AMERICAN UNIVERSITY OF BEIRUT

# AN INFLATIONARY SCENARIO IN MIMETIC GRAVITY

by

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# Abstract of the Thesis of

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Title: An Inflationary Scenario in Mimetic Gravity

We study the Starobinsky potential model of inflation in mimetic gravity where the box of the mimetic field (d'Alembertian) is coupled to the inflaton potential, and then note that the observable quantities, namely the tensor-to-scalar ratio in perturbations and the spectral index agree with observations. To that end, a review of cosmology and inflation in ordinary General Relativity is presented. The results of Higgs inflationary scenario and Starobinsky/$R^2$ model are noted, specifically the tensor-to-scalar ratio and the spectral index which are proportional to the inverse square of the number of e-folds $N$ and to the inverse of $N$ respectively during inflation. Then Mimetic gravity, a modified theory of gravity which incorporates dark matter in the underlying geometry of spacetime is revisited. This incorporation is due to writing the physical metric as an auxiliary metric multiplied by a conformal mode. We then proceed to setup the theory of mimetic inflation by coupling a linear function of the d'Alembertian of the mimetic field with the inflaton potential in the action. We get the equations of motion, analyze them in the modified slow roll approximation, and show that an inflationary epoch takes place. After proving that a reheating phase indeed occurs in our model, we then carry cosmological perturbations to the background equations of motion to get the scalar and tensor power spectrum. It is shown afterwards that for our Starobinsky potential model the observable quantities (spectral index and tensor-to-scalar ratio) agree with observations and with Higgs inflationary scenario / $R^2$ inflation. Finally, a very brief look on self-reproduction/eternal inflation and how to possibly avoid it based on this work is mentioned.

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# Chapter 1

# A brief introduction to General Relativity

The objective of this chapter is to present a brief review on General Relativity starting from its assumptions and finishing off with the Einstein field equations.

## 1.1 General Theory of Relativity

### 1.1.1 *Principles*

As it is well known, the equality of the gravitational and inertial masses is what drove Einstein to think of his classic elevator thought experiment from which the equivalence principle stemmed:

*In a freely falling (non-rotating) laboratory occupying a small region of spacetime, the laws of physics are those of special relativity.* [1]

This principle suggests that a 4 dimensional spacetime, even if curved, is locally indistinguishable from the flat Minkowski spacetime (that of special relativity). Mathematically speaking, if we have a metric $g_{\mu\nu}$ describing a curved spacetime, then we can always find local inertial coordinates (also called normal coordinates) such that at each point P of spacetime [2]:

$$g_{\mu\nu}(P) = \eta_{\mu\nu} \quad \text{and} \quad \partial_\sigma g_{\mu\nu}|_P = 0 \tag{1.1}$$

The second principle worth mentioning is the general principle of relativity:

*All systems of reference are equivalent with respect to the formulation of the fundamental laws of physics.* [3]
Mathematically it asserts that the laws of physics are the same in all coordinate systems, i.e. they are invariant under general coordinate transformations. In General Relativity specifically, the physics should be expressed in Tensor equations as they are invariant under coordinate transformations.

We now present a brief yet concrete General Relativity machinery. We start with the metric tensor.

### 1.1.2 *The Metric Tensor*

Since we are dealing with 4 dimensional "curved" spacetimes now, their description is accompanied by the introduction of the metric tensor which is defined on the manifold (spacetime) itself. The metric tensor is usually denoted by $g_{\mu\nu}$, and it is used to correct for angles and distances in curved spacetimes. The spacetime line element is therefore:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.2}$$

The metric is also symmetric, $g_{\mu\nu} = g_{\nu\mu}$ which can be easily seen from $ds^2$ invariance. Not only does the metric play a crucial role in "measuring" distances and angles in curved spacetimes, but also constitutes the basis of quantities which tell us whether spacetimes are curved or not, like the Rienmann Tensor which is derived from the metric as will be shown below. The metric tensor is also used to lower and raise indices in the following manner

$$v^\mu = g^{\mu\nu} v_\nu \quad \text{and} \tag{1.3}$$

$$v_\mu = g_{\mu\nu} v^\nu \tag{1.4}$$

where $v^\mu$ are the contravariant components of the vector $v = v^\mu \mathbf{e}_\mu$ which is itself invariant. $\mathbf{e}_\mu$ are the basis vectors and one can define a set of "dual basis vectors" $\mathbf{e}^\mu$ defined by the relation:

$$\mathbf{e}^\alpha \mathbf{e}_\beta = \delta^\alpha_\beta \tag{1.5}$$

so $\mathbf{v}$ can also be written in terms of its covariant components as $\mathbf{v} = v_\mu \mathbf{e}^\mu$. In (1.3) we used the inverse metric which is defined via:

$$g^{\alpha\mu} g_{\mu\beta} = \delta^\alpha_\beta \tag{1.6}$$

The metric tensor can also raise and lower Tensor indices by the same manner. The volume element of spacetime is $\sqrt{|g|} d^4x = \sqrt{-g} d^4x$ where $g$ is the determinant of the metric tensor which is negative due to the mixed signature.

### 1.1.3 *Christoffel symbols and Covariant derivatives*

In a curved spacetime, one cannot simply pick 2 vectors at two different points and add/subtract them and do usual calculus, this is due to the curvature of the manifold since every vector lie in the tangent space of the corresponding point. Not only that, even basis vectors change when going from point A to B. To that end, *Christoffel symbols* (or the *affine connection*) are defined as follows:

$$\Gamma^\rho_{\mu\nu} = \mathbf{e}^\rho \frac{\partial \mathbf{e}_\mu}{\partial x^\nu} \tag{1.7}$$

Geometrically, christoffel symbols describe how much of the rate of change in the basis vector $\mathbf{e}_\mu$ with respect to the $x^\nu$ coordinate is projected onto $\mathbf{e}^\rho$. In terms of the metric, we have:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu} \right) \tag{1.8}$$

where we make the definitions

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \quad \text{and} \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} \tag{1.9}$$

In (1.8), we defined the christoffel symbols of the second kind. To get those of the first kind, recall that the metric can be used to raise and lower indices, hence:

$$\Gamma_{\mu\nu\rho} = g_{\mu\sigma} \Gamma^\sigma_{\nu\rho} \tag{1.10}$$

From now on, we assume that our manifolds are *torsionless* and this leads to the symmetry of the christoffel symbols in the last two indices:

$$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} \tag{1.11}$$

One can also write the christoffel symbols with contracted first two indices in terms of the determinant of the metric $g$ as:

$$\Gamma^\mu_{\mu\nu} = \partial_\nu \ln \sqrt{-g} = \frac{1}{\sqrt{-g}} \partial_\nu \sqrt{-g} \tag{1.12}$$

As addressed above, one cannot simply subtract vectors at two different points and form a derivative in the limit where the separation is negligible since we have a curved spacetime, let alone that the basis vectors change along curved spaces. As such, the notion of parallel transport comes in rescue, and from it a new kind of derivative emerges, the so-called *Covariant Derivative.* We will not delve into the mechanism of parallel transport and how it facilitates doing calculus on curved spacetimes for the "lack of space and time" at the time of writing this thesis, and one can refer to any introductory General relativity book at their leisure. From a practical point, this "update" of our ordinary partial derivatives to covariant derivatives can be thought of as a necessity to get some kind of derivative which is invariant under general coordinate transformations, as partial derivatives are not! The covariant derivative of contravariant vector components is:

$$\nabla_\mu u^\nu = \partial_\mu u^\nu + \Gamma^\nu_{\mu\rho} u^\rho \tag{1.13}$$

and that of covariant components is:

$$\nabla_\mu u_\nu = \partial_\mu u_\nu - \Gamma^\rho_{\mu\nu} u_\rho \tag{1.14}$$

One might ask what about the covariant derivative of a scalar? By understanding that the covariant derivative differs from the ordinary partial derivative simply because the coordinate basis vectors change with position in the manifold, it should then be clear to see that the covariant derivative of a scalar is the same as its partial derivative since scalars have no dependence on coordinate basis vectors unlike vectors. That is

$$\nabla_\mu \phi = \partial_\mu \phi \tag{1.15}$$

To conclude this section, we write the generalization of vector operators on 4 dimensional manifolds.

*Gradient*

The gradient of a scalar field is simply given by:

$$\nabla \phi = (\nabla_\mu \phi)\mathbf{e}^\mu = (\partial_\mu \phi)\mathbf{e}^\mu \tag{1.16}$$

recall that the covariant derivative of a scalar field is the same as its partial derivative.

*Divergence*

$$\nabla \cdot \mathbf{v} = \nabla_\mu v^\mu = \partial_\mu v^\mu + \Gamma^\mu_{\mu\nu} v^\nu \tag{1.17}$$

as we used (1.14) after the second equality. Using the result (1.12), we can rewrite the divergence as:

$$\nabla \cdot \mathbf{v} = \nabla_\mu v^\mu = \frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}\,v^\mu) \tag{1.18}$$

where g is the determinant of the metric.

*Laplacian*

If we replace $\mathbf{v}$ by $\nabla\mathbf{v}$ in $\nabla.\mathbf{v}$ we get the updated Laplacian $\nabla^2\phi$ which is oftenly called the "box operator" or the d'Alembertian. This, accompanied by (1.18) results in:

$$\Box\phi = \nabla^2\phi = (\nabla_\mu \nabla^\mu \phi) = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\phi\right) \tag{1.19}$$

### 1.1.4 *Geodesic equation*

Newton's laws of motion fall short when we consider strong gravitational fields and high velocities, and they are replaced by the Geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{1.20}$$

where $\tau$ is the proper time. Eq.(1.20) describes how particles move under gravitational fields of different spacetimes, the properties of which are incorporated in the christoffel symbols. The paths the particles follow are therefore called "Geodesics". This equation is verified experimentally and can be taken as a postulate on its own (as newton's laws). It can also be derived from Hamilton's principle by requiring that particles follow paths of minimal proper time between events. This can be

achieved by varying with respect to the affine parameter in the corresponding action then setting it to be the proper time.
For the ambitious reader, I refer him to section 8.8 in Hobson's book [1] where he discusses how one can get the geodesic equation from Einstein equations, and hence one needs to only start form the Einstein-Hilbert action alone without postulating another equation aside from it.

### 1.1.5 *Riemann curvature tensor and Ricci scalar*

We alluded above that the metric can lead us to quantities which contain the necessary and sufficient information related to curvature. It happens that this quantity is the Riemann Tensor (or curvature tensor) which reads in coordinates:

$$R^{\alpha}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}_{\beta\nu} - \partial_{\nu}\Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\lambda\mu}\Gamma^{\lambda}_{\beta\nu} - \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\beta\mu} \tag{1.21}$$

In a flat region of the manifold, we may choose coordinates such that christoffel symbols and their derivatives vanish, retaining

$$R^{\alpha}_{\beta\mu\nu} = 0$$

at every point in this region. Conversely, if $R^{\alpha}_{\beta\mu\nu} = 0$ at every point in some region of the manifold, then it implies that there exists a set of coordinates which lead to the metric being flat. Thus when the curvature tensor vanishes in some region of the manifold, it is then a necessary and sufficient condition that that region is flat.
The symmetry properties of the Riemann tensor are (after loweing the first index using the metric):

$$\begin{aligned} R_{\mu\nu\sigma\rho} &= -R_{\nu\mu\sigma\rho} && \text{(antisymmetry in the first two indices)} \\ R_{\mu\nu\sigma\rho} &= -R_{\mu\nu\rho\sigma} && \text{(antisymmetry in the last two indices)} \\ R_{\mu\nu\sigma\rho} &= R_{\sigma\rho\mu\nu} && \text{(symmetry in the first and last pairs of indices)} \end{aligned} \tag{1.22}$$

We may easily deduce the *cyclic identity*:

$$R_{\mu\nu\sigma\rho} + R_{\mu\sigma\rho\nu} + R_{\mu\rho\nu\sigma} = 0 \tag{1.23}$$

Moreover, the *Bianchi identity* reads:

$$\nabla_{\lambda}R_{\mu\nu\rho\sigma} + \nabla_{\rho}R_{\mu\nu\sigma\lambda} + \nabla_{\sigma}R_{\mu\nu\lambda\rho} = 0 \tag{1.24}$$

This is a tensor relation and thus holds in all coordinate systems. Using the Riemann tensor, one can define the Ricci Tensor:

$$R_{\alpha\beta} = R^{\mu}_{\alpha\beta\mu} \tag{1.25}$$

which is symmetric. A further contraction gives the Ricci scalar (or curvature scalar)

$$R = g^{\mu\nu}R_{\mu\nu} = R^{\mu}_{\mu} \tag{1.26}$$

### 1.1.6 *Einstein Field Equations*

Having defined the relevant quantities describing *curvature*, we need to relate them to *energy and matter* in a covariant equation, since in the end as John Wheeler put it: "Spacetime tells matter how to move; matter tells spacetime how to curve". Consequently, in General Relativity matter and energy have their own tensor, the *Stress-Energy Tensor* $T_{\mu\nu}$. One can get the field equations from the Einstein-Hilbert action:

$$S_{EH} = \int_M d^4x\sqrt{-g}\,R \tag{1.27}$$

where R is the Ricci scalar. This action is considered a function of the metric and its first and/or second order derivatives, and varying it with respect to the metric gives

$$\delta S_{EH} = \int_M (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}\sqrt{-g}d^4x \tag{1.28}$$

Demanding that $\delta S_{EH} = 0$ for an arbitrary variation in the metric (Hamilton's principle of least action), we immediately find that

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{1.29}$$

which represents the field equations in vacuum, and where $G_{\mu\nu}$ is the Einstein tensor. We note that $G_{\mu\nu}$ is symmetric as Ricci tensor and the metric are. Now in the presence of matter, one needs to add an extra term in the action

$$S = \frac{1}{2\kappa}S_{EH} + S_{Matter} = \int_M (\frac{1}{2\kappa}L_{EH} + L_{Matter})d^4x \tag{1.30}$$

where $\kappa = \frac{8\pi G}{c^4}$ Varying with respect to the (inverse) metric one gets

$$\frac{1}{2\kappa}\frac{\delta L_{EH}}{\delta g^{\mu\nu}} + \frac{\delta L_M}{\delta g^{\mu\nu}} = 0 \tag{1.31}$$

From (1.28) we see that

$$\frac{\delta L_{EH}}{\delta g^{\mu\nu}} = \sqrt{-g}G_{\mu\nu} \tag{1.32}$$

Thus if we make the assertion that the energy-momentum tensor is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta L_M}{\delta g^{\mu\nu}} \tag{1.33}$$

Then we recover the full Einstein field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.34}$$

These are the Einstein field equations in the presence of matter. The left hand side represents curvature while the right hand side contains matter. It might seem that the assertion (1.33) is a bold one and unjustified, nevertheless, this tensor has all the properties required of an energy-momentum tensor, the most important of which is

$$\nabla_\mu T^{\mu\nu} = 0 \tag{1.35}$$

implying local conservation of energy and momentum.

# Chapter 2

# Cosmology

## 2.1 The FRW metric

Starting from the cosmological principle, which states that at any particular time the universe looks the same from all positions in space and all directions in space at any point are equivalent, we have good physical reasons to study cosmological models in which the universe is postulated to be homogeneous and isotropic. Other than this assumption, Weyl's postulate is adopted and before stating it we state that one should intuitively think of the geometry of "space" evolving over "time" by introducing a family of non-intersecting three dimensional spatial hypersurfaces parametrized by a paremeter $t$, which makes "time" a parameter that is valid globally upon slicing spacetime into these hypersurfaces labeled by $t$. In general, there is no preferred "slicing" and hence no preferred "time" coordinate. Now the Weyl's postulate comes in and states that a fundamental observer's worldline (that who has no motion to the overall cosmological fluid) should be hypersurface orthogonal. In other words, there *is* a preferred slicing, namely the one constructed by taking the unique spacelike hypersurfaces to be orthogonal to the worldlines of the fundamental observers (fluid particles). The metric embodying this postulate is:

$$ds^2 = dt^2 - g_{ij}dx^i dx^j \quad (\text{for } i, j = 1, 2, 3) \tag{2.1}$$

where $g_{0i} = g_{i0} = 0$, $g_{00} = 1$ and $t$ may be taken to be the proper time along the worldline of any fundamental observer. This metric ensures that each spacelike hypersurface $t = constant$ is orthogonal to the observer's worldine (via $g_{0i} = 0$), and the coordinate system used is the so-called Synchronous coordinate system (or synchronous gauge). Combining these together one can obtain the standard form of the *Friedmann-Robertson-Walker* line element:

$$ds^2 = dt^2 - a^2(t)[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{2.2}$$

where $a(t)$ is the scale factor, the "radius" of the universe, and $k$ is the spatial curvature constant. If $k = 1$, the universe is said to be "closed" or "spherical", if $k = 0$, it is said to be flat, and if $k = -1$, it is said to be "open" or "hyperbolic".

We know from observations that the observable universe is flat, hence $k = 0$. [4] [5] [6] [7]. Generally this metric can be written as

$$ds^2 = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j \tag{2.3}$$

where $\gamma_{ij}$ takes care of the coordinate transformation factors. For example, when $k = 0$ and we are in cartesian coordinates, the metric yields

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$$

For the FRW ansatz the evolution of the homogeneous isotropic universe boils down to the behavior of the single function $a(t)$. Its form is dictated by the matter content of the universe via the Einstein field equations.

## 2.2 The Cosmological Field Equations

The energy-momentum tensor of a perfect fluid, in a local cartesian inertial frame, that is the inertial rest frame of the fluid has the form (in units where c=1):

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \tag{2.4}$$

which also could be written as:

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p\eta^{\mu\nu} \tag{2.5}$$

To obtain the expression in any coordinate system, one only replaces the minkowski metric by $g^{\mu\nu}$ and hence we get a fully covariant expression for the stress-energy tensor components of a perfect fluid

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} \tag{2.6}$$

The Ricci tensor components of the FRW metric (2.2) are:

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} \\ R_{22} &= (a\ddot{a} + 2\dot{a}^2 + 2k)r^2 \\ R_{33} &= (a\ddot{a} + 2\dot{a}^2 + 2k)r^2 \sin^2\theta \end{aligned} \tag{2.7}$$

One can obtain another form of the Einstein equations by taking the trace of equation (1.34) to get:

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \tag{2.8}$$

Plugging (2.7), the metric components and the covariant components of (2.6) in (2.8) where we adopt the comoving coordinate system where the four velocity is $u^\mu = (1, 0, 0, 0)$, we get the *Friedmann Equations*:

$$H^2 \equiv (\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{2.9}$$
$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{2.10}$$

where $H \equiv \frac{\dot{a}}{a}$ is the *Hubble parameter* which represents the expansion rate of the observable universe. These two differential equations determine the time evolution of the scale factor $a(t)$.

## 2.3 Equation of motion of the cosmological fluid and some cosmological models

For any particular model of the universe, the above friedmann equations are sufficient to determine $a(t)$. Nevertheless, we can derive one further equation from the fact that the energy-momentum conservation requires

$$\nabla_\mu T^{\mu\nu} = 0$$

Using this and the normalization condition on the four-velocity, we get the equations of continuity and motion for the cosmological fluid:

$$\nabla_\mu(\rho u^\mu) + p\nabla_\mu u^\mu = 0 \tag{2.11}$$
$$(\rho + p)u^\mu \nabla_\mu u^\mu = (g^{\mu\nu} - u^\mu u^\nu)\nabla_\mu p \tag{2.12}$$

The second equation is satisfied identically since both sides are zero, which confirms that the fluid particles (galaxies) follow geodesics. This is also expected since $p$ is a function of time alone and so there are no pressure gradients to push them off geodesics. The continuity equation (2.11) can be written as

$$(\partial_\mu \rho)u^\mu + (\rho + p)(\partial_\mu u^\mu + \Gamma^\mu_{\nu\mu} u^\nu) = 0$$

and remembering that $\rho$ is a function of $t$ alone with $u^\mu = \delta^\mu_0$, this reduces to

$$\dot{\rho} + (\rho + p)\frac{3\dot{a}}{a} = 0 \tag{2.13}$$

which expresses energy conservation. This equation can be in fact derived directly from the field equations by eliminating $\ddot{a}$. Therefore, only two of the three equations (2.9), (2.10), and (2.13) are independent, and one may choose the two equations that are most convenient for calculations. Equation (2.13) can be rearranged into

$$\frac{d(\rho a^3)}{dt} = -3p\dot{a}a^2 \tag{2.14}$$

and by writing the derivative with respect to $t$ as one with respect to $a$, one obtains

$$\frac{d(\rho a^3)}{da} = -3pa^2 \tag{2.15}$$

Now we note that for a fluid, its density and pressure are related by its equation of state. In cosmology it is usual to assume that each component of the fluid has an equation of state of the form

$$p = w\rho \tag{2.16}$$

where the *equation of state parameter* $w$ is a constant. The energy equation (2.15) can then be written

$$\frac{d(\rho a^3)}{da} = -3w\rho a^2$$

This equation has the immediate solution

$$\rho \propto a^{-3(1+w)} \tag{2.17}$$

which gives the evolution of the density $\rho$ as a function of the scale factor. We move on to describe some cosmological scenarios for a flat universe $k = 0$.

**-Matter dominated:**

Matter here is considered "dust", i.e. non-relativistic and moving at low speeds. For such a model, $w = 0$, this comes from statistical mechanics since $\frac{p}{\rho} \propto \frac{v^2}{c^2}$ and $\frac{v}{c} \simeq 0$ for non-relativistic speeds. Therefore,

$$\rho \propto \frac{1}{a^3}$$

which is intuitive and says that the density falls off proportionally with "volume". Substituting this density in the friedmann equation (2.9) with $k = 0$ one gets:

$$a(t) \propto t^{\frac{2}{3}} \tag{2.18}$$

and

$$\rho \propto \frac{1}{t^2} \tag{2.19}$$

while the hubble parameter decreases as:

$$H(t) \equiv \frac{\dot{a}}{a} = \frac{2}{3t} \tag{2.20}$$

**-Radiation dominated:**

The equation of state parameter here is $w = 1/3$ [8]:

$$\rho \propto \frac{1}{a^4}$$

Utilizing the same friedmann equation one gets

$$a(t) \propto t^{\frac{1}{2}} \tag{2.21}$$

and

$$\rho \propto \frac{1}{t^2} \tag{2.22}$$

The radiation dominated universe expands at a slower rate than that of matter dominated mainly due to the positive pressure.

**-Vacuum (cosmological constant Λ)** This scenario suggests a constant density which is the same if one includes a cosmological constant Λ. The equation of state is $p = -\rho$ which helps in expansion due to the negative pressure representing a "repulsive" gravitational force. The Friedmann equation becomes:

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\rho_0} = H_0 \tag{2.23}$$

which yields the exponential solution

$$a(t) \propto e^{H_0 t} \tag{2.24}$$

If we have more than one species contributing significantly to the density and pressure, and they do not interact except via their mutual gravitation, then the multi-component fluid can be modeled as a single fluid with

$$\rho \equiv \sum_i \rho_i\,, \qquad p \equiv \sum_i p_i \tag{2.25}$$

and the corresponding equations of state are

$$w_i \equiv \frac{p_i}{\rho_i} \tag{2.26}$$

For each species $i$ the present ratio of the energy density relative to the *critical density* $\rho_{crit} \equiv \frac{3H_0^2}{8\pi G}$ is called a density parameter and is defined as

$$\Omega_i \equiv \frac{\rho_0^i}{\rho_{\rm crit}} \tag{2.27}$$

Now the subscript 0 refers to a quantity which is evaluated at the present time $t_0$. Upon normalizing the scale factor such that $a_0 = a(t_0) \equiv 1$, the Friedmann equation (2.9) can be written as

$$\left(\frac{H}{H_0}\right)^2 = \sum_i \Omega_i a^{-3(1+w_i)} + \Omega_k a^{-2}, \tag{2.28}$$

where $\Omega_k \equiv -k/a_0^2 H_0^2$ is the *curvature density parameter* Evaluating equation (2.28) at the present time we obtain

$$\sum_i \Omega_i + \Omega_k = 1 \tag{2.29}$$

The second Friedmann equation (2.10) evaluated at $t = t_0$ yields

$$\frac{1}{a_0 H_0^2}\frac{d^2 a_0}{dt^2} = -\frac{1}{2}\sum_i \Omega_i (1 + 3w_i) \tag{2.30}$$

and defines the condition for expansion today.

## 2.4 Issues with standard cosmology

Standard cosmology was triumphant in explaining the expansion of the universe but fell short to address what are known to be the flatness and horizon problems. As such, we will address these problems briefly and in the next chapter we will see how inflation solves them.

### 2.4.1 *Horizon problem*

We start by defining the *comoving particle horizon* $\eta$, which is the causal horizon or the maximum comoving distance light can travel between $t = 0$ and $t$

$$\eta \equiv \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{da}{Ha^2} = \int_0^a d\ln a \left(\frac{1}{aH}\right) \tag{2.31}$$

where we expressed it in terms of the *comoving Hubble radius* $(aH)^{-1}$ which will play an important role in the discussion. We note that the physical size of the particle horizon is $d = a(t)\eta$. Using the Friedmann equation (2.9), we get that for a universe dominated by a fluid with an equation of state parameter $w$ we have

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)} \tag{2.32}$$

During the conventional expansion in standard cosmology, $w \gtrsim 0$ and $(aH)^{-1}$ increases monotonically; plugging eq.(2.32) in eq.(2.31) we find that the comoving particle horizon (the fraction of the universe which is in causal contact) increases with time

$$\eta \propto a^{\frac{1}{2}(1+3w)} \tag{2.33}$$

For example, for radiation dominated (RD) and matter dominated (MD) universes we find

$$\eta = \int_0^a \frac{da}{Ha^2} \propto \begin{cases} a & \text{RD} \\ a^{1/2} & \text{MD} \end{cases} \tag{2.34}$$

which means that the comoving horizon grows monotonically with time, and this implies that the comoving scales entering the horizon *today* have been far outside the horizon at CMB decoupling. But if they are outside the horizon, this means they have not been in causal contact before, yet the near-homogeneity of the CMB tells us that there are regions which have equilibrated temperatures and uniformity across them! In other words, regions that should not have been in causal contact at the time of photon decoupling possess uniform properties as if they were in contact before! How is this possible?

### 2.4.2 *Flatness problem*

Defining

$$\Omega \equiv \sum_i \Omega_i = 1 - \Omega_k \tag{2.35}$$

we write the Friedmann equation (2.9) as

$$\Omega_k(a) = 1 - \Omega(a) = \frac{-k}{(aH)^2} \tag{2.36}$$

where

$$\Omega(a) \equiv \frac{\rho(a)}{\rho_{\rm crit}(a)}\,, \qquad \rho_{\rm crit}(a) \equiv 3H(a)^2\,. \tag{2.37}$$

Note that $\Omega(a)$ here is time dependent whereas in the previous section $\Omega's \equiv \Omega(a_0)$ were constants. We have shown above that in standard cosmology the comoving hubble radius $(aH)^{-1}$ increases with time, and from equation (2.36) the quantity $|\Omega - 1|$ must diverge with time. But we know nowadays that the universe is nearly flat and hence $k \simeq 0$ (equivalently $\Omega(a_0) \simeq 1$), combining this with the fact that the critical value $\Omega = 1$ is an *unstable fixed point* [9], then the near-flatness observed today requires an extremely fine tuned value of $\Omega$ close to 1 in the early universe! To illustrate the idea further, one finds that the deviation from flatness at Big Bang Nucleosynthesis (BBN), during the GUT era, and at the Planck scale, respectively has to satisfy the following conditions [9]:

$$|\Omega(a_{\rm BBN}) - 1| \;\leq\; \mathcal{O}(10^{-16})\,, \tag{2.38}$$
$$|\Omega(a_{\rm GUT}) - 1| \;\leq\; \mathcal{O}(10^{-55})\,, \tag{2.39}$$
$$|\Omega(a_{\rm pl}) - 1| \;\leq\; \mathcal{O}(10^{-61})\,. \tag{2.40}$$

We see that the further we rewind time backwards, the more extreme the unexplainable initial fine tuned value of $\Omega$ being close to 1!

Before closing this chapter, I should emphasize that the flatness and horizon problems are not strict inconsistencies in standard cosmology. One could get away with them if one just assumes that the initial value of $\Omega$ was extremely close to one and that the homogeneity over superhorizon scales is just an initial condition when the universe began. These problems therefore are really just severe shortcomings of the predictive power of the standard cosmological model, in which one must assume them to be initial conditions since they are beyond explanation - prediction in standard cosmology. Yet, a theory which explains these initial conditions dynamically is indeed very attractive and hence we delve into studying inflation in the next chapter.

# Chapter 3

# Inflation

## 3.1 Solving the standard cosmological puzzles

In the last chapter, we saw the critical role that the comoving hubble radius $(aH)^{-1}$ plays in the horizon and flatness problems. These problems arise mainly because $(aH)^{-1}$ increases in the standard cosmological model; this suggests that these problems might be resolved if one *inverts the behavior of the comoving hubble radius, i.e. make it sufficiently decrease in the very early universe*. Recall the definition of the comoving particle horizon (conformal time with units where $c = 1$) as a logarithmic integral of the comoving Hubble radius

$$\eta = \int_0^a d\ln a' \, \frac{1}{a' H(a')} \tag{3.1}$$

Let us make the important yet subtle distinction between the comoving Hubble radius $(aH)^{-1}$ and the comoving particle horizon $\eta$ quoting Baumann and Dodelson [9] [10]:

> If particles are separated by distances greater than $\eta$, they *never* could have communicated with one another; if they are separated by distances greater than $(aH)^{-1}$, they cannot talk to each other *now*! This distinction is crucial for the solution to the horizon problem which relies on the following: It is possible that $\eta$ is much larger than $(aH)^{-1}$ now, so that particles cannot communicate today but were in causal contact early on. From Eqn. (3.1) we see that this might happen if the comoving Hubble radius in the early universe was much larger than it is now so that $\eta$ got most of its contribution from early times. Hence, we require a phase of decreasing Hubble radius. Since $H$ is approximately constant while $a$ grows exponentially during inflation we find that the comoving Hubble radius decreases during inflation just as advertised.

Equipped with this, we move on to address the aforementioned problems.

### 3.1.1 *Flatness problem revisited*

Recalling the Friedmann equation (2.36) for a non-flat universe of spatial curvature $k$

$$\Omega(a) - 1 = \frac{k}{(aH)^2} \tag{3.2}$$

Following the proposal that the comoving Hubble radius decreases in the very early universe, we see that this drives $\Omega(a)$ towards 1 (and hence $k$ towards 0); i.e. it drives the universe towards flatness rather than away from it! $\Omega = 1$ is an attractor solution during inflation; this easily solves the flatness problem.
Intuitively speaking, inflation "flattens" the universe regardless of any initial curvature due to the rapid and vast expansion. The exponential growth smooths out the geometry of spacetime, making the observable portion of it appear very flat regardless of its initial curvature.

### 3.1.2 *Horizon problem revisited*

When the comoving hubble horizon decreases, it means that the large scales entering the universe were inside the causal horizon before inflation. The figure below will help us understand this in more detail

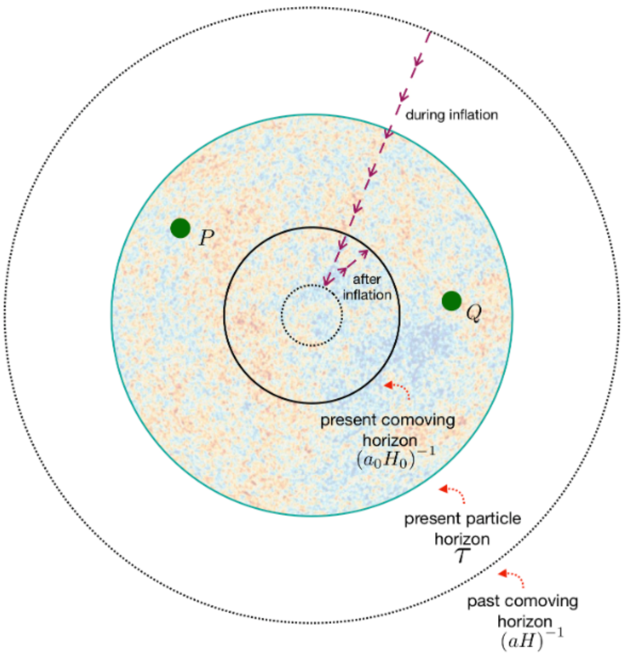


Figure 3.1: Evolution of the comoving Hubble radius $(aH)^{-1}$. $(aH)^{-1}$ starts large then shrinks during inflation, when it reaches its smallest size inflation ends and it starts increasing again. All scales that are relevant to cosmological observations today were larger than the Hubble radius until $a \simeq 10^{-5}$. Yet at sufficiently earlier times, these scales were smaller than the comoving Hubble radius and within the comoving particle horizon and therefore causally connected.

One can see that most of the contribution of today's comoving particle horizon comes from the inflationary epoch since we start with a large $(aH)^{-1}$ which decreases till inflation ends then increases again. The integral (3.1) tells us then that indeed most of the contribution of today's particle horizon comes from the inflationary era and hence $\eta$ is fairly large today since we started at a large $(aH)^{-1}$ before inflation. We must note that the colored patch here represents the causal horizon and the black dotted circle represents the comoving Hubble radius (which we utilize for measurements, i.e. everything inside could be measured). In other words, at early times the horizon was large enough so that observable scales were in causal contact. As inflation took place, the horizon shrank and scales came out to disconnected regions. Inflation then finished and the horizon started to grow to the present size. Two casually-disconnected CMB regions, $P$ and $Q$, were then in thermal equilibrium at some point in the past, thus resolving the horizon problem.
Physically, one can think of this as inflation (being a brief but extreme period of exponential expansion of the universe) could have stretched small, causally connected regions to much larger scales far beyond the observable universe. As a result, regions that appear casually disconnected in the CMB were actually in thermal contact (causal contact) before inflation (inside the horizon) which explains the observed uniformity.

## 3.2 Number of e-folds at the end of inflation

One can ask about "how much inflation" the universe should undergo so that the theory is consistent with observations, and now we address this question. Recalling that the comoving Hubble radius decreases during inflation and looking at figure 3.2, we have

$$(a_0H_0)^{-1} < (a_iH_i)^{-1} \tag{3.3}$$

where the subscripts $0, i$ refer to time to be fixed at "present" in an idealized scenario (look below) and the start of inflation respectively. Now let us relate $(a_0H_0)^{-1}$ to $(a_eH_e)^{-1}$, i.e. the comoving Hubble radius at the end of inflation. This can be done by considering the universe to be radiation dominated for simplicity and ignore the relatively recent periods of matter and dark energy domination. Remembering that $H \propto a^{-2}$ during the radiation-dominated era, we have

$$\frac{a_0H_0}{a_RH_R} \simeq \frac{a_0}{a_R}(\frac{a_R}{a_0})^2 = \frac{a_R}{a_0} \simeq \frac{T_0}{T_R} \simeq \frac{10^{-4}ev}{T_R} \simeq \frac{10^{-13}Gev}{T_R} \tag{3.4}$$

where in the third equality we utilized $a \propto T^{-1}$ since $\rho \propto a^{-4}$ and $\rho \propto T^4$ in a radiation-dominated universe. We will also assume that the energy density at the end of inflation was converted quickly into thermal plasma particles so that the Hubble radius did not experience growth between the end of inflation and the beginning of the hot Big Bang (radiation), $(a_eH_e)^{-1} \simeq (a_RH_R)^{-1}$ and therefore we will assume $T_e \simeq T_R$ hence

$$\frac{a_0H_0}{a_eH_e} \simeq \frac{10^{-13}Gev}{10^{15}Gev} = 10^{-28} \tag{3.5}$$

where we used that $T_e \simeq 10^{15} Gev$ [11]. From (3.3) we now get

$$(a_i H_i)^{-1} > 10^{28}(a_e H_e)^{-1} \tag{3.6}$$

which immediately leads to

$$N \equiv ln(\frac{a_e}{a_i}) > ln(10^{28}\frac{H_i}{H_e}) \simeq ln(10^{28}) \simeq 64 \tag{3.7}$$

where we defined $N$ to be the number of e-folds during inflation and we used the fact that $H$ is approximately constant during inflation. Equation (3.7) is the famous statement that the solution of the horizon problem requires about 60 e-folds of inflation, where it should be noted that it imposes only a lower bound on $N$, but 60 is the accepted number since it abides by observations as we will see later on.

## 3.3 Definition of Inflation

Inflation, as we saw before, refers to a fraction of a second where the comoving Hubble radius decreased with time. We can get more equivalent conditions for inflation to take place via the Friedmann equations

$$\frac{d}{dt}\left(\frac{H^{-1}}{a}\right) < 0 \quad \Rightarrow \quad \frac{d^2 a}{dt^2} > 0 \quad \Rightarrow \quad \rho + 3p < 0 \tag{3.8}$$

Requiring that the time derivative of $(aH)^{-1}$ is negative, and evaluating the time derivative to be

$$\frac{d}{dt}(aH)^{-1} = \frac{-\ddot{a}}{(aH)^2} \tag{3.9}$$

we see that it leads to the second equivalent condition which is $\ddot{a} > 0$. This explains why inflation is often referred to as a period of accelerated expansion of the universe. Now to get the third equivalent condition, we consult the Friedmann equation (2.10) (with units $8\pi G = c = 1$) which reads

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) \tag{3.10}$$

and imposing that $\ddot{a} > 0$ we immediately see

$$p < -\frac{1}{3}\rho \tag{3.11}$$

In other words, we need the "matter" during inflation to have negative pressure resulting in a "repulsive" gravitational force and thus explaining the accelerated expansion. One could ask about the basic mechanism in which one obtains these conditions from a physical theory, we will see that if a *scalar field* exists in the early universe it can act as an effective cosmological constant and can meet the inflationary conditions.

## 3.4 Inflation formalism

### 3.4.1 *A scalar field as a cosmological fluid*

For simplicity, consider a scalar field $\varphi$ present in the very early universe. This is the so-called *'inflaton' field* where the dynamics of this field is encoded in the action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - V(\varphi)\right] \tag{3.12}$$

where $L_\varphi = \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - V(\varphi)$ is the lagrangian of the inflaton field. The corresponding field equation for the inflaton is obtained via Euler-Lagrange equations (or varying with respect to the inflaton) and reads

$$\Box\varphi + \frac{dV}{d\varphi} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\varphi) + \frac{dV}{d\varphi} = 0 \tag{3.13}$$

where $\Box \equiv \nabla^\mu\nabla_\mu$ is the d'Alembertian operator. The energy-momentum tensor for the scalar field can be calculated from

$$T^{(\varphi)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta L_\varphi}{\delta g^{\mu\nu}} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\varphi\partial_\sigma\varphi + V(\varphi)\right) \tag{3.14}$$

Now recall the energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu} \tag{3.15}$$

and upon restricting the case to a homogeneous inflaton field $\varphi \equiv \varphi(t)$ and comparing (3.14) and (3.15) we get that the inflaton energy-momentum tensor is that of a perfect fluid provided that

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)\,, \tag{3.16}$$

$$p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)\,. \tag{3.17}$$

with the resulting equation of state

$$w_\phi \equiv \frac{p_\varphi}{\rho_\varphi} = \frac{\frac{1}{2}\dot{\varphi}^2 - V}{\frac{1}{2}\dot{\varphi}^2 + V} \tag{3.18}$$

which shows that the scalar field can lead to negative pressure and the condition of inflation ($w_\varphi < -\frac{1}{3}$) if $V$ dominates over kinetic energy $\frac{1}{2}\dot{\varphi}^2$. So a fourth equivalent condition for inflation to happen in this formalism is $V > \frac{1}{2}\dot{\varphi}^2$.
Now assuming the FRW geometry, equation (3.13) reduces to

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0 \tag{3.19}$$

which can also be attained if one substitutes equations (3.16) and (3.17) in (2.13). (3.19) is often referred to as the inflation equation. Let us further assume that the scalar field dominates the energy density during inflation and that its energy density is sufficiently large that we may neglect the curvature term in the Friedmann equation

$$H^2 = \frac{1}{3}\rho - \frac{k}{a^2}$$

(although ignoring the k term is not necessary since inflation will render it negligible very soon - flatness problem). Using (3.16), we then have

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) \tag{3.20}$$

Equations (3.19) and (3.20) provide a set of coupled differential equations in $\varphi$ and $H$ which determine completely the evolution of the inflaton field and the scale factor of the universe during the epoch of the inflaton domination (inflation).

It is worthy to focus on the form of the the inflation equation. If one thinks of the plot of the potential $V$ versus the inflaton $\varphi$ as defining some curve, then the motion of the inflaton is identical to that of a ball rolling (or sliding) under gravity along the curve, subject to a frictional force proportional to its speed and to the value of the Hubble parameter.

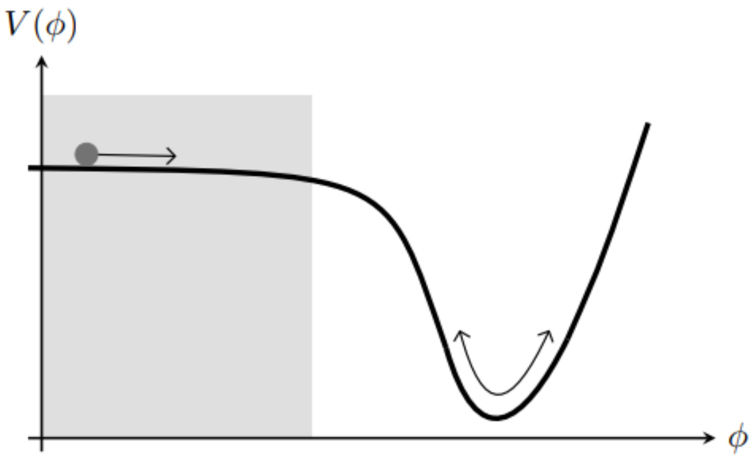


Figure 3.2: Example of some inflaton potential. Inflation occurs in the shaded part of the potential. It ends when the kinetic energy becomes comparable to the potential.

As can be seen, the shaded part represents the acceleration phase of the early universe. After inflation ends and the inflaton field reaches the minimum, the energy density of the inflaton is then converted into radiation via the process of reheating (look section 3.6) after several oscillations about the minimum and we gracefully exit into a radiation-dominated universe.

### 3.4.2 *The slow-roll approximation*

In general, an analytical solution for equations (3.19) and (3.20) is possible in the *slow-roll approximation* in which it is assumed that $\dot{\varphi}^2 << V(\varphi)$. On differentiating with respect to time this implies that $\ddot{\varphi} << \frac{dV}{d\varphi}$ and $\ddot{\varphi}$ can then be neglected in the inflation equation yielding

$$3H\dot{\varphi} = -\frac{dV}{d\varphi} \tag{3.21}$$

Moreover, equation (3.20) becomes

$$H^2 = \frac{1}{3}V(\varphi) \tag{3.22}$$

Differentiating (3.22) with respect to $t$ and using (3.21) we obtain

$$\dot{H} = -\frac{1}{2}\dot{\varphi}^2 \tag{3.23}$$

The conditions for inflation in the slow-roll approximation can be expressed in a dimensionless form. Using (3.21) and (3.22) it easy to show that

$$\epsilon \equiv \frac{1}{2}(\frac{V'}{V})^2 = -\frac{\dot{H}}{H^2} << 1 \tag{3.24}$$

where $V' \equiv \frac{dV}{d\varphi}$. Differentiating this with respect to $\varphi$ we find

$$\eta \equiv \frac{V''}{V} << 1 \tag{3.25}$$

We stressed above that inflation ends when kinetic energy becomes comparable to the potential, this condition can then be written as

$$\epsilon(\varphi_{end}) \simeq 1 \tag{3.26}$$

The above two conditions help us specify the shape of the potential and the rolling behavior. They require the potential to be sufficiently "flat" that the inflaton "rolls" slowly enough for inflation to occur. Before closing this section, it is worth noting that if $V$ is indeed sufficiently flat then it remains roughly a constant during inflation. Then using (3.22) we see that $H$ is also roughly constant and the scale factor grows *exponentially*

$$a(t) \propto e^{Ht} \tag{3.27}$$

This is why we refer to an *exponential* expansion of the universe during inflation. At last, we can express the number of e-folds during slow-roll as

$$\begin{aligned} N(\varphi) &\equiv \ln\frac{a_{\rm end}}{a} \\ &= \int_t^{t_{\rm end}} H dt = \int_{\varphi}^{\varphi_{\rm end}} \frac{H}{\dot{\varphi}} d\varphi \approx \int_{\phi_{\rm end}}^{\varphi} \frac{V}{V'} d\varphi\,, \end{aligned} \tag{3.28}$$

where we used the slow-roll results (3.21) and (3.22).

To check whether an inflationary model agrees with observations at observable scales, two observable quantities can be theoretically predicted, namely the perturbation tensor-to-scalar ratio and the spectral index. We know that these are related to the slow-roll parameters via [12]

$$n_s - 1 = -6\epsilon + 2\eta, \tag{3.29}$$

$$r = 16\epsilon \tag{3.30}$$

One can usually relate these to the number of e-folds $N$ and check for $N \simeq 60$ if $r$ and $n_s$ agree with observations. To demonstrate the idea, the following figure depicts the observational ranges for $r$ and $n_s$ for several inflationary models and shows how well they agree with observations:

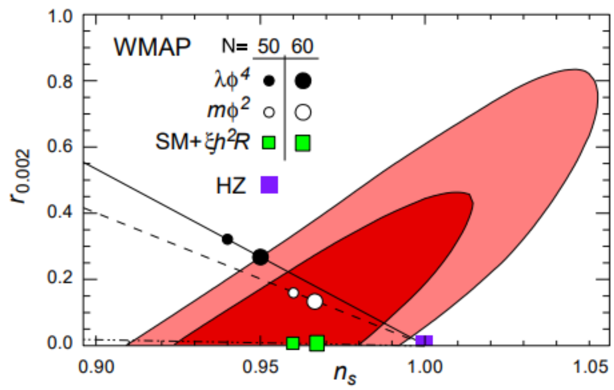


Figure 3.3: The allowed ranges for $r$ and $n_s$. The green boxes are the predictions of the Higgs model, the small box is for $N = 50$ e-folds and the bigger one corresponds to $N = 60$. Plotted are other models as well, HZ being Harrison-Zeldovich spectrum. We see that the higgs model and the $\phi^2$ model are in agreement with observations.

It should be noted that there are many inflationary models, this figure with the models considered only serves as a demonstration of the idea.

## 3.5 A (very) brief look at reheating

Most of the energy density during inflation is in the form of the inflaton potential $V(\varphi)$. Inflation ends when the potential steepens and the field picks up kinetic energy. The energy in the inflaton sector then has to be transferred to the particles of the Standard Model. This process is called **reheating** and starts the hot Big Bang [11]. After the inflaton field reaches the minimum, it begins to oscillate around it where we can model the potential as $V(\varphi) \simeq \frac{1}{2}m^2\varphi^2$. The corresponding equation of motion of the inflaton becomes:

$$\ddot{\varphi} + 3H\dot{\varphi} + m^2\varphi = 0$$

The energy density evolves according to the continuity equation (2.13):

$$\dot{\rho_\varphi} + 3H\rho_\varphi = -3Hp_\varphi = \frac{3}{2}H(m^2\varphi^2 - \dot{\varphi}^2)$$

where we have used eq.(3.17). Averaging over one oscillation, the right hand side becomes zero after ignoring the Hubble friction term in the inflaton's equation of motion. This is justified on timescales much shorter than the expansion time, i.e. $\frac{1}{m} \ll \frac{1}{H}$, and we get oscillating solutions from the equation of motion which then average to zero [11]. The solution of the above equation is then the averaged energy density going like $\rho_\varphi \propto a^{-3}$ (, behaving like pressureless matter or dust. As this energy density drops, the amplitude of the oscillations decreases.

To avoid that the universe ends up empty, the inflaton has to couple ot Standard Model fields. The energy stored in the inflaton will then transfer to ordinary particles via

$$\dot{\rho_\varphi} + 3H\rho_\varphi = -\Gamma_\varphi\rho_\varphi$$

where $\Gamma_\varphi$ is the inflaton decay rate [9] [13]. This decay rate depends on complicated and model-dependent physical processes that we will not review here, yet refer the interested reader to the review by Basset *et al.* [14] and [15] to delve more into the rich yet intricate reheating theories. Eventually the inflaton's energy density is converted into standard model degrees of freedom (the corresponding radiation energy) and the hot Big Bang commences. It is worthy to mention that we can also see that the inflaton behaves like dust at the end of inflation by noting that the equation of state parameter (3.18) is no longer close to $-1$ in the reheating stage yet more close to zero (dust-like), so that the universe has transitioned to an epoch of decelerated expansion.

## 3.6 Higgs/$R^2$ inflaionary models

### 3.6.1 *The Higgs boson as the inflaton*

Throughout the formalism, we did not specify the physical nature of the inflaton field $\varphi$, but simply used it as a parameter (or 'clock') to parameterize the time evolution of the inflationary energy-density. In the Standard Model (SM), there is a *unique* candidate for the inflaton, namely the Higgs boson. We will closely yet briefly follow the treatment of [16] here in setting up the higgs inflationary model. The basic idea is to couple the higgs field and gravity in a non-minimal way as apparent in the following action

$$S_J = \int d^4x\sqrt{-g}\left\{ -\frac{M^2 + \xi h^2}{2}R + \frac{\partial_\mu h \partial^\mu h}{2} - \frac{\lambda}{4}\left(h^2 - v^2\right)^2 \right\} \tag{3.31}$$

The first term describes the non-minimal coupling, $\xi$ and $\lambda$ are parameters of the theory, and $M \simeq M_p$ (planck scale mass). The third term is the mexican-hat potential which arises from the standard model and is responsible for the Higgs mechanism. The subscribt $J$ on the action means that we are in the so-called Jordan frame, i.e. the frame in which the scalar field (here higgs field) non-minimally couples to the Ricci scalar, as opposed to the Einstein frame in which the scalar field minimally couples to gravity and the action resembles that of the Einstein-Hilbert action. That being said, one can get rid of the non-minimal coupling by utilizing the conformal transformation from the Jordan to Einstein frame:

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \ , \quad \Omega^2 = 1 + \frac{\xi h^2}{M_P^2} \ . \tag{3.32}$$

This however leads to a non-minimal kinetic term of the Higgs. Thus it is convenient to introduce the change of variable to the new scalar field $\chi$

$$\frac{d\chi}{dh} = \sqrt{\frac{\Omega^2 + 6\xi^2 h^2/M_P^2}{\Omega^4}} \ . \tag{3.33}$$

and then we get the action in the Einstein frame

$$S_E = \int d^4x \sqrt{-\hat{g}} \left\{ -\frac{M_P^2}{2}\hat{R} + \frac{\partial_\mu \chi \partial^\mu \chi}{2} - U(\chi) \right\} , \tag{3.34}$$

where the hat denotes evaluation with respect to the metric $\hat{g}_{\mu\nu}$ and the potential is

$$U(\chi) = \frac{1}{\Omega(\chi)^4} \frac{\lambda}{4} \left( h(\chi)^2 - v^2 \right)^2 \ . \tag{3.35}$$

Now using (3.33), we see that for small higgs field values, $\frac{d\chi}{dh} = \frac{1}{\Omega}$ but $\Omega \simeq 1$ (from equation 3.32) hence $h \simeq \chi$. So $\chi$ and $h$ have the same potential in this limit. For large $h \gg M_P/\sqrt{\xi}$ (or $\chi \gg \sqrt{6} M_p$) we get using equation (3.32)

$$h \simeq \frac{M_P}{\sqrt{\xi}} \exp\left( \frac{\chi}{\sqrt{6} M_P} \right) \ . \tag{3.36}$$

Plugging this in (3.35) and noting that $\Omega^4 = \frac{\xi^2 h^4}{M^4}$ in this limit the potential becomes

$$U(\chi) = \frac{\lambda M_P^4}{4\xi^2} \left( 1 + \exp\left( -\frac{2\chi}{\sqrt{6} M_P} \right) \right)^{-2} \ . \tag{3.37}$$

Now the analysis of inflation in the Einstein frame can be performed as usual in the slow-roll approximation. The slow roll parameters take the form

$$\epsilon = \frac{M_P^2}{2} \left( \frac{dU/d\chi}{U} \right)^2 \simeq \frac{4 M_P^4}{3\xi^2 h^4} \ , \tag{3.38}$$

$$\eta = M_P^2 \frac{d^2U/d\chi^2}{U} \simeq -\frac{4 M_P^2}{3\xi h^2} \ , \tag{3.39}$$

where $M_p^2$ is set to one in equations (3.24) and (3.25). Using (3.26) we see that inflation ends when $h_{end} \simeq (\frac{4}{3})^{\frac{1}{4}} \frac{M_p}{\sqrt{\xi}}$. The number of e-folds is then (using 3.28)

$$N = \int_{h_{\rm end}}^{h} \frac{1}{M_P^2} \frac{U}{dU/d\chi} d\chi = \int_{h_{\rm end}}^{h} \frac{1}{M_P^2} \frac{U}{dU/dh} \left( \frac{d\chi}{dh} \right)^2 dh \simeq \frac{6}{8} \frac{h^2 - h_{\rm end}^2}{M_P^2/\xi} \ . \tag{3.40}$$

Plugging $h_{end} \simeq (\frac{4}{3})^{\frac{1}{4}}\frac{M_p}{\sqrt{\xi}}$ in (3.40) and solving for $h$ in terms of $N$, then plugging back in the formulas of $r$ and $n_s$ (3.29,3.30), we get for $N = 60$

$$n \simeq 1 - \frac{8(4N+9)}{(4N+3)^2} \simeq 0.9662 \tag{3.41}$$

$$r = 16\epsilon \simeq \frac{192}{(4N+3)^2} \simeq 0.0033 \tag{3.42}$$

which are well within the observational range as depicted by WMAP measurements in the figure (3.3). To that end, the higgs model gives an inflationary scenario which agrees well with observations, signifying that the Higgs boson is a legitimate candidate for the inflaton field in the early universe.

### 3.6.2 *The Starobinsky model*

There is yet another "equivalent" model to that of Higgs which was due to Starobinsky. We will not delve into the details of the model as it is equivalent to the model which we considered carefully above; one can refer to [17] [18] [19] for a detailed calculation. The model starts from an $R^2$ theory with the action

$$S = \frac{M_p^2}{2}\int d^4x\sqrt{-g}(R + \frac{R^2}{\mu^2}) \tag{3.43}$$

where $M_p^2$ denotes the mass scale of gravity and $\mu$ is a mass parameter depending on conformal field content. Another description of the same theory can be obtained by using conformal transformations and linearizing the $R^2$ term via the introduction of lagrange multiplier [18]. This enables us to cast the action into the Einstein-frame which would then have the form:

$$S = \int d^4x\sqrt{-\hat{g}}\left\{ -\frac{M_P^2}{2}\hat{R} + \frac{\partial_\mu\varphi\partial^\mu\varphi}{2} - V(\varphi)\right\}, \tag{3.44}$$

where $V(\varphi)$ is now the Starobinsky potential:

$$V(\varphi) = \frac{\mu^2 M_p^2}{8}(1 - e^{-\sqrt{\frac{2}{3}}\frac{\varphi}{M_p}}) \tag{3.45}$$

As a result, the inflationary scenario associated to this potential corresonding to an action in Einstein-frame or to an action including an $R^2$ term in Jordan-frame are referred to as Starobinsky inflation. The Higgs and Starobinsky models are therefore "equivalent" because as the higgs model, this model yields:

$$r \simeq \frac{12}{N^2} \simeq 0.0033;$$
$$n_s \simeq 1 - \frac{2}{N} \simeq 0.9666$$

for $N = 60$ [19]. The values of $r$ and $n_s$ are approximately the same and their behavior as a function of $N$ is $r \propto \frac{1}{N^2}$ and $n_s \propto 1 - \frac{1}{N}$ in both models, showing their equivalence. Another feature is that the potential of the transformed Higgs field for large field values can be approximated as (by using the binomial expansion on eq.(3.37))

$$U(\chi) \simeq 1 - 2e^{\frac{2\chi}{\sqrt{6}M_p}}$$

Now if one notes that for large $\chi$

$$(1 - e^{\frac{2\chi}{\sqrt{6}M_p}})^2 \simeq 1 - 2e^{\frac{2\chi}{\sqrt{6}M_p}}$$

then one can write equation (3.37) as

$$U(\chi) \simeq \frac{\lambda M_P^4}{4\xi^2}(1 - e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_p}})^2$$

which resembles the starobinsky potential. Hence the Higgs potential for the transformed field at large values is the same as Starobinsky's, adding another layer of resemblence or equivalence between the two inflationary models.

Among many models, Higgs inflation, or equivalently Starobinsky's inflation with an $R^2$ term attract special attention as they provide the best fit data to the astrophysical and cosmological observations [20] [21]. As such, we now seek to investigate if we can reproduce the Starobinsky model in a modified theory of gravity, namely Mimetic gravity. To address this, a treatment of mimetic gravity is due.

# Chapter 4

# Mimetic gravity

In 2013, Chamseddine and Mukhanov proposed a modified theory of gravity, Mimetic gravity [22] by decomposing the physical metric into an auxiliary metric and a scalar field. The resulting field equations have an additional term containing the first derivatives of the scalar field which suggests that it *mimics* dark matter even in the absence of actual matter. It then can be thought of as the velocity potential of dark matter as will be demonstrated in this chapter.

## 4.1 Equations of motion

Consider a parametrization of the physical metric in the following manner:

$$g_{\mu\nu} = (\widetilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)\widetilde{g}_{\mu\nu} \equiv P\widetilde{g}_{\mu\nu} \tag{4.1}$$

this is a conformal transformation as P is a scale factor. It also should be obvious that the physical metric is invariant under the conformal transformation of the auxiliary metric, that is:

$$\begin{aligned} \widetilde{g}_{\mu\nu} &\to \Omega^2\widetilde{g}_{\mu\nu} \\ \implies \widetilde{g}^{\mu\nu} &\to \frac{1}{\Omega^2}\widetilde{g}^{\mu\nu} \end{aligned}$$

replacing these in (4.1) we get $g_{\mu\nu} \to g_{\mu\nu}$.
Now one can proceed usually by setting up the action but should pay attention to the fact that the metric now is a function of the auxiliary metric and the scalar field:

$$S = -\frac{1}{2}\int_M \sqrt{-g(\widetilde{g}_{\mu\nu},\phi)}\left(R(g_{\mu\nu}(\widetilde{g}_{\mu\nu},\phi)) + L_{\text{matter}}\right) d^4x \tag{4.2}$$

Varying the action with respect to the metric gives

$$\delta S = \int_M d^4x \frac{\delta L_{\text{total}}}{\delta g_{\alpha\beta}}\delta g_{\alpha\beta} = -\frac{1}{2}\int_M d^4x\sqrt{-g}(G^{\alpha\beta} - T^{\alpha\beta})\delta g_{\alpha\beta} \tag{4.3}$$

where $L_{\text{total}}$ is the total lagrangian, i.e. containing the $R$ and $L_{\text{matter}}$ contributions, $G^{\mu\nu}$ is the Einstein tensor, and $T^{\mu\nu}$ is the energy-momentum tensor for the matter.

We notice now that the variation $\delta g_{\alpha\beta}$ must be written in terms of the variation of the auxiliary metric $\delta\widetilde{g}_{\alpha\beta}$ and $\delta\phi$

$$\delta g_{\alpha\beta} = P\delta\widetilde{g}_{\alpha\beta} + \widetilde{g}_{\alpha\beta}\delta P$$

where

$$\begin{aligned}\delta P &= \delta(\widetilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi) \\ &= \delta\widetilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + 2\widetilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \\ &= -\widetilde{g}^{\kappa\mu}\widetilde{g}^{\lambda\nu}\delta\widetilde{g}_{\mu\nu}\partial_\kappa\phi\partial_\lambda\phi + 2\widetilde{g}^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi\end{aligned}$$

where in the third line we used $\delta\widetilde{g}^{\alpha\beta} = -\widetilde{g}^{\alpha\mu}\widetilde{g}^{\alpha\nu}\delta\widetilde{g}_{\mu\nu}$ and relabeled indices by replacing $\alpha$ by $\kappa$ and $\beta$ by $\lambda$. Now we get

$$\begin{aligned}\delta g_{\alpha\beta} &= P\delta\widetilde{g}_{\alpha\beta} + \widetilde{g}_{\alpha\beta}(-\widetilde{g}^{\kappa\mu}\widetilde{g}^{\lambda\nu}\delta\widetilde{g}_{\mu\nu}\partial_\kappa\phi\partial_\lambda\phi + 2\widetilde{g}^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi) \\ &= P\widetilde{g}_{\mu\nu}\delta^\mu_\alpha\delta^\nu_\beta + \frac{1}{P}g_{\alpha\beta}(-P^2g^{\kappa\mu}g^{\lambda\nu}\widetilde{g}_{\mu\nu}\partial_\kappa\phi\partial_\lambda\phi + 2g_{\alpha\beta}Pg^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi)\end{aligned}$$

where in the second line we used $\widetilde{g}_{\mu\nu} = \frac{1}{P}g_{\mu\nu}$ and $\widetilde{g}^{\mu\nu} = Pg^{\mu\nu}$ (can be easily proved from (4.1)). Finally we get

$$\delta g_{\alpha\beta} = P\widetilde{g}_{\mu\nu}(\delta^\mu_\alpha\delta^\nu_\beta - g_{\alpha\beta}g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi) + 2g_{\alpha\beta}g^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi \tag{4.4}$$

which implies that

$$\delta S = -\frac{1}{2}\int_M d^4x\sqrt{-g}(G^{\alpha\beta}-T^{\alpha\beta})\times(P\widetilde{g}_{\mu\nu}(\delta^\mu_\alpha\delta^\nu_\beta - g_{\alpha\beta}g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi) + 2g_{\alpha\beta}g^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi) \tag{4.5}$$

and therefore we obtain the corresponding equations of motion where the modified Einstein field equations are

$$(G^{\mu\nu} - T^{\mu\nu}\ ) - (G - T\ )\, g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\partial_\beta\phi = 0, \tag{4.6}$$

and the continuity equation is

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,(G - T\ )\, g^{\mu\lambda}\partial_\lambda\phi\right) = \nabla_\mu\left((G - T\ )\,\partial^\mu\phi\right) = 0, \tag{4.7}$$

where $\nabla_\mu$ denotes the covariant derivative with respect to the metric $g_{\mu\nu}$. It is noteworthy to state that the scalar field appears explicitly in the equations while the auxiliary field does not as it appears implicitly only via the physical metric. As we noted above, using $\widetilde{g}_{\mu\nu} = \frac{1}{P}g_{\mu\nu}$ one can get the constraint equation for the scalar field:

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1. \tag{4.8}$$

If we take the trace of (4.7) we get

$$(G - T\ )\,(1 - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) = 0, \tag{4.9}$$

which is identically satisfied due to the constraint equation, even for non zero values of $(G - T)$. To work out the equations, one starts with an ansatz of the physical metric, get $\phi$ from the constraint equation, plug in the continuity equation and get $G - T$, then finally plug both back in the field equations to solve for the metric.

Later, it was demonstrated in [23] that an equivalent approach to derive the same set of equations is to begin with the action

$$S = \int_M \sqrt{-g}\,\left(R + \lambda(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + 1) + L_{\text{matter}}\right)\,d^4x \tag{4.10}$$

where $\lambda$ is a Lagrange multiplier. The modified field equations are obtained by varying with respect to the metric $g_{\mu\nu}$, the variation with respect to $\phi$ yields its continuity equation, and finally one gets the constraint equation upon varying with respect to $\lambda$. Then one identifies $\lambda$ with $G - T$ to get the exact equations above.

## 4.2 Physical interpretation of the equations

Since $(G - T)$ is generally non-zero, one can get more general solutions than in ordinary general relativity. To get the intuition behind this model, lets start by writing the field equations as follows:

$$G^{\mu\nu} = T^{\mu\nu}\ + \tilde{T}^{\mu\nu}, \tag{4.11}$$

where

$$\tilde{T}^{\mu\nu} = (G - T)\partial^\mu\phi\partial^\nu\phi \tag{4.12}$$

We now compare this expression with the stress energy tensor of a perfect fluid

$$T^{\mu\nu} = (\varepsilon + p)\,u^\mu u^\nu - pg^{\mu\nu} \tag{4.13}$$

where $\varepsilon$ is the energy density, $p$ is pressure and $u^\mu$ is four-velocity satisfying the normalization condition $u^\mu u_\mu = 1$. Upon making the following identification:

$$\varepsilon \equiv G - T, \quad u^\mu \equiv \partial^\mu\phi, \quad p = 0 \tag{4.14}$$

we get that the stress energy tensor of a pressure-less perfect fluid (dust) becomes equivalent to $\tilde{T}^{\mu\nu}$. Thus, we get a stress energy tensor in the modified equations which imitates the potential motion of "dust" (which is the standard model for cold dark matter) with energy density $G-T$ and velocity potential $\phi$. Hence we get "cold dark matter" without actually having dark matter present in a matter lagrangian in the thoery! In summary, even in the absence of matter, we get equations of the form $G^{\mu\nu} = \tilde{T}^{\mu\nu}$ meaning that the incorporation of dust in the theory is there without having to introduce actual dust in a matter lagrangian; hence we can say that dark matter is incorporated in the geometry of spacetime (via 4.1) instead of actual matter lagrangians. It is also easily seen now that with the identification above, the constraint equation for the scalar field plays the role of the normalization condition of the four-velocity.

Local conservation of energy and momentum in General relativity is automatically built in the theory via the *Bianchi identity*

$$\nabla^{\mu} G_{\mu\nu} = 0 \tag{4.15}$$

which immediately implies (through the field equations)

$$\nabla^{\mu} T_{\mu\nu} = 0 \tag{4.16}$$

In the Mimetic dark matter model, the Bianchi identity gives

$$\nabla^{\mu}(T_{\mu\nu} + (G - T)\partial_{\mu}\phi\partial_{\nu}\phi) = 0 \tag{4.17}$$

If we prove that the mimetic stress-energy tensor satisfies the conservation law, we can then conclude the conservation of energy and momentum for the matter stress-energy tensor. This is achieved by writing

$$\nabla^{\mu}((G - T)\partial_{\mu}\phi\partial_{\nu}\phi) = \nabla^{\mu}((G - T)\partial_{\mu}\phi)\partial_{\nu}\phi + (G - T)\partial_{\mu}\phi\nabla^{\mu}(\partial_{\nu}\phi) \tag{4.18}$$

and noting that the first term is zero due to the continuity equation, while the second is zero because the constraint equation gives (after differentiation)

$$\partial_{\mu}\phi\nabla^{\nu}(\partial_{\mu}\phi) = 0$$

and

$$\nabla^{\nu}\partial_{\mu}\phi = \nabla^{\mu}\partial_{\nu}\phi$$

via the symmetry of christoffel symbols.

## 4.3 Application to Cosmology

To see the implications of this theory on standard cosmology and find a solution, we work in the synchronous coordinate system where the metric takes the form

$$ds^2 = d\tau^2 - \gamma_{ij} dx^i dx^j \tag{4.19}$$

where $\gamma_{ij}$ is a three dimensional metric. Now if we take $\phi = t$ which means that the hypersurfaces of constant $\phi$ are the same of those of constant time, we satisfy the constraint equation. The continuity equation therefore becomes

$$\partial_0 \left( \sqrt{\det \gamma}\, (G - T\ ) \right) = 0 \tag{4.20}$$

which yields

$$G - T = \frac{C\left(x^i\right)}{\sqrt{\det\gamma}} \tag{4.21}$$

Now if we consider the Firedmann universe where we have $\gamma_{ij} = a^2\left(\tau\right)\delta_{ij}$, then this equation takes the form

$$G - T = \frac{C(x^i)}{a^3}$$

which is the exact density profile of dust at cosmological scales. Therefore, $G - T$ represents the density profile of dust in standard cosmology without having to resort to introduce matter densities in our theory, i.e. we have dark matter which is imitated by the extra degree of freedom of the gravitational field identified by the conformal mode. The constant of integration $C(x^i)$ determines the amount of this "mimetic" dark matter.

This modified theory of gravity has been expanded upon in several works. For example, if one modifies the action above to include a potential term depending on the mimetic field, it can be shown that one can produce quintessence, bouncing universes, and essentially any type of background cosmology one desires [24]. By further introducing another terms in the action it has been already shown that one can resolve singularities in the universe and blackholes [25] [26]. Cynlindrical solutions in mimetic gravity has been studied in [27], as well as static spherically symmetric spacetimes in [28]. Deriving the Tolman-Oppenheimer-Volkoff equations in mimetic gravity and studying the corresponding solutions for quark and neutron stars is found in [29]. Finally, one can refer to [30] for a review of developments and applications of this theory.

# Chapter 5

# A Mimetic Inflationary Scenario

## 5.1 Setting up the problem

The mimetic field introduced in chapter 4 reproduces the dark matter component of the universe. Recall that this mimetic field satisfies the constraint equation

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1 \tag{5.1}$$

and when applied to cosmology with the metric

$$ds^2 = dt^2 - \gamma_{ij}dx^i dx^j \tag{5.2}$$

we get $\phi = t$ as a solution for the constraint equation. In the synchronous coordinate system, the spacelike hypersurfaces are labeled by the time coordinate, and now they can be labeled by $\phi$. Moreover, one notices that in this coordinate system, the invariant quantity

$$\kappa \equiv \Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi \tag{5.3}$$

yields (using (1.19)):

$$\begin{aligned}
\kappa &= (\nabla_\mu\nabla^\mu\phi) \\
&= \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\phi\right) \\
&= \frac{1}{\sqrt{det(\gamma_{ik})}}\,\partial_0\left(\sqrt{det(\gamma_{ik})}\,g^{00}\,\partial_0\phi\right) \\
&= \frac{1}{\sqrt{det(\gamma_{ik})}}\,\frac{\partial}{\partial t}\left(\sqrt{det(\gamma_{ik})}\right) \\
&= \frac{1}{2det(\gamma_{ik})}\,\frac{\partial}{\partial t}\left(det(\gamma_{ik})\right)
\end{aligned}$$

which gives

$$\kappa = \frac{1}{2}\frac{\partial}{\partial t}\ln(det\gamma_{ik}) \tag{5.4}$$

Hence $\kappa$ represents the trace of the extrinsic curvature of the hypersurfaces with constant $\phi$ (the spacelike hypersurfaces). The extrinsic curvature quantifies the curvature of a submanifold embedded in a higher dimensional manifold. Thus it describes how a three dimensional hypersurface is curved within a four dimensional spacetime. More concretely, the extrinsic curvature measures how the normal vector to the hypersurface changes as one moves along the hypersurface. If the normal vector changes, we conclude that the hypersurface is curved in the higher dimensional spacetime, while if it is zero, then the hypersurface is flat.

In synchronous coordinates, this heavily simplifies because the hypersurfaces are spacelike hypersurfaces of constant t (and equivalently of constant $\phi$), hence the normal vector is simply $(1, 0, 0, 0)$. So if one thinks of a series of spacelike slices through spacetime evolving with $t$ the extrinsic curvature tells us how these slices are "bending" as time progresses.

As for the trace of the extrinsic curvature, it simply provides a measure of the average expansion or contraction rate of the volume of a spatial hypersurface embedded in a 4-dimensional spacetime. A positive $\kappa$ (trace of extrinsic curvature) says that the volume of the spacelike hypersurfaces is increasing with time, while a negative $\kappa$ signifies that the volume is decreasing. $\kappa = 0$ just states that the volume does not change without tackling the specific geometry as the volume might still be deforming according to what extrinsic curvature it has. We will see below that $\kappa = 3H = 3\frac{\dot{a}}{a}$ for a homogeneous isotropic universe indicating that the extrinsic curvature of our 3-d spacelike universe is positive, denoting that the expansion rate of our universe aligns with our intuition about $\kappa$.

Lets close this brief interlude and proceed with setting up the theory with the following action [31]:

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}R + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - C\left(\kappa\right)V(\varphi)\right] \tag{5.5}$$

where $8\pi G$ is set to be 1. The $R$ term constitutes usual GR, the $\lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1)$ term is the addition of mimetic gravity where $\phi$ is the mimetic field, the $\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$ term is the introduction of the inflaton field $\varphi$ which signifies that we'll work in an inflationary epoch, and the final term is a coupling of a function of $\kappa \equiv \Box\phi$ with the inflaton potential.
The introduction of the d'Alembertian of the mimetic field in the action stems from previous works where it has been used to resolve black hole and cosmological singularities [26] [32] [33] [25]. Here, we will use it investigate the Starobinsky model with linear coupling of $\kappa$ in mimetic gravity.

We proceed to get the equations of motion, the variation of the action with respect to the lagrange multiplier $\lambda$ leads to the constraint equation (5.1). Varying with respect to the metric yields the modified Einstein field equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \tag{5.6}$$

where

$$\begin{aligned} T_{\mu\nu} &= 2\lambda\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left(\left(C - \kappa C'\right)V - g^{\rho\sigma}\partial_\rho\left(C'V\right)\partial_\sigma\phi\right) \\ &+ \left(\partial_\mu\left(C'V\right)\partial_\nu\phi + \partial_\nu\left(C'V\right)\partial_\mu\phi\right) + \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\left(g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi\right) \end{aligned} \tag{5.7}$$

Note that the prime here denotes the derivative of the corresponding function with respect to its arguments, i.e., $C' \equiv dC/d\kappa$ and $V' \equiv dV/d\varphi$.
The variation with respect to the mimetic field $\phi$ yields the continuity equation:

$$\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\left(2\lambda\partial_\nu\phi + \partial_\nu\left(C'V\right)\right)\right] = 0 \tag{5.8}$$

or equivalently $\nabla_\mu(g^{\mu\nu}\left(2\lambda\partial_\nu\phi + \partial_\nu\left(C'V\right)\right) = 0$ upon multiplying both sides by $\frac{1}{\sqrt{-g}}$.
Finally, the variation with respect to the inflaton field $\varphi$ leads to the inflation equation:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi\right) + CV' = 0, \tag{5.9}$$

equivalently, $\Box\phi + CV' = 0$ (a modified Klein-Gordon equation).

## 5.2 Background Equations of Motion

Consider a homogeneous isotropic flat universe with the metric

$$ds^2 = dt^2 - a^2\left(t\right)\delta_{ik}dx^i dx^k. \tag{5.10}$$

This is saying that during our inflationary epoch, we are working in a flat universe which is homogeneous and isotropic. Inhomogeneities will be treated later on. In such a universe, the inflaton field is only a function of $t$, $\varphi = \varphi(t)$ and the solution to the constraint equation (5.10) is still $\phi = t$.
The $0-0$ Einstein equation becomes:

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R = T_{00}$$

For the (5.1) metric, I quote the Ricci tensor components and the Ricci scalar:

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{11} &= a\ddot{a} + 2\dot{a}^2 \\ R_{22} &= a\ddot{a} + 2\dot{a}^2 \\ R_{33} &= a\ddot{a} + 2\dot{a}^2 \\ \implies R = g^{\mu\nu}R_{\mu\nu} &= -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \end{aligned}$$

Therefore

$$G_{00} = 3\frac{\dot{a}^2}{a^2}$$

which is equal to $\frac{1}{3}\kappa^2$ since $\kappa = 3H$ where $H$ is the Hubble parameter:

$$\kappa = 3\frac{\dot{a}}{a} \tag{5.11}$$

this can be easily seen from

$$\begin{aligned}\kappa &= \frac{1}{2}\frac{\partial}{\partial t}\ln(det\gamma_{ik}) \\ &= \frac{1}{2}\frac{\partial}{\partial t}\ln(a^6) \\ &= 3\frac{\partial}{\partial t}\ln(a) \\ &= 3\frac{\dot{a}}{a}\end{aligned}$$

So the $0-0$ component of the Einstein tensor becomes

$$G_{00} = \frac{1}{3}\kappa^2$$

As for $T_{00}$, it evaluates to:

$$\begin{aligned}T_{00} &= 2\lambda\partial_0 t\partial_0 t + g_{00}\left(\left(C - \kappa C'\right)V - g^{00}\partial_0\left(C'V\right)\partial_0 t\right) \\ &+ \left(\partial_0\left(C'V\right)\partial_0 t + \partial_0\left(C'V\right)\partial_0 t\right) + \partial_0\varphi\partial_0\varphi - \frac{1}{2}g_{00}\left(g^{00}\partial_0\varphi\partial_0\varphi\right) \\ &= 2\lambda + [(C-\kappa C')V - \partial_0(C'V) + 2\partial_0\left(C'V\right) + \dot{\varphi}^2 - \frac{1}{2}\dot{\varphi}^2] \\ &= 2\lambda + (C-\kappa C')V + (\dot{C'V}) + \frac{1}{2}\dot{\varphi}^2\end{aligned}$$

and the $0-0$ Einstein equation yields

$$\frac{1}{3}\kappa^2 = 2\lambda + \left(C - \kappa C'\right)V + (\dot{C'V}) + \frac{1}{2}\dot{\varphi}^2, \tag{5.12}$$

The continuity equation (5.8) can be solved to give

$$\begin{aligned}&\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\left(2\lambda\partial_\nu\phi + \partial_\nu\left(C'V\right)\right)\right] = 0 \\ \Longrightarrow\ &\partial_0\left[\sqrt{a^6}g^{00}\left(2\lambda\partial_0 t + \partial_0\left(C'V\right)\right)\right] = 0 \\ \Longrightarrow\ &a^3(2\lambda + \partial_0(C'V)) = B(x^i) \\ \Longrightarrow\ &2\lambda = \frac{B(x^i)}{a^3} - (\dot{C'V})\end{aligned} \tag{5.13}$$

where B is an integration constant quantifying “mimetic dust” which we set to zero (after all, the amount of mimetic dust will decay rapidly during the exponential expansion where $a \propto e^t$ amidst inflation). Then the $0-0$ Einstein equation (5.12) simplifies to

$$\frac{1}{3}\kappa^2 = (C - \kappa C')\, V + \frac{1}{2}\dot{\varphi}^2, \tag{5.14}$$

The $i-i$ Einstein equations will not be independent equations and ultimately we need two independent equations which determine the evolution of $a(t)$ and $\varphi(t)$. We can choose the inflation equation to be the second independent equation, akin to what one does in cosmology where one can use the energy conservation equation alongside the $0-0$ Einstein equation, since it can be derived directly from the field equations.
The inflation equation (5.9) becomes:

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) + CV' = 0 \\ \implies &\frac{1}{a^3}\partial_0(a^3 g^{00}\partial_0\varphi) + CV' = 0 \\ \implies &\frac{1}{a^3}(3a^2\dot{a}\partial_0\varphi + a^3\partial_0^2\varphi) + CV' = 0 \end{aligned}$$

which yields

$$\ddot{\varphi} + \kappa\dot{\varphi} + CV' = 0. \tag{5.15}$$

As stated above, equations (5.14) and (5.15) are enough to determine the behavior of the scale factor $a(t)$ and the inflaton field $\varphi(t)$. After some algebra, one can also find the rate of change of the Hubble constant $H$ - or $\kappa$ since $\kappa = 3H$ - by taking the time derivative of equation (5.14) and using equation (5.15):

$$\dot{\kappa} = -\frac{3}{2}(\dot{\varphi}^2 + (C'\dot{V})) \tag{5.16}$$

or, equivalently,

$$\dot{\kappa} = -\frac{3}{2}\frac{\dot{\varphi}\,(\dot{\varphi} + C'V')}{1 + \frac{3}{2}C''V}. \tag{5.17}$$

## 5.3 The Model

### 5.3.1 *The inflationary epoch*

Lets take the function corresponding to the mimetic interactions to be

$$C(\kappa) = 1 + \frac{\kappa}{m}, \tag{5.18}$$

and assume that $m << 1$ which will be apparent later on. This linear choice of $C(\kappa)$ is motivated by Occam's razor principle, i.e. one tries the simplest possible choice and by this choice of $C$ the $(C - \kappa C')$ term in (5.14) becomes unity, hence a suitable choice to simplify the equations.
Let us also take the potential to be that of the Starobinsky model presented in section 3.6.2:

$$V(\varphi) = \frac{1}{2} m^2 M_p^2 (1 - e^{-\frac{\varphi}{M_p}})^2 \tag{5.19}$$

where $M_p \equiv (8\pi G)^{-\frac{1}{2}}$ is the Planck mass and $m$ is a small parameter to be constrained later on. The argument of the exponential henceforth is dimensionless as it should be, and we set $M_p = c = 1$ to be consistent with (5.6) yielding the potential:

$$V(\varphi) = \frac{1}{2} m^2 (1 - e^{-\varphi})^2$$

For small $\varphi < 1$, it describes a massive scalar field and we will check whether it allows for reheating after inflation (check section 5.3.2). The Starobinsky model first insures that inflation will be driven by the potential alone which is reminiscent of ordinary inflation, as the kinetic part will be much smaller than it (see below). It also ensures that the potential is 'flat enough' to support slow-roll behavior as it approaches its minimum at 0, where inflation would end, and the inflaton's energy density would then be converted to radiation, as will be discussed in section (5.3.2). The potential is always positive too.
$V'$ takes the form :

$$V'(\varphi) \equiv \frac{dV}{d\varphi} = m^2 e^{-\varphi}(1 - e^{-\varphi}) \tag{5.20}$$

The background equations of motion (5.14) and (5.15) become:

$$\kappa^2 = \frac{3}{2} m^2 (1 - e^{-\varphi})^2 + \frac{3}{2}\dot{\varphi}^2 \tag{5.21}$$

$$\ddot{\varphi} + \kappa\dot{\varphi} + \left(1 + \frac{\kappa}{m}\right)(m^2 e^{-\varphi}(1 - e^{-\varphi})) = 0 \tag{5.22}$$

Now we need to check that the modified slow-roll approximation is valid, i.e. if the second derivative in the inflation equation can be ignored. We start by assuming that we can neglect $\ddot{\varphi}$, in other words $|\ddot{\varphi}| \ll |\kappa\dot{\varphi}|$, and get the condition on $V$ such that this holds, then check if the potential model satisfies the condition. I should also note that the slow roll condition here is modified and is not the same as in ordinary inflation due to the mimetic-inflaton coupling.

Starting by ignoring $\ddot{\varphi}$, the inflation equation becomes

$$\kappa\dot{\varphi} + \left(1 + \frac{\kappa}{m}\right) V' = 0$$

$$\implies \dot{\varphi} = -(\frac{1}{\kappa} + \frac{1}{m})V' \simeq -\frac{1}{m} V'$$

because $\kappa \simeq m$ for $\varphi > 1$ (more precisely, $O(1)$) as can be seen from (5.21) (basically we assume slow roll works for $\varphi > 1$ and that $V$ dominates $\dot{\varphi}^2$ and check for self consistency). Now $\ddot{\varphi}$ yields

$$\begin{aligned}\ddot{\varphi} &\simeq -\frac{1}{m}(\dot{V'}) \\ &= -\frac{1}{m}V''(-\frac{1}{m}V') \\ &= \frac{1}{m^2}V'V''\end{aligned}$$

For a general V with $C$ specified by (5.18), the field equation is

$$\begin{aligned}\kappa &= \sqrt{3V + \frac{3}{2}\dot{\varphi}^2} \\ &= \sqrt{3V + \frac{3}{2}\frac{1}{m^2}V'^2}\end{aligned}$$

hence

$$|\kappa\dot{\varphi}| = \frac{1}{m}V'\sqrt{3V + \frac{3}{2}\frac{1}{m^2}V'^2}$$

therefore,

$$\begin{aligned}\frac{|\ddot{\varphi}|}{|\kappa\dot{\varphi}|} &= \frac{\frac{1}{m^2}V'V''}{\frac{1}{m}V'\sqrt{3V + \frac{3}{2}\frac{1}{m^2}V'^2}} \\ &= \frac{\frac{1}{m^2}V''}{\frac{1}{m}\sqrt{3V + \frac{3}{2}\frac{1}{m^2}V'^2}}\end{aligned}$$

To justify the approximation where we neglected $\ddot{\varphi}$, this ratio must be significantly less than 1. For our model, equations (5.19) and (5.20) provide $V$ and $V'$. Lets calculate $V''$ and plug it in the ratio above to check whether our model satisfies the modified slow-roll condition and, if so, identify the regions of $\varphi$ where this holds true:

$$V'' = m^2(e^{-2\varphi} - e^{-\varphi}(1 - e^{-\varphi}))$$

Substituting $V$, $V'$, and $V''$ above we get

$$\frac{|\ddot{\varphi}|}{|\kappa\dot{\varphi}|} = \frac{|(e^{-2\varphi} - e^{-\varphi}(1 - e^{-\varphi}))|}{\sqrt{\frac{3}{2}(1 - e^{-2\varphi})^2 + \frac{3}{2}e^{-2\varphi}(1 - e^{-\varphi})^2}}$$

which is indeed much less than 1 for $\varphi > 1$ (achieving self consistency) as shown by the plot below:

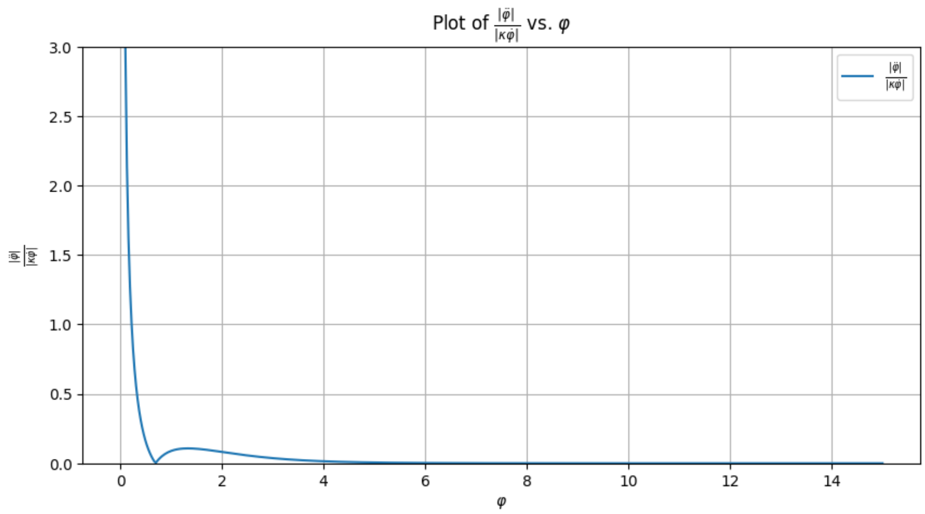


Figure 5.1: Determining the range for which the slow-roll approximation is valid. It can be seen that for $\varphi > O(1)$ the approximation is justified

It should be also noted that if we started from $\kappa = \sqrt{3V}$, i.e. demanded the slow roll condition holds and inflation is driven by the potential alone, then solved for the range for which this takes place, we would get the same result; here we were more careful by not neglecting the kinetic term in the derivation but we still deduce in section (5.3.3) that it can be neglected for this range above and inflation is driven by the potential alone, achieving self-consistency.
Therefore, we can apply the slow-roll approximation for our model when $\varphi > 1$ (when I say this I mean order 1, around 1), and the approximation will break down for $\varphi < 1$ which will mark the end of inflation and a conversion of the inflaton's density to radiation. This will be shown in section (5.3.2) below. We will only focus on the inflationary epoch here, and this begs the question, are we really in an inflationary epoch now? One can argue that the slow roll condition already suffices, but to strengthen our claim let us show that $\ddot{a} > 0$ which is the sufficient condition for inflation.
Recall that

$$\dot{H} + H^2 = \frac{\ddot{a}}{a}$$

In terms of $\kappa$ this yields:

$$\frac{1}{3}\dot{\kappa} + \frac{\kappa^2}{9} = \frac{\ddot{a}}{a} \tag{5.23}$$

using (5.16) we evaluate $\dot{\kappa}$

$$\begin{aligned}\dot{\kappa} &= -\frac{3}{2}(\dot{\varphi}^2 + (\dot{C'V})) \\ &= -\frac{3}{2}(\dot{\varphi}^2 + \frac{1}{m}\dot{V}) \\ &= -\frac{3}{2}(\dot{\varphi}^2 + \frac{1}{m}V'\dot{\varphi})\end{aligned}$$

where in the third line we used the chain rule on $V$. In the modified slow-roll approximation we got

$$\begin{aligned}\dot{\varphi} &\simeq -\frac{1}{m}V' \\ \implies V' &\simeq -m\dot{\varphi}\end{aligned}$$

and hence $\dot{\kappa}$ evaluates to

$$\dot{\kappa} \simeq -\frac{3}{2}(\dot{\varphi}^2 + \frac{1}{m}(-m\dot{\varphi}\dot{\varphi})) \simeq 0^-$$

and we immediately see that $\ddot{a} > 0$ from equation (5.23).

### 5.3.2 *Reheating phase*

After demonstrating that $\varphi > 1$ corresponds to an inflationary epoch, let us now examine whether a reheating phase occurs when the inflaton crosses $\varphi = 1$ from the right, signaling the transition to a decelerated expansion in mimetic gravity. After the inflaton reaches the minimum, the potential can be approximated as $V(\varphi) = \frac{1}{2}m^2\varphi^2$. The equation of motion of the inflaton becomes

$$\ddot{\varphi} + \kappa\dot{\varphi} + (1 + \frac{\kappa}{m})m^2\varphi = 0$$

If this yields a damped oscillating $\varphi(t)$, then we show that the inflaton is decaying towards the minimum and hence its energy density $\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$ decays too since the inflaton will have a decaying kinetic energy term and will oscillate with a decreasing amplitude to eventually settle at $V = 0$; then we can conclude that indeed an exit to a decelerated expansion occurs after the end of inflation (just like in section 3.5). Unlike in section 3.5, we are not able to get an analytical solution here due to the $\kappa = 3H$ term being time-dependent (decreasing after inflation), so we will investigate this numerically. Note that around the origin

$$\kappa^2 = 9H^2 \simeq \frac{3}{2}m^2\varphi^2 + \frac{3}{2}\dot{\varphi}^2$$

Substituting this into the inflaton eq. of motion we get

$$\ddot{\varphi} + (3\dot{\varphi} + 3m\varphi)\sqrt{\frac{1}{6}m^2\varphi^2 + \frac{1}{6}\dot{\varphi}^2} + m^2\varphi = 0$$

which can be studied using the phase space diagram. The behavior of the solutions in the $\varphi - \dot{\varphi}$ is shown in the figure below:

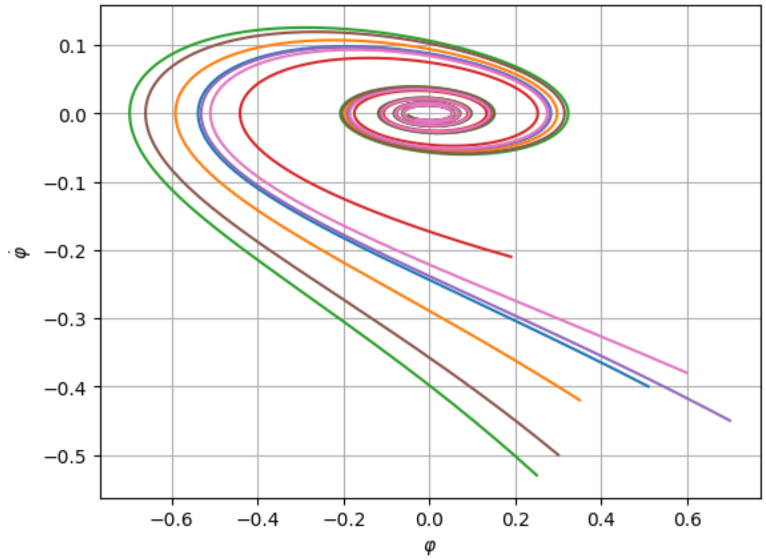


Figure 5.2: The $\varphi - \dot{\varphi}$ phase space. $(0, 0)$ is an attractor solution to which all solutions converge in time

As can be seen, we get an attractor at $(0, 0)$, meaning that the inflaton undergoes damped oscillations around the minimum at $\varphi = 0$ to eventually settle there in the end. Due to this ocillation with a decreasing kinetic term and amplitude, the inflaton's energy density $\rho_\varphi = \frac{1}{2}\dot{\varphi} + V(\varphi)$ is easily seen to decay when the inflaton converges to $\varphi = 0$ with $\dot{\varphi} = 0$ as seen from figure 5.2, hence $\rho_\varphi$ will eventually decay to zero (as $\dot{\varphi} = 0$ and $V(0) = 0$ at the minimum) transforming into the radiation energy density via conservation of energy (continuity equation). Thus we prove that a reheating phase is attainable in this theory and a graceful exit to radiation-dominated cosmology takes place.

### 5.3.3 *Potential-driven inflation*

At last, let us investigate whether inflation is driven solely from the potential or not. In our modified slow-roll approximation, $\dot{\varphi}$ becomes

$$\dot{\varphi} \simeq -\frac{1}{m}V' = -me^{-\varphi}(1 - e^{-\varphi}) \tag{5.24}$$

comparing $V$ and $\dot{\varphi}^2$ we can see that $V$ dominates for $\varphi > 1$ as shown in the figure below:

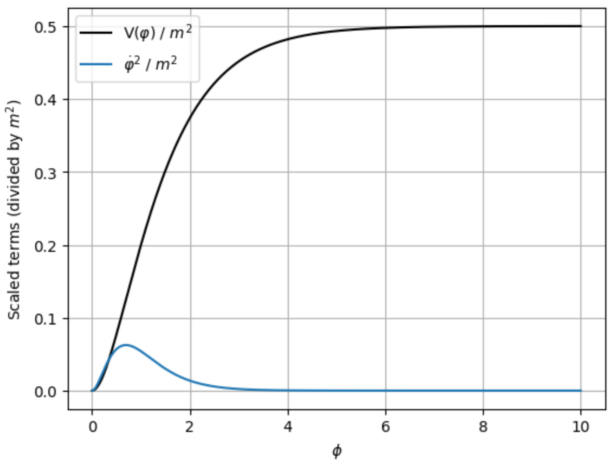


Figure 5.3: Potential dominates kinetic energy for $\varphi > 1$ meaning that $V$ is driving inflation in the slow-roll approximation. $m$ is a positive small parameter to be fixed later on in section 5.5, hence the plot is of the rescaled terms since both of them contain $m^2$ as a multiplicative factor

and hence in the slow roll approximation we get:

$$\kappa^2 = 3V = \frac{3}{2}m^2(1 - e^{-\varphi})^2 \tag{5.25}$$

and inflation is dominantly driven by the potential. This model is a bit reminiscent of ordinary inflation where inflation is also driven by the potential since the condition for inflation there can be restated as $\dot{\varphi}^2 < V(\varphi)$. One can also show now that the universe undergoes exponential expansion using (5.25) for the inflationary epoch by noting that our $V$ is roughly flat when inflation occurs and then $\kappa = 3H \simeq$ *constant* leading to $a \propto e^{\sqrt{V}t}$. Equations (5.24) and (5.25) are our main focus now during the inflationary epoch which is driven by the potential. We do not have the same relations (3.29) and (3.30) for observable quantities applied here since the perturbation theory in mimetic inflation will not be as that in ordinary inflation. So to see whether this inflationary scenario agrees with observations, a perturbative treatment of the equations of motion is due. Vacuum (quantum) fluctuations in the inflaton field in the early universe cause fluctuations in the corresponding energy density, which in turn cause fluctuations in the metric describing the otherwise flat, homogeneous, and isotropic universe. These fluctuations are the seeds of structure formation and utilizing them we can get observable quantities and check whether our model is realistic or not.

## 5.4 Cosmological Perturbations

Metric perturbations can be categorized into 3 distinct types: *scalar*, *vector*, and *tensor* perturbations. This classification is based on the symmetry properties of the

isotropic homogeneous background, which is obviously invariant with respect to the group of spatial rotations and translations at a given moment of time. [34]
The metric perturbations can be decomposed into these three irreducible pieces which evolve independently and therefore can be studied separately to linear order perturbations.
*Scalar perturbations* are induced by energy density inhomogeneities. These are the most important since they exhibit gravitational instability and may lead to structure formation in the universe.
*Vector perturbations* are related to the rotational motion of the fluid (matter). They decay very quickly hence they are not interesting from a cosmological point of view.
*Tensor perturbations* describe gravitational waves which are the gravitaional field's degrees of freedom.

### 5.4.1 *Scalar Perturbations*

We start with scalar perturbations. We reiterate that upon splitting the perturbed metric into the three categories, one can treat each split alone to linear order in perturbations.
For the scalar perturbations, the metric takes the form:

$$ds^2 = a^2[(1+2\Phi)d\eta^2 + 2\partial_i B dx^i d\eta - ((1-2\psi)\delta_{ij}dx^i dx^j - 2\partial_i\partial_j E)dx^i dx^j] \quad (5.26)$$

where $\eta$ is the conformal time:

$$d\eta = \frac{dt}{a(t)} \quad (5.27)$$

and $\Phi$, $\psi$, $B$, and $E$ are scalars. By choosing the conformal Newtonian gauge, the scalars $E$ and $B$ vanish and the metric reduces to:

$$ds^2 = (1+2\Phi)dt^2 - a^2(1-2\psi)\delta_{ij}dx^i dx^j \quad (5.28)$$

Since the spatial part of the energy momentum tensor satisfies $\delta T^i_j = \delta^i_j$ we have $\Phi = \psi$ [34] then the metric becomes:

$$ds^2 = (1+2\Phi)dt^2 - a^2(1-2\Phi)\delta_{ij}dx^i dx^j \quad (5.29)$$

Then we have a metric depending on a generalized "Newtonian" potential, hence the name Newtonian guage. Now we carry out the scalar perturbations in this mimetic inflationary scenario mostly following the treatment in [31]. To linear order in perturbations, the constraint equation (5.1) gives:

$$\dot{\delta\phi} = \Phi, \quad (5.30)$$

where $\delta\phi$ is the perturbation in the mimetic field. To get the perturbed $0-i$ Einstein equations, we note that

$$\delta T^0_i = (2\lambda + (\dot{C'V}))\delta\phi_{,i} + \delta\left(C'V\right)_{,i} + \dot{\varphi}\delta\varphi_{,i} \quad (5.31)$$

where the notation $_{,i}$ represents differentiation with respect to $i^{th}$ coordinate. The first term is zero via (5.13). After a long calculation one gets

$$\begin{aligned}\delta\left(C'V\right) &= C''V\delta\kappa + C'V'\delta\varphi \\ &= -3C''V\left(\dot{\Phi} + H\Phi + \frac{1}{3a^2}\Delta\delta\phi\right) + C'V'\delta\varphi,\end{aligned} \tag{5.32}$$

where (5.30) was used to express the gravitational potential in terms of derivatives of the perturbed mimetic field. Combining (5.31) and (5.32) with the perturbed $G^i_0$ components:

$$\delta G^i_0 = 2(\Phi + H\dot{\Phi})_{,i} \tag{5.33}$$

we get the perturbed $0-i$ Einstein equations

$$\begin{aligned}\dot{\Phi} + H\Phi &= \frac{1}{2\left(1+\frac{3}{2}C''V\right)}\left[\left(\dot{\varphi} + C'V'\right)\delta\varphi - \frac{C''V}{a^2}\Delta\delta\phi\right] \\ &= -\frac{\dot{H}}{\dot{\varphi}}\delta\varphi - \frac{C''V}{\left(2+3C''V\right)a^2}\Delta\delta\phi,\end{aligned} \tag{5.34}$$

where $\Delta \equiv \partial_i\partial_i$, the spatial laplacian, and in the second line equation (5.17) was used. The perturbed inflation equation (5.9) is

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} - \frac{1}{a^2}\Delta\delta\varphi - 4\dot{\varphi}\dot{\Phi} - 2\left(\ddot{\varphi} + 3H\dot{\varphi}\right)\Phi + \delta\left(CV'\right) = 0. \tag{5.35}$$

We evaluate $\delta(CV')$ to be:

$$\delta\left(CV'\right) = -3C'V'\left(\dot{\Phi} + H\Phi + \frac{1}{3a^2}\Delta\delta\phi\right) + CV''\delta\varphi. \tag{5.36}$$

Now we need to evaluate $CV''$, we take the time derivative of (5.15) and get:

$$CV'' = -\frac{1}{\dot{\varphi}}\left(\dddot{\varphi} + 3H\ddot{\varphi} + 3\dot{H}\dot{\varphi} + 3C'V'\dot{H}\right) \tag{5.37}$$

Substituting this into (5.36) then (5.36) into (5.35) and making use of (5.34), we can rewrite equation (5.35) as:

$$\begin{aligned}&\ddot{\delta\varphi} + 3H\dot{\delta\varphi} - \frac{1}{a^2}\Delta\left(\delta\varphi + \frac{2C'V' - 4\dot{\varphi}C''V}{2+3C''V}\delta\phi\right) \\ &-\left(\frac{\dddot{\varphi}}{\dot{\varphi}} + 3H\frac{\ddot{\varphi}}{\dot{\varphi}} - \dot{H}\right)\delta\varphi - 2\left(\ddot{\varphi} + H\dot{\varphi}\right)\Phi = 0.\end{aligned} \tag{5.38}$$

Above we mentioned that $\delta T^i_j = \delta^i_j$ and hence the $i-j$ equations contribute nothing for $i \neq j$. Also upon inspection, the $0-0$ and $i-i$ perturbed equations provide nothing new. Substituting the expression for $\delta\varphi$ in terms of $\Phi$ and $\delta\phi$ from (5.34) into $i-i$ Einstein equation, we find that it is identically satisfied. The $0-0$ Einstein equations can determine $\delta\lambda$ which has no interest to us. Hence the three equations (5.30), (5.34), and (5.38) are a complete set to determine the unknowns $\delta\varphi$, $\Phi$ and $\delta\phi$.

As it is the case with such perturbations, we go to Fourier space now and consider a plane wave with co-moving wave-number $k$:

$$\begin{aligned} \delta\varphi &\propto \int \delta\varphi_k e^{i\overrightarrow{k}.\overrightarrow{x}} d^3\overrightarrow{k} \\ \Phi &\propto \int \delta\Phi_k e^{i\overrightarrow{k}.\overrightarrow{x}} d^3\overrightarrow{k} \\ \delta\phi &\propto \int \delta\phi_k e^{i\overrightarrow{k}.\overrightarrow{x}} d^3\overrightarrow{k} \end{aligned} \tag{5.39}$$

Analyzing the behavior of these modes depends on whether the physical wavelength $\lambda_{ph}$ is much smaller or larger than the Hubble scale $H^{-1}$. For short-wavelength perturbations, $\lambda_{ph} \ll H^{-1}$ or $k \gg Ha$, (5.38) is simplified to:

$$\delta\ddot{\varphi}_k + 3H\delta\dot{\varphi}_k + \frac{k^2}{a^2}\delta\varphi_k \simeq 0, \tag{5.40}$$

which has the solution

$$\delta\varphi_k \simeq \frac{A_k}{a}\exp\left(\pm ik\int\frac{dt}{a}\right), \tag{5.41}$$

where $A_k$ is a constant of integration. One can refer to [31] for verification that in this limit the skipped terms in (5.38) are really negligible. $\Phi$ and $\delta\phi$ also oscillate in this limit.

As for the other limit, i.e. where the inhomogeneity crosses the Hubble scale at the time determined by the equation $k \sim H_k a_k$, the spatial derivatives terms decay as $1/a^2$ and the Laplacian in (5.34) and (5.38) can be neglected resulting in

$$\begin{aligned} &\dot{\Phi}_k + H\Phi_k \simeq -\frac{\dot{H}}{\dot{\varphi}_k}\delta\varphi_k \\ &\delta\ddot{\varphi}_k + 3H\delta\dot{\varphi}_k - \left(\frac{\dddot{\varphi}}{\dot{\varphi}} + 3H\frac{\ddot{\varphi}}{\dot{\varphi}} - \dot{H}\right)\delta\varphi_k - 2\left(\ddot{\varphi} + H\dot{\varphi}\right)\Phi \simeq 0, \end{aligned} \tag{5.42}$$

for $k \ll Ha$. By direct substitution, one can prove that the exact solutions for these equations are:

$$\delta\varphi_k = A\frac{\dot{\varphi}}{a}\int a dt, \quad \Phi_k = A\frac{d}{dt}\left(\frac{1}{a}\int a dt\right), \tag{5.43}$$

where $A$ is an integration constant, and as follows from (5.30), the perturbation of the mimetic field is

$$\delta\phi_k = A\frac{1}{a}\int a dt. \tag{5.44}$$

These are the solutions we will use to get the scalar perturbation's power spectrum. It should be noted that we can use the solutions of the superhorizon equations to

conclude an observed quantity after inflation ends due to pertubations “freezing out” after horizon exit. These frozen out perturbations are then treated as initial conditions when they re-enter the horizon, and we can solve for them based on observations nowadays; this is how we can relate measurements happening now to primordial perturbations value at the end of inflation! In intuitive terms, we can think of the perturbation no longer able to feel its own self-gravity at superhorizon scales, since it is larger than the characteristic scale over which the physical processes in the universe operate coherently [1]. Therefore the solutions at superhorizon scales (before re-entry) are conserved (till re-entry) and hence one can conclude the value of the perturbations at the horizon crossing (essentially at end of inflation too) using the super horizon solutions.
I should note that there are proofs that the gauge invariant perturbations freeze out, but in our case this perturbation leads to the gravitational potential perturbation when we choose the Newtonian gauge, hence this also freezes out in this gauge. One can refer to [9] [13] to check the proof of the conservation of the gauge-invariant curvature perturbation on super-horizon scales.

For superhorizon scales, the solution is (5.43) which we will evaluate now. Lets start by evaluating $\frac{1}{a}\int adt$; we know that $-\dot{H} \ll H^2$ during inflation from eq. (3.24) which still holds in our case if one inspects eqs. (5.16) and (5.25), the integral can then be approximated as:

$$\frac{1}{a}\int adt = \frac{1}{a}\int \frac{da}{H} = \frac{1}{H}\left(1 - (-\frac{\dot{H}}{H^2}) + ...\right) + \frac{D}{a} \simeq \frac{1}{H}, \tag{5.45}$$

This can be achieved using multiple integration by parts steps, and we neglected the $\frac{D}{a}$ term since it decays exponentially fast. For the inflaton perturbation with co-moving wavenumber k, we know that the amplitude of the typical quantum fluctuations is $\sqrt{\delta\varphi_k^2 k^3} \simeq H_{k=Ha}$ at time $t_k$ [34] [9] and hence one can easily see using the first equation in (5.43) that the integration constant $A$ is

$$A \simeq \left(\frac{H^2}{|\dot{\varphi}|}\right)_{k=Ha}, \tag{5.46}$$

where the subscript means that this has to be evaluated at the horizon crossing when the perturbation got stretched out during inflation and crossed the horizon. It follows from the second equation in (5.43) that

$$\Phi_k \simeq \left(\frac{H^2}{|\dot{\varphi}|}\right)_{k=Ha} \frac{d}{dt}\left(\frac{1}{a}\int adt\right). \tag{5.47}$$

Since we want to get the perturbation spectrum after inflation ends (or more accurately after they leave the Hubble scale during inflation), we note first that $\frac{d}{dt}\left(\frac{1}{a}\int adt\right)$ is of order one after the end of inflation; one can easily compute this for radiation-dominated or matter dominated universes with their corresponding $a(t)$.

The after-inflation spectrum of gravitational potential $\delta_\Phi$ for perturbations that left the Hubble scale during inflation therefore is defined by

$$< \Phi_k \Phi_{k'} >= \frac{2\pi^2}{k^3}\delta_\Phi^2 \delta^{(3)}(\overrightarrow{k} - \overrightarrow{k}')$$
$$\implies \delta_\Phi \equiv \sqrt{\Phi_k^2 k^3}$$

which gives

$$\delta_\Phi \simeq \left(\frac{H^2}{|\dot{\varphi}|}\right)_{k=Ha}. \tag{5.48}$$

### 5.4.2 *Tensor Perturbations*

The consideration of tensor perturbations (gravitational waves) in mimetic inflation is exactly the same as in ordinary inflation. This can be seen because the gravitational waves equation has the same solutions in mimetic gravity and ordinary GR, also the mimetic stress energy tensor cannot be decomposed into true tensor perturbations, i.e. the perturbations that come from it are due to a scalar field (mimetic field) and then are considered scalar perturbations. We simply quote the result for the tensor perturbations spectrum [34]:

$$\delta_h \simeq H_{k=Ha}. \tag{5.49}$$

Equations (5.48) and (5.49) will constitute the basis section to compute the observable quantities.

## 5.5 Observable quantities

Recall that inflation in our model is driven solely by the potential and in the slow roll approximation we get equations (5.24) and (5.25). Plugging those in the spectrum of the scalar perturbation (5.48) we get (upon noticing that $H^2 = \frac{1}{9}\kappa^2 = \frac{1}{3}V$)

$$\begin{aligned}\delta_\Phi &\simeq \left(\frac{H^2}{|\dot{\varphi}|}\right)_{k=Ha}\\ &\simeq \left(\frac{\frac{m^2}{6}(1-e^{-\varphi})^2}{me^{-\varphi}(1-e^{-\varphi})}\right)_{k=Ha}\\ &\simeq \left(\frac{m}{6}e^{\varphi}(1-e^{-\varphi})\right)_{k=Ha}\\ &\simeq \left(\frac{m}{6}(e^{\varphi}-1)\right)_{k=Ha}\end{aligned} \tag{5.50}$$

To be able to relate $\varphi_k$ to the number of e-folds, we note that the number of e-folds with a given $k$ such that the perturbation starts to exceed the Hubble scale ($k = Ha$) is

$$N_k \simeq \ln\left(\frac{Ha_f}{k}\right) \tag{5.51}$$

which one can easily derive from the equation defining the number of e-folds, i.e. $a \simeq a_f e^{-N}$.We know that for the scales covered by the CMB observations the range of $N_k$ is between 50 and 60. We also note the following relation (see 3.28):

$$H = -\frac{dN}{d\varphi}\dot{\varphi}, \tag{5.52}$$

from where it follows that

$$\begin{aligned} N &= -\int \frac{H}{\dot{\varphi}} d\varphi \\ &\simeq \int \frac{\frac{m}{\sqrt{6}}(1-e^{-\varphi})}{me^{-\varphi}(1-e^{-\varphi})} d\varphi \\ &\simeq \frac{1}{\sqrt{6}} \int e^{\varphi} d\varphi \\ &\simeq \frac{1}{\sqrt{6}} e^{\varphi} \end{aligned} \tag{5.53}$$

Hence $N_k \simeq \frac{1}{\sqrt{6}} e^{\varphi_k}$. Inverted, this yields

$$\varphi_k = ln(\sqrt{6}N_k) \tag{5.54}$$

and now we write (5.50) in terms of the number of e-folds

$$\delta_\Phi \simeq \frac{m}{6}(\sqrt{6}N_k - 1) \tag{5.55}$$

To obtain the correct amplitude at observable scales which corresponds to $\delta_\Phi$ being of order $10^{-5}$ at the pivot scale $k_p = 0.05Mpc^{-1}$ [10], we must take $m \simeq 10^{-6}$ since $N_k \simeq 60$ at observable scales.
Lets get the tensor perturbation spectrum now. Since it is of order $H$ as seen from equation (5.49), we can simply get it from the equation $H = \frac{1}{\sqrt{3}}\sqrt{V}$ or from equation (5.52), both yielding:

$$\delta_h \simeq H \simeq \frac{m}{\sqrt{6}}(1 - e^{-\varphi}) \tag{5.56}$$

which satisfies the condition $\delta_h < \delta_\Phi$ (from observations).
Now we get the observable quantities. We start by evaluating the spectral index

$$\begin{aligned} n_s - 1 &\equiv \frac{d\ln\delta_\Phi^2}{d\ln k} \\ &= -\frac{d\ln\delta_\Phi^2}{dN_k} \\ &\simeq -\frac{d}{dN_k}(2ln(\frac{1}{6}m(\sqrt{6}N_k - 1))) \\ &\simeq -\frac{2\frac{\sqrt{6}}{6}m}{\frac{1}{6}m(\sqrt{6}N_k - 1)} \\ &\simeq -\frac{2\sqrt{6}}{\sqrt{6}N_k - 1} \end{aligned} \tag{5.57}$$

where in the second equality we used (5.51). For $N_k = (50, 60)$, $n_s = (0.96, 0.9664)$, in agreement with observations by referring to figures (3.3) and (5.4) [12].

As for the tensor-to-scalar ratio, it evaluates to

$$r \equiv \frac{\delta_h^2}{\delta_\Phi^2} \simeq \frac{\frac{m^2}{6}(1-e^{-\varphi})^2}{\frac{m^2}{6^2}e^{2\varphi}(1-e^{-\varphi})^2} \simeq \frac{1}{N_k^2} \tag{5.58}$$

This is also in agreement with observations as for $N_k = (50, 60)$, $r = (4\times10^{-4}, 2.777\times 10^{-4})$ agrees with the observational upper bound on it as depicted by the figures (3.3) and (5.4).

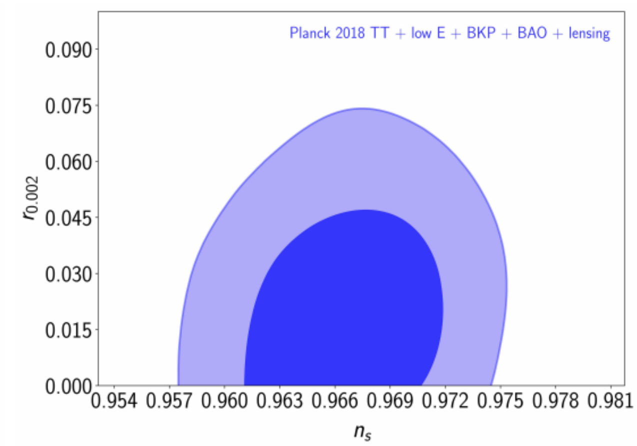


Figure 5.4: 2D probability constraints on $n_s$ and $r$ for the Planck dataset (2018). These are plotted with $1\sigma$ and $2\sigma$ confidence contours

A summary of the $n_s$ and $r$ constraints from different measurements are referred to in the following table [12]:

| Parameter | Limits | Data set |
|---|---|---|
| $n_s$<br>$r$ | $0.9661 \pm 0.0037$<br>$< 0.065$ | Planck 2018 TT + low E + BKP<br>+ BAO + lensing |
| $n_s$<br>$r$ | $0.9683 \pm 0.0059$<br>$< 0.0660$ | Planck TT + lowP + lowTE + BKP<br>+ BAO + lensing |
| $n_s$<br>$r$ | $0.9666 \pm 0.0062$<br>$< 0.103$ | Planck TT+lowP |
| $n_s$<br>$r$ | $0.9711 \pm 0.0099$<br>$< 0.17$ | SPT+WMAP7+BAO+$H_0$ |
| $n_s$<br>$r$ | $0.970 \pm 0.012$<br>$< 0.19$ | ACT+WMAP7+BAO+$H_0$ |
| $n_s$<br>$r$ | $0.973 \pm 0.014$<br>$< 0.24$ | WMAP7 + BAO +$H_0$ |
| $n_s$<br>$r$ | $0.982\pm^{+0.020}_{-0.019}$<br>$< 0.36$ | WMAP7 ONLY |
| $n_s$<br>$r$ | $0.968 \pm 0.015$<br>$< 0.22$ | WMAP5+BAO+SN |
| $n_s$<br>$r$ | $0.986 \pm 0.022$<br>$< 0.43$ | WMAP5 ONLY |
| $n_s$<br>$r$ | $0.97 \pm 0.04$<br>$< 0.31$ | WMAP3 + SDSS |
| $n_s$<br>$r$ | $0.99 \pm 0.05$<br>$< 0.60$ | WMAP3 ONLY |

Figure 5.5: Summary of the $n_s$,$r$ constraints from various measurements.

Finally, we got to demonstrate that our mimetic inflationary model produced an inflationary scenario which agrees with observations at observable scales. On top of that, even for non-zero lagrange multiplier, the Starobinsky potential model is preserved in the mimetic setting with the choice of $C(\kappa)$ to be linear in $\kappa$. In other words, we were able to find a specific (linear) coupling of a function of the d'Alembertian of the mimetic field with the Starobinsky potential so that the results of the $R^2$ inflationary model are preserved in a mimetic setting ($r \propto \frac{1}{N^2}$ and $n_s \propto 1-\frac{1}{N}$). The success of the Starobinsky model can now be traced back to its $R^2$ formalism or the mimetic framework. Our analysis shows that the model we considered is in full agreement with both the Starobinsky/$R^2$ and Higgs models, suggesting that the presence of the mimetic field naturally gives rise to the "mimicking" of certain physical phenomena! Ultimately, this connection underscores the versatility and robustness of the mimetic framework in capturing essential aspects of these models.

A cool feature to shine the light upon is the fact than we can relate measurements nowadays to primordial perturbation spectra, how so? The key point to note is that via inflation, super horizon scale fluctuations in the early universe were produced. These later on provide the seeds of galaxy formation and perturbations in the CMB that we observe today. By studying the distribution of galaxies and CMB perturbations as a function of scale, we can obtain an idea about the underlying primordial spectrum of perturbations that produced them (at the end of inflation, which are then treated as initial conditions of the matter perturbations we see now). Thus, by predicting this primordial spectrum, we can relate it to an inflationary scenario which happened about 14 billion years ago![1] [10]

For the meticulous reader, they might have an issue with the exponentially growing scalar perturbation. I should stress that this model works for *observable scales*, for which the scalar perturbation is much less than one. For example if inflation ends at $\varphi = 1$, one could solve for what $\varphi$ it started by demanding $N = 60$ in the expression of $N(\varphi)$ if one evaluates the definite integral for the range $[\varphi_{start}, \varphi_{end}]$. For our model specifically, we get that $\varphi_{start} \simeq 5$; and we can then see that for our observable universe the range from which inflation started till it ended is small. The scalar perturbation in this range remains very small too, hence there should be no problem at all at observable scales.
As to non observable scales, i.e. the parts of the universe where inflation started at a much bigger $\varphi$, the number of e-folds would be much larger than 60 and our analysis might need to be reformulated to take care of these regions to ensure that the perturbations are still relatively small up to a planck scale or sub planckian scale where we impose the maximum inflaton value (farthest event in the past corresponding to the planck scale). If done properly, we get a theory suggesting that the tensor-to-scalar ratio goes like $\frac{1}{N^2}$ for all scales with no issues in regards of perturbations! This off course cannot be confirmed observationally, what can be confirmed are properties related to the observable scales. It should be noted that this might be resolved if one uses higher order perturbation theory (above linear) as one cannot use only linear order if the perturbations are not small anymore.

# Chapter 6

# Conclusion and Future work

In this thesis, a review on GR, cosmology, inflation, and mimetic gravity was presented. Then we studied a mimetic inflationary scenario based on the coupling of the d'Alembertian of the mimetic field with the inflaton potential. After getting the background equation of motion, we applied the modified slow-roll approximation for the inflationary epoch and proposed to investigate the Starobinsky potential model. To check whether this model produces an inflationary scenario which agrees with current observations, a pertubative treatment was revised where we got the solution for the scalar and tensor perturbations at super-horizon scales. Using our model, the observable quantities (namely the spectral index and the tensor to scalar ratio) were in agreement with observations.

As for future work, we stressed a bit on the difficulty which arises due to the analysis concerning the non-observable scales. Our model did not address the eternal inflation issue for some unobservable scales (practically when our scalar peturbation starts exceeding 1). Many models also do not avoid this "issue". I refer to it as an issue rather than a problem because, in my view, it is not fundamentally significant. This is due to the fact that one cannot observationally check or verify if there are regions undergoing self-reproduction (eternal inflation) since they lie within the range of unobservable scales.
For people who would like to get rid of this issue and hence make their theory more "elegant", a proposition of a small modification is due. The idea is to add more parameters to be constrained by the theory later on so one can get to a planck scale which could be determined via $H \simeq 1$ [31]. For such a scale, one demands that the analysis starts there (which should correspond to the largest $\varphi$), then one needs to check that the tensor perturbation is less than the scalar perturbation, which in turn should be less than one for all scales below the planck scale (i.e. for all $\varphi$ values below $\varphi_{planck}$). For example, a model worth investigating would be

$$V = m^2\beta^2(1 - e^{-\frac{\varphi}{\sqrt{\beta}}})^2$$

My idea is to firstly constrain $m^2\beta^2$ to be of order 1, which would lead to $\beta$ being a large number, and then to stretch out the function via the $\frac{1}{\sqrt{\beta}}$ in the exponential so one can reach a "*planck scale*" at a relatively large $\varphi$. Note that this is merely a suggestion to clarify the idea. One could explore alternative forms of the potential or increase the number of parameters to allow for greater flexibility when constraining them within the framework of the theory. One can even modify the choice of $C(\kappa)$, but I believe this will make the analysis considerably harder. If these parameters can be adjusted such that the ratio $r$ and the spectral index $n_s$ align well with observations at observable scales (determined by solving for $N = 60$ and identifying where inflation ends), and if the tensor perturbation remains smaller than the scalar perturbation, which itself must stay below 1 across all scales up to the Planck scale, then the issue of self-reproduction would be resolved. Readers are encouraged to refer to [31] [35] for a more in-depth understanding of self-reproducing universes and how they can be avoided.

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