# THE CONCEPT OF NUMBER
# IN PIAGET

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ABSTRACT

The works of Jean Piaget on the formation of concepts mark a new frontier in child developmental psychology. He gives a new impetus to epistemology by basing his works on experimental studies he either did alone or in collaboration with others. His originality lies in the methodology he adopted to solve problems concerning the growth of knowledge.

In the field of mathematics his contribution can be looked upon from two angles; his contribution to the understanding of some mathematical concepts *qua* concepts, and his contribution to describing their formation from a psychological point of view.

His works on the formation of numbers clarified the controversy between Russell and Whitehead on the one hand and Poincaré on the other.

He concluded that number is neither reducible to purely logical notions such as the " class of classes " nor to purely intuitive ones such as those held by Poincaré. Piaget conceives of number as a synthesis of the two logical operations of seriation and classification.

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# CHAPTER I

## INTRODUCTION

> " For Frege this was the alternative: either we are dealing with strokes of the pen - this would not give an arithmetic - or we must grant that the symbols have a meaning, and then the meaning exists independently of the symbols." 1

Our interest is in the meaning of number, i.e. the meaning of these symbols. Some think that this is a function of an inherent faculty in man:

> " Man, even in the lower stages of development, possesses a faculty which, for want of a better name, I shall call Number Sense". 2

Whether we consider the number sense is best explained in terms of an inherent faculty in man or an ability acquired through teaching, the study of the concept as such is worthy of close study.

### PURPOSE

Of all other mathematical concepts why study the number concept? And if so, why the number concept of Jean Piaget?

The purpose of the study of the number concept is of double importance. On the one hand it is highly useful to understand the nature of number when we are teaching it. On the other hand, number happens to be the first concept that a child faces in his first year at school. The current revisions of the curricula of mathematics in

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1. F. Waismann, Introduction to Mathematical Thinking (N.Y. : Harper 1959), p. 244.

2. T. Dantzig, Number The Language Of Science ( N.Y. : Macmillan Co., 1954), p. 1.

some fifty countries would substantiate this claim.[3]

Jean Piaget's original work on the number concept is widely hailed. His originality, however, lies more in his epistemological method rather than in content. This study will be oriented towards the development of Piaget's number concept.

The choice of studying Piaget's number concept rests on the hope that this concept can be used as an epistemological 'model' for other concepts as well.

## METHOD

For this purpose, a review of his general theory is indispensable. In this introductory chapter some guide lines of Piaget's general theory are developed. Then, in the second chapter those aspects of the general theory that are relevant to the number concept will be exposed. The third chapter is devoted to the evolution of the concept of number. And finally the educational implications of teaching mathematics will be reviewed using the number concept as a model.

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3. Introduction to Mathematics in Primary Schools, XIIIth Inter. Conf. On Public Education, UNESCO and I.B.E. ( Geneva, 1950).

## GENERAL ORIENTATION

> " It is a great mistake to suppose that a child acquires the notion of number and other mathematical concepts just from teaching. On the contrary, to a remarkable degree he develops them himself, independently and spontaneously. When adults try to impose mathematical concepts on a child prematurely, his learning is merely verbal: true understanding of them comes only with his mental growth." 4

This puts us directly in front of two important issues. On the one hand the question is raised as to the efficiency of teaching mathematical concepts. If it is true that these concepts are not acquired just from teaching, then how is it that the child acquires them? The answer to this question would direct us to one side of Piaget's theory of knowledge. On the other hand, if the spontaneous appearance of these concepts is a function of the mental growth of the child, how are these concepts acquired? To put it differently, one question asks about the nature of the acquisition in terms of the functional and structural complex of mathematical knowledge, while the other asks whether these concepts just shine on the child ( or in the child, it doesn't matter which in this context) at a certain level of mental growth. Does the child learn these concepts in wholes at certain times in his development or not?

These will be the two main questions that will be dealt with in this study.

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4. J. Piaget, " How Children Form Mathematical Concepts", Sci. Amer., ( 189 No. 5, 1953 ), p. 74.

## Biographical Background*

Jean Piaget was born in Neuchâtel, Switzerland on August 9, 1896. He had his secondary education there and attended the University of Neuchâtel, from which he received his doctorate in the year 1918. As a child he was interested in biology and zoology, and wrote his first scientific paper at the age of ten about a sparrow he saw in the park. He was also interested in philosophy, religion, and psychology among other things. The question of epistemology concerned him at a very young age and accompanied him all his life. While working with Dr. Simon, who was in charge of the Binet laboratory in Paris, he became very much interested in child psychology. Then Claparède offered him a job as the director of J.J. Rousseau Institute in Geneva. From there he launched his studies on child psychology that brought him world fame before he was thirty. Up till 1963 the books and articles that he wrote alone, or in collaboration with others, amount to around 150. It is only lately, in the last five to ten years, that his works received considerable attention as experimental works. A review of the titles of these books and articles would show his diversity of interests.

Three major aspects of Piaget's theory will be discussed now; namely, Piaget the scientist, Piaget the logician and finally Piaget the epistemologist - Genetic Epistemologist.

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* The bibliographical comments in this section are wholly taken from J.H. Flavell, op.cit.

## Piaget The Scientist

Since all of Piaget's works can be described as scientific works one should feel justified to ask the question about his specific scientific aim. Flavell describes his aim as follows:

> " The theoretical and experimental investigation of the qualitative development of intellectual structures". 5

Another point could be made as evidence for Piaget's scientific spirit. Those who are already acquainted with the criticisms charged against Piaget's method of inquiry and its validity*, will certainly appreciate the implications behind Piaget's formulation of his theory of stages as a function of environmental factors:

> " It is evident that the succession of these stages may be accelerated or retarded by the 'milieu' and that the average ages vary from one 'milieu' to another. 6

From a purely methodological point of view, this statement can be considered as a concession to the possibility of variations in modifications of learning as will be dictated by the prevailing conditions. This concession is extremely important. The succession of stages of development, he claims, is dependent on milieu. With regard to intelligence testing as a methodology, Piaget's opinion,

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5. J.H. Flavell, op.cit., p. 261.

6. J. Piaget,"Les Developpements Intellectuel chez les Enfants", Mind, ( XL 1941), 137-160.

* The criticisms against his use of the clinical method in experimentation, and experiments done in one society.

seem to retain generality by implication. Talking of Piaget, Mays says:

> " He criticizes the method of intelligence testing as a tool for investigating child thought, since (a) the test-questions and test-conditions under which the test is given are stereotyped in character, and (b) the use of a fixed questionaire neglects the spontaneous answers of the child". 7

The spontaneous answers of the children, and the spontaneous formation of concepts in their minds are key words to Piaget's theory. What he wanted to say is that these tests, if they tell anything, tell very little because they miss a great deal of what is going on in the mind of the child. Now it would seem that Piaget had an important subject matter to study; and he applied to it a method which may be as reliable as the existing methods. Of course the criticisms were made on his early research, which he considered later as a first draft for more sophisticated research.*

## Piaget The Logician

Piaget's works in this field range from the literary interpretations of logic in connection with other disciplines to a detailed study of formal logic. His major works in the latter aspect comprise two volumes:

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7. W.Mays., " Development of Logical and Mathematical Concepts", <u>Nature</u>, ( 174, 1954), p. 625.

* Cf., Flavell, p. viii.

Traité de Logique*, and Essai Sur Les Transformation Des Operations Logiques*. His interdisciplinary remarks appear mostly in his work; Logic and Psychology ***.

What is of concern, here, is his theory of the relation between logic and epistemology, which will develop later into a coherent theory of number.

Piaget draws a clear distinction between two types of logic and (in his terminology) one is the Logique Formelle and the other is the Logique Appliquée. This distinction is to emphasize his interest in the second type of logic. And this he thinks comes before formal logic, historically and genetically:

> Historically and genetically, the so called applied logic always preceeds the formal logic. 8

The subject matter of formal logic is to receive,

> ....simply, as given a certain number of statements, that are qualified as some false and some true, and its proper work starts with the formal composition of these statements, true or false by hypothesis.9

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* (Colin, Paris, 1949).

* (Paris, presses univers., France, 1952).

*** (N.Y.: Basic Books, 1957).

8. J. Piaget., Traité de Logique (Paris: Colin, 1949), p. 6.

9. Ibid, p. 7.

And this kind of logic is applicable to pure mathematics:

> " Bernays held, for example, and from the standpoint of a perfectly formalized anxiomatic logic he is undoubtedly right, that logical relations are strictly applicable only to mathematical deduction, since every other form of thought merely has an approximate character."10

These two quotations would now constitute a reference to Piaget's use of formal logic. There is some doubt as to whether Piaget wanted to speak of this kind of logic in his <u>Growth of Logical Thinking From Childhood to Adolescence</u>. Parsons [11] reviews this work and points out the ambiguity in Piaget's logical symbolism; and he also makes an evaluation of Piaget's works on stages corresponding to logical structures. Then asks the question on how to characterize the behavior of a subject that cannot be explained by the 'four'* group, if it is more complicated. The question, though interesting on the theoretical hypothetical level, might not have much weight when looked upon empirically. Other remarks pertaining to the purely formal type of logic are brought up by Donaldson in her revision of <u>La Gènèse des Structures Logiques Elémentaires: Classifications et Seriations.</u> [12] The important notion in the research of Inhelder and Piaget is the notion of the null class. Donaldson believes that a set of empty cards doesn't really constitute a null class. And if

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10. J. Piaget, <u>Logic and Psychology</u> ( N.Y.: Basic Books, 1957) pp. 2-3.

11. Charles Parsons, " The Growth of Logical Thinking, A logician's Point of View " <u>British Journal of Psychology</u>," (LI, 1960), 75-84.

* This type of groups will be discussed later.

12. Margaret Donaldson, "La Genese des Structures Logiques Elémentaires: Classifications et Seriations" <u>British Journal of Psychology</u>, (LI,1960) pp. 181-184.

it does, it is certainly an ambiguous conception of the null class. Here again, this is an interesting discussion on the theoretical level; but one would tend to think that the null class as defined by the axioms of the 'group' of classes is more of an abstraction of the 'intuitive' notion encountered on cards with no drawings on them. If the other classes formed are formed by applying a certain property-criterion to the drawings on them, then the empty card would seem to be the closest approximation to the concept of an empty class. Many other issues can be investigated experimentally in this connection as to whether children classify, i.e. form classes of cards or of pictures on cards. In the first case it would be absurd to say that the empty class is the class of empty cards.

The most interesting aspect of Piaget, is that he uses a sophisticated logical analysis to describe the behavior of the child. The other type of logic is logique appliquée.

To Piaget, there are two types of experience; the immediate experience and the scientific one which is related to logic:

"En premier lieu, il y a ce que nous désignerons sous le nom d'expérience immédiate, c'est à dire le contact direct et non critiquée avec les faits....

"En second lieu, il y a ce que nous appelerons l'expérience scientifique, c'est à dire l'acte par lequel la raison interroge les faits lorsqu'elle a mis les faits en relations suffisament précises les uns avec les autres pour qu'ils répondent par oui ou par non". [13]

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13. J. Piaget., Mind, op. cit., p.147.

A step further, when all the judgments derived from experience can be classified in such a way that they yield questions answerable by yes or by no, we then have formal logic. This would lead to a classification of statements described as true or false. Formal logic then, will be in the final abstraction if this procedure is continued.

Another way of looking at logic within a psychological perspective makes logic a function of society. And certainly it is not the formal discipline that is intended here:

"Si la societé ne crée pas l'intelligence, elle est donc tout au moins régulatrice de la raison. La logique, du point de vue psychologique, n'est pas autre chose que cette régulation." 14

Here again the question on how far does the society 'paternize' the thought of its members may be a question to be settled by empirical investigation; if not totally, at least in parts. [15]

Let us now examine logic still under a different light. Since philosophical investigations would deal with the relationship between logic in form and logic in content, in some of their aspects, and since there is no need to keep any form that doesn't help the philosopher pass

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14. *Ibid.*, p. 158.

15. A specific question that may be interesting to investigate would deal with the first appearance of contrapositive inference in the child. And then see how influential are the social factors.

judgments about its contents, it would be reasonable to expect the logical forms to undergo continuous modification till they become adequate enough. It is also obvious that, in the long run, this process of "purification" would lead to an axiomatic abstract logic. But

> "the very fact that philosophical interpretations leave its [symbolic logic] internal technique unchanged shows that the latter has reached the axiomatic level; symbolic logic thus constitutes, if for no other reasons, an ideal 'model' of thought". 16

It is these "other reasons" that Piaget is interested in. And this would clearly illustrate how Piaget thinks of logic in connection with the empirical facts when he is the observer of a subject under experimental conditions. As will be illustrated later, the relation he thinks of is a descriptive relation that helps us imagine what is going on in the mind of the child. Though logic is used as a "model" of cognitive systems, it would be a mistake to anticipate psychological facts on the basis of logical analysis:

> "One can no more try to settle questions of cognitive facts by appeal to logic as an ideal model than one can prove a logical theorem by observing how people think." 17

The two disciplines should keep their independent autonomy.

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16. J. Piaget, Psychology of Intelligence (N.Y.: Harcourt, Brace & Co. 1950), p. 29.

17. Flavell, op.cit., p. 170.

This would bring us face to face with the relation between logic and epistemology. Though being an intricate relation one must keep in mind that these two fields deal with different sides of human thought.

"Nous conviendrons donc d'appeler épistémologie l'étude de la connaissance en tant que rapport entre le sujet et l'objet et de réserver le term de logique pour l'analyse formelle de la connaissance." 18

It is obvious that he is talking of formal logic at this level, for one would not expect the "logique appliquée" to be so sophisticated in abstraction. By way of summing up, a concluding note to stress the functions of the two disciplines will act as a guiding line on the limitations of the scope of logic.

" C'est en ce sens que la logique demeure exclusivement relative aux activités du sujet et ne s'occupe pas des interactions entre le sujet et l'objet, lesquelles concernent seulement l'épistémologie." 19

Since the question of epistemology is going to occupy us now some introductory remarks on Piaget's psychology will be of a great help.

Only those aspects of his psychology that are necessary for the understanding of his epistemology will be dealt with here. More specifically, Piaget believes that there is an intricate relation among logic, psychology and epistemology.

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18. J. Piaget., Traité de Logique, op.cit., p. 4.
19. Ibid., p.5.

" Piaget believes that most logicians and mathematicians do possess some assumptions implicit or explicit of what their elementary terms and operations express in the way of human cognitive activity, and the student of intellectual development may be of assistance here by exploring the developmental history of these terms and operations."[20]

The important point that should be kept in mind as we proceed concerns the nature of these operations; They are dynamic and in constant development. And this is precisely the point where he disagrees with the Gestalt psychologists in their conception of cognition. The following are the points of disagreement - (using "schemas" as Operations).

(1) The "schemas", to Piaget, are dynamic in the genetic sense. While those of the Gestaltists seem to come out of vacuum.

(2) The "schemas" are generalizing structures, while Gestaltists don't have such a history of 'experience'.

(3) "Schemas" are relatively "good" in the sense that they compare with the continuous early ones and the ones to come; and hence less pretentious than the 'forms' groping in an extra intelligent trial and error to reach good Gestalten. [21]

Keeping in mind, therefore, that logic deals with the formal constitution of cognition, epistemology with the interrelation between

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20. Flavell, op.cit., p. 170.

21. Flavell, op.cit., pp. 73-74.

object and the cognizer, and that these cognitive schemas are developmental, in a dynamic sense, we may proceed to examine different aspects of his epistemological theory.

## Genetic Epistemology - Piaget the Epistemologist

Piaget confines himself to an original epistemological question by asking " ... comment s'accroissent les connaissances? " [22]. And that would constitute his primary concern. Speaking of this question as original does not mean that philosophers haven't dealt with it before:

> "What is really important about Piaget's approach is that it gives us for the first time a method of testing experimentally many of the concepts and principles which philosophers have discussed in the past on a purely a priori level." 23

The originality then, lies in the method or approach, more than in the question itself. And this orientation towards the development of knowledge would explain why the child attracts Piaget's attention. On the other hand, to start with the child is important for another good reason.

> "And Piaget would argue that a careful study of logical systems at this simple level might also lead to a better understanding of the structure of adult thought". 24

Now a sketch of Piaget's epistemological works can be outlined.

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22. J. Piaget, <u>Traité de Logique</u>, <u>op.cit</u>., p. 5.

23. W. Mays., <u>Nature</u>, <u>op.cit</u>., p. 625.

24. W. Mays., "The Epistemology of Prof. Piaget", <u>Proc. Arist. Society</u> ( LIV, 1953 - 1954), 49 - 76, p. 75.

On the one hand, if concepts develop with age an examination of their starting point is essential to know how they grow. On the other hand, the study of the starting point will help us understand the operations in the adults thought. This concession, however, will answer the questions asked above on whether concepts are taken in 'wholes'; for now we see that they 'grow'.

And now, to study this growth will involve leaning on other sciences and drawing relations among them. That is why Piaget thinks of the 'circle of sciences' [25] as he approaches this problem. An example of this circle, one can start with logic - mathematics, which is a human activity hence psychological, and then ending up by cognition being modeled by structures will lead back to logic - mathematics. This is really an over-simplification of the relations involved; but it does add some clarity to the interrelations among sciences.

How does his main epistemological position contrast with other schools of thought? First, Piaget would reject the empiricist thesis that the subject gets to know reality in simple direct contact. To him the relation is subtle and complex, somewhat similar to Kant's concession on not knowing the 'ding an sich'.

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25. Refer to Flavell p.254 and to Piaget, Introduction a l'Epistémologie Génétique, Vol. III Conclusions (Paris: Presses Univer.1950).

> " The apprehension of reality is ever and always as much an assimilatory construction by the subject as it is an accomodation of the subject. This is the epistemological restatement of the notion that the twin invariants assimilation and accommodation are indissociably involved in all contacts with reality." 26

This rejection of empiricism would become clearer when we come to explain the concepts of assimilation and accommodation. Let us now accept it, though tentatively, for the sake of gaining more appreciation of the general epistemological position at this stage.

Second, Piaget would disagree with the school of intellectualism on the concept of intellectual structures by which we apprehend reality in an organic manner, for he believes that these structures change.[27] He would side more with the Gestalt psychologists on this issue in particular. But before going too far we should note here, that the Gestaltists' pure groupings' is not consistant with Piaget's frame of mind. He might accept, though, a continuum of undirected - directed behavior,which will lead to a cognition which is equilibrated - disequilibrated in manner. This position also, will become clearer when we discuss the functional aspect of his theory of genetic development. And once more, let us restrain ourselves to the general lines. By way of a brief answer to satisfy the eager questioner who still wants to know how does cognition begin, one could say: Cognition starts at the boundary between the

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26. Flavell, op.cit., p. 69.

27. Ibid., p. 71.

cognizer and the object as development proceeds alternatingly through assimilation - accommodation.[28]

Looking now with a different perspective , i.e. in terms of what comes first in cognition, the reality as such, or the functional operations, we may give a clearer idea of the answer just given, as well as put the general position in a more current terminology.

> " In other words, to use current terminology, 'knowing how' preceeds 'knowing that' in child's thought. As an illustration Piaget finds that intellectual behaviour consists at first of simple classificatory activities in which the child compares, distinguishes and orders the objects around him, and that logic and mathematics as we ordinarily know them,develop out of this." 29

Is this intellectual progression, which seems as if it is a longitudinal study of one concept through the life of a child, enough to guarantee the explanation of the concept-complex in man? Here again to satisfy the eager mind, I would asśure him that Piaget does have a key concept, that of 'Décalages' through which he answers this question.[30]

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28. Cf. J. Piaget, The Construction of Reality in The Child ( N.Y.: Basic Books, 1954), pp. 354-366. and Flavell, op.cit., pp. 61-62.

29. W. Mays, Proc. of Arist. Society, op.cit., p. 50.

30. Cf. Flavell, op.cit., p. 264.

What of the human cognizer as such? He has been partly characterized above as an adaptive organizm. And in analogous terms the cognizer is still " construed to be an ever organized entity, which accommodates its schemas (the basic units of this organization) to external reality as it assimilates the reality to the schemas." 31

In one sense this consideration of the cognizer would reduce him to a product of an active entity and the environment. What is of interest here, however, is the active entity - the entity being postulated. In realistic terms:

> " A real subtle and penetrating accommodation to reality - really being 'realistic' about reality - is simply not possible without an assimilatory framework which, to substantiate assimilation once more, tells the organism where to look and how to organize what it finds." 32

This would now define the lines of the organism's activities.

To sum up, the original question of growth of knowledge can be answered through the conception of an adaptive organism. Now turning to a different aspect of Piaget's theory the answer to the same question will be sought from a different perspective.

The method applied here is 'historico-developmental'- a method that Piaget refers to very often. In brief, this method would amount to considering the child developing on parellel lines to the

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31. Ibid., pp. 262 - 263.

32. Ibid., p. 71.

development of science itself. For he says:

> Or, the history of science shows us, and that is important for me to interpret correctly the psychology of the child, that it is just these 'purified' structures, i.e. born from the reflection of the mind on itself and also opposed to the phenomenological experience, that are the most adaptable to explain finally the real experience. 33

As science develops the concepts within its subject matter get more and more "purified". And this is a very close analogy to the way the concepts develop in the mind of the child. This purification process does not negate the other adaptive approach, but rather illustrates it. Moreover, these purified concepts will form part of the subject's experience irrespective of whether they have been purified historically - as science develops - or genetically - as the child grows. And to Piaget's mind the 'scientific experience' referred to earlier is richer than the immediate experience. This would constitute an explanation as to why Einstein's physics within the 'experience' of non-Euclidean space could explain the facts more than the Euclidean-space-bound Newtonian physics.[34]

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33. J. Piaget, Mind, op.cit., p. 153.

34. Ibid., p. 152.

The general theme of this paper is the development of Piaget's mathematical concepts, specifically the number concept. We will start by considering mathematical thought as a 'model' for other thought operations. And then we will turn to the growth of the number concept.

" L'évolution des mathématiques est un admirable exemple de ces structures que la raison crée au fur et à mesure qu'elle prend conscience de son activité et sans arriver jamais à 'axiomatiser' vraiment, d'une maniere adéquate, ce fonctionnement intime. De même chez l'enfant, les formes successives de la causalité ne sont pas seulement des accomodations à l'expérience: elles sont aussi des essais progressifs de réflection sur la causalité elle-même en tant qu'activité fonctionelle de la raison." 35

In a very general way we can consider the growth of mathematical concepts from simple activities of the child, after which they get 'purified' becoming abstract concepts used in abstract 'axiomatized' systems.

As stated earlier, the logical concepts come first, then come the mathematical concepts. This can be illustrated by contrasting the logical proportions versus the metrical ones. Talking of the child Piaget says:

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35. Ibid.

" He then coordinates the inversions and reciprocities, and thus arrives at a qualitative statement of the porportion which he verifies by measuring, and in this way finally discovers the metrical proportions". 36

The formation of all mathematical concepts is not as easy as that. Though this illustration contains the rudiments of mathematical thinking it is not sufficient to describe the general process without some modifications and new interpretations.

" In short, each field of experience (that of shape and size, weight, etc.) is in turn given a structure by the group of concrete operations, and gives rise in its turn to the construction of invariants (or concepts of conservation). But these operations and invariants can not be generalized in all fields at once; this leads to a progressive structuring of actual things, but with a time-lag of several years between the different fields of subject-matters". 37

Different structures are required for different subject matters before these structures can form a generalized whole.

But if logic as such is taken as appearing in the mind of the child before mathematics, we would be generalizing too much and missing the important interrelation. Considering logic in terms of operations on the one hand, and then in terms of propositional system on the other

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36. J. Piaget, Logic and Psychology, op.cit., p. 44.

37. Ibid., p. 17.

hand, we would then easily see why concepts appear as relatively isolated subsystems before they are 'integrated' into a formal system.

> " And in fact, logic appears relatively late in the thinking of children: the first operations dealing with classes occur between 7 and 8 , on the average, and those concerned with propositions between 11 and 12". 28

This would entail that the operations that require the operations of classes as prerequisites would not appear before seven or eight. Number is one of those operations.

But it would not imply that the operations the child performs before he is eleven or twelve are illogical. The distinction is rather on the nature of these operations. The young child can 'operate' logically <u>on</u> reality, at different levels and different ages, but will not 'operate' <u>about</u> reality till a relatively old age.

> " Problems which relate, for example, to the size of objects and which a child could solve in a concrete form at 7, cannot be solved when stated in verbal terms until the age of 12". 39

To go back and support the idea of systems being an abstraction over and above their subsystem counterparts,

> " Piaget finds that the distinction so familiar to philosophers between arithmetic as a physical system of objective activities, and arithmetic as an abstract mathematical system does not on the average occur before adolescence." 40

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38. <u>Ibid</u>., p. 6.

39. W. Mays., "Epist of Prof. Piaget", <u>op.cit</u>., p. 67.

40. <u>Ibid</u>., p. 56.

Finally Piaget claims the notion of number is a subsystem of arithmetic, and number itself may involve subsystems in its formation. Here the harmony in Piaget's treatment of concept formation can be really appreciated:

> " In short children must grasp the principle of conservation of quantity before they can develop the concept of number. Now conservation of quantity of course is not in itself a numerical notion: rather it is a logical concept." 41

One further distinction should be made to explain and illustrate all this peculiar terminology. This is a distinction between the concept of enumeration and the concept of number. To put the distinction negatively, the number we are studying here, is not one of the numbers 1,2,3,... that the child utters when he is four or five years old. It is not also the symbol that the adult uses to denote the concept and the child memorizes with his language-symbol 'répertoire'.

> " Although the child knows the names of the numbers, he has not yet grasped the essential idea of numbers; namely, that the number of objects in a group remains the same, is 'conserved', no matter how they are shuffled or arranged." 42.

Piaget is talking about the child of five or six. And the essential idea of number will be our main guiding line as we discuss its acquisition in the following chapters. But before doing that we need some exposition of certain particular aspects of his theory, that will help us develop this epistemological issue into a Unit.

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41. J. Piaget, Scientific American, op.cit., p. 75.

42. Ibid., p. 74.

# CHAPTER II

## THE GENERAL DEVELOPMENTAL THEORY

In the previous chapter there were two main issues underlying the epistemological analysis. It was concluded that cognition started at the boundary between the object and the cognizer. The importance of this issue in relation to the present theme is clear. For the question is, after all, how does the child 'cognize' the number? Piaget looks at this problem from two different perspectives. On the one hand concepts are what he calls groupings, that is, they are systems of relations among existing operations. On the other hand these concepts are states of equilibrium. To put it more clearly, the internal or psychological aspect of cognition is described in terms of organized structures. When cognition is looked upon as an object-cognizer relation it is described in terms of adaptation which is described in turn in terms of equilibrium.

But it would be a mistake to think that either one of these can be discussed in isolation from the other. Like their biological counterparts, they are inseparable:

> " Du point de vue biologique, l'organisation est inséparable de l'adaptation: ce sont les deux processus complémentaires d'un mécanisme unique, le premier étant l'aspect interne du cycle dont l'adaptation constitue l'aspect extérieur". 1

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1. J. Piaget, Naissance de l'Intelligence Chez L'Enfant, (Neuchatel, Delachaux et Niestlé, 1948), p. 13.

THE FUNCTIONAL ASPECT

The child of Piaget is an 'active' organism who is continuously trying to 'conquer' the environment around him. The environment of this child is not the same as that of the adult due to the excess of egocentricity in the child. And with that, it can be imagined how the child grows within an environment, how he assimilates as he is accommodating himself to it. Put in other words, the child's understanding of the environment - i.e. his cognitive organization of it - is a function of his decentralization. He loses egocentricity. He does not even 'know' himself as he is involved with his purely subjective view of the world. But slowly he gets to identify himself as an entity among other entities. And this takes place by a gradual development which proceeds in a logical step-by-step succession. As an example:

> " When the child repeats the movements which led him to the interesting result, he no longer repeats them just as they are but graduates and varies them, in such a way as to discover fluctuations in the result. The 'experiment in order to see', consequently, from the very beginning, has the tendency to extend to the conquest of the external environment." 2

This "experiment in order to see", was an advance over the secondary circular reactions in which the child does things just for doing them. (And that was the meaning of the step-by-step succession).

It was stated earlier that the theory of genetic epistemology could be explained in terms of a dynamic development of the organism

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2. J. Piaget, The Origins of Intelligence in Children ( N.Y.: Inter. univer.Press. 1952), cited in Flavell, op.cit., p. 115.

without referring to the structural part of the operations involved. But what gives rise to groupings, or concepts in general? When achieved, what guarantees the movements from one grouping to another, or to combine subsystems into a total system arriving at the concept?[3]

These are, obviously, two different questions that will be answered within the frame of a developmental theory. The concept of equilibrium is required for these answers.

### The Concept Of Equilibrium

To Piaget, psychological equilibrium is not different from the physical one. As in physics, equilibrium is a description of a state of affairs. Piaget also talks of psychological equilibrium in terms of forces counteracting each other. In his own words:

> " A state of equilibrium, it should be remembered, is one in which all the virtual transformations compatible with the relationships of the system compensate each other". 4

This would amount to describing equilibrium as preserving the 'status quo'. But not all preservations are the same. In physical systems there are different kinds of equilibrium; e.g. stable, unstable, and indifferent. And if the analogy holds, there are parallel

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3. Cf. Jean Piaget, Mécanisme du Developement Mental (Geneve, Librairie Naville & Cie. 1942), p. 7.

states in the psychological system. And indeed the child moves from one equilibrium state to a better one as he progresses from one stage to another.[5] And in every stage the child starts all over again, but not from scratch; for now he has in his " répertoire" the equilibrium of the previous stage. The same thing takes place in terms of structures - that is, on the other side of the coin. For, when Piaget asks about the mechanisms at the pre-operational level that are the ancestors of the next operations, he says:

> " I term these mechanisms 'regulations'. They are to be conceived of as partial compensations or partial returns to the starting-point, with compensatory adjustments accompanying changes in the direction of the original activity." 6

In functional terms, these regulations are similar in nature to the idea of a feed-back in an electrical system that are present when disequilibrium prevails. Once these partial returns to the starting-point are integrated into an equilibrated system, they become constituent components of the 'group'. The 'group' here is a system in equilibrium.

Up till now it has been suggested that the child moves from one state of equilibrium to another. The nature of each state as such

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5. Flavell, op.cit., p. 244.

6. J. Piaget, Logic and Psychology, op.cit., p. 46.

is not yet described. That is, what goes on during the process of equilibration as the child starts from a 'disturbed' system and moves to an equilibrated one? An example will illustrate this process. Suppose we have a child who can not see that the quantity in a ball of clay is still the same when rolled into a rod. This incapability may be because he is either concentrating on the length of the rod or on its thickness by giving corresponding answers in each case. His second step is the alternation between the rod and the ball, but forgetting all the time his previous concentration. In the third step - as the rod is rolled further - the child sees the length and thickness in a compensatory relation, but not yet in relation to the ball. In the last step he sees, that what the ball lost in thickness - as it is turned to a rod - had gained in length. Hence the quantity of clay is conserved. This last step is the equilibrium state. The question, why the child would move in these four determined steps, is answered by Piaget in probabilistic terms.[7] That is, it is more probable that the child would do so. And since this procedure is practically the same in all problems of conservation, to study any one of them is to illustrate how the others would develop, (conservation of weight, volume, number, etc.)[8].

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7. Cf. Flavell, op.cit., pp. 245-249.

8. Ibid., p. 245.

It is obvious that this study of equilibration, emphasizes the structure of the process. But how does this process proceed? And to this question the twin concepts of assimilation and accommodation are introduced. For then, cognition - concept formation - will be defined as the equilibrium between assimilation and accommodation. And the state of equilibrium will be the formed concept.

Assimilation

The relationship between the organism and the environment - milieu, has been alluded to above. But it is here that the milieu is emphasized as a key concept in understanding assimilation. Granting that the organism can not live without the environment it would be, in effect, concluded that the actions in the environment are conceived in relation to the organism. That is, one event in the environment will have different consequences on different organisms or on different states of the same organism; and that is to be called assimilation.[9] The analogy with its biological counterpart is helpful in understanding the mental process; if mental 'reactions' are conceived of as parallel to the physio-chemical ones that take place during biological assimilation.

But here again, the organism can not receive what the environment offers him indiscriminently. That is, a child of two months can only suck his food and would starve to death if given adult food. It seems as if there is a sort of readiness in the organism to assimilate

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9. J. Piaget, Mind, op. cit., p. 149.

certain things and not others. As explained in Herbatian terms, the organism will assimilate what he is equipped to assimilate. And probably this basic equipment is biological in nature. But, nevertheless, the fact remains that the organism does not assimilate in vacuum.[10]

This is one of the relationships between the organism and the environment. Another would deal with the consequences within the organism. In other words, is the organism such a passive entity that it becomes helpless in face of an environment that it can not assimilate? To a certain extent he is helpless but not in all situations. We notice, after some time, that the child would turn the bottle right side up before he starts sucking. Piaget describes this phenomenon as the accommodation process.

## Accommodation

This concept is referred to more than once in Play, Dreams, and Imitations in Childhood ranging in complexity from the typically 'sensori-motor' accommodations to the operational ones at the adult level. In that work of Piaget, there was no urgent need to define it in a precise language. But as the discussion of equilibrium gets more and more in touch with cognition as such, the need for an operational

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10. Cf. Flavell, op. cit. p. 50.

definition - in terms of the equilibrium process - becomes more and more obvious. In Le Mécanisme du Développement Mental he introduced the concept of the "recherche de l'équilibre" , in which the organism is in a state of continuous re-equilibration, as we mentioned above. Then accommodation would be equivalent to the readjustments that the organism undergoes in his "recherche de l'équilibre". The re-adjustments are conceived both in terms of circumstances arising in the environment and innovations in the goal of the organism.[11]

A question may be raised here about the possibility of accommodation to any situation. And if not, what would be the degree to which the organism would respond? But this question could have been raised in connection with assimilation as well. And the answer to any one of them would imply the other. And the degree to which the organism assimilates determines in a sense the degree to which he will accommodate and vice a versa. In this context equilibrium becomes a determining factor in adaptation; or, probably the only determining factor. For then adaptation will be the degree of equilibrium between accommodation and assimilation. [12]

By abstracting these concepts and focusing them on the mental life a new perspective should be possible. The analogy with the biological process provides some insight. Equilibration in the biological

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11. J. Piaget, Le Mécanisme du Développement Mental, op. cit., p. 15.

12. J. Piaget, La Naissance de l'Intelligence chez l'Enfant, op. cit., p. 12.

world is a necessary concept to explain survival. And if the analogy is carried far enough, would then equilibration become necessary for mental life? There seems to be no clear-cut answer to this question. Two main questions that can not be answered in terms of the above mentioned concepts alone are: first, if the child is in a state of re-equilibrating all the time, is there a general track that binds these equilibrations into a directed process? And, second, if so, what is the relation between the first equilibrium state and the one next to it? Or, in general terms, what binds one equilibrium state with another irrespective whether they are at different levels of operational difficulty or on the same level?

The first question can be answered in terms of egocentricity and phenomenology, and the second one will be answered in terms of a theory of "Décalages".

## Dynamics Of Mental Growth

It was stated earlier that Piaget's child goes step-by-step from one equilibrium state to another. And to ask why he does so is still a valid question. In general terms he leaves one state of equilibrium when he discovers that it is no longer sufficient for the new situation. Then he would substitute another system for the solution. And the striking phenomenon is that, taken at a certain stage, the child seems to be so sure of his behavior; and in fact, he won't modify it unless he is pushed to. The need for modification arises when the child reaches the limits of the field he is operating within.[13] The child reaches these limits by one of two possible ways: Either the environment is rearranged in such a way that he can not handle it with the mechanisms he already possesses, or, there is a factor of development within the child that enables him to conceive the limit and beyond. Chance factors, or external agents may be responsible for the first alternative. As for the second alternative, this driving force, is somewhat mysterious.

Piaget postulates for the inner factor the concept of "la prise de conscience". That is, the child is capable of thinking <u>about</u> reality, which is not a phenomenon of enlightenment only, but rather a translation of actions into notions, hence resulting in transformations.[14] Had it not been for the introduction of this "prise de conscience", it would have been very difficult to explain learning. For the child learns as he meets new situations by experimenting and recoordinating. The degree to which his 'hypotheses' are reducible to empirical investigations determines the degree to which he can assimilate - accommodate. In fact his learning is comparable to that of the adult.[15]

At the level of representational thought one can say that the child develops his own symbols - terminology - before he 'socializes' them. For he may reach an internal state of equilibrium before he

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13. Cf. J. Piaget, <u>Psychology of Intelligence</u>, <u>op. cit</u>. p. 7. And <u>Sci. Amer. op. cit.</u>, p. 77.
14. J. Piaget, <u>Mind</u>, <u>op. cit</u>., p. 152.
15. <u>Ibid</u>., p. 160.

unfolds it for modification and socialization.[16] Now this unfoldment can be tied up with the inner "prise de conscience" in one main track that the process of equilibration is following.

## Egocentricity And Phenomenology

There is a difference between the child's environment and the adult's. For a child of less than two years old, his world is probably a flow of unconnected pictures in which he is immersed without being able to organise it. This phenomenon of being self-centered in a field of unrelated phenomena is described as egocentricity. And unless the child loses this egocentricity he will not be able to structure the world around him. I think this is a key concept, because any order would necessarily require an observer. And if the child can not put himself in isolation from his environment, he will not be able to order it. And yet he has to see himself within it before he can claim that he comes to grips with it. This simultaneous differentiation and integration is in the background of every equilibrium; and hence underlying every cognitive activity. This dual aspect reappears in the discussion of the formation of the number concept.

> "... It is precisely when the subject is most self-centered that he knows himself the least, and it is to the extent that he discovers himself that he places himself in the universe and constructs it by virtue of that fact."[17]

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16. Cf. Flavell, op. cit., pp. 152 - 155.

17. Cf. J. Piaget, The Construction of Reality, op. cit., p.xii, and Flavell, op. cit., p. 60.

The phenomenon of egocentricity is easily verified in the first stages of language acquisition of the child. The child does not account for his listeners in his first speeches and does not care to make himself clear.[18]

In very general terms it can be said that the development of the mind and hence the development of concepts involves basically a continuous struggle against egocentricity and phenomenology.[19] Described in other words, the child is the slave of himself; and as he frees himself through stages he gets to comprehend reality as such and conceptualize about it. And this continuous struggle takes place at every new stage. That is, the child would recoil in the face of every new situation - gets possessed by his egocentricity - and sees the event as a phenomenon by itself. But with time and under his "recherche de l'équilibre", he gets to structure the new event in terms of coordination of his experience and experiments. With this, the flow of egocentricity and phenomenology can be considered as an oversimplified restatement of equilibration.[20] And this, in general, constitutes a cognitive model which may be considered as the main track that equilibration is following.

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18. Cf. J. Piaget, Language And Thought of The Child, (London: Kegan Paul, 1932), p. 32, and
J. Church, Language and The Discovery of Reality, (N.Y.: Random House, 1961), p. 72.

19. Cf. J. Piaget, Mind, op. cit., p. 153, and
Flavell, op. cit., p. 135.

20. Cf. Flavell, op. cit, p. 244.

"Decalages"

In discussing the relation between two states of equilibrium with reference to the concept of "décalages", a distinction should be made between two states of equilibrium on the same developmental level, on the one hand, and two states of equilibrium on different developmental levels on the other hand. Piaget refers to the first relation as a relation of horizontal "décalages", while the second, he terms as a relation of vertical "décalages".

With the horizontal "décalages" the child can re-establish equilibrium when faced with a situation that requires a certain stage of development. Given two situations X and Y, for which the child is equipped with enough structures to handle them when they are on the same developmental level, the child will handle one after the other by virtue of horizontal "décalages".[21]

While, on the other hand, the child who has achieved a state of equilibrium at a certain level will be able to reach a similar state of equilibrium at a different level by virtue of vertical "décalages".[22] As an example, the child who knows that the equilibrium of a balance is a function of the length of the arms and the weights in the pans, will be able to represent this situation on the operational level by handling symbols and concepts <u>about</u> reality; this is achieved by vertical "décalages".

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21. Cf. J. Piaget, <u>Le Mécanisme du Developpement Mental</u>, <u>op. cit.</u>, pp.49-56 and Flavell, <u>op. cit.</u>, pp.21-24.
22. Cf. J. Piaget, Le Mécanisme du Developpement Mental, <u>op. cit.</u>, pp.37-49. and Flavell, <u>op. cit.</u>, pp. 21-24.

Another example will illustrate the concept of "décalages" once for all. If a child at a stage A can not find the ball after he sees you put it behind a screen, he will have the same difficulty when he reaches a higher stage B where he is required to find the ball after it rolls under a bed. In other words, the difficulty he had in coping with visible displacements at stage A will be repeated when he has to cope with invisible displacements at a higher stage B.[23]

In general, Piaget does not seem to have a logical explanation of the theory of "décalages". He seems to deal with them as empirical facts that are simply there. Up till now there is no logical justification as to why they exist, especially at the early periods of the child's life.

By way of summing up, the cognizer is first faced with an environment that seems to be a flow of unorganized phenomena, including himself. Then, as time passes he begins to decentralize himself by freeing himself from his egocentricity and starts making the right accomodations for the right assimilations. And as he develops in time and stages he begins to think about reality acquiring in this way different stages of equilibrium - or sub-systems - which get coordinated thereafter in a total integrated system amounting to a concept. The guiding force in this development is the cognizer's struggle with his egocentricity; always trying to see himself isolated from his environment, and yet within it in order to structure it and understand its guiding laws.

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23. Cf. Flavell, op. cit., p. 134.

## THE STRUCTURAL ASPECT

The study of structures is very important for several reasons. For, in Piaget's words:

> " The structured wholes, considered as the form of equilibrium of the subject's operational behavior, is therefore of fundamental psychological importance, which is why the logical (algebraic) analysis of such structures gives the psychologist an indispensable instrument of explanation and prediction." 24

The structures have a theoratical justification, for without them the creativity* of the mind would be difficult to explain. [25] And more basically, when analyzed carefully, they would constitute the elements of intellectual assimilations.[26]

Assuming that these structures have a kind of plasticity among their defining features, they would account for the theory of mental development. But there would still remain a major difficulty that can not be overcome by completely isolating this treatment from the functional aspect of man's intellect discussed above. For, if given a purely logical treatment, there would ultimately be a deficiency in the formal logic, which is axiomatic in nature and which would not yield itself to wholistic explanations. And again the whole structures, that

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24. J. Piaget, Logic And Psychology, op. cit., p. 45
25, J. Piaget, Mind, op. cit., p. 151.
26, Ibid.

* Creativity is explained in terms of the varying experience versus the patterned structures, when the latter are assumed to be Plastic.

are interconnected, would be reduced to simple linear deductions. Piaget would want to posit a psychological existance for these structures:

> " Operational mechanisms, however, have a psychological existence, and are made up of 'structured wholes', the elements of which are connected in the form of a cyclical system irreducible to a linear deduction." 27

## The Concept Of Groupings

At the very outset, Groupings can be considered as patterns of cognition. They are similar to 'spider-webs' in the role they play in assimilation. In fact, they are the elements of assimilation. And from this perspective accommodation becomes simply a rearrangement of the 'web' to catch more material for assimilation.

From a purely theoretical point of view, the question can be raised whether the mind really operates through groupings. But this question could also be asked to the physicist when he tries to describe a pattern of the sub-atomic world. The notion of patterns may explain the data that we have, irrespective of their real nature. And in this respect, Piaget can be credited with originality.

In what way can it be said that these structures are given to the subject? To such a question Piaget offers four alternatives and accepts only one of them. [28]

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27. J. Piaget, Logic And Psychology, op. cit., p. 24.

28. Cf. J. Piaget, Logic And Psychology, op. cit., pp. 38-40.

First, the subject could get these groupings as a product of his cumulative experience. But this alternative is ruled out when we take as a criterion the idea that the individual is aware of these structures. That is, the person may be a good runner without being aware of the laws of mechanics involved which is not the case in cumulative experience. Hence the person, not being aware of these structures, could not get them through cumulative experience.

A second possibility is that they are in the mind a *priori* . But why, then, do they appear so late? And the answer to this question rests on other *a priori* judgments.

Third, they may be a result of the maturation of neural connections. Then they should appear in wholes in the act of thinking, which is not the case. They appear in parts of structures.

> Fourth, and this is the alternative that Piaget accepts:"...the lattice and the group I N C R are regarded as structures belonging to the simple forms of equilibrium attained by thought activity." 29

Therefore, the person acquires these patterns as he is equilibrating and re-equilibrating. And he attains them in parts and keeps on building them up till they get integrated into a 'group-lattice' form of equilibrium.

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29. *Ibid*., p. 40.

An intuitive description of the genesis of these groupings would seem helpful at this point. During the interaction between the organism and his environment, the organism finds himself continuously urged to coordinate his operations into coherent wholes. On the other side this coordination obeys the laws of equilibrium. And then " the grouping is therefore a form of equilibrium of inter-individual actions, and it thus regains its autonomy at the very core of social life." [30]

In an attempt to clarify the genesis of these groupings, reference is made to more elementary concepts; such as, schemas - or scemata, the conservation concept, invariants, and operations.

Schemas

Following Piaget, a schema is a structured unit of coordinated actions. The sucking schema, for example, coordinates the movements of the hands, the mouth, the bottle, into a unit act called sucking. And we speak of a child assimilating the nipple to his sucking schema. Such units are, then, the ancestors of the integrated systems that will be called groupings. And in fact, these are considered as invariants throughout the race. In their very primitive forms the distinction between biological schemas and cognitive ones is a hair-split difference, such that, it will be difficult to tell the one from

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30. Cf. J. Piaget, Psychology of Intelligence, op. cit., pp. 163-4. and Flavell, op. cit., p. 201.

the other. But as the child grows, the cognitive schemas will be best characterized as plans for action. This perspective would make the explanation of class of actions as related to one plan much easier. That is, it could be said that knowledge is a result of our grasping schema. [31]

No matter how a schema is conceived (whether it is a plan, a coordinated unit of action, a pattern, etc.), it is always a structured unit. And the more actions - behaviours -, related to a schema, the more it is the unit that binds together the different actions in a class of actions. As Joseph Church puts it: "

" A schema is an implicit principle by which we organize experience." [32]

In their latest phases, i.e. at the formal operational level, they become very much like attitudes, methods of attack, etc... And these methods can be generalized to many problems [33] - very much in line with the theory of "décalages". Put in other words, these schemas, once integrated into a total system, become specialized cognitive instrumentalities which the subject uses as he meets new problems.

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31. Cf. Flavell, op. cit., p. 54.

32. J. Church, op. cit., p. 36.

33. Cf. Flavell, op. cit., p.222 As an example, the schema of proportions derived mainly from the INCR group yields $p^*/p = q^*/q$ in the problem of balance.

But once again, it should be noticed that groupings, schemas as their ancestors, still rest on more basic notions of 'experience'. Thus it becomes necessary to clarify the nature of the 'object' as it exists in time and space.

Concept of Conservation

The importance of the idea that number, quantity, volume, etc. are conserved, can not be under-estimated. And when it is recalled that the adult's environment is different from that of the child. Adult's knowledge of conservation must be different from the child's. And as a matter of fact, it is established empirically in that the child thinks that the ball of clay when rolled into a sausage changes in quantity. This is one example, that helps re-construct the child's world in which even simple perceptual conservations are built up step-by-step. From a structural point of view, the concept of conservation still assumes the concept of reversibility. For Piaget, reversibility is a key step in conservation.[34] For, if the child can not perceive that he can roll the sausage back to a ball, he will not perceive that the quantity of clay in the ball is conserved.

In the study of Elkind, the discovery of conservation in children is found to agree with the results found by Piaget; and hence reversibility is considered as instrumental in the development of quantity conservation.[35]

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34. D. Elkind, "Children's Discovery Of Conservation", Journal Of Genetic Psychology, (98 1961), pp. 224-5.

35. Ibid., p.226; Elkind goes further to explain Piaget's theory from the point of view of being a nurture-nature theory.

Therefore, reversibility is necessary for the concept of conservation in general (be it conservation of number, quantity, volume etc.). When related to the object, the concept of conservation would give the object a character of permanence, which is necessary for any formation of a concept.

The object concept

Piaget studied this concept thoroughly in his work: The Construction of Reality In The Child. And his study seem to rest on three main assumptions.[36] (1) There can be found an appropriate criterion to tell that X has the Concept. (2) The perception of objects of the young is different from that of the adult. (3) The concept is constructed in stages and not in take-it-all or leave-it-all process.

Piaget believes that for a mature formation of an object concept, is the ability to perceive objects in space existing qua objects; i.e. the ability to see the object as an entity that moves in space and time, acts and can be acted upon.[37] And as the experiments establish, this conception is reached after a development through different stages,[38]

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36. Cf. Flavell, op. cit., p. 130.
37. Ibid., p. 129.
38. J. Piaget, Construction Of Reality, op. cit., p. 4.

ranging from the simple phenomenological attitude of the child, in which pictures flow infront of his eyes and disappear for no reason and with no order, to the formation of images even of absent objects and their displacements.

In another place, he talks of this permanence of the object as an invariant, which is not effected by displacements.[39] Moreover, when the child can search and find an object transferred from A to B and then from B to C, and he finds the object in C, then the child is said to have the object concept.[40]

Now with those concepts defined (the concept of conservation and the object concept) permanence is achieved in a changing world. Without these terms it would be difficult to construct an orderly world through continuously changing entities.

A companion to the concept of permanence, is the concept of invariance, or invariants.

## Invariants

The biological analogy has been relied upon to help in understanding mental life. If it is true that certain functions are basic to the life of the race, then it would be expected that their mental counterparts will occupy the same position. These assumptions partly explain intellectual communication. If people didn't have some basic

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39. J. Piaget, Logic And Psychology, op. cit., p. 10.

40. Flavell, op. cit., p. 134.

structures or functions that are common to them, their chances of making themselves clear and comprehending others would be greatly reduced. It was stated above, in connection with egocentricity, that the child does not apparently care to make himself understood in his first speech. But being a product of his environment, he soon realizes the necessity to conform to certain common grounds on which he meets other members of his society. This view should not be surprising, if it is recalled that the child's social life is his only life.

From the individual's perspective, there is a need to postulate certain general patterns through which the individual orders his experience into coherent wholes. And these general patterns need not be static organized units, but rather, functional and dynamic. Piaget finds in his research a constancy of transformation rather than constancy of units transformed, which he refers to as "functional invariants".

Piaget finds several types of invariants as suggested by the biological analogy.

> " Du point de vue biologique, par example, il est évident que certaines fonctions, telles que l'assimilation, la réproduction, etc., sont absolument générales et communes a toutes les etres." 41

In another place he draws attention to the applicability of this biological analogy to the mental life; for he says:

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41. J. Piaget, Mind, op. cit., p. 150.

" Pour décrire le mécanisme fonctionnel de la pensée en termes biologiques vrais, il suffira dès lors de dégager les invariants communs à toutes les structurations dont la vie est capable." 42

From this he concludes that it is necessary for their counterparts in the mental life to exist.[43] For without this assumption, it would be difficult to explain creativity conceived as the offspring of the friction between the variable structures and the constant ones.

Therefore, entities themselves as well as certain functional structures gain permanence as the child grows. Even in the simple process of classification - putting entities into classes - the permanence of the entities as well as the permanence of classification i.e. the process, is required.

But when this view is generalized, it gives rise to some problems. For examples: How is re-classification explained? Consider an element a in A and by empirical evidence, or otherwise, it should be in B. How, then, can that happen if A is permanent? Moreover, we also know that this continuous rearrangement of objects in the universe is an essential part of the process of mental development; for, without it, the process of repeated experimental verification would become meaningless. This question can always be settled considering the notion of reversibility - the return to the start.

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42. J. Piaget, La Naissance de l'Intelligence chez l'Enfant, op.cit., p. 10.

43. Ibid., p. 11. " Il existe, en effet, dans le développement mental, des élements variables et d'autres invariants."

<u>Reversibility</u>

In very general and simple terms, an operation is said to be reversible if its effects can be cancelled by retracing the steps back. Suppose there are a number of scattered objects on a plane, they can always be lined up in a straight line. This operations is reversible because the objects can be scattered back.

As mentioned above the concepts of feed-back, returning to the original point, etc., are essential concepts for cognition. The reversibility concept is needed in the most elementary form of cognition - cognition through trial and error. For one can not try again after he errs, if he can not go back to the original point. That is why Piaget conceives of reversibility as the very core of cognition and not merely as a property of the grouping.[44]

In its primitive form, just the concept of being able to "undo" what one does, is enough to explain the genesis of reversibility. In more sophisticated terms the child loses egocentricity as he starts to see his point of view simultaneously with its contradictory - which is a main feature of reversibility. The more he loses egocentricity, the more he grows to be realistic about himself and the world, and the more he comprehends the world. And this is achieved by the child's being sociable and continuously interacting with his peers and environment.[45]

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44. Cf. Flavell, <u>op.cit.</u>, p. 189; and for the more formal treatment of this concept see Piaget, <u>Traité de Logique</u>, <u>op.cit.</u>, p. 274 and p. 303.

45. Cf. Flavell, <u>op.cit.</u>, p. 201.

One might jump to an unjustified conclusion concerning the loss of egocentricity if he concludes that it is desirable that the child loses his egocentricity completely. This may be true of the conception of egocentricity in infants; but certainly does not hold when egocentricity is conceived of as concentrations at the higher levels of cognition. For, at the level just before the child gets engaged in formal operational statements and solutions of problems, midway between centrations and reversible operations become a main feature of intellectual structure.[46] And in later stages, this flexibility in holding obstinately to one's point of view and yet ready to change it in view of contradictory facts may become the essential character of a practicing scientist.[47]

## Operations

Whenever logical operations or arithmetical operations are referred to they usually mean a certain act - a transformation from one state to another. And usually this act has certain guiding rules that are observed. More systematically, an operation is a representational act which is an "integral part of an organized net-work of related acts."[48] In this definition the nature of the act and the

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46. Cf. Flavell, op. cit., p. 163.

47. Cf. M. Cohen & E. Nagel, An Introduction to Logic and Scientific Method.(N.Y.: 1934, Harcourt, Brace & Co.), p. 394.

48. Flavell, op. cit., p. 166.

nature of the net-work of acts are not specified, to leave room for interpreting these acts as psychological and still hold the same definition.

Piaget, in his Logic and Psychology, tries to analyze psychological operations in terms of their constructability into genetic structures and compare them with logical operations after they are analyzed in terms of structured systems of elements defined by a set of laws. And on the grounds of this operationalism, logic and psychology may meet.[49] In that context operations are conceived of as actual psychological activities, which constitute the core of any effective knowledge.[50]

If this is the case, concepts will be formulated, in the long run, in terms of net-works of operations. Piaget from this perspective, does not differ much from the 'operationalists' for they also believe that concepts are nothing but a set of operations performed over a certain domain.[51] And since these concepts are bound to their domain, different concepts in different fields emerge.

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49. Cf. J. Piaget, Logic And Psychology, op. cit., p. 7.

50. Ibid.

51. Cf. P.W. Bridgeman, The Logic of Modern Physics, (N.Y: Macmillan Co. 1928), p. 5. and
Carlson, Dimensions Of Behavior (Lund, C.W.K. Gleerup 1949) p. 27.

In an operation, like addition before the child can add one and two to get three, he should be able to identify - realize the identity - of one and of two. More generally, the child must first discover that A and A' are conserved before he can add them into A + A' = B. Such an operation rests on some more basic - primary - logical concepts; i.e. such an operation involves a 'group' structure which must be investigated in connection with the general theory.

## Groupings

In technical terms, there is a difference between the concept of grouping and the concept of group. Though the structure of groupings assumes the structure of the group the group is still further limited. The definition of a grouping is any closed and reversible system in which all the operations remain within the whole, and which seem as if it is not susceptible of progression.[52]

The definition of a lattice is also required because its properties are among the properties of the grouping. In general, a lattice "consists of a set of elements and relation which can hold or 'relate' two or more of these elements." 53 More technically, any two elements of a lattice have one least upper bound ( l.u.b.) and

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52. J. Piaget, Le Mécanisme Du Developpement Mental, op.cit., p.3.
53. Flavell, op. cit., p. 172.

one greatest lower bound (g.l.b.).[54] The l.u.b. of two or more classes is the smallest class that contains them; And the g.l.b. is the greatest class that they contain - or which is contained in all of them. The integers for example, form a lattice; and so do hierarchical classifications.[55]

In such a system, every class is its own g.l.b. and l.u.b., which is called by Piaget the property of tautology.

Piaget differentiates between eight types of groupings.*

1) Grouping I

If we take the elements of a grouping to be logical addition equations such that; $A + A' = B$ , $B - A' = A$ etc.

$B + B' = C$ , $C - B = B'$ etc.

$C + C' = D$ , $D - C' = C$ etc.

then we define grouping I by the following properties:

1- Composition: $(B - A' = A) + (C - B' = B) = (C - B' - A' = A)$

That is, if we take B' from C, we get B from which we have already taken A' ; Hence, the final result is A.

2- Associativity : $[(B-A'=A)+(C-B'=B)]+(D-C'=C) = (B-A'=A)+[(C-B'=B)+(D-C'=C)] = (D-C'-B'-A' = A)$

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54. Ibid.

55. Take class B - Mammals - that contains A - Dogs, C - Cats, Etc.
Then l.u.b. of A and B is $A \cup B = B$
And g.l.b. of A and B is $A \cap B = A$

* Cf. Flavell, op.cit., pp. 174-187. And J. Piaget, Traité de Logique, op.cit., pp. 96-187.

3- General Identity: $(0+0=0) + (C - B' = B) = (C - B' = B)$ where 0 is the null class.

4- Reversibility: $(D - C' = C) + (-B - B' = -C) = (0+0=0)$

5- Special identity: Every class plays the role of identity element with itself. i.e. $A + A = A$ (Tautology). And it is also the identity element of all its subclasses. i.e. If $A \subseteq B$, then $A + B = B$ Or $A \cup B = B$

2) Grouping II

In this grouping the elements are equations of secondary classes and hence like grouping ( 1) except for the new tautologies added by the new situation. That is, if B is now conceived to be made of two new classes $A_2$ and $A'_2$ such that, $B = A + A' = A_2 + A_2'$ Where $A_2 \neq A$ and $A_2 \subseteq A'$ then the new tautology $A + A_2'$ is added to the other ones, viz. $A + A = A$ and $A + B = B$. And similarly for $A + A_3' = A_3'$ if $B = A_3 + A_3'$ and so on.

3) Grouping III ( Bi-Univocal multiplication of classes )

Take a class of people $D_1$, divide its members with respect to colour: $A_1$ = White, $B_1$ = Black, $C_1$ = Yellow. Now take the same class, call it $D_2$, and divide it with respect to residence : $A_2$ = Urban, B2 = Suburban, $C_2$ = Rural.

Then $D_1 \times D_2 = D_1D_2$ is the general logical product described with the lattice and equal to the matrix of nine entities. ( It is important to note that we can still divide them into other subclasses and get a bigger matrix, but for simplicity we chose to define only the nine-matrix product).

1- Composition; $A_1 \times A_2 = A_1A_2$ a new class (white - urban)

2- Associativity; $A_1 \times ( B_2 \times B_3) = ( A_1 \times B_2) \times B_3$

3- Special Identity; $A_1 \times A_1 = A_1$ . $B_1B_2 \times B_1B_2 = B_1B_2$ where the general identity is Z ( the universe. $A_1 \times Z = Z$ . $A_2 \times Z = Z$ )

4- Absorption; $D_1 \times A_1$ // $A_1$ ( Notice that $A_1$ is smaller than $D_1$ )

5- Division; $A_1A_2 \div A_1 = A_2$ ( If whiteness is extracted from white-urban then $A_2$__Urban __ is greater than $A_1A_2$__ white-urban.)

4) Grouping IV (Co-Univocal multiplication of classes)

The multiplication now is one-to-many instead of one-to-one as in grouping III. As an example, take the class $K_1$ to comprise: $A_1$ = Sons of x, $B_1$ = Grandsons of x. $C_1$ = Greatgrandsons of x. And take $K_2$ to comprise : $A_2$ = Brothers, $A_2'$ = First cousins of $A_2$, and $B_2'$ = Second cousins of $A_2$.

$$\text{Then } K_1 \times K_2 = ( A_1 + B_1 + C_1) \times ( A_2 + A_2' + B_2' )$$
$$= A_1A_2 + A_1A_2' + A_1B_2' + B_1A_2 + B_1A_2' + B_1B_2' + C_1A_2 + C_1A_2' + C_1B_2'$$

But $A_1A_2' = A_1B_2' = B_1B_2' = 0$

From another point of view, the class $A_2$ is the only class of $K_2$ which can be applied to $A_1$, i.e. have intersection with it; for, no one of the sons of x can be the first cousin of his brother, nor his second cousin. Similarly no one of the grandsons of x can be the second cousin of anyone of those who are brothers.

Simplifying this multiplication then, we get:

$$K_1 \times K_2 = A_1A_2 + B_1 ( A_2 + A'_2 ) + C_1 ( A_2 + A'_2 + B'_2 )$$
$$= A_1A_2 + B_1A_2 + B_1A'_2 + C_1A'_2 + C_1A'_2 + C_1B'_2$$

Up till now _ i.e. Groupings I to IV _ all the operations were performed on logical classes as such. In the next four groupings we will have the operations on relations between members of classes; relations such as: ' a is the father of b' (symbolized $a > b$ or $a \rightarrow b$ etc.), or ' a is the brother of b ' ( symbolized by $a \leftrightarrow b$ ).

5) Grouping V ( Addition of assymetrical relations )

A word about the nature of these assymetrical relations may be in place here, to help us understand the operations done on them. Formally they can be defined as: if for two elements a and b $a > b$ and $b > a$ , then $a = b$.

More intuitively, two elements are in an assymetrical relation to each other, if there is a marked difference between them that implies order, i.e. If $x > y$ it can not be the case that $y > x$. The "father of" relation is such a relation.

These relations have the following properties:

a) Intensive quantification property. That is, the actual magnitude of a and a' is not to be considered in the case where $a < b$

and $a' \prec b$.

E.G. $0 \xrightarrow{a} A \xrightarrow{a'} B$, $0 \xrightarrow{b} B$   In opposition to $A \subsetneq B$ & $A' \subsetneq B$ In $A + A' = B$

b) The composition principle holds in the usual way. i.e.

$(0 \xrightarrow{a} A) + (A \xrightarrow{a'} B) + (B \xrightarrow{b'} C) = (0 \xrightarrow{c} C)$ and

$(A \xrightarrow{a'} B) + (B \xrightarrow{b'} C) = (A \xrightarrow{a'b'} C)$   or simply

$a + a' = b.\quad a + a' + b' = c.\quad a' + b' = a'b'.$

c) Associativity; $a + (a' + b') = (a + a') + b'$

d) Tautology; $a + a = a$

c) Resorption; $a + b = b$

f) Reversibility; is now inverse instead of negation;

$$(A \xrightarrow{a} B) + (B \xleftarrow{a'} A) = A \xrightarrow{o} A \text{ or } A = A$$

g) The general identity is now equivalence.

$\xrightarrow{a} + \xleftarrow{a} = \overline{\ o\ }$ and

$\xrightarrow{b} + \xleftarrow{a'} = \xrightarrow{a}$ and so on.

6) Grouping VI (Addition of symetrical relations)

The symmetrical relations are relations of the form of: " is the brother of", " has the same grandfather as ", all symbolyzed by $\xleftrightarrow{a}$.

Take the following example: Let

(A) " $x \xleftrightarrow{o} x$ " signify " $x = x$ "

(B) " $x \xleftrightarrow{a} y$ " " " " " " x brother of y "

(C) " $x \xleftrightarrow{a'} z$" " " " " " " x first cousin to z"

(D) " $x \xleftrightarrow{b} z$ " " " " " " x has the same grandfather as z "

a) Composition property;

$( x \overset{a}{\longleftrightarrow} y ) + ( y \overset{a}{\longleftrightarrow} z ) = x \overset{a}{\longleftrightarrow} z$ and

$( x \overset{a}{\longleftrightarrow} y ) + ( y \overset{b}{\longleftrightarrow} z ) = x \overset{b}{\longleftrightarrow} z$ which asserts only the grand-father relation since x and z may be brothers or first cousins.

$( x \overset{a}{\longleftrightarrow} y ) + ( y \overset{a'}{\longleftrightarrow} z ) = ( x \overset{a'}{\longleftrightarrow} z )$; if z is first cousin to one brother he must be first cousin to the other as well.

$( x \overset{a'}{\longleftrightarrow} y ) + ( y \overset{a'}{\longleftrightarrow} z ) = ( x \overset{b}{\longleftrightarrow} z )$; i.e. x is either the brother of z or his first cousin and in both cases they will have the same grandfather.

$( x \overset{a}{\longleftrightarrow} y ) + ( y \overset{\bar{b}}{\longleftrightarrow} z ) = ( x \overset{\bar{b}}{\longleftrightarrow} z )$; If x and y are brothers, and y and z don't have the same grandfather, then x and z will not have the same grandfather.

b) Associativity is always true.

$$\left[ ( r \overset{a}{\longleftrightarrow} x ) + ( x \overset{a'}{\longleftrightarrow} y ) \right] + ( y \overset{a'}{\longleftrightarrow} z ) = ( r \overset{a}{\longleftrightarrow} x ) + \left[ ( x \overset{a'}{\longleftrightarrow} y ) + ( y \overset{a'}{\longleftrightarrow} z' ) \right] = r \overset{b}{\longleftrightarrow} z$$

c) The inverse is the reciprocal relation obtained by just permuting the terms. That is, inverse of $x \overset{a}{\longleftrightarrow} y$ is $y \overset{a}{\longleftrightarrow} x$

d) The general identity is $x \overset{0}{\longleftrightarrow} x$ or $x = x$. Thus,

$( x \overset{a}{\longleftrightarrow} y ) + ( y \overset{a}{\longleftrightarrow} x ) = x \overset{0}{\longleftrightarrow} x$ and

$( x \overset{0}{\longleftrightarrow} x ) + ( y \overset{b}{\longleftrightarrow} z ) = ( y \overset{b}{\longleftrightarrow} z )$

e) The special identity is the tautology.

$( x \overset{a}{\longleftrightarrow} y ) + ( x \overset{a}{\longleftrightarrow} y ) = ( x \overset{a}{\longleftrightarrow} y )$

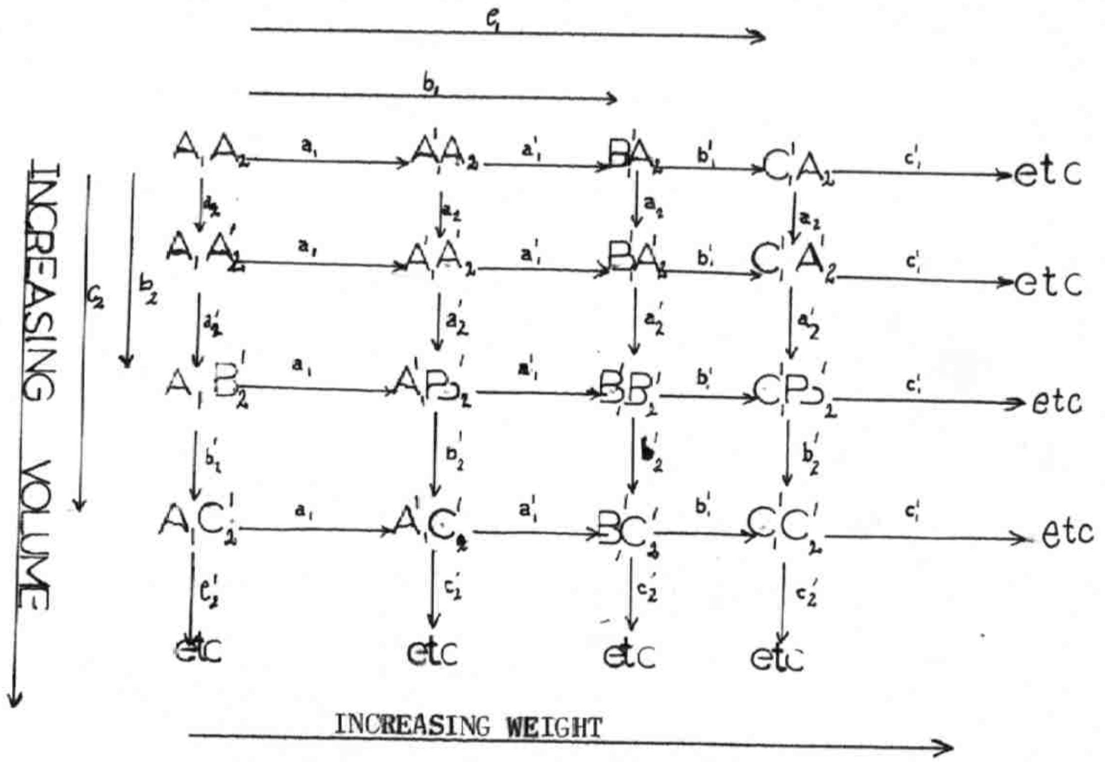
f) Resorption;

$( x \overset{a}{\longleftrightarrow} y ) + ( x \overset{b}{\longleftrightarrow} y ) = ( x \overset{b}{\longleftrightarrow} y )$, because the relation $\overset{b}{\longleftrightarrow}$ contains already $\overset{a}{\longleftrightarrow}$ as one of its alternatives.

7) Grouping VII (Bi-Univocal multiplication of relation)

Analogous to grouping III as opposed to grouping I, so is grouping VII as opposed to grouping V.

To understand the properties of this grouping, one may refer to the following table as a guide:




To simplify the grasp of this matrix, we note that entities on the same row have the same volume, and those on the same column have the same weight. But by the property of intensive quantification $a_1$ is not necessarily equal to $a_1'$.

a) Composition;

1) $(A_1A_2 \xrightarrow{c_1} C_1A_2) \times (C_1'A_2 \downarrow a_2 \; C_1'A_2') = (A_1A_2 \xrightarrow{c_1} \downarrow a_2 \; C_1'A_2')$

2) $(A_1A'_2 \xrightarrow{b_1} \downarrow a'_2\ B'_1B'_2) \times (B'_1B'_2 \xrightarrow{b'_1} \downarrow b'_2\ CC'_1C'_2) = (A_1A'_2 \xrightarrow{c_1} \downarrow a'_2 + b'_2\ C'_1C'_2)$

Also

3) $(A_1A'_2 \xrightarrow{b_1} \downarrow a'_2\ B'_1B'_2) \times (B'_1B'_2 \xleftarrow{a'_1} \uparrow a'_2\ A'_1A'_2) = (A_1A'_2 \xrightarrow{a_1} \downarrow o\ A'_1\ A'_2)$

Or simply $(A_1A'_2 \xrightarrow{a_1} A'_1\ A'_2)$

b) Associativity; Let a, b, c be of the type $a = (A_1A_2 \xrightarrow{b_1} \downarrow c_2 B'_1C'_2)$,

then a (bc) = (ab) c in all instances.

c) The inverse operation is the logical division as in grouping III & IV.

d) The general identity is the null difference;

$(A_1A_2 \xrightarrow{a_1} \downarrow a_2\ A'_1A'_2) \div (A_1A_2 \xrightarrow{a_1} \downarrow a_2\ A'_1\ A'_2)$ or in alternative form;

$(A_1A_2 \xrightarrow{a_1} \downarrow a_2\ A_1A'_2) \times (A_1A_2 \xleftarrow{a_1} \uparrow a_2\ A'_1A'_2) = (A_1A_2 \xrightarrow{o} \downarrow o\ A_1A_2)$

e) The special identity is the tautology;

$(A_1A_2 \xrightarrow{a_1} \downarrow a_2\ A'_1A'_2) \times (A_1A_2 \xrightarrow{a_1} \downarrow a_2\ A'_1A'_2) = (A_1A_2 \xrightarrow{a_1} \downarrow a_2\ A'_1A'_2)$

f) Absorption;

$(A_1A_2 \xrightarrow{a_1} \downarrow a_2 A'_1A'_2) \times (A_1A_2 \xrightarrow{b_1} \downarrow b_2\ B'_1B'_2) = (A_1A_2 \xrightarrow{a_1} \downarrow a_2\ A'_1A'_2)$

8) Grouping VIII (Co-Univocal multiplication of relations)

Let $\xleftrightarrow{o}$ = " The same person as "

$\xleftrightarrow{a}$ = " Is the son of the same father as "

$\xleftrightarrow{a'}$ = " Is the first cousin to "

$\xleftrightarrow{b'}$ = " Is the grandson of the same grandfather as " be our defined symmetrical relations in a family hierarchy. And

Let

$\downarrow$a = " Is the father of " (with $\uparrow$a = " Is the son of " )

$\downarrow$b = " Is the grandfather of " (with $\uparrow$b = " Is the grandson of ")

Then the properties of this grouping are:

a) Composition: ( A $\downarrow$a B ) x ( B $\xleftrightarrow{a'}$ C ) = A $\downarrow$a $\xleftrightarrow{a'}$ C'

Or A is the uncle of C

b) Associativity holds among three or more products of the type (A $\leftrightarrow\downarrow$B ).

c) Piaget asserts that the special identity " conform to the usual rule" [56] but does not illustrate it. But one may consider it in the form of the following tautology.

d) Tautology; ( A $\xleftrightarrow{a'}$ $\downarrow$b B) x ( A $\xleftrightarrow{a'}\downarrow$b B ) = ( A $\xleftrightarrow{a'}\downarrow$b B )

e) " One would presume that he [Piaget] intends absorption to be of the form " [57]: ( A $\xleftrightarrow{a}\downarrow$b C ) x ( A $\xleftrightarrow{a}$ $\downarrow$a C ) = ( A $\xleftrightarrow{a}$ $\downarrow$a C )

f) Inverse operation is logical division; and a product divided by itself gives the identity. (A $\xleftrightarrow{a_1}$ $\downarrow$a B) ÷ ( A $\xleftrightarrow{a_1}$ $\downarrow$a B) = A $\xleftrightarrow{\cdot}$ $\downarrow\cdot$A

One more grouping, that of equalities, may be considered seperately, though it appears practically as underlying all the other groupings. It can be also considered as a special case of symmetrical relations.

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56. Flavell, op. cit., p. 187.

57. Ibid.

The properties of this grouping are the following:

a) Composition; $(A = B) + (B = C) = (A = C)$

b) Association; $(A = B) + [(B = C) + (C = D)] = [(A = B) + (B = C)] + (C = D) = (A = D)$

c) Inverse of $A = B$ is $B = A$

d) Each equality plays the role of a special identity with itself and with any other equality; $(A = B) + (A = B) = (A = B)$

$(A = B) + (C = D) = (A = B) = (C = D)$

e) The general identity is $A = A$

Groupings can be observed when two children at different ages are proceeding to solve a given problem. Suppose we take two children; one, at the " concrete operational level ", and one, at the " formal operational level ", and give them a problem that involves checking one combination out of four combinations. Suppose the combinations are; $A \times B$, $A' \times B$, $A \times B'$, and $A' \times B'$. The concrete operational child will <u>discover</u> these combinations as he is experimenting on reality.

Suppose further that the concrete operational child has a rectangle that he can change its dimensions; and you ask him to make it as big as another rectangle that is fixed. Suppose the variables are symbolyzed by; p - ' increase length', q - ' increase width '. Then he will discover the following combinations; $pq$, $p\bar{q}$, $\bar{p}q$, $\bar{p}\bar{q}$.

The formal operational child will intuitively see these combinations and will also see the combinations of these combinations. That is, he will be able to formulate all the possible combinations and then proceed to check which ones really exist. In brief he will see the following lattice.

(1) $\overline{pq \vee \bar{p}q \vee p\bar{q} \vee \bar{p}\bar{q}}$ (5) $\bar{p}\bar{q}$ (9) $p\bar{q} \vee \bar{p}q$

(2) $pq$ (6) $pq \vee p\bar{q}$ (10) $p\bar{q} \vee \bar{p}\bar{q}$

(3) $p\bar{q}$ (7) $pq \vee \bar{p}q$ (11) $\bar{p}q \vee \bar{p}\bar{q}$

(4) $\bar{p}q$ (8) $pq \vee \bar{p}\bar{q}$ (12) $pq \vee \bar{p}q \vee p\bar{q}$

(13) $pq \vee \bar{p}q \vee \bar{p}\bar{q}$ (14) $pq \vee p\bar{q} \vee \bar{p}\bar{q}$ (15) $p\bar{q} \vee \bar{p}q \vee \bar{p}\bar{q}$

(16) $pq \vee \bar{p}q \vee p\bar{q} \vee \bar{p}\bar{q}$

If these combinations are taken as elements of a Lattice then each pair of them will have one and only one g.l.b. and l.u.b. This lattice is the result of integrating groupings III, II, I in the concrete operational level.

When used as descriptions of what goes on in the mind of the child, groupings are discovered one after the other as the child grows; and then starts to integrate them and, finally, be able to reason with them, in hypothetical manner.[58]

## Groups ( The I N C R Group )

The above mentioned groupings are general, and cover practically every sort of experience. That is, most solutions can be analyzed and found to involve one of these groupings or an integration of two of them or more.

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58. For the empirical evidence for these groupings, refer to Flavell, op. cit., p. 190, where he gives a list of references in Piaget's works.

Piaget also considers formal groups. This group is supposedly found at all levels of development but gets really purified into the I N C R logical group in the latest stages. The treatment of this group is both physical and psychological, for Piaget would not hesitate, for example, to describe the behaviour of the child who is trying to re-establish equilibrium on a balance, in terms of the I N C R group. He thinks that, at least on the formal operational level, such a group can give account of the reasoning process involved in finding a solution to a problem. And in fact, reference to this group - to its properties - is made at all levels of development.[59]

In a certain sense this group can be thought of as one of the very primary 'gifts' given to the thinking organism. For, when Piaget discusses intelligence in contrast to perception, he finds that:

> "... it is necessary to remember that we shall attribute the widest meaning to this concept [group] for if, as recent works has shown, the logical definition of the group is inexhaustible and involves the most essential process of thought, it is possible, purely from our psychological point of view, to consider a group every system of operations capable of permitting a return to the point of departure. Considered thus, it is self-evident that practical groups exist prior to any perception or awareness of any group whatever." 60

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59. Cf. Flavell, op. cit., p. 135.

60. J. Piaget, The Construction Of Reality In The Child, op. cit., pp. 100-101.

Therefore, with some modification, any 'feedback' behaviour is considered as a return to the starting point; and hence, constitutes the reversibility property of the group. Also " all circular reactions can in a sense be said to possess at least the reversibility characteristic.[61]

These circular reactions can be of the type of scratching the floor over and over, putting the hand forward and withdrawing it, and many other actions typical of the early life. But one should not conclude hastily that the group is now adopted to explain behaviour without modification. For, as a matter of fact, Piaget talks of three types of groups that are revealed by the behaviour of the growing child, namely; the "practical group", the "subjective group", and the "objective group". [62]

In the practical group-behaviour the child can not see the group-like character of his behaviour for, then, he is still mixing up the object with his action and with himself. For the child does not yet (in early childhood) have an object concept.

In the objective group the child can see, for example, the organization in space of which he is a part. That is, he can now see - due to the associativity of the group - that he can go from A to C or from A to B and then from B to C.

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61. Flavell, op.cit., p. 136.

62. Ibid., p. 137.

The subjective group is the intermediary one in which the child partly sees himself as an entity in the universe but still under the influence of his strong egocentricity. The action and the object are still intermingled together.

But late in life, i.e. at the operational level, the group becomes a definite pattern of behaviour. And at the same time, the properties of the group become well defined. The child who possesses the sixteen-elements lattice, can now notice that certain operations on a logical sum don't change the sum. That is the sum is in a sense conserved. This is a striking parallelism with the operation of increasing the space between the powker chips that does not change their number.

In a similar manner, all the other properties of the special group I N C R will become part of the child's 'répertoire'. Piaget defines the properties of the I N C R group as follows.[63]

Suppose we are given a certain logical sum $p \vee q$. The identity operation (1) is defined by $I ( p \vee q ) = p \vee q$. The negation operation is the familiar De Morgan one, i.e. $N ( p \vee q ) = \bar{p} . \bar{q}$ . Then there remain the two operations C - Correlate, and R - Reciprocity defined by: $C ( p \vee q ) = p.q$ and $R ( p \vee q ) = \bar{p} \vee \bar{q}$.

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63. Cf. Flavell, op. cit., pp. 215 - 222, and J. Piaget, Logic And Psychology, op. cit., p. 33.

Now these operations taken as elements of a group, will satisfy the conditions for a mathematical group.[64] i.e.

NCR = I, NC = R , NR = C , RC = N , and INC = R

Or more simply, any transformation done by two or more transformations of the group can be done by one transformation alone.

Piaget's interest is not in this formal group as such, but rather in its physical counterpart. For, he sees its applications in the physical world in problems like that of the balance, the hydraulic press,etc., where he asserts that if p is taken to mean " add weight in one pan ", then to re-establish equilibrium one can do one of two things; namely, take off the weight - $\bar{p}$ , or counterbalance it by another weight in the other pan - $p^{*}$.[65]

The concept of grouping and groups can be summarized by answering the question raised earlier. The subject in his interaction with the environment develops certain methods of handling it which take the shapes of patterns that are flexible to accommodate new problems. With the operation of reversibility - which is genetic - and some primary concepts such as conservation, object concept, the subject clusters

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64. Cf. W. Lederman, <u>Introduction To The Theory of Finite Groups.</u> (Edinburgh: Oliver and Boyd Ltd., 1957), pp. 2-3.

65. For a further analysis of this point see Flavell, <u>op.cit.</u>, pp. 218 - 222.

other operations and concepts slowly as he progresses, moving from one grouping to another, to achieve the highly abstract level of formal operational thought.

The child does this in distinct steps.

## Stage Development

Piaget's psychology had been called developmental after this aspect of his theory. In fact, it is one of the major contributions to the epistemological question of concept acquisition. As pointed out earlier, the question of concepts as whether taken in wholes or in parts is finally answered by Piaget by postulating a genetic developmental structure.[66]

In Piaget's classification of these stages, he refers to four main ones. It should be noted, however, that for certain discussions it may be found more convenient to divide them and sub-divide them further. The most common divisions are four and six.[67]

### Stage I ( Sensori-motor period, 0 -- 2 years )

Within this period Piaget distinguishes between six different, but interrelated, stages. They range from complete egocentrism in

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66. A brief summary of each of these stages is presented by Robert Thomson, in his book The Psychology of Thinking ( Edinburgh: Penguin, 1959), pp. 94-103.

67. We will refer to the same divisions that Piaget himself enumerates in his Logic and Psychology, op. cit., namely, the division of four.

stages one and two[68] to the "invention of new means through internal, mental eoordination." [69] In the second stage the child responds to actions and events that he is directly involved with, i.e. actions such as; sucking, turning his head to the source of the sound, grasping a finger offered to him, etc. [70] In the third stage there is a marked difference; for, now, the child can respond with anticipation in such a way as to cry if he sees his mother put a hat on her head in preparation to leave.[71] Piaget would explain this behaviour in terms of coordination of schemata which may be considered in a sense as an explanation of the theory of associationism. In brief, this third stage may be considered as a stage of reaching out - an outward movement into the unknown.[72] In the fourth stage the child starts to lose his egocentrism. He starts to see relations among objects instead of subject-object relations all the time.[73] In the last two stages, the child acquires the mechanism of symbols - such as language, facial expressions (smiling, frowning) - and he can, now, discriminate them and employ them.[74]

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68. Cf. Flavell, op. cit., p. 149.
69. Ibid., p. 118.
70. Ibid., p. 112.
71. Ibid.
72. Ibid., p. 149.
73. Ibid., p. 150.
74. Cf. Flavell, op. cit., p. 150.

The child in this period, passes from a purely chaotic phenomenological world to an immediate environment that can now be ordered with some skills and tools.

Stage II ( Pre-Operational Thought, 2 -- 7 years)

In this period the child starts to grasp collections of events. He can now order these events in a sweeping moment and still relating the past to the present and anticipating the future.[75] In fact, we have seen the similar anticipatory behaviour in the sensori-motor period - the child crying when his mother puts her hat on. But here, though the reaction is almost the same in nature it is much quicker and includes a variety of anticipatory reactions. The concept of vertical "décalages" gains more meaning now.

When this period is looked upon in general terms it reveals its transitory character. This period is basically a transition between a sensori-motor functioning to another where the child functions intelligently by a symbolic inner manipulation of reality.[76]

More specifically, the pre-operational child can not yet see the reversible reactions. As an example, he can not yet see that the quantity of water is still the same when poured into smaller containers.[77] His reasoning at this period is still "transductive" , i.e. neither

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75. Ibid., pp. 151 - 2.
76. Ibid., p. 151.
77. Ibid., p. 159.

inductive nor deductive - it goes from particular to particular. In brief, he just "juxtaposes" elements instead of connecting them in an implication form.[78]

From the experiments on this period, Piaget concludes that the child is still " animistic" and is possessed by a generalized immature tendency, that is revealed in his conception of space, volume, quantity, number, time, causality, etc.[79]

The pre-operational child is the son of the sensori-motor one, and it would be surprizing if he didn't resemble the sensori-motor child; for then he would be arising " ex nihilo".[80] But this period can be 'credited' with the gain the child makes in moving to the limit of his egocentric field where he finds his tools of adjustment insufficient. And in a sense every stage could be credited with this movement towards disequilibrium. But the achievement of this period, seems to be the realization that transductive logic is not enough to explain phenomena.

Stage III ( Concrete operations, 7 -- 11 years )

This period starts at the jump the child makes from "juxtaposition" to systematic behaviour. When compared with the pre-operational child one finds that:

> " It is simply that the older child seems to have at his command a coherent and integrated cognitive system with which he organizes and manipulates the world around him." 81

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78. *Ibid.*, p. 160.
79. *Ibid.*, p. 161.
80. *Ibid.*, p. 162.
81. *Ibid.*, p. 165.

Here the concept of groupings is starting to appear as a 'whole' of simultaneous operations.

This period should be compared with the next period (the formal operational). Though most concepts that have an empirical basis get formed in this period, the child still lacks the intellectual strategy which would be necessary for generalizations and eventual inventions. Here his concepts are somehow bound by the physical reality. In brief, three main strategic limitations characterize the concrete operational child.[82] In comparison with the older child the following conclusions may be drawn:

1) What the concrete operational child doesn't do (and the older one does) is act on the real as a special case of the possible and not the other way around. (Refer to grouping VIII).
2) The child may see, for example, conservation of weight, of volume, but not of weight and volume. He is still attached to the reality and does not, as a rule, show the 'horizontal décalages'.
3) The second limitation is also present on an intellectual level; i.e. He might have the separate intellectual structures that may be enough to solve problems pertaining to them separately, but he is not able to integrate them in a whole to solve the more difficult problems. One may say that he has the sub-systems, but not yet the system.

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82. Cf. Flavell, op. cit., pp. 203-204.

Stage IV (Formal Operations 11 -- 12 to 14 -- 15 years)

Just as the child in the pre-operational period would move to concrete operational (under the pressure of problems raised as the preoperational structure gets sharper) so the formal operational child, facing gaps and difficulties in the concrete operational period, moves to formal operational thinking to fill-in the gaps.

Now he can make use of the reciprocity which helps him keep one variable constant instead of annulling it by a direct operation - negation.[83]

This transition marks a move to genuinely scientific analysis. That is, the child no more thinks of reality he is facing directly. He is now liberated and can think of combinations of experiments to verify his hypotheses.[84]

One can point out three main features that characterize this period.[85]

1) The child now possesses a hypothetico-deductive strategy. He can now formulate hypothesis and 'invent' experiments related to them.

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83. Ibid., p. 209.

84. It might be interesting to note,here, that at this stage only the child can formulate a hypothesis in the scientific sense. And, at this stage, the child gains the strategy of hypothesis verification or falsification.

85. Cf. Flavell, op. cit., pp. 204-205.

2) He can now make use of statements "containing" the concrete operational results to combine them in propositional systems. By virtue of that, the formal-operational thought is sometimes referred to as the second degree operational or 'operations to the second power'.

3) Variables are easily isolated now (for he has already developed a concept of causality if the situation calls for such a relation). They can also be combined in all possible combinations to be checked for reality.

In summing up the last three periods it can be claimed that:[86]

First, the pre-operational child is an object of wonder. He is fanciful about the world; and things look as if they are parts of a game that he plays with no conception of rule or a binding law.

Second, the concrete operational child is the organizer who arranges and orders things in 'pigeon-holes'. He is totally taken by the real and distrustful of the possible. All his actions are of such a nature that he can act upon the real and react to it.

Third, the formal operational child has the organizing mind of the previous one, but is daring to move into the 'wonderful'. He is taken by the wonderful; but, unlike the pre-operational child, he moves into it with a sober mind that experiments and controls to discover the underlying laws.

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86. Cf. Flavell, op. cit., p. 211.

The last stage of mental development seems to be a stage of propositional behaviour emerging from interrelated logical systems. And two marked acquisitions seem to appear 'suddenly' in this period:

First, the propositions are interrelated irrespective of their contents. Second, and strikingly, there appears "a series of operational schemata which have no apparent connection with each other nor with the logic of propositions." [87]

These operational schemata are of the type of mechanical equilibrium, proportions, permutations, combinations, aggregations. They tend to group themselves in isolated systems, and apparently, do not get integrated in a whole system. And there is no apparent justification for their compartmental isolation.

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87. J. Piaget, Logic And Psychology, op. cit., p. 22.

## THEORY OF PERCEPTION

The role played by perception in cognitive structures can not be over emphasized. And any scientific theory of epistemology will have to start with perception; because, at the very foundation of knowledge - objective knowledge of the universe - the starting point is in the face-to-face contact with reality. Some people would explain scientific knowledge as an ' objectivation ' of the elaborated sensations through a series of progressive abstractions from the empirical content.[88]

As far as Piaget's theory is concerned, perception is to be explained in so far as it is related to intellectual-operational developments involved in intelligence. And like all other concepts, the developmental analysis of perception is the backbone of his works. Perception is then explained in terms of an integral summation of micro-intervals of perceptive " encounters ".[89] This suggests the development of concepts as integral systems of sub-systems. There is no description of the nature of these encounters between the eye element and the stimulus element. Only the abstract assumption of encounters - " rencontres " - is supposed to explain the origin of the perceptive phenomenon. And then the eye is supposed to encounter ' bits ' of a line before it perceives the straight line in totality.

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88. René Hubert, Le sens du Réel (Paris: Alcan, 1930), p. 103.
89. Cf. Flavell, op. cit., pp. 226 - 27.

Maybe the question about the nature of these encounters is analogous to the question raised about the nature of the primary schemata. And maybe both questions shouldn't be asked to a scientist who is trying to explain - describe - the phenomenon. Such a scientist is probably not supposed to answer the " why " question at this level; and may be perfectly justified if he does not go that far, and is satisfied in describing events as they " seem " to be - and this would not be an easy compromise.

Piaget is more thorough than that; for, to him, perception is only a sub-system of intelligence. Though the two concepts are inter-dependent, one can not trust his perception alone. For it is only through intelligence that we may have the absolute knowledge of $A = C$ from $A = B$ and $B = C$.[90]

Perception on the other hand, may give us $A = C$, but at its best, and it may also give us $A \neq C$ under certain conditions. In general, perception is always probabilistic in nature, in contrast with intelligence - and in opposition to the Gestalt concept of perception.[91]

With this perspective a distinction between perceptive and intellectual processes may be drawn. And one marked difference is in the fact that, perceptual structures never achieve equilibrium states in contrast with intellectual structures.

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90. Ibid., pp. 232 - 233.

91. Cf. Flavell, op. cit., pp. 232 - 3. And W. Mays, Proc. Of Arist. Society, op. cit., pp. 50 - 51.

The perceived object at any moment of time is supposed to depend on duration of " exposure " and the fixation ability of the subject. The investigation on the perception of lines shows the following results.[92]

a) As the length of a line is a function of the accumulation of encounters, all lines are subject to " what can be called an absolute overestimation". This would constitute " Elementary Error I ". In other words, as the subject accumulates encounters, with the passage of time, he tends to develop a habit of it and somehow tends to over-accumulate. The results will be an overestimation of the length of the line. One can easily conclude to a certain extent, that the longer the line the more the overestimation. But then, a corrective factor gets in if the line is too long.

b) In line with the above observation, if the subject is presented with two unequal lines, he would tend to overestimate the longer one. This " Elementary Error II " is slightly different from I; for now, the subject may add encounters with the smaller line to encounters with the longer one. By virtue of this relative situation this error is referred to as " relative overestimation ".

c) Then the concept of couplage - " couplages" - between encounters is explained in terms of spatio-temporal concepts. It is a kind of spatio-visual transportation of one element to another and

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92. Cf. Flavell, op. cit., pp. 228 - 231.

reciprocally. Then encounters-accumulation gets coordinated with a decentration through couplage to give rise to a complete or incomplete " couplings ". The more complete are the couplings between encounters, the more is the probability that the two lines will be perceived as they really are.

Probabilistic elements thus make their way into perception. And this problem may become a fundamental one if measurement - physical - is conceived of as essentially being a comparison between the length of two lines. But, this may be an oversimplification. For as Piaget puts it: " ...a sequence of perceptions does not necessarily imply a perception of sequence." 93

Which means, even though the couplage is similar to the one-to-one process there is no guarantee that the subject will be aware of it.

In summary, " our perception of figures is built up as a result of a series of random eye and other muscular movements which are gradually corrected ".[94] It is the function of intelligence to correct these gradual movements. And in fact, our intelligence is the dependable reference in this context.

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93. Cf. J. Piaget, The Construction Of Reality In The Child, op. cit., p. 325, and Flavell, op. cit., 147.

94. W. Mays, Nature, op. cit., p. 626.

## THEORY OF INTELLIGENCE

In a way, all the above exposition could be called a theory of intelligence. In very general terms, an intelligent act is the act by which one structures experience through a " prise de conscience ".[95]

This perspective immediately points out three main factors involved in intelligent functioning; namely; experience, structure, " prise de conscience ". This is still very much in line with the theory of adaptation discussed above. In fact, this definition may be regarded as a major outline of the whole theory.

There may be a parallel between this theory of intelligence and the theory of adaptation.[96] Intelligence begins at a biological substratum, i.e. it begins whenever actions begin. Then through certain functional invariants, such as assimilation-accommodation, the acts get organized into wholes. These wholes take the form of primary schemata born at the interaction between assimilation and accommodation. Then as the adaptive process proceeds, the relations between accommodation and assimilation get integrated into systems, and by a " prise de conscience " they become the core of intellectual functioning. This is essentially the nature of cognition.

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95. Cf. J. Piaget, Mind , op. cit., pp. 137 - 160.
96. Cf. Flavell, op. cit., p. 67.

Intelligence then - and hence intellectual functioning - starts with a flow of unrelated pictures and proceeds to " a sort of an ideal stage that is never achieved".[97] which integrates our experience into structured wholes in perfect equilibrium.

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97. Cf. J. Piaget, Mind, op. cit., p. 135; "Pour notre part,.... la raison est une sorte d'idéal, jamais atteint en fait, et vers lequel tend l'intelligence a travers mille péripéties."

# CHAPTER III

## THE NUMBER CONCEPT

Regarding the question concerning the relation between logical groupings and the theme of this study, the functional aspect of the general theory might be recalled. According to that notion the child develops from a stage completely governed by his perception to a stage where his intelligence tends toward ideal patterns used as assimilatory mechanisms. These ideal patterns are the logical groupings discussed previously. Piaget uses this logical approach to cognition, in a trial to exhaust every possible operation that could be performed on classes of objects or relations among them.[1] After laying this foundation he goes on to analyze how different concepts arise out of it.

In fact, he uses the logical groupings for more than one purpose.[2] First, he conceives of them as " parsimonious " structures characterizing " ideal " cognition. Second, he uses them as a framework to interpret the changes taking place as the child develops into stages. Third, these groupings are used as diagnostic guide lines within each stage of development. And, finally, the over-all purpose is to see how mathematical concepts " spring spontaneously " from them.[3]

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1. Cf. Flavell, op. cit., p. 188.

2. Ibid., p. 190.

3. Cf. Jean Piaget, Scientific American, op. cit., p. 79.

## EPISTEMOLOGICAL ANALYSIS

In general, and in contrast to the empirical position, the child in his act of thinking co-ordinates operations done on reality. From the simple operations of gathering, classifying, dissociating, etc., the child begins to discover spontaneously that some of these operations can be co-ordinated. Then he slowly discovers the logical groupings. But by further abstraction ( purification ), he begins to realize that the groupings can be further ' grouped ', hence giving rise to mathematical groups; such as the additive group of natural numbers. This change from groupings to groups gives the number concept.

### The Construction Of Number

While the number formation is described in this general way[4], it should not be supposed that the grouping is " given " to the child so easily. It has been established experimentally that the perceptual intuition of the five-to-six years old child is still irreversible: he can not yet grasp the idea of inclusion, and he still thinks that the " brown beads are more than the wooden ones ".[5]

The concept of conservation ( the twin of reversibility ) is also necessary for the formation of number. For, as Mays says of Piaget :

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4. Jean Piaget, Introduction à L'Epistémologie Génétique, op. cit., p. 84.

5. Cf. 1) Jean Piaget, Child's Conception of Number (London, Routledge Kegan Paul, 1952), chp. vii.
2) " " , Intr. à L'Epist. Génétique, op. cit., pp. 84-85.

" He finds that it is difficult to speak of the intuition of an integer before the child's logical concepts of invariance have developed at about 7." [6]

And similarly, it would be difficult to talk of the child's conception of conservation without referring to the notion of reversibility. For, unless the child sees that he can pour the liquid split into two containers back in the original, he would not be able to see the quantity conserved.[7] It is these twin concepts that marks the beginning of the notion of an invariant whole. In other words, these concepts form the beginning of a reversible system characterized as a grouping.

The invariant whole is undoubtedly the backbone of the number concept; but it does not tell the whole story. For, in the long run, it will be seen that the number would depend on the invariance of the whole under a certain specific consideration. In other words, the child may realize that a quantity of liquid in container B is conserved when poured into the two containers $A_1$ and $A_2$, but it would take him some time before he can abstract the qualitative differences between $A_1$ and $A_2$ to be able to consider them as quantitative units.

But it may be safely stated that once the conservation of number is reached ( at about seven years of age ), the child will unmistakably

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6. W. Mays, Nature, op. cit., p.626.

7. Ibid., p. 625.

realize that a row of beads does not change in number when the beads are spread wider apart; for, now he is aware ( at least intuitively ) that the increase of the length of the row is proportional to the length of the spaces between the beads.[8] That is, he now possesses the reversible operation as well.

And, moreover, the child is able to perform logical multiplication and conceive of one operation as compensating the other. In the long run, these operations have to be co-ordinated in such a way as to help the child perceive the class structure. That is, he will have the grasp of number only when he grasps the essential operations involved in classes of objects and in relations among objects;[9]

> "... the implication is that a genuine operational grasp of number is scarcely possible without the prior acquisition of the logical operations of which it is a synthesis." 10

In essense, what the child needs to realize is the possibility of adding $A_1$ to $A_2$ in such a way that they can be considered as $A+A = 2A$. Stated differently, what he needs is the concept of a unit that will enable him to reduce both $A_1$ and $A_2$ into A.

The concept of unit

In general, two objects $A_1$ and $A_2$ will be classed under B if they have the B property in common. But among themselves they will be

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8. Ibid., pp. 625 - 626.

9. Cf. Flavell, op. cit., p. 198.

10. Ibid.

distinguishable due to certain differences. To reduce $A_1$ and $A_2$ into units such that B = 2A involves the operation of disregarding the differences, but not completely. At least they must be related to each other in such a way that they can be chosen one after the other. If their differences are disregarded up to the point that the only difference remaining between them is the order with which they are chosen, then $A_1$ and $A_2$ can be considered as units in the whole B = 2A. If the class contained more objects in it, then the difference between any two objects will be the same, i.e. a difference of order of choice. This would amount to a naive interpretation of what is meant by equating the differences between two objects grouped together in a logical class,so that they become " extensively quantified ". And

> " To explain this evolution we must understand that, for instance, half a quantity is not only a quantity which is equal to another quantity and which when added to it constitutes the original quantity, but also that it is equal to the difference between the whole and the other half." 11

This equating of differences, as mentioned earlier,is the result of reducing the differences between the objects in a class into a difference of order of choice.[12] Or, the equating of differences would amount to " seriation of units seen as equal in all respects except the temporary relative positions of each one in the series."[13]

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11. Jean Piaget, The Child's Conception of Number, op. cit., p. 23.

12. It should be noted here that ordination as a process is essentially inherent in cardination as a process.

13. Jean Piaget, The Child's Conception Of Number, op. cit., pp. 84-85.

It had been verified empirically that in the case of correspondence and in the case of relationships inherent in the conservation of quantities, the arithmetical operations are explained in terms of an implicit or explicit introduction of the concept of unit.[14] The correspondence intended here ( as an arithmetical operation ) is of the type where any element of one set would correspond to any other element in another set, as opposed to correspondence between a certain side of a triangle with a certain side in the other triangle. The two triangles are still said to be in correspondence.

The passage from logical grouping to numerical quantification lies in the possibility of ' iterating ' the logical elements as units. And then " the conversion of intensive logical elements into iterable units with numerical quantification yields arithmetic operations. Similarly, when infralogical elements ( e.g. the parts of a whole ) become iterable units, one gets measurement operations.".[15]

The relationship between quantification as number and measurement is obviously in the notion of unit that is common to both of them. And hence, the unit of measurement is conceived of as developing in the same way as the unit of number.

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14. Ibid.

15. Flavell, op. cit., p. 198.

> " It shows more particularly that the construction of the unit, necessary for measurement, involves equalization of differences, and this we now see to be the essential condition for the transition from compositions of purely qualitative relations to truly numerical compositions." 16

Construction of positive numbers

The iteration principle involved in the formation of the unit is, in fact, sufficient to explain the series of positive numbers as such. Because, we notice that by the time the child realizes that certain elements can be iterated as units, he then possesses the group structure which is free of the tautology property. Now, two operations would get composed to form a new one. To reach to this level however the child must have had fully grasped the logical structure which serves as basis for the principle of iteration; and, hence, the level at which the child can generalize the iteration process is a level involving a further abstraction over the logical structures. That is why it was found that the construction of the series of numbers is not possible at the age of six to seven years.[17]

In general, therefore, it could be safely stated that the construction of the unit concept is essentially the construction of the positive numbers; because, as we have seen, the equalization of differences is exactly the process by which the notion of the unit is reached.

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16. Jean Piaget, Child's Concept of Number, op. cit., p. 238.

17. Cf. Jean Piaget, Intr. a l'Epist. Génétique, op. cit., p. 93.

But the idea of the operational notion involved in the formation of the unit, and hence the positive numbers, is undoubtedly the binding notion that stands distinguished in this respect. For, after all, it will be the notion that will facilitate the explanation of the number series. It is not the existence of six objects in a class that gives the concept of the number six, but rather the co-ordination of the operations of classification and differentiation, i.e. seriation, that is the number six. The number therefore, is nothing more than the operations done on the objects.

The negative numbers certainly assert the operational character of the number and serve as an illustration as to what is meant by the number being defined by the operations done on objects rather than by the objects themselves. It is the intention of to include or not to include that makes the difference between positive and negative numbers. And if that was not the case, we would find it very difficult to explain the negative numbers by perception of objects as such.

More evidence as to the difficulty involved in accepting the argument of direct perception of objects as the origin of the notion of numbers, is revealed by the notion of the zero. For the zero would have to be explained as a perception of no perception. While with the operational approach the zero becomes parallel to the notion of non-operation as an operation by itself.[18]

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18. Ibid., p. 114.

The fractions also emerge from the abstraction involved in considering a whole as continuous and yet discontinuous at the same time.[19]

But the numbers that explain the ideas involved in number are the transfinite numbers. With the transfinite numbers most of the confusion resulting in the case of the finite numbers due to the fusion of ordination and cardination is avoided. For with the transfinite the two operations are separated. The number immediately loses its iterable character and there is a return to the logical notions by denoting a logical class qualitatively and at most " intensively ". For , the transfinite is a class of all denumerable classes, qualified by the common quality of denumerability and joined simply by their logical union.[20] And in fact, this way of thinking is a return to the tautology property replacing the iteration principle; for now one has $\aleph_0 \times \aleph_0 = \aleph_0$ , $\aleph_0 + \aleph_0 = \aleph_0$ . To show the separation of the process of cardination from that of ordination, one may consider the number of ways by which this cardinal $\aleph_0$ can be ordered. It can be ordered in an infinite number of ways, which was not the case with the finite numbers.[21] This would be an illustration of the idea of one to many correspondence hinted at in connection with co-univocal multiplication of classes and relations which asserts the return from the group to the groupings.

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19. Ibid.

20. Ibid., pp. 128-129.

21. Ibid., p. 130.

Conclusion

In this terminology, Zeno's famous paradox may be explained by the way of thinking of number as denoting a perceived reality rather than denoting an operation done on reality. It was shown that " number is indeed possible to the extent that the elements A, A' , B' , ... , are viewed no longer as being either equivalent or non-equivalent, but as being at the same time equivalent and non-equivalent."[22] But, that would amount to considering any cardinal number as a class of objects whose members are treated as units.[23]

It was also shown above that objects in a class will be considered as units only when the difference between any two of them is reduced to a difference in the temporal order of choice. That is, any two units will be different only in the sense that they are chosen at different times, otherwise they are identical.

This notion is possible only after abstracting the qualitative differences between objects, which marks the passage from qualitative logical groupings into quantitative mathematical ones.[24] In this sense only, the concept of number as being the ' class of classes ' could be considered as a logical concept.

But that would be an oversimplification of the operations involved in the formation of number. To consider number as a ' class of classes ' would be leaning too much on the idea of one-to-one corres-

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22. Jean Piaget, Child's Concept of Number, op. cit., p. 156.

23. Ibid., p. 157.

24. Jean Piaget, Intr. a L'Epist. Génétique, op. cit., pp. 101-102.

pondence which will not necessarily lead to mathematical quantification.[25] The correspondence that is involved in the formation of number is the one drawn between any member of a class and any other member of another class. As an example, one can think of this correspondence as the one drawn between the disciples of Christ and the generals of Napoleon; both twelve.

In the long run, mathematical correspondence will be formed as a result of correspondence between classes, but after equating the differences between their elements. This equating of differences between the elements is " the criterion for transition from an arbitrary qualitative series to the corresponding numerical series...[26]".

Hence, the relation between logic of classes and logic of relations when brought together, irrespective of the inclusion and the seriation relation, will form the series of numbers " which are indissociably cardinal and ordinal".[27]

It was also shown, above, that the two operations, namely cardination and ordination, are at the foundation of the iteration principle. For, in the case of the transfinite numbers, when the two operations were dissociated, the numbers lost their iterable quality and the introduction of the tautology, as a feature of the logical grouping took place. On the other hand, the finite numbers prove

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25. <u>Ibid</u>., p. 90.

26. Jean Piaget, <u>Child's Concep. of Number</u>, <u>op. cit</u>., p. 95.

27. <u>Ibid.</u>, p. viii.

to have the special quality of fusing cardination and ordination in their formation.[28] And the only difference between the Nth element and the ( N + 1 ) th element is that the first has ( n - 1 ) elements before it while the second has n elements before it.

From such a notion the idea of the numerical succession becomes necessary for the grasp of number.[29] In general the principle of iteration together with the notion of a unit will be sufficient to define any integer n+1 from the definition of the integer n.[30] But this is the other side of the coin; for it was shown that the concept of unit presupposes the concept of number. And what is referred to as mathematical induction ( and Peano's fifth axiom in specific ) can be considered to assume an implicit reference to both the cardinal value of a number as well as its ordinal value; which will reduce, in the long run, to the concept of number as being a synthesis of classes and assymmetrical relations.[31]

To sum up, a number is therefore a class of objects considered as identical, so they can be iterated, and yet different only in their temporal arrangement in the class.

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28. Ibid., p. 113.

29. Cf. Jean Piaget, Logic and Psychology, op. cit., p. 8.

30. Cf. A.E. Taylor, Plato (London, Methuem, 1949), p. 513.

31. Jean Piaget, Intr. a L'Epist. Génétique, op. cit., p. 106.

## Number As An Epistemological Model

The claim that number can be used as an epistemological model for the development of other concepts is based on the assumption that the number concept presupposes certain operations and notions that are common to other concepts as well. It was shown that the concept of measurement is a twin concept of number and can be conceived of as a synthesis of some basic operations:

> " One may therefore say that measurement is a synthesis of division into parts and of substitution, just as number is a synthesis of the inclusion of categories and of serial order." 32

And the operation involved in measurement is so closely related to the one involved in the formation of number such that they can be considered as two sides of the same coin.

But to show the parallelism between the concept of number and concepts in general, an investigation of the essential operations that are involved in other concepts, seem unavoidable. To fill the gap between mathematical concepts and other ones, a reference should be made to Piaget's contention as to the nature of the mathematical concepts:

> " For Piaget, then, mathematics has its origin in neither the contemplation of eternal numerical essences, nor in partial abstractions from sense data, but from simple actions such as assembling, dividing and ordering. It is these actions which preceed thought and logicomathematical operation." 33

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32. Cf. 1) Jean Piaget, Scien.Amer., op. cit., p. 78.
2) " " , Intr. a L'Epist. Génétique, op.cit., pp. 116-117.

33. W. Mays, Proc. of Arist. Society, op.cit., p. 70.

Therefore, number and concepts in general start from the same set of givens; namely actions. It was shown that number presupposes certain logical conceptions as a foundation for its formation. But so do other concepts; the geometrical concepts can be used as an illustration.[34] In essence, the parallel would be complete if it can be shown that the logical operations assumed in the formation of the number concept are the same as the ones involved in the formation of other concepts. Specifically, the operations of seriation and classification are the required ones.[35]

In a marginal way, it is noticed that " every notion whether it be scientific or merely a matter of common sense, presupposes a set of principles of conservation, either explicit or implicit."[36] Since this concept of conservation seem to be underlying intellectual activities at all levels, it could be assumed as a kind of a functional a priori from a psychological point of view.[37]

What are the binding operations involved in the formation of any concept? A concept of any nature can be determined by its denotation and its connotation. That is, a concept is defined by its extension and

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34. Cf. Jean Piaget, Scientific American, op.cit., pp. 76 - 77.

35. CF. W. Mays, Proc. of Arist. Society, op.cit., p. 73.

36. Jean Piaget, Child's Concept. of Number, op.cit.,p. 3.

37. Ibid., pp. 3-4.

a criterion. But what is the extension of a concept if not the class of objects to which the criterion is applicable? The binding criterion itself is a qualitative relation that connects the members of the same class among themselves as similar and members of different classes as different. Up till now the concept of number is parallel to any other concept in its construction. Yet,a concept results in the synthesis between its extension and its connotation; for they are interdependent, and are conceived one in terms of the other. But so is number; a synthesis between classification and seriation, i.e. a synthesis between class and relation. Number, being quantity " par excellence ", is seen to emerge from qualitative logical conceptions.[38]

This contention is probably the clue that led Russell to consider number as the logical concept of " class of classes ". The number One would then be the class of ' ones '. The Two the class of pairs, and so on. Or in other terms 'one' would be the argument a to the function θ (x) such that θ (a) is true; and if there exists an argument b such that θ (b) is true then b is identical to a.

The famous objection raised by Poincaré to this concept is based on the idea that Russel's argument contains a <u>petitio principii</u>. For the idea of ' a man ' already implies the concept of ' one ' , and then Russel is defining ' one ' in terms of ' one '.

But Couturat would still cling to Russel's definition and claims that ' a man ' is of the same logical nature as ' all men ', ' some men ', ' no man ' which do not imply any numerical quantity.

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38. Jean Piaget. <u>Intr. a L'Epist. Génétique</u>. <u>op.cit</u>., p. 75.

Piaget offers the following alternative: when ' a man ' is used to compare man to ' 2 men' or ' n men', it has the idea of one. But when used in a logical context denoting an example of inclusion of one class in another like ' all ', ' some ', ' non ', it does not have any numerical significance.

In sum, it is the context that determines the nature of a concept. This is another way of saying that the concept is determined by the grouping to which it belongs.[39]

The number is seen to be the synthesis of two logical operations, namely classification and seriation. That is, number is the result of integrating the logical groupings of classes with those of relations simultaneously. It was seen that any concept is logically conceived of as the synthesis of its extension and the defining criterion, i.e. relation.

To study the evolution of the number concept is, therefore, to study the essential phases in the evolution of any concept.

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39. For reference to the above points of view and further remarks, refer to Jean Piaget, Intr. a L'Epist. Génétique, op.cit., p. 88.

## EXPERIMENTAL ANALYSIS

The analysis of the experimental work done in this field will be divided into three main sections. The first section will be an exposition of Piaget's works reported in his book; The Child's Conception of Number.* The second section will be an exposition of other research related to Piaget's works and reported in a number of journals. The third section will be an evaluation and will include a suggested modification on the method of research and interpretation of results.

### Piaget's works

The works of Piaget in this field dealt with three main problems; namely, the problem of conservation, the problem of correspondence, and the problem of ordination and cardination. The general question investigated under these several headings is the relation between classes and numbers.

The problem of conservation was treated under two main headings; namely, conservation of continuous quantities ( liquids ) and conservation of discontinuous quantities ( beads ). The general line of investigation was to see whether the child can tell that a quantity stays the same when divided into a number of parts. If the child can not see that the amount of lemonade stays the same when poured out in two smaller glasses, then he was asked some questions as to point to the bigger quantity, in a trial to see why he is misled.

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* Jean Piaget, The Child's Conception of Number, op.cit.

The results found in this experiment, and supported in the other experiments also, point to a genetic development of the concept of conservation in three main stages.

In the first stage, the children show that they are incapable of grasping the fact that a quantity stays conserved when divided into parts.[40] The reason for that, was found to be due to the child's total dependence on his perception. He would think that the original quantity is more than the quantity in the two smaller glasses because it reaches a greater height in the original glass. By the same systematic error, some children think that the quantity in the two smaller glasses is more because the number of glasses is larger.

The interpretation of the reactions of the children rests on the assumption that qualities per se do not in fact exist.[41] They are always compared and differentiated, i.e. put in asymmetrical relations (which is the germ of quantitative thinking). In the long run the child's qualitative reactions are interpreted in such a way as to reflect on his quantitative conceptions.

In the second stage the child hesitates in giving the correct answer. He might give a spontaneous response to show his grasp of conservation when the variations are slight, but fails to do so when

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40. Jean Piaget, The Child's Conception of Number, op.cit., p. 13.
41. Ibid., pp. 10 - 11.

they are greater; which reflects that he understands the questions asked to him, but is not yet ready to assume conservation as <u>a priori</u>.[42] The main observation one can make at this level, is the child's hesitation as due to a difficult period he is passing through where he is tortured between his perception and his intuition.

The hesitation is explained by the fact that the child will be able to perform the logical multiplication when the variations are small, but when the variations are greater " one or other of the relations becomes all-important and he is left hesitating between them".[43]

In the third stage, the child at once postulates conservation of quantity and can grasp the proportionality of differences and therefore acquires the notion of extensive quantification. Then the child is said to have the conservation concept. It should be noted here, that he also masters the logical multiplication operation.

The same triple - stage development is also found in the case of discontinuous quantities.[44]

The second problem that was investigated is the problem of correspondence. Two types of correspondence were specifically studied; namely, provoked and spontaneous correspondence. The provoked correspondence was studied by the use of flowers and flower vases, eggs

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42. <u>Ibid</u>., p. 15.

43. <u>Ibid</u>., p. 16.

44. <u>Ibid</u>., p. 25.

and egg-cups, and glasses and bottles. The spontaneous correspondence was studied by the use of two sets of beads arranged either in rows or in figures.

The study of this problem was motivated by Cantor's works where it was shown that such an operation is fundamental to the formation of the integers.[45]

In connection with the first problem the child was asked to put flowers in flower vases or eggs in egg-cups. Then he was asked whether the two sets, the flowers and the flower vases or the eggs and egg-cups, were equal after taking out the flowers or the eggs and grouping them together. Three stages were found in this experiment in support of the results found earlier.

At the first stage, the children can not make the correspondence yet, and are satisfied with their global comparison of the length of the two rows; for they still think that a row of six elements is equal to a row of five elements if they are of the same length.

At the second stage, the child thinks that " quantification is reducible neither to number ( most of the children can count up to ten ), nor to one-to-one correspondence, but to an intuitive correspondence depending on the perceptual configuration of the set."[46]

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45. Ibid., p. 41.

46. Ibid., p. 35.

At the third stage, the child infers immediately the equivalence of the two classes from the one-to-one correspondence and is no more influenced by the spatial relations. This stage may be characterized by the victory of intelligence over perception.

The interpretation of the children's reactions at the three stages is based on operation co-ordination. That is, in the first stage he behaves in line with his irreversible perception and can not construct even the intuitive correspondence. In the second stage he is capable of co-ordination, but on the intuitive level, and not until the third stage, will he be capable of true co-ordination after he discovers that any modification in the spatial arrangements of the elements can be corrected by an inverse operation.[47]

The same experiment was repeated with the child exchanging objects for pennies, once counting aloud and once without counting. The same results were found to support the previous ones. The general result found in this experiment indicates" that the importance given to the procedure of one for one exchange, in which so many authors have attempted to see the beginnings of cardination, is unjustified, since the procedure does not in itself result in the notion of necessary equivalence of exchanged sets."[48]

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47. Ibid., p. 56.

48. Ibid., p. 61.

And the more important result found in this experiment is that " counting aloud has no effect on the mechanism of numerical thought."[49] The same results were reaffirmed in the case of spontaneous correspondence.[50]

To sum up, the general conclusion that explains the lack of conservation seems to rest on the fact that " the elementary relationships inherent in the global perceptions are merely juxtaposed instead of being co-ordinated."[51]

✱ The third problem investigated is the problem of seriation ( or ordination ) as related to cardination. And in very general terms it was found that " the evolution of ordination goes hand in hand with that of cardination ."[52] This is because ordination is an intrinsic correspondence in the set.

In the experiment related to this problem, the child was given a series of sticks of different lengths and was asked to seriate them. Then he was given a series of dolls to match the sticks. And he was asked to tell the doll that would match with a certain stick after disturbing the order ( perceptual correspondence ).

Here again, there were three distinguishable stages; starting with the first stage where the child can not re-establish correspondence

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49. Ibid., p. 63.

50. Ibid., pp. 73-74.

51. Ibid., p. 87.

52. Ibid., p. 97.

once it has been disturbed, ending with the stage where the child sees that the position of one element in the series ordered by an asymmetrical relation assumes a certain cardinal number which is the number of elements that have preceeded it.[53]

A remarkable piece of behaviour of the child at the second stage is that he <u>discovers</u> the series, if he succeeds in reaching it at all. In trying to seriate sticks 10 , 9 and 8 he measures 10 against 8 and 9 against 8 but not 10 against 9.[54]

This behaviour suggests that the child does not yet conceive of all the possibilities at once, but rather discovers them one after the other.

Therefore, the child starts at the first stage, by a cardination that is global, non-conserved and in no one-to-one correspondence, and by an ordination which is just juxtaposed, and develops into a stage where the ordinal $n+1$ means to him the element that comes after n cardinal elements.[55]

The last part of the book deals with the relation between classes, relations and numbers. And it is this part that throws most light on the theme of this study and validates the attempt to consider the formation of the number concept as an epistemological model for the formation of other concepts. For, in Piaget's words:

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53. *Ibid.*, p. 112.

54. *Ibid.*, p. 104.

55. *Ibid.*, p. 153.

" If we regard the extension of concepts as being inseparable from their comprehension, every notion thus corresponding to a class, it becomes obvious that concepts and numbers have an important common basis, namely the additive operation, which brings together the scattered elements into a whole, or divides these wholes into parts." 56

And instead of considering number and class as two separate notions, such that one of them is reducible to the other, or as incomparable notions, it might be easier to consider them as interdependent developing together. Or, as a matter of fact, both class and number can be proved to " result from the same operational mechanism of grouping".[57]

It was also observed that the difficulty experienced by the seven-to-eight years old children is that they think of parts as just juxtaposed without any synthesis, and not as parts of a whole.[58] And in this case, both the operation of inclusion essential to the understanding of class, and the operation of addition in numbers are not genuinely grasped.[59]

And when addition of classes resulting in a conserved whole, is investigated further, it will be found to assume the properties of a lattice. For in adding class A to class A' we define the smallest class that contains them both, namely B, as their sum or l.u.b.[60].

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56. Ibid., pp. 161 - 2.

57. Ibid.,

58. Ibid., p. 171.

59. Ibid., p. 175.

60. Ibid., p. 177.

The point that is explicitely made in this connection is the rejection of Russell's concept of number as a " class of classes ". For to Piaget:

> " In order to transform classes F and $F_1$ into numbers, the first essential condition is that their terms shall be regarded as equivalent from all points of view simultaneously." 61

And this is done by an operation brought from outside, i.e. disregarding quality, and not a " class of classes ".

In few words, the formation of two objects from A + A' and three objects from A + A' + A'' results from considering these classes as identical and yet distinct; and this can not be reduced to simple " additive composition of classes without some further operation ".[62]

In sum, it is therefore the child's ability to ' operate ' that gives rise to the concept of number and counting.

Once more, if the operation of addition is investigated further it will be found that the concept of conservation is essential to its formation, as hinted to in the previous chapter. Moreover, it can be seen that the child might learn some formulas like 2 + 2 = 4, 2 + 3 = 5, etc., but it is not until he grasps that 4 + 4 = 7 + 1 = 5 + 3 = 6 + 2 = 8, that he knows addition. To reach to this level the child is supposed to be capable of mobility of thought; i.e. capable of real operation. Moreover, he should be capable of grasping at once all the possible compositions of the number eight; and this formal thought was seen to appear

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61. Ibid., p. 183.

62. Ibid.

at the age of eleven to twelve.

To sum up, it can be demonstrated that the addition and the subtraction operations will be completed only when the child identifies them as operations done on the additive ' group ' of integers; " ... apart from which there can be nothing but unstable intuition".[63] The same holds true of division and multiplication.

Other related works

The works of Wohlwill, Estes, and Elkind will be investigated in this section. In general they all support the results found by Piaget except for the research done by Estes.

To investigate the conservation of number, Wohlwill took 72 kindergarten children at the average age of five years ten months.[64] This age is supposed to be the age between non-conservation and conservation.

The children were asked three key questions. First, they were asked to give six chips from a bag, to test their identification of cardinal number. Second they were asked to reproduce as many chips as there were on a table, to test whether they can establish the numerical equivalence relationship between two collections. Third they were asked to establish the dimensions of number irrespective of per-

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63. Ibid., p. 195.

64. Wohlvill, " Conservation of Number ", Child Development, ( 1962 33), pp. 153-167.

ceptual cues, and that was done by disturbing the perceptual correspondence between the two collections.

The results found agree with the findings of Piaget. First, it was found that conservation depends on perceptual cues. And there is a general lack of conservation as due to irrelevant perceptual disturbances. But it was found, that, to establish absolute conservation is independent from establishing relative conservation, i.e. conservation by comparing two sets of objects. If one analyses this finding further, he might be able to relate it to the one-to-one correspondence problem of Piaget, and still find the two researches in agreement.

Second, it was found that children may achieve conservation by a process of inference. And this deserves to be investigated further.

Third, and this is an important result from the educational point of view, there is a gap between counting and the true concept of number.

Elkind's research on children's discovery of conservation is more directed to the problem of conservation in general.[65] But obviously the findings of such a research are closely related to the concept of number conservation.

The research was carried on 175 children at Claflin School, Newston Massachusetts, at an age level of five to eleven years.

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65. Elkind, " Children's discovery of Conservation " , Journal of genetic Psychology, ( 1961, 98 ) pp. 219-227.

The findings of this research show that the discovery of conservation varies with age and depends on the type of quantity conserved, which is in agreement with Piaget's findings.

But in his other research on the development of quantitative thinking, Elkind tries to replicate Piaget's studies, using the statistical F test to analyse the data.[66]

He used the same materials, i.e. sticks, liquids (orangeade) and beads, and carried the research on 80 children at the ages of four, five and six-to-seven, in Pittsfield, Massachusetts.

The findings of this research agree with those of Piaget in showing that comparing quantities depended jointly on age and type of quantity compared.

It was also found that gross quantities were easiest to compare, i.e. comparison such as A is larger than B, A is smaller than B, etc. While intensive quantities were intermediate in terms of the difficulty of comparing them, i.e. comparisons such as A is longer and wider than B, A is taller and thinner than B, etc. And finally the extensive quantities were the most difficult to compare, i.e. comparisons such as $X = \frac{1}{2}Y$, $X = 2Z$, etc.. These results are in complete agreement with Piaget's findings; for, as it was shown above, numerical quantification is only possible after the mastery of the logical operations involved.

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66. Elkind, " Development of Quantitative Thinking ", Journal of Genetic Psychology, ( 1961 98 ) pp. 37 - 46.

Two other findings of this research, which may prove to be very useful later on, are the following:

First, the assumption, hinted at by Piaget, that liquids are more difficult to compare because of their indivisibility to perceptual parts is supported in this research.

Second, the assumption that a common conceptualizing ability underlies children's success in comparing quantities, is also supported by these findings.

The last research that will be discussed in this connection is that of B.W. Estes, on the development of number concept.[67]

The research was done on 52 American children of ages; four, five and six. And the questions asked to the children were to count the number of objects arranged in a pattern, and to compare the quantities of marbles in two jars filled simultaneously.

The findings of the research were the following:

> " In summary, no evidence was obtained which supported Piaget's theories as to the development of stages or age levels in acquisition of 'mathematical and logical concepts'. This study found (a) that if the children could count, they counted correctly whatever the arrangement of objects; (b) they did not confuse extension of line with increase in number; (c) they did not mistake an apparent increase with a true increase in number..."[68]

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67. B.W. Estes, " Development of Number Concept", <u>Journal of Genetic Psychology</u>, ( 1956 88 ), pp 219-223.

68. <u>Ibid</u>., p. 221.

Evaluation

Recalling the experiments of Piaget and of Estes, the problem of counting as associated with number presents itself as a classical problem. Piaget wouldn't interpret the results of Estes as they were interpreted. Since counting does not necessarily mean an understanding of number the conflict in the two results is obviously unjustified.

As to the conflict in the results found in the experiment about conservation, there is much evidence to support Piaget's results, which may suggest a reconsideration of the research of Estes.

But irrespective of the conflicts that seem to be implied by the research of Estes, a revision of the experiments of Piaget would reveal some interesting issues that might be analyzed further.

Piaget's general line of questioning seem to be directed to the perceptual judgments of the children. Very rarely does he ask for predictions or explanations. The questions are of the following type: " Are there more orangeade or lemonade? ", " Are there more eggs or egg-cups?" etc.. Irrespective of the many justifications given for the terminological difficulty, there are times when one feels that the child either doesn't understand the question, or he expresses himself in a clumsy way.

Probably the general attitude to be taken in interpreting the results, is to give more weight for the answers of the child that have predictable nature. When the child doesn't have the misleading perceptual cues infront of him, he will be compelled to use his intelligence to solve the problem. As an example, the child was given two containers L ( narrow and tall ) and A ( wide and short ), and was asked to tell the level the beads would reach in L when they are poured from A into L.

> " If we put the beads in there (L), what would happen? - They would be higher. Would there be the same amount? - No ..." 69.

This response is classified as a typical response of the stage I children tested for the conservation of discontinuous quantities. This classification is probably made on the basis of the answer to the second question. But a closer investigation of the first response may lead to the classification of this child ( Port: 5;0 ) with the third stage group, or at least with the second stage.

The first response gives the feeling that probably the child knows the relation between length and width, though maybe intuitively. Just his expectation that the beads will reach higher in container L suggests that the child might have the intuitive grasp of conservation but failed to express this notion in his second answer.

Piaget's argument as to the terminological difficulty, is based on the assumption that the child can not yet distinguish between amount and perceptual judgment of height and width; and that is due to a lack in his intellectual structure. Put in other words, Piaget would seem to be testing how far is the intelligence of the child

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69. Jean Piaget, The Child's Conception of Number, op. cit., p. 26.

capable of correcting his perception. Probably if the child was asked to explain his responses, as to the answer quoted above, his answer would explain something about his intelligence.

Or probably by modifying the conditions of the experiment such that the perceptual difference is very large, the child might be able to operate intelligently uninfluenced by his perception ; i.e. he would realize that his perception is irrelevant and resort to his intelligence. If the child were given a container L, much narrow and tall, and asked to put water in it as much as there is in A, which is wide and short, then one could probably tell more about his intellectual structure.

All in all, Piaget's result seem to be justified in as far as the relation between intelligence and perception is asserted. Or stated differently, Piaget's general result could be safely framed as follows: As far as the adult intelligence can correct erroneous perception,the child seem to reach this level through stages that outline his intellectual development. This would not say anything about the nature of the intelligence of the child, nor about the true relation between perception and intelligence. Had the child's intelligence been like that of the adult, or at least comparable to it, and had his corrective operations been like those of the adult, or at least comparable to them, then the results of Piaget would follow.

In light of the above comments one may still accept Piaget's results in a hypothetical way and proceed to test them with a modified method that avoids the above mentioned pittfalls.

The two major problems that should be solved, and which are not adequately handled in Piaget's research, are : (a) The separation of perceptual judgments from intelligent ones, and (b) the magnification of the perceptual error and testing at what stage would intelligence start correcting it.

In the first problem one would have a better understanding of the child's intelligent judgments. While in the second problem one will be able to determine the role of terminology. If the child corrects his perception at all he will be forced to use a certain terminology to express his correction.

To supplement these remarks the following plan of research is proposed:

By taking three examplary hypothesis that Piaget investigated and modifying the method for their verification, one might reach certain results that will enable him to evaluate Piaget's works on number on more solid grounds. The guide line, however, is to try, as much as possible to separate the child's judgments where reasoning is involved from those judgments influenced by perception.

One can take three groups of children ranging in age from four years to eight years and a half; which is the period in which most concepts reach a concrete operational level. These groups will be composed as follows: (a) one hundred between the age of four years and the age of five years and a half, (b) one hundred between the age of five years and a half and the age of seven years, and (c) one hundred between the age of seven and the age of eight and a half.

The notions to be investigated are the notion of conservation, correspondence and addition. The hypothesis to be tested is the same as that of Piaget, namely, these notions develop in stages.

A. The notion of conservation can be tested by the use of two sets of material; namely;

a) two identical containers filled with the same amount of orangeade and lemonade, together with two small containers. And

b) three containers A ( short and wide ), B ( long and thin ) and C ( which is significantly taller and thinner than B ).

1. Take the two containers K ( orangeade ) and L ( lemonade ) and ask the child the following question: If we pour K in $K_1$ and $K_2$ ( the two smaller containers ), will there be orangeade as much as there is lemonade? After asking the child to explain his answer, K should be actually poured in $K_1$ and in $K_2$. To test for interference of perception the child should be asked again: Is there as much orangeade as there is lemonade?

If the child answers both questions correctly with satisfactory explanations, then he is given the second test. If he answers the first question correctly and fails to answer the second, then he is given the second test for further evidence as to the discrepency between his intellectual judgments and those influenced by perception.

The hypothesis is that the child will not answer the second question correctly and fail to answer the first. But if he does, he is given the second test to reduce chance responses.

If the child fails to answer both questions correctly, he is classified as first group ( non-conservation ).

2. The second test will require the three containers A, B and C. A should be filled with a colored liquid ( colored water will do ).

The child is asked the following question: If we pour A in B, where will the liquid reach? If he points to a level higher than that of A, the same question is repeated using container C in place of B. If he points to a still higher level, then he is classified as passing the item; if he can explain his answers in a satisfactory way.

The test for perceptual interference consists of the following: Pour A into B and ask whether B has as much liquid as a container identical with A ( A may be refilled for this experiment ). Then pour A into C and ask the same question with respect to C. The second question should be asked irrespective of the answer to the question about B.

If the child answers the first question correctly, then the second question will give further evidence as to the child's grasp of the logical relationships; and at the same time will reduce the chance responses. If he doesn't answer the first question correctly then the second question might reveal some evidence as to the possibility of establishing the existence of a corrective mechanism ( at a time when the perceptual error is very great ).

If the child gives satisfactory answers and explanations to all the questions in both tests, he is classified as third group ( conservation ). But if the child gives satisfactory performance on one of the tests only, he is classified as second group ( intuitive conservation ). The results of such an experiment might reveal the necessity for further groupings, depending on the interpretation of the data at hand.

Correlations between all possible combinations of classes of responses must be calculated when analysing the data. In case one finds positive correlations between perceptual judgments and intelligent judgments, one may conclude that Piaget's line of questioning is valid and his results concerning intelligence and perception are verified; i.e. the intelligence of the child is limited by his perception, and if it corrects erroneous perceptual judgments, it does so in a way comparable to that of the adult. If the positive correlations increase with the age level then Piaget's results as to the age development are also verified.

B. The notion of correspondence can be tested in two tests similar to those performed by Piaget. The main hypothesis is also the same as that of Piaget; namely, the child develops the notion of correspondence through stages. And the line of questioning in this experiment is also motivated by the hope to separate between the child's intelligent judgments and his perceptual ones.

1. The child is asked to fill one of two opaque jars with marbles simultaneously with the experimenter who is filling the other one, one at a time. Then after dropping fifteen to sixteen marbles in each jar the child is asked the following question: Do the two jars contain the same number of marbles now?

The same procedure is repeated with two transparent jars different in shape. And the child is asked the same question as in the previous case.

If the child answers the two questions satisfactorily he is classified as third group ( correspondence ). If he answers one and fails to answer the other, he is classified as second group ( intuitive correspondence ). If he does not answer any one of the questions correctly he is classified as first group ( non-correspondence ).

One further test may be performed in this connection that might prove helpful in clarifying the relation between intelligence and perception is the following:

Two transparent jars of different shapes such that they really magnify the perceptual interference are taken and the child is asked the same question as in the above test.

2. A row of six flower vases is put on a table infront of the child. He should be told that each vase should have only one flower in it. Then he is asked to take as many flowers as there are vases from a basket containing many flowers. He is also instructed not to put the flowers in the vases until he is told to do so. If the child fails to get exactly six flowers he should be helped by the experimenter, after recording his failure.

In the second test the flowers are put in the vases; And the child is asked whether there are as many flowers as there are vases. If the child fails to perform satisfactorily on this question he is immediately dropped and classified as first group ( non-correspondence ). But if he responds correctly he is allowed to continue the experiment for further verification.

Now the flowers are taken out off the vases and arranged infront of them in one-to-one correspondence. Then the child is asked the following question: If the flowers are grouped together, will they have the same number as the vases? Then the flowers are actually grouped and the following question is put to the child: Are there as many flowers as there are vases?

If the child predicts that the flowers will have the same number as the vases before the flowers are grouped, he is considered as passing the item. Then the flowers are actually grouped and his response is recorded once more.

The situation in test (2) is as follows: (a) the child is given the chance to produce the cardinal number six with the help of provoked correspondence. If he fails to see this hint the experimenter helps him establish the correspondence ( put the flowers in the vases ) and asks him whether the flowers and the vases that hold them are of the same number. Then to make sure that he grasps the idea of correspondence, the flowers are arranged in front of the vases, and the child is asked about the equivalence of their numbers. Then (b) the child is asked to predict, on the bases of correspondence, whether perceptual changes are going to effect the number. Finally (c), the perceptual difference is actually achieved and the child's reaction to it is recorded.

The child will be considered as third group ( correspondence ) if he answers all the questions satisfactorily. But he will be classified as second group ( intuitive ) if he responds correctly to two situations out of the three ( namely a, b, and c ) without any help. If he is helped in establishing correspondence he will be classified as first group even if he answers the prediction correctly; because he might be predicting on the bases of the present perceptual correspondence.

If the children are found to move in these three stages as they grow in age then Piaget's result will be considered as verified.

C. The last notion that could be verified is the notion of inclusion which amounts to additive composition of classes. Here again perceptual judgments are to be kept separated from intelligent ones as much as possible.

The experiment performed by Piaget required the children to pass a judgment as to which class is bigger; the class of brown beads or the class of wooden ones. The total class of wooden beads was composed of eighteen brown beads and two white ones.

It can be noticed easily that the differentiation between the two classes is an adjective differentiation, which seems to require a certain amount of abstraction before it can be performed. Piaget's result seems to confuse the ability to abstract with the ability for inclusion. Perhaps ' wooden beads ' are not as familiar to the child as ' brown beads ' or beads in general. The assumption is that the child might not be familiar with the class of wooden beads and hence would not understand the question and falls back to compare classes that he is familiar with ( which do not require him to abstract ), namely the brown beads with the white ones.

One can avoid this confusion by giving the child a class that is equally familiar to him as its composing constituents. Then the judgments of the child would reveal his ability to perform the operation of inclusion. That is, one should find a class whose name is as familiar as the name of its constituents. With such a situation the interference of abstraction or terminological difficulty will be considerably reduced. And the question will then be asking to compare two classes that have the inclusion relation as their major relation.

The child is presented with a class of twenty cars ( toys , that he is familiar with ) composed of eighteen trucks and two station wagons ( or normal small cars ).

The child is asked to identify each part alone ( call it by name ) and to identify the total class, by asking him: What do you call these ? ( the trucks ), and these ? ( the station wagons ), and these all? ( the cars ). If the child uses a different terminology than the one used here, the remaining questions will be put to him in his own terms.

The questions put to the child are the following; Are there as many trucks as there are cars? ( Or one could use the child's terminology for trucks and cars ). If the answer is in the negative then he is asked: Which ones are more?

One may still object, here, and claim that the child's perceptual judgment is interfering with his intelligent one. To clarify this issue ( i.e. detect a causal relation between intelligence and perception if there is any), one could change the number of trucks and station wagons keeping the total number of cars in the class as constant. After four or five changes one may be able to decide on the relation of perception to intelligence in this context. As to the question asked to the child after each change, it should be the same as the question asked in the original situation. It should be noted however that such a step was not performed by Piaget.

The interpretation of the results should be as follows: (a) If the child answers all questions ( together with those asked after the changes in constituents ) correctly, he will be classified as third group ( inclusion ). (b) If his answer is affirmative to the first question, but he corrects it either on the second question or after the first change in constituents, then he is classified as second group ( intuitive inclusion ). If the child does not fall into any one of these categories he is classified as first group ( non-inclusion ).

Here again, if the results point to a group classification changing with age, the results of Piaget would be considered as verified.

## GENERAL CONCLUSIONS

The two classical approaches to the formulation of the number concept, i.e. counting and one-to-one exchange are definitely ruled out by Piaget's results. And the two modern views that are investigated are: The concept of " class of classes " claimed by Russell to be at the foundation of the number concept, and hence the claim that number can be reduced to logical concepts. Then there is the concept of number as a correspondence between two sets as claimed by Cantor. Piaget's findings amount to taking the two modern concepts as complementary in such a way that Poincaré's objection will also be taken into consideration.

Piaget finds that concepts like conservation, reversibility, one-to-one correspondence, and inclusion are all necessary to the formation of number. For, after all, number is nothing but the fusion of two basic operations underlying all the above concepts; namely, classification and seriation. In the long run, what Piaget would say is: " ... classer et sérier a la fois, c'est précisement dénombrer ".[70]

Piaget concludes that the formation of number results in co-ordination of logical relations into a formal structure. It is not, for example, the notion of conservation that gives rise to logical multiplication of relations but the contrary.[71]

A formal structure is only an equilibrium state reached at a certain level; and the child is in constant re-equilibration, which amounts to a constant re-coordination involving new notions.[72]

These logical co-ordinations start to be applicable on the very simple sensori-motor actions; in the form of applying the same scheme of action to analoguous situations.

Then during the act of interiorization of actions ( i.e. assimilating actions by a mental " prise de conscience ") that are

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70. Jean Piaget, Intr. a L'Epist.Génétique, op.cit., p. 102.

71. Cf. Jean Piaget, The Child's Conception of Number, op.cit., p. 19.

72. Ibid., p. 204.

successive and classified in units of actions, they become representational and capable of anticipation. This constitutes the intuitive thinking.

In the concrete operational level, these representational acts become reversible and gain the property of a logical grouping.

Finally they lose their empirical contents and become abstractions that are capable of integrating and differentiating at the same time to synthesise as a number concept on the hypothetico-deductive level. These representations are then the form of intellectual operations.

But when describing this evolution of number we find a striking parallel between number and any notion or concept, on the common ground of logical operations from which every concept emerges. Hence, to study the formation of the number concept is to get acquainted with the synthesis between the various logical operations that give rise to all concepts.

CHAPTER IV

# EDUCATIONAL IMPLICATIONS

The general theory of Piaget is focussed on the question of the growth of knowledge in the child. Much research needs to be done in the future before there will be a satisfactory answer for the educator as to the possibility of applying this theory in educational practice; for, up till now, there is no one work that outlines an educational theory based on the works of Piaget. Its possible influence in some educational fields can, however, be anticipated.

Though it may seem premature, an attempt could be made to draw some educational implications specifically concerning the teaching of number.

## GENERAL IMPLICATIONS

Since it is not certain as to what is the best method to be followed in school to insure the development of concepts in the mind of the child, the teacher is advised to improve on the two external factors that might influence his educational experience; namely, (a) the teacher's acquaintance with the laws governing the growth of knowledge in the child as described by Piaget, and (b) the teacher's full grasp of his subject matter. For, " in the schoolroom, children are likely to be taught concepts ( if this can be done !) according to the way the teacher thinks they develop in his pupils."[1]

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[1] K. Lovell, The Growth of Basic Mathematical and Scientific Concepts in Children ( N.Y., Philosophical Library, 1961), p. 54.

Knowledge of the laws governing the growth of the child will enable the teacher to make the child's experience fruitful and effective. This can only take place if the grounds for the new experience are prepared in such a way that they form together with the new experience a comprehensible unit from the logical point of view. If the transformation from one task to the other is determined by the rules of the child's logic, then it is easier for the child to cope with it. The belief is based on the fact that the child can not assimilate in vacuum.[2]

The major assumption that Piaget stresses in all his works is that the child is an active entity, in the fullest sense of the word. Consequently, knowledge is not ' transmitted ' to the child as though he is a passive recipient. In Flavell's words:

> " Stable and enduring cognition about the world around us can come about only through a very active commerce with this world on the part of the knower"3.

An appropriate Piagian slogan would be : " penser c'est opérer".[4] And hence the child is to be looked upon as the conqueror of the world rather than just the observer.

In general, then, a school of Piaget will approach the real life as an ideal. For to educate is to help the child form the appropriate patterns of operations, if that is possible.

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2. Cf. Flavell, op.cit., p. 155.

3. Ibid., p. 367.

4. Ibid.

## Piaget's Views On Education

In line with the education for real life, Piaget adheres to the ideal of a democratic education; for he thinks that the child's logic will be corrected and developed in an environment of free discussion and cooperation. He would be in favor of training the students in the democratic way of thinking and encourage them to form their own governments[5].

Piaget's belief is that the child can only learn through an active process of habit formation. For, habits are nothing but patterns of operations resulting from the process of interiorization of actions. That is why he thinks that education should start whenever action starts, so that the child's actions are modified and corrected before they are interiorized into crystallized habits. In his own words:

> " During the first months of an infant's life, its manner of taking the breast, of laying its head on the pillow, etc., becomes crystallized into imperative habits. This is why education must begin in the cradle." 6

The child should, therefore, be taught in such a way that he can operate with his knowledge and put it in practice. Otherwise, to educate the child in an ivory tower would be a waste of time and efforts. The child can only learn in a context that makes sense to

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5. Cf. Jean Piaget, The Moral Judgment of The Child ( London: Kegan Paul & Co., 1932), p. 216.

6. Ibid., p. 80.

him; and this should be the foundations on which education should build. For, Piaget thinks that:

> " It is somewhat humilating, in this connection, to see how heavily traditional education sets about the task of making spelling enter into brains that assimilate with such ease the mnemonic contents of the game of marbles."7

Suppose that all these views are accepted and the general theory verified, what would be some of the educational imperatives Piaget would endorse?

## Applications In Specific Fields

The specific educational fields in which Piaget's theory may be applicable are not yet fully identified. It is possible that the field of educational testing will receive the greatest impact, both in intelligence testing and in placement testing.

The stage development theory will certainly give the people interested in intelligence testing a frame of reference as to the ability of the child at a certain stage. The theory of groupings will also help them in determining the contents of the items for their tests. And finally the functional approach to intellectual development will help them, in the long run, define intelligence on the basis of states of equilibrium between accommodation and assimilation.

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7. Ibid., pp. 40 - 41.

The field of curriculum planning may also make use of Piaget's theory. An immediate result of the general theory is the possibility of postponing the teaching of arithmetical operations till the child reaches the age of seven or eight. While at the earlier stages the child will be taught to appreciate the logical foundations of these operations, starting with the practical plane and abstracting step-by-step.

Unfortunately both fields, testing and curriculum planning, have not yet made wide spread use of these recommendations, not even in the way of experimental research.[8]

In the field of methods, the theory may have practical applications in two major areas: (a) The contents of the curriculum could be given to the child in such a way that he is allowed to play an active role in its formation, order of presentation, and selectivity. That is, the child will learn what he wants to learn, up to a certain extent. The general frame may be decided by the curriculum planner, but the details will be filled by the student himself with the assistance of his teacher. Put more concretely, the child will be given a choice, at the beginning of each period for example, to join any one of several tasks performed in groups in the classroom. The tasks should be of such an attractive nature that the child will actively indulge in them. The teacher will pass around the several groups now and then and ask the individual child the appropriate question

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8. Cf. Flavell, op.cit., p. 366.

that may help him realize certain economical ways of thinking. As an illustration, the child may chose to copy a certain pattern ( say a square made of four beads ), and after he finishes the teacher may ask him whether the copy has as much beads in it as the original pattern. If the patterns in the different tasks grow in complexity as the child grows, he may be able, then, to reach a fuller grasp of the idea of correspondence between two sets.

(b) The second application is in the group procedure that may be followed. The children may continuously perform tasks in supervised groups; for, it is the group interaction that teaches the child probably as much as the teacher-leader method, if not more.

To sum up, Piaget's views on education seem to favor the democratic education. But unfortunately, his theory is not yet put in actual use. Much research need to be done before passing a final recommendation on the educational implications of such a theory. The feasible areas in educational practice where this theory will have fruitful applications are the areas of educational testing and curriculum planning, and methods of teaching. And in Flavell's words:

> " It is a safe bet, however, that it [ the general theory ] is going to follow closely on the heels of the test construction programs."[9]

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9. Ibid., p. 365.

## TEACHING NUMBER

The application of the general theory to the teaching of number was chosen for the following three purposes: First , the number is the concept that was described earlier in this study. Second, number is undoubtedly the backbone of mathematical knowledge. Finally, number was chosen from the very beginning because it is the first concept introduced in schools, and certainly among the last to be studied in itself.

Piaget rightly asserts that elementary mathematics can be taught " on the basis of the content aspects of the development of number in children".[10]

But it will be the task of the teacher to decide on the way of presentation of these contents. For, if they are presented correctly, the teaching of number and of arithmetical operations will go hand to hand. What the teacher must keep in mind, are the logical operations involved in notions like conservation, reversibility, correspondence, inclusion, etc., while he is preparing the tasks in which the children will participate. And this, in fact, often demands considerable ingenuity.[11]

The teacher will have to find tasks such that the child will be able to familiarize himself with their structure as he is performing

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10. *Ibid*., p. 19.

11. Cf. Flavell, *op.cit*., p. 368.

them. They should be of such a nature as to allow the child to participate in them and at the same time facilitate the interiorization process of actions. In brief, the tasks should develop in form and in content as to allow the process of interiorization of actions that starts with the direct contact with objects and moves into abstraction till it requires the pure formal presentation of actions.

The essential idea involved in the formation of number, as well as in all other concepts, is the ability to carry simultaneously the two fundamental processes of integration and differentiation. The teacher will do well if he makes it his job to give the child enough practice, through varied situations, in this decision formation process. He should be able to use tasks that will encourage the child to make such judgments.

But the ability to see things as similar and yet different at the same time is in essence the process of generalization,which is in turn the object of mathematical thought. Therefore, the teacher should consider the ability to generalize as the main aim of his teaching of mathematics.[12]

The suggestion to the teacher of mathematics is to give the children " materials that can be made up into different collections according to different criteria; children should establish correspondence, order, include one class within a more general class, etc.

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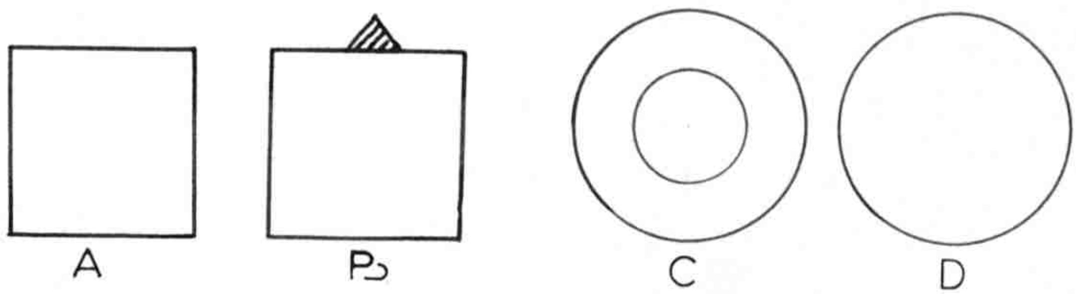
12. Cf. Lovell, op.cit., p. 146.

For example, a child should establish correspondence between his row of shells and that of his comrade".[13]

To put all these ideas in a practical frame, a task is suggested as an example that will be used to train the child in identifying similarities and differences at the same time. The following pictures will be a helpful reference:




The child will be asked to tell the difference between the square A and the square B. After he tells the difference he is asked to point the difference between the two circles C and D. Then finally, he is required to tell the difference between A and B on the one hand and C and D on the other hand.

In the last question the child will have to forget the minor differences between the squares themselves and the circles themselves and concentrate on the major similarities. A great variety of such exercises will help the child, in the long run, make similar decisions,

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13. Ibid., p. 52.

that will give him a deeper insight into the concept of the unit.

Another task that will help the child practice the use of correspondence and at the same time seriate, consists in asking one child to pass chocolate bars and biscuits to his friends such that he should give the ones on one row the chocolates and those behind them the biscuits and so on, alternatingly until every one gets his biscuits or his chocolates. To make this game more fruitful, a child standing at one of the back rows is asked, when the distribution has just started, in which row he should stand so that he gets a biscuits.

Finally one could invent games so that there is one principle or another for which the children will continuously practice. For example, they may be told to go and buy sweets from their friends such that the sellers are instructed to give their comrades one biscuit and one chocolate for each penny.

With such games the children will be able to appreciate the principles underlying concepts like one-to-one correspondence, seriation, ordination, etc., before they are formally asked to explain what they are doing. Then the teaching of number after long years of such training may become much more facilitated and the child will have a better grasp of the operations involved in elementary arithmetic.

## CONCLUSION

The two important assumptions underlying all the works of Piaget are: (a ) The assumption that the organism is an active entity in his environment, and (b) the essential condition for the development of any concept is the ability to integrate and differentiate at the same time.

The first assumption would definitely deny the educational practices where the child is considered as a passive entity who is supposed to receive knowledge rather than grasp or assimilate knowledge. To be more precise, Piaget would tend to think that a student will assimilate a lecture in as much as it can be constructed in terms of his experience in such a way that it forms with it a continuous logical unit.

The child's thinking is essentially a co-ordination of operations he performs in terms of actions carried on reality, at the first stages. Then he transforms his activity and modifies it and develops in ability of abstraction till he reaches a level where his operations become a formal representation of reality, which is then the stage of formal operations.

During this process of development the child will have the essential scheme of action. At certain times he finds himself in a harmonious correspondence with reality that he can accommodate himself to it in as much as he can assimilate it. This is the state of equilibrium that Piaget refers to. But quite often the child will find that his approach to problems is still insufficient due to the presence of certain conditions in the environment. He will, then, be obliged to change his course of operations, but for a more appropriate one. In this " recherche de l'équilibre" the child will accommodate to the regulations of his environment. The patterns of behavior accepted by his society will have their marked influence on his formation of schemas, whose crystallized form will be the child's habits.

The importance of having the child in a continuous interaction with his society is of a great influence on educational practice. For, then, the educational planning will have to reach to the group where the child is getting 'educated'. The free discussion and the group interaction will then form the basis of the child's educational experience. His friction with his peers will then be the means to his clear thinking.

The group projects and the 'free' class, where the child can do what he wants to do, seem to be the appropriate outcome of a theory that claims the active organism and the ' regulating ' society.

The second assumption seems to imply that the child should learn how to make judgments about things that are different but yet similar at the same time. The greater the variety of things the higher is the probability that the child will make the right judgments faster.

The educational implication of such a principle seem to require the teacher to prepare enough tasks of varied nature and complexity so that they will lead the child step-by-step to improve in the art of making such judgments. The tasks should be well constructed that the right judgments will be the only ones the child will favor. That is, by reducing the alternatives that the child will face and constructing the task such that it favors the right judgments, the probability that the child will respond correctly will be higher. As an example, the decision that A and B are different from C and D is because the first ones are squares while the other ones are circles. The difference is unique; and the probability that the child will realize it is high.

On the other hand the difference between square A and square B is also unique and of the same nature, in terms of probable identification.

If such tasks are, then, varied and logically interconnected then the child may achieve the skill of identification.

Therefore, the teacher must keep in mind this over-all aim and try to build his tasks such that they require the minimum efforts and at the same time be representative units of a logical operation.

The teacher must also remember that the child can not perform an operation unless he has assimilated its ingredients. In this context, the ingredients of an operation are the logical operations.

of which it is a synthesis. The logical operations, however, are reached after a long acquaintance with actions done directly on objects.

The teacher who knows the logical construction of his subject matter, and the laws governing the behavior of the child would probably be the one that can best apply these educational recommendations.

## WORKS CITED

### Books

Bridgeman, P.W. The Logic of Modern Physics. New York: Macmillan, 1928.

Carlson, Dimension of Behavior. Lund C.W.X. Gleerup, 1949.

Church, J. Language and The Discovery of Reality. New York : Random House, 1961.

Cohen, M., and Nagel, E. An Introduction To Logic And Scientific Method. New York : Harcourt Brace & Co., 1934.

Dantzig, T. Number The Language Of Science. New York : Macmillan , 1954.

Flavell, J.H. The Developmental Psychology of Jean Piaget. Princeton New Jersey, D. Van Nostrand Co., 1963.

Hubert, René. Le Sens du Réel. Paris : Alcan, 1930 .

Lederman, Walter. Introduction To The Theory Of Finite Groups. Edinburgh : Oliver and Boyd Ltd., 1957.

Lovell, K. The Growth Of Basic Mathematical And Basic Scientific Concepts. New York : Philosophical Library, 1961.

Piaget, Jean . Language And Thought of The Child . New York : Harcourt Brace and Co., 1926.

__________.The Moral Judgment of The Child. London : Kegan Paul and Co., 1932.

__________. Le Mécanisme du Developpement Mental. Geneve : Librairie Naville, 1942.

________. Naissance de l'Intelligence chez l'Enfant . Neuchatel, Delachaux et Niestlé, 1948.

________. Traité de Logique. Paris : Colin, 1949.

Piaget, Jean. Psychology of Intelligence . New York : Harcourt, Brace and Co., 1950.

________. Introduction a l'Epistémologie Génétique. Paris : Presses Univer., 1950.

________. The Origins of Intelligence in Children . New York : Inter. Univer. Press, 1952.

________. The Child's Construction of Number . London : Routledge and Kegan Paul, 1952.

________. The Construction of Reality in The Child. New York : Basic Books , 1954.

________. Logic and Psychology. New York : Basic Books, 1957.

Thomson, Robert. The Psychology of Thinking . Edinburgh : Penguin, 1959.

Waismann, F. Introduction to Mathematical Thinking . New York : Harper, 1959.

Introduction to Mathematics in Primary Schools . XIIIth Inter. Confer. on Public Education, UNESCO and I.B.E., Geneva , 1950.

## Articles

Donaldson, Margaret. " La Genese des Structures Logiques Elémentaires: Classifications et seriations." British Journal of Psychology, LI ( 1960 ), 181 - 184.

Elkind, D. " Development of Quantitative Thinking," Journal of Genetic Psychology, XCVIII ( 1961 ), 37-47.

_________. " Childrens Discovery of Conservation," Journal of Genetic Psychology, XCVIII ( 1961 ), 219-227.

Estes, B.W. " Development of Number Concept," Journal of Genetic Psychology, LXXXVIII ( 1956 ), 219-222.

Mays, W. " Development of Logical and Mathematical Concepts : Piaget's Recent psycho-logical Studies. " Nature , CLXXIV ( October 1954 ), 625-626.

_________. " The Epistemology of Professor Piaget". Proceedings of The Aristotelian Society , LIV ( 1953-1954) , 49-76.

Parsons, Charles . " Inhelder and Piaget's : The Growth of Logical Thinking. II. A Logicians Point of View. " British Journal Of Psychology, LI ( 1960 ), 75-84.

Piaget, Jean. "Les Developpement Intellectuels chez les Enfants", Mind, XL ( 1931 ), 137-160.

_________. " How Children Form Mathematical Concepts". Scientific American, CLXXXIX, No.5, ( 1953 ), 74-79.

Wohlwill, J.F., and Lowe, R.C. " An Experimental Analysis of The Development of the Conservation of Number," Child Development, XXXIII ( 1962 ), 153-167.