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Confluence of pattern recognition and signal processing: application of Al-Alaoui pattern recognition algorithm to digital filters design

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Abstract: A weighted mean square error (WMSE) approach to optimising digital filters is delineated. It is applied in the current work to optimising the classical Al-Alaoui IIR differentiators to obtain new improved wideband differentiators of varying orders. These can be directly used for analog to digital conversion and in many digital signal processing applications. The weighted MSE approach is motivated by the Al-Alaoui WMSE approach to pattern recognition. In addition, the differentiators are converted to integrators of similar orders that are further optimised. Various examples and comparisons are presented to demonstrate the viability of the proposed approach.

1 Introduction

This work targets optimising digital filters using a novel optimisation approach that is motivated by the Al-Alaoui algorithm for pattern recognition which employs the generalised inverse and single layer neural networks [1–3]. This can be explained for a two-class case. We determine a discriminant function $d(x) = w^{\mathrm{T}}\boldsymbol{x}$ where $\boldsymbol{w}$ is a weight vector and $\boldsymbol{x}$ is a sample vector to be classified as belonging to one of the two classes. A sample x_j is classified correctly if $w^{\mathrm{T}}x_j > 0$ and x_j is in class 1 or if $w^{\mathrm{T}}x_j < 0$ and x_j is in class 2. In the latter case, we observe that x_j is classified correctly if $w^{\mathrm{T}}(-x_j) > 0$. The mean-squared-error, MSE, between $w^{\mathrm{T}}\boldsymbol{x}$ and the optimal Bayes classifier $d_{\mathrm{o}}(x)$ is given by [1]

$$\varepsilon^2 = \int [w^{\mathrm{T}}\boldsymbol{x} - d_{\mathrm{o}}(x)]^2 P(x)\mathrm{d}x \tag{1}$$

Thus, the mean-square-error criterion places emphasis on points where $P(x)$, the probability of the sample is large, rather than on the points near the decision surface $d_{\mathrm{o}}(x) = 0$. Although by repeating the misclassified samples, we are in effect increasing their probabilities and thus placing the emphasis where it should be placed, on points near the decision surface.

This suggests an iterative procedure which, starting with the MSE solution, repeats the misclassified samples then tests the resulting solution on all the samples and repeats the resulting misclassified samples and so on. The procedure could be terminated either after the error was reduced to a certain value or by specifying the number of iterations to be carried out and keeping the one that results with the least number of errors. The algorithm always works because of its strong statistical underpinning.

Ho and Kashyap [4] noted that the generalised inverse provides a closed form MSE solution. The problem formulation for the general multiclass case takes the following forms: given $(\boldsymbol{A})$ and $(\boldsymbol{B})$, find the MSE solution for $AW = B$. The matrix $(\boldsymbol{A})$ of the augmented training samples is augmented by adding a component of value that equals to one, with each sample occupying a row. In addition, we have a target matrix $(\boldsymbol{B})$ of 1′s and 0′s and a weight matrix $(\boldsymbol{W})$ which is to be determined. The columns of $(\boldsymbol{W})$ are the weight vectors for the classes.

The MSE solution is [5–7]

$$\hat{W} = (A)^{\dagger}B \tag{2}$$

where † designates the pseudo or generalised inverse. If $(\boldsymbol{A})$ is of full column rank, which arises in practice, then the solution takes the form [7]

$$\hat{\mathrm{W}} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}B \tag{3}$$

Concerning weighted MSE, we mean minimising [6, 7]

$$||Aw - b||^2_{R^{-1}} \triangleq (Aw - b)^{\mathrm{T}}R^{-1}(Aw - b) \tag{4}$$

where $(^{\mathrm{T}})$ denotes transpose and $(\boldsymbol{R})$ is a positive definite matrix, hence there exists a matrix $(\boldsymbol{Q})$ such that $Q^{\mathrm{T}}Q = R^{-1}$. To do this, it is only necessary to consider a matrix equation of the form $QAW = QB + QE$ instead of $AW = B + E$. Hence, the least squares solution for W is given by the so-called weighted generalised inverse solution [6]

$$\hat{W} = (QA)^{\dagger}QB \tag{5}$$

If (A) is of full column rank, this becomes [7]

$$\hat{W} = (A^{\mathrm{T}}R^{-1}A)^{-1}A^{\mathrm{T}}R^{-1}B \tag{6}$$

In the single pattern adaptation, changes are carried out only if the pattern is erroneously classified. Three equivalent forms were developed in [1, 2] and the approach was shown to be equivalent to a weighted MSE. Form 1 simply replicates the pattern that is in error and appends it to the training set, and thus both $(\boldsymbol{A})$ and $(\boldsymbol{B})$ are modified. Form 2 multiplies the row of the sample that is in error in matrix $(\boldsymbol{A})$, and the corresponding row in matrix $(\boldsymbol{B})$ by $\sqrt{n+1}$, where (n) is the number of times that the sample was erroneously classified during the iteration process. Form 3 modifies only the $(\boldsymbol{B})$ matrix by modifying the rows corresponding to the sample that is in error, by an amount proportional to the error and in the direction of the gradient of $\|AW - B\|$ with respect to the corresponding row of $(\boldsymbol{B})$. There are three batch forms corresponding to the three single pattern adaptation forms [1, 2].

In this work, the weighted MSE algorithm in [1, 2] is modified by iteratively increasing only the weight corresponding to the output with the highest inaccuracy instead of increasing the weights of all the outputs that fall short in meeting the specified required accuracy. The modified algorithm is implemented with simulated annealing, (SA) [8]. The metric to minimise for the differentiators

is defined as follows

$$E_{\mathrm{w}} = \sum_{i=1}^{N_{\mathrm{int}}} w_i \int_{\omega_{\mathrm{l}i}}^{\omega_{\mathrm{h}i}} (|H(\omega)| - \omega)^2 \mathrm{d}\omega \tag{7}$$

where E_{w} is the weighted sum of the squares of the errors, N_{int} is the number of intervals, $\omega_{\mathrm{l}i}$ and $\omega_{\mathrm{h}i}$ are the lower and higher limits, respectively, of interval i and w_i is the weight corresponding to interval i in the weighted error function.

The weights are initially selected according to the following approach

$$\begin{aligned} & w_i \sqrt{\left(\left|H\left(\frac{\omega_{\mathrm{l}i} + \omega_{\mathrm{h}i}}{2}\right)\right| - \left(\frac{\omega_{\mathrm{l}i} + \omega_{\mathrm{h}i}}{2}\right)\right)^2} \\ & = w_j \sqrt{\left(\left|H\left(\frac{\omega_{\mathrm{l}j} + \omega_{\mathrm{h}j}}{2}\right)\right| - \left(\frac{\omega_{\mathrm{l}j} + \omega_{\mathrm{h}j}}{2}\right)\right)^2} \end{aligned} \tag{8}$$

That is the error at the middle point of each interval is considered, and the weights are selected such that the weighted absolute errors at the interval centres are made of comparable magnitude.

However, the selection in (8) is not used exactly as it is in this paper. There was some fine tuning and rounding off done via trial and error to further enhance the performance of SA. That is why the weights are rounded integers. However, the selection of weights as in (8) can be used as a guideline and starting point: The reason behind it is to harmonise the weighted sum in (7) by making the terms of the sum of similar order of magnitude.

It should be noted that the MSE approach acts as a sort of equaliser by reducing the large errors and increasing the small errors. When the results are not acceptable, weighted mean square error (WMSE) may be employed to improve the outcome of the MSE. In the proposed weighted mean-squared error, the frequency range was subdivided into several intervals, and the error in each interval was assigned a different weight. The weights updating was inspired by modifying the Al-Alaoui Weighted MSE Algorithm, where for each set of weights the optimisation algorithm was run and the weight of the area that showed the highest error was increased. The process was repeated until the proposed results were obtained. Starting from the second order low-pass differentiators, using their coefficients as the starting point for optimisation, we were able to obtain good approximations of the ideal differentiator.

The organisation of the rest of the paper is as follows: Section 2 follows on the previous discussion and presents ideas related to tuning and the optimisation problem to aid in the description of the problem. Section 3 further discusses other related work. Section 4 considers the optimised differentiators. Section 5 discusses the optimised integrators. Section 6 compares the proposed filters with state of art filters of similar order and Section 7 presents the conclusion. The Appendix presents the MSE derivation of α as noted in Section 2.

2 Interactive design approach (IDA)

For an optimisation problem, a true optimal solution could be obtained by exhaustive search which is often prohibitively costly computationally. Instead, a criterion of optimality is chosen, often MSE or equivalently least squares, and an optimal solution is accordingly obtained. Instead of exhaustive search, IDA is employed in this research. IDA first locates the regions of interest by employing the chosen criterion of optimality, and than carries out local search in those regions. This is delineated in the design of differentiators and integrators [9–13] and best illustrated in the development of the Al-Alaoui operator which, because of page limitations, was not delineated at the time [14]. Differentiators and integrators are basic components and building blocks for filters and systems [9]. Al-Alaoui, motivated by his work in designing active analog differentiators and integrators [10, 11], proposed an approach to design a digital differentiator starting from an integrator, which will work as well to design an integrator starting from a differentiator, by inverting the transfer function of an integrator [12]. In [12], the approach was applied to the Tick integrator to obtain Al-Alaoui-Tick differentiator. In [13], the approach of [12] was applied to the Simpson integrator to obtain Al-Alaoui-Simpson differentiator. Linear interpolation of two systems as in (1) was implemented in [14–19] for integrators. However, it can be applied to interpolate two differentiators or any systems where appropriate, with $H_1(z)$ and $H_2(z)$ representing the transfer functions of the two systems to be interpolated and $H_{\mathrm{opt}}(z)$ representing the resulting transfer function from interpolation

$$H_{\mathrm{opt}}(z) = \alpha H_1(z) + (1 - \alpha) H_2(z), \quad 0 \le \alpha \le 1 \tag{9}$$

In [13], $H_1(z)$ and $H_2(z)$ are the transfer functions of the forward rectangular rule and the trapezoidal rule, respectively. Although α may take varying values in the range, $0 \le \alpha \le 1$. The choice of $\alpha = 0.75$ obtained a new integrator which better approximates the desired properties of (9), [14]. The choice of $\alpha = 0.75$ will be delineated next. The application of the approach of [13] to the resulting integrator yielded the corresponding Al-Alaoui differentiator. Equations (10) and (11) correspond, respectively, to the transfer functions of the Al-Alaoui first order integrator and differentiator with $\alpha = 0.75$ in (9) [14, 20]

$$H_I(z) = \frac{T}{8} \frac{(z+7)}{(z-1)} \tag{10}$$

$$H_a(z) = \frac{8(z-1)}{7T(z + (1/7))} \tag{11}$$

The derivation of the parameter α in the Al-Alaoui integrator [14] was reached as follows: the author looked at the error plotted as a function of α in Fig. 1*a*. Starting from $\alpha = 0$, the error is negative. At high values of α (towards $\alpha = 1$), the error is positive. The error has a negative minimum value in intermediate frequency ranges and has positive values at the high frequencies, and it goes to zero at DC. Fig. 1*a* clearly shows that the ranges of $(0.7 < \alpha)$ and $(\alpha > 0.8)$ should be eliminated if the purpose is to obtain a differentiator with an extended high frequency range. In [14], the author chose α in (9) using the MSE of the obtained integrator with the ideal integrator while constraining the minimum to occur at $\omega = \pi/2$ and obtained $\alpha = 0.7881$. By fine tuning α between 0.7–0.8, the author obtained $\alpha = 0.75$ which resulted in the celebrated Al-Alaoui operator.

The absolute magnitude error plots of the MSE filter with $\alpha = 0.7881$, the Al-Alaoui integrator with $\alpha = 0.75$, in addition to the trapezoidal and rectangular integrators are shown in Fig. 1*b*. It can be seen that both the Al-Alaoui and the MSE integrators perform significantly better than the trapezoidal and rectangular integrators. Fig. 1*b* shows that the MSE integrator performs better than Al-Alaoui at lower frequencies but is outperformed at higher frequencies. Besides this, other possibilities exist as illustrated in [21].

3 Related work

In [15], linear interpolation as in (9) was carried out to obtain new numerical integration rules by interpolating two Newton-Cotes rules to obtain a new integration rule that cancels out terms in the corresponding Euler–MacLaurin error formula. In [22], Papamarkos and Chamzas developed wideband IIR integrators by employing linear programming optimisation. In [20], a further elaboration of the work in [14] was carried out by interpolating the trapezoidal integration rule with the backward rectangular integration rule, which yielded the same stable differentiator of [14] without the need of implementing the stabilising step of [14]. In addition, prewarping equations similar to those of the bilinear,

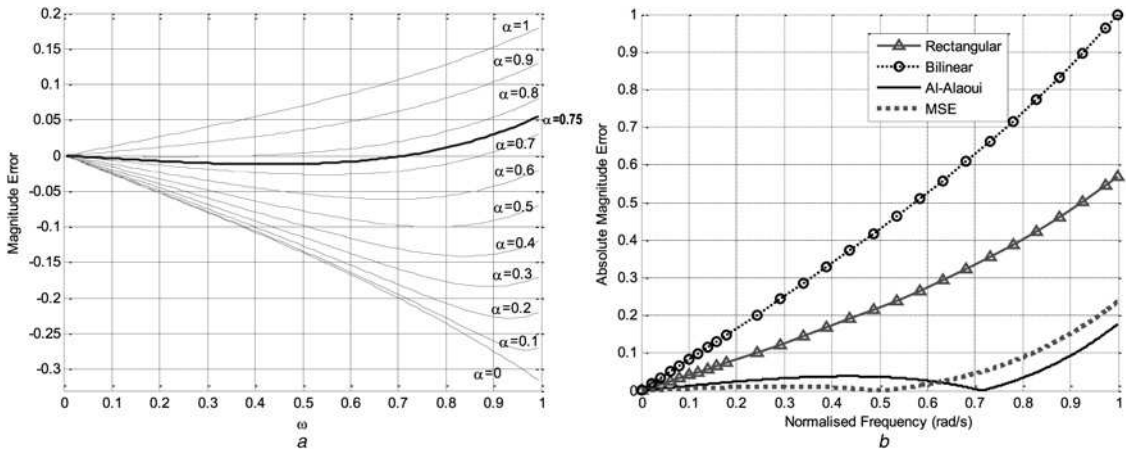

Fig. 1 *Error plotted as a function of α*

a Magnitude error of the Al-Alaoui integrator as a function of α
b Absolute magnitude error of the Rectangular, Bilinear, Al-Alaoui with $\alpha = 0.75$, and the MSE integrators

Tustin, *s*-to-*z* transform were proposed by employing the resulting differentiator of [14, 20], with variable α, to obtain *s*-to-*z* transforms that fill the gap between the bilinear and the backward difference transforms.

The differentiator developed in [14, 20] with fixed $\alpha = 0.75$ was designated as Al-Alaoui operator and popularised by Chen and Moore in the development of fractional order differentiators and integrators [23]. Eventually, the term Al-Alaoui operator was used to refer to the integrator and differentiator developed in [14, 20] for both the fixed $\alpha = 0.75$ case and to the variable α case. The case of fixed $\alpha = 0.75$ is often designated as the 'classical' or 'conventional' Al-Alaoui operator, and the variable α case as the 'generalised' Al-Alaoui operator [14, 20, 23, 24].

In [16], linear interpolation of (9) using the Simpson and the trapezoidal integration rules was carried out to obtain a class of second order integrators and low-pass differentiators. The differentiators developed in [16] were employed in [25] to obtain higher order *s*-to-*z* transforms. In [26, 27], genetic algorithms (GA) were implemented to, respectively, generate new integrators and to optimise the choice of α for the Al-Alaoui integrator. GA is a random search/optimisation technique that is adopted from the principles of gene mechanics and was first proposed by Holland [28] with a more practical implementation developed in [29]. Although this work employs SA, other heuristic algorithms can be used. However, SA lends itself for easier implementation than, for example, GA in this specific case. In our scenario, the variables to be optimised can have any real value, and hence there are infinite possibilities, not a fixed number selected from a discrete set in which GA would be more practical to implement since the concepts of "crossover", "mutation" etc… can be applied more easily.

In [17, 18], the similarities of Al-Alaoui operator to other proposed approaches were delineated. In [30], novel analog to digital transforms were proposed. The approach applied the Boxer–Thaler expansion to a leaky differentiator or to a leaky integrator instead of an ideal differentiator or integrator. In [31, 32], discretisation of fractional order systems utilising a modified Al-Alaoui operator and the transforms of [30] were proposed. In [33, 34], fractional delay was employed to improve the magnitude and/or the phase responses of integrators and differentiators.

Substantial recent research aimed at employing different optimisation techniques to design wideband IIR digital differentiators and integrators [19, 20, 35–44]. However, a decade earlier, Papamarkos and Chamzas [22] employed linear programming to develop wideband second order integrators. In [35], Richardson extrapolation and fractional delay were employed. In [36], two methods were proposed to obtain novel low-pass IIR differentiators with linear phases in the passband regions. Ngo in [37] provided impetus to recent research on wideband digital integrators and differentiators by his innovative approach of applying *z*-transform to the closed form Newton-Cotes integration formula and obtaining a novel wideband third-order Trapezoidal integrator and the corresponding differentiator by employing the inverting approach of [12]. Pei and Hsu derived by means of fractional delay a low order wideband digital differentiator [38]. Al-Alaoui [19, 20] obtained digital integrators from the numerical integration rules that he developed in [15] and applied the inverting approach of [12] to obtain the corresponding differentiators. He presented 2-segment, optimised 3-segment and optimised 4-segment wideband IIR integrators and differentiators. SA was employed in the optimisation phase. It is to be noted that the optimised 4-segment reduces to 2-segment, second-order, differentiator and integrator because of pole-zero cancelation; however, the 4-segment designation was retained to avoid possible confusion with the other two-segment integrator and differentiator. In [39, 40], Gupta *et al.* successfully developed wideband IIR Integrators by interpolating some of the popular and the recently developed integrators. Gupta–Jain–Gupta in [39] successfully extended the approach of (9) to interpolating three integrators, by introducing a second parameter and thus a second degree of freedom.

In [41], Gupta *et al.* interpolated the SKG rule [25] and the trapezoidal integration rule to obtain wideband recursive digital integrators. In [42], Upadhyay obtained nearly linear phase responses over the full band by optimising, over different frequency bands, a fourth order IIR filter. He employed pole-zero placement in [43–45]. In [44], Upadhay and Singh proposed new first order integrators and differentiators by applying the concept of time-constant analysis. Jain-Gupta-Jain employed GA in designing linear phase IIR Integrators and differentiators [46]. Recently, Tseng and Lee proposed a novel wideband integrator design using the interlaced sampling method and the Simpson integration rule [47]. Al-Alaoui and Baydoun in [21] presented new IIR digital differentiators of first and higher orders based on optimising the Al-Alaoui integrator using SA, Genetic Algorithms and Fletcher and Powell Optimisations. Gupta and Yadav in [48] optimised second and third order digital integrators using a modified particle swarm optimisation (PSO).

The relationship of Al-Alaoui algorithm to boosting, which was introduced about 20 years after Al-Alaoui algorithm [3], was noted by Andrew Webb [49]. Later modifications of the Al-Alaoui pattern recognition algorithm employed multi-layer neural networks and introduced Al-Alaoui backpropagation algorithm and Al-Alaoui–Levenberg–Marquardt Algorithm [50–52].

4 Optimisation of the Al-Alaoui differentiators

This section presents the optimisation approach used for obtaining the different proposed differentiators.

4.1 Optimizing the Al-Alaoui-Tick differentiator

The transfer function of the Al-Alaoui-Tick differentiator is given by

$$H_T(z) = \frac{0.852}{T}\frac{(z-1)(z+1)}{(z+0.5531)(z+0.0579)} \tag{12}$$

Which has the form

$$H(z) = \frac{1}{T}\frac{b_2z^2 + b_1z + b_0}{a_2z^2 + a_1z + a_0} \tag{13}$$

Optimising (13) using SA after obtaining the initial coefficients from (12), we obtain

$$H_{\text{Opt}-\text{AT}}(z) = \frac{1.1549}{T}\frac{(z-1.0005)(z+0.4959)}{(z+0.5781)(z+0.0964)} \tag{14}$$

The weights were determined as shown in Table 1.

Although the Al-Alaoui-Tick differentiator is a low pass differentiator, the WMSE optimisation led to a wideband differentiator filter having a very small error over the whole frequency range.

The differentiator does not have a zero at $z = 1$, which allows for the zero placement by replacing the zero at $z = 1.0005$ with $z = 1$, or the constrained optimisation approach constraining the zero to $z = 1$. The zero placement yields (15)

$$H_{\text{Opt}-\text{AT}-2}(z) = \frac{1.1549}{T}\frac{(z+0.4960)(z-1)}{(z+0.5781)(z+0.0964)} \tag{15}$$

The constrained optimisation approach yields (16)

$$H_{\text{Opt}-\text{AT}-3}(z) = \frac{1.1536}{T}\frac{(z+0.4973)(z-1)}{(z+0.5773)(z+0.0964)} \tag{16}$$

The magnitude responses of the Al-Alaoui-Tick based differentiators are shown in Fig. 2.

Table 1 Weights for Optimising the Al-Alaoui-Tick differentiator

Interval	0–0.1π	0.1π–0.25π	0.25π–0.5π	0.5π–0.75π	0.75π–π
weight for MSE	11 000	2000	100	4	1

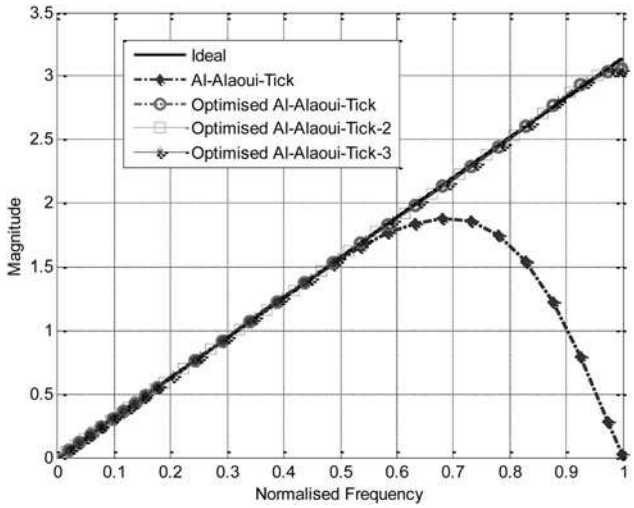

Fig. 2 *Magnitude of the Al-Alaoui-Tick differentiators*

Table 2 Weights for Optimising Al-Alaoui-Simpson differentiator

Interval	0–0.1π	0.1π–0.25π	0.25π–0.5π	0.5π–0.75π	0.75π–π
weight for MSE	11 000	2000	100	4	1

4.2 Optimising the Al-Alaoui-Simpson differentiator

The transfer function of the Al-Alaoui-Simpson differentiator is given by

$$H_S(z) = \frac{0.8039(z-1)(z+1)}{T(z+0.2679-0.0054i)(z+0.2679+0.0054i)} \tag{17}$$

Which has the same form as (13).

Optimising (13) after obtaining the initial coefficients from (17), we obtain

$$H_{\text{Opt}-\text{AS}}(z) = \frac{1.1524}{T}\frac{(z-1.0044)(z+0.4970)}{(z+0.5798)(z+0.0955)} \tag{18}$$

The weights were determined as shown in Table 2.

Also and like in the case of the Tick differentiator, the WMSE optimisation led to a filter having a very small error over the whole frequency range.

A differentiator can also be obtained with a zero placement at $z = 1$ in place of the zero at $z = 1.0044$. This leads to equation

$$H_{\text{Opt}-\text{AS}-2}(z) = \frac{1.1524}{T}\frac{(z-1)(z+0.4970)}{(z+0.5798)(z+0.0955)} \tag{19}$$

The constrained optimisation approach, with the constraint of a zero at $z = 1$, yields (20)

$$H_{\text{Opt}-\text{AS}-3}(z) = \frac{1.1523}{T}\frac{(z-1)(z+0.4973)}{(z+0.5861)(z+0.0879)} \tag{20}$$

The Magnitude responses of the different differentiators based on the Al-Alaoui-Simpson differentiator are similar to Fig. 2 of the Al-Alaoui-Tick differentiator and are omitted for brevity.

5 Optimisation of the Al-Alaoui integrators

This section presents the corresponding integrators. Two approaches are used for finding the integrators starting from the differentiators. The first one is the transfer function inversion approach as in [12]. The second method employs the inversion followed by performing WMSE optimisation.

5.1 Direct conversion

The optimised Al-Alaoui-Tick integrators, shown next as (21) followed a pole placement approach that yields (22) and constrained optimisation (23) are obtained by applying the transfer function inversion procedure of [12] to (14), (15) and (16)

$$I_{\text{Opt}-\text{AT}}(z) = 0.8655T\frac{(z+0.5781)(z+0.0964)}{(z-0.9995)(z+0.4959)} \tag{21}$$

$$I_{\text{Opt}-\text{AT}-2}(z) = 0.8659T\frac{(z+0.5781)(z+0.0964)}{(z-1)(z+0.4959)} \tag{22}$$

$$I_{\text{Opt}-\text{AT}-3}(z) = 0.8669T\frac{(z+0.5773)(z+0.0964)}{(z-1)(z+0.4973)} \tag{23}$$

The optimised Al-Alaoui-Simpson integrators, presented next as (24), (25) and (26) are obtained by applying the procedure of [12]

to (18), (19) and (20)

$$I_{\text{Opt-AS}}(z) = 0.8640T\frac{(z+0.5798)(z+0.0955)}{(z-0.9956)(z+0.4970)} \quad (24)$$

$$I_{\text{Opt-AS-2}}(z) = 0.8678T\frac{(z+0.5798)(z+0.0955)}{(z+0.4970)(z-1)} \quad (25)$$

$$I_{\text{Opt-AS-3}}(z) = 0.8678T\frac{(z+0.5861)(z+0.0879)}{(z-1)(z+0.4973)} \quad (26)$$

5.2 Optimised integrators

Starting from the integrators obtained through the conversion of the differentiators, we further optimise the digital filters using the same WMSE approach used for the differentiators. In all the cases, we maintain a pole at $(z=1)$. For an interesting discussion on the digital integrator with a pole $z=1$, please refer to the article by Dattorro [53]. In the considered cases, the weights were determined according to Table 3.

Equation (21) is optimised to yield equation

$$I_{\text{Opt-AT-O}}(z) = 0.8644T\frac{(z+0.5759)(z+0.0964)}{(z-1)(z+0.4936)} \quad (27)$$

Equation (24) is optimised to yield equation

$$I_{\text{Opt-AS-o}}(z) = 0.8639T\frac{(z+0.5893)(z+0.094)}{(z-1)(z+0.5021)} \quad (28)$$

The figures related to these integrators are presented in the next section. It is possible to start the optimisation from the other integrators as in (23) and (26), and the results were similar. In addition, the resulting integrators lead to a new set of differentiators that can be obtained by inverting the integrators as discussed in [12].

Equation (27) leads to equation

$$H_{\text{Opt-AT-O}}(z) = \frac{1.1569}{T}\frac{(z-1)(z+0.4936)}{(z+0.5759)(z+0.0964)} \quad (29)$$

Equation (28) yields equation

$$H_{\text{Opt-AS-O}}(z) = \frac{1.1575}{T}\frac{(z-1)(z+0.5021)}{(z+0.5893)(z+0.094)} \quad (30)$$

The figures related to these differentiators are presented in the next section.

6 Comparisons with previous results

In this section we compare the results of the optimised filters to those obtained in [39, 43].

Table 3 Weights for Optimising the Different Al-Alaoui integrators

Interval	0–0.25π	0.25π–0.5π	0.5π–0.75π	0.75π–π
weight for MSE	2	1	1	4

6.1 Comparisons with differentiators

The results of [39, 43] will be denoted according to the equations

$$H_{\text{Upadhyay}}(z) = \frac{(0.5805*1.987)(z+0.5033)(z-1)}{T(z+0.569)(z+0.11)} \quad (31)$$

$$H_{\text{Gupta-Yadav-PSO}}(z) = \frac{1.156(z-0.992)(z+0.5547)}{T(z+0.1042)(z+0.627)} \quad (32)$$

Since the results are close, we display the percentage absolute relative error results according to (33)

$$\%\ \text{Absolute Relative Error} = \frac{|\text{Proposed Value} - \text{Analog Value}|}{\text{Analog Value}} \times 100 \quad (33)$$

Fig. 3 shows the percentage absolute relative magnitude errors (PARE) of the first set of proposed differentiators which do not have a zero *at* $z=1$. Hence, Fig. 3 compares the differentiators of (14) and (18) with those of (31) and (32).

The figures indicate that the optimised second order differentiators presented in this paper are very close to the ideal response. In addition, Fig. 3 illustrates that choosing WMSE over MSE yields a better response and justifies the high weights at low frequencies, since the MSE based optimisation of the Al-Alaoui-Tick differentiator performs poorly at low frequencies. Over the frequency range $0.01\pi \le \omega \le 0.99\pi$, the filter $H_{\text{Opt-AT}}$ of (14) has a maximum PARE = 2.58 with a mean PARE = 0.26, and the filter $H_{\text{Opt-AS}}$ of (18) has a maximum PARE = 2.72 with a mean PARE = 0.27. On the other hand, the maximum PARE = 4.73 for H_{Upadhyay} of (32), with a mean PARE = 0.43, and the maximum PARE = 4.83 for $H_{\text{Gupta-Yadav-PSO}}$ of (33) with a mean PARE = 0.43.

In addition to the first set of proposed differentiators, we need to compare the performance of the optimised differentiators who have a zero *at* $z=1$ that are considered in (16) and (20). Fig. 4 shows the PARE of the proposed differentiators and compares them with those form the literature. For these differentiators, over the frequency range $0.01\pi \le \omega \le 0.99\pi$, the filter $H_{\text{Opt-AT-3}}$ of (16) has a maximum PARE = 3.14 and a mean PARE = 0.27, and the filter $H_{\text{Opt-AS-3}}$ of (20) has a maximum PARE = 2.12 and a mean PARE = 0.44.

In addition, we consider the performance of the differentiators of (29) and (30) obtained after the inversion of the integrators. Fig. 5

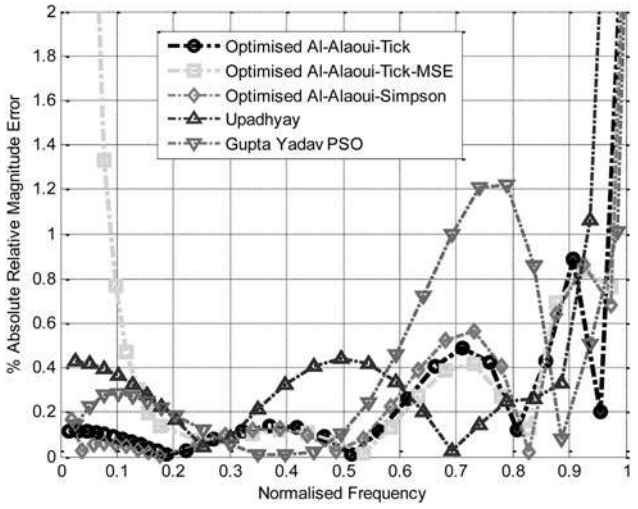

Fig. 3 *Percentage absolute relative magnitude error comparison of differentiators of (14) and (18) with those of (31) and (32)*

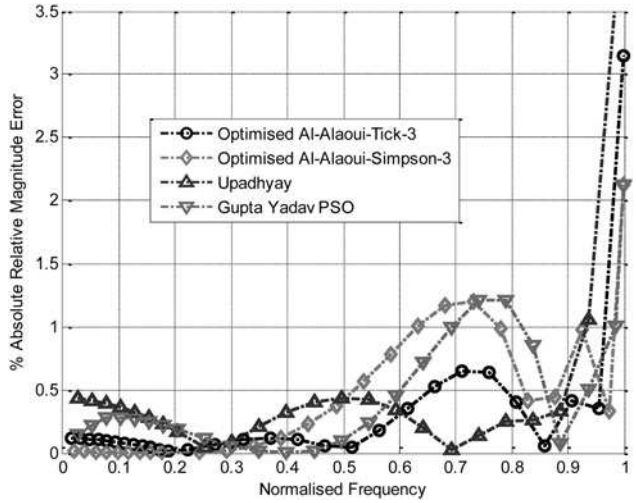

Fig. 4 *Percentage absolute relative magnitude error comparison of differentiators of (16) and (20) with those of (31) and (32)*

shows the PARE of these filters and compares them with those form the literature. For these filters and considering the frequency range $0.01\pi \le \omega \le 0.99\pi$, the filter $H_{\text{Opt}-\text{AT}-\text{O}}$ of (29) has a maximum PARE = 2.50 and a mean PARE = 0.28, and the filter $H_{\text{Opt}-\text{AS}-\text{O}}$ of (30) has a maximum PARE = 1.87 with a mean PARE = 0.35.

The different proposed differentiators clearly outperform the state of the art filters of the same orders.

6.2 Comparisons with integrators

Concerning the integrators, we compare them with two state of the art second order integrators. The first one is the Gupta-Yadav PSO [48] optimised second order integrator with equation

$$I_{\text{Gupta}-\text{Yadav}-2\text{nd}}(z) = 0.8651T\frac{(z+0.1042)(z+0.6276)}{(z-1)(z+0.5547)} \tag{34}$$

It is interesting to note that (34) is almost the inverse of (32) except for the fact that the inverse would yield a pole at (z = 0.992) instead of at (z = 1) which indicates pole placement.

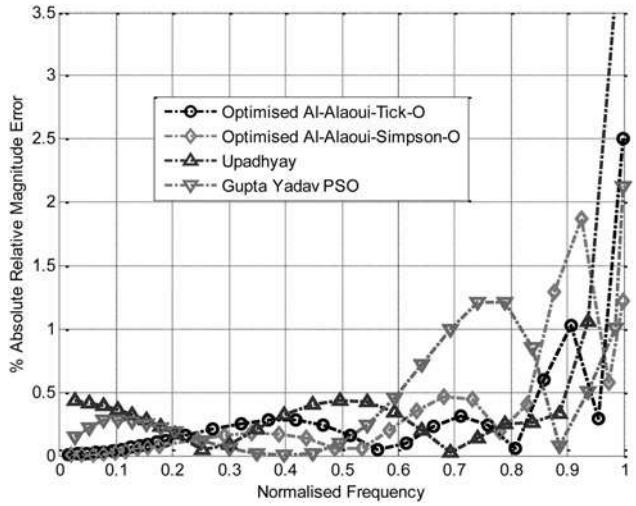

Fig. 5 *Percentage absolute relative magnitude error comparison of differentiators obtained after the integrators' inversion with various differentiators*

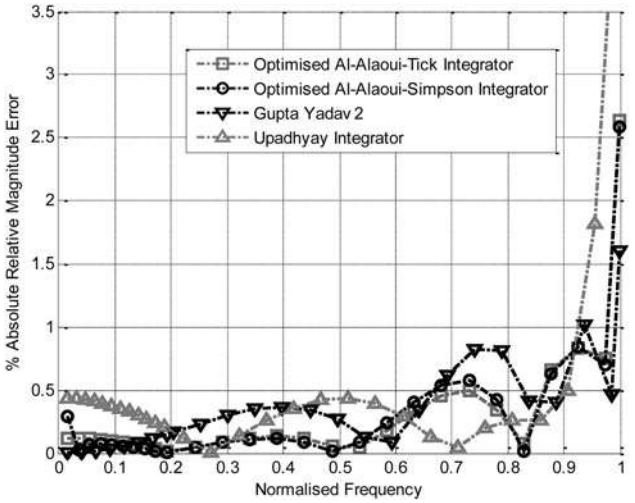

Fig. 6 *Percentage absolute relative magnitude error comparison of directly converted integrators (without pole at (z = 1)) with various integrators*

The second integrator is from [43] and has equation

$$I_{\text{Upadhyay}}(z) = \frac{0.569T(z+1.7575)(z+0.11)}{(0.5805*1.987)(z+0.5033)(z-1)} \tag{35}$$

Fig. 6 displays the comparison between the initially obtained integrators denoted in (21) and (24) with the ones from the literature. Over the frequency range $0.01\pi \le \omega \le 0.99\pi$, the filter $I_{\text{Opt}-\text{AT}}$ of (21) has a maximum PARE = 2.64 with a mean PARE = 0.26, and the filter $I_{\text{Opt}-\text{AS}}$ of (24) has a maximum PARE = 2.58 while the mean PARE = 0.27. In comparison, the maximum PARE = 1.6 for $I_{\text{Gupta}-\text{Yadav}-2\text{nd}}$ of (34) while the mean PARE = 0.35, and the maximum PARE = 4.97 for I_{Upadhyay} of (35) with a mean PARE = 0.43.

Moreover, Fig. 7 shows the comparison of the integrators that have a pole at (z = 1) with (23) and (26) with the ones from the literature. For these integrators, over the frequency range $0.01\pi \le \omega \le 0.99\pi$, the filter $I_{\text{Opt}-\text{AT}-3}$ of (23) has a maximum PARE = 3.2780 with a mean PARE = 0.28, and the filter $I_{\text{Opt}-\text{AS}-3}$ of (26) has a maximum PARE = 2.17 with a mean PARE = 0.44.

In addition to the directly converted integrators, we compare the performance of those obtained after optimising them while ensuring a pole at (z = 1). Fig. 8 shows the error comparison. For these filters and considering the frequency range $0.01\pi \le \omega \le$

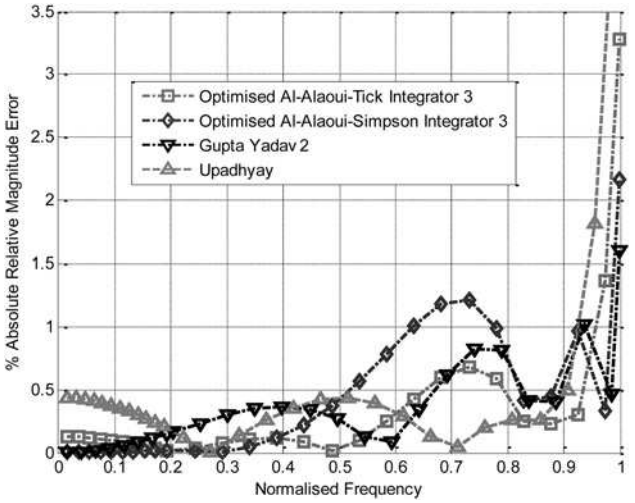

Fig. 7 *Percentage absolute relative magnitude error comparison of directly converted integrators (with pole at (z = 1)) with various integrators*

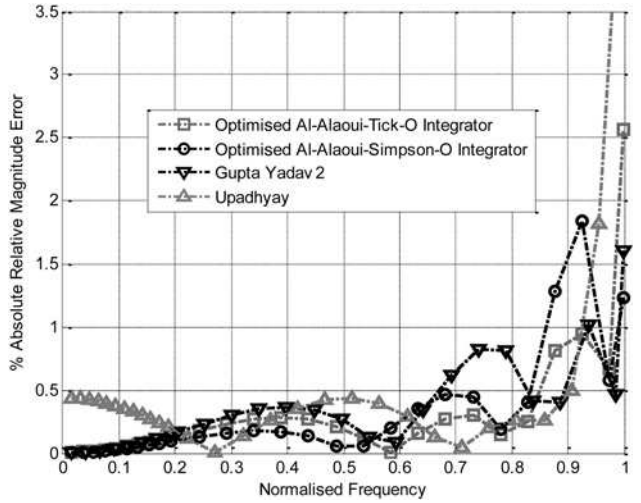

Fig. 8 *Percentage absolute relative magnitude error comparison of optimised integrators with various integrators*

0.99π, the filter $I_{\mathrm{Opt-AT-O}}$ of (27) has a maximum PARE = 2.57 with a mean PARE = 0.28, and the filter $I_{\mathrm{Opt-AS-O}}$ of (28) has a maximum PARE = 1.83 with a mean PARE = 0.34.

The performance of the proposed integrators is comparable with those of the state of the art filters of the same degree.

7 Conclusion

This paper introduces a weighted mean squared to designing digital filters employing IDA: An Interactive Design Approach. The approach was motivated by the Al-Alaoui weighted generalised inverse algorithm for pattern recognition [1–3, 50–52]. The approach was applied to the classical Al-Alaoui low-pass differentiators to obtain wide-band differentiators. The optimised second order differentiators performed better than several state of the art filters. Above all, the approach provides an intelligently controlled interactive design environment where one can better achieve the desired accuracy over the frequency range of interest. Although the approach concentrated on differentiators, various integrators of similar orders and of acceptable performance were obtained. The proposed ideas may be used to design general digital filters.

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9 Appendix

9.1 Derivations for minimising the magnitude MSE in (8)

In (1), $H_1(z)$ is the rectangular rule and $H_2(z)$ is the trapezoidal rule

$$H_1(z) = \frac{T}{z-1} \tag{36}$$

$$H_2(z) = \frac{T}{2}\frac{z+1}{z-1} \tag{37}$$

Hence, (1) becomes

$$H_2(z) = \frac{T}{2}\frac{(1-\alpha)z+\alpha+1}{z-1} \tag{38}$$

To minimise the squared magnitude error with the ideal integrator, we set the derivative to zero

$$\frac{\partial}{\partial\alpha}\left[\left\|\frac{T}{2}\frac{(1-\alpha)z+\alpha+1}{z-1}\right\| - \left\|\frac{1}{j\omega}\right\|\right]^2 = 0 \tag{39}$$

where $\|\cdot\|$ indicates the magnitude operation. Performing the derivation, we obtain

$$2\left(\left\|\frac{T}{2}\frac{(1-\alpha)z+\alpha+1}{z-1}\right\| - \left\|\frac{1}{j\omega}\right\|\right)\cdot\left\|\frac{1-z}{2}\right\| = 0 \tag{40}$$

To find the value of α, we set

$$\begin{aligned}&\left\|\frac{T}{2}\frac{(1-\alpha)z+\alpha+1}{z-1}\right\| - \left\|\frac{1}{j\omega}\right\| = 0 \Rightarrow\\ &T\|(1-\alpha)z+\alpha+1\| = 2\frac{\|z-1\|}{\omega}\end{aligned} \tag{41}$$

Considering the normalised period $T=1$, and setting the derivative to zero at the middle value of $\omega=\pi/2$, we obtain

$$\begin{aligned}&\|(1-\alpha)e^{j\pi/2}+\alpha+1\| = 2\frac{\|e^{j\pi/2}-1\|}{\pi/2} \Rightarrow\\ &\|[(1-\alpha)\cos(\pi/2)+\alpha+1]+j[(1-\alpha)\sin(\pi/2)]\|\\ &= \frac{2}{\pi/2}\|[\cos(\pi/2)-1]+j\sin(\pi/2)\| \Rightarrow\\ &\|(\alpha+1)+j(1-\alpha)\| = \frac{2}{\pi/2}\|-1+j\|\end{aligned} \tag{42}$$

Taking the magnitude, we obtain

$$\begin{aligned}&\sqrt{(1+\alpha)^2+(1-\alpha)^2} = \frac{4}{\pi}\sqrt{2} \Rightarrow\\ &\sqrt{2(1+\alpha^2)} = \frac{4}{\pi}\sqrt{2} \Rightarrow\\ &1+\alpha^2 = \frac{16}{\pi^2}\end{aligned} \tag{43}$$

Finally

$$\alpha = \sqrt{\frac{16}{\pi^2}-1} = 0.7881 \tag{44}$$