_2$ or a_3.

5. <u>Nature of the t-conic when $t \to 0$, $t \to a_i$.</u>

When $t \to 0$, P_1 and $Q_3 \to A_2$

P_2 and $Q_1 \to A_3$

P_3 and $Q_2 \to A_1$

Thus the six points go to the vertices of $A_1A_2A_3$. Although three points do not determine a unique conic, the t-conic approaches a limiting position in which the tangent at each vertex is parallel to the opposite side, since the chord P_iQ_j is always parallel to A_iA_j. Its equation is obtained simply by setting $t = 0$ in the equation of the t-conic, which gives:

$$a_1a_2x_1x_2 + a_2a_3x_2x_3 + a_1a_3x_1x_3 = 0.$$

As noted earlier, the center is the centroid, $(\frac{1}{a_i})$. This conic is known as Steinmer's circumscribed ellipse.

When $t = a_i$, the t-conic is reduced into a degenerate case of two straight lines and the center of the conic becomes their intersection. For example $t = a_1$ gives:

$$x_1[(a_2-a_1)(a_3-a_1)x_1 - a_2(a_3-a_1)x_2 - a_3(a_2-a_1)x_3] = 0$$

which is obviously the two lines $x_1 = 0$ (the line A_2A_3) and

$$(a_2-a_1)(a_3-a_1)x_1 - a_2(a_3-a_1)x_2 - a_3(a_2-a_1)x_3 = 0.$$

Their intersection is

$$(0,\ a_3(a_2-a_1),\ a_2(a_1-a_3)).$$

The intersection of the t-conic and the line at infinity is given by the equation:

$$a_1^3a_3(a_2-t)x_1^2 + a_2^3a_3(a_1-t)x_2^2$$
$$+ a_1a_2(a_1a_2a_3-(a_2a_3+a_1a_3-a_1a_2)t)x_1x_2=0 \quad (1)$$

If we consider $A_1A_2A_3$ to be equilateral, then equation (1) takes the form $x_1^2 + x_1x_2 + x_3^2 = 0$ which shows that the t-conic and line at infinity intersect at imaginary points and hence the t-conic is an ellipse. In fact the intersection is given by:

$$\left(\frac{x_1}{x_2}, 1, \frac{x_3}{x_2}\right) \sim (-1 \pm \sqrt{3}i, 2, -1 \mp \sqrt{3}\, i), \text{ where } i = \sqrt{-1}.$$

Consider $A_1A_2A_3$ to be isosceles such that $a_2 = a_3 \neq a_1$. Equation (1) takes the form:

$$a_1^3(a_3-t)x_1^2 + a_3^3(a_1-t)x_2^2 + a_1a_3^2(a_1-t)x_1x_2 = 0 \text{ ....(2)}$$

The discriminant of (2) is:

$$d = a_1^2a_3^4(a_1-t)^2 - 4a_1^3a_3^3(a_3-t)(a_1-t)$$

i.e.

$$d = a_1^2a_3^3(a_1-t)[(4a_1-a_3)t - 3a_1a_3].$$

In theory three cases may be considered:

a. $d = 0$

b. $d > 0$

c. $d < 0$

Let us see where these cases lead us:

a. Let $d = 0$, then

$$a_1^2a_3^3(a_1-t)(4a_1-a_3)t - 3a_1a_3] = 0$$

which is true if $t = a_1$

or $$t = \frac{3a_1a_3}{4a_1-a_3}$$

in the first case when $t = a_1$, $P_3 \to A_2$ and $Q_2 \to A_3$. Thus

P_1, Q_1, P_3, Q_2 lie on $x_1 = 0$ and therefore a degenerate case arises such that two intersecting lines determine the conic.

In the second case if we are able to choose the points P_i, Q_i so that the length of each parallel segments is

$$t = \frac{3a_1a_3}{4a_1-a_3},$$

then we will have a parabola through the six points P_i, Q_i $i = 1, 2, 3$. This is always possible unless $a_3 = 4a_1$. In this case P_1 has the coordinates $(0, (a_1-a_3), 3a_1)$; similar expressions may be found for the other points.

We discuss now the third case i.e. $d < 0$, omitting the second case.

c. $d < 0$

Then

$$a_1^2a_3^3(a_1-t)[(4a_1-a_3)t - 3a_1a_3] < 0 \quad \ldots\ldots\ldots\ldots(1)$$

(1) may be put in the form:

$$(a_3-4a_1)t^2 + 2a_1(2a_1+a_3)t - 3a_1^2a_3 < 0 \quad \ldots\ldots\ldots(2)$$

(2) is a quadratic inequation in t. The sign of a quadratic equation is the same as the sign of the coefficient of the second degree term outside the roots and opposite to it inside the roots. But the roots of (2) are

$$t = a_1 \quad \text{and} \quad t = \frac{3a_1a_3}{4a_1-a_3} .$$

Therefore the t-conic is an ellipse if and only if:

(i) $4a_1 - a_3 < 0$ and t between a_1 and $\dfrac{3a_1a_3}{4a_1-a_3}$ or

(ii) $4a_1 - a_3 > 0$ and t is not between a_1 and $\dfrac{3a_1a_3}{4a_1-a_3}$.

i.e.

$$t < \frac{3a_1a_3}{4a_1-a_3} \text{ or } t > a_1.$$

6. Restrictions on t and Conclusions.

In this section we will not take the trouble of deriving the results, since the procedure will be similar to that in chapter III of this paper.

(i) If

$$t = \frac{a_1a_2a_3}{a_1a_2+a_2a_3+a_3a_1},$$

then $P_1Q_1P_2Q_2P_3Q_3$ has equal sides.

(ii) Again if

$$t = \frac{a_1a_2a_3}{a_1a_2+a_2a_3+a_3a_1},$$

then the hexagon $P_1Q_1P_2Q_2P_3Q_3$ touches a proper conic. This is because P_1Q_2, P_3Q_1 and P_2Q_3 are concurrent if and only if

$$t = \frac{a_1a_2a_3}{a_1a_2+a_2a_3+a_3a_1}.$$

Thus under this condition Brianchon's theorem applies and the hexagon touches a proper conic [2,p.110]. Also when $t = a_i$, $i = 1, 2, 3$, P_1Q_2, P_3Q_1 and P_2Q_3 are concurrent. But in this case the hexagon is a degenerate case, two vertices on one line and four on another.

(iii). Intersection of P_1Q_2, P_3Q_1, P_2Q_3:

Denote by (x_1, x_2, x_3) the intersection, we find that

$$(x_1, x_2, x_3) \sim \left(\frac{a_1 - t}{a_1^2}, \frac{a_2 - t}{a_2^2}, \frac{a_3 - t}{a_3^2} \right),$$

which reduces to $(a_2+a_3,\ a_3+a_1,\ a_1+a_2)$.

The point of intersection is known as Brianchon point. This point is also the center of the t-conic when $P_1Q_1 = P_3Q_2$ i.e.

$$t = \frac{a_1a_2a_3}{a_1a_2+a_2a_3+a_3a_1} .$$

(iv). Intersection of the parallel segments, P_1Q_3, P_2Q_1, P_3Q_2: If

$$t = \frac{2a_1a_2a_3}{a_1a_2+a_2a_3+a_3a_1},$$

then the parallel segments intersect at the point:

$$\left(a_2+a_3 - \frac{a_2a_3}{a_1},\ a_1+a_3 - \frac{a_1a_3}{a_2},\ a_2+a_1 - \frac{a_1a_2}{a_3} \right).$$

7. <u>The conic inscribed in the hexagon $P_1Q_1P_2Q_2P_3Q_3$.</u>

I. In part (iii) above, it is mentioned that $P_1Q_1P_2Q_2P_3Q_3$ circumscribes a conic when

$$t = \frac{a_1a_2a_3}{a_1a_2+a_2a_3+a_3a_1} .$$

In this section we proceed to find this conic. For convenience, we denote this conic by B-conic.

In chapter III, it is mentioned that the I-conic takes the form:

$$\sum_{i=1}^{3}\left(\frac{x_i}{u_i}\right)^2 - 2 \sum_{i<j} \frac{x_i x_j}{u_i u_j} = 0 \quad \text{..........(1)}$$

where (u_i) as defined before.

The center of the B-conic is the intersection of P_1Q_2 and P_2Q_3 which is $(a_2+a_3,\ a_3+a_1,\ a_1+a_2)$.

Applying the relation between u_i and x_i, $i = 1, 2, 3$, which is given in chapter III, we get:

$$u_i = \frac{1}{2a_1a_2a_3t}\ ,\ i = 1, 2, 3.$$

Substituting the values for u_i in (1), and simplifying we get:

$$x_1^2+x_2^2+x_3^2-2x_1x_2-2x_2x_3-2x_1x_3 = 0 \quad \text{..........(2)}$$

as the B-conic touching the sidesof the hexagon.

Rewriting (2)

$$\sum_{i=1}^{3} (x_i)^2 - 2 \sum_{i<j} x_i x_j = 0.$$

The B-conic touches the sides of $A_1A_2A_3$ at $C_1(0,1,1)$, $C_2(1,0,1)$, $C_3(1,1,0)$.

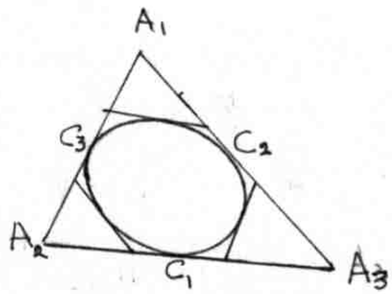

Fig. 7

The equations of A_iC_i $i = 1, 2, 3$, are

$$x_2 - x_3 = 0$$

$$x_1 - x_3 = 0$$

$$x_1 - x_2 = 0$$

respectively. These equations represent the bisectors of the angles of $A_1A_2A_3$. They concur at the incenter (1,1,1) of $A_1A_2A_3$.

8. <u>Intersection of the B-conic and the line at infinity.</u>

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

is the line of infinity

$$x_3 = - \frac{a_1x_1+a_2x_2}{a_3} .$$

Substitute in equation (2) for x_3, we get:

$$(x_1-x_2)^2 + \frac{(a_1x_1+a_2x_2)^2}{a_3^2} + \frac{2}{a_3}(a_1x_1+a_2x_2)(x_1+x_2) = 0.$$

Simplifying we get:

$$\left(\frac{a_1+a_3}{a_3}\right)^2 x_1^2 - 2\left(1 - \frac{a_1}{a_3} - \frac{a_2}{a_3} - \frac{a_1a_2}{a_3^2}\right)x_1x_2 + \left(\frac{a_2+a_3}{a_3}\right)^2 x_2^2 = 0 \quad ...(3)$$

The discriminant of (3) is

$$4\left[\frac{-4a_1}{a_3} - \frac{4a_2}{a_3}\right] = - 16 \quad \frac{a_1+a_2}{a_3}$$

which is obviously negative.

Therefore the B-conic is an ellipse.

In fact, it is tangent to the sides of $A_1A_2A_3$ at the points where they are cut by the angle bisectors.

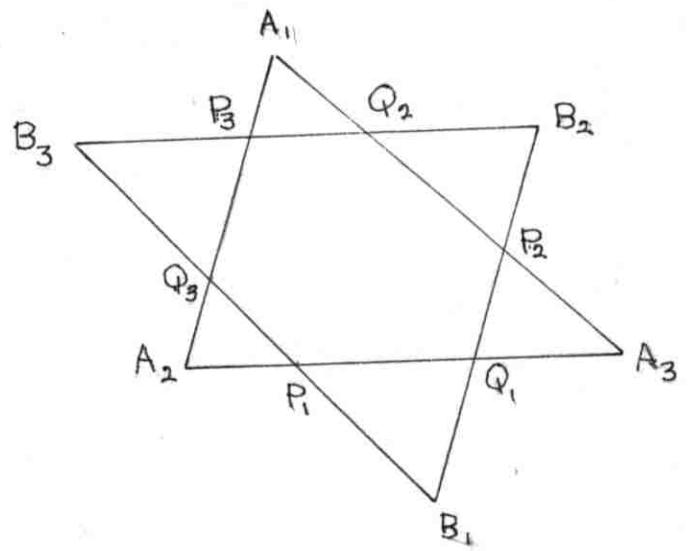

Fig. 8

9. <u>Coordinates of B_i; $i = 1, 2, 3$.</u>

Let

$$B_1 = P_1Q_3 \cdot P_2Q_1; \; B_2 = P_3Q_2 \cdot P_2Q_1; \; B_3 = P_1Q_3 \cdot P_3Q_2$$

The coordinates of B_i may be obtained by the same procedure as in Chapter III.

B_1 $(a_2a_3((a_2+a_3)t-a_2a_3),\; a_1a_3^2(a_2-t),\; a_1a_2^2(a_3-t))$

B_2 $(a_2a_3^2(a_1-t),\; a_1a_3((a_3+a_1)t-a_1a_3),\; a_1^2a_2(a_3-t))$

B_3 $(a_2^2a_3(a_1-t),\; a_1^2a_3(a_2-t),\; a_1a_2((a_1+a_2)t-a_1a_2))$

10. The Conic Determined by A_1, A_2, B_1, A_3, B_2: Denote this Conic by R.

The general equation of a conic is $\sum_{i,j}^{3} a_{ij}x_ix_j = 0$, $a_{ij}=a_{ji}$. Since the R-conic passes through A_i, $i = 1, 2, 3$, it is reduced to the form

$$\sum_{i<j} a_{ij}x_ix_j = 0 \qquad\qquad (1)$$

Proceeding as in chapter III, we find that the R-conic is represented by:

$$\frac{(a_3-t)t}{a_3^2}\, x_1x_2 + \frac{(a_1-t)(a_2a_3-a_2t-a_3t)}{a_1^2a_2}\, x_2x_3$$

$$+ \frac{(a_2-t)(a_1a_3-a_3t-a_1t)}{a_1a_2^2}\, x_1x_3 = 0.$$

11. Condition for the R-conic to pass through B_3.

Let us assume that the R-conic passes through B_3, then the coordinates of B_3 satisfy the equation of the R-conic.

If we substitute the coordinates of B_3 for x_i, $i = 1, 2, 3$ in the R-conic and simplify we get:

$$t^2(a_2a_3+a_1a_3+a_1a_2)^2 - 3t(a_2a_3+a_1a_3+a_1a_2) + 2 = 0$$

i.e.

$$t = \frac{a_1a_2a_3}{a_2a_3+a_1a_3+a_1a_2}, \text{ or } \frac{2a_1a_2a_3}{a_2a_3+a_1a_3+a_1a_2}$$

as the condition on t so that the R-conic passes through B_3.

When

$$t = \frac{2a_1a_2a_3}{a_2a_3+a_1a_3+a_1a_2}, \ P_1Q_3, P_3Q_2$$

and P_2Q_1 are concurrent at $\frac{a_i - t_i}{a_i^2}$ and the R-conic becomes degenerate. But

$$t = \frac{a_1a_2a_3}{a_2a_3+a_1a_3+a_1a_2}$$

puts the R-conic in the form:

$$\frac{a_2+a_3}{a_1} x_2x_3 + \frac{a_1+a_3}{a_2} x_1x_3 + \frac{a_2+a_1}{a_3} x_1x_2 = 0 \ \ldots\ldots\ldots\ldots(1)$$

Equation (1) represents the conic that passes through $A_1, B_3, A_2, B_1, A_3, B_2$.

The line at infinity $\Sigma\, a_ix_i = 0$ intersects (1) in

$$\frac{a_1(a_1+a_3)}{a_2a_3} x_1^2 - 2x_1x_2 + \frac{a_2(a_2+a_3)}{a_1a_3} x_2^2 = 0 \ \ldots\ldots\ldots(2)$$

The discriminant d of (2) is:

$$d = -\frac{4(a_1a_2+a_1a_3+a_2a_3)}{a_3^2},$$

which is negative.

Therefore, (1) represents an ellipse.

12. <u>Statement:</u>

The hexagon $A_1B_3A_2B_1\ A_3B_2$ touches a proper conic (i.e., a conic which has a unique tangent at each point).

PROOF: Let us find the equations of A_iB_i, i = 1, 2, 3.

$$A_1B_1:\ 0.x_1 - a_2^2(a_3-t)x_2 + a_3^2(a_2-t)x_3 = 0 \quad \ldots\ldots\ldots\ldots (1)$$

$$A_2B_2:\ a_1^2(a_3-t)x_1 + 0.x_2 - a_3^2(a_1-t)x_3 = 0 \quad \ldots\ldots\ldots\ldots (2)$$

$$A_3B_3:\ -a_1^2(a_2-t)x_1 + a_2^2(a_1-t)x_2 + 0.x_3 = 0 \quad \ldots\ldots\ldots\ldots (3)$$

Consider the determinant of the coefficients of equations A_iB_i, $i = 1, 2, 3$.

$$\begin{vmatrix} 0 & -a_2^2(a_3-t) & a_3^2(a_2-t) \\ a_1^2(a_3-t) & 0 & -a_3^2(a_1-t) \\ -a_1^2(a_2-t) & a_2^2(a_1-t) & 0 \end{vmatrix}$$

Expansion of the determinant gives:

$$-a_1^2a_2^2a_3^2(a_1-t)(a_2-t)(a_3-t) + a_1^2a_2^2a_3^2(a_1-t)(a_2-t)(a_3-t)$$

which is zero.

Therefore, the straight lines A_iB_i are concurrent. Hence, by the converse of Brianchon's Theorem [2, p.110] the hexagon touches a proper conic.

13. The Conic Inscribed in the Hexagon $A_1B_3A_2B_1A_3B_2$.

(This conic will be denoted by the s-conic).

We use $[u_1, u_2, u_3]$ to denote the line coordinates.

$A_1B_3 \cdot A_1B_2 = A_1(1, 0, 0)$, the line equation of A_1 is $u_1 = 0$

$A_2B_3 \cdot A_2B_1 = A_2(0, 1, 0)$, the line equation of A_2 is $u_2 = 0$

$A_1B_3 \cdot A_2B_3 = B_3(a_2^2a_3(a_1-t),\ a_1^2a_3(a_2-t), a_1a_2((a_1+a_2)t-a_1a_2))$

the line equation of B_3 is:

$$\frac{a_2^2 a_3}{a_2-t} u_1 + \frac{a_1^2 a_3}{a_1-t} u_2 + \frac{a_1 a_2((a_1+a_2)t-a_1a_2)}{(a_1-t)(a_2-t)} u_3 = 0.$$

Let $A_1B_2 . A_2B_1 = Z$. To find Z, we find equations of A_1B_2, A_2B_1 and solve them simultaneously to get

$Z(a_3((a_2+a_3)t-a_2a_3),\ a_3((a_3+a_1)t-a_1a_3),\ a_1a_2(a_3-t))$.

The line equation of Z is:

$$a_3((a_2+a_3)t-a_2a_3)u_1 + a_3((a_3+a_1)t-a_1a_3)u_2 + a_1a_2(a_3-t)u_3 = 0.$$

The line equation of the s-conic is given by: [6, p. 86].

$$\left[\frac{a_2^2 a_3}{a_2-t} u_1 + \frac{a_1^2 a_3}{a_1-t} u_2 + \frac{a_1 a_2((a_1+a_2)t-a_1a_2)}{(a_1-t)(a_2-t)} u_3\right]\Big[a_3((a_2+a_3)t-a_2a_3)u_1$$
$$+ a_3((a_3+a_2)t-a_1a_3)u_2+a_1a_2(a_3-t)u_3\Big] + \lambda u_1u_2 = 0. \quad \text{---} (1)$$

A_3B_1 is represented as:

$$- a_1a_3(a_2-t)x_1 + a_2((a_2+a_3)t-a_2a_3)x_2 = 0.$$

Therefore its line coordinates are:

$$[- a_1a_3(a_2-t),\ a_2((a_2+a_3)t-a_2a_3),\ 0].$$

Since A_3B_1 is tangent to the s-conic, its coordinates should satisfy (1). [2, p.26].

Substituting in equation (1) and simplifying we get:

$$\lambda = \frac{a_3[-2a_1a_2a_3 + (a_2a_3+a_1a_2+a_1a_3)t]^2}{(a_1-t)(a_2-t)}.$$

Substituting λ in equation (1), and simplifying we get:

$$\sum \frac{1}{a_1^2}(a_1-t)(a_2a_3-a_2t-a_3t)u_1^2$$
$$-\frac{1}{a_1a_2a_3}\sum[(a_3a_1-a_3t-a_1t)(a_1a_2-a_1t-a_2t)+a_1^2(a_2-t)(a_3-t)u_2u_3=($$

as the line equation of the s-conic. The subscripts are actual integers and the meaning of Σ given in Chapter I is applied.

To convert the line equation of the s-conic to its point equation involves tedious computations. Here we give the outline of the method and the result only.

Assume that the line equation of a conic is given by:

$$A_{11}U_1^2 + A_{22}U_2^2 + A_{33}U_3^2 + 2A_{12}U_1U_2 + 2A_{23}U_2U_3+2A_{13}U_1U_3 = 0$$

where A_{ij} is the cofactor of a_{ij} in the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

which is obtained from $\Sigma\, a_{ij}x_ix_j = 0$, $a_{ij} = a_{ji}$, the point equation of the conic. Consider the determinant

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$$

then $\Delta\, a_{ij}$ = cofactor of A_{ij} [6, p.79].

Using this relation we find that point equation of the s-conic is given by:

$$t^2 \sum_i x_i^2 - 2 \sum \frac{1}{a_1} [(a_1+2a_2+2a_3)t^2 - 2(a_2a_3+a_3a_1+a_1a_2)t + 2a_1a_2a_3] x_2x_3 = 0$$

where the subscripts are actual integers.

or

$$a_1a_2a_3t^2\left(\sum_i^3 x_i\right)^2 - 4[t^2 \sum_i^3 (a_i) - (\sum_{i<j} a_ia_j)t + a_1a_2a_3] \times$$

$$\sum_{i<j} a_{ij}x_ix_j = 0 \;.......(1)$$

When $t = 0$, we have

$$\sum_{i<j} a_{ij}x_ix_j = 0,$$

which is the Steiner's circumscribed ellipse about $A_1A_2A_3$. But in this case the hexagon $A_1B_3A_2B_1A_3B_2$ becomes the triangle $B_1B_2B_3$. Also when

$$t = \frac{2a_1a_2a_3}{a_1a_2+a_2a_3+a_3a_1},$$

B_1, B_2, B_3 go to the point

$$\left(a_2+a_3-\frac{a_2a_3}{a_1},\ a_3+a_1-\frac{a_3a_1}{a_2},\ a_1+a_2-\frac{a_1a_2}{a_3}\right),$$

which is the intersection of P_3Q_2, P_1Q_3, P_2Q_1. If we denote this point by G, then the hexagon becomes $A_1GA_2GA_3G$, for this case the converse of Brianchon's theorem does not apply since three sides of this hexagon concur. [2, p.110].

14. Concurrence point of A_iB_i.

Let this point be H. Solving simultaneously

$$0.x_1 - a_2^2(a_3-t)x_2 + a_3^2(a_2-t)x_3 = 0 \quad \dots\dots\dots\dots\dots(1)$$

$$a_1^2(a_3-t)x_1 + 0.x_2 - a_3^2(a_1-t)x_3 = 0 \quad \dots\dots\dots\dots\dots(2)$$

we get:

$$\frac{x_2}{x_3} = \frac{a_3^2(a_2-t)}{a_2^2(a_3-t)}$$

$$\frac{x_1}{x_3} = \frac{a_3^2(a_1-t)}{a_1^2(a_3-t)}$$

Therefore
$$H(x_i) \sim \left(\frac{a_i-t}{a_i^2}\right)$$

When $t = \dfrac{a_1a_2a_3}{a_2a_3+a_3a_1+a_1a_2}$, H is given by $(a_2+a_3, a_3+a_1, a_1+a_2)$.

This point is the point of concurrency of P_1Q_2, P_2Q_3 and P_3Q_1. It is also the center of the B-conic, the R-conic, the t-conic and the s-conic.

15. Locus of H.

$$X_1 = -\frac{1}{a_1^2}t + \frac{1}{a_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$X_2 = -\frac{1}{a_2^2}t + \frac{1}{a_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$X_3 = -\frac{1}{a_3^2}t + \frac{1}{a_3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Let

$$K_1 = \begin{vmatrix} -\frac{1}{a_2^2} & -\frac{1}{a_3^2} \\ \frac{1}{a_2} & \frac{1}{a_3} \end{vmatrix} \qquad K_2 = \begin{vmatrix} -\frac{1}{a_3^2} & -\frac{1}{a_1^2} \\ \frac{1}{a_3} & \frac{1}{a_1} \end{vmatrix}$$

$$K_3 = \begin{vmatrix} -\frac{1}{a_1^2} & -\frac{1}{a_2^2} \\ \frac{1}{a_1} & \frac{1}{a_2} \end{vmatrix}$$

Then $K_1X_1 + K_2X_2 + K_3X_3 = 0$, so the locus of H is a straight line given by

$$\left(\frac{1}{a_2a_3^2} - \frac{1}{a_3^2a_2^2}\right)x_1 + \left(\frac{1}{a_1^2a_3} - \frac{1}{a_1a_3^2}\right)x_2 + \left(\frac{1}{a_2^2a_1} - \frac{1}{a_1^2a_2}\right)x_3 = 0.$$

Further simplification gives:

$$a_1^2(a_2-a_3)x_1 + a_2^2(a_3-a_1)x_2 + a_3^2(a_1-a_2)\,x_3 = 0$$

as the equation of the locus of H.

The following points lie on the locus of H:

1. $\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}\right)$

2. $\left(0, a_3^2(a_1-a_2), a_2^2(a_1-a_3)\right)$

3. $\left(a_3^2(a_2-a_1), 0, a_1^2(a_2-a_3)\right)$

4. $\left(a_2^2(a_3-a_1), a_1^2(a_3-a_2), 0\right)$

5. $\left((a_2+a_3), (a_3+a_1), (a_1+a_2)\right)$,

We end this section by the following remarks:

1. $\Delta\ B_1B_2B_3$ is congruent to $\Delta\ A_1A_2A_3$ when

$$t = \frac{a_1a_2a_3}{a_1a_2+a_2a_3+a_1a_3}$$ and triangle $B_1B_2B_3$

may be obtained from triangle $A_1A_2A_3$ by a rotation of π about the point $(a_2+a_3, a_3+a_1, a_1+a_2)$.

2. When A_iB_i concur at the centroid, we have

$$\frac{a_i-t}{a_i^2} = \frac{\lambda}{a_i} \quad i = 1,2,3, \ \lambda \neq 0$$

therefore

$$t = a_i(1-\lambda)$$

If the triangle is not equilateral, these expressions are equal only if $\lambda = 1$ and $t = 0$. Then $A_iB_i (i = 1, 2, 3)$ are the medians of $A_1A_2A_3$. This is clear from the equations of A_iB_i given in Section 13 of this Chapter.

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INDEX OF SYMBOLS

$A_1A_2A_3$	Triangle of Reference
A_i	Vertices of $A_1A_2A_3$
a_i	Sides of $A_1A_2A_3$, a_i opposite A_i
δ_i, ∂_i	Angle at A_i
d	Discriminant, $B^2 - 4AC$
Δ	Determinant or triangle
L_1L_2	Intersection of two lines
t_i	Line segments, t_i parallel to a_i
$\sim$	Similar or proportional
$=$	Congruent
[k, p.N]	Reference book whose number is k in the entries of the bibliography; page N.