AMERICAN UNIVERSITY OF BEIRUT

# ON THE CAPACITY OF LINEAR ADDITIVE CHANNELS WITH THE NOISE SPANNING HERMITE FUNCTIONS

by

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submitted in partial fulfillment of the requirements
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# An Abstract of the Thesis of

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Title: On the Capacity of Linear Additive Channels with the Noise Spanning Hermite Functions

We consider a linear additive noise channel where the input is average-power constrained and the noise probability law is not necessarily Gaussian, but is rather in the finite span of even Hermite Functions. We study the nature of the capacity achieving input distribution of such a channel.

It's well known, by Shannon's Theorem, that the capacity achieving distribution of the described channel is of a continuous type, namely Gaussian, whenever the noise is Gaussian.

In our study, we present some sample case analysis and develop a general procedure that proves the discreteness of the capacity achieving distribution whenever the noise is in the finite span of even Hermite Functions with the exception of the Gaussian.

**Keywords: Capacity, Linear Additive Channel, Gaussian Channel, Non-Gaussian Noise, Hermite Functions**

# Contents

# Chapter 1

# Introduction

## 1.1 Overview

In his 1948 paper, Shannon [1] proved that the capacity achieving distribution of an average power constrained AWGN channel is of a continuous type. In fact, the optimal input distribution is Gaussian distributed.

Later, Smith [2] provided necessary and sufficient conditions to be satisfied by the optimal inputs . Shamai and al. [3] extended the work of Smith to complex Gaussian channels, Abou-Faycal and al. [4] considered a non-deterministic average power constrained Rayleigh-fading channel and adapted the techniques used by Smith to their problem. Recently [5] investigated non-linear AWGN under even moment as well as finite support constraints and concluded that the input capacity achieving distribution is discrete.

In most of the papers cited above, the authors investigated channels where the noise is assumed to be Gaussian distributed.

In fact, the noise of an additive linear channel was historically modeled with a Gaussian Probability Density Function ( pdf ) for mainly two reasons. First, the Gaussian distribution maximizes the entropy of a random variable with finite variance constraint. Second, the noise resulting from multiple independent sources asymptotically approaches a Gaussian distribution due to the Central Limit Theorem ( CLT ).

However, research results and studies assured that modeling the noise as Gaussian doesn't always really capture the noise characteristics especially in cases where the noise is impulsive. J. Lin and Evans [6] showed that the RF noise in wireless communication systems is too complicated to be modeled as Gaussian distributed. The author argues that modeling the noise as a mixture of Gaussian Distributions, i.e. a weighted sum of Gaussian PDFs with zero mean and different variances leads to a better performance and models more accurately the impulsive nature of the noise.
Thus, channels where the noise is not assumed to be Gaussian distributed turned out to be more practical in some settings and need to be further investigated.

In his paper, Tchamkerten [7] derived certain conditions or criteria on the noise distribution that guarantees the discreteness of the capacity-achieving distribution under input-amplitude constraint. Das [8] investigated average power constrained non-Gaussian additive noise channel and showed that the capacity achieving distribution has bounded (resp. unbounded) support when the noise PDF decays at a rate slower (resp. faster) than a Gaussian.

This study is concerned about the nature of the capacity achieving input distribution of a linear additive channel $Y = X + N$ where the noise pdf is in the finite span of even Hermite Functions and the input is average-power constrained. This setup is different from that of [8] and [7]. In [7] the input is assumed to be amplitude constrained and the problem of average-power constrained is suggested as an interesting problem that hasn't been solved. Also, the noise distributions we consider don't satisfy the conditions in [8] and the characterization of the optimal input we seek is more exhaustive than the result in [8].

## 1.2 Problem Definition

We are interested in studying the capacity of linear additive noise channels modeled as $Y = X + N$ where $X$ is the input, $Y$ is the output of the channel and $N$ is the noise which is independent of $X$ and absolutely continuous with pdf:

$$p_N(n) = \left[\alpha_0 H_0(n) + \alpha_2 H_2(n) + \alpha_4 H_4(n) + \ldots + \alpha_{2k} H_{2k}(n)\right] e^{-n^2/2},$$

where $k \in \mathbb{N}^*$ , $\alpha_0, \alpha_2, \ldots, \alpha_{2k} \in \mathbb{R}$ and $H_k(.)$ is the probabilist's Hermite polynomial of order $k$; defined as:

$$H_k(x) = (-1)^k \, e^{x^2/2} \frac{d^k}{dx^k} \left[ e^{-x^2/2} \right].$$

For example,

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= x, \\ H_2(x) &= x^2 - 1 \\ &\vdots \end{aligned}$$

We exclude from this study noise pdfs, $p_N(.)$, that satisfy:
$\exists w_0 \in \mathbb{R}$ s.t. $p_N\big|_{\mathcal{F}}(w_0) = p_N\big|_{\mathcal{F}}^{'}(w_0) = p_N\big|_{\mathcal{F}}^{''}(w_0) = 0$ for technical reasons that will appear shortly in this study.
In addition, we impose on the input an average power constraint :

$$\mathsf{E}\left[X^2\right] \leq a,$$

where $a$ is a positive fixed parameter. We also assume, without loss of generality, that the noise is 0-mean.

Note that for $p_N(.)$ to be a valid pdf then the following two conditions should hold:

$$\begin{aligned} \int \sum_{i=0}^{k} \alpha_{2i} H_{2i}(n) e^{-n^2/2}\, dn &= 1 \\ \sum_{i=0}^{k} \alpha_{2i} H_{2i}(n) &\geq 0 \quad \forall n \in \mathbb{R}, \end{aligned}$$

and we assume in the remainder of this thesis that the $\alpha_i$ 's are chosen accordingly. For example, if $k = 1$, then a necessary and a sufficient condition to guarantee $p_N(n) \geq 0 \quad \forall n \in \mathbb{R}$ is:

$$0 < \alpha_2 \leq \alpha_0$$

and $\alpha_0 = \dfrac{1}{\sqrt{2\pi}}$ is necessary and sufficient in order to have: $\int p_N(n)\, dn = 1$.

# Chapter 2

# Sample Case Analysis

## 2.1 Noise in the span of $\psi_2$ and $\psi_0$

In this chapter we will investigate in detail a particular example where the noise is given by:

$$p_N(n) = [\beta_1 H_2(n) + \beta_1 H_0(n)]\, e^{-n^2/2},$$

The input is subject to an average power constraint:

$$\mathsf{E}\left[X^2\right] \leq a,$$

where $a$ is a positive fixed parameter, and $\beta_1$ is chosen so that $\displaystyle\int_{-\infty}^{+\infty} p_N(n)\, dn = 1$. Since,

$$\int_{-\infty}^{+\infty} n^2 e^{-n^2/2}\, dn = -ne^{-n^2}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} -e^{-n^2/2}\, dn = \int_{-\infty}^{+\infty} e^{-n^2/2}\, dn = \sqrt{2\pi};$$

we choose $\beta_1 = \dfrac{1}{\sqrt{2\pi}}$.

Before proceeding we determine the first and second moments of the noise:

$$\mathsf{E}\left[N\right] = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} n^3 e^{-n^2/2}\, dn = 0.$$

$$\mathsf{E}\left[N^2\right] = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} n^4 e^{-n^2/2}\, dn = 3.$$

## KKT Conditions

Since the channel transition probability density function is given by:

$$p_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}}(y-x)^2 e^{-(y-x)^2/2}, \tag{2.1}$$

given $X$, $Y$ is an absolute continuous random variable with pdf (2.1) . One can establish that for any probability distribution $F_X$ on X , $Y$ is also absolutely continuous and has a pdf denoted by $p_Y(y; F_X)$.

Using the theory of convex optimization it can be shown that an input random variable $X^*$ with CDF $F^*$ acheives the capacity $C$ of an average power limited channel if and only if there exists $\gamma \geq 0$ such that,

$$\gamma(x^2 - a) + C - \int p(y|x) \ln \frac{p(y|x)}{p(y; F^*)}\, dy \geq 0, \tag{2.2}$$

for all $x$, with equality whenever $x$ is a point of increase of $F^*$ .

Substituting the expression of $p(y|x)$ in (2.2) we obtain,

$$\begin{aligned}\gamma(x^2 - a) + C + \frac{1}{2}\ln 2\pi - \int_{-\infty}^{+\infty} \ln{(y-x)^2} \frac{1}{\sqrt{2\pi}}(y-x)^2 e^{-(y-x)^2/2}\, dy \\ + \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}(y-x)^4 e^{-(y-x)^2/2}\, dy + \int p(y|x) \ln p(y; F^*)\, dy \geq 0,\end{aligned} \tag{2.3}$$

where we assumed that the various integrals exist which is formally proven in the following lemma.

**Lemma 1.**

$$\int_{-\infty}^{+\infty} \ln{(y-x)^2} \frac{1}{\sqrt{2\pi}} (y-x)^2 e^{-(y-x)^2/2}\, dy \ , \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} (y-x)^4 e^{-(y-x)^2/2}\, dy$$

*and* $\int p(y|x) \ln p(y; F^*)\, dy$ *exist.*

*Proof.*

$$\int_{-\infty}^{+\infty} \ln{(y-x)^2} \frac{1}{\sqrt{2\pi}} (y-x)^2 e^{-(y-x)^2/2}\, dy = \int_{-\infty}^{+\infty} \ln{(u^2)} \frac{1}{\sqrt{2\pi}} u^2 e^{-u^2/2}\, du$$

which *is independent of* $x$ and thus it remains to prove that this integral converges to a constant value and doesn't diverge.
In fact, since $\ln{(u^2)} u^2 e^{-u^2/2}$ is even , it's enough to prove that $\int_0^{+\infty} \ln{(u^2)} u^2 e^{-u^2/2}\, du$ is finite. Note first that by L'hopital's rule , we have:

$$\begin{aligned} \lim_{u\to 0} \left[\ln{(u^2)} u^2 e^{-u^2/2}\right] &= 0 \\ \lim_{u\to\infty} \left[\ln{(u^2)} u e^{-u^2/4}\right] &= 0. \end{aligned}$$

Thus, given any $\epsilon > 0 \quad \exists u_0 \quad s.t. \left|\ln{(u^2)} u e^{-u^2/4}\right| < \epsilon \quad \forall u > u_0$
Thus,

$$\begin{aligned} \frac{2}{\sqrt{2\pi}} \left|\int_0^{+\infty} \ln{(u^2)} u^2 e^{-u^2/2}\, du\right| &= \frac{2}{\sqrt{2\pi}} \left|\int_0^{u_0} \ln{(u^2)} u^2 e^{-u^2/2}\, du + \int_{u_0}^{+\infty} \ln{(u^2)} u^2 e^{-u^2/2}\, du\right| \\ &\le \frac{2}{\sqrt{2\pi}} \left[\left|\int_0^{u_0} \ln{(u^2)} u^2 e^{-u^2/2}\, du\right| + \left|\int_{u_0}^{+\infty} \ln{(u^2)} u^2 e^{-u^2/2}\, du\right|\right] \end{aligned}$$

The first integral is finite since the function $\ln{(u^2)} u^2 e^{-u^2/2}$ is continous and since the interval $[0, u_0]$ is compact, then it's bounded over that interval by some constant A.

$$\begin{aligned}
\frac{2}{\sqrt{2\pi}}\left|\int_0^{+\infty}\ln(u^2)u^2e^{-u^2/2}\,du\right| &\leq \frac{2}{\sqrt{2\pi}}\left[Au_0+\left|\int_{u_0}^{+\infty}\ln(u^2)ue^{-u^2/4}ue^{-u^2/4}\,du\right|\right]\\
&\leq \frac{2}{\sqrt{2\pi}}\left[Au_0+\int_{u_0}^{+\infty}\left|\ln(u^2)ue^{-u^2/4}\right|\left|ue^{-u^2/4}\right|\,du\right]\\
&\leq \frac{2}{\sqrt{2\pi}}\left[Au_0+\epsilon\int_{u_0}^{+\infty}ue^{-u^2/4}\,du\right]\\
&\leq \frac{2}{\sqrt{2\pi}}\left[Au_0+\epsilon(2e^{-u_0^2/4})\right]
\end{aligned}$$

In conclusion,

$$\int_{-\infty}^{+\infty}\ln(y-x)^2\frac{1}{\sqrt{2\pi}}(y-x)^2e^{-(y-x)^2/2}\,dy$$

is equal to a constant which is denoted by $\beta$ ($\approx 1.8$) hereafter. □

When it comes to the second integral in (2.3), it's finite and equal to:

$$\frac{1}{2}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}(y-x)^4e^{-(y-x)^2/2}\,dy=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}(u)^4e^{-(u)^2/2}\,dy=\frac{3}{2}.$$

Finally, the third integral is finite by Lemma 2 of [9]. Substituting in (2.3) we get:

$$\gamma(x^2-a)+C-\beta+\frac{1}{2}\ln 2\pi+\frac{3}{2}+\int p(y|x)\ln p(y;F^*)\,dy\geq 0 \qquad (2.4)$$

**Theorem 1.** *The optimal distribution of the input of the channel described above is even.*

*Proof.* $I(X;Y)=I(-X;-Y)$ because $p_{Y|X}=p_{-Y|-X}$. Also, $I(-X;-Y)=I(-X;Y)$ since the mapping from $Y$ to $-Y$ is bijective . In conclusion, $I_{F_X}=I_{F_{-X}}$ and since the mutual information is concave in the input distribution then the optimal distribution is even. □

**Theorem 2.** *The optimal output distribution (induced by the optimal input distribution) is even.*

*Proof.*

$$\begin{aligned}
p_Y(y;F^*) &= \int p_{Y|X}(y|x)dF^*(x) \\
&= \frac{1}{\sqrt{2\pi}}\int (y-x)^2 e^{-(y-x)^2/2}\, dF^*(x)
\end{aligned}$$

$$\begin{aligned}
p_Y(-y;F^*) &= \frac{1}{\sqrt{2\pi}}\int (-y-x)^2 e^{-(-y-x)^2/2}\, dF^*(x) \\
&= \frac{1}{\sqrt{2\pi}}\int (y+x)^2 e^{-(y+x)^2/2}\, dF^*(x) \\
&= \frac{1}{\sqrt{2\pi}}\int (y-x)^2 e^{-(y-x)^2/2}\, dF^*(-x) \\
&= \frac{1}{\sqrt{2\pi}}\int (y-x)^2 e^{-(y-x)^2/2}\, dF^*(x) \\
&= p_Y(y;F^*)
\end{aligned}$$

□

Now, since $\ln p_Y(y; F^*)$ is a continuous function in $y$ and integrable w.r.t. $e^{-y^2/2}$, then using Fourier Hermite Series Expansion:

$$\ln p_Y(y; F^*) = \sum_{n=0}^{\infty} c_n H_n(y)$$

Thus,

$$\begin{aligned}
\int p(y|x) \ln p(y; F^*)\, dy &= \int \frac{1}{\sqrt{2\pi}} (y-x)^2 e^{-(y-x)^2/2} \ln p(y; F^*)\, dy \\
&= \int \frac{1}{\sqrt{2\pi}} (y-x)^2 e^{-(y-x)^2/2} \sum_{n=0}^{\infty} c_n H_n(y)\, dy \\
&= \int \frac{1}{\sqrt{2\pi}} (u)^2 e^{-(u)^2/2} \sum_{n=0}^{\infty} c_n H_n(u+x)\, du \\
&= \int \frac{1}{\sqrt{2\pi}} (u)^2 e^{-(u)^2/2} \sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} \binom{n}{k} x^k H_{n-k}(u)\, du \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} \binom{n}{k} x^k \int (u)^2 e^{-(u)^2/2} H_{n-k}(u)\, du \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} \binom{n}{k} x^k \int \left[H_2(u) + H_0(u)\right] e^{-(u)^2/2} H_{n-k}(u)\, du \\
&= \sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} \binom{n}{k} x^k (\delta_{n-k} + 2\delta_{n-k,2}) \\
&= \sum_{n=0}^{\infty} c_n x^n + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \\
&= \sum_{n=0}^{\infty} \Big[c_n + (n+1)(n+2) c_{n+2}\Big] x^n,
\end{aligned}$$

where we interchanged integral and sum by Fubini's Theorem.

The same expression can be deduced using Parseval's Theorem. In fact,

$$
\begin{aligned}
\int p(y|x) \ln p(y; F^*)\, dy &= \int \frac{1}{\sqrt{2\pi}} \left[H_2(y-x) + H_0(y-x)\right] e^{-(y-x)^2/2} \ln p(y; F^*)\, dy \\
&= \int \frac{1}{\sqrt{2\pi}} \left[H_2(u) + H_0(u)\right] e^{-u^2/2} \ln p(u+x; F^*)\, du \\
&= \int \frac{1}{\sqrt{2\pi}} \left[H_2(u) + H_0(u)\right] e^{-u^2/2} \sum_{n=0}^{\infty} c_n H_n(u+x)\, du \\
&= \int \frac{1}{\sqrt{2\pi}} \left[H_2(u) + H_0(u)\right] e^{-u^2/2} \sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} \binom{n}{k} x^k H_{n-k}(u)\, du \\
&= \int \left[H_2(u) + H_0(u)\right] \frac{e^{-u^2/2}}{\sqrt{2\pi}} \left\{ \sum_{n=0}^{\infty} c_n x^n H_0(u) + \sum_{n=2}^{\infty} \binom{n}{n-2} c_n x^{n-2} H_2(u) + \dots \right\} \\
&= \sum_{n=0}^{\infty} c_n x^n + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \\
&= \sum_{n=0}^{\infty} \Big[ c_n + (n+1)(n+2) c_{n+2} \Big] x^n
\end{aligned}
$$

Substituting the above expression in (2.4) we get:

$$
\gamma(x^2 - a) + C - \beta + \frac{1}{2} \ln 2\pi + \frac{3}{2} + \sum_{n=0}^{\infty} \Big[ c_n + (n+1)(n+2) c_{n+2} \Big] x^n = 0 \quad (2.5)
$$

$$
\iff -\gamma x^2 + \left[ \gamma a - C + \beta - \frac{1}{2} \ln 2\pi - \frac{3}{2} \right] = \sum_{n=0}^{\infty} \Big[ c_n + (n+1)(n+2) c_{n+2} \Big] x^n,
$$

whenever $x$ is a point of increase of $F^*$.

## Extension to Complex Domain

Assume the points of increase of $F_X^*$ have an accumulation point. Then the following points hold:

- equality in (2.5) holds on a set with an accumulation point.
- extending the function to the complex plane, yields an analytic function zero on a set with accumulation point.

- identity theorem implies that the function is zero everywhere.

In solving (2.5) for the coefficients $c_n$, we will distinguish between the two cases: $n$ is even or $n$ is odd.

<u>*Case 1: n is odd*</u>

$c_n + (n+1)(n+2)c_{n+2} = 0 \quad \forall n$, and hence,

$$c_3 = \frac{-c_1}{2\times 3} = \frac{-c_1}{3!} \qquad c_5 = \frac{-c_3}{4\times 5} = \frac{c_1}{5!} \qquad ....... \qquad c_n = c_1\frac{(-1)^{\frac{n+3}{2}}}{n!},$$

where n is odd. Finally,

$$c_1 = \frac{1}{\sqrt{2\pi}}\int \ln\Big[p_Y(y)\Big] y e^{-(y)^2/2}\, dy = 0,$$

since $p_Y(y)$ is an even function according to theorem 2. In conclusion, $c_{2k+1} = 0 \quad \forall\, k \in \mathbb{N}$.

<u>*Case 2: n is even*</u>

Now,

$$\begin{aligned} c_0 + 2c_2 &= \left[\gamma a - C + \beta - \frac{1}{2}\ln 2\pi - \frac{3}{2}\right] \\ c_2 + (3\times 4)c_4 &= -\gamma \\ c_n + (n+1)(n+2)c_{n+2} &= 0, \quad \text{when} \quad n \geq 4. \end{aligned}$$

Thus,

$$c_6 = \frac{-c_4}{5\times 6} \qquad c_8 = \frac{-c_6}{7\times 8} = \frac{c_4 4!}{8!} \qquad ....... \qquad c_{2k} = \frac{c_4(-1)^k 4!}{(2k)!},$$

where $k \geq 2$.

Also,

$$\begin{aligned} c_2 &= \frac{A - c_0}{2}, \quad \text{where} \quad A = \gamma a - C + \beta - \frac{1}{2}\ln 2\pi - \frac{3}{2} \\ c_2 &= B - 3.4c_4, \quad \text{where} \quad B = -\gamma \quad \Leftrightarrow c_4 = \frac{2B - A + c_0}{4!} \\ &\vdots \\ c_{2k} &= \frac{(-1)^k \Big[2B - A + c_0\Big]}{(2k)!}, \text{ where } k \geq 2 \end{aligned}$$

Thus,

$$\ln p_Y(y) = c_0 H_0(y) + \frac{A - c_0}{2} H_2(y) + [2B - A + c_0] \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k)!} H_{2k}(y) \qquad (2.6)$$

But,

$$\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} H_{2k}(x) = \text{even}(e^{xt - t^2/2}) = \frac{e^{xt - t^2/2} + e^{-xt - t^2/2}}{2} = e^{-t^2/2}\Big[\cosh xt\Big]$$

Setting $t^2 = -1$ , then we get:

$$\begin{aligned} \ln p_Y(y) &= c_0 H_0(y) + \frac{A - c_0}{2} H_2(y) + [2B - A + c_0][e^{\frac{1}{2}} \cos y - 1 + \frac{1}{2!}(y^2 - 1)] \\ &= -3B + A + By^2 + [2B - A + c_0][e^{\frac{1}{2}} \cos y] \\ &= 3\gamma + A - \gamma y^2 + [-2\gamma - A + c_0][e^{\frac{1}{2}} \cos y] \end{aligned}$$

In summary,

$$p_Y(y; F^*) = K e^{K' \cos y} e^{-\gamma y^2}, \qquad (2.7)$$

where $K = e^{3\gamma + A}$, $K' = [-2\gamma - A + c_0][e^{\frac{1}{2}}]$ and it remains to check whether $p_Y(y; F^*)$ is inducible by an input pdf $p_X(.)$. We start by examining possible values of $\gamma$.

**Theorem 3.** *The value of* $\gamma$ *in the expression of* $p_Y(y)$ *(2.7) satisfies:* $0 \leq \gamma \leq \frac{1}{2}$.

*Proof.*

$$\begin{aligned}
|M_N(w)| &= |p_N\big|_{\mathcal{F}}(-w)| = \frac{1}{\sqrt{2\pi}}|1-w^2|e^{-w^2/2} \leq d(1+w^2)e^{-w^2/2} \leq d(1+|w|)^2 e^{-w^2/2} \\
|M_Y(w)| &= |M_X(w)| \times |M_N(w)| \leq |M_N(w)| \leq (1+|w|)^2 e^{-w^2/2} \\
p_Y(y) &= Ke^{K' \cos y} e^{-\gamma y^2} \leq ce^{-\gamma y^2} \leq c(1+|y|)^2 e^{-\gamma y^2}
\end{aligned}$$

By the extension of Hardy's theorem [10], [11] we get:

$$\begin{aligned}
\gamma \times \frac{1}{2} &\leq \frac{1}{4} \\
\Leftrightarrow \gamma &\leq \frac{1}{2}
\end{aligned}$$

□

**Theorem 4.** $p_Y(y; F^*)$ *is not inducible by any input pdf* $p_X(.)$.

*Proof.* By Appendix[B] , there exists a sequence $a_n$ and a sequence of input distributions each with accumulation point $F_n$ s.t. $\lim_{n\to\infty} a_n = 0$ and $F_n$ converges to 0 in the weak sense.

Now,

$$p_Y(y; F_n) = \int p_N(y-x)\, dF_n(x)$$

Fix $y \in \mathbb{R}$:

$$\begin{aligned}
\int p_N(y-x)\, dF_n(x) &\longrightarrow \int p_N(y-x)\, dF(x) \\
&= \int p_N(y-x)\, \delta(x) \\
&= p_N(y),
\end{aligned}$$

where we used weak convergence and the fact that $p_N(y-x)$ is continuous and bounded .

Thus,

$$\begin{aligned} & p_Y(y;F_n) \longrightarrow p_N(y) \\ \Leftrightarrow & K_n e^{K_n' \cos y} e^{-\gamma_n y^2} \longrightarrow \frac{1}{\sqrt{2\pi}} y^2 e^{-y^2/2}, \end{aligned} \tag{2.8}$$

pointwise $\forall y \in \mathbb{R}$.

Evaluate (2.8) at $y=0: K_n e^{K_n'} \longrightarrow 0 \quad$ as n $\longrightarrow \infty$.

Thus, given an $\epsilon > 0, \quad \exists n_0 \in \mathbb{N}^*$ s.t. $\left|K_n e^{K_n'}\right| < \epsilon \quad \forall n \geq n_0$.

Evaluate (2.8) at $y = 2\pi : K_n e^{K_n'} e^{-\gamma_n 4\pi^2} \longrightarrow \frac{4}{\sqrt{2\pi}} \pi^2 e^{-2\pi^2} > 0$ as n $\longrightarrow \infty$.

But,

$$0 \leq K_n e^{K_n'} e^{-\gamma_n 4\pi^2} \leq K_n e^{K_n'}$$

Thus, by the sandwich theorem,

$$\lim_{n \to \infty} K_n e^{K_n'} e^{-\gamma_n 4\pi^2} = 0,$$

which is a contradiction. Thus, our assumption that the points of increase of $F_X^*$ have an accumulation point is invalid. Thus, the input capacity achieving distribution is discrete. □

# Chapter 3

# Limitations of the Described Procedure and Proposing Another One

## Motivation

The procedure of decomposing $\ln p_Y(y; F^*)$ over Hermite polynomials described in *Chapter 2*, i.e. writing $\ln p_Y(y; F^*) = \sum_{n=0}^{\infty} c_n H_n(y)$, and seeking to determine the corresponding coefficients $c_n$ turns out to be cumbersome when the noise pdf is in a large span of Hermite functions.

In fact, as the span of Hermite functions increase, the recurrence relation involving the coefficients $c_n$ gets more and more complicated. To elaborate on this, we will consider the following example.

**Example 1.**

$$\begin{aligned} p_N(n) &= K_2 \left[ H_4(n) + 6H_2(n) + 3H_0(n) \right] e^{-n^2/2} \\ &= K_2 n^4 e^{-n^2/2}, \end{aligned}$$

where $K_2$ is a normalizing constant.

Using the same procedure as in Chapter 2, we get the following recurrence relation:

$$\sum_{n=0}^{\infty}\Big[(n+1)(n+2)(n+3)(n+4)c_{n+4}+6(n+1)(n+2)c_{n+2}+3c_n\Big]x^n$$
$$=\gamma(x^2-a)+C-\beta$$

It's clear that solving this recurrence relation is very difficult. Thus, another approach should be investigated. Note that the proof of the discreteness of the capacity achieving distribution whenever the noise is in the span of $H_0(.), H_2(.)$, and $H_4(.)$ is provided in the Appendix.

Using the theory of convex optimization and writing the KKT expression as in the previous chapter, we get the expression of (2.2):

$$\gamma(x^2-a)+C-\int p(y|x)\ln\frac{p(y|x)}{p(y;F^*)}\,dy\geq 0,$$

for all $x$, with equality whenever $x$ is a point of increase of $F^*$.
This is equivalent to:

$$\gamma(x^2-a)+C-\int p(y|x)\ln p(y|x)\,dy+\int p(y|x)\ln p(y;F^*)\,dy\geq 0,$$

for all $x$, with equality whenever $x$ is a point of increase of $F^*$.
Now, $\int p(y|x)\ln p(y|x)\,dy$ exists and is finite since it's the entropy of the noise, denoted by $H$. Thus,

$$s(x)=\gamma(x^2-a)+C-H+\int p_N(y-x)\ln p(y;F^*)\,dy\geq 0, \qquad (3.1)$$

for all $x$, with equality whenever $x$ is a point of increase of $F^*$ .

**Lemma 2.** $h(x;F^*)=\int p_N(y-x)\ln p(y;F^*)\,dy$ *has an analytic extension to* *the complex domain, i.e. the mapping*

$$h(.;F):\mathbb{C}\to\mathbb{C} \quad \textit{defined by}$$
$$z\to h(z;F)=\int p_N(y-z)\ln p(y;F)\,dy \tag{3.2}$$

*is analytic.*

*Proof.* We refer the reader to Lemma 2 of [9]. □

However, instead of decomposing $\ln[p_Y(y;F^*)]$ over Hermite polynomials as before, we propose to take Distributional Fourier Transform after extending $s(x)$ to the complex domain.
Extending $s(x)$ to the complex domain,

$$s(z)=\gamma(z^2-a)+C-H+\int p_N(y-z)\ln p(y;F^*)\,dy$$

yields an analytic function by Lemma 2.

*Assume now that the points of increase of $F^*$ have an accumulation point*, then by the identity theorem [12], $s(\cdot)$ is identically null . Thus,

$$s(x)=\gamma(x^2-a)+C-H+\int p_N(y-x)\ln p(y;F^*)\,dy=0,$$

for all $x\in\mathbb{R}$. Recognizing that $x^2=\int y^2p_N(y-x)\,dy-\sigma_N^2$, this is equivalent to[]

$$\begin{aligned}0&=[C-H-\gamma a-\gamma\sigma_N^2]+\int p_N(y-x)\ln p(y;F^*)dy+\gamma\int y^2p_N(y-x)\,dy\\&=\kappa+\int p_N(y-x)\ln\left[e^{\gamma y^2}p(y;F^*)\right]\,dy\\&=\kappa+p_N(-y)*\ln\left[e^{\gamma y^2}p(y;F^*)\right]\Big|_x,\end{aligned}$$

where $\kappa=[C-H-\gamma a-\gamma\sigma_N^2]$. Taking the distributional Fourier transform on both sides, we get

$$p_N\big|_{\mathcal{F}}(-w) \times \ln\left[e^{\gamma y^2} p(y;F^*)\right]\big|_{\mathcal{F}}(w) = -\kappa\delta(w) \tag{3.3}$$

To prove the last assertion, we note that $\ln\left[e^{\gamma y^2} p(y;F^*)\right]$ is a tempered function, insured by the upper bound on $-\ln p(y;F)$, see (Lemma 2 of [9]) , and hence its Fourier transform exists. We need to distinguish between two cases:

*Case1:* $p_N\big|_{\mathcal{F}}(w) = M_N(-jw) \neq 0, \forall\, w \in \mathbb{R}$.
If this is the case, we proceed as follows:
Equation ( 3.3) is equivalent to:

$$\ln\left[e^{\gamma y^2} p(y;F^*)\right]\big|_{\mathcal{F}}(w) = \frac{-\kappa\delta(w)}{M_N(0)}, \tag{3.4}$$

Taking the inverse Fourier transform of equation (3.4) yields,

$$p(y;F^*) = e^{\frac{-\kappa}{M_N(0)}} \cdot e^{-\gamma y^2}. \tag{3.5}$$

Equation (3.5) shows that under the assumption that the optimal input has an accumulation point, the output PDF, $p(y;F^*)$, of the channel must be necessarily Gaussian which is not possible unless the input $X^*$ and the noise $N$ are both Gaussian according to Cramer's decomposition theorem [13, Th.19, p.53]. Therefore, unless the noise is Gaussian, $F^*$ has no accumulation points and therefore it is discrete. It remains to investigate the nature of the capacity achieving distribution when the Fourier transform of the noise has zeros.

*Case2:* $\exists w_0 \in \mathbb{R}: \quad p_N\big|_{\mathcal{F}}(w_0) = M_N(-jw_0) = 0$.
If this is the case, we proceed as follows:

Since $p_N(.)$ is in the finite span of even Hermite functions, it's in the form of:

$$p_N(n) = r(n)e^{-n^2/2},$$

where $r(n)$ is an even polynomial of some degree $2k$.

Thus, $p_N\big|_{\mathcal{F}}(w)$ is of the form of:

$$p_N\big|_{\mathcal{F}}(w) = r_1(w)e^{-w^2/2},$$

where $r_1(w)$ is an even polynomial of the same degree $2k$.[1] Thus, the zeros of $p_N\big|_{\mathcal{F}}(w)$ are isolated and finite in number. Let's denote by:

$$\begin{aligned} Z &= \{w_i \in \mathbb{R} : p_N\big|_{\mathcal{F}}(w_i) = 0\} \\ G(w) &= \ln\left[e^{\gamma y^2} p(y; F^*)\right]\Bigg|_{\mathcal{F}} \end{aligned}$$

Now, using (3.3) we have the product of two functions, an unknown function $G(w)$ that needs to be determined and $p_N\big|_{\mathcal{F}}(w)$ which is a known Schwartz function with zeros such that its product with $G(w)$ is $\delta(w)$.

Thus, $G(w)$ is zero *almost everywhere*. In fact $G(w) = \mu\delta(w) + f(w)$, where $f(w) = 0$ except possible on the set $Z$ and $\mu$ is a constant. [2]

We will deal with the case $f(w_i) = \mu_i\delta(w - w_i)$ ,$w_i \in Z$, and we will prove in Chapter [5] that the only possibility for $f(w_i)$ among functions/distributions that are supported at one point is being a $\delta$. Thus, this assumption together with the fact that $\ln\left[e^{\gamma y^2} p(y; F^*)\right]$ is a real and even function and thus $\ln\left[e^{\gamma y^2} p(y; F^*)\right]\big|_{\mathcal{F}}(w)$

[1]Fourier Transform of a Gaussian is another Gaussian and multiplication by polynomial corresponds to differentiation in frequency domain.

[2]The case $G(w_i) < \infty \, \forall \, w_i \in Z$ is not sensible as a distribution since it's the same as the all-zero distribution.

is real and even give us:

$$G(w) = \mu\delta(w) + k_1\delta(w - w_1) + k_2\delta(w - w_2) + \ldots k_m\delta(w - w_m)$$
$$+ k_1\delta(w + w_1) + k_2\delta(w + w_2) + \ldots k_m\delta(w + w_m),$$

where $k_1, k_2, \ldots k_m \in \mathbb{R}$; $w_1, w_2, \ldots w_m \in Z$; $2m$ is the cardinality of $Z$.
Now, we will use Inverse Distributional Fourier Transform, weak convergence as well as other techniques to reach a contradiction and prove that $p_Y(., F^*)$ is not inducible by any input pdf $p_X(.)$. This will be the subject of the next chapter.

# Chapter 4

# Proof of the Invalidity of the Output Law

As discussed in *Chapter 3*, after extending the KKT expression to the complex plane, assuming that the points of increase of $F^*$ have an accumulation point, taking Distributional Fourier Transorm and determining $G(w)$ we get:

$$\begin{aligned} G(w) = \mu\delta(w) + k_1\delta(w - w_1) + k_2\delta(w - w_2) + \ldots k_m\delta(w - w_m) \\ + k_1\delta(w + w_1) + k_2\delta(w + w_2) + \ldots k_m\delta(w + w_m), \end{aligned}$$

where $k_1, k_2, \ldots k_m \in \mathbb{R}$; $w_1, w_2, \ldots w_m \in Z$ and $2m$ is the cardinality of $Z$.
Now, taking Inverse Distributional Fourier Transform:

$$G(y) = \ln\left[e^{\gamma y^2} p(y; F^*)\right] = \mu + \frac{1}{2}k_1 \cos(w_1 y) + \frac{1}{2}k_2 \cos(w_2 y) + \ldots \frac{1}{2}k_m \cos(w_m y)$$

Thus, this yields:

$$p(y; F^*) = K_0 e^{K_1 \cos(w_1 y) + K_2 \cos(w_2 y) + \ldots K_m \cos(w_m y)} e^{-\gamma y^2}, \tag{4.1}$$

where $K_0 = e^{\mu}, K_i = \frac{k_i}{2}, i = 1, \dots, m.$
Now , using the same procedure as in proof of Theorem 4 and taking a sequence $a_n = \frac{1}{n}$, we get:

$$
\begin{aligned}
p_Y(y; F_n) &\longrightarrow p_N(y) \\
\Leftrightarrow K_{n,0} e^{K_{n,1}\cos(w_1 y) + K_{n,2}\cos(w_2 y) + \dots K_{n,m}\cos(w_m y)} e^{-\gamma_n y^2} &\longrightarrow p_N(y) \\
\Leftrightarrow K_{n,0} e^{K_{n,1}\cos(w_1 y) + K_{n,2}\cos(w_2 y) + \dots K_{n,m}\cos(w_m y)} e^{-\gamma_n y^2} &\longrightarrow r(y) e^{-y^2/2} \quad , \quad \text{as } n \to \infty,
\end{aligned} \tag{4.2}
$$

pointwise $\forall y \in \mathbb{R}$, where $r(.)$ is in the finite span of even Hermite polynomials.
Now, (4.2) is equivalent to:

$$
K_{n,0} e^{K_{n,1}\cos(w_1 y) + K_{n,2}\cos(w_2 y) + \dots + K_{n,m}\cos(w_m y)} e^{(-\gamma_n + \frac{1}{2}) y^2} \longrightarrow r(y) \ , \quad \text{as } n \to \infty \tag{4.3}
$$

## 4.1

We will split the zeros in $Z$ into two categories: the rationals and the irrationals, and we will denote by $m$ their total number.
Let $w_1, w_2, \dots w_{k-1}$ be the rational zeros; and let $\eta = \{w_k, w_{k+1}, \dots, w_m\}$ be the irrational zeros.
Define $t$ to be the Least Common Multiple(LCM) of $\frac{1}{w_1}, \frac{1}{w_2}, \dots, \frac{1}{w_{k-1}}$.

**Theorem 5.** *There exist $A, y_k, \dots, y_m \in \mathbb{R}$ such that*

$$
e^{K_{n,k}\cos(w_k y_k) + \dots + K_{n,m}\cos(w_m y_m)} \longrightarrow A \ , \quad \textit{as } n \to \infty
$$

*where*

$$\begin{cases} A \neq 0 \\ \cos(w_k y_k), \ldots, \cos(w_m y_m) \neq 0 \end{cases}$$

*and*

$$\lim_{n\to\infty} K_{n,i} < \infty \quad \forall \quad 1 \leq i \leq m, \;\; \textit{whenever } 1 \leq |\eta| < m.$$

*Proof.* We will proof the above theorem by induction over the cardinality of $\eta$ and we will split the values of the range of $i$ into two categories.

1. For $k \leq i \leq m$ :

   <u>*Base Case:*</u> $m = k,\ i.e.\ |\eta| = 1.$

   Now evaluating (4.3) at $y_1 = 4\pi b_1 t$, $y_2 = 2\pi b_1 t$ ,where $b_1$ is some integer value to be determined later, and taking the ratio we get:

$$e^{K_{n,k}[\cos(w_k 4\pi b_1 t)-\cos(w_k 2\pi b_1 t)]} e^{(-\gamma_n+\frac{1}{2})\left[(4\pi b_1 t)^2-(2\pi b_1 t)^2\right]} \longrightarrow \frac{r(4\pi b_1 t)}{r(2\pi b_1 t)}$$

   Now,

   $\gamma_n$ is increasing(being the slope of $C$ versus $a$) and $0 \leq \gamma_n \leq \frac{1}{2}$ , thus the limit of $\gamma_n$ exists.

   Let

$$\begin{aligned} \lim_{n\to\infty} \gamma_n &= \gamma_L \\ A &= \frac{r(4\pi b_1 t)}{r(2\pi b_1 t)} \frac{1}{e^{(-\gamma_L+\frac{1}{2})[(4\pi b_1 t)^2-(2\pi b_1 t)^2]}} \\ y_k &= \frac{1}{w_k} \arccos\left[\cos(w_k 4\pi b_1 t) - \cos(w_k 2\pi b_1 t)\right] \end{aligned}$$

   Thus, we have:

$$e^{K_{n,k}[\cos(w_k y_k)]} \longrightarrow A\ , \quad \text{as } n \to \infty \tag{4.4}$$

   Note the following:

   - $A \neq 0$ and finite for some integer values $b_1$ . This is the case since the number of zeros of $r(.)$ are finite , while the possible choices of $b_1$ are infinite.

- $\cos(w_k y_k) \neq 0$ , since $w_k$ and $w_1$ are relatively irrational.
  - Suppose it's zero, then:

$$\begin{aligned} w_k y_1 &= \pm w_k y_2 + 2p\pi \\ w_k \left[ 2\pi(2 \pm 1) LCM \left( \frac{1}{w_1}, \frac{1}{w_2}, \ldots, \frac{1}{w_k} \right) \right] &= 2p\pi \\ \frac{w_k}{w_1} &= \frac{p}{q}, \end{aligned}$$

Contradiction since $w_k$ and $w_1$ are relatively irrational .

So, the setup of the problem is satisfied.

Taking the logarithm on both sides of equation (4.4), the result follows.

*Inductive Case:*

Now, assume the proposition is true for $|\eta| = l - k, (m = l - 1)$, and we prove it for $|\eta| = l - k + 1\,(m = l)$.

Let $w_u = max\{w : w \in \eta\}$ .

Let:

$$\begin{cases} y_1 = \pi b_1 t - \frac{\pi}{w_u} \\ y_2 = \pi b_1 t + \frac{\pi}{w_u}, \end{cases}$$

where $b_1$ is an integer value to be determined later.

Now, $\cos(w y_1) = \cos(w y_2)$ for $w = w_1, \ldots, w_{k-1}$ (the rational zeros).

Also, $\cos(w_u y_1) = \cos(w_u y_2)$.

Evaluate (4.3) at $y_1, y_2$ and take the ratio:

$$e^{K_{n,k}[\cos(w_k y_1) - \cos(w_k y_2)] + \ldots + K_{n,u-1}(\ldots) + K_{n,u+1}(\ldots) + K_{n,l}[\cos(w_l y_1) - \cos(w_l y_2)]} \longrightarrow A,$$

where we let:

$$\begin{aligned} A &= \frac{r(y_1)}{r(y_2)} \frac{1}{e^{(-\gamma_L + \frac{1}{2})[(y_1)^2 - (y_2)^2]}}. \\ y_j &= \frac{1}{w_j} \arccos\left[\cos(w_j y_1) - \cos(w_j y_2)\right], \quad \text{where} \\ j &= k, \ldots, u-1, u+1, \ldots, l \end{aligned}$$

It's clear that $A \neq 0$ and finite , through choosing smartly $b_1$ that satisfies $r(y_1) \neq 0$ and $r(y_2) \neq 0$ .

Now, suppose $\cos(w_j y_j) = 0$, then: $w_j y_1 = \pm w_j y_2 + 2p\pi$ where $p \in \mathbb{Z}$.

- 1.

$$\begin{aligned} w_j y_1 &= -w_j y_2 + 2p\pi \\ 2b_1 w_j \pi t &= 2p\pi \\ w_j t &= \frac{p}{b_1} \\ \frac{w_j}{w_1} &= \frac{p}{qb_1}, \end{aligned}$$

  Contradiction since $w_j$ and $w_1$ are relatively irrational .

- 2.

$$\begin{aligned} w_j y_1 &= w_j y_2 + 2p\pi \\ -2\pi \frac{w_j}{w_u} &= 2p\pi \\ \frac{w_j}{w_u} &= -p \geq 1, \end{aligned}$$

  Contradiction since $\frac{w_j}{w_u} < 1$, recall that $w_u = max\{w : w \in \eta\}$ .
  Note that the assertion above $-p \geq 1$ is justified since $0 \notin Z$ and all $w's \in Z$ are positive.

In conclusion, the problem reduces to $|\gamma| = l-k$ and we found $A, y_k, \ldots, y_{u-1}$,

$y_{u+1}, \dots, y_l \in \mathbb{R}$, such that :

$$e^{K_{n,k}\cos(w_k y_k)+\ldots+K_{n,l}\cos(w_l y_l)} \longrightarrow A$$

and

$$\begin{cases} A \neq 0 \\ \cos(w_k y_k), \dots, \cos(w_l y_l) \neq 0 \end{cases}$$

Finally, by the induction step, we get :

$$\lim_{n\to\infty} K_{n,j} < \infty, \quad j = k, k+1, \dots, u-1, u+1, \dots, l$$

Evaluating (4.3) again at two values and taking the ratio, since all the coefficients have a (finite) limit, we get:

$$\lim_{n\to\infty} K_{n,j} < \infty, \quad k \leq j \leq l$$

This proves that $\lim_{n\to\infty} K_{n,i} < \infty \ \forall \ k \leq i \leq m$, and it remains to prove that $\lim_{n\to\infty} K_{n,i} < \infty \ \forall \ 1 \leq i \leq k-1$.

2. For $1 \leq i \leq k-1$ :
   We use the fact that, as proved above, $\lim_{n\to\infty} K_{n,j} < \infty \ \forall \ k \leq j \leq m$ . Also, we use induction as before and we choose:

$$\begin{cases} y_1 = \frac{\pi}{w_{irr}} - \frac{\pi}{w_u} \\ y_2 = \frac{\pi}{w_{irr}} + \frac{\pi}{w_u} \end{cases}$$

   where $w_{irr}$ is a real number that is irrational with the rational zeros $w_1, \dots, w_{k-1}$ such that $r(y_1) \neq 0$ and $r(y_2) \neq 0$ and where $w_u = max\{w_1, \dots, w_{k-1}\}$ .

We define :

$$\begin{aligned}
A &= \frac{r(y_1)}{r(y_2)} \frac{1}{e^{(-\gamma_L + \frac{1}{2})[(y_1)^2 - (y_2)^2]}} \\
y_j &= \frac{1}{w_j} \arccos\left[\cos(w_j y_1) - \cos(w_j y_2)\right], \quad \text{where} \\
j &= 1, \ldots, u-1, u+1, \ldots, k-1
\end{aligned}$$

It's clear that $A \neq 0$ and finite.

Now, suppose $\cos(w_j y_j) = 0$, then: $w_j y_1 = \pm w_j y_2 + 2p\pi$ where $p \in \mathbb{Z}$.

- 1.

$$\begin{aligned}
w_j y_1 &= -w_j y_2 + 2p\pi \\
\frac{2\pi w_j}{w_{irr}} &= 2p\pi \\
\frac{w_j}{w_{irr}} &= p,
\end{aligned}$$

  Contradiction since $w_j$ and $w_{irr}$ are relatively irrational .

- 2.

$$\begin{aligned}
w_j y_1 &= w_j y_2 + 2p\pi \\
-2\pi \frac{w_j}{w_u} &= 2p\pi \\
\frac{w_j}{w_u} &= -p \geq 1,
\end{aligned}$$

  Contradiction since $\frac{w_j}{w_u} < 1$, recall that $w_u = max\{w_1, \ldots, w_{k-1}\}$ .

In conclusion, the problem reduces to $|Z - \eta| = k - 2$, $Z - \eta$ is the set of rational zeros, and using the induction step we get :

$$\lim_{n \to \infty} K_{n,i} < \infty \;\; \forall \;\; 1 \leq i \leq k-1.$$

$\square$

Now, we will prove that the convergence in equation (4.3) is impossible. Evaluating (4.3) at $y_1 = \pi\left(\frac{k'}{w_1} + t\right)$ and $y_2 = \pi\left(\frac{k'}{w_1} - t\right)$ and taking the ratio, we get :

$$e^{K_{n,k}[\cos(w_k y_1)-\cos(w_k y_2)]+\ldots+K_{n,m}[\cos(w_m y_1)-\cos(w_m y_2)]} e^{\left(-\gamma_n+\frac{1}{2}\right)\left(y_1^2-y_2^2\right)} \longrightarrow \frac{r(y_1)}{r(y_2)}$$

Thus,

$$\lim_{n\longrightarrow\infty} e^{K_{n,k}[\cos(w_k y_1)-\cos(w_k y_2)]+\ldots+K_{n,m}[\cos(w_m y_1)-\cos(w_m y_2)]} e^{\left(-\gamma_n+\frac{1}{2}\right)(2\pi t)\left(\frac{2\pi k'}{w_1}\right)} = \frac{r(y_1)}{r(y_2)} \tag{4.5}$$

Now, the R.H.S. of equation (4.5) is a ratio of polynomials in $k'$ of equal degree and thus :

$$\lim_{k'\longrightarrow\infty} \frac{r(y_2)}{r(y_1)} = 1 \tag{4.6}$$

However,
*Assuming* $\gamma_L < \frac{1}{2}$ *for the moment* ; we treat the case $\gamma_L = \frac{1}{2}$ later in (4.2.1); and using Theorem 5 we have:

$$\lim_{n\longrightarrow\infty} K_{n,j} < \infty \quad \forall k \le j \le m$$
$$\Rightarrow \lim_{n\longrightarrow\infty} \{K_{n,k}\left[\cos(w_k y_2) - \cos(w_k y_1)\right] + \ldots + K_{n,m}\left[\cos(w_m y_2) - \cos(w_m y_1)\right]\} < \infty$$

and hence

$$\lim_{k'\longrightarrow\infty}\lim_{n\longrightarrow\infty} e^{K_{kn}[\cos(w_k y_2)-\cos(w_k y_1)]+\ldots+K_{mn}[\cos(w_m y_2)-\cos(w_m y_1)]} e^{\left(-\gamma_n+\frac{1}{2}\right)(2\pi t)\left(\frac{2\pi k'}{w_1}\right)}$$
$$= \lim_{k'\longrightarrow\infty} e^{\left(-\gamma_L+\frac{1}{2}\right)(2\pi t)\left(\frac{2\pi k'}{w_1}\right)} \lim_{n\longrightarrow\infty} e^{K_{kn}[\cos(w_k y_2)-\cos(w_k y_1)]+\ldots+K_{mn}[\cos(w_m y_2)-\cos(w_m y_1)]}$$
$$= \infty \tag{4.7}$$

Equation (4.6) and equation (4.7) yield a Contradiction.

## 4.2 Remarks

In this section we will address the different possible scenarios that were left out in Section(4.1) . We prove that a contradiction arises under all such possibilities.

### 4.2.1

What happens if :

$$\gamma_L = \lim_{n \to \infty} \gamma_n = \frac{1}{2} \quad ?$$

Then (4.2) is equivalent to:

$$K_{n,0} e^{K_{n,1}\cos(w_1 y) + K_{n,2}\cos(w_2 y) + \ldots + K_{n,m}\cos(w_m y)} \longrightarrow r(y) \ , \quad \text{as } n \to \infty \tag{4.8}$$

Well, Theorem 5 proves that $\lim_{n \longrightarrow \infty} K_{n,j} < \infty \ \forall \ 1 \leq j \leq m$ .
Now, one argues that L.H.S. in (4.8) is bounded in $y$ while the R.H.S. is a polynomial in $y$ which is impossible.

### 4.2.2

What happens if :

$$|\eta| = 0, \quad \text{(all the zeros of } r(y) \text{ are rational)?}$$

**Lemma 3.** *Let $f_n(x)$ be a sequence of functions that is periodic with period $T$. Then the limit function $f(x)$ is periodic with the same period $T$.*

*Proof.*

$$\begin{aligned} f_n(x+T) &= f_n(x) \quad \forall n \in \mathbb{N}. \\ f(x) &= \lim_{n\to\infty} f_n(x) \\ &= \lim_{n\to\infty} f_n(x+T) \\ &= f(x+T) \end{aligned}$$

□

We have:

$$K_n e^{K_{n,1}\cos(w_1 y)+K_{n,2}\cos(w_2 y)+\ldots+K_{n,m}\cos(w_m y)} e^{-\gamma_n y^2} \longrightarrow r(y)e^{-y^2/2} \ , \quad \text{as } n \to \infty$$

This is equivalent to:

$$K_{n,0} e^{K_{n,1}\cos(w_1 y)+K_{n,2}\cos(w_2 y)+\ldots+K_{n,m}\cos(w_m y)} \quad \longrightarrow \quad r(y)e^{\left(\gamma_L - \frac{1}{2}\right)y^2} \ , \quad \text{as } n \to \infty$$

Now, the L.H.S. is a periodic function $\forall n \in \mathbb{N}$. Thus, using Lemma 3 the limit function should be a periodic function with the same period; however, the R.H.S. is not periodic.

### 4.2.3

*What happens if all the zeros of $r(y)$ are irrational ?*

The same reasoning in the above theorem can be used to prove that:

$$\lim_{n\longrightarrow\infty} K_{n,j} < \infty \ \forall \ 1 \leq j \leq m$$

In fact, instead of choosing

$$\begin{cases} y_1 = \pi b_1 t - \frac{\pi}{w_u} \\ y_2 = \pi b_1 t + \frac{\pi}{w_u}, \end{cases}$$

where $w_u = max\{w : w \in \eta\}$ .

We choose:

$$\begin{cases} y_1 = b_1 \frac{\pi}{3} - \frac{\pi}{w_u} \\ y_2 = b_1 \frac{\pi}{3} + \frac{\pi}{w_u} \end{cases}$$

$$\begin{aligned} A &= \frac{r(y_2)}{r(y_1)} \frac{1}{e^{(-\gamma_L + \frac{1}{2})[(y_2)^2 - (y_1)^2]}} \\ y_j &= \frac{1}{w_j} \arccos\left[\cos(w_j y_2) - \cos(w_j y_1)\right], \quad \text{where} \\ j &= 1, \ldots, u-1, u+1, \ldots, m \end{aligned}$$

Suppose $\cos(w_j y_j) = 0$, then: $w_j y_1 = \pm w_j y_2 + 2p\pi$ where $p \in \mathbb{Z}$

- 1.

$$\begin{aligned} w_j y_1 &= -w_j y_2 + 2p\pi \\ \frac{2\pi b_1 w_j}{3} &= 2p\pi \\ w_j &= \frac{3p}{b_1}, \quad \text{Contradiction: } w_j \text{ is irrational} \end{aligned}$$

- 2.

$$\begin{aligned} w_j y_1 &= w_j y_2 + 2p\pi \\ -2\pi \frac{w_j}{w_u} &= 2p\pi \\ \frac{w_j}{w_u} &= -p, \quad \text{Contradiction: } w_u \text{ is the max} \end{aligned}$$

Thus, the problem reduces to $|\eta| = m - 1$ and using the induction step we prove that $\lim_{n \longrightarrow \infty} K_{n,j} < \infty \;\forall\; 1 \leq j \leq m$.

Once we establish this, we proceed as in Section(4.1) to prove that the convergence in equation (4.3) is impossible.

## 4.3 Conclusion

Based on this *exhaustive* study which covers all the possible cases, the convergence of $p_n(y)$ described in (4.3) is impossible and thus the capacity achieving distribution is discrete.

# Chapter 5

# Addressing the assumption of deltas in Chapter 3

As discussed in Chapter 3 ; after writing the KKT expression, extending it to the complex domain, assuming the points of increase of $F^*$ have an accumulation point and taking the Distributional Fourier Transform; we have:

$$p_N\Big|_{\mathcal{F}}(-w) \times G(w) = -\kappa\delta(w) \tag{5.1}$$

where

$$G(w) = \ln\left[e^{\gamma y^2} p(y; F^*)\right]\Bigg|_{\mathcal{F}}(w)$$

Since the zeros of $p_N\big|_{\mathcal{F}}(w)$ are isolated and finite in number , then $G(w)$ is zero *almost everywhere.* In fact $G(w) = \mu\delta(w) + f(w)$, where $f(w) = 0$ except possibly on the set $Z$ and $\mu$ is a constant.

**Theorem 6.** *Suppose that* $\delta, \delta', \delta'', \ldots, \delta^{(N)}$ *are the only functions or distributions whose support is only one point.*
*Assume w.l.o.g. that* $w = \pm w_0$ *are the zeros of* $p_N\big|_{\mathcal{F}}(w)$.
*Then,*

$$G(w) = \mu\delta(w) + \mathbf{C}\left[\delta(w - w_0) + \delta(w + w_0)\right]$$

*This is equivalent to:*

$$f(w) = \mathbf{C}\left[\delta(w - w_0) + \delta(w + w_0)\right],$$

*where* $\mathbf{C}$ *is a constant.*

*Proof.* $g(y) = \ln\left[e^{\gamma y^2} p(y; F^*)\right]$ is a real and even function since $p_Y(y)$ is even by Theorem 2. Thus, $G(w)$ is real and even being the Fourier Transform of a real and even function.

By the assumption we get:

$$\begin{aligned} G(w) = \mu\delta(w) \quad &+ \quad a_0\delta(w - w_0) + a_0\delta(w + w_0) \\ &+ \quad a_1\delta^{'}(w - w_0) + a_1\delta^{'}(w + w_0) \\ &\vdots \\ &+ \quad a_N\delta^{(N)}(w - w_0) + a_N\delta^{(N)}(w + w_0), \end{aligned}$$

where $a_0, a_1, \ldots a_n \in \mathbb{R}$ and we used the fact that $G(w)$ is even.

Taking Inverse Distributional Fourier Transform, we get:

$$\begin{aligned} g(y) = \ln\left[e^{\gamma y^2} p(y; F^*)\right] \quad &= \quad \mu + 2\sum_{k=0}^{N} a_k(-j)^k y^k \cos(w_0 y) \\ &= \quad \mu + q(y)\cos(w_0 y), \end{aligned}$$

where $q(y)$ is a real even polynomial of degree $N$ using the fact that $g(y)$ is even.

Thus,

$$p(y; F^*) = e^{\mu} e^{q(y)\cos(w_0 y)} e^{-\gamma y^2}$$

Now, since $p(y; F^*)$ is a pdf then $q(.)$ is of at most degree 2. Otherwise, $p(y; F^*)$ doesn't integrate to 1 and $p(y; F^*) \to \pm\infty$ as $y \to \infty$.

Thus,

$$G(w) = \mu\delta(w) + a_0\delta(w-w_0) + a_0\delta(w+w_0) + a_2\delta^{''}(w-w_0) + a_2\delta^{''}(w+w_0)$$

Now, it's easy to prove that $a_2 = 0$ given the fact that, as stated in the problem formulation, the case where $\exists w_0 \in \mathbb{R}$ s.t. $p_N\big|_{\mathcal{F}}(w_0) = p_N\big|_{\mathcal{F}}^{'}(w_0) = p_N\big|_{\mathcal{F}}^{''}(w_0) = 0$ is excluded.

Since $p_N\big|_{\mathcal{F}}(w_0) = 0$, then $p_N\big|_{\mathcal{F}}^{'}(w_0) \neq 0$ or $p_N\big|_{\mathcal{F}}^{''}(w_0) \neq 0$. Let's assume $p_N\big|_{\mathcal{F}}^{'}(w_0) \neq 0$.

$$\int p_N\big|_{\mathcal{F}}(w)G(w)S(w)\,dw = S(0)\ \forall\ S(.)\text{ being Schwartz.}$$

Also,

$$\int p_N\big|_{\mathcal{F}}(w)e^{-jwT}G(w)S(w)\,dw = e^{-jwT}S(w)\bigg|_0 = S(0)\ \forall\ T \in \mathbb{R}$$

Thus,

$$\int G(w)p_N\big|_{\mathcal{F}}(w)\left(1-e^{-jwT}\right)S(w)\,dw = 0$$

Now,

$$
\begin{aligned}
& \int G(w) p_N\big|_{\mathcal{F}}(w)\left(1-e^{-jwT}\right) S(w)\, dw \\
= & \int \Big[\mu\delta(w)+a_0\delta(w\pm w_0)+a_2\delta^{''}(w\pm w_0)\Big]\, p_N\big|_{\mathcal{F}}(w)\left(1-e^{-jwT}\right) S(w)\, dw \\
= & \; a_0\, p_N\big|_{\mathcal{F}}(w_0)\left(1-e^{-jw_0T}\right) S(w_0)+a_0\, p_N\big|_{\mathcal{F}}(-w_0)\left(1-e^{jw_0T}\right) S(-w_0) \\
& + a_2\, p_N\big|^{''}_{\mathcal{F}}(w_0)\left(1-e^{-jw_0T}\right) S(w_0)+a_2\, p_N\big|^{''}_{\mathcal{F}}(-w_0)\left(1-e^{jw_0T}\right) S(-w_0) \\
& + 2a_2\, p_N\big|^{'}_{\mathcal{F}}(w_0)\left(jTe^{-jw_0T}\right) S(w_0)+2a_2\, p_N\big|^{'}_{\mathcal{F}}(-w_0)\left(jTe^{jw_0T}\right) S(-w_0) \\
& + 2a_2\, p_N\big|^{'}_{\mathcal{F}}(w_0)\left(1-e^{-jw_0T}\right) S^{'}(w_0)+2a_2\, p_N\big|^{'}_{\mathcal{F}}(-w_0)\left(1-e^{jw_0T}\right) S^{'}(-w_0) \\
& + a_2\, p_N\big|_{\mathcal{F}}(w_0)\left[\left(1-e^{-jwT}\right) S(w)\right]^{''}\Big|_{w_0}+a_2\, p_N\big|_{\mathcal{F}}(-w_0)\left[\left(1-e^{-jwT}\right) S(w)\right]^{''}\Big|_{-w_0} \\
= & \; 2a_2\, p_N\big|^{'}_{\mathcal{F}}(w_0)\,(jT)\, S(w_0)-2a_2\, p_N\big|^{'}_{\mathcal{F}}(w_0)\,(jT)\, S(-w_0) \\
= & \; 2a_2 jT\, p_N\big|^{'}_{\mathcal{F}}(w_0)\left[S(w_0)-S(-w_0)\right],
\end{aligned}
$$

where we choose $T=\frac{2\pi}{w_0}$ and used the fact that $p_N\big|^{'}_{\mathcal{F}}(.)$ is odd . □

Thus,

$$2a_2 jT\, p_N\big|^{'}_{\mathcal{F}}(w_0)\left[S(w_0)-S(-w_0)\right]=0$$

Since this is true for all Schwartz functions, we choose a Schwartz function $S(.)$ that is not even. This yields $a_2=0$.
Note that if it's the case that $p_N\big|^{'}_{\mathcal{F}}(w_0)=0$ while $p_N\big|^{''}_{\mathcal{F}}(w_0)\neq 0$, then we proceed as above to prove that $a_2=0$ through choosing $T=\frac{\pi}{2}$.

Also, suppose that $p_N\big|_{\mathcal{F}}(w)$ has more than two zeros but finitely many. Let's assume there are $2k$ zeros which exist in pairs: $\pm w_0,\ldots,\pm w_{k-1}$ .
Then, since the zeros are isolated we can write $G(w)$ as :

$$
\begin{aligned}
G(w) &= \mu_1\delta(w)+f(w) \\
&= \mu_1\delta(w)+h_0(w)+h_1(w)+\ldots+h_{k-1}(w)
\end{aligned}
$$

where **supp** $h_i(w) \in \{\pm w_i\}$ for $i = 0, \ldots, k-1$.

Now, from equation (5.1) we have:

$$p_N\big|_{\mathcal{F}}(-w) \times G(w) = -\kappa\delta(w)$$

Thus,

$$p_N\big|_{\mathcal{F}}(w) \times h_i\,(w) = 0 \;\; \forall\; i = 0, \ldots, k-1$$

In each of the above equations, the same procedure as in the proof of Theorem 6 can be used to yield:

$$G(w) = \mu_1\delta(w) + \sum_{i=0}^{k-1} \mathbf{C_i}\left[\delta(w - w_i) + \delta(w + w_i)\right]$$

This justifies the assumption in the above theorem that the zeros of $p_N\big|_{\mathcal{F}}(w)$ are: $\pm w_0$.

# Chapter 6

# Conclusion

We have proved in this study that the capacity achieving distribution of an average power constrained linear channel is discrete, whenever the noise is in the finite span of even Hermite functions. However, we have the intuition that this result will still hold even if the noise PDF is in the infinite span of Hermite functions. It might be the case that through some modifications to our suggested approach, we can generalize our result to the case of the noise being in the infinite span of Hermite functions. We plan to investigate this problem soon.

The major millstone that we faced in our approach was to rigorously prove the intuition that the only possibility for $G(w)$ is being a combination of shifted deltas, one centered at 0 and each of the others centered at one of the zeros of $p_N|_{\mathcal{F}}(.)$. This was a challenging task and we were able to prove that intuition through making use of the fact that $p_N(.)$ is even which leads to $G(w)$ is even. That result in a sense establishes the following theorem.

**Theorem 7.** *If* $F|_{\mathcal{F}} \times G|_{\mathcal{F}} = \delta(w)$*, where the zeros of* $F|_{\mathcal{F}}$ *denoted by* $w_i \neq 0$ *are isolated and countable and* $G|_{\mathcal{F}}$ *is even, then* $G|_{\mathcal{F}}(w) = c\delta(w) + \sum_i c_i \delta(w - w_i)$ .

The question that arises is whether the same result that $G(w)$ is a combination of shifted deltas still holds when $G(w)$ is not even. The setup of this problem is in a sense complementary to Weiner's Tauberian Theorem [14] since in Weiner's theorem we have:

(a) The convolution of the two functions is zero.

(b) The Fourier Transform of $F$ has no zeros.

However, in our problem the convolution of the two functions is zero (*we know that* $G(0) = \delta(w)$ *so we can cast the problem as the product of the Fourier Transform of the functions being zero instead of* $\delta(w)$) , but the Fourier Transform of $F$ has zeros. Thus, by the Tauberian Theorem [15] the translates of $F$ are not dense in $\mathcal{L}_2(\mathbb{R})$ which makes the problem harder to solve. Our next objective is to investigate this problem further and determine whether the above theorem can be generalized to such scenarios.

# Appendix A

# Noise is in the Span of $\psi_0$ , $\psi_2$ and $\psi_4$

In this chapter, we will prove that the capacity achieving distribution is discrete whenever the noise is in the span of $\psi_0$ , $\psi_2$ and $\psi_4$. In particular, the motivation example discussed in Chapter 3 falls in this category.
Since $p_N(.) \in$ span of $\{\, \psi_0(.), \psi_2(.), \psi_4(.) \,\}$ , then $p_N(u)$ is given by:

$$p_N(u) = (\alpha u^4 + \beta u^2 + \gamma) e^{-u^2/2}$$

Now, taking the sequence $a_n = \frac{1}{n}$ which in turn will generate a sequence of input distributions or random Variables $X_n$, we get:

$$p_Y(y; F_n) = \int p_N(y - x)\, dF_n(x)$$

Using Proposition **??** and the fact that $p_N(y - x)$ is continuous and bounded, we have:

Fix $y \in \mathbb{R}$:

$$\begin{aligned}\int p_N(y-x)\,dF_n(x) &\longrightarrow \int p_N(y-x)\,dF(x)\\ &= \int p_N(y-x)\,\delta(x)\\ &= p_N(y), \quad \text{as} \quad n \to \infty\end{aligned}$$

As in Chapter 3, let's denote by:

$$Z = \{\pm w_1,\ \pm w_2 \in \mathbb{R} \colon p_N\big|_{\mathcal{F}}(\pm w_1) = p_N\big|_{\mathcal{F}}(\pm w_2) = 0\}$$

Using equation (4.1) we get:

$$p(y; F_n^*) = K_n e^{K_{1,n}\cos(w_1 y) + K_{2,n}\cos(w_2 y)} e^{-\gamma_n y^2}$$

Thus, we have:

$$K_n e^{K_{1,n}\cos(w_1 y) + K_{2,n}\cos(w_2 y)} e^{-\gamma_n y^2} \longrightarrow (\alpha y^4 + \beta y^2 + \gamma) e^{-y^2/2}, \quad \text{as} \quad n \to \infty \tag{A.1}$$

Assume that $w_1$ and $w_2$ are relatively irrational.

Choose:

$$\begin{aligned} y_1 &= \pi\left(\frac{l}{w_2} - \frac{k}{w_1}\right)\\ y_2 &= \pi\left(\frac{k}{w_1} + \frac{l}{w_2}\right)\end{aligned}$$

Notice that this choice of $y_1$ and $y_2$ yields:

- $\cos(w_1 y_1) = \cos(w_1 y_2)$

- $\cos(w_2 y_1) = \cos(w_2 y_2)$

Thus, choosing $l = 1$ and evaluating (A.1) at $y_1$ and $y_2$ and taking the ratio :

$$e^{\left(-\gamma_n+\frac{1}{2}\right)\left(y_2^2-y_1^2\right)} \longrightarrow \frac{(\alpha y_2^4 + \beta y_2^2 + \gamma)}{(\alpha y_1^4 + \beta y_1^2 + \gamma)}$$
$$e^{\left(-\gamma_n+\frac{1}{2}\right)\left(\frac{2\pi k}{w_1}\right)\left(\frac{2\pi}{w_2}\right)} \longrightarrow \frac{(\alpha y_2^4 + \beta y_2^2 + \gamma)}{(\alpha y_1^4 + \beta y_1^2 + \gamma)}, \quad \text{as} \quad n \to \infty \tag{A.2}$$

We know that, $0 < \gamma_n \leq \dfrac{1}{2}$ and that $\gamma_n$ is increasing. Thus, $\lim\limits_{n\to\infty} \gamma_n$ exists and is denoted by $\gamma_L$.

*Case1:* $\lim\limits_{n\to\infty} \gamma_n < \dfrac{1}{2}$

The left hand side of (A.2) increases exponentially with $k$ while the right hand side is a polynomial in $k$, which is a *Contradiction.*

In fact,

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty} e^{\left(-\gamma_n+\frac{1}{2}\right)\left(\frac{2\pi k}{w_1}\right)\left(\frac{2\pi}{w_2}\right)} \\
&= \lim_{k\to\infty} e^{\left(-\gamma_L+\frac{1}{2}\right)\left(\frac{2\pi k}{w_1}\right)\left(\frac{2\pi}{w_2}\right)} \\
&= \infty
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty} e^{\left(-\gamma_n+\frac{1}{2}\right)\left(\frac{2\pi k}{w_1}\right)\left(\frac{2\pi}{w_2}\right)} \\
&= \lim_{k\to\infty} \frac{(\alpha y_2^4 + \beta y_2^2 + \gamma)}{(\alpha y_1^4 + \beta y_1^2 + \gamma)} \\
&= 1
\end{aligned}$$

*Case2:* $\lim\limits_{n\to\infty} \gamma_n = \dfrac{1}{2}$

Then, (A.1) is equivalent to:

$$K_n e^{K_{1,n}\cos(w_1 y)+K_{2,n}\cos(w_2 y)} \longrightarrow (\alpha y^4 + \beta y^2 + \gamma), \quad \text{as} \quad n \to \infty \tag{A.3}$$

It's easy to prove that $\lim_{n\to\infty} K_{1,n} < \infty$ , and $\lim_{n\to\infty} K_{2,n} < \infty$. Thus, the left hand side of (A.3) is bounded in $y$ while the right hand side is polynomial in $y$.

Thus $p_Y(y; F^*)$ is not inducible by any input PDF $p_X(.)$. So, our assumption; that the points of increase of $F_X^*$ have an accumulation point; is invalid. Thus, the input capacity achieving distribution is discrete.

# Appendix B

# Continuity of Optimal dist in $a$

In this chapter, we will prove that if the capacity achieving distribution satisfying $\mathsf{E}[x^2] \leq a$ is continuous in a. Also, we claim that there exists a sequence of continuous distributions $F_n$ s.t. $E_{F_n}[x^2] \to 0$ as $n \to \infty$.

Fix $a > 0$. Let $\zeta_a$ denote the input distributions satisfying $\mathsf{E}[x^2] \leq a$ and let $F_a^* \in \zeta_a$ denote the optimal distribution.

**Property 1.** *If $F_\epsilon \to F$ in the weak sense and $F$ has an accumulation point $x_0$, then $\exists$ a subsequence that has an accumulation point.*

*Proof.* $F_\epsilon \to F$ in the weak sense, thus:

$$\lim_{\epsilon \to 0} \int f\, dF_\epsilon = \int f\, dF \;\; \forall\, f \text{ being continuous and bounded}$$

Now, fix $\delta > 0$ and consider the interval $I = (x_0 - \delta, x_0 + \delta)$ and let $supp f = I$. Then,

$$\lim_{\epsilon \to 0} \int_I f\, dF_\epsilon = \int_I f\, dF \quad > 0$$

Thus, $\exists$ a subsequence $F_{\epsilon'}$ s.t. $F_{\epsilon'}$ has an accumulation point and $\epsilon' \to 0$.

□

Now, consider the sequence of CDFs :

$$F_\epsilon = \frac{\epsilon}{a}\delta(x) + \left(1 - \frac{\epsilon}{a}\right) F_a^*$$

Then $F_\epsilon \in \zeta_{a-\epsilon}$. In fact,

$$\begin{aligned} \int x^2\, dF_\epsilon &= \int x^2 \frac{\epsilon}{a}\, d\delta(x) + \int x^2 \left(1 - \frac{\epsilon}{a}\right) dF_a^*(x) \\ &= \left(1 - \frac{\epsilon}{a}\right) a \;=\; a - \epsilon \end{aligned}$$

Also, $F_\epsilon \to F_a^*$ in the weak sense as $\epsilon \to 0$ . In fact,

$$\lim_{\epsilon\to 0} \int f\, dF_\epsilon = \int \lim_{\epsilon\to 0} f\, dF_\epsilon = \int f\, dF_a^* \;\; \forall\, f \text{ being continuous and bounded,}$$

where we interchanged limit and integral by DCT.
Now, assume $F_a^*$ has an accumulation point, then using property 1 there $\exists$ a subsequence of CDFs $F_{\epsilon'} \in \zeta_{a-\epsilon}$ s.t. $F_{\epsilon'}$ has an accumulation point and $\epsilon^{'} \to 0$.

Now, consider the sequence :

$$G_{\epsilon'} = \epsilon^{'} F_{\epsilon'} + \left(1 - \epsilon^{'}\right) \delta$$

Then, $G_{\epsilon'}$ is a sequence of CDFs where each has an accumulation point. Also,

$$\begin{aligned} \int x^2\, dG_{\epsilon'} &= \int x^2 \epsilon^{'}\, dF_{\epsilon'}(x) + \int x^2 \left(1 - \epsilon^{'}\right) \delta(x) \\ &= \epsilon^{'} (a - \epsilon) \to 0 \end{aligned}$$

Thus, $G_{\epsilon'} \to 0$ in the Mean Square sense. Thus, $G_{\epsilon'} \to 0$ in distribution and since we are working on $\mathbb{R}$ convergence in distribution implies weak convergence.

Finally, we claim that $F^*$ is continuous in $a$. We present below the main steps of the proof.

*Elements of the proof of the claim:*

1. $I(F)$ is continuous.

2. $J = ]I(F^*) = C - \delta, C + \delta[$ is open

3. $V =$ inverse image of $J$ is open

4. $V \cap \zeta_{a-\epsilon} \neq$ for some $\epsilon > 0$.

5. $F^*_{a-\epsilon} \in V$

# Bibliography


[1] C. E. Shannon, "A mathematical theory of communication, parts i & ii," *Bell Syst. Tech. J.*, vol. 27, pp. 379–423; 623–656, 1948.

[2] J. G. Smith, "The information capacity of peak and average power constrained scalar Gaussian channels," *Inform. Contr.*, vol. 18, pp. 203–219, 1971.

[3] S. Shamai and I. Bar-David, "The Capacity of Average and Peak-Power-Limited Quadrature Gaussian channels," *IEEE Trans. Inf. Theory*, vol. 41, pp. 1060–1071, July 1995.

[4] I. Abou-Faycal, M. D. Trott, and S. Shamai, "The capacity of discrete-time memoryless Rayleigh-fading channels," *IEEE Trans. Inf. Theory*, vol. 47, pp. 1290–1301, May 2001.

[5] J. Fahs and I. Abou-Faycal, "On the Detrimental Effect of Assuming a Linear Model for Non-Linear AWGN Channels," *in Proceedings IEEE International Symposium on Information Theory*, p. 1693, August 2011. St. Petersburg, Russia.

[6] M. N. J. Lin and B. Evans, "Non Parametric Impulsive Noise Mitigation in OFDM Systems Using Sparse Bayesian Learning,"

[7] A. Tchamkerten, "On the Discreteness of Capacity-Achieving Distributions," *IEEE Trans. Inf. Theory*, vol. 50, pp. 2773–2778, November 2004.

[8] A. Das, "Capacity-Achieving Distributions for Non-Gaussian Additive Noise Channels," *in Proceedings IEEE International Symposium on Information Theory*, p. 432, June 2000. Sorrento, Italy.

[9] J.Fahs, N.Ajeeb, and I.Abou-Faycal, "The capacity of average power constrained additive non-gaussian noise channels," *in 19th International Conference on Telecommunications*, April 2012. Jounieh, Lebanon.

[10] S. Thangavelu, "On Theorems of Hardy, Gelfand-Shilov and Beurling for Semisimple Groups," *Publ. RIMS, Kyoto Univ.*, vol. 40, pp. 311–344, 2004.

[11] C. Pfannschmidt, "A Generalization of the Theorem of Hardy: A Most General Version of the Uncertainty Principle for Fourier Integrals," *Math. Nachr.*, vol. 182, p. 317327, 1996.

[12] H. Silverman, *Complex Variables.* Houghton Mifflin Company, 1975.

[13] H. Cramer, *Random Variables and Probability Distributions.* Cambridge University Press, 1970.

[14] W. Rudin, *Functional Analysis.* Theorem 9.5, pp. 211.

[15] Wikipedia, "Wiener's tauberian theorem." `http://en.wikipedia.org/wiki/Wiener's_tauberian_theorem/`.

[16] M. Katz and S. Shamai, "On the capacity-achieving distribution of the discrete-time noncoherent and partially coherent AWGN channels," *IEEE Trans. Inf. Theory*, vol. 50, pp. 2257–2270, October 2004.

[17] R. Gallager, *Information Theory and Reliable Communication.* John Wiley & Sons, November 1968.

[18] A. Shiryaev, *Probability.*