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Stellar dynamics around a massive black hole – II. Resonant relaxation

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ABSTRACT

We present a first-principles theory of resonant relaxation (RR) of a low-mass stellar system orbiting a more massive black hole (MBH). We first extend the kinetic theory of Gilbert to include the Keplerian field of a black hole of mass $M_\bullet$. Specializing to a Keplerian stellar system of mass $M \ll M_\bullet$, we use the orbit-averaging method of Sridhar & Touma to derive a kinetic equation for RR. This describes the collisional evolution of a system of $N \gg 1$ Gaussian rings in a reduced 5-dim space, under the combined actions of self-gravity, 1 post-Newtonian (PN) and 1.5 PN relativistic effects of the MBH and an arbitrary external potential. In general geometries, RR is driven by both apsidal and nodal resonances, so the distinction between scalar RR and vector RR disappears. The system passes through a sequence of quasi-steady secular collisionless equilibria, driven by irreversible two-ring correlations that accrue through gravitational interactions, both direct and collective. This correlation function is related to a 'wake function', which is the linear response of the system to the perturbation of a chosen ring. The wake function is easier to appreciate, and satisfies a simpler equation, than the correlation function. We discuss general implications for the interplay of secular dynamics and non-equilibrium statistical mechanics in the evolution of Keplerian stellar systems towards secular thermodynamic equilibria, and set the stage for applications to the RR of axisymmetric discs in Paper III.

Key words: Galaxy: centre – galaxies: kinematics and dynamics – galaxies: nuclei.

1 INTRODUCTION

Star clusters around massive black holes (MBHs) in galactic nuclei are dense stellar systems in which stellar orbital motions are fast enough for relaxation processes to have significantly modified their structure. Relaxation is driven by gravitational 'collisions' between stars, leading to exchanges of orbital energies and angular momenta. Classical two-body relaxation results in the diffusion of both energy and angular momentum, over the two-body relaxation time-scale $T_{\rm relax}$ (Chandrasekhar 1942, 1943a,b; Binney & Tremaine 2008). In its fully developed form, two-body relaxation can feed the MBH with stars, and lead to the formation of a stellar density cusp around the MBH (Bahcall & Wolf 1976; Cohn & Kulsrud 1978). But for the stellar systems observed in galactic nuclei, $T_{\rm relax}$ can exceed the Hubble time; then two-body relaxation would have had negligible effect. Rauch & Tremaine (1996, hereafter RT96) proposed a more efficient mechanism, *resonant relaxation* (RR), that could be in operation in stellar systems with globally degenerate (or resonant) orbital frequencies. RR does not enhance the energy relaxation rate but promotes more efficient relaxation of orbital angular momentum distribution. When this happens in a low-mass stellar system orbiting an MBH, stars whose orbits have become sufficiently eccentric can fall into the MBH, feeding it mass, energy and angular momentum. RR is particularly important in a low-mass stellar system around an MBH, and is a result of the degeneracy of the Kepler problem. As the radial and azimuthal periods of a Keplerian ellipse are equal, orbital precession is suppressed; only the smaller non-Keplerian forces, such as cluster self-gravity, general relativistic effects and external gravitational sources contribute to orbital precession. This gives rise to a pattern of interactions between Keplerian orbits which persists over the longer time-scales of precession. It is this extended period of interplay, or coherence, between Keplerian orbits that makes RR more efficient than classical two-body relaxation.

Sketch of the Rauch–Tremaine model. Let us consider a stellar system of size R and total mass M, consisting of $N \gg 1$ stars (each of mass $m_\star = M/N$), orbiting an MBH of mass $M_\bullet$. Stellar orbits are determined by the combined gravitational effects of the Keplerian potential of the MBH and cluster self-gravity, general relativistic effects and any external gravitational sources. The stellar system will be referred to as

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Keplerian (Sridhar & Touma 2016a, hereafter Paper I) when the non-Keplerian forces are much smaller than the Keplerian field of the MBH. Therefore, for an isolated Keplerian stellar system, the mass ratio must be small, $\varepsilon \equiv M/M_\bullet \ll 1$, and its size must be large enough, $R \gg r_\bullet$, where $r_\bullet \equiv 2GM_\bullet/c^2$ is the Schwarzschild radius of the MBH. In two-body relaxation random gravitational encounters between stars, with a coherence time of the order of the short Kepler orbital time-scale $T_{\rm kep} = 2\pi(R^3/GM_\bullet)^{1/2}$, build up to order unity over times $T_{\rm relax} = N\varepsilon^{-2}T_{\rm kep}$.[1] As noted earlier, $T_{\rm relax}$ can exceed the Hubble time, whereas RR works more efficiently because of the longer coherence times which we now estimate. Following RT96, we consider RR in a non-relativistic Keplerian stellar system, for which $(r_\bullet/R) \ll \varepsilon \ll 1$. Then orbits evolve slowly over secular time-scales $T_{\rm sec} = \varepsilon^{-1}T_{\rm kep}$, which is longer than Kepler orbital times. Over several $T_{\rm sec}$, stellar orbits can be thought of as *Gaussian rings* (see Paper I), which are Keplerian ellipses of fixed semimajor axes that can deform in the mean self-gravitational field of the cluster. A secular collisionless theory of this system was worked out in Paper I, and applies in the limit $N \to \infty$ while $M = Nm_\star$ is held constant. For realistic systems, N is large but finite, say $N \sim 10^5$–10^7, and it is this granularity that drives RR.

As the semimajor axes of Gaussian rings do not change, there is no exchange of Keplerian energies during secular gravitational encounters.[2] But mutual torquing can exchange angular momentum between rings leading to diffusion of angular momenta. The diffusion of a vectorial quantity like angular momentum can be complicated, and depends on the geometry and orbital structure of the stellar system. According to RT06, the physics of this process can be understood by considering two extreme cases: *scalar RR* which is the diffusion of the magnitude of angular momenta (and hence the eccentricity of stellar orbits); and *vector RR* which is the diffusion of the direction of angular momenta. Diffusion of the magnitude of angular momenta occurs as a result of the mutual torquing of Gaussian rings whose apsides precess over times $T_{\rm sec}$. Hence, the coherence time for these secular encounters is $\sim T_{\rm sec}$, which is much longer than $T_{\rm kep}$. The magnitude of the angular momentum can be thought of as varying linearly in time for times shorter than $T_{\rm sec}$, and random walking over longer times. The time-scale for scalar RR turns out to be $T_{\rm res} = NT_{\rm sec} = \varepsilon T_{\rm relax}$, which is shorter than $T_{\rm relax}$. Vector RR can occur in near-spherical systems, even in the absence of an MBH, when the apsides of stellar orbits precess fast (say, when the stellar system is immersed in an external density cusp) and apsidal resonances have negligible effect. Then orbital eccentricities are conserved, in addition to the semimajor axes. Averaging over the fast apsidal precession, each stellar orbit can be replaced by an axisymmetric annulus whose inner/outer radii are equal to the (constant) peri/apocentre distances. Diffusion of the orientation of orbits occurs as a result of the mutual torquing of the stellar annuli with coherence times, $\sim N^{1/2}T_{\rm sec}$. In a Keplerian stellar system, it turns out that the vector RR time-scale is also of the order of the coherence time, giving $T^{\rm v}_{\rm res} = N^{1/2}T_{\rm sec} = N^{-1/2}T_{\rm res}$, which is shorter than even the scalar RR time-scale. Thus, there emerges the notion of a hierarchy of relaxation processes, with longer coherence times corresponding to shorter relaxation times: (a) the Keplerian orbital time-scale and classical two-body relaxation; (b) the slow secular apse precession time-scale and scalar RR; (c) the slower secular nodal precession time-scale and vector RR. For general stellar distributions, the distinction between scalar RR and vector RR can get blurred.

RR over the years. RR has been explored extensively through both numerical simulations and stochastic modelling, a trend that was also initiated by RT96: restricted N-body simulations (splitting stars into field and test stars) and N-wire simulations were used to demonstrate that RR is more efficient than two-body relaxation; the efficiency was characterized in both coherent/linear and diffusive regimes, by fitting for parameters that were left free in their random walk model. Hopman & Alexander (2006) revisited the problem of the feeding of an MBH with RR acting as the driving mechanism, instead of the classical two-body relaxation considered by Bahcall & Wolf (1976). Using the RT96 model, they parametrized the strength of RR in a Fokker–Planck equation (which is averaged over angular momentum) for the distribution of energies, with loss-cone boundary conditions. They could constrain the fluxes into the MBH, the energies of stars that contribute to it, and enhancement in gravitational wave events. They also studied the implications of the process for the Galactic Centre and concluded that the kinematics of stars was consistent with the workings of RR. Gürkan & Hopman (2007) used the wire approximation of RT96 and random sampling to explore the RR time-scale as a function of eccentricity, and concluded that RR is more efficient on eccentric orbits. Kupi, Eilon & Alexander (2010) used small-scale Newtonian N-body simulations to determine the efficiency of RR with regard to the inspiral of compact remnants into the MBH and emission of gravitational waves, and concluded that RR will increase the rates of inspiral events by a factor of a few over two-body relaxation.

Madigan, Hopman & Levin (2011) generalized Hopman & Alexander (2006) in two significant ways by using (a) an autoregressive moving average model for RR (instead of using the RT06 random walk model) that is constrained by restricted N-body simulations; (b) a Fokker–Planck equation that kept track of both energy and angular momentum. Their solution allowed them to account for a density core in the Galactic Centre, and to conclude that binary disruption makes for resonantly relaxed orbits which are too eccentric for the observed S-stars. Kocsis & Tremaine (2011) explored an analytical model of vector RR to investigate possible warps in the distribution of stars at the Galactic Centre. Vector RR has also been studied by Kocsis & Tremaine (2015) through numerical simulations of annuli–annuli interactions, and a random walk model of orbit normals on a sphere. They found that the dominant torques were between stars with radially overlapping orbits, and that the vector RR rate increased rapidly for highly eccentric ($e \gtrsim 0.8$) orbits. Touma & Tremaine (2014) determined maximum entropy equilibria of self-gravitating Keplerian discs, for the case where all the particles have equal semimajor axes. Their numerical investigations revealed broken-symmetry states which could be lopsided and uniformly precessing.

Using extensive post-Newtonian (PN) N-body simulations, Merritt et al. (2011) argued that general relativistic precession offers a severe barrier to the capture of stars by the MBH, thereby quenching RR. They considered dynamical mechanisms for barrier penetration and

[1] Logarithmic corrections to $T_{\rm relax}$ have been neglected.
[2] The Keplerian orbital energies diffuse over the longer time-scale $T_{\rm relax}$.

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assessed the rate of gravitational wave events they enable. This is the most serious attempt at studying the workings of RR through N-body simulations, and stands as a benchmark for confrontation with any fundamental theory of RR in the presence of general relativistic corrections. Bar-Or & Alexander (2014, 2015) have also studied the problem of the quenching of RR torques by general relativistic precession; they model the background potential as a correlated Gaussian noise, and study the stochastic process through a Fokker–Planck equation and Monte Carlo simulations. Hamers, Portegies Zwart & Merritt (2014) made extensive PN, restricted N-body simulations with a view to understanding the behaviour of S-stars in the Galactic Centre. They identified three regimes of relaxation: non-resonant, resonant and anomalous, and recovered functional forms for the diffusion coefficients in all three regimes. These results were used in a Fokker–Planck equation to obtain the steady-state distribution of angular momentum for orbits near the MBH. Merritt (2015a,b,c) developed a numerical algorithm of the Cohn–Kulsrud type for solving a Fokker–Planck equation in both energy and angular momentum, which used diffusion coefficients describing the effects of two-body relaxation, RR, anomalous relaxation and energy loss due to emission of gravitational waves; loss of stars to the MBH was also included. The algorithm was applied to recover time-evolving distribution functions (DFs) in galactic nuclei.

Approach to a kinetic theory of RR. The brief overview of developments in RR given above is, admittedly, far from exhaustive; rather it is meant to be illustrative of what we perceive as some major trends over the past two decades. Numerical simulations have grown in sophistication: in quality and size, as well as in their ability to include physical effects such as general relativistic precession and emission of gravitational waves. In contrast, theoretical understanding of RR is based on various parametrized stochastic models whose parameters are recovered from the numerical simulations. It is necessary to develop a new framework for RR, one whose foundations are based on rigorous theories of non-equilibrium statistical mechanics. We begin by recalling the theoretical basis underlying the standard treatment of classical two-body relaxation.

Two-body relaxation has been studied extensively, both analytically and numerically. The historical beginnings of the kinetic theory of stellar systems are rooted in the problem of the evolution of globular clusters. Chandrasekhar (1942, 1943a,b) derived a Fokker–Planck equation by considering gravitational encounters in an infinite homogeneous system of stars. The derivation of the diffusion and friction coefficients was simplified by Cohen, Spitzer & Routly (1950), Gasiorowicz, Neuman & Riddell (1956) and Rosenbluth, MacDonald & Judd (1957). The general kinetic equation for inhomogeneous stellar systems was derived by Gilbert (1968); this is rigorous and based on a systematic $1/N$ expansion of the BBGKY (Bogoliubov–Born–Green–Kirkwood–Yvon) equations of non-equilibrium statistical mechanics. Heyvaerts (2010) revisited the problem using action–angle variables (assuming an integrable system), and proved that the kinetic equation satisfies an H-theorem. The most difficult parts of the theory are certain polarization terms arising from collective interactions. A first simplification of the kinetic equation was achieved by Polyachenko & Shukhman (1982) through the neglect of the polarization terms – see also Chavanis (2012, 2013).[3] There has been some work on the applications of the kinetic equation (in the Gilbert form, or the reduced Polyachenko–Shukhman form), but we do not review this literature. We note, however, that the kinetic equation applies to inhomogeneous systems for which computing interactions between realistic stellar orbits can be complicated. One can pass from Polyachenko & Shukhman (1982) to the more tractable and well-investigated Fokker–Planck description of classical two-body relaxation. This involves the step taken originally by Chandrasekhar, in which gravitational encounters are estimated by assuming that stars move with constant velocities. Thus we are comforted that the journeyman theory of two-body relaxation, which is textbook material (Binney & Tremaine 2008), rests on the firm foundations of Gilbert (1968) through descent via Polyachenko & Shukhman (1982).

But we have seen that two-body relaxation is not efficient in galactic nuclei with MBHs, and we must consider RR. To the best of our knowledge, the stochastic models used to interpret numerical simulations of RR are all ad hoc. Our goal is to develop a first-principles theory of RR, which is based on Gilbert (1968). For simplicity, we consider a stellar system consisting of equal-mass stars; the extension to a stellar system with a range of masses is straightforward, but this will not be treated here. Our focus is on presenting the derivation of the kinetic equation governing RR in detail, so all the developments in this paper are of a formal nature. Applications of our theory will be presented in separate papers. We begin in Section 2 with an outline of Gilbert's theory of the collisional relaxation of general stellar systems, adding to it arbitrary external sources of gravity. The theory is based on a $1/N$ expansion: at order unity we get the collisionless Boltzmann equation (CBE), and collision terms emerge at order $1/N$. We cast the equations in Poisson bracket (PB) form by defining various potentials. In Section 3, the external potential is taken to be that of an MBH, and a canonical transformation is effected to coordinates centred on the MBH. Another canonical transformation expresses the Gilbert equations in the Delaunay variables, which are the natural action–angle variables for the Kepler problem. No approximations have been made and the equations apply to a stellar system around an MBH for any mass ratio ε. In Section 4, we consider the application to Keplerian stellar systems for which $\varepsilon \ll 1$. We begin with the collisionless theory and recall the results of Paper I on the secular limit, obtained using the method of multiple scales. Then we move to the collisional theory and orbit-average the Gilbert equations; the calculations are straightforward, with details given in Appendix A. The result is a kinetic equation for RR as envisioned by RT96. In Section 5, we supplement the mean self-gravity of the system by secular relativistic corrections (up to 1.5 PN order) and external sources of gravity. Thus, we arrive at a general kinetic equation for RR. Some general physical features of this equation are discussed in Section 6, and concluding remarks are offered in Section 7.

[3] This simplification is analogous to the passage from the equations of Balescu (1960) and Lenard (1960) to the Landau (1936) equations, for electrostatic plasmas.

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2 GENERAL SETTING FOR COLLISIONAL RELAXATION

2.1 Outline of Gilbert's theory

We begin with a brief account of the theory of collisional relaxation of Gilbert (1968). The stellar system (or cluster) consists of a large number N of stars, each of mass $m_\star$, and total mass $M = Nm_\star$. Let $\boldsymbol{x}_i$ and $\boldsymbol{v}_i$ be the position and velocity of the ith star (for $i = 1, \ldots, N$), with respect to an inertial frame. The phase space of the system is $6N$-dim with coordinates $\Gamma_i \equiv (\boldsymbol{x}_i, \boldsymbol{v}_i)$. We also use $\mathrm{d}\Gamma_i = \mathrm{d}^3\boldsymbol{x}_i \mathrm{d}^3\boldsymbol{v}_i$ and write the $6N$-dim volume element as $\mathrm{d}\Gamma_1 \ldots \mathrm{d}\Gamma_N$. We recall that, in the standard statistical formulation (Lifshitz & Pitaevskii 1981), the system is described by an N-particle DF $f^{(N)}(\Gamma_1, \ldots, \Gamma_N, t)$ which is a symmetric function of $\Gamma_1, \ldots, \Gamma_N$. Reduced s-particle DFs $f^{(s)}(\Gamma_1, \ldots, \Gamma_s, t)$ are derived for $s = 1, \ldots, (N-1)$, by integrating $f^{(N)}$ over the coordinates $\Gamma_{s+1}, \ldots, \Gamma_N$. The N-particle DF is a probability distribution in the $6N$-dim phase space $\{\Gamma_1, \ldots, \Gamma_N\}$; hence, all the s-particle DFs are probability distributions in their respective $6s$-dim phase spaces $\{\Gamma_1, \ldots, \Gamma_s\}$. Evolution equations for the $f^{(s)}$ can be obtained, by integrating the Liouville equation over these very coordinates. This is the BBGKY hierarchy of equations, in which the equation for $f^{(1)}$ depends on $f^{(2)}$, and so on up to $f^{(N)}$. For a large N system, the chain of equations is very long, and is usually truncated through ordering in a physically relevant small parameter.

Drawing on earlier work in plasma physics (Balescu 1960; Lenard 1960), Gilbert achieved closure of the BBGKY equations for stellar systems through power-series expansion in the small parameter $(1/N)$. With notation $\Gamma = (\boldsymbol{x}, \boldsymbol{v})$, $\Gamma' = (\boldsymbol{x}', \boldsymbol{v}')$ and $\Gamma'' = (\boldsymbol{x}'', \boldsymbol{v}'')$, the two-particle DF can be expanded as $f^{(2)}(\Gamma, \Gamma', t) = f^{(1)}(\Gamma, t) f^{(1)}(\Gamma', t) + (1/N) f^{(2)}_{\rm irr}(\Gamma, \Gamma', t) + \cdots$, where $f^{(2)}_{\rm irr}$ is the irreducible part of the two-particle correlation, satisfying $\int f^{(2)}_{\rm irr}\, \mathrm{d}\Gamma = \int f^{(2)}_{\rm irr}\, \mathrm{d}\Gamma' = 0$. It is straightforward to verify that $f^{(3)}$ is determined uniquely in terms of $f^{(1)}$ and $f^{(2)}_{\rm irr}$, to first order in $1/N$. Hence, the BBGKY hierarchy closes at this order, and provides two coupled equations for $f^{(1)}$ and $f^{(2)}_{\rm irr}$ which describe the collisional evolution of a self-gravitating system. Irreversibility arises through the 'adiabatic turn-on' initial conditions, for which $f^{(2)}_{\rm irr}$ vanishes in the distant past. Henceforth, we drop the subscript '1', and write the one-particle DF as $f(\Gamma, t)$. The acceleration of a test particle at $(\boldsymbol{x}, t)$ by the mean field is

$$\boldsymbol{a}(\boldsymbol{x}, t) = \int f(\Gamma', t)\, \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{x}')\, \mathrm{d}\Gamma', \qquad \text{where} \qquad \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{x}') = GM \frac{\boldsymbol{x}' - \boldsymbol{x}}{|\boldsymbol{x}' - \boldsymbol{x}|^3} \tag{1}$$

is the (scaled) Newtonian 'bare' inter-particle acceleration. External sources of gravity (an MBH, an infalling gas cloud or star cluster) can also be included in the formalism, although these are absent in Gilbert (1968). Let $\boldsymbol{a}^{\rm e}(\boldsymbol{x}, t)$ be the acceleration of the test star due to external sources. Then the equation for $f(\boldsymbol{x}, \boldsymbol{v}, t)$ can be written as

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + (\boldsymbol{a} + \boldsymbol{a}^{\rm e}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \frac{1}{N} \boldsymbol{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}} - \frac{1}{N} \int \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{x}') \cdot \frac{\partial f^{(2)}_{\rm irr}(\Gamma, \Gamma', t)}{\partial \boldsymbol{v}}\, \mathrm{d}\Gamma'. \tag{2}$$

Mean-field theory corresponds to the limiting case when $N \to \infty$, $m_\star \to 0$ with $Nm_\star = M =$ mass of the system which is held constant. The accelerations $\boldsymbol{b}(\boldsymbol{x}, \boldsymbol{x}')$, $\boldsymbol{a}(\boldsymbol{x}, t)$ and $\boldsymbol{a}^{\rm e}(\boldsymbol{x}, t)$ are all well defined in this limit. However, both terms on the right-hand side vanish in the limit. Hence, the mean-field theory of a stellar system is completely described by the one-particle DF, $f(\boldsymbol{x}, \boldsymbol{v}, t)$, whose time evolution is governed by the CBE. When N is large but not infinite, the accrual of small changes induced in f, by the right-hand side of equation (2), cannot be ignored over long times. The first term corrects the mean-field acceleration of a star by a small amount, from $\boldsymbol{a}$ to $(1 - 1/N)\boldsymbol{a}$, taking note of the fact that the gravitational force on a star is due only to the other $(N-1)$ stars. It is the second term that is of importance for collisional evolution, where $f^{(2)}_{\rm irr}$ acts as the driving term.

The equation for $f^{(2)}_{\rm irr}(\Gamma, \Gamma', t)$ depends on both $f(\Gamma, t)$ and $f(\Gamma', t)$, and is difficult to deal with in the general case. However, we are interested in describing the quasi-steady collisional relaxation of f. The stellar system can be thought of as evolving through a series of collisionless equilibria, for which $\{\boldsymbol{v} \cdot \partial/\partial \boldsymbol{x} + ((1 - 1/N)\boldsymbol{a} + \boldsymbol{a}^{\rm e}) \cdot \partial/\partial \boldsymbol{v}\} f \simeq 0$, being of order $1/N$ or smaller. This implies that $\partial f/\partial t$ is first order in $1/N$, due solely to the collision term. It turns out that, in this case, $f^{(2)}_{\rm irr}$ can be expressed in terms of a conditional probability function $\mathcal{W}(\Gamma \,|\, \Gamma', t)$, which will henceforth be referred to as the wake of Γ' at Γ. The wake can be defined through the following *gedanken* experiment: from the N stars that are distributed in 6-dim phase space according to f, select one star and place it at the phase space location Γ'; this will induce a small perturbation $(1/N)\mathcal{W}$ at every location Γ in the entire phase space. At this point, it is useful to define two kinds of acceleration at any point $\boldsymbol{x}$. The wake acceleration,

$$\boldsymbol{a}^{\rm w}(\boldsymbol{x}, \Gamma', t) = \int \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{x}'') \mathcal{W}(\Gamma'' \,|\, \Gamma', t)\, \mathrm{d}\Gamma'', \tag{3}$$

is the acceleration at $\boldsymbol{x}$ due to the wake of Γ'. We also define the perturbing acceleration,

$$\boldsymbol{a}^{\rm p}(\boldsymbol{x}, \boldsymbol{x}', t) = \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{x}') - \boldsymbol{a}(\boldsymbol{x}, t), \tag{4}$$

where the mean-field acceleration is subtracted from the bare inter-particle acceleration. The wake $\mathcal{W}(\Gamma \,|\, \Gamma', t)$ must be calculated by integrating an equation for the related wake function $\mathcal{W}(\Gamma \,|\, \Gamma'(t'), t')$, where $\Gamma'(t') = (\boldsymbol{x}'(t'), \boldsymbol{v}'(t'))$ is the location at time $t' \le t$, of the star that arrives at the desired location Γ' at time t. The equation for $\mathcal{W}(\Gamma \,|\, \Gamma'(t'), t')$ is

$$\frac{\partial \mathcal{W}}{\partial t'} + \boldsymbol{v} \cdot \frac{\partial \mathcal{W}}{\partial \boldsymbol{x}} + \left\{\boldsymbol{a}(\boldsymbol{x}, t') + \boldsymbol{a}^{\rm e}(\boldsymbol{x}, t')\right\} \cdot \frac{\partial \mathcal{W}}{\partial \boldsymbol{v}} + \boldsymbol{a}^{\rm w}(\boldsymbol{x}, \Gamma'(t'), t') \cdot \frac{\partial f}{\partial \boldsymbol{v}} = -\boldsymbol{a}^{\rm p}(\boldsymbol{x}, \boldsymbol{x}'(t'), t') \cdot \frac{\partial f}{\partial \boldsymbol{v}}, \qquad \text{for } t' \le t. \tag{5}$$

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This equation needs to be solved with the adiabatic turn-on initial condition $\mathcal{W}(\Gamma \mid \Gamma'(t'), t') \to 0$ as $t' \to -\infty$, corresponding to a vanishingly small wake in the distant past. The right-hand side identifies the perturbing acceleration $\boldsymbol{a}^{\rm p}(\boldsymbol{x}, \boldsymbol{x}'(t'), t')$ as the driver of the wake. The wake $\mathcal{W}(\Gamma \mid \Gamma', t)$ can be obtained by (formally) integrating equation (5) from $t' = -\infty$ to $t' = t$, and is therefore a functional of the quasi-steady f. It may be verified that equation (5) preserves $\int \mathcal{W}(\Gamma \mid \Gamma', t)\, {\rm d}\Gamma = 0$, because the wake contains no net mass.

Extending the work of Rostoker (1964) in plasma physics to the case of an inhomogeneous stellar system, Gilbert (1968) used non-trivial operator identities to derive the fundamental relation between $f^{(2)}_{\rm irr}$ and $\mathcal{W}$:

$$f^{(2)}_{\rm irr}(\Gamma, \Gamma', t) = \mathcal{W}(\Gamma \mid \Gamma', t)\, f(\Gamma', t) + \mathcal{W}(\Gamma' \mid \Gamma, t)\, f(\Gamma, t) + \int \mathcal{W}(\Gamma \mid \Gamma'', t)\, \mathcal{W}(\Gamma' \mid \Gamma'', t)\, f(\Gamma'', t)\, {\rm d}\Gamma''. \tag{6}$$

This decomposition of $f^{(2)}_{\rm irr}$ means that the irreducible two-particle correlation at (Γ, Γ') gets three contributions from the wake function: (a) the wake of Γ' at Γ; (b) the wake of Γ at Γ'; (c) the product of the wake values at the points Γ and Γ' of a third ring at Γ'', summed over all locations $\mathcal{R}''$. All three contributions come with suitable f-weighting. The third term accounts for the contribution of collective effects ('gravitational polarization') to the microscopic processes driving RR. This is the nature of the full theory at $O(1/N)$. Since $\int f^{(2)}_{\rm irr}(\Gamma, \Gamma', t)\, {\rm d}\Gamma' = 0$, equation (6) implies that $\int \mathcal{W}(\Gamma \mid \Gamma', t)\, f(\Gamma', t)\, {\rm d}\Gamma' = 0$.

Together, equations (2) and (6) give a closed set of kinetic equations governing the collisional relaxation of $f(\Gamma, t)$. This can also be written more explicitly as

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + \left\{ \left(1 - \frac{1}{N}\right) \boldsymbol{a} + \boldsymbol{a}^{\rm ext} \right\} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \mathcal{C}^{\rm dis}[f] + \mathcal{C}^{\rm fluc}[f], \tag{7a}$$

$$\mathcal{C}^{\rm dis}[f] = -\frac{1}{N} \int \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{x}') \cdot \frac{\partial}{\partial \boldsymbol{v}} \left\{ f(\Gamma, t) \mathcal{W}(\Gamma' \mid \Gamma, t) \right\} {\rm d}\Gamma', \tag{7b}$$

$$\mathcal{C}^{\rm fluc}[f] = -\frac{1}{N} \int f(\Gamma', t) \left\{ \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{x}') + \boldsymbol{a}^{\rm w}(\boldsymbol{x}, \Gamma', t) \right\} \cdot \frac{\partial \mathcal{W}(\Gamma \mid \Gamma', t)}{\partial \boldsymbol{v}} {\rm d}\Gamma'. \tag{7c}$$

Here $\mathcal{C}^{\rm dis}[f]$ and $\mathcal{C}^{\rm fluc}[f]$ are the dissipative and fluctuating parts of the collision term. As the notation indicates both are functionals of f, because they depend on the wake $\mathcal{W}(\Gamma \mid \Gamma', t)$ which, as discussed earlier, can be thought of as a functional of f.

2.2 Gilbert's equations in PB form

Gilbert's kinetic theory is expressed in the physically direct language of the various forces acting on a star. However, it is more convenient to exchange accelerations for corresponding potentials, and rewrite the kinetic equations in PB form. We have

$$\{\boldsymbol{b},\ \boldsymbol{a},\ \boldsymbol{a}^{\rm e},\ \boldsymbol{a}^{\rm w},\ \boldsymbol{a}^{\rm p}\} = -\frac{\partial}{\partial \boldsymbol{x}} \{P,\ \Pi,\ \Pi^{\rm e},\ \Pi^{\rm w},\ \Pi^{\rm p}\}, \tag{8}$$

where the potential functions on the right-hand side are defined by

$$P(\boldsymbol{x}, \boldsymbol{x}') = -\frac{GM}{|\boldsymbol{x} - \boldsymbol{x}'|}, \qquad \text{(scaled) Inter-particle Poisson kernel;} \tag{9a}$$

$$\Pi(\boldsymbol{x}, t) = \int f(\Gamma', t) P(\boldsymbol{x}, \boldsymbol{x}')\, {\rm d}\Gamma', \qquad \text{Mean-field potential of the cluster;} \tag{9b}$$

$$\Pi^{\rm e}(\boldsymbol{x}, t) = \text{External potential;} \tag{9c}$$

$$\Pi^{\rm w}(\boldsymbol{x}, \Gamma', t) = \int P(\boldsymbol{x}, \boldsymbol{x}'') \mathcal{W}(\Gamma'' \mid \Gamma', t)\, {\rm d}\Gamma'', \quad \text{Wake potential at } \boldsymbol{x} \text{ due to } \Gamma'; \tag{9d}$$

$$\Pi^{\rm p}(\boldsymbol{x}, \boldsymbol{x}', t) = P(\boldsymbol{x}, \boldsymbol{x}') - \Pi(\boldsymbol{x}, t), \qquad \text{Perturbing potential.} \tag{9e}$$

The mean-field Hamiltonian is

$$H^{(1)}(\Gamma, t) = H^{(1)}(\boldsymbol{x}, \boldsymbol{v}, t) = \frac{v^2}{2} + \Pi(\boldsymbol{x}, t) + \Pi^{\rm e}(\boldsymbol{x}, t), \tag{10}$$

with the corresponding mean-field equations of motion,

$$\frac{{\rm d}\boldsymbol{x}}{{\rm d}t} = \frac{\partial H^{(1)}}{\partial \boldsymbol{v}} = \boldsymbol{v}, \qquad \frac{{\rm d}\boldsymbol{v}}{{\rm d}t} = -\frac{\partial H^{(1)}}{\partial \boldsymbol{x}} = \boldsymbol{a} + \boldsymbol{a}^{\rm e}. \tag{11}$$

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The PB between two phase space functions $\chi_1(\Gamma, t)$ and $\chi_2(\Gamma, t)$ is

$$[\chi_1, \chi_2]_{(\Gamma)} \stackrel{\text{def}}{=} \frac{\partial \chi_1}{\partial \boldsymbol{x}} \cdot \frac{\partial \chi_2}{\partial \boldsymbol{v}} - \frac{\partial \chi_1}{\partial \boldsymbol{v}} \cdot \frac{\partial \chi_2}{\partial \boldsymbol{x}}. \tag{12}$$

Using this we can rewrite equation (5) for the wake function $\mathcal{W}(\Gamma \mid \Gamma'(t'), t')$ as

$$\frac{\partial \mathcal{W}}{\partial t'} + \left[\mathcal{W}, H^{(1)}(\Gamma, t')\right]_{(\Gamma)} + \left[f(\Gamma, t'), \Pi^{\mathrm{w}}(\boldsymbol{x}, \Gamma'(t'), t')\right]_{(\Gamma)} = \left[\Pi^{\mathrm{p}}(\boldsymbol{x}, \boldsymbol{x}'(t'), t'), f(\Gamma, t')\right]_{(\Gamma)}, \qquad \text{for } t' \le t,$$
$$\text{with adiabatic turn-on initial condition} \quad \lim_{t' \to -\infty} \mathcal{W}(\Gamma \mid \Gamma'(t'), t') = 0. \tag{13}$$

There are two ways to write the Gilbert kinetic equations in PB form. The first is based on equations (2) and (6), which is closer to the original BBGKY equations:

$$\frac{\partial f}{\partial t} + \left[f(\Gamma, t), H^{(1)}(\Gamma, t) - \frac{\Pi(\boldsymbol{x}, t)}{N}\right]_{(\Gamma)} = \frac{1}{N} \int \left[P(\boldsymbol{x}, \boldsymbol{x}'), f^{(2)}_{\mathrm{irr}}(\Gamma, \Gamma', t)\right]_{(\Gamma)} \mathrm{d}\Gamma',$$
$$f^{(2)}_{\mathrm{irr}}(\Gamma, \Gamma', t) = \mathcal{W}(\Gamma \mid \Gamma', t)\, f(\Gamma', t) + \mathcal{W}(\Gamma' \mid \Gamma, t)\, f(\Gamma, t) + \int \mathcal{W}(\Gamma \mid \Gamma'', t)\, \mathcal{W}(\Gamma' \mid \Gamma'', t)\, f(\Gamma'', t)\, \mathrm{d}\Gamma''. \tag{14}$$

The second is the explicit form given by Gilbert, displaying the dissipative and fluctuating contributions to the collision integral. From equations (7a)–(7c), we have

$$\frac{\partial f}{\partial t} + \left[f(\Gamma, t), H^{(1)}(\Gamma, t) - \frac{\Pi(\boldsymbol{x}, t)}{N}\right]_{(\Gamma)} = \mathcal{C}^{\mathrm{dis}}[f] + \mathcal{C}^{\mathrm{fluc}}[f], \tag{15a}$$

$$\mathcal{C}^{\mathrm{dis}}[f] = \frac{1}{N} \int \left[P(\boldsymbol{x}, \boldsymbol{x}'), f(\Gamma, t)\mathcal{W}(\Gamma' \mid \Gamma, t)\right]_{(\Gamma)} \mathrm{d}\Gamma', \tag{15b}$$

$$\mathcal{C}^{\mathrm{fluc}}[f] = \frac{1}{N} \int f(\Gamma', t) \left[P(\boldsymbol{x}, \boldsymbol{x}') + \Pi^{\mathrm{w}}(\boldsymbol{x}, \Gamma', t), \mathcal{W}(\Gamma \mid \Gamma', t)\right]_{(\Gamma)} \mathrm{d}\Gamma'. \tag{15c}$$

Either form of the kinetic equations applies to the collisional evolution of f whenever it satisfies

$$\left[f(\Gamma, t), H^{(1)}(\Gamma, t) - \frac{\Pi(\boldsymbol{x}, t)}{N}\right]_{(\Gamma)} \simeq 0, \quad \text{being of order } 1/N \text{ or smaller.} \tag{16}$$

Therefore, $f(\Gamma, t)$ may be thought of as passing through a sequence of collisionless equilibria in a quasi-static manner.

3 KINETIC EQUATION FOR A STELLAR SYSTEM ORBITING AN MBH

We now specialize the kinetic theory to a stellar system of total mass $M = Nm_\star$, orbiting an MBH of mass $M_\bullet$. The mass ratio $\varepsilon = M/M_\bullet$ is a natural ordering parameter; the smaller it is, the more 'Keplerian' is the stellar system. We begin with the kinetic equation in the BBGKY form, equation (14), and make two canonical transformations. The first transformation is from the inertial frame to a (generally non-inertial) frame centred on the MBH. Then we transform to the Delaunay variables, which are natural action–angle variables for the Kepler problem. The formulae we derive are valid for arbitrary ε; however, the second transformation proves really useful only when ε is small.

3.1 Transformation to coordinates centred on the MBH

The Keplerian force of the MBH on a star is described by taking the external potential of equation (9c) as

$$\Pi^{\mathrm{e}}(\boldsymbol{x}, t) = -\frac{GM_\bullet}{|\boldsymbol{x} - \boldsymbol{x}_\bullet(t)|}, \tag{17}$$

where $\boldsymbol{x}_\bullet(t)$ is the position of the MBH with respect to the inertial frame. It is natural to transform to a non-inertial frame centred on the MBH. Let $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_\bullet(t)$ and $\boldsymbol{u} = \boldsymbol{v} - \dot{\boldsymbol{x}}_\bullet(t)$ be the position and velocity, respectively, of a test star relative to the MBH. It is evident that $(\boldsymbol{r}, \boldsymbol{u})$ are a canonically conjugate pair of 6-dim phase space coordinates. Using notation $\Gamma_\star = (\boldsymbol{r}, \boldsymbol{u})$, we have $\mathrm{d}\Gamma = \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{v} = \mathrm{d}^3\boldsymbol{r}\mathrm{d}^3\boldsymbol{u} = \mathrm{d}\Gamma_\star$. We write the (one-particle) DF as $f(\Gamma_\star, t)$ and the wake as $\mathcal{W}(\Gamma_\star \mid \Gamma'_\star, t)$. We take as an $O(1)$ reference the Kepler potential of the MBH potential $\Pi^{\mathrm{ext}} = -GM_\bullet/r$ given in equation (17). Then the other potentials in equations (9a), (9b), (9d) and (9e) are all quantities of $O(\varepsilon)$. Hence, we rescale these and write $\{P, \Pi, \Pi^{\mathrm{w}}, \Pi^{\mathrm{p}}\} = \varepsilon\{p, \varphi, \varphi^{\mathrm{w}}, \varphi^{\mathrm{p}}\}$, where

$$p(\boldsymbol{r}, \boldsymbol{r}') = -\frac{GM_\bullet}{|\boldsymbol{r} - \boldsymbol{r}'|}, \qquad \text{(rescaled) Poisson kernel;} \tag{18a}$$

$$\varphi(\boldsymbol{r}, t) = \int f(\Gamma'_\star, t) p(\boldsymbol{r}, \boldsymbol{r}')\, \mathrm{d}\Gamma'_\star, \qquad \text{(rescaled) Mean-field potential;} \tag{18b}$$

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$$\varphi^{\rm w}(\boldsymbol{r},\Gamma'_\star,t)=\int p(\boldsymbol{r},\boldsymbol{r}'')\mathcal{W}(\Gamma''_\star\,|\,\Gamma'_\star,t)\,{\rm d}\Gamma''_\star,\qquad\text{(rescaled) Wake potential;}\tag{18c}$$

$$\varphi^{\rm p}(\boldsymbol{r},\boldsymbol{r}',t)=p(\boldsymbol{r},\boldsymbol{r}')-\varphi(\boldsymbol{r},t),\qquad\text{(rescaled) Perturbing potential.}\tag{18d}$$

All the PBs in equations (13) and (15a)–(15c) remain invariant, because $(\boldsymbol{x},\boldsymbol{v})\to(\boldsymbol{r},\boldsymbol{u})$ is a canonical transformation, i.e. for any two phase space functions χ_1 and χ_2, the PB defined earlier in equation (12) is

$$[\chi_1,\chi_2]_{(\Gamma)}=[\chi_1,\chi_2]_{(\Gamma_\star)}\stackrel{\rm def}{=}\frac{\partial\chi_1}{\partial\boldsymbol{r}}\cdot\frac{\partial\chi_2}{\partial\boldsymbol{u}}-\frac{\partial\chi_1}{\partial\boldsymbol{u}}\cdot\frac{\partial\chi_2}{\partial\boldsymbol{r}}.\tag{19}$$

The first term on the left-hand side of the wake equation (13) is $(\partial\mathcal{W}/\partial t')_{(\Gamma)}$ where we have now indicated explicitly that $\Gamma=(\boldsymbol{x},\boldsymbol{v})$ is to be held constant. We need to express this in terms of functions of $\Gamma_\star=(\boldsymbol{r},\boldsymbol{u})$ and t. The chain rule gives

$$\left(\frac{\partial\mathcal{W}}{\partial t'}\right)_{(\Gamma)}=\left(\frac{\partial\mathcal{W}}{\partial t'}\right)_{(\Gamma_\star)}-\dot{\boldsymbol{x}}_\bullet\cdot\frac{\partial\mathcal{W}}{\partial\boldsymbol{r}}-\ddot{\boldsymbol{x}}_\bullet\cdot\frac{\partial\mathcal{W}}{\partial\boldsymbol{u}}=\left(\frac{\partial\mathcal{W}}{\partial t'}\right)_{(\Gamma_\star)}+\left[\mathcal{W},\boldsymbol{r}\cdot\ddot{\boldsymbol{x}}_\bullet(t)-\boldsymbol{u}\cdot\dot{\boldsymbol{x}}_\bullet(t)-\frac{1}{2}|\dot{\boldsymbol{x}}_\bullet(t)|^2\right]_{(\Gamma_\star)}.\tag{20}$$

Then equation (13) for the wake function $\mathcal{W}(\Gamma_\star\,|\,\Gamma'_\star(t'),t')$ is

$$\frac{\partial\mathcal{W}}{\partial t'}+\left[\mathcal{W},H_{\rm org}(\Gamma_\star,t')\right]_{(\Gamma_\star)}+\varepsilon\left[f(\Gamma_\star,t'),\varphi^{\rm w}(\boldsymbol{r},\Gamma'_\star(t'),t')\right]_{(\Gamma_\star)}=\varepsilon\left[\varphi^{\rm p}(\boldsymbol{r},\boldsymbol{r}'(t'),t'),f(\Gamma_\star,t')\right]_{(\Gamma_\star)},\qquad\text{for } t'\le t,$$

$$\text{with adiabatic turn-on initial condition}\quad\lim_{t'\to-\infty}\mathcal{W}(\Gamma_\star\,|\,\Gamma'_\star(t'),t')=0.\tag{21}$$

We have written $H_{\rm org}(\boldsymbol{r},\boldsymbol{u},t)=H^{(1)}+\{\boldsymbol{r}\cdot\ddot{\boldsymbol{x}}_\bullet(t)-\boldsymbol{u}\cdot\dot{\boldsymbol{x}}_\bullet(t)-|\dot{\boldsymbol{x}}_\bullet(t)|^2/2\}$, and dropped the subscript $\Gamma_\star$ in the time derivative. The acceleration of the MBH, $\ddot{\boldsymbol{x}}_\bullet(t)$, is due to the gravitational forces of all the stars, and is an $O(\varepsilon)$ quantity. Rescaling, we write $\ddot{\boldsymbol{x}}_\bullet(t)=\varepsilon\boldsymbol{A}_\bullet(t)$, where

$$\boldsymbol{A}_\bullet(t)=GM_\bullet\int f(\Gamma_\star,t)\,\frac{\hat{\boldsymbol{r}}}{r^2}\,{\rm d}\Gamma_\star.\tag{22}$$

Using equation (10) for $H^{(1)}$, we can write

$$H_{\rm org}(\Gamma_\star,t)=H_{\rm org}(\boldsymbol{r},\boldsymbol{u},t)=\frac{u^2}{2}-\frac{GM_\bullet}{r}+\varepsilon\varphi(\boldsymbol{r},t)+\varepsilon\boldsymbol{r}\cdot\boldsymbol{A}_\bullet(t)\tag{23}$$

explicitly as a function of $(\boldsymbol{r},\boldsymbol{u},t)$. This is the Hamiltonian governing the mean-field dynamics of $\boldsymbol{r}$ and $\boldsymbol{u}$ through the equations of motion:

$$\frac{{\rm d}\boldsymbol{r}}{{\rm d}t}=\frac{\partial H_{\rm org}}{\partial\boldsymbol{u}}=\boldsymbol{u},\qquad\frac{{\rm d}\boldsymbol{u}}{{\rm d}t}=-\frac{\partial H_{\rm org}}{\partial\boldsymbol{r}}=-\frac{GM_\bullet}{r^2}\hat{\boldsymbol{r}}-\varepsilon\frac{\partial\varphi}{\partial\boldsymbol{r}}-\varepsilon\boldsymbol{A}_\bullet,\tag{24}$$

as can be verified from the definitions of $\boldsymbol{r}$ and $\boldsymbol{u}$. Note that both $H_{\rm org}$ and the equations of motion are identical to those given in section 2 of Paper I. Similarly, equation (14) can be rewritten in the MBH variables. Then $f(\Gamma_\star,t)$ satisfies

$$\frac{\partial f}{\partial t}+\left[f(\Gamma_\star,t),H_{\rm org}(\Gamma_\star,t)-\frac{\varepsilon}{N}\varphi(\boldsymbol{r},t)\right]_{(\Gamma_\star)}=\frac{\varepsilon}{N}\int\left[p(\boldsymbol{r},\boldsymbol{r}'),f^{(2)}_{\rm irr}(\Gamma_\star,\Gamma'_\star,t)\right]_{(\Gamma_\star)}{\rm d}\Gamma'_\star,$$
$$f^{(2)}_{\rm irr}(\Gamma_\star,\Gamma'_\star,t)=\mathcal{W}(\Gamma_\star\,|\,\Gamma'_\star,t)\,f(\Gamma'_\star,t)+\mathcal{W}(\Gamma'_\star\,|\,\Gamma_\star,t)\,f(\Gamma_\star,t)+\int\mathcal{W}(\Gamma_\star\,|\,\Gamma''_\star,t)\,\mathcal{W}(\Gamma'_\star\,|\,\Gamma''_\star,t)\,f(\Gamma''_\star,t)\,{\rm d}\Gamma''_\star.\tag{25}$$

3.2 Transformation to the Delaunay variables

We now transform from the MBH-centric canonical coordinates, $\Gamma_\star=(\boldsymbol{r},\boldsymbol{u})$, to the Delaunay variables $\mathcal{D}\equiv\{I,L,L_z;w,g,h\}$, which are action–angle variables for the Kepler problem (Plummer 1960; Murray & Dermott 1999; Binney & Tremaine 2008). The three actions are $I=\sqrt{GM_\bullet a}$; $L=I\sqrt{1-e^2}$ the magnitude of the angular momentum; and $L_z=L\cos i$ the z-component of the angular momentum. The angles conjugate to them are, respectively, w the orbital phase, g the argument of the periapse and h the longitude of the ascending node. The transformation is exact and no restriction is placed on the magnitude of ε (although this is preparatory to the orbit-averaging that follows in the next section for $\varepsilon\ll1$). In the Hamiltonian $H_{\rm org}$ of equation (23), the Keplerian orbital energy $E_{\rm k}=(u^2/2-GM_\bullet/r)$ will be taken as $O(1)$. The other two terms, $\varepsilon\varphi$ and $\varepsilon\boldsymbol{r}\cdot\boldsymbol{A}_\bullet$, are evidently of $O(\varepsilon)$.

The Kepler orbital energy $E_{\rm k}(I)=-1/2(GM_\bullet/I)^2$ depends only on the action I. Therefore, for the pure Kepler problem (i.e. when $\varepsilon=0$) all the Delaunay variables, excepting w, are constant in time; w itself advances at the rate

$$\Omega_{\rm k}(I)=\frac{{\rm d}E_{\rm k}}{{\rm d}I}=\frac{(GM_\bullet)^2}{I^3}.\tag{26}$$

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In order to rewrite the various physical quantities in terms of the $\mathcal{D}$ variables, we need to express the position vector $\boldsymbol{r}(\mathcal{D}) = (x, y, z)$ in terms of the $\mathcal{D}$ variables (Plummer 1960; Murray & Dermott 1999; Sambhus & Sridhar 2000):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} C_g C_h - C_i S_h S_g & -S_g C_h - C_i S_h C_g & S_i S_h \\ C_g S_h + C_i C_h S_g & -S_g S_h + C_i C_h C_g & -S_i C_h \\ S_i S_g & S_i C_g & C_i \end{pmatrix} \begin{pmatrix} a(C_\eta - e) \\ a\sqrt{1-e^2}\, S_\eta \\ 0 \end{pmatrix}, \tag{27}$$

where S and C are shorthand for sine and cosine of the angles given as subscript. Here a is the semimajor axis, η is the eccentric anomaly, related to the orbital phase through $w = (\eta - e\sin\eta)$, $e = \sqrt{1 - L^2/I^2}$ is the eccentricity and i is the inclination angle determined by $\cos i = (L_z/L)$. It is also useful to note that $r = \sqrt{x^2 + y^2 + z^2} = a(1 - e\cos\eta)$.

The DF is written as $f(\mathcal{D}, t)$ and the wake as $\mathcal{W}(\mathcal{D} \mid \mathcal{D}', t)$. The transformation $\Gamma_\star \to \mathcal{D}$ being canonical, $\mathrm{d}\Gamma_\star = \mathrm{d}\mathcal{D}$. Then the potentials of equations (18a)–(18d) and $\boldsymbol{A}_\bullet$ can be written as

$$p(\mathcal{D}, \mathcal{D}') = -\frac{GM_\bullet}{|\boldsymbol{r} - \boldsymbol{r}'|}, \qquad \text{(rescaled) Poisson kernel;} \tag{28a}$$

$$\varphi(\mathcal{D}, t) = \int f(\mathcal{D}', t) p(\mathcal{D}, \mathcal{D}')\, \mathrm{d}\mathcal{D}', \qquad \text{(rescaled) Mean-field potential;} \tag{28b}$$

$$\boldsymbol{A}_\bullet(t) = GM_\bullet \int f(\mathcal{D}, t)\, \frac{\hat{\boldsymbol{r}}}{r^2}\, \mathrm{d}\mathcal{D}, \qquad \text{(rescaled) MBH acceleration;} \tag{28c}$$

$$\varphi^{\mathrm{w}}(\mathcal{D}, \mathcal{D}', t) = \int p(\mathcal{D}, \mathcal{D}'') \mathcal{W}(\mathcal{D}'' \mid \mathcal{D}', t)\, \mathrm{d}\mathcal{D}'', \qquad \text{(rescaled) Wake potential;} \tag{28d}$$

$$\varphi^{\mathrm{p}}(\mathcal{D}, \mathcal{D}', t) = p(\mathcal{D}, \mathcal{D}') - \varphi(\mathcal{D}, t), \qquad \text{(rescaled) Perturbing potential.} \tag{28e}$$

The mean-field Hamiltonian is

$$H_{\mathrm{org}}(\mathcal{D}, t) = E_{\mathrm{k}}(I) + \varepsilon\varphi(\mathcal{D}, t) + \varepsilon \boldsymbol{r} \cdot \boldsymbol{A}_\bullet(t). \tag{29}$$

Since the transformation $\Gamma_\star \to \mathcal{D}$ is canonical, all the PBs remain invariant. The PB between any two phase space functions $\chi_1(\mathcal{D})$ and $\chi_2(\mathcal{D})$ is

$$[\chi_1, \chi_2]_{(6)} \stackrel{\mathrm{def}}{=} \left(\frac{\partial\chi_1}{\partial w}\frac{\partial\chi_2}{\partial I} - \frac{\partial\chi_1}{\partial I}\frac{\partial\chi_2}{\partial w}\right) + \left(\frac{\partial\chi_1}{\partial g}\frac{\partial\chi_2}{\partial L} - \frac{\partial\chi_1}{\partial L}\frac{\partial\chi_2}{\partial g}\right) + \left(\frac{\partial\chi_1}{\partial h}\frac{\partial\chi_2}{\partial L_z} - \frac{\partial\chi_1}{\partial L_z}\frac{\partial\chi_2}{\partial h}\right). \tag{30}$$

From equation (21), we see that the wake function $\mathcal{W}(\mathcal{D} \mid \mathcal{D}'(t'), t')$ satisfies

$$\frac{\partial\mathcal{W}}{\partial t'} + \Omega_{\mathrm{k}}\frac{\partial\mathcal{W}}{\partial w} + \varepsilon\,[\mathcal{W}, \varphi + \boldsymbol{r}\cdot\boldsymbol{A}_\bullet]_{(6)} + \varepsilon\left[f(\mathcal{D}, t'), \varphi^{\mathrm{w}}(\mathcal{D}, \mathcal{D}'(t'), t')\right]_{(6)} = \varepsilon\left[\varphi^{\mathrm{p}}(\mathcal{D}, \mathcal{D}'(t'), t'),\, f(\mathcal{D}, t')\right]_{(6)}, \quad \text{for } t' \le t,$$

$$\text{with adiabatic turn-on initial condition} \quad \lim_{t' \to -\infty} \mathcal{W}(\mathcal{D} \mid \mathcal{D}'(t'), t') = 0. \tag{31}$$

Similarly, from equation (25), the kinetic equation for $f(\mathcal{D}, t)$ is

$$\frac{\partial f}{\partial t} + \Omega_{\mathrm{k}}\frac{\partial f}{\partial w} + \varepsilon\left[f, \left(1 - \frac{1}{N}\right)\varphi + \boldsymbol{r}\cdot\boldsymbol{A}_\bullet\right]_{(6)} = \frac{\varepsilon}{N}\int \left[p(\mathcal{D}, \mathcal{D}'), f^{(2)}_{\mathrm{irr}}(\mathcal{D}, \mathcal{D}', t)\right]_{(6)} \mathrm{d}\mathcal{D}',$$

$$f^{(2)}_{\mathrm{irr}}(\mathcal{D}, \mathcal{D}', t) = \mathcal{W}(\mathcal{D} \mid \mathcal{D}', t)\, f(\mathcal{D}', t) + \mathcal{W}(\mathcal{D}' \mid \mathcal{D}, t)\, f(\mathcal{D}, t) + \int \mathcal{W}(\mathcal{D} \mid \mathcal{D}'', t)\, \mathcal{W}(\mathcal{D}' \mid \mathcal{D}'', t)\, f(\mathcal{D}'', t)\, \mathrm{d}\mathcal{D}''. \tag{32}$$

4 ORBIT-AVERAGING FOR A KEPLERIAN STELLAR SYSTEM

The stellar system is said to be Keplerian when the mass ratio $\varepsilon = M/M_\bullet \ll 1$. In this case, the dominant force on a star is the inverse-squared Newtonian force of the MBH, and ε is the natural small parameter for a perturbation theory of Keplerian systems. We want to simplify the description of the previous section and obtain an $O(\varepsilon)$ description of the quasi-steady collisional evolution.

4.1 Elements of secular collisionless dynamics

The first step is to understand the general features of the collisionless limit, $N \to \infty$ and $m_\star \to 0$ with $M =$ constant, of the kinetic equation (32) for $f(\mathcal{D}, t)$:

$$\frac{\partial f}{\partial t} + \Omega_{\mathrm{k}}\frac{\partial f}{\partial w} + \varepsilon\,[f, \varphi + \boldsymbol{r}\cdot\boldsymbol{A}_\bullet]_{(6)} = 0. \tag{33}$$

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We begin by recalling results from the theory presented in Paper I, which is accurate to $O(\varepsilon)$. The natural time-scales in the problem are (a) the Kepler orbital period $T_{\rm kep} = 2\pi/\Omega_{\rm k}$, on which the phase w varies, and (b) the longer secular time-scale $T_{\rm sec} = \varepsilon^{-1} T_{\rm kep}$ over which the other five Delaunay variables $\mathcal{R} \equiv \{I, L, L_z; g, h\}$ evolve in the mean.[4] The CBE describing secular evolution was derived by using the method of multiple scales, to orbit-average the full CBE over the fast orbital phase w. Two time variables t and a slow time $\tau = \varepsilon t$ were introduced; these are natural measures of time over $T_{\rm kep}$ and $T_{\rm sec}$, respectively. We argued that the secular description cannot depend on the fast orbital phase w. The remaining five Delaunay variables $\mathcal{R} \equiv \{I, L, L_z; g, h\}$ are the coordinates of a Gaussian ring (or just ring) which is a Keplerian ellipse of given semimajor axis, eccentricity, inclination, periapse angle and nodal longitude. The secular evolution of $N \gg 1$ stars can be thought of as the evolution of $N \gg 1$ Gaussian rings, so the $O(1)$ part of the DF must be a function of the slow variables $(\mathcal{R}, \tau)$, and independent of the fast variables (w, t). It turns out that the appropriate expansion of the DF, to $O(\varepsilon)$ accuracy, is

$$f(\mathcal{D}, t, \tau) = \frac{1}{2\pi}\{F(\mathcal{R}, \tau) + \varepsilon f_0(\mathcal{R}, \tau)\} + \frac{\varepsilon}{2\pi}\sum_{n\neq 0} f_n(\mathcal{R}, t, \tau)\exp[inw] + O(\varepsilon^2), \tag{34}$$

where $F(\mathcal{R}, \tau)$ will be referred to as the ring DF. The DF f is a probability distribution function in $\mathcal{D}$-space satisfying $\int f\,{\rm d}\mathcal{D} = 1$ in the absence of loss of stars to the MBH – see equation 1 of Paper I. Using ${\rm d}\mathcal{D} = {\rm d}\mathcal{R}\,{\rm d}w$, we have

$$\int F(\mathcal{R}, \tau)\,{\rm d}\mathcal{R} + \varepsilon\int f_0(\mathcal{R}, \tau)\,{\rm d}\mathcal{R} + O(\varepsilon^2) = 1. \tag{35}$$

The method of multiple scales provided a solution to $O(\varepsilon)$. We returned to the usual single-time description, and presented the following solution to the full CBE equation (33):

$$f(\mathcal{D}, t) = \frac{1}{2\pi}\{F(\mathcal{R}, \varepsilon t) + \varepsilon f_0(\mathcal{R}, \varepsilon t)\} + \frac{\varepsilon}{2\pi}\Delta F(\mathcal{D}, \varepsilon t) + \frac{\varepsilon}{2\pi}\Lambda(\mathcal{R},\ w - \Omega_{\rm k} t,\ \varepsilon t) + O(\varepsilon^2). \tag{36}$$

It turns out that $\boldsymbol{A}_\bullet = \boldsymbol{0}$ to the required accuracy (see equation A3), and $F(\mathcal{R}, \tau)$ satisfies the ring CBE:

$$\frac{\partial F}{\partial \tau} + [F, \Phi] = 0, \tag{37}$$

where $\Phi(\mathcal{R}, \tau)$ is the ring Hamiltonian (coming from stellar self-gravity),

$$\Phi(\mathcal{R}, \tau) = \oint \varphi(\mathcal{D}, \tau)\,\frac{{\rm d}w}{2\pi}, \tag{38}$$

which is equal to the orbit-averaged mean-field potential. The PB without subscript is the 4-dim PB,

$$[\chi_1, \chi_2] = \left(\frac{\partial\chi_1}{\partial g}\frac{\partial\chi_2}{\partial L} - \frac{\partial\chi_1}{\partial L}\frac{\partial\chi_2}{\partial g}\right) + \left(\frac{\partial\chi_1}{\partial h}\frac{\partial\chi_2}{\partial L_z} - \frac{\partial\chi_1}{\partial L_z}\frac{\partial\chi_2}{\partial h}\right), \tag{39}$$

whose action is restricted to the 4-dim $I =$ constant surfaces in the 5-dim $\mathcal{R}$-space. Under the self-consistent ring dynamics of equation (37), $F(\mathcal{R}, \tau)$ can have significant time variations over times $T_{\rm sec}$.

As shown in equation (36), there are three kinds of $O(\varepsilon)$ corrections to the ring description. $f_0(\mathcal{R}, \tau)$ is a secular correction term to the ring DF, which remains undetermined in our $O(\varepsilon)$ theory. The other two correction terms are purely fluctuating in w. Of these, $\Delta F(\mathcal{D}, \varepsilon t)$ is a known function of the ring DF F and w; every ring DF $F(\mathcal{R}, \varepsilon t)$ in the 5-dim $\mathcal{R}$-space is accompanied by this small fluctuating (in w) distortion in the full 6-dim $\mathcal{D}$-space. Λ is an arbitrary function of its arguments $\mathcal{R}$, $(w - \Omega_{\rm k} t)$ and εt, with zero mean over w. The arbitrariness of form is due to the limitations of our $O(\varepsilon)$ theory. Yet it has physical significance, in that it emphasizes the requirement that there are no instabilities growing on the fast orbital times $T_{\rm kep}$ – as it must be, for $F(\mathcal{R}, \tau)$ to be a physical $O(1)$ description.

The $O(1)$ self-consistent secular collisionless description is obtained by taking the limit $\varepsilon \to 0$, while the time-scales of interest are large enough to make τ a sensible $O(1)$ measure of time. This limit is described completely by the ring DF, $F(\mathcal{R}, \tau)$, which satisfies the ring CBE (equation 37). The normalization of equation (35) reduces to

$$\int F(\mathcal{R}, \tau)\,{\rm d}\mathcal{R} = 1, \tag{40}$$

so $F(\mathcal{R}, \tau)$ may be thought of as a probability distribution function in the 5-dim $\mathcal{R}$-space. The orbit of every ring is restricted to its 4-dim $I =$ constant surface. The dynamics of the remaining four ring variables $\{L, L_z; g, h\}$ is given by the Hamiltonian equations,

$$\frac{{\rm d}L}{{\rm d}\tau} = -\frac{\partial\Phi}{\partial g}, \qquad \frac{{\rm d}g}{{\rm d}\tau} = \frac{\partial\Phi}{\partial L}; \qquad \frac{{\rm d}L_z}{{\rm d}\tau} = -\frac{\partial\Phi}{\partial h}, \qquad \frac{{\rm d}h}{{\rm d}\tau} = \frac{\partial\Phi}{\partial L_z}, \tag{41}$$

which will be referred to as the ring equations of motion. Using these, it is straightforward to write the ring CBE (equation 37) as ${\rm d}F/{\rm d}\tau = 0$, justifying the term 'collisionless'. The ring description makes no reference to the orbital phase w, and is self-consistent and closed in 5-dim $\mathcal{R}$-space. However, we will later need to take account of the evolution of w to $O(1)$. It turns out that this is slaved to the ring degrees of

[4] The relevant time-scales for the two best-known Keplerian systems, the Galactic Centre and the nucleus of M31, are discussed at the end of the introductory part of section 2 in Paper I.

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freedom, and given by

$$\frac{\mathrm{d}w}{\mathrm{d}\tau} = \frac{\Omega_{\rm k}(I)}{\varepsilon} + \frac{\partial \Phi}{\partial I}. \tag{42}$$

The orbital phase increases steadily over times $T_{\rm kep}$, and also has modulations over times $T_{\rm sec}$.

4.2 Orbit-averaged kinetic equation

We now return to the problem of quasi-steady collisional evolution of Keplerian stellar systems, and simplify the kinetic equation of the previous section through first-order perturbation theory in ε. In quasi-steady evolution, it is clear that the DF f cannot depend on t, and hence must be of the form $f(\mathcal{D}, \tau)$. Moreover, in the collisionless limit, the DF must go smoothly to the form given in equation (34). Therefore, we write

$$f(\mathcal{D}, \tau) = \frac{1}{2\pi} F(\mathcal{R}, \tau) + \frac{\varepsilon}{2\pi} \sum_{n=-\infty}^{\infty} f_n(\mathcal{R}, \tau) \exp\left[\mathrm{i}nw\right] + O(\varepsilon^2), \tag{43}$$

where $F \geq 0$ and $f_{-n} = f_n^\star$. We recall from Section 2 that the wake is a 'response function' characterizing the collisionless system which, to $O(1)$, is described completely by the secular DF $F(\mathcal{R}, \tau)$. Hence to $O(1)$ the wake must describe the perturbation in $\mathcal{R}$-space due to the selection of a ring at a given location. Therefore, the power series in ε for the wake can be written as

$$\mathcal{W}(\mathcal{D} \,|\, \mathcal{D}', \tau) = \frac{1}{2\pi} W(\mathcal{R} \,|\, \mathcal{R}', \tau) + \frac{\varepsilon}{2\pi} \sum_{n=-\infty}^{\infty} \varpi_n(\mathcal{R} \,|\, \mathcal{D}', \tau) \exp\left[\mathrm{i}nw\right] + O(\varepsilon^2), \tag{44}$$

where $(1/N)W(\mathcal{R} \,|\, \mathcal{R}', \tau)$ can be thought of as the additional perturbation at $\mathcal{R}$, resulting from the selection of a ring at $\mathcal{R}'$. We will refer to $W(\mathcal{R} \,|\, \mathcal{R}', \tau)$ as the ring wake function.

The Gilbert equations (31) and (32) are valid for any value of the mass ratio ε. Orbit-averaging is a systematic, perturbative method that can be applied to a Keplerian stellar system for which $\varepsilon \ll 1$. This proceeds by substituting the multiple-scale forms, equation (43) for the DF f and equation (44) for the wake $\mathcal{W}$, in the Gilbert equations and developing them perturbatively in the small parameter ε. The calculations are straightforward but lengthy, and we provide the details in Appendix A. Henceforth, we discuss the $O(1)$ theory for the collisional evolution of $F(\mathcal{R}, \tau)$ and $W(\mathcal{R} \,|\, \mathcal{R}', \tau)$ in 5-dim $\mathcal{R}$-space – the fluctuating quantities, f_n and ϖ_n, contribute only at higher order. In contrast to Gilbert's kinetic theory of point masses, this is a kinetic theory of Gaussian rings, and is the appropriate framework for describing the RR of RT96. Analogous to the various potentials that arise in the PB form of Gilbert's equations, various ring potentials make their appearance. These are all derived in Appendix A – see equations (A17b), (A18), (A20) and (A21) – and reproduced below:

$$\Psi(\mathcal{R}, \mathcal{R}') = -GM_\bullet \oint \oint \frac{\mathrm{d}w}{2\pi} \frac{\mathrm{d}w'}{2\pi} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

is the 'bare' inter-ring potential in $\mathcal{R}$-space, and plays a role analogous to the Poisson kernel, $p(\mathcal{D}, \mathcal{D}')$, in Gilbert's theory. The other ring potentials are

$$\Phi(\mathcal{R}, \tau) = \int F(\mathcal{R}', \tau)\, \Psi(\mathcal{R}, \mathcal{R}')\, \mathrm{d}\mathcal{R}', \quad \text{Ring mean-field potential;}$$

$$\Phi^{\rm w}(\mathcal{R}, \mathcal{R}', \tau) = \int W(\mathcal{R}'' \,|\, \mathcal{R}', \tau)\, \Psi(\mathcal{R}, \mathcal{R}'')\, \mathrm{d}\mathcal{R}'', \quad \text{Ring wake potential;}$$

$$\Phi^{\rm p}(\mathcal{R}, \mathcal{R}', \tau) = \Psi(\mathcal{R}, \mathcal{R}') - \Phi(\mathcal{R}, \tau), \quad \text{Ring perturbing potential.}$$

All of these have the same significance as the corresponding potentials in Gilbert's theory, as is clear from their appearance in the governing equations for W and F given below.

From equation (A22) the wake function $W(\mathcal{R} \,|\, \mathcal{R}'(\tau'), \tau')$ is determined by

$$\frac{\partial W}{\partial \tau'} + \left[W(\mathcal{R} \,|\, \mathcal{R}'(\tau'), \tau'), \Phi(\mathcal{R}, \tau')\right] + \left[F(\mathcal{R}, \tau'), \Phi^{\rm w}(\mathcal{R}, \mathcal{R}'(\tau'), \tau')\right] = \left[\Phi^{\rm p}(\mathcal{R}, \mathcal{R}'(\tau'), \tau'), F(\mathcal{R}, \tau')\right], \qquad \text{for } \tau' \leq \tau,$$

$$\text{with adiabatic turn-on initial condition} \quad \lim_{\tau' \to -\infty} W(\mathcal{R} \,|\, \mathcal{R}'(\tau'), \tau') = 0. \tag{45}$$

From equations (A23) and (A2), the kinetic equation for the ring DF $F(\mathcal{R}, \tau)$ is

$$\frac{\partial F}{\partial \tau} + \left(1 - \frac{1}{N}\right)[F, \Phi(\mathcal{R}, \tau)] = \frac{1}{N} \int \left[\Psi(\mathcal{R}, \mathcal{R}'), F^{(2)}_{\rm irr}(\mathcal{R}, \mathcal{R}', \tau)\right] \mathrm{d}\mathcal{R}',$$

$$F^{(2)}_{\rm irr}(\mathcal{R}, \mathcal{R}', \tau) = W(\mathcal{R} \,|\, \mathcal{R}', \tau)\, F(\mathcal{R}', \tau) + W(\mathcal{R}' \,|\, \mathcal{R}, \tau)\, F(\mathcal{R}, \tau) + \int W(\mathcal{R} \,|\, \mathcal{R}'', \tau)\, W(\mathcal{R}' \,|\, \mathcal{R}'', \tau)\, F(\mathcal{R}'', \tau)\, \mathrm{d}\mathcal{R}''. \tag{46}$$

Equations (45) and (46) are the basic equations governing the RR of a non-relativistic and isolated Keplerian stellar system. In Appendix A we discuss the $O(\varepsilon)$ fluctuations and argue that they are bounded and hence equations (45) and (46) form a closed and consistent set of equations in 5-dim $\mathcal{R}$-space. An important feature of the evolution of $F(\mathcal{R}, \tau)$, as described by equations (45) and (46), is the absence of the

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mass ratio ε in all the physical quantities, except in the time variable $\tau = \varepsilon t$. As pointed out by RT96, this implies the following property of mass invariance: if the mass of every star in a Keplerian cluster is changed from $m_\star$ to $m_{\star 1}$, then the collisional evolution of the transformed cluster is identical to the original one, so long as the time variable τ is replaced by the new time variable $\tau_1 = (m_{\star 1}/m_\star)\tau$.

5 RESONANT RELAXATION

Here we provide a self-contained description of the equations governing RR, after generalizing to include orbit-averaged relativistic corrections and external perturbations; these will, of course, break the property of mass invariance of time evolution mentioned above.

5.1 Collisionless limit

The ring mean-field Hamiltonian $H(\mathcal{R}, \tau)$ can be obtained by supplementing the ring mean-field potential $\Phi(\mathcal{R}, \tau)$ with general relativistic corrections and external perturbations. *Relativistic corrections* to the gravity of the MBH cause precession of both the apse and node of a Gaussian ring. The relativistic (secular) Hamiltonian governing the 1 PN Schwarzschild precession of the apses and the 1.5 PN Lense–Thirring precession of the apses and nodes for a spinning MBH is given by

$$H^{\rm rel}(I, L, L_z) = H^{\rm PN1}(I, L) + H^{\rm PN1.5}(I, L, L_z), \qquad H^{\rm PN1}(I, L) = -\,B_1\,\frac{1}{I^3 L} \quad \text{with} \quad B_1 = \frac{3(GM_\bullet)^4}{c^2}\,\frac{M_\bullet}{M},$$
$$H^{\rm PN1.5}(I, L, L_z) = B_{1.5}\,\frac{L_z}{I^3 L^3}, \quad \text{with} \quad B_{1.5} = \frac{2(GM_\bullet)^5\,\chi}{c^3}\,\frac{M_\bullet}{M}, \tag{47}$$

where $0 \le \chi \le 1$ is the spin parameter. Here we have assumed that the MBH spin angular momentum points along the z-axis; factors $(M_\bullet/M) = \varepsilon^{-1}$ have been included in the constants B_1 and $B_{1.5}$, because we measure time in terms of the slow time $\tau = \varepsilon t$. *External perturbations* due to nuclear density cusps and/or slowly moving distant masses can also be included. Let $\varepsilon\Phi^{\rm ext}(\mathcal{R}, \tau)$ be the orbit-averaged potential and $\varepsilon \boldsymbol{A}_\bullet^{\rm ext}(\tau)$ be the acceleration of the MBH. These external sources will contribute a (scaled) orbit-averaged tidal potential,

$$\Phi^{\rm tid}(\mathcal{R}, \tau) = \Phi^{\rm ext}(\mathcal{R}, \tau) + \boldsymbol{X}(\mathcal{R}) \cdot \boldsymbol{A}_\bullet^{\rm ext}(\tau), \tag{48}$$

$$\text{where} \qquad \boldsymbol{X}(\mathcal{R}) = \oint \frac{{\rm d}w}{2\pi}\,\boldsymbol{r}(\mathcal{D}) = \text{centre of mass of a Gaussian ring.} \tag{49}$$

Then the general mean-field ring Hamiltonian is

$$H(\mathcal{R}, \tau) = \Phi(\mathcal{R}, \tau) + H^{\rm rel}(I, L, L_z) + \Phi^{\rm tid}(\mathcal{R}, \tau), \tag{50}$$

where we note that the mean-field potential is, as given in equation (A18), determined self-consistently by the ring DF:

$$\Phi(\mathcal{R}, \tau) = \int F(\mathcal{R}', \tau)\,\Psi(\mathcal{R}, \mathcal{R}')\,{\rm d}\mathcal{R}', \qquad \text{Ring mean-field potential;} \tag{51}$$

where

$$\Psi(\mathcal{R}, \mathcal{R}') = -GM_\bullet \oint\oint \frac{{\rm d}w}{2\pi}\,\frac{{\rm d}w'}{2\pi}\,\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}, \qquad \text{'bare' inter-ring potential.} \tag{52}$$

The (mean-field) ring equations of motion are

$$I = \sqrt{GM_\bullet a} = \text{constant},$$
$$\frac{{\rm d}L}{{\rm d}\tau} = -\,\frac{\partial H}{\partial g}, \qquad \frac{{\rm d}g}{{\rm d}\tau} = \frac{\partial H}{\partial L}; \qquad \frac{{\rm d}L_z}{{\rm d}\tau} = -\,\frac{\partial H}{\partial h}, \qquad \frac{{\rm d}h}{{\rm d}\tau} = \frac{\partial H}{\partial L_z}. \tag{53}$$

Therefore, in the collisionless limit, the appropriate generalization of equation (37) for the $F(\mathcal{R}, \tau)$ is

$$\frac{\partial F}{\partial \tau} + [F, H] = 0. \tag{54}$$

This is the general form of the ring CBE, whose properties are discussed in more detail in Paper I.

5.2 Kinetic equation for RR

We now need to generalize equation (45) for the ring wake function, and equation (46) for the ring DF, to account for the change in the mean-field Hamiltonian from $\Phi(\mathcal{R}, \tau)$ to $H(\mathcal{R}, \tau)$. The new wake equation can be determined from the collisionless response of the stellar system, by extending the *gedanken* experiment of Gilbert (1968) for point mass stars to Gaussian rings: from the N rings that are distributed in 5-dim ring space according to the DF $F(\mathcal{R}, \tau)$, select one ring and place it at the phase space location $\mathcal{R}'$; this will induce a small perturbation $(1/N)W$ at every location $\mathcal{R}$ in the 5-dim $\mathcal{R}$-space. Then the net perturbed DF at any instant $\tau' < \tau$ is

$$F_1(\mathcal{R}, \tau') = -\,\frac{F(\mathcal{R}, \tau')}{N} + \frac{\delta(\mathcal{R} - \mathcal{R}'(\tau'))}{N} + \frac{W(\mathcal{R}\,|\,\mathcal{R}'(\tau'), \tau')}{N}, \tag{55}$$

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where $\mathcal{R}'(\tau')$ is the location at time τ' of the ring which arrives at $\mathcal{R}'$ at time τ. Since $N \gg 1$ the perturbation is effectively infinitesimal, so F_1 must satisfy the linearized CBE discussed in Paper I. This then gives the general ring wake equation we seek:

$$\frac{\partial W}{\partial \tau'} + \left[W(\mathcal{R}\,|\,\mathcal{R}'(\tau'),\tau'), H(\mathcal{R},\tau')\right] + \left[F(\mathcal{R},\tau'), \Phi^{\rm w}(\mathcal{R},\mathcal{R}'(\tau'),\tau')\right] = \left[\Phi^{\rm p}(\mathcal{R},\mathcal{R}'(\tau'),\tau'), F(\mathcal{R},\tau')\right], \qquad \text{for } \tau' \le \tau,$$

$$\text{with adiabatic turn-on initial condition} \quad \lim_{\tau'\to-\infty} W(\mathcal{R}\,|\,\mathcal{R}'(\tau'),\tau') = 0. \tag{56}$$

Here $\Phi^{\rm w}$ is the gravitational potential due to the wake which was derived in equation (A20):

$$\Phi^{\rm w}(\mathcal{R},\mathcal{R}',\tau') = \int W(\mathcal{R}''\,|\,\mathcal{R}',\tau')\,\Psi(\mathcal{R},\mathcal{R}'')\,{\rm d}\mathcal{R}'', \qquad \text{Ring wake potential.} \tag{57}$$

$\Phi^{\rm p}$ is the difference between the 'bare' inter-ring interaction potential and the mean-field potential, which was derived in equation (A21):

$$\Phi^{\rm p}(\mathcal{R},\mathcal{R}',\tau') = \Psi(\mathcal{R},\mathcal{R}') - \Phi(\mathcal{R},\tau'), \qquad \text{Ring perturbing potential.} \tag{58}$$

As earlier the ring wake function satisfies the two identities

$$\int W(\mathcal{R}\,|\,\mathcal{R}',\tau)\,{\rm d}\mathcal{R} = 0, \qquad \text{Zero mass in the wake of ring } \mathcal{R}';$$

$$\int W(\mathcal{R}\,|\,\mathcal{R}',\tau) F(\mathcal{R}',\tau)\,{\rm d}\mathcal{R}' = 0, \qquad \text{Zero net wake at } \mathcal{R} \text{ due to all the rings.} \tag{59}$$

When we compare equation (56) with the earlier equation (45) for the ring wake function, we see that the only change is in the second term on the left-hand side where Φ has been replaced by H. This is expected because the mean-field ring orbits are now governed by H. However, on the right-hand side, the perturbing potential $\Phi^{\rm p} = \Psi - \Phi$ remains unchanged, because this is the driver of the irreversible statistical evolution of the system, and cannot be affected by deterministic agencies like relativistic corrections and external gravitational fields.

The new kinetic equation for the DF $F(\mathcal{R},\tau)$ is obtained by replacing, on the left-hand side of equation (46), the quantity $(\Phi - \Phi/N)$ by $(H - \Phi/N)$, because the Φ/N term corrects for the fact that only $(N-1)$ of the N stars exert forces on any given star. The right-hand side, which represents irreversible statistical evolution, remains unaltered. Therefore, the general ring kinetic equation is

$$\frac{\partial F}{\partial \tau} + \left[F, H - \frac{\Phi(\mathcal{R},\tau)}{N}\right] = \frac{1}{N}\int \left[\Psi(\mathcal{R},\mathcal{R}'), F^{(2)}_{\rm irr}(\mathcal{R},\mathcal{R}',\tau)\right]\,{\rm d}\mathcal{R}',$$

$$F^{(2)}_{\rm irr}(\mathcal{R},\mathcal{R}',\tau) = W(\mathcal{R}\,|\,\mathcal{R}',\tau)\,F(\mathcal{R}',\tau) + W(\mathcal{R}'\,|\,\mathcal{R},\tau)\,F(\mathcal{R},\tau) + \int W(\mathcal{R}\,|\,\mathcal{R}'',\tau)\,W(\mathcal{R}'\,|\,\mathcal{R}'',\tau)\,F(\mathcal{R}'',\tau)\,{\rm d}\mathcal{R}''. \tag{60}$$

By eliminating $F^{(2)}_{\rm irr}$ and manipulating the resulting expressions, we can also write the ring kinetic equation in a form where the dissipative and fluctuating contributions to the ring collision term are displayed explicitly:

$$\frac{\partial F}{\partial \tau} + \left[F, H - \frac{\Phi(\mathcal{R},\tau)}{N}\right] = C^{\rm dis}[F] + C^{\rm fluc}[F], \tag{61a}$$

$$C^{\rm dis}[F] = \frac{1}{N}\int \left[\Psi(\mathcal{R},\mathcal{R}'), F(\mathcal{R},\tau) W(\mathcal{R}'\,|\,\mathcal{R},\tau)\right]\,{\rm d}\mathcal{R}', \tag{61b}$$

$$C^{\rm fluc}[F] = \frac{1}{N}\int F(\mathcal{R}',\tau)\left[\Psi(\mathcal{R},\mathcal{R}') + \Phi^{\rm w}(\mathcal{R},\mathcal{R}',\tau), W(\mathcal{R}\,|\,\mathcal{R}',\tau)\right]\,{\rm d}\mathcal{R}'. \tag{61c}$$

The kinetic equation – either equation (60) or (61a)–(61c) – applies to the collisional evolution of F whenever it satisfies

$$\left[F(\mathcal{R},\tau), H(\mathcal{R},\tau) - \frac{\Phi(\mathcal{R},\tau)}{N}\right] \simeq 0, \quad \text{being of order } 1/N \text{ or smaller.} \tag{62}$$

Therefore, $F(\mathcal{R},\tau)$ may be thought of as passing through a sequence of collisionless equilibria in a quasi-static manner.

Boundary conditions. Equation (56) and either equation (60) or (61a)–(61c) need to be supplied with boundary conditions in $\mathcal{R}$-space. There are two main cases of interest.

(1) 'Lossless' stellar systems in which the MBH is not considered to be a sink of stars. Then the DF is normalized as $\int F(\mathcal{R},\tau)\,{\rm d}\mathcal{R} = 1$. Subject to this normalization $F(\mathcal{R},\tau)$ can take any positive value at any location in $\mathcal{R}$-space; in other words, the domain of F is all of $\mathcal{R}$-space. Hence, the domain of the wake function $W(\mathcal{R}\,|\,\mathcal{R}',\tau)$ is all of $\mathcal{R}$- and $\mathcal{R}'$-spaces.

(2) Stellar systems in which stars that come too close to the MBH are either tidally shredded or swallowed whole. Since the system loses stars, we can only require that the normalization of the DF at some initial time τ_0 be

$$\int F(\mathcal{R},\tau_0)\,{\rm d}\mathcal{R} = 1, \qquad \text{Initial normalization with loss of stars to MBH.} \tag{63}$$

At later times $\tau > \tau_0$, we will have $\int F(\mathcal{R},\tau)\,{\rm d}\mathcal{R} \le 1$. The simplest model of the loss is the assumption that a star is lost to the MBH when its pericentre distance is smaller than some fixed value $r_{\rm lc}$, the *loss-cone radius*. When the loss cone is empty, stars belonging to the cluster

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must necessarily have pericentre radii $a(1-e)$ larger than $r_{\rm lc}$. Hence, the domain of $F(\mathcal{R},\tau)$ is restricted to regions of $\mathcal{R}$-space in which I and L are large enough:

$$I_{\rm lc} < I \quad \text{where} \qquad I_{\rm lc} = \sqrt{GM_\bullet r_{\rm lc}}; \qquad L_{\rm lc}(I) < L \le I, \quad \text{where} \quad L_{\rm lc}(I) = I_{\rm lc}\left[2-\left(\frac{I_{\rm lc}}{I}\right)^2\right]^{1/2}. \tag{64}$$

As I increases from its minimum value of $I_{\rm lc}$, the function $L_{\rm lc}(I)$ monotonically increases from its minimum value of $I_{\rm lc}$ and approaches its asymptotic value of $\sqrt{2}I_{\rm lc}$. The ring DF and wake functions satisfy the 'empty loss-cone' (or 'absorbing') boundary conditions: that for all τ,

$$F(\mathcal{R},\tau)=0 \qquad \text{for } I \le I_{\rm lc} \text{ and } L \le L_{\rm lc}(I), \qquad W(\mathcal{R}\,|\,\mathcal{R}',\tau)=0 \qquad \text{for } I, I' \le I_{\rm lc} \text{ and } L \le L_{\rm lc}(I),\ L' \le L_{\rm lc}(I'). \tag{65}$$

Equation (56) and either equation (60) or (61a)–(61c), together with suitable boundary conditions on the DF and wake, are the fundamental equations governing RR.

6 DISCUSSION

It should come as no surprise that there is a structural similarity between the ring kinetic equation and the PB form of Gilbert's equation.[5] This invariance of form is a natural consequence of averaging over a (fast) angle variable, a procedure that preserves canonical structure in phase space. Gilbert's equations apply to a general stellar system in 6-dim phase space, whereas the RR equations apply to a Keplerian stellar system orbiting an MBH in a reduced 5-dim ring space. What we have demonstrated is that the ring equations can be obtained from the 6-dim Gilbert equations by replacing (a) 'point mass star' with 'Gaussian ring', (b) all 6-dim PBs with 4-dim PBs, and (c) the time variable t by the slow time $\tau=\varepsilon t$. Thus, we have corresponding (scaled) potentials: $\Psi(\mathcal{R},\mathcal{R}')$ which is the 'bare' inter-ring potential between rings $\mathcal{R}$ and $\mathcal{R}'$; the ring mean-field potential $\Phi(\mathcal{R},\tau)$ at location $\mathcal{R}$; the potential $\Phi^{\rm w}(\mathcal{R},\mathcal{R}',\tau)$ felt by a ring at $\mathcal{R}$ due to the wake of ring $\mathcal{R}'$; and $\Phi^{\rm p}(\mathcal{R},\mathcal{R}',\tau)=\Psi(\mathcal{R},\mathcal{R}')-\Phi(\mathcal{R},\tau)$ which is the perturbing potential felt by a ring at $\mathcal{R}$ due to a ring at $\mathcal{R}'$. All these potentials are related to the DF or the wake through formulae that are analogous to those in Gilbert's theory.

Ring wake as the driver of RR. The kinetic equation in the BBGKY form (equation 60) tells us that irreversible collisional evolution is driven by the irreducible part of the two-particle correlation:

$$F^{(2)}_{\rm irr}(\mathcal{R},\mathcal{R}',\tau) = W(\mathcal{R}\,|\,\mathcal{R}',\tau)\,F(\mathcal{R}',\tau) + W(\mathcal{R}'\,|\,\mathcal{R},\tau)\,F(\mathcal{R},\tau) + \int W(\mathcal{R}\,|\,\mathcal{R}'',\tau)\,W(\mathcal{R}'\,|\,\mathcal{R}'',\tau)\,F(\mathcal{R}'',\tau)\,{\rm d}\mathcal{R}''. \tag{66}$$

This decomposition of $F^{(2)}_{\rm irr}$ means that the irreducible two-ring correlation at $(\mathcal{R},\mathcal{R}')$ gets three kinds of contributions from the wake function: (a) the wake of $\mathcal{R}'$ at $\mathcal{R}$; (b) the wake of $\mathcal{R}$ at $\mathcal{R}'$; (c) the product of the wake values at the points $\mathcal{R}$ and $\mathcal{R}'$ of a third ring at $\mathcal{R}''$, summed over all locations $\mathcal{R}''$. All three contributions come with suitable F-weighting. The third term accounts for the contribution of collective effects ('gravitational polarization') to the microscopic processes driving RR. This is the nature of the full theory at $O(1/N)$.

Correlations build through ring–ring collisions: at very early times $\tau\to-\infty$, the wake $W\to 0$ which implies that $F^{(2)}_{\rm irr}\to 0$. As the wake of every ring builds up over time, so does $F^{(2)}_{\rm irr}$, hence we can think of the wake as the fundamental driver of collisional evolution. The ring wake equation (56) is

$$\frac{\partial W}{\partial\tau'} + \left[W(\mathcal{R}\,|\,\mathcal{R}'(\tau'),\tau'),H(\mathcal{R},\tau')\right] + \left[F(\mathcal{R},\tau'),\Phi^{\rm w}(\mathcal{R},\mathcal{R}'(\tau'),\tau')\right] = \left[\Phi^{\rm p}(\mathcal{R},\mathcal{R}'(\tau'),\tau'),F(\mathcal{R},\tau')\right], \quad \text{for } \tau'\le\tau,$$

with adiabatic turn-on initial condition $\lim_{\tau'\to-\infty} W(\mathcal{R}\,|\,\mathcal{R}'(\tau'),\tau')=0,$

where

$$\Phi^{\rm w}(\mathcal{R},\mathcal{R}',\tau') = \int W(\mathcal{R}''\,|\,\mathcal{R}',\tau')\,\Psi(\mathcal{R},\mathcal{R}'')\,{\rm d}\mathcal{R}'', \qquad \text{Ring wake potential;}$$

$$\Phi^{\rm p}(\mathcal{R},\mathcal{R}',\tau') = \Psi(\mathcal{R},\mathcal{R}')-\Phi(\mathcal{R},\tau'), \qquad \text{Ring perturbing potential.}$$

Note that $\Phi^{\rm w}$ is linear in W, whereas $\Phi^{\rm p}$ is independent of W. If the right-hand side happened to be zero, i.e. $[\Phi^{\rm p},F(\mathcal{R},\tau')]=0$, then the wake equation would be a linear integral equation which is homogeneous in W. The solution that is compatible with the adiabatic turn-on initial condition is $W=0$ for all time. Since there is no wake, the collision term in equations (61a)–(61c) vanishes. Then the ring DF satisfies the ring CBE, and there is no relaxation. Therefore,

$$\left[\Phi^{\rm p}(\mathcal{R},\mathcal{R}'(\tau'),\tau'),F(\mathcal{R},\tau')\right] = \left[\Psi(\mathcal{R},\mathcal{R}'(\tau'))-\Phi(\mathcal{R},\tau'),F(\mathcal{R},\tau')\right] \tag{67}$$

is the 'source term' for the ring wake function, and for the entire RR process. In general, this drives changes in both the magnitude and direction of angular momenta through apsidal and nodal resonances, and there is no strict separation into scalar RR and vector RR.

[5] Compare the ring wake equation (56) with equation (13); the ring kinetic equation in the BBGKY form (equation 60) with equation (14); or the ring kinetic equation in fluctuation–dissipation form (equation 61a–61c) with equations (15a)–(15c).

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Time evolution. The ring kinetic equation is for RR what the Boltzmann equation is for the kinetic theory of gases. Just as the Boltzmann equation follows the DF over times that are longer than the duration of collisions between gas molecules, the ring kinetic equation tracks changes in the DF of the system over times that are longer than $T_{\rm sec}$ which is the typical coherence time of secular stellar encounters. Over times ∼several $T_{\rm sec} \ll NT_{\rm sec}$, the collision term $\{C^{\rm dis}[F] + C^{\rm fluc}[F]\}$ makes only a small contribution to the change in F. Then the system is effectively collisionless and is well approximated by the ring CBE (equation 54). In this phase, the ring DF can display significant variations over times $T_{\rm sec}$: these could be due to (a) secular instabilities, or (b) secular collisionless relaxation (i.e. violent relaxation of Gaussian rings), or (c) excitation of secular modes by an external perturber, and possible loss of stars to the MBH. In the absence of continuing forcing on the secular time-scale by external sources, it is expected that the stellar system would settle in a stationary state $\partial F/\partial\tau = 0$ which implies that $[F, H] = 0$. By the secular Jeans theorem of Paper I, F must be a function of the isolating integrals of motion of H. These ring DFs are necessarily linearly stable to perturbations.

Over a longer time-scale $\gg T_{\rm sec}$, the effect of the collision term $\{C^{\rm dis}[F] + C^{\rm fluc}[F]\}$ can no longer be ignored. The ring DF will evolve slowly due to the slow accrual of the relaxing effect of angular momentum exchanges between the rings. We can think of F as passing slowly through a sequence of secular Jeans equilibria for which $[F, H - \Phi/N]$ is of smaller order than $1/N$. Then equations (61a)–(61c) imply that $\partial F/\partial\tau \simeq \left\{C^{\rm dis}[F] + C^{\rm fluc}[F]\right\} \sim O(1/N)$. Therefore, the time over which F deforms (quasi-steadily) by order unity is the RR time-scale $T_{\rm res} = NT_{\rm sec} = (N/\varepsilon)T_{\rm kep}$. During this phase of evolution, we can allow for the external tidal potential $\Phi^{\rm tid}$ to also vary slowly – as might be the case when a galactic nuclear cusp adjusts adiabatically to the slowly varying ring DF. All through the relaxation process, rings exchange angular momentum, so a ring that has lost enough angular momentum to have its periapse decrease to $r_{\rm lc}$ will be lost to the MBH. It is of great interest to determine the mass, energy and angular momentum lost by the Keplerian star cluster to the MBH.

7 CONCLUSIONS

We have derived the fundamental kinetic equation governing the RR of low-mass stellar systems around an MBH. This includes the effects of stellar self-gravity, general relativistic corrections up to 1.5 PN order, as well as external deterministic sources of gravity that vary on the secular time-scale. We considered a stellar system consisting of stars of equal mass, but this is not a serious limitation. Mass-segregation effects in RR can be studied by straightforward generalization to a system with a range of stellar masses. We have seen in the discussion of the previous section that the wake function is the driver of RR, and that the source term for the wake has contributions from both apsidal and nodal resonances which may be equally important in general geometries. Therefore, the traditional, physically motivated split of RR into scalar RR and vector RR dissolves, in general, into a seamless blend of both processes, with the latter dominating when fast apse precession promotes apse-averaged dynamics. We noted the structural similarity between the ring kinetic equation and Gilbert's equation. Heyvaerts (2010) proved an H-theorem for the latter, for an integrable stellar system admitting global action–angle variables. It seems reasonable to expect that a similar exercise would be successful for RR. Our theory of RR is valid for stellar systems with arbitrary geometry and figure rotation, so long as these are dynamically stable quasi-steady secular equilibria. The orbital structure can be regular, chaotic or mixed;[6] the formalism presented in this paper applies to all of these.

Below is a brief guide for the application of the ring kinetic equation to any Keplerian stellar system. These are followed by remarks on future directions and useful ways of comparison with numerical simulations.

(1) Secure an 'explicit' form for the orbit-averaged Poisson kernel Ψ in equation (52). Although desirable, a closed form expression is difficult to come by; more often than not the orbit-averaged 'bare' ring potential is expressed as a Fourier series in the apses and nodes, with the Fourier coefficients depending on the eccentricities and the inclinations (or related Delaunay variables). One could also work with a pre-computed table of the kernel in the physical limit of interest.

(2) Using Ψ construct an equilibrium pair, $\{F, \Phi\}$, of DF F and mean-field potential, Φ, which is related to F through the integral of equation (51). Secular dynamics is governed by H, the mean-field Hamiltonian of equation (50), which is the sum of the mean-field potential Φ, and contributions from 1.5 PN general relativistic corrections and any external gravitational potential. By the secular Jeans theorem of Paper I, F must be a function of the secular integrals of motion. In general, one needs to solve an integral equation; it would be very interesting to see the development of Schwarzschild-like iterative numerical methods. Moreover, it is also necessary that the DF be dynamically stable.

(3) With orbit-averaged kernel in hand, and an initially stable equilibrium model, solve the wake equation (56) for W, with adiabatic turn-on initial condition.

(4) Use the pair $\{F, W\}$ to compute the collision integral, which is the right-hand side of the ring kinetic equation (61a)–(61c) driving relaxation.

(5) Solve the kinetic (equations 61a–61c) for F: the global solution of equation (61a) couples all the three steps given above, with the ring mean-field potential Φ recovered from F, then wake W updated with F and Φ as one progresses towards a relaxed state.[7]

[6] Averaging over the fast Kepler orbital phase is independent of whether the resulting secular dynamics is integrable or not; this is as true for planetary systems as Keplerian stellar systems. When the secular dynamics is non-integrable, chaotic diffusion through broken tori is restricted to the 4-dim $\{L, L_z, g, h\}$ subspace of the full 6-dim phase space.

[7] In the course of relaxation, the linear secular stability of the evolving quasi-equilibria must be monitored. In case one encounters a dynamical instability, fast (i.e. on the secular time-scale) collisionless processes will intervene. Once the instability has saturated and the system settled in a new secular collisionless equilibrium, the subsequent RR evolution can be resumed by following steps 2–5.

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The steps above are explicitly laid out in a companion paper Sridhar & Touma (2016b, hereafter Paper III), where we formulate and study the physical kinetics of the RR of zero-thickness, flat, axisymmetric discs. Working with the Polyachenko & Shukhman (1982) approximation of neglecting 'gravitational polarization', we derive a Fokker–Planck equation for the DF with diffusion coefficients that are given self-consistently in terms of contributions from apsidal resonances between pairs of stellar orbits. For 'lossless' stellar discs, we prove an H-theorem and conservation laws for the disc mass, energy and angular momentum. When stars are lost to the MBH through an empty loss cone, the loss rates of mass, energy and angular momentum can be expressed in terms of the diffusion flux at the loss-cone boundary.

It is hard to overemphasize the need for comparison with numerical simulations, and we can see three different routes of exploration: (a) The diffusion coefficients and angular momentum diffusion statistics predicted by our RR theory could be compared with results obtained from refined *N*-body simulations designed for that same purpose, such as Merritt et al. (2011) for RR in a spherical cluster. (b) The ring kinetic equation (61a)–(61c) can be solved for the thermally relaxed equilibrium of any chosen initial stable equilibrium. The equilibrium and associated statistics could be compared with the *N*-body evolution of (a sample of) the same initial state. (c) The time-evolving DF of our formalism can be compared with related solutions of various Fokker–Planck equations proposed by others (Hopman & Alexander 2006; Madigan et al. 2011), ad hoc though their assumptions may be. This should allow calibration of the performance of ad hoc recipes for RR vis-a-vis the fundamental formulation we have presented.

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REFERENCES

Bahcall J. N., Wolf R. A., 1976, ApJ, 209, 214
Balescu R., 1960, Phys. Fluids, 3, 52
Bar-Or B., Alexander T., 2014, Class. Quantum Grav., 31, 244003
Bar-Or B., Alexander T., 2015, preprint (arXiv:1508.01390)
Binney J., Tremaine S., 2008, Galactic Dynamics, 2nd edn. Princeton Univ. Press, Princeton, NJ
Chandrasekhar S., 1942, Principles of Stellar Dynamics. Univ. Chicago Press, Chicago, IL
Chandrasekhar S., 1943a, ApJ, 97, 255
Chandrasekhar S., 1943b, ApJ, 97, 263
Chavanis P.-H., 2012, Physica A, 391, 3680
Chavanis P.-H., 2013, A&A, 556, A93
Cohen R. S., Spitzer L., Routly P. M., 1950, Phys. Rev., 80, 230
Cohn H., Kulsrud R. M., 1978, ApJ, 226, 1087
Gasiorowicz S., Neuman M., Riddell R. J., 1956, Phys. Rev., 101, 922
Gilbert I. H., 1968, ApJ, 152, 1043
Gürkan M. A., Hopman C., 2007, MNRAS, 379, 1083
Hamers A. S., Portegies Zwart S. F., Merritt D., 2014, MNRAS, 443, 355
Heyvaerts J., 2010, MNRAS, 407, 355
Hopman C., Alexander T., 2006, ApJ, 645, 1152
Kocsis B., Tremaine S., 2011, MNRAS, 412, 187
Kocsis B., Tremaine S., 2015, MNRAS, 448, 3265
Kupi G., Eilon E., Alexander T., 2010, J. Phys.: Conf. Ser., 228, 012056
Landau L. D., 1936, Phys. Z. Sowj. Union, 10, 154
Lenard A., 1960, Ann. Phys., 10, 390
Lifshitz E. M., Pitaevskii L. P., 1981, Physical Kinetics, 1st edn. Pergamon Press, Oxford
Madigan A.-M., Hopman C., Levin Y., 2011, ApJ, 738, 99
Merritt D., 2015a, ApJ, 804, 52
Merritt D., 2015b, ApJ, 804, 128
Merritt D., 2015c, ApJ, 810, 2
Merritt D., Alexander T., Mikkola S., Will C. M., 2011, Phys. Rev. D, 84, 044024
Murray C. D., Dermott S. F., 1999, Solar System Dynamics. Cambridge Univ. Press, Cambridge
Plummer H. C., 1960, An Introductory Treatise on Dynamical Astronomy. Dover, New York
Polyachenko V. L., Shukhman I. G., 1982, SvA, 26, 140
Rauch K. P., Tremaine S., 1996, New Astron., 1, 149 (RT96)
Rosenbluth M. N., MacDonald W. M., Judd D. L., 1957, Phys. Rev., 107, 1
Rostoker N., 1964, Phys. Fluids, 7, 491
Sambhus N., Sridhar S., 2000, ApJ, 542, 143
Sridhar S., Touma J. R., 2016a, MNRAS, 458, 4129 (Paper I)
Sridhar S., Touma J. R., 2016b, preprint (arXiv:1602.05763) (Paper III)
Touma J., Tremaine S., 2014, J. Phys. A: Math. Theor., 47, 292001

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APPENDIX A: DERIVATION OF THE ORBIT-AVERAGED KINETIC EQUATION

Here we derive the orbit-averaged kinetic equation, beginning with the exact Gilbert equations (equations 31 and 32) given in Delaunay variables. Orbit-averaging is a systematic procedure, of perturbative development of these equations in the small parameter ε.

The first step is to substitute in equation (32) the multiple-scale forms, equation (43) for the DF f and equation (44) for the wake $\mathcal{W}$. Then all the $O(1)$ terms vanish. The $O(\varepsilon)$ terms give the following equation for $F(\mathcal{R},\tau)$ and all the $f_n(\mathcal{R},\tau)$:

$$\frac{\partial F}{\partial \tau} + \sum_{n\neq 0} in\Omega_{\rm k}\, f_n \exp\left[inw\right] + \left[F, \left(1-\frac{1}{N}\right)\varphi + \boldsymbol{r}\cdot\boldsymbol{A}_\bullet\right]_{(6)} = \frac{1}{N}\int \left[p(\mathcal{D},\mathcal{D}'), F^{(2)}_{\rm irr}(\mathcal{R},\mathcal{R}',\tau)\right]_{(6)} {\rm d}\mathcal{R}'\,\frac{{\rm d}w'}{2\pi}, \tag{A1}$$

where

$$F^{(2)}_{\rm irr}(\mathcal{R},\mathcal{R}',\tau) = W(\mathcal{R}\,|\,\mathcal{R}',\tau)\,F(\mathcal{R}',\tau) + W(\mathcal{R}'\,|\,\mathcal{R},\tau)\,F(\mathcal{R},\tau) + \int W(\mathcal{R}\,|\,\mathcal{R}'',\tau)\,W(\mathcal{R}'\,|\,\mathcal{R}'',\tau)\,F(\mathcal{R}'',\tau)\,{\rm d}\mathcal{R}'' \tag{A2}$$

is the irreducible part of the two-ring correlation function. Note that only the $O(1)$ part of the wake W contributes directly to the $O(\varepsilon)$ kinetic theory. The Poisson kernel $p(\mathcal{D},\mathcal{D}') = -GM_\bullet/|\boldsymbol{r}-\boldsymbol{r}'|$ is a specified $O(1)$ function of its arguments and is independent of ε. The other quantities, φ and $\boldsymbol{A}_\bullet$, are needed only to $O(1)$ accuracy. From equation (28c), we have to $O(1)$:

$$\boldsymbol{A}_\bullet(\tau) = GM_\bullet \int F(\mathcal{R},\tau)\,{\rm d}\mathcal{R} \oint \frac{{\rm d}w}{2\pi}\,\frac{\hat{\boldsymbol{r}}}{r^2} = \boldsymbol{0}, \tag{A3}$$

because $\oint {\rm d}w\,\hat{\boldsymbol{r}}/r^2 = \boldsymbol{0}$, by the conservation of angular momentum along a Kepler orbit; just as in the collisionless theory of Paper I, the MBH does not accelerate to $O(\varepsilon)$ accuracy. From equation (28b), the rescaled mean potential to $O(1)$ is

$$\varphi(\mathcal{D},\tau) = \int F(\mathcal{R}',\tau)\,{\rm d}\mathcal{R}' \oint p(\mathcal{D},\mathcal{D}')\,\frac{{\rm d}w'}{2\pi}. \tag{A4}$$

The PB on the right-hand side of equation (A1) is over the $\mathcal{D}$ variables, so the integral over w' can be taken inside the PB. Since $p(\mathcal{D},\mathcal{D}')$ is the only quantity inside the PB that depends on w', the w'-integral operates only on $p(\mathcal{D},\mathcal{D}')$. Then equation (A1) simplifies to

$$\frac{\partial F}{\partial \tau} + \sum_{n\neq 0} in\Omega_{\rm k}\, f_n \exp\left[inw\right] + \left(1-\frac{1}{N}\right)\left[F,\varphi(\mathcal{D},\tau)\right]_{(6)} = \frac{1}{N}\int \left[\oint p(\mathcal{D},\mathcal{D}')\frac{{\rm d}w'}{2\pi}, F^{(2)}_{\rm irr}(\mathcal{R},\mathcal{R}',\tau)\right]_{(6)} {\rm d}\mathcal{R}'\,. \tag{A5}$$

Averaging over w, we get the following equation for $F(\mathcal{R},\tau)$:

$$\frac{\partial F}{\partial \tau} + \left(1-\frac{1}{N}\right)\oint \left[F,\varphi(\mathcal{D},\tau)\right]_{(6)}\frac{{\rm d}w}{2\pi} = \frac{1}{N}\int \left[\oint\oint p(\mathcal{D},\mathcal{D}')\frac{{\rm d}w}{2\pi}\frac{{\rm d}w'}{2\pi}, F^{(2)}_{\rm irr}(\mathcal{R},\mathcal{R}',\tau)\right]_{(6)} {\rm d}\mathcal{R}'\,. \tag{A6}$$

Subtracting equation (A6) from equation (A5), and dropping the extremely small terms proportional to $1/N$ in comparison to $O(1)$ terms, we get the following equation for the $f_n(\mathcal{R},\tau)$:

$$\sum_{n\neq 0} in\Omega_{\rm k}\, f_n \exp\left[inw\right] + \left[F,\varphi(\mathcal{D},\tau)\right]_{(6)} - \oint \left[F,\varphi(\mathcal{D},\tau)\right]_{(6)}\frac{{\rm d}w}{2\pi} = 0, \tag{A7}$$

whose solution is

$$f_n(\mathcal{R},\tau) = \frac{\rm i}{n\Omega_{\rm k}} \oint \left[F,\varphi(\mathcal{D},\tau)\right]_{(6)} \exp\left[-inw\right]\frac{{\rm d}w}{2\pi}, \qquad n\neq 0. \tag{A8}$$

Hence all the $f_n(\mathcal{R},\tau)$ for $n\neq 0$ are slaved to the quasi-steady collisional evolution of $F(\mathcal{R},\tau)$. As in Paper I, the function $f_0(\mathcal{R},\tau)$ remains undetermined in our $O(\varepsilon)$ calculations.

Similarly, when the expansions of equations (43) and (44) for the DF and the wake are substituted in equation (31), all the $O(1)$ terms vanish. The $O(\varepsilon)$ terms give the following equation for $W(\mathcal{R}\,|\,\mathcal{R}'(\tau'),\tau')$ and all the $\varpi_n(\mathcal{R}\,|\,\mathcal{D}'(\tau'),\tau')$:

$$\begin{aligned}\frac{\partial W}{\partial \tau'} + \sum_{n\neq 0} in\Omega_{\rm k}\,\varpi_n \exp\left[inw\right] + \left[W,\varphi(\mathcal{D},\tau')\right]_{(6)} + \left[F(\mathcal{R},\tau'),\varphi^{\rm w}(\mathcal{D},\mathcal{R}'(\tau'),\tau')\right]_{(6)} \\ = \left[\varphi^{\rm p}(\mathcal{D},\mathcal{D}'(\tau'),\tau'), F(\mathcal{R},\tau')\right]_{(6)}, \qquad \text{for } \tau'\leq\tau.\end{aligned} \tag{A9}$$

Note that we have written the rescaled wake potential as $\varphi^{\rm w}(\mathcal{D},\mathcal{R}',\tau)$, instead of $\varphi^{\rm w}(\mathcal{D},\mathcal{D}',\tau)$. This is because, when the expansion of equation (44) is substituted in equation (28d), we find that $\varphi^{\rm w}$ is independent of w' to $O(1)$ accuracy:

$$\varphi^{\rm w}(\mathcal{D},\mathcal{R}',\tau) = \int W(\mathcal{R}''\,|\,\mathcal{R}',\tau)\,{\rm d}\mathcal{R}'' \oint p(\mathcal{D},\mathcal{D}'')\,\frac{{\rm d}w''}{2\pi}. \tag{A10}$$

In equation (A9) $\mathcal{D}'(\tau') = \{\mathcal{R}'(\tau'), w'(\tau')\}$ is the location of the star in 6-dim phase space at time τ', which arrives at the phase space location $\mathcal{D}' = \{\mathcal{R}', w'\}$ at time τ. In general, this orbit is governed by the Hamiltonian $H_{\rm org}$ of equation (29), but we need it in equation (A9) only to $O(1)$ accuracy. This is just the secular collisionless dynamics described above by equations (41) and (42): $\mathcal{R}'(\tau')$ is such that $I' = \text{constant}$ and $\{L'(\tau'), L'_z(\tau'); g'(\tau'), h'(\tau')\}$ obey the Hamiltonian equations of motion (equation 41) with $\Phi(\mathcal{R}',\tau')$ acting as the Hamiltonian. Integrating

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equation (42) over time, we have

$$w'(\tau') = w' + \frac{\Omega_{\rm k}(I')}{\varepsilon}\left(\tau' - \tau\right) + \int_{\tau}^{\tau'} \frac{\partial \Phi(\mathcal{R}'(\tau''), \tau'')}{\partial I}\, {\rm d}\tau''. \tag{A11}$$

The ring orbit $\mathcal{R}'(\tau')$ is independent of $w'(\tau')$, which is slaved to it.

Since only the W part of the wake contributes to the equation for F and f_n, we can obtain an equation for it by averaging equation (A9) over both w and w'. All the terms with ϖ_n vanish in the w-averaging, and play no further role. Then the equation for $W(\mathcal{R}\,|\,\mathcal{R}'(\tau'), \tau')$ is

$$\begin{aligned}\frac{\partial W}{\partial \tau'} + \oint\!\!\oint \left[W, \varphi(\mathcal{D}, \tau')\right]_{(6)} \frac{{\rm d}w}{2\pi}\frac{{\rm d}w'}{2\pi} + \oint\!\!\oint \left[F(\mathcal{R}, \tau'), \varphi^{\rm w}(\mathcal{D}, \mathcal{R}'(\tau'), \tau')\right]_{(6)} \frac{{\rm d}w}{2\pi}\frac{{\rm d}w'}{2\pi} \\ = \oint\!\!\oint \left[\varphi^{\rm p}(\mathcal{D}, \mathcal{D}'(\tau'), \tau'), F(\mathcal{R}, \tau')\right]_{(6)} \frac{{\rm d}w}{2\pi}\frac{{\rm d}w'}{2\pi}, \qquad \text{for } \tau' \le \tau.\end{aligned}$$

$$\text{with adiabatic turn-on initial condition} \quad \lim_{\tau' \to -\infty} W(\mathcal{R}\,|\,\mathcal{R}'(\tau'), \tau') = 0. \tag{A12}$$

Equation (A12) for W, equation (A6) for F, equation (A2) for $F^{(2)}_{\rm iir}$ and equation (A8) for the f_n are the orbit-averaged equations describing the $O(\varepsilon)$ kinetic theory. However, they are in a raw form and need to be processed further such that all direct dependences on the orbital phases w and w' are eliminated. In other words, we seek a kinetic description that is self-consistent and closed in 5-dim $\mathcal{R}$-space. This is done in below by (a) expressing the 6-dim PBs in terms of 4-dim PBs, and (b) defining suitable potential functions in $\mathcal{R}$-space.

For any two phase space functions, $\chi_1(\mathcal{D})$ and $\chi_2(\mathcal{D})$, the 6-dim PB can be expressed in terms of the 4-dim PB (without subscript) as

$$[\chi_1, \chi_2]_{(6)} = \left(\frac{\partial \chi_1}{\partial w}\frac{\partial \chi_2}{\partial I} - \frac{\partial \chi_1}{\partial I}\frac{\partial \chi_2}{\partial w}\right) + [\chi_1, \chi_2]. \tag{A13}$$

For functions $\chi_1(\mathcal{R})$ and $\chi_2(\mathcal{R})$, the 6-dim PB equals the 4-dim PB: $[\,\chi_1\,, \chi_2\,]_{(6)} = [\,\chi_1\,, \chi_2\,]$.

We begin with the three PBs in the wake equation (A12):

$$\left[W(\mathcal{R}\,|\,\mathcal{R}'(\tau'), \tau'), \varphi(\mathcal{D}, \tau')\right]_{(6)} = -\frac{\partial W}{\partial I}\frac{\partial \varphi}{\partial w} + \left[W(\mathcal{R}\,|\,\mathcal{R}'(\tau'), \tau'), \varphi(\mathcal{D}, \tau')\right],$$

$$\left[F(\mathcal{R}, \tau'), \varphi^{\rm w}(\mathcal{D}, \mathcal{R}'(\tau'), \tau')\right]_{(6)} = -\frac{\partial F}{\partial I}\frac{\partial \varphi^{\rm w}}{\partial w} + \left[F(\mathcal{R}, \tau'), \varphi^{\rm w}(\mathcal{D}, \mathcal{R}'(\tau'), \tau')\right],$$

$$\left[\varphi^{\rm p}(\mathcal{D}, \mathcal{D}'(\tau'), \tau'), F(\mathcal{R}, \tau')\right]_{(6)} = +\frac{\partial \varphi^{\rm p}}{\partial w}\frac{\partial F}{\partial I} + \left[\varphi^{\rm p}(\mathcal{D}, \mathcal{D}'(\tau'), \tau'), F(\mathcal{R}, \tau')\right].$$

The first term on the right-hand side of all three equations can be written in the form $\partial\{\}/\partial w$, and will vanish when averaged over w. So the three double integrals over the 6-dim PBs in equation (A12) reduce to double integrals over the 4-dim PBs. Since the 4-dim PBs act only in $\mathcal{R}$-space, the integrals over w and w' can be taken inside the 4-dim PBs. Of the two functions in each of the PBs, it is only the potentials φ, $\varphi^{\rm w}$ and $\varphi^{\rm p}$ that depend on the orbital phases w and w'. In particular, φ and $\varphi^{\rm w}$ are functions of w but independent of w', so we get

$$\oint\!\!\oint \left[W(\mathcal{R}\,|\,\mathcal{R}'(\tau'), \tau'), \varphi(\mathcal{D}, \tau')\right]_{(6)} \frac{{\rm d}w}{2\pi}\frac{{\rm d}w'}{2\pi} = \left[W(\mathcal{R}\,|\,\mathcal{R}'(\tau'), \tau'), \oint \varphi(\mathcal{D}, \tau')\frac{{\rm d}w}{2\pi}\right],$$

$$\oint\!\!\oint \left[F(\mathcal{R}, \tau'), \varphi^{\rm w}(\mathcal{D}, \mathcal{R}'(\tau'), \tau')\right]_{(6)} \frac{{\rm d}w}{2\pi}\frac{{\rm d}w'}{2\pi} = \left[F, \oint \varphi^{\rm w}(\mathcal{D}, \mathcal{R}'(\tau'), \tau')\frac{{\rm d}w}{2\pi}\right].$$

However, $\varphi^{\rm p}$ depends on both w and w', so we are left with a double integral inside the 4-dim PB:

$$\oint\!\!\oint \left[\varphi^{\rm p}(\mathcal{D}, \mathcal{D}'(\tau'), \tau'), F(\mathcal{R}, \tau')\right]_{(6)} \frac{{\rm d}w}{2\pi}\frac{{\rm d}w'}{2\pi} = \left[\oint\!\!\oint \varphi^{\rm p}(\mathcal{D}, \mathcal{D}'(\tau'), \tau')\frac{{\rm d}w}{2\pi}\frac{{\rm d}w'}{2\pi}, F(\mathcal{R}, \tau')\right].$$

In the integral over w' on the right-hand side, we note that w' occurs in the integrand only through the function $w'(\tau')$ which is given in equation (A11). Since w' appears here only as a linear additive term, we can replace $w'(\tau')$ by w' in the potential $\varphi^{\rm p}$:

$$\oint\!\!\oint \left[\varphi^{\rm p}(\mathcal{D}, \mathcal{D}'(\tau'), \tau'), F(\mathcal{R}, \tau')\right]_{(6)} \frac{{\rm d}w}{2\pi}\frac{{\rm d}w'}{2\pi} = \left[\oint\!\!\oint \varphi^{\rm p}(\mathcal{D}, \mathcal{R}'(\tau'), w', \tau')\frac{{\rm d}w}{2\pi}\frac{{\rm d}w'}{2\pi}, F(\mathcal{R}, \tau')\right],$$

where the dependence of $\varphi^{\rm p}$ on the arguments $\mathcal{R}'(\tau')$ and w' is now indicated explicitly.
Then equation (A12) for $W(\mathcal{R}\,|\,\mathcal{R}'(\tau'), \tau')$ can be written as

$$\begin{aligned}\frac{\partial W}{\partial \tau'} + \left[W(\mathcal{R}\,|\,\mathcal{R}'(\tau'), \tau'), \oint \varphi(\mathcal{D}, \tau')\frac{{\rm d}w}{2\pi}\right] + \left[F(\mathcal{R}, \tau'), \oint \varphi^{\rm w}(\mathcal{D}, \mathcal{R}'(\tau'), \tau')\frac{{\rm d}w}{2\pi}\right] \\ = \left[\oint\!\!\oint \varphi^{\rm p}(\mathcal{D}, \mathcal{R}'(\tau'), w', \tau')\frac{{\rm d}w}{2\pi}\frac{{\rm d}w'}{2\pi}, F(\mathcal{R}, \tau')\right], \qquad \text{for } \tau' \le \tau.\end{aligned}$$

$$\text{with adiabatic turn-on initial condition} \quad \lim_{\tau' \to -\infty} W(\mathcal{R}\,|\,\mathcal{R}'(\tau'), \tau') = 0. \tag{A14}$$

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We now turn to equation (A6) for the ring DF $F(\mathcal{R},\tau)$. Since

$$[F(\mathcal{R},\tau),\varphi(\mathcal{D},\tau)]_{(6)} = -\frac{\partial F}{\partial I}\frac{\partial \varphi}{\partial w} + [F(\mathcal{R},\tau),\varphi(\mathcal{D},\tau)],$$

we have

$$\oint [F(\mathcal{R},\tau),\varphi(\mathcal{D},\tau)]_{(6)}\,\frac{\mathrm{d}w}{2\pi} = \oint [F(\mathcal{R},\tau),\varphi(\mathcal{D},\tau)]\,\frac{\mathrm{d}w}{2\pi} = \left[F(\mathcal{R},\tau),\oint \varphi(\mathcal{D},\tau)\frac{\mathrm{d}w}{2\pi}\right].$$

Also,

$$\left[\oint\!\!\oint p(\mathcal{D},\mathcal{D}')\frac{\mathrm{d}w}{2\pi}\frac{\mathrm{d}w'}{2\pi}, F^{(2)}_{\rm irr}(\mathcal{R},\mathcal{R}',\tau)\right]_{(6)} = \left[\oint\!\!\oint p(\mathcal{D},\mathcal{D}')\frac{\mathrm{d}w}{2\pi}\frac{\mathrm{d}w'}{2\pi}, F^{(2)}_{\rm irr}(\mathcal{R},\mathcal{R}',\tau)\right].$$

Then equation (A6) for $F(\mathcal{R},\tau)$ can be written as

$$\frac{\partial F}{\partial \tau} + \left(1-\frac{1}{N}\right)\left[F,\oint \varphi(\mathcal{D},\tau)\frac{\mathrm{d}w}{2\pi}\right] = \frac{1}{N}\int \left[\oint\!\!\oint p(\mathcal{D},\mathcal{D}')\frac{\mathrm{d}w}{2\pi}\frac{\mathrm{d}w'}{2\pi}, F^{(2)}_{\rm irr}(\mathcal{R},\mathcal{R}',\tau)\right]\mathrm{d}\mathcal{R}'. \tag{A15}$$

Similarly, the solution (equation A8) for the $O(\varepsilon)$ correction to the DF can be simplified to

$$f_n(\mathcal{R},\tau) = \frac{1}{\Omega_{\rm k}}\frac{\partial F}{\partial I}\oint \varphi(\mathcal{D},\tau)\exp[-\mathrm{i}nw]\frac{\mathrm{d}w}{2\pi} + \frac{\mathrm{i}}{n\Omega_{\rm k}}\left[F,\oint \varphi(\mathcal{D},\tau)\exp[-\mathrm{i}nw]\frac{\mathrm{d}w}{2\pi}\right], \qquad n\neq 0. \tag{A16}$$

We have now eliminated all the 6-dim PBs and written equation (A14) for W, equation (A15) for F and equation (A16) for the f_n in terms of 4-dim PBs operating in $\mathcal{R}$-space. Only the integrals of various potential functions over w and w' remain to be simplified. Using equation (A4) for φ, equation (A10) for $\varphi^{\rm w}$ and $\varphi^{\rm p} = p-\varphi$, the integrals needed are

$$\Phi(\mathcal{R},\tau) \overset{\rm def}{=} \oint \varphi(\mathcal{D},\tau)\frac{\mathrm{d}w}{2\pi} = \int F(\mathcal{R}',\tau)\,\mathrm{d}\mathcal{R}' \oint\!\!\oint p(\mathcal{D},\mathcal{D}')\frac{\mathrm{d}w}{2\pi}\frac{\mathrm{d}w'}{2\pi},$$

$$\varphi_n(\mathcal{R},\tau) \overset{\rm def}{=} \oint \varphi(\mathcal{D},\tau)\exp[-\mathrm{i}nw]\frac{\mathrm{d}w}{2\pi} = \int F(\mathcal{R}',\tau)\,\mathrm{d}\mathcal{R}' \oint\!\!\oint p(\mathcal{D},\mathcal{D}')\exp[-\mathrm{i}nw]\frac{\mathrm{d}w}{2\pi}\frac{\mathrm{d}w'}{2\pi},$$

$$\Phi^{\rm w}(\mathcal{R},\mathcal{R}',\tau) \overset{\rm def}{=} \oint \varphi^{\rm w}(\mathcal{D},\mathcal{R}',\tau)\frac{\mathrm{d}w}{2\pi} = \int W(\mathcal{R}''\,|\,\mathcal{R}',\tau)\,\mathrm{d}\mathcal{R}'' \oint\!\!\oint p(\mathcal{D},\mathcal{D}'')\frac{\mathrm{d}w}{2\pi}\frac{\mathrm{d}w''}{2\pi},$$

$$\Phi^{\rm w}(\mathcal{R},\mathcal{R}',\tau) \overset{\rm def}{=} \oint\!\!\oint \varphi^{\rm p}(\mathcal{D},\mathcal{D}',\tau')\frac{\mathrm{d}w}{2\pi}\frac{\mathrm{d}w'}{2\pi} = \oint\!\!\oint p(\mathcal{D},\mathcal{D}'')\frac{\mathrm{d}w}{2\pi}\frac{\mathrm{d}w''}{2\pi} - \oint \varphi(\mathcal{D},\tau)\frac{\mathrm{d}w}{2\pi}.$$

A quantity that is common to all these integrals is the partially orbit-averaged Poisson kernel, whose Fourier series we write as

$$\oint p(\mathcal{D},\mathcal{D}')\frac{\mathrm{d}w'}{2\pi} = \Psi(\mathcal{R},\mathcal{R}') + \sum_{n\neq 0}\psi_n(\mathcal{R},\mathcal{R}')\exp[\mathrm{i}nw], \tag{17a}$$

$$\text{where} \qquad \Psi(\mathcal{R},\mathcal{R}') = -GM_\bullet \oint\oint \frac{\mathrm{d}w}{2\pi}\frac{\mathrm{d}w'}{2\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{17b}$$

$$\psi_n(\mathcal{R},\mathcal{R}') = -GM_\bullet \oint\oint \frac{\mathrm{d}w}{2\pi}\frac{\mathrm{d}w'}{2\pi}\frac{\exp[-\mathrm{i}nw]}{|\boldsymbol{r}-\boldsymbol{r}'|}, \qquad n\neq 0, \tag{17c}$$

are the ring–ring interaction potential functions introduced in Paper I. The functions Ψ and ψ_n will be treated as known functions of their arguments. Using equations (A17a)–(A17c), we get

$$\Phi(\mathcal{R},\tau) = \int F(\mathcal{R}',\tau)\,\Psi(\mathcal{R},\mathcal{R}')\,\mathrm{d}\mathcal{R}', \tag{A18}$$

$$\varphi_n(\mathcal{R},\tau) = \int F(\mathcal{R}',\tau)\,\psi_n(\mathcal{R},\mathcal{R}')\,\mathrm{d}\mathcal{R}', \tag{A19}$$

$$\Phi^{\rm w}(\mathcal{R},\mathcal{R}',\tau) = \int W(\mathcal{R}''\,|\,\mathcal{R}',\tau)\,\Psi(\mathcal{R},\mathcal{R}'')\,\mathrm{d}\mathcal{R}'', \tag{A20}$$

$$\Phi^{\rm p}(\mathcal{R},\mathcal{R}',\tau) = \Psi(\mathcal{R},\mathcal{R}') - \Phi(\mathcal{R},\tau). \tag{A21}$$

We are now ready to cast equation (A14) for W, equation (A15) for F and equation (A16) in a form that is determined entirely by $\mathcal{R}$-space quantities. The equation for the wake function $W(\mathcal{R}\,|\,\mathcal{R}'(\tau'),\tau')$ is

$$\frac{\partial W}{\partial \tau'} + \left[W(\mathcal{R}\,|\,\mathcal{R}'(\tau'),\tau'),\Phi(\mathcal{R},\tau')\right] + \left[F(\mathcal{R},\tau'),\Phi^{\rm w}(\mathcal{R},\mathcal{R}'(\tau'),\tau')\right] = \left[\Phi^{\rm p}(\mathcal{R},\mathcal{R}'(\tau'),\tau'),F(\mathcal{R},\tau')\right], \qquad \text{for } \tau'\le\tau.$$

$$\text{with adiabatic turn-on initial condition} \quad \lim_{\tau'\to-\infty} W(\mathcal{R}\,|\,\mathcal{R}'(\tau'),\tau') = 0. \tag{A22}$$

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The kinetic equation for the ring DF $F(\mathcal{R}, \tau)$ is

$$\frac{\partial F}{\partial \tau} + \left(1 - \frac{1}{N}\right)[F, \Phi(\mathcal{R}, \tau)] = \frac{1}{N}\int \left[\Psi(\mathcal{R}, \mathcal{R}'), F^{(2)}_{\rm irr}(\mathcal{R}, \mathcal{R}', \tau)\right] \, {\rm d}\mathcal{R}'. \tag{A23}$$

The Fourier coefficients of the $O(\varepsilon)$ fluctuations in the DF f_n are given by

$$f_n(\mathcal{R}, \tau) = \frac{1}{\Omega_{\rm k}}\left\{\frac{\partial F}{\partial I}\varphi_n + \frac{\rm i}{n}[F, \varphi_n]\right\}, \qquad n \neq 0. \tag{A24}$$

As earlier the function $f_0(\mathcal{R}, \tau)$ remains undetermined by our $O(\varepsilon)$ theory. Using equation (A24) for the f_n in equation (43), we get the following expression for the full DF to first order in ε:

$$f(\mathcal{D}, \tau) = \frac{1}{2\pi}F(\mathcal{R}, \tau) + \frac{\varepsilon}{2\pi\Omega_{\rm k}}\sum_{n\neq 0}\left(\frac{\partial F}{\partial I}\varphi_n + \frac{\rm i}{n}[F, \varphi_n]\right)\exp[{\rm i}nw] + \frac{\varepsilon}{2\pi}f_0(\mathcal{R}, \tau). \tag{A25}$$

Once the ring DF $F(\mathcal{R}, \tau)$ has been determined by solving equations (45) and (46), the $O(\varepsilon)$ fluctuations in w are completely determined by equation (A25). To determine the function $f_0(\mathcal{R}, \tau)$, it is necessary to work to higher order in ε. With this caveat in mind, the orbit-averaged kinetic description presented here is self-consistent and closed in 5-dim $\mathcal{R}$-space.

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