Mathematika 66 (2020) 649–680 doi:10.1112/mtk.12028

NOWHERE-ANALYTIC SMOOTH CURVES WITH NON-TRIVIAL ANALYTIC ISOTROPY

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Abstract. We study the smoothness properties of planar curves $\gamma \subset \mathbb{R}^2$, $0 \in \gamma$, which are invariant under a local real-analytic diffeomorphism ψ fixing the origin. Under certain conditions, depending on the first-order jet (if the eigenvalues of $d\psi(0)$ are not both of modulus one) or on a higher order jet (if ψ is tangent to the identity) of ψ and γ, we show that γ must be real analytic as soon as it is smooth enough — in particular, if it is of class C^∞. On the other hand, when these conditions are not verified we can construct examples of nowhere-analytic curves of class C^∞, whose Taylor expansion is divergent at 0, which are invariant under non-trivial real-analytic local diffeomorphisms (either tangent to the identity or not).

§1. *Introduction.* This paper is devoted to the following basic question: let $\gamma \subset \mathbb{R}^2$ be (the germ of) a smooth curve passing through 0, and let ψ be a locally defined real-analytic diffeomorphism of $\mathbb{R}^2$ fixing 0. Suppose that γ is invariant under ψ. Does it follow that γ, or at least $\gamma \setminus \{0\}$, is real analytic?

It is easily seen that the answer, in general, is negative. For instance, we can take as ψ the reflection $(x, y) \to (-x, y)$ (respectively, the central symmetry $(x, y) \to (-x, -y)$) and as γ the graph of any even (respectively, odd) smooth function. Examples of the same kind can be given for any real-analytic involution ψ: for this reason, we will consider the involution case as the "trivial" one and we will exclude it from our considerations.

Leaving involutions aside, the answer turns out not to be so simple. Our aim is to show that it is possible to provide sufficient, purely formal conditions (i.e. only depending on the Taylor expansion of ψ and γ up to a certain order) which ensure that γ must be C^ω whenever it is ψ-invariant. Furthermore, we will show how to construct examples (in which those formal conditions are not satisfied) of (non-involutive) ψ and nowhere-analytic curves γ of class C^∞ which are ψ-invariant.

One of the motivations behind this question comes from CR geometry, that is, the study of real submanifolds $M \subset \mathbb{C}^n$. In this context, the analysis of the automorphism group of M is a natural subject and has a long history: the automorphisms involved are (local) biholomorphisms leaving M invariant and M is often, though not always, assumed to be a real-analytic CR manifold. We refer, more specifically, to automorphisms which fix a point $p \in M$ as *isotropies*. For a real analytic M, the isotropy group tends to be well behaved: under suitable non-degeneracy conditions, it turns out to be a finite-dimensional Lie group (see [**14**]). On the other hand, if M is only of class C^∞, this group need not have such a structure, even in the strongly pseudo-convex case (see [**16**]). However, also for certain classes of smooth submanifolds, finite jet determination results (see [**11, 15**]) imply that the isotropy group embeds injectively into a Lie group. It is reasonable to expect that a manifold of class C^∞ but

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Received 4 July 2019, published online 4 May 2020.
MSC (2010): 37F99, 32H50, 32B15 (primary).
The author was supported by the Austrian Science Fund FWF grant P24878 N25.

not C^ω should admit fewer (holomorphic or real analytic) automorphisms than a real-analytic one: this should be true regardless of the presence of a CR structure.

It seems to us worthwhile to seek to understand which phenomena only depend on smoothness conditions and not on the presence of CR structures. To this end, we study the action of real-analytic transformations on a smooth submanifold $M \subset \mathbb{R}^n$: this is, however, an interesting setting in its own right, and appears to contain several open questions. Among them there is the characterization of real-analytically homogeneous submanifolds, see [**18, 20**]. The study of this problem (at least within certain approaches) leads quite naturally to the question of determining the isotropy group of nowhere-analytic smooth curves. For example, in [**9**] it is shown that a curve $\gamma \subset \mathbb{C}$ which is homogeneous under local holomorphic diffeomorphisms is necessarily of class C^ω. The proof is based on the fact that the isotropy of a nowhere-analytic curve contains at most involutive elements of $\mathrm{Hol}(\mathbb{C}, 0)$. The constructions in Examples 3.6 and 4.8 of this paper, however, show that this approach cannot be directly generalized to curves which are homogeneous under, more in general, local real-analytic transformations.

Another motivation comes from the theory of local, discrete dynamical systems. Although, of course, invariant manifolds play a central role in it, the focus is typically on proving existential results, showing that (analytic) invariant manifolds can be constructed for various classes of analytic diffeomorphism germs. Our point of view is slightly different, in that we either already assume that an invariant manifold exists or try to construct invariant manifolds which are nowhere-analytic. Nevertheless, as one can expect our methods rely heavily on the known results about dynamical systems in low dimensions.

As mentioned above, in the present paper we are principally interested in the case when M is a smooth planar curve $\gamma \subset \mathbb{R}^2$ passing through 0, and $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ is a real-analytic isotropy for γ. To show that the problem is indeed non-trivial, we start with the following remark: unless γ (or at least $\gamma \setminus \{0\}$) is of class C^ω, ψ cannot be "any" diffeomorphism germ. For instance, it cannot be (involutions aside) a conformal germ $\psi \in \mathrm{Hol}(\mathbb{C}, 0)$, nor a linear transformation $\psi \in GL(2, \mathbb{R})$ or an "analytic shear" of the form $\psi(x, y) = (h(x), y + g(x))$ (see [**9**, Lemmas 4.5, 4.7 and 4.10]). The proof of such results relies on the examination of the local dynamics of the germs involved.

Our general aim is thus to find out which elements of $\mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ can appear in the isotropy group of a non-analytic smooth curve. It is easily seen that if a germ ψ admits an invariant curve γ, then the tangent space $T_0(\gamma)$ is an eigenspace for $d\psi(0)$, hence its eigenvalues λ_1, λ_2 are both real. We always take λ_1 to be the one associated to $T_0(\gamma)$, and we distinguish two main situations: when $|\lambda_1| \neq 1$, and when $|\lambda_1| = 1$.

The first case is considered in § 3. This case might be possibly well known, since it is made easy by the existence of normal forms for ψ (as provided in [**17**]). However, we chose to follow a somewhat more elementary approach. If $|\lambda_1| \neq 1$, one can show that a smooth invariant curve γ for ψ is necessarily real analytic: furthermore, the "maximal smoothness" of a non-real-analytic ψ-invariant curve turns out to be $\log |\lambda_2| / \log |\lambda_1|$ (Theorems 3.2 and 3.3). On the other hand, in Example 3.6, we show that the "worst" possible kind of a non-analytic smooth curve, namely one which is nowhere-analytic and whose Taylor expansion is divergent at 0, can admit germs with $\lambda_1 = 1$, $|\lambda_2| \neq 1$ in its isotropy group.

When instead $|\lambda_1| = |\lambda_2| = 1$, we can always take ψ to be tangent to the identity by squaring it and applying a blow-up. Of course, the behavior of invariant curves γ for such a ψ is linked to the higher order jets of both ψ and γ: in § 4.1, we define a pair of integer invariants (d_1, d_2) (with $d_1 \geqslant 2$, $1 \leqslant d_2 \leqslant d_1 - 1$) which depend on ψ and on the formal ideal I of $\mathbb{R}[[x, y]]$ generated by the Taylor development at 0 of a defining function for γ (it is in fact sufficient to know the jet of order d_1 of ψ and I). The integer d_1 is simply the order to which the restriction of ψ to γ is tangent to the identity at 0. The definition of d_2 is less transparent, but roughly speaking its geometrical interpretation is linked to the behavior of ψ in the direction transverse to γ.

Our main results are the following: assume $d_1 < \infty$. If $d_2 = d_1 - 1$, then $\gamma \setminus \{0\}$ is of class C^ω, and in fact γ is of class C^ω if I admits a convergent generator (see Theorem 4.2). On the other hand, there exist nowhere-analytic smooth curves γ whose Taylor expansion is divergent at 0 and which are invariant under a $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$, tangent to the identity, such that $d_2 < d_1 - 1$; see Example 4.8.

Like for the results in [9], the proof of Theorems 3.2 and 4.2 depends on the local dynamics of (holomorphic) diffeomorphism germs. Our general method is analogous in both cases: first, we lift the diffeomorphism ψ and the curve γ by a certain number of blow-ups. At every step, the center of the subsequent blow-up is the point of the exceptional divisor which corresponds to the tangent line of the previous iterated lift of γ. After sufficiently many iterations, the dynamical properties of the lift of ψ allow to either apply the classical stable manifold theorem (in Theorem 3.2) or the theory of parabolic curves developed by Hakim in [12, 13] (for Theorem 4.2) to show that the corresponding lift of γ is of class C^ω.

In the case of $|\lambda_1| \neq 1$, the eigenvalues of the linear part of the lifts of ψ change at every step (cf. [1]), until eventually one of them is attractive and the other repulsive, so that the stable manifold is a one-dimensional real-analytic curve. As previously remarked, when the eigenvalues of $d\psi(0)$ all lie inside the unit disc (i.e. ψ belongs to the so-called Poincaré domain) polynomial normal forms have been given by Reich [17]. Since we are interested in two-dimensional germs (which can admit only few resonances), using these normal forms allows a rather straightforward proof of Theorem 3.2. We chose to employ the blow-up procedure to exemplify the method used later for germs tangent to the identity (where normal forms are not available), so that the treatments in §§ 3 and 4 are part of a common framework. Also note that in Theorem 3.2 we are concerned with some germs outside of the Poincaré domain as well. For completeness, we record in Theorem 3.3 the more precise smoothness results that can be obtained for germs in the Poincaré domain by normalizing them, and we show in Examples 3.4 and 3.5 that these are essentially optimal.

If ψ is tangent to the identity, the use of repeated blow-ups to simplify the dynamics at the expense of the geometry of the space has been introduced by Abate [2] to show the existence of parabolic curves, on the basis of the method used by Camacho and Sad [7] to prove the existence of a separatrix for singular vector fields. The formal logarithm of ψ has been later employed in [6, 8] to provide a direct link between the results for singular formal vector fields and those for diffeomorphisms tangent to the identity. In our situation, if γ satisfies the assumption in Theorem 4.2, the tangent space of a suitable lift of γ defines a non-degenerate characteristic direction for the corresponding lift of ψ after sufficiently many blow-ups (this is what allows the application of the theory of parabolic curves). As a matter of fact, the condition $d_2 = d_1 - 1$ is nothing else but a reformulation of this property (see Lemma 4.4).

When the assumptions in Theorems 3.2 and 4.2 are not satisfied, the use of repeated blow-ups does not "improve" the local dynamics of the lifts of ψ: In the first case, the eigenvalues do not change, whereas in the second one the tangent line of any lift of γ always defines a degenerate characteristic direction. Indeed, one can check that the non-analytic smooth curve $\gamma = \{y = f(x)\}$ with $f(x) = 0$ for $x \leqslant 0$ and $f(x) = e^{-\frac{1}{x}}$ for $x > 0$ admits such germs in its isotropy group (see Example 2.2). In Examples 3.6 and 4.8 we show that nowhere-analytic curves can also admit non-trivial isotropy, either tangent or non-tangent to the identity. In each case, the diffeomorphism ψ involved has an open basin of attraction U around the origin, and the construction of the invariant curve consists in joining a suitable point $p \in U$ with $\psi(p)$ by a smooth, nowhere-analytic arc contained in U and then extending this arc in a unique way by ψ-invariance. The properties of ψ allow then to prove that the curve γ so obtained is of class C^∞ around 0, and that the Taylor development of its defining function is divergent.

§2. *Notation, background, preliminaries.*

2.1. *Germs of analytic maps.* We denote by $\mathrm{Diff}^\omega(\mathbb{R}^n, 0)$ the group of germs at 0 of real-analytic maps $\psi : U \to \mathbb{R}^n$, where U is a neighborhood of 0 in $\mathbb{R}^n$, such that $\psi(0) = 0$ and $J\psi(0) \neq 0$. We define $\mathrm{Hol}(\mathbb{C}^n, 0)$ in a similar way in the category of holomorphic maps.

For any $\phi \in \mathrm{Diff}^\omega(\mathbb{R}^n, 0)$ and $j \in \mathbb{Z}$ we denote by $\phi^{\circ j}$ the composition of ϕ with itself performed j-times; nevertheless we will usually write ϕ^{-1} for $\phi^{\circ -1}$. We will denote by $Id \in \mathrm{Diff}^\omega(\mathbb{R}^n, 0)$ the germ of the identity map, so that $\phi^{\circ 0} = Id$. If $d\phi(0) = Id$, we say that ϕ is *tangent to the identity*.

Let $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^n, 0)$, and let $(M, 0)$ be the germ at 0 of an embedded submanifold $M \subset U \subset \mathbb{R}^n$. We call M an *invariant submanifold* for ψ if there is a neighborhood V of 0 such that $\psi(p) \in M$ for all $p \in M \cap V$. With a slight abuse of language, in the paper we will refer to properties of the germ of M at 0 as properties of M. In particular, we will say that M is real analytic if it is of class C^ω in a small enough neighborhood of 0.

In the following, we will be mostly be concerned with the case when $n = 2$ and $M = \gamma$ is a curve in $\mathbb{R}^2(x, y)$. We will usually write $\gamma = \{y = f(x)\}$ with f of class at least C^1, $f(0) = f'(0) = 0$. If $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$, $\psi(x, y) = (g(x, y), h(x, y))$, then for any $p = (x, f(x)) \in \gamma$ we have $\psi(p) = (g(x, f(x)), h(x, f(x)))$. The ψ-invariance of γ corresponds to the requirement $\psi(p) \in \gamma$, that is,

$$f(g(x, f(x))) = h(x, f(x))$$

for x close to 0; we will often refer to it as the "mapping equation."

2.2. *Analytic isotropy.* If $M \subset \mathbb{R}^n$ is a smooth submanifold and $p \in M$, the set of the germs $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^n, p)$ such that M is ψ-invariant is a subgroup; we will call it the *analytic isotropy group* of M at p. Clearly, if M is real analytic, its isotropy group at any point is very large and in fact isomorphic to the isotropy group of a plane of the same dimension. However, for a generic smooth submanifold M of $\mathbb{R}^n$ we should expect the analytic isotropy group to be trivial at any point $p \in M$. One of probably many ways to see this is given by the following lemma, showing that such a situation is generic in the topological sense. We recall that a subset of a topological space is called *residual* if its complement is of first category:

LEMMA 2.1. *Let $\Omega \Subset \mathbb{R}^n$ be a domain and for any $f \in C^\infty(\Omega)$ let $\Gamma(f) \subset \Omega \times \mathbb{R}$ be its graph. Let $C^\infty(\overline{\Omega})$ be equipped with the Fréchet space topology. Then the set of the $f \in C^\infty(\overline{\Omega})$ such that the analytic isotropy of $\Gamma(f)$ reduces to the identity at every point contains a residual subset; in particular, it is dense in $C^\infty(\overline{\Omega})$.*

Proof. For a fixed $f \in C^\infty(\overline{\Omega})$ and $p \in \Gamma(f)$, we denote by $E_p(f)$ the set of those $q \in \Gamma(f)$ such that there is a local real-analytic diffeomorphism, defined in a neighborhood of p in $\mathbb{R}^{n+1}$, sending p to q and preserving $\Gamma(f)$. We call this set *equivalence locus* of p in $\Gamma(f)$. In [10, Theorem 4.1] it is shown that the set $\mathcal{T}$ of the functions $f \in C^\infty(\overline{\Omega})$ such that $E_p(f) = \{p\}$ for all $p \in \Gamma(f)$ contains a residual subset.

Let $f \in \mathcal{T}$ (note that in particular f is nowhere-analytic): then the analytic isotropy at any $p \in \Gamma(f)$ is trivial. Indeed, if $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^n, p)$ preserves $\Gamma(f)$, for all $q \in \Gamma(f)$ in the domain of ψ we have $\psi(q) \in E_q(f) = \{q\}$, that is, $\psi(q) = q$. Since the set of fixed points of ψ locally contains the nowhere-analytic hypersurface $\Gamma(f)$, it follows that $\psi = Id$. □

On the other hand, some non-analytic smooth hypersurfaces also have an analytic isotropy which does not reduce to the identity. This is very easy to see by considering, for instance, non-analytic curves that are invariant under linear transformations of the kind $(x, y) \to (x, -y)$ or $(x, y) \to (-x, -y)$. More in general, it is easy to show that every diffeomorphism germ of order two is C^ω-conjugated to its linear part (see, e.g., [9]), and thus admits many (even nowhere-analytic) invariant curves. For our purposes, we consider the set of involutions as the "trivial class."

A less trivial case is given by the graph of the common textbook example of a non-analytic smooth function:

Example 2.2. Define $f \in C^\infty(\mathbb{R})$ by $f(x) = 0$ for $x \leqslant 0$, $f(x) = e^{-\frac{1}{x}}$ for $x > 0$ and let $\gamma = \{y = f(x)\} \subset \mathbb{R}^2$. The analytic isotropy group of γ at 0 is non-trivial.

Indeed, define the one-parameter group $\{\psi_{1,t}\}_{t\in\mathbb{R}} \subset \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ by $\psi_{1,t}(x, y) = (\frac{x}{1+tx}, \frac{1}{e^t}y)$: the mapping equation

$$f\left(\frac{x}{1+tx}\right) = \frac{f(x)}{e^t}$$

is clearly satisfied for $-1 < x \leqslant 0$, while for $x > 0$ we have

$$f\left(\frac{x}{1+tx}\right) = e^{-\frac{1+tx}{x}} = e^{-\frac{1}{x}}e^{-t} = \frac{f(x)}{e^t}.$$

Taking this further, we can define $\{\psi_{n,t}(x, y)\}_{t\in\mathbb{R}}$ by $\psi_{n,t}(x, y) = (\frac{x}{1+tx^n}, ye^{-tx^{n-1}})$: the associated mapping equation

$$f\left(\frac{x}{1+tx^n}\right) = f(x)e^{-tx^{n-1}}$$

is again clearly satisfied for $x \leqslant 0$, and for $x > 0$ we have

$$f\left(\frac{x}{1+tx^n}\right) = e^{-\frac{1+tx^n}{x}} = e^{-\frac{1}{x}}e^{-tx^{n-1}} = f(x)e^{-tx^{n-1}}.$$

Though non-analytic, γ possesses thus quite a large isotropy group. This is not surprising, since the example arises naturally as an integral curve of a singular analytic vector field. Of course such a curve is forced to be real analytic outside the origin. We will show later that much "worse" curves of class C^∞ can admit a non-trivial analytic isotropy group, and that, on the other hand, not all types of germs can appear in the isotropy group of a non-analytic curve.

2.3. *Stable manifold theorem.* We will now recall some well-known results about the dynamics of germs of diffeomorphisms around a fixed point. For our purposes, it will be sufficient to consider the two-dimensional case: let then $\psi \in \mathrm{Hol}(\mathbb{C}^2, 0)$, and let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the eigenvalues of $d\psi(0)$. We will only be interested in the case where $\lambda_1 \neq \lambda_2$: let $E_{\lambda_1}, E_{\lambda_2}$ be the associated eigenspaces. Put

$$E^s = \bigoplus_{\lambda\in\{\lambda_1,\lambda_2\}, |\lambda|<1} E_\lambda;$$

the space E^s is called the *stable subspace* of $d\psi(0)$. Of course, in our situation either $E^s = \{0\}$, $E^s = \mathbb{C}^2$ or it coincides with one of the eigenspaces E_{λ_j}. We can define the *unstable subspace* E^u as the stable space associated to the inverse of ψ.

Assume now that $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$, and let $\lambda_1, \lambda_2 \in \mathbb{C}$ be again the eigenvalues of $d\psi(0)$. We still denote by ψ, $d\psi(0)$ the extension of the germ and its differential to $\mathbb{C}^2$, and for any $\lambda \in \{\lambda_1, \lambda_2\}$ we denote by $E_\lambda \subset \mathbb{C}^2$ the (complex) eigenspace associated to λ, and put $E_{\lambda,\bar{\lambda}} = (E_\lambda \oplus E_{\bar{\lambda}}) \cap \mathbb{R}^2$. The (real) stable space $E^s \subset \mathbb{R}^2$ for $d\psi(0)$ is defined as follows:

$$E^s = \bigoplus_{\lambda\in\{\lambda_1,\lambda_2\},|\lambda|<1,\lambda\in\mathbb{R}} (E_\lambda \cap \mathbb{R}^2) \oplus \bigoplus_{\lambda\in\{\lambda_1,\lambda_2\},|\lambda|<1,\lambda\in\mathbb{C}\setminus\mathbb{R}} E_{\lambda,\bar{\lambda}}.$$

If the eigenvalues of $d\psi(0)$ are both real, the previous expression reduces to $E^s = \bigoplus_{|\lambda|<1}(E_\lambda \cap \mathbb{R}^2)$. As before, we can also define the unstable space E^u by considering the inverse ψ^{-1}. We will need the following version of the stable manifold theorem in $\mathbb{C}^2$:

THEOREM 2.3. *Let $\psi \in \mathrm{Hol}(\mathbb{C}^2, 0)$, let λ_1, λ_2 be the eigenvalues of $d\psi(0)$, $\lambda_1 \neq \lambda_2$, and let E^s be the stable subspace defined above. Then there exists a ψ-invariant local embedded complex submanifold W^s_{loc}, whose tangent space at 0 is E^s, and $\delta > 0$ such that the set of the $p \in B(0, \delta)$ whose orbit is exponentially convergent to 0 coincides with $W^s_{loc} \cap B(0, \delta)$. W^s_{loc} is called the* stable manifold *of ψ through* 0.

Analogously, there exists an unstable manifold *W^u_{loc} tangent to the unstable space E^u and behaving in the same way for the orbits of ψ^{-1}.*

If $d\psi(0)$ has no eigenvalue of modulus 1, then $E^s \oplus E^u = \mathbb{C}^2$ and W^s_{loc} contains all the points sufficiently close to 0 whose orbits converge to 0. In this case, ψ is topologically conjugated to its linearization.

The previous result collects several particular cases of the stable-unstable-center manifold theorems (as well as the Grobman–Hartman theorem) for diffeomorphism germs, as formulated, for example, in [**3**, Theorems 3.1.1, 3.1.3 and subsequent remark, Theorem 5.3.5].

COROLLARY 2.4. *Let $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ and let λ_1, λ_2, E^s be as before with $\lambda_1, \lambda_2 \in \mathbb{R}$. Then the analogous statement to Theorem 2.3 holds for ψ in the real-analytic setting.*

Indeed, the previous corollary can be proved by extending ψ to a map in $\mathrm{Hol}(\mathbb{C}^2, 0)$, applying Theorem 2.3 and then considering the real-analytic submanifold $W^s_{\mathrm{loc}} \cap \mathbb{R}^2$. Note that under the assumption that $\lambda_1, \lambda_2 \in \mathbb{R}$ the tangent space $E^s \subset \mathbb{C}^2$ of $W^s_{\mathrm{loc}} \subset \mathbb{C}^2$ is generated (over $\mathbb{C}$) by vectors belonging to $\mathbb{R}^2$, hence we have $\dim_{\mathbb{R}}(W^s_{\mathrm{loc}} \cap \mathbb{R}^2) = \dim_{\mathbb{C}} W^s_{\mathrm{loc}}$.

2.4. *Blow-up.* In what follows we describe the very well-known construction of the blow-up of a point in a surface; we focus on the case of $(0, 0) \in \mathbb{R}^2$ although of course the construction is coordinate-invariant and can be performed in any field. Our main purpose here is to fix the notation we are going to use in the rest of the paper.

Consider coordinates (x, y) on $\mathbb{R}^2$ and homogeneous coordinates $[u : v]$ on the projective space $\mathbb{RP}^1$. Define a hypersurface $M \subset \mathbb{R}^2 \times \mathbb{R}E\mathbb{P}^1$ in the following way:

$$M = \{(x, y, [u : v]) \in \mathbb{R}^2 \times \mathbb{RP}^1 : xv = yu\}.$$

We will denote by $\pi : M \to \mathbb{R}^2$ the restriction of the projection $\pi : \mathbb{R}^2 \times \mathbb{RP}^1 \to \mathbb{R}^2$ on the first factor. The hypersurface M (we will also write $M = \widetilde{\mathbb{R}^2}$) is called the *blow-up* of $\mathbb{R}^2$ at the origin, and $S = \pi^{-1}(0) \subset M$ is the *exceptional divisor*; note that $S \cong \mathbb{RP}^1$. One can check that the restriction of π to $M \setminus S$ is a diffeomorphism $M \setminus S \to \mathbb{R}^2 \setminus \{0\}$. Moreover, there is a natural identification between S and the projectivization $\mathbb{P}(T_0(\mathbb{R}^2))$ of the tangent space of $\mathbb{R}^2$ at the origin, given by $S \ni (0, 0, [u : v]) \to p(u\partial/\partial x + v\partial/\partial y) \in \mathbb{P}(T_0(\mathbb{R}^2))$ (here p is the projection $p : T_0(\mathbb{R}^2) \to \mathbb{P}(T_0(\mathbb{R}^2))$).

We consider the two local charts on M that correspond to the charts $\{u \neq 0\}$ and $\{v \neq 0\}$ of $\mathbb{RP}^1$. In $\{u \neq 0\}$ (respectively, $\{v \neq 0\}$), the hypersurface M can be expressed as a graph over $\mathbb{R}_x \times \mathbb{RP}^1$ (respectively, $\mathbb{R}_y \times \mathbb{RP}^1$) as $\{y = xV\}$ (respectively, $\{x = yU\}$) where $V = v/u$ (respectively, $U = u/v$). Accordingly, on the chart $M \cap \{u \neq 0\}$ we consider coordinates (x', y') on M which are induced by the pull-back of the coordinates (x, V) of $\mathbb{R}_x \times \mathbb{RP}^1$ by the projection $M \to \mathbb{R}_x \times \mathbb{RP}^1$, that is, we put $x' = x$, $y' = V$. We define coordinates (x'', y'') on $M \cap \{v \neq 0\}$ in an analogous way.

In such coordinates, the exceptional divisor S is defined by $S = \{x' = 0\}$ (respectively, $S = \{y'' = 0\}$) and the projection π is expressed as $\pi(x', y') = (x', x'y')$ (respectively, $\pi(x'', y'') = (x''y'', y'')$).

The construction of iterated blow-ups has been employed by several authors (cf. [**2, 6, 8**]) to study the dynamics of germs $\psi \in \mathrm{Hol}(\mathbb{C}^2, 0)$ which are tangent to the identity, based on the method developed by Seidenberg [**19**] and Camacho and Sad [**7**] for the reduction of singularities of vector fields. The main ingredient is the fact that, although π is not a diffeomorphism around S, the lift of such germs by π is well defined (see, e.g., [**1, 2**]):

LEMMA 2.5. *Let $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$. Then there exists a (unique) mapping $\widetilde{\psi} : \widetilde{U} \to \widetilde{\mathbb{R}^2}$, where $\widetilde{U}$ is a neighborhood of S in $\widetilde{\mathbb{R}^2}$, such that $\psi \circ \pi = \pi \circ \widetilde{\psi}$. The exceptional divisor $S \cong \mathbb{RP}^1$ is $\widetilde{\psi}$-invariant and the action of $\widetilde{\psi}$ on S is conjugated to the action of $d\psi(0)$ on $\mathbb{P}(T_0(\mathbb{R}^2)) \cong \mathbb{RP}^1$. In particular, if ψ is tangent to the identity, then all the points of S are fixed by $\widetilde{\psi}$.*

Suppose that $\partial/\partial x \cong [1{:}0]$ is an eigenvector of $d\psi(0)$, then $\xi = [1 : 0] \in S$ is a fixed point for $\widetilde{\psi}$ and, denoting by (g, h) the components of ψ, the components $(\widetilde{g}, \widetilde{h})$ of $\widetilde{\psi}$ in the chart $\{u \neq 0\}$ containing $[1{:}0]$ are given by

$$\widetilde{g}(x', y') = g(x', x'y'), \quad \widetilde{h}(x', y') = \frac{h(x', x'y')}{g(x', x'y')} \in \mathbb{R}\{x', y'\}.$$

The germ of $\widetilde{\psi}$ at ξ defines again an element of $\mathrm{Diff}^\omega(\mathbb{R}^2, 0)$.

Smooth curves defined in a neighborhood of 0 admit a smooth lift as well:

LEMMA 2.6. *Let γ be a curve of class C^k $(k \geqslant 1)$ in $\mathbb{R}^2$, $0 \in \gamma$. Then there exists a (unique) curve $\widetilde{\gamma} \subset \widetilde{\mathbb{R}^2}$ of class C^{k-1} such that $\pi(\widetilde{\gamma}) = \gamma$; indeed, one has $\widetilde{\gamma} = \overline{\pi^{-1}(\gamma \setminus \{0\})}$. Furthermore, $\widetilde{\gamma}$ intersects S at the point corresponding to $T_0(\gamma)$ in the identification $\mathbb{P}(T_0(\mathbb{R}^2)) \cong S$.*

Proof. We can assume that $T_0(\gamma) = \langle \partial/\partial x \rangle$, hence γ is locally parametrized by $\mathbb{R} \ni t \to (t, f(t)) \in \mathbb{R}^2$ with f of class C^k and $f(0) = f'(0) = 0$. Working in the chart $\{u \neq 0\}$, for any $t \neq 0$ we have $\pi^{-1}(t, f(t)) = (t, f(t)/t)$. Since $g(t) = f(t)/t$ extends continuously through $t = 0$ to a function of class C^{k-1}, we have that $\overline{\pi^{-1}(\gamma \setminus \{0\})}$ is the curve of class C^{k-1} locally defined by $\widetilde{\gamma} = \{y' = g(x')\}$. The second assertion of the lemma follows from the fact that $g(0) = f'(0) = 0$. □

2.5. *Nowhere-analytic functions.* Later, in our constructions of invariant curves we will need some standard facts about the existence of nowhere-analytic functions with prescribed behavior. We start with the following:

LEMMA 2.7. *Let* $a_1 < a_2 \in \mathbb{R}$ *and let* $\{d_n^1\}, \{d_n^2\}$ *be arbitrary real sequences. Then there exists* $\eta \in C^\infty([a_1, a_2])$ *such that* η *is nowhere-analytic in the interval* (a_1, a_2) *and* $\eta^{(j)}(a_1) = d_j^1$, $\eta^{(j)}(a_2) = d_j^2$ *for all* $j \in \mathbb{N}$.

Proof. By the well-known Borel–Ritt Theorem, fixed any real sequence $\{c_n\}_{n \geqslant 0}$ there exists a real function g, of class C^∞ on a neighborhood U of 0 and real analytic on $U \setminus \{0\}$, such that $g^{(j)}(0) = c_j$ for all $j \geqslant 0$. Define a cut-off function $\chi \in C^\infty([-1, 1])$ by $\chi(x) = 1 - e^{-\frac{1}{\tan^2(\pi x/2)}}$; note that χ is flat at $x = -1, 0, 1$ and it is of class C^ω on $(-1, 1) \setminus \{0\}$. Taking $g_\epsilon(x) = \chi(x/\epsilon)g(x)$ for small $\epsilon > 0$, we again have $g_\epsilon^{(j)}(0) = c_j$ for $j \geqslant 0$ and $g_\epsilon \in C^\infty([-\epsilon, \epsilon])$ is of class C^ω on $(-\epsilon, \epsilon) \setminus \{0\}$ and flat at $x = \pm\epsilon$.

Let now $\widetilde{\eta}$ be any nowhere-analytic smooth function defined in a neighborhood of $[a_1, a_2]$. Define two sequences $\{c_n^1\}_{n \geqslant 0}, \{c_n^2\}_{n \geqslant 0}$ by putting $c_j^1 = d_j^1 - \widetilde{\eta}^{(j)}(a_1)$, $c_j^2 = d_j^2 - \widetilde{\eta}^{(j)}(a_2)$ for $j \geqslant 0$. For $\epsilon > 0$ small enough, associate to $\{c_n^1\}_{n \geqslant 0}, \{c_n^2\}_{n \geqslant 0}$ two functions $g_\epsilon^1, g_\epsilon^2 \in C^\infty([-\epsilon, \epsilon])$ as above, and define

$$\eta(x) = \widetilde{\eta}(x) + g_\epsilon^1(x - a_1) + g_\epsilon^2(x - a_2)$$

for x in a neighborhood of $[a_1, a_2]$. It is easy to check that η satisfies the conditions required by the lemma. □

LEMMA 2.8. *Fix* $a_1 < a_2 \in \mathbb{R}$, $0 < \lambda < 1$. *For any sufficiently small* $\epsilon > 0$ *and* $0 < \delta_1, \delta_2 < 1$, *there exists a nowhere-analytic function* $\eta \in C^\infty([a_1, a_2])$ *with the following properties:* $\eta(a_2) = \epsilon$, $\eta(a_2) = \lambda\epsilon$, $\eta'(a_2) = \delta_2$, $\eta'(a_1) = \delta_1$, $\eta^{(j)}(a_2) = \eta^{(j)}(a_1) = 0$ *for* $j \geqslant 2$, $0 < \eta'(x) < 1$ *for all* $x \in [a_1, a_2]$.

Proof. Start by choosing $\widetilde{\eta} \in C^\infty([a_1, a_2])$, not necessarily nowhere-analytic, satisfying the requirements of the lemma: this is easy to achieve by first choosing a piecewise-affine function fulfilling the desired conditions and then smoothening out its corners. Note that the $\widetilde{\eta}$ so constructed is real analytic everywhere except for a finite set of points. Let $I_1, \ldots, I_n$ be the intervals where $\widetilde{\eta}$ is analytic, and for any j choose a function $\theta_j \in C^\infty(\overline{I}_j)$ which is nowhere-analytic on I_j and flat at its endpoints. For small enough $\varepsilon > 0$, the function $\eta = \widetilde{\eta} + \varepsilon \sum_{j=1}^n \theta_j$ is nowhere-analytic on (a_1, a_2) and has the desired properties. □

Remark 1. We note here that not only the set of nowhere-analytic functions, but also the set of the functions whose Taylor series is divergent at every point is dense in $C^\infty([a_1, a_2])$ (see [5]). In the statement of Lemma 2.7, we may thus require this stronger property on η.

2.6. *Formal mappings.* Out of convenience, we will now recall some well-known facts about homomorphisms and ideals of the ring of formal power series.

Let $\mathbb{R}[[x, y]]$ be the ring of formal power series in the variables x, y with coefficients in $\mathbb{R}$, and let $\mathfrak{m}$ be its maximal ideal. For any $\alpha \in \mathbb{R}[[x, y]]$, its *order* $\nu(\alpha)$ is the largest $n \in \mathbb{N}$ such that $\alpha \in \mathfrak{m}^n$ ($\nu(\alpha) = 0$ if α is invertible and $\nu(0) = \infty$).

Let $\psi(x, y) = (g(x, y), h(x, y)) \in (\mathbb{R}[[x, y]])^2$, and suppose $\psi(0, 0) = (0, 0)$: then ψ induces a ring homomorphism $\psi : \mathbb{R}[[x, y]] \to \mathbb{R}[[x, y]]$ by composition, that is,

$\psi(\alpha) = \alpha \circ \psi = \alpha(g(x, y), h(x, y))$ for all $\alpha \in \mathbb{R}[[x, y]]$. We will always consider homomorphisms defined by composition in this way. We define the order of ψ as the largest $n \in \mathbb{N}$ such that $\psi(\mathbb{R}[[x, y]]) \subset \mathfrak{m}^n$; note that $\nu(\psi) = \min\{\nu(g), \nu(h)\}$.

If the Jacobian of ψ at 0 is non-vanishing, the induced ring homomorphism is invertible. If $d\psi(0) = Id$, we also say that the induced ring homomorphism is tangent to the identity to order $d \in \mathbb{N}$ if $\nu(\psi - Id) = d$.

We call a proper ideal $I \subset \mathbb{R}[[x, y]]$ a *formal curve ideal* if it is of the form $I = (\rho)$ with $\nu(\rho) = 1$. Under this assumption the quotient $\mathbb{R}[[x, y]]/I$ is isomorphic to $\mathbb{R}[[t]]$, and for any $\alpha \in \mathfrak{m} \setminus I$ of order 1 we have that $\alpha + I$ generates the maximal ideal $\mathfrak{n}$ of $\mathbb{R}[[x, y]]/I$.

Assuming that $\frac{\partial\rho}{\partial y}(0) \neq 0$, there exists a (uniquely determined) generator ρ' of I of the form $\rho'(x, y) = y - f(x)$, $f \in \mathbb{R}[[x]]$. An element $\beta \in \mathbb{R}[[x, y]]$ is thus in I if and only if $\beta(x, f(x)) \equiv 0$; furthermore, $x + I$ generates $\mathfrak{n}$.

Given a ring homomorphism ψ, a formal curve ideal $I = (\rho)$ is invariant by ψ if $\psi(I) \subset I$, that is, if $\rho \circ \psi \in I$. Thus, ψ induces a ring homomorphism ψ^I on $\mathbb{R}[[x, y]]/I$ of composition type; the formal conjugacy class of ψ^I does not depend on the identification $\mathbb{R}[[x, y]]/I \cong \mathbb{R}[[t]]$, that is, on the choice of $\alpha \in \mathfrak{m} \setminus I$ of order 1. In particular, the order of tangency of ψ^I with the identity is well defined.

If $\rho(x, y) = y - f(x)$ and $\psi(x) = g(x, y)$, $\psi(y) = h(x, y)$, the ψ-invariance condition on I is equivalent to the formal mapping equation $f(g(x, f(x))) = h(x, f(x))$, valid in $\mathbb{R}[[x]]$. Taking $x + I$ as the generator of $\mathfrak{n}$ and thus identifying $\mathbb{R}[[x, y]]/I \cong \mathbb{R}[[x]]$, we can then write ψ^I as $\mathbb{R}[[x]] \ni \beta(x) \xrightarrow{\psi^I} \beta(g(x, f(x))) \in \mathbb{R}[[x]]$.

§3. *Germs which are not tangent to the identity.*

In this section we are going to look at germs $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ such that the eigenvalues λ_1, λ_2 of $d\psi(0)$ do not both lie on the unit circle. Of course, $|\lambda_1| = |\lambda_2| = 1$ does not imply that $d\psi(0) = Id$; however, the lift of (a power of) ψ through a blow-up is tangent to the identity and can be treated as in § 4.

3.1. *The case* $|\lambda_1| \neq 1$.

The simplest case to consider is $\log|\lambda_2|/\log|\lambda_1| < 0$; this situation can be treated by a direct application of the stable manifold theorem and the invariant curve γ can be assumed to be just continuous. More precisely, we have the following

LEMMA 3.1. *Let $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ and suppose that the eigenvalues λ_1, λ_2 of $d\psi(0)$ satisfy* $\log|\lambda_2|/\log|\lambda_1| < 0$. *Let $\gamma \ni 0$ be a ψ-invariant curve of class C^0, that is, it locally admits a continuous, injective parametrization. Then $\gamma \setminus \{0\}$ is of class C^ω. If γ is of class C^1, then it is also real analytic around* 0.

Proof. Note that $d\psi(0)$ is represented by a matrix with real coefficients, hence its eigenvalues are either both real or they are complex conjugates. In the second case we would have $|\lambda_1| = |\lambda_2|$ against our assumption that $\log|\lambda_2|/\log|\lambda_1| < 0$, hence we have $\lambda_1, \lambda_2 \in \mathbb{R}$.

The result follows from the stable manifold theorem, in particular from the last part of Theorem 2.3. Indeed, let $(-t_0, t_0) \ni t \to \gamma(t) \in \mathbb{R}^2$ be a continuous injective parametrization of γ with $\gamma(0) = 0$. Choosing a small enough $t_1 > 0$, for any $t \in (-t_1, t_1)$ we have that $\psi(\gamma(t)) \in \gamma$ (and $\psi^{-1}(\gamma(t)) \in \gamma$), thus there exists a unique $\chi(t) \in (-t_0, t_0)$ such that $\gamma(\chi(t)) = \psi(\gamma(t))$. The map $\chi : (-t_1, t_1) \to \mathbb{R}$ is continuous and (since ψ is invertible) injective. Moreover $\chi(0) = 0$ and $\chi(t) \neq t$ for $t \neq 0$ because $0 \in \mathbb{R}^2$ is an isolated fixed point for ψ (which follows from the fact that ψ is topologically conjugated to its linear part).

The function χ is either strictly increasing or strictly decreasing: suppose the latter. Then by considering $\psi^{\circ 2}$ instead of ψ we can (possibly taking a smaller t_1) replace χ by $\chi^{\circ 2}$, hence we can assume that χ is strictly increasing. Focus now on $t > 0$ (the case of $t < 0$ is analogous): we either have $\chi(t) > t$ or $\chi(t) < t$ for all $0 < t < t_1$. Suppose that the latter is true. Then we claim that $\gamma(t') \in W^s_{\text{loc}}$ for any $0 < t' < t_1$, where W^s_{loc} is given by Theorem 2.3. Indeed, since $0 < \chi(t) < t$ we have $\chi^{\circ n}(t') \to 0$ as $n \to \infty$, hence $\psi^{\circ n}(\gamma(t')) = \gamma(\chi^{\circ n}(t')) \to 0$ as $n \to \infty$. By Theorem 2.3 this implies that $\gamma(t') \in W^s_{\text{loc}}$. If $\chi(t) > t$ for all $0 < t < t_1$ we replace ψ by ψ^{-1} and we repeat the same argument.

From the previous discussion follows that $\gamma([-\epsilon, \epsilon] \setminus \{0\})$ is contained in the set $W^s_{\text{loc}} \cup W^u_{\text{loc}}$ for some small $\epsilon > 0$: since $\gamma(t) \neq 0$ for $t \neq 0$ this means that $\gamma \setminus \{0\}$ is of class C^ω. If γ is of class C^1, it must coincide with either W^s_{loc} or W^u_{loc} and thus it must be real analytic. □

From now on we will assume that the invariant curve γ for the diffeomorphism $\psi \in \text{Diff}^\omega(\mathbb{R}^2, 0)$ is at least of class C^1. In suitable coordinates we have $T_0(\gamma) = \langle \frac{\partial}{\partial x} \rangle$, so that $\frac{\partial}{\partial x}$ is an eigenvector for $d\psi(0)$. This in turn implies that the eigenvalues $\lambda_1, \lambda_2 \neq 0$ of the linear part of ψ are both real. We will always take λ_1 to be the eigenvalue associated to the eigenvector $\frac{\partial}{\partial x}$.

For any $\psi \in \text{Diff}^\omega(\mathbb{R}^2, 0)$ with a distinguished eigenvalue λ_1 such that $|\lambda_1| \neq 1$, we now define $\tau = \tau(\psi) \in \mathbb{R}$ as $\tau(\psi) = \log|\lambda_2| / \log|\lambda_1|$. It is easy to check that the ratio $\log|\lambda_2| / \log|\lambda_1|$ is constant along the group generated by ψ in $\text{Diff}^\omega(\mathbb{R}^2, 0)$, that is, $\tau(\psi) = \tau(\psi^{\circ n})$ for all $n \in \mathbb{Z}$ (as long as the eigenvalue in the denominator is always associated to the same eigenvector).

THEOREM 3.2. *Let* $\psi \in \text{Diff}^\omega(\mathbb{R}^2, 0)$ *and let* γ *be a* ψ*-invariant curve,* $0 \in \gamma$*. Suppose that* $|\lambda_1| \neq 1$ *and that* γ *is of class* C^k*,* $k > \max\{0, \tau(\psi)\}$*. Then* γ *is a real-analytic curve.*

Proof. By the observations above we can replace ψ by $\psi^{\circ 2}$ (which still admits γ as an invariant curve) without changing $\tau(\psi)$, to ensure that the eigenvalues are both positive. Furthermore, possibly taking the inverse of ψ we can normalize λ_1 in such a way that $0 < \lambda_1 < 1$. Performing a linear change of coordinates, we will also assume that $d\psi(0)$ is in its Jordan form (note that, since $d\psi(0)$ has real eigenvalues, we can normalize it by conjugating in $GL(2, \mathbb{R})$). If $g, h \in C^\omega(\mathbb{R}^2, 0)$ are the components of ψ, that is, $\psi(x, y) = (g(x, y), h(x, y))$, we have $h(x, y) = \lambda_2 y + O(2)$ and either $g(x, y) = \lambda_1 x + O(2)$ or $g(x, y) = \lambda_1 x + y + O(2)$ (where in the latter case $\lambda_1 = \lambda_2$).

If $\lambda_2 > 1$, then the conclusion of the theorem is given by Lemma 3.1 with γ of class C^1. Suppose then that $0 < \lambda_2 \leqslant 1$. Our aim is to reduce to a situation in which we can apply the stable manifold theorem in a way similar to Lemma 3.1. In order to achieve this, we will perform a suitable sequence of blow-ups. In a small neighborhood of 0, we can express γ as a graph over $\mathbb{R}_x$: $\gamma = \{y = f(x)\}$, where f is of class C^k around 0 and $f(0) = f'(0) = 0$. Denote by

$$T_k(x) = \sum_{j=2}^{k} \frac{f^{(j)}(0)}{j!} x^j$$

the polynomial expressing the kth order Taylor expansion of f about 0, and define a local change of coordinates $\phi \in \text{Diff}^\omega(\mathbb{R}^2, 0)$ as $\phi(x, y) = (x, y - T_k(x))$. We will denote again by $\gamma = \{y = f(x)\}$ and ψ, respectively, the image $\phi(\gamma)$ of the invariant curve and the conjugate $\phi \circ \psi \circ \phi^{-1}$ of ψ through the diffeomorphism ϕ. After this change of coordinates we have $f^{(j)}(0) = 0$ for $0 \leqslant j \leqslant k$.

The invariance of γ with respect to ψ is expressed by the mapping equation

$$f(g(x, f(x))) = h(x, f(x)), \tag{1}$$

valid for all x in a neighborhood of 0 in $\mathbb{R}$. We can check that the mapping equation implies

$$h(x, y) = y(\lambda_2 + Q(x, y)) + r(x),$$

where $r(x) = O(x^{k+1})$ and $Q = O_1$ (here and in the following, we are going to write O_ℓ for a power series or function which is of order ℓ jointly in the variables (x, y)), by taking the k-jet of both sides in (1). The k-jet of the left-hand side is, in fact, vanishing, and since $h(x, f(x)) = r(x) + O(x^{k+1})$ this is only possible if $r(x)$ is of order $k + 1$.

Let $\pi(x_1, y_1) = (x_1, x_1 y_1)$ be the blow-up at 0, and let $\widetilde{\psi} = (\widetilde{g}, \widetilde{h})$ be the lift of ψ satisfying $\pi \circ \widetilde{\psi} = \psi \circ \pi$ (see Lemma 2.5). Furthermore, let $\widetilde{\gamma}$ be the lift of γ through the blow-up. The curve $\widetilde{\gamma}$ is the closure of $\pi^{-1}(\gamma \setminus \{0\})$, and can be expressed as $\widetilde{\gamma} = \{y_1 = \widetilde{f}(x_1)\}$ with $\widetilde{f}(t) = f(t)/t$ (see Lemma 2.6). Hence $\widetilde{f}$ is of class C^{k-1} and $\widetilde{f}^{(j)}(0) = 0$ for $0 \leqslant j \leqslant k - 1$. Observe that f is of class C^ω if and only if that is the case for $\widetilde{f}$.

Next, we perform a computation of the lift of ψ in the coordinate chart $\{u \neq 0\}$ of $\widetilde{\mathbb{R}^2}$ in which $[1 : 0] \cong 0 \in S$. Let $g(x, y) = \lambda_1 x + P(x, y)$ or $g(x, y) = \lambda_1 x + y + P(x, y)$ with $P(x, y) = O_2$. In either case,

$$\widetilde{g}(x_1, y_1) = g(x_1, x_1 y_1) = \lambda_1 x_1 + O_2.$$

For the second component we get

$$\widetilde{h}(x_1, y_1) = \frac{h(x_1, x_1 y_1)}{g(x_1, x_1 y_1)} = \frac{\lambda_2 y_1 + y_1 Q(x_1, x_1 y_1)}{\lambda_1 + O(1)} + \frac{r(x_1)}{\lambda_1 x_1 + P(x_1, 0) + y_1 x_1 S(x_1, y_1)}$$

for a suitable $S(x_1, y_1)$. It follows that

$$\widetilde{h}(x_1, y_1) = \frac{\lambda_2}{\lambda_1} y_1 + y_1 \widetilde{Q}(x_1, y_1) + \widetilde{r}(x_1),$$

where $\widetilde{r}(x_1) = \frac{r(x_1)}{x_1} / (\lambda_1 + \frac{P(x_1, 0)}{x_1}) = O(x_1^k)$ and $\widetilde{Q}(x_1, y_1) = O_1$.

In summary, we have that the linear part of $\widetilde{\psi} \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ is either diagonal or (possibly, if $k = 1$) triangular. In any case its eigenvalues are $(\widetilde{\lambda}_1, \widetilde{\lambda}_2) = (\lambda_1, \lambda_2/\lambda_1)$. Note that $\tau(\widetilde{\psi}) = \frac{\log \widetilde{\lambda}_2}{\log \widetilde{\lambda}_1} = \frac{\log \lambda_2 - \log \lambda_1}{\log \lambda_1} = \tau(\psi) - 1$.

Iterating the blow-up operation k times we obtain a finite sequence $(\widetilde{\psi}^j, \widetilde{\gamma}^j)$, $j = 0, \ldots, k$, such that $(\widetilde{\psi}^0, \widetilde{\gamma}^0) = (\psi, \gamma)$ and each $\widetilde{\psi}^{j+1}$, respectively, $\widetilde{\gamma}^{j+1}$, is the lift of $\widetilde{\psi}^j$, respectively, $\widetilde{\gamma}^j$, by the blow-up at 0. Each $\widetilde{\gamma}^j$ is a $\widetilde{\psi}^j$-invariant curve that can be locally written as $\widetilde{\gamma}^j = \{y_j = \widetilde{f}_j(x_j)\}$ with $\widetilde{f}_j$ of class C^{k-j}, and $\tau(\widetilde{\psi}^j) = \tau(\psi) - j$.

For $j = k$ we have $\tau(\widetilde{\psi}^k) = \tau(\psi) - k < 0$ and $\widetilde{\gamma}^k$ is of class C^0, hence by Lemma 3.1 we have that $\widetilde{\gamma}$ is locally contained in the union of the stable manifold W^s_{loc} and the unstable manifold W^u_{loc} for $\widetilde{\psi}^k = (\widetilde{g}_k, \widetilde{h}_k)$ at 0. However, we have that W^u_{loc} coincides with the exceptional divisor $S_k = \{x_k = 0\}$ for the last blow-up $(x_k, y_k) \to (x_k, x_k y_k)$. Indeed, by Theorem 2.3 W^u_{loc} is the only $\widetilde{\psi}_k$-invariant 1-manifold which is tangent to $\partial/\partial y_k$, and S_k fulfills these properties (alternatively, one can apply Lemma 3.1 with $\gamma = S_k$). Since $\widetilde{\gamma}^k$ only intersects S_k at 0, it must be completely contained in W^s_{loc}, hence it is real analytic. □

3.2. *Curves of class $C^{k,\alpha}$.* In order to prove Theorem 3.2, it is in most cases simpler to conjugate ψ in $\mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ to its normal form, as given by Reich [17]. Indeed, when $0 < \lambda_1, \lambda_2 < 1$ there is at most one resonant monomial and the mapping equation for the normalized germ is quite simple.

However, the proof via blow-ups — which relies on relatively more "classical" tools like the stable manifold theorem — is suited to being generalized to situations where a normal form is not available, or would not provide a significant simplification of the mapping equation. This is what we are going to do in § 4 for germs which are tangent to the identity. Furthermore, in Theorem 3.2 non-hyperbolic germs can also appear since it is possible that $\lambda_2 = 1$, so the application of at least one blow-up or of the version of the center-stable manifold theorem for non-hyperbolic germs seems necessary anyway.

It seems worthwhile to state a refinement of Theorem 3.2 in which the smoothness requirement on γ is weakened, at least in the case $\tau(\psi) \geqslant 1$, which implies $0 < \lambda_2 \leqslant \lambda_1$ and thus that ψ belongs to the Poincaré domain. The proof can be obtained by passing to the normal form and we shall only sketch it. We consider Hölder continuity conditions on the derivatives by taking γ of class $C^{k,\alpha}$, $k \geqslant 1$, $0 \leqslant \alpha < 1$, where we set $C^{k,0} = C^k$.

THEOREM 3.3. *Let ψ, γ be as in Theorem 3.2 with $\tau(\psi) \geqslant 1$. Let γ be of class $C^{k,\alpha}$, where we assume $k \geqslant \tau(\psi)$ if $1 < \tau(\psi) \in \mathbb{N}$ and $k + \alpha > \tau(\psi)$ otherwise. Then γ is real analytic.*

Proof. Applying the results in [**17**] to our situation, we can distinguish the following cases: if $\tau(\psi) \notin \mathbb{N}$, we can put ψ in the linear normal form $\psi(x, y) = (\lambda_1 x, \lambda_2 y)$, whereas if $\tau(\psi) = 1$ we can either put it in the form $\psi(x, y) = (\lambda x, \lambda y)$ or $\psi(x, y) = (\lambda x + y, \lambda y)$. In both cases, the result follows by differentiating the mapping equation enough times. When $\tau(\psi) = k \in \mathbb{N}$, we can choose coordinates in which either $\psi(x, y) = (\lambda x, \lambda^k y)$ or $\psi(x, y) = (\lambda x, \lambda^k y + x^k)$: in the latter case the mapping equation reads

$$f(\lambda x) = \lambda^k f(x) + x^k,$$

differentiating which k times we get the relation $f^{(k)}(\lambda x) = f^{(k)}(x) + (k!/\lambda^k)$ for the kth derivative of f, which gives an immediate contradiction since f is assumed to be of class C^k around 0. □

3.3. *Nowhere-analytic invariant curves of finite smoothness.* We will now give some examples showing that the regularity assumptions in Theorem 3.3 are essentially sharp. As a matter of fact, for $k \geqslant 1$, $0 < \alpha < 1$ the curve $\gamma = \{y = |x|^{k+\alpha}\}$ is not analytic and is invariant under $(x, y) \to (\lambda_1 x, \lambda_2 y)$ for any $0 < \lambda_2 < \lambda_1 < 1$ such that $\log \lambda_2 / \log \lambda_1 = k + \alpha$. However, nowhere-analytic examples are also possible:

Example 3.4. For any $\tau > 1$, $\tau \notin \mathbb{N}$, there exist a (linear) diffeomorphism ψ with $\tau(\psi) = \tau$ and a ψ-invariant, nowhere-analytic curve $\gamma \ni 0$ of class $C^{k,\alpha}$ such that $k + \alpha = \tau(\psi)$. Furthermore, $\gamma \setminus \{0\}$ is of class C^∞.

Proof. Choose $0 < \lambda_2 < \lambda_1 < 1$ such that $\log \lambda_2 / \log \lambda_1 = \tau$ and define $\psi(x, y) = (\lambda_1 x, \lambda_2 y)$. Let $\eta \in C^\infty([\lambda_1, 1])$ be nowhere-analytic in $(\lambda_1, 1)$ and $\eta^{(j)}(\lambda_1) = \eta^{(j)}(1) = 0$ for all $j \in \mathbb{N}$ (see also Lemma 2.7). We define $\gamma = \{y = f(x)\}$ by putting $f(0) = 0$, $f(x) = \lambda_2^n \eta(x/\lambda_1^n)$ for all $x \in [\lambda_1^{n+1}, \lambda_1^n]$, $n \in \mathbb{N}$, and $f(x) = f(-x)$ for $x \in [-1, 0]$. By construction it is clear that $f \in C^\infty([-1, 1] \setminus \{0\})$ is nowhere-analytic, and that γ is ψ-invariant. Moreover, putting $M_j = \max_{[\lambda_1, 1]} |\eta^{(j)}|$, for any $x \in [-\lambda_1^n, \lambda_1^n]$ we have $f^{(j)}(x) = (\frac{\lambda_2}{\lambda_1^j})^m \eta^{(j)}(x/\lambda_1^m)$ for some $m \geqslant n$, hence $|f^{(j)}(x)| \leqslant (\frac{\lambda_2}{\lambda_1^j})^n M_j$. Since $\frac{\lambda_2}{\lambda_1^j} < 1$ for $j \leqslant k$, this estimate shows that f is of class C^k around 0. We need to verify that $f^{(k)}$ is Hölder continuous with exponent $\alpha = \tau(\psi) - k < 1$. Since f is an even function it is sufficient to estimate $|f^{(k)}(x'') - f^{(k)}(x')|$

for $0 \leqslant x' < x''$. Let $\ell \leqslant m$ be such that $\lambda_1^{m+1} \leqslant x' \leqslant \lambda_1^m$, $\lambda_1^{\ell+1} \leqslant x'' \leqslant \lambda_1^\ell$, and let $x_0' = x'/\lambda_1^m$, $x_0'' = x''/\lambda_1^\ell$. One has

$$|f^{(k)}(x'') - f^{(k)}(x')| = \left(\frac{\lambda_2}{\lambda_1^k}\right)^\ell |\eta^{(k)}(x_0'') - \left(\frac{\lambda_2}{\lambda_1^k}\right)^{m-\ell} \eta^{(k)}(x_0')|. \tag{2}$$

Suppose first that $\ell < m-1$, so that $|x'' - x'| \geqslant \lambda_1^{\ell+1} - \lambda_1^m = \lambda_1^{\ell+1}(1 - \lambda_1^{m-\ell-1}) \geqslant A\lambda_1^\ell$ with $A = \lambda_1 - \lambda_1^2$. Using (2) we obtain $|f^{(k)}(x'') - f^{(k)}(x')| \leqslant 2M_k(\frac{\lambda_2}{\lambda_1^k})^\ell = 2M_k\lambda_1^{\alpha\ell} \leqslant (2M_k/A^\alpha)|x'' - x'|^\alpha$.

If $\ell = m-1$, we write $x'' - x' = \lambda_1^\ell((x_0'' - \lambda_1) + \lambda_1(1 - x_0'))$, and using (2) together with the flatness of η at $x = \lambda_1, 1$ we have

$$\begin{aligned}|f^{(k)}(x'') - f^{(k)}(x')| &\leqslant \left(\frac{\lambda_2}{\lambda_1^k}\right)^\ell \left(M_{k+1}(x_0'' - \lambda_1) + \left(\frac{\lambda_2}{\lambda_1^k}\right)\frac{M_{k+1}}{\lambda_1}\lambda_1(1 - x_0')\right)\\ &\leqslant \lambda_1^{\alpha\ell}\frac{M_{k+1}}{\lambda_1}(x_0'' - \lambda_1 + \lambda_1(1 - x_0'))^\alpha = \frac{M_{k+1}}{\lambda_1}|x'' - x'|^\alpha.\end{aligned}$$

Finally, if $\ell = m$, we have $x'' - x' = \lambda_1^\ell(x_0'' - x_0')$ and

$$\begin{aligned}|f^{(k)}(x'') - f^{(k)}(x')| &= \left(\frac{\lambda_2}{\lambda_1^k}\right)^\ell |\eta^{(k)}(x_0'') - \eta^{(k)}(x_0')| \leqslant \lambda_1^{\alpha\ell}M_{k+1}(x_0'' - x_0')^\alpha\\ &= M_{k+1}|x'' - x'|^\alpha.\end{aligned}$$

□

Next, we provide an example with $\tau(\psi) = 1$:

Example 3.5. Fixed $0 < \lambda < 1$, let $\psi(x, y) = (\lambda x + y, \lambda y)$. Then there is a nowhere-analytic curve $\gamma \ni 0$ of class C^1 which is invariant for ψ. Furthermore, $\gamma \setminus \{0\}$ is of class C^∞.

Proof. As in the previous example, we start by choosing a function $\eta \in C^\infty([\lambda + \epsilon, 1])$ for a certain $0 < \epsilon < 1 - \lambda$, nowhere-analytic on $(\lambda + \epsilon, 1)$, satisfying the following properties (see Lemma 2.8). We will require that $\eta(1) = \epsilon$, $\eta(\lambda + \epsilon) = \lambda\epsilon$, and for some small $\delta > 0$ that $\eta'(1) = \delta$, $\eta'(\lambda + \epsilon) = \lambda\delta/(\lambda + \delta)$. Furthermore $\eta^{(j)}(\lambda + \epsilon) = \eta^{(j)}(1) = 0$ for all $j \geqslant 2$. Finally, $|\eta(x)| \leqslant 1$ and $0 < \eta'(x) \leqslant 1$ for all $x \in [\lambda + \epsilon, 1]$.

We define $f(x) = \eta(x)$ for $\lambda + \epsilon \leqslant x \leqslant 1$; in particular $\lambda + f'(x) > 0$ on $[\lambda + \epsilon, 1]$, hence the map $x \to \lambda x + f(x)$ sends the interval $[\lambda + \epsilon, 1] = [\xi_1, \xi_0]$ injectively into $[\lambda^2 + 2\lambda\epsilon, \lambda + \epsilon] = [\xi_2, \xi_1]$. Thus we can extend the function f to $[\xi_2, \xi_1)$ by putting $f(\lambda x + f(x)) = \lambda f(x)$ for all $x \in [\xi_1, \xi_0)$, which implies $|f(x)| \leqslant \lambda$ for all $x \in [\xi_2, \xi_1)$. Differentiating the defining relation we get $f'(\lambda x + f(x)) = \lambda f'(x)/(\lambda + f'(x))$ for all $x \in [\xi_1, \xi_0)$ and for $j \geqslant 2$

$$f^{(j)}(\lambda x + f(x)) = \frac{\sum_{\ell=2}^j f^{(\ell)}(x)P_{j\ell}(f'(x), \dots, f^{(j)}(x))}{(\lambda + f'(x))^{2j-1}}$$

for suitable polynomials $P_{j\ell}$, as can be verified inductively. We deduce that $\lim_{x\to\xi_1^-} f(x) = \lambda f(1) = \lambda\epsilon$, $\lim_{x\to\xi_1^-} f'(x) = \lambda f'(1)/(\lambda + f'(1)) = \lambda\delta/(\lambda + \delta)$ and

$$\lim_{x\to\xi_1^-} f^{(j)}(x) = \frac{\sum_{\ell=2}^j f^{(\ell)}(1)P_{j\ell}(f'(1), \dots, f^{(j)}(1))}{(\lambda + f'(1))^{2j-1}} = 0$$

for all $j \geqslant 2$. This shows that $f \in C^\infty([\xi_2, \xi_0])$.

Let $h(x) = x/(1 + x/\lambda)$ and for $x_0 > 0$ define $x_j = h^{\circ j}(x_0)$. Note that for every $n \in \mathbb{N}$, $0 < h(x) \leqslant 1/(n+1)$ if $0 < x \leqslant 1/n$. Since $f'(\lambda x + f(x)) = h(f'(x))$, we have in particular $0 < f'(x) \leqslant 1/2$ for $x \in [\xi_2, \xi_1]$.

Suppose now that the function f has been defined on the interval $[\xi_n, \xi_0]$, $n \geqslant 2$, and that $\xi_n = \lambda\xi_{n-1} + f(\xi_{n-1})$, $0 < \xi_n < (\lambda + \epsilon)^n$, $f(\xi_n) = \lambda^n\epsilon$, $0 < f(x) < \lambda^{n-1}$ and $0 < f'(x) \leqslant 1/n$ for $x \in [\xi_n, \xi_{n-1}]$. Then $\lambda + f'(x) > 0$ for $x \in [\xi_n, \xi_{n-1}]$, hence again the map $\lambda x + f(x)$ sends the interval $[\xi_n, \xi_{n-1}]$ injectively into the interval $[\xi_{n+1}, \xi_n]$ where $\xi_{n+1} = \lambda\xi_n + f(\xi_n)$. We have $0 < \xi_{n+1} < \lambda(\lambda+\epsilon)^n + \lambda^n\epsilon < \lambda(\lambda+\epsilon)^n + (\lambda+\epsilon)^n\epsilon = (\lambda+\epsilon)^{n+1}$. As before, we extend f to $[\xi_{n+1}, \xi_n)$ by putting $f(\lambda x + f(x)) = \lambda f(x)$ for all $x \in [\xi_n, \xi_{n-1})$, which immediately implies $f(\xi_{n+1}) = \lambda^{n+1}\epsilon$ and $0 < f(x) < \lambda^n$ for $x \in [\xi_{n+1}, \xi_n)$.

Moreover, since $f'(\lambda x + f(x)) = h(f'(x))$ we have $0 < f'(x) \leqslant 1/(n+1)$ for $x \in [\xi_{n+1}, \xi_n)$. The smoothness of f at ξ_n follows automatically from the smoothness of f around ξ_{n-1}, since the relation $f(\lambda x + f(x)) = \lambda f(x)$ holds for all x in a neighborhood of ξ_{n-1}. Thus $f \in C^\infty([\xi_{n+1}, \xi_0])$, which concludes the inductive step.

Since $\xi_n \to 0^+$ as $n \to \infty$, the procedure described above gives a nowhere-analytic function $f \in C^\infty((0,1])$ such that $f(\lambda x + f(x)) = \lambda f(x)$ for all $x \in (0,1]$. Furthermore, $f(x) \to 0$ and $f'(x) \to 0$ as $x \to 0^+$, hence we can extend f in a C^1 fashion to $[-1,1]$ by putting $f(0) = 0$ and $f(x) = -f(-x)$ for $x \in [-1, 0)$. The resulting $\gamma = \{y = f(x)\}$ is ψ-invariant: the mapping equation is satisfied by construction for $x > 0$, and for $x < 0$ we have $f(\lambda x + f(x)) = -f(-\lambda x - f(x)) = -f(\lambda \cdot (-x) + f(-x)) = -\lambda f(-x) = \lambda f(x)$. □

3.4. *The case of* $\lambda_1 = 1$, $|\lambda_2| \neq 1$. Although the conclusion of Theorem 3.2 holds for some germs which have the eigenvalue λ_2 of modulus 1, the diffeomorphisms $\psi_{1,t}$ in Example 2.2 show that it does not hold for all germs which are not tangent to the identity. Now we want to show that it is possible to construct analogous examples in which the curve γ is of class C^∞ but nowhere-analytic. To this end, we define $\phi \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ by

$$\phi(x, y) = \left(\frac{x}{1 - x^2}, 2y - x^2\right)$$

whose inverse $\phi^{-1} = \psi \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ is given by

$$\psi(x, y) = \left(\frac{\sqrt{1 + 4x^2} - 1}{2x}, \frac{1}{2}y + \frac{2x^2 + 1 - \sqrt{1 + 4x^2}}{4x^2}\right).$$

Example 3.6. Let ψ, ϕ be as above. There exists a curve (and indeed a cardinality of continuum of curves) $\gamma \subset \mathbb{R}^2$, $0 \in \gamma = \{y = f(x)\}$, such that

- γ is of class C^∞ and ψ- (and ϕ-)invariant;
- f is nowhere real analytic, and the radius of convergence of the Taylor series of f at the origin is 0.

To prove the claim in Example 3.6 we will need some preliminary results:

LEMMA 3.7. *Let ϕ be as above: then there is a uniquely determined ϕ-invariant formal curve $I \subset \mathbb{R}[[x, y]]$ of the form $I = (y - u(x))$ with $u \in \mathbb{R}[[x]]$ and $u(0) = 0$. Furthermore, I is purely formal, that is, $u \notin \mathbb{R}\{x\}$.*

Proof. Any such $u = \sum_{j \geqslant 1} u_j x^j$ must satisfy the mapping equation

$$u\left(\frac{x}{1 - x^2}\right) = 2u(x) - x^2,$$

that is,

$$\sum_{j\geqslant 1} u_j \left(\sum_{k\geqslant 0} x^{2k+1}\right)^j = 2\sum_{j\geqslant 1} u_j x^j - x^2. \tag{3}$$

By examining (3), one can easily check inductively that $u_{2k+1} = 0$ for all $k \geqslant 0$ and that the sequence $\{u_{2k}\}$ must satisfy $u_2 = 1$ and

$$u_{2k} = \sum_{\ell=1}^{k-1} \binom{k+\ell-1}{2\ell-1} u_{2\ell} \tag{4}$$

for all $k \geqslant 2$; in particular $u \in \mathbb{R}[[x]]$ is uniquely determined. To see that $u \notin \mathbb{R}\{x\}$, we note that (4) implies $u_{2k} \geqslant (2k-2)u_{2k-2}$ for all $k \geqslant 2$, hence $u_{2k} \geqslant 2^{k-1}(k-1)!$ for all $k \geqslant 1$ and the radius of convergence of u is 0. $\square$

LEMMA 3.8. *Let* $0 < \lambda < 1$, $R, C, \ell > 0$ *and* $b \in \mathbb{R}$ *be fixed. Then there exists* $C' > 0$ *with the following property. Given any two real sequences* $\{a_n\}$, $\{b_n\}$ *such that* $|a_n - \lambda| \leqslant C/n^\ell$, $|b_n - b| \leqslant C/n^\ell$ *for all* $n \in \mathbb{N}$, *define* $\{y_n\}$ *iteratively by choosing* $y_1 \in [-R, R]$ *and putting* $y_{n+1} = a_n y_n + b_n$ *for all* $n \geqslant 0$*: then* $|y_n - \frac{b}{1-\lambda}| \leqslant C'/n^\ell$ *for all* $n \in \mathbb{N}$.

Proof. Defining $y'_n = y_n - \frac{b}{1-\lambda}$, the sequence $\{y'_n\}$ satisfies the recursive relation $y'_{n+1} = a_n y'_n + b'_n$ with $b'_n = b_n - b\frac{1-a_n}{1-\lambda}$, so that

$$|b'_n| \leqslant |b_n - b| + |b|\frac{|a_n - \lambda|}{1-\lambda} \leqslant \frac{|b| + 1 - \lambda}{1-\lambda} \cdot \frac{C}{n^\ell} = \frac{D}{n^\ell}$$

for all $n \in \mathbb{N}$. Thus, up to replacing $\{y_n\}$ by $\{y'_n\}$ we can assume $b = 0$ in the statement of the lemma.

Consider now the (non-negative) sequence $\{z_n\}$ defined recursively by $z_1 = R$ and

$$z_{n+1} = \left(\lambda + \frac{C}{n^\ell}\right) z_n + \frac{D}{n^\ell}$$

for $n \in \mathbb{N}$; this is a majorant sequence for $\{y'_n\}$ since $|y'_n| \leqslant z_n \Rightarrow |y'_{n+1}| \leqslant |a_n||y'_n| + |b'_n| \leqslant (\lambda + C/n^\ell)z_n + D/n^\ell = z_{n+1}$. Put $\lambda' = \frac{1+\lambda}{2}$, $\lambda'' = \frac{1+\lambda'}{2}$ and choose an integer $n_0 > \max\{1 + \frac{1}{1-(\lambda'')^{1/\ell}}, (\frac{2C}{1-\lambda})^{1/\ell}\}$ (so that $(\frac{n}{n+1})^\ell > \lambda''$ and $\lambda + \frac{C}{n^\ell} < \lambda'$ for all $n \geqslant n_0$). Finally, define $E = \max_{n\leqslant n_0} z_n n^\ell$ and $C' = \max\{E, \frac{D}{\lambda''-\lambda'}\}$; note that C' depends only on λ, b, R, C and ℓ as claimed. By construction $z_n \leqslant E/n^\ell \leqslant C'/n^\ell$ for all $n \leqslant n_0$. Arguing by induction, then, we get $z_{n+1} \leqslant \lambda' C'/n^\ell + D/n^\ell \leqslant \lambda'' C'/n^\ell \leqslant C'/(n+1)^\ell$ for all $n \geqslant n_0$, which concludes the proof. $\square$

Put now $g(x) = \frac{\sqrt{1+4x^2}-1}{2x}$, $r(x) = \frac{2x^2+1-\sqrt{1+4x^2}}{4x^2}$ so that $\psi(x, y) = (g(x), \frac{1}{2}y + r(x))$; observe that $g(x) = x - x^3 + O(x^4)$ and $r(x) = O(x^2)$. For any $x_1 \in \mathbb{R}$ and $n \in \mathbb{N}$, let $x_n = g^{\circ(n-1)}(x_1)$ as long as it is defined. From the Fatou theory of one-dimensional maps tangent to the identity we derive the following: there are $\varepsilon, A > 0$ such that for any $x_1 \in \mathbb{R}$ with $|x_1| \leqslant \varepsilon$ we have $x_n \to 0$ (monotonically) as $n \to \infty$ and, furthermore, $|x_n| \leqslant A/\sqrt{n}$ for all $n \in \mathbb{N}$.

Let us turn to the construction of $\gamma = \{y = f(x)\}$. Since $g(-x) = -g(x)$ and $r(x) = r(-x)$, the mapping ψ commutes with the y-axis reflection $(x, y) \to (-x, y)$, thus we can define $f(x)$ for small enough $x > 0$ and then extend f as an even function. Fix $\xi_0 > 0$ with $|\xi_0| < \varepsilon$, and put $\xi_n = g^{\circ n}(\xi_0)$. We can choose a function η like in Lemma 2.7, with $a_1 = \xi_1$, $a_2 = \xi_0$ and the sequences $\{d^1_j\}$, $\{d^2_j\}$ defined by $d^2_j \equiv 0$, $d^1_0 = (\xi_1)^2/2$, $d^1_1 = \xi_1$, $d^1_2 = 1$ and $d^1_j = 0$ for $j \geqslant 3$. We start then by putting $f(x) = \eta(x)$ for all $x \in [\xi_1, \xi_0]$.

Next, we define f inductively over any interval $[\xi_{n+1}, \xi_n)$, $n \geqslant 1$, by putting

$$f(x) = \frac{1}{2} f\left(\frac{x}{1-x^2}\right) + \frac{x^2}{2} \tag{5}$$

for all $x \in [\xi_{n+1}, \xi_n)$; we remark that by construction $x \to x/(1-x^2)$ maps $[\xi_{n+1}, \xi_n]$ into $[\xi_n, \xi_{n-1}]$. Finally, we put $f(0) = 0$ and $f(x) = f(-x)$ for $-\xi_0 < x < 0$. With this definition the curve $\gamma = \{y = f(x)\}$ is invariant under ϕ as (5) corresponds precisely to the ϕ-mapping equation.

It is clear that f is of class C^∞ on any open interval (ξ_{n+1}, ξ_n). To check that f is of class C^∞ at $x = \xi_1$, it is sufficient to note that, because of the choice of $\{d^2_j\}$ above, f is flat at $x = \xi_0$ (i.e. all its derivatives vanish). The defining relation (5) then implies that all the derivatives of $f(x)$ converge to the derivatives of $x^2/2$ for $x \to \xi_1^-$, hence from the choice of the sequence $\{d^1_j\}$ follows that $f(x)$ is smooth at $x = \xi_1$. Using (5) recursively we see that f is of class C^∞ on the whole interval $\{0 < x < \xi_0\}$. What remains to be shown is the following:

LEMMA 3.9. *The function f defined above is of class C^∞ in a neighborhood of* 0, *and $f^{(j)}(0) = j!u_j$ for all $j \in \mathbb{N}$ where $u(x) = \sum_{j\geqslant 1} u_j x^j$ is given by Lemma* 3.7.

Proof. It is sufficient to show that f is smooth around 0, since in that case the Taylor expansion Tf of f at 0 gives a formal ϕ-invariant curve of the form $\{y = Tf(x)\}$, therefore it must coincide with u by Lemma 3.7.

To prove the smoothness of f around 0, we will use the following claim: for all $j \geqslant 0$ there exist functions $r_j(x)$, $q_{j,k}(x)$ $(0 \leqslant k \leqslant j)$, real analytic in a neighborhood of 0 containing $[-\xi_0, \xi_0]$, such that $q_{j,j}(0) = 1/2$ and

$$f^{(j)}(g(x)) = r_j(x) + \sum_{k=0}^{j} q_{j,k}(x) f^{(k)}(x) \tag{6}$$

for all $x \neq 0$, where we set $f^{(0)}(x) := f(x)$. Indeed, for $j = 0$ the ψ-invariance of γ amounts to the equation $f(g(x)) = \frac{1}{2} f(x) + r(x)$. Assuming that (6) is satisfied for a certain $j \in \mathbb{N}$, by differentiating we obtain

$$f^{(j+1)}(g(x))g'(x) = r'_j(x) + \sum_{k=0}^{j+1} (q'_{j,k}(x) + q_{j,k-1}(x)) f^{(k)}(x)$$

(where we set $q_{j,j+1} = q_{j,-1} \equiv 0$), and since $g'(x) = 1 + O(x^2)$ is invertible we have that (6) holds for $j+1$ with $q_{j+1,k}(x) = (q'_{j,k}(x) + q_{j,k-1}(x))/g'(x)$ for all $0 \leqslant k \leqslant j+1$ and $r_{j+1}(x) = r'_j(x)/g'(x)$. Note that $q_{j+1,j+1}(0) = q_{j,j}(0)/g'(0) = 1/2$.

Next, we need to show the following: for any $j \in \mathbb{N}$ there exist $v_j \in \mathbb{R}$ and $C_j, \ell_j > 0$ such that $|f^{(j)}(x_n) - v_j| \leqslant C_j/n^{\ell_j}$ for all $x_1 \in [\xi_1, \xi_0]$, $n \in \mathbb{N}$. Arguing by induction, let us start with $j = 0$. The mapping equation for γ gives $f(g(x)) = \frac{1}{2} f(x) + r(x)$, from which we deduce, given any $x_1 \in [\xi_1, \xi_0]$, the relation $f(x_{n+1}) = \frac{1}{2} f(x_n) + r(x_n)$ for all $n \in \mathbb{N}$. Moreover, as noted above, we have $x_n \leqslant A/\sqrt{n}$ for all $n \in \mathbb{N}$, where the constant $A > 0$ does not depend on x_1; since $r(x) = O(x^2)$, this implies $|r(x_n)| \leqslant A'/n$, $n \in \mathbb{N}$, for some constant $A' > 0$ independent of x_1. We can now apply Lemma 3.8 with $\lambda = 1/2$, $C = A'$, $\ell = 1$, $b = 0$, $R = \max_{x\in[\xi_1,\xi_0]} |f(x)|$, $a_n \equiv 1/2$, $b_n = r(x_n)$, $y_n = f(x_n)$. We obtain a constant $C' > 0$, independent of $y_1 = f(x_1)$ (and thus independent of $x_1 \in [\xi_1, \xi_0]$) such that $|f(x_n)| \leqslant C'/n$ for all $n \in \mathbb{N}$; hence the claim is proved for $j = 0$ with $v_0 = 0$, $C_0 = C'$ and $\ell_0 = 1$.

Assume that the claim holds up to a certain $j \in \mathbb{N}$. Let $r_j(x)$ and $q_{j,k}(x)$, $0 \leqslant k \leqslant j$, be defined by (6), and choose $A', A'_k > 0, l, l_k \geqslant 1, 0 \leqslant k \leqslant j$, such that $|r_j(x) - r_j(0)| \leqslant A'|x|^l$, $|q_{j,k}(x)| \leqslant A'_k$ and $|q_{j,k}(x) - q_{j,k}(0)| \leqslant A'_k|x|^{l_k}$ for all $x \in [-\xi_0, \xi_0]$. Putting $s(x) = r_j(x) + \sum_{k=0}^{j-1}(q_{j,k}(x)f^{(k)}(x))$ and $b = r_j(0) + \sum_{k=0}^{j-1}(q_{j,k}(0)v_k)$, we have

$$|s(x_n) - b| \leqslant |r_j(x_n) - r_j(0)| + \sum_{k=0}^{j-1} |f^{(k)}(x_n) - v_k| \cdot |q_{j,k}(x_n)| + |q_{j,k}(x_n) - q_{j,k}(0)| \cdot |v_k|$$

$$\leqslant \frac{A'A^l}{n^{l/2}} + \sum_{k=0}^{j-1} \frac{A'_k C_k}{n^{\ell_k}} + |v_k| \frac{A'_k A^{l_k}}{n^{l_k/2}} \leqslant \frac{A''}{n^{\ell''}}$$

for every $x_1 \in [\xi_1, \xi_0]$, $n \in \mathbb{N}$, where $A'' > 0$ is large enough and $\ell'' > 0$ small enough. With a suitable choice of A'', ℓ'' we can ensure that $|q_{j,j}(x_n) - 1/2| = |q_{j,j}(x_n) - q_{j,j}(0)| \leqslant A''/n^{\ell''}$ as well for every $x_1 \in [\xi_1, \xi_0]$, $n \in \mathbb{N}$. We can now evaluate (6) at $x = x_n$ to get $f^{(j)}(x_{n+1}) = q_{j,j}(x_n)f^{(j)}(x_n) + s(x_n)$, and again apply Lemma 3.8 with the b defined above, $\lambda = 1/2$, $C = A''$, $\ell = \ell''$, $R = \max_{x \in [\xi_1, \xi_0]} |f^{(j)}(x)|$, $a_n = q_{j,j}(x_n)$, $b_n = s(x_n)$, $y_n = f^{(j)}(x_n)$ to obtain an estimate of the kind required by the claim.

Applying Lemma 3.8 with $\lambda = 1/2$ and $b = r_j(0) + \sum_{k=0}^{j-1}(q_{j,k}(0)v_k)$ we get $v_j = \lim_{n\to\infty} f^{(j)}(x_n) = 2(r_j(0) + \sum_{k=0}^{j-1} q_{j,k}(0)v_k)$, which only depends on the previous v_k and on the functions r_j, $q_{j,k}$ appearing in the expression (6). Using this and the fact that f is an even function we can see that repeating the same inductive argument for $x < 0$ we obtain exactly the same estimate: $|f^{(j)}(x_n) - v_j| \leqslant C_j/n^{\ell_j}$ for all $x_1 \in [-\xi_0, -\xi_1]$, $n \in \mathbb{N}$ with the same constants v_j, C_j, ℓ_j.

To finish the proof, fix $j \in \mathbb{N}$, $\epsilon > 0$ and let $n_0 \in \mathbb{N}$ such that $n_0 > (\frac{C_j}{\epsilon})^{1/\ell_j}$. For any $0 < x < \xi_{n_0}$, there exists $n \geqslant n_0$ such that $x \in [\xi_n, \xi_{n-1})$, that is, $x = x_n = g^{\circ(n-1)}(x_1)$ for some $x_1 \in [\xi_1, \xi_0)$. By the previous paragraphs we have $|f^{(j)}(x) - v_j| \leqslant C_j/n^{\ell_j} \leqslant C_j/n_0^{\ell_j} < \epsilon$. Arguing in the same way for $x < 0$ we get $|f^{(j)}(x) - v_j| < \epsilon$ for all $-\xi_{n_0} < x < \xi_{n_0}$, $x \neq 0$. It follows that $f^{(j)}$ extends continuously across 0, and since j is arbitrary we conclude that f is of class C^∞. ☐

§4. *Germs which are tangent to the identity.*

4.1. *Definition of integer invariants and statement of the result.* Let $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ be tangent to the identity and admit an invariant curve $\gamma \ni 0$ of class C^∞. Let $\{r = 0\}$, $r \in C^\infty(\mathbb{R}^2, 0)$, be any defining equation for γ in a neighborhood of $0 \in \mathbb{R}^2$. The Taylor expansion of r at 0 defines a formal series $\rho \in \mathbb{R}[[x, y]]$; clearly the formal curve ideal $I = (\rho)$ (see § 2.6) only depends on the germ of γ at 0 and not on the choice of the defining function.

Denote still by ψ the ring automorphism of $\mathbb{R}[[x, y]]$ induced by the germ ψ. Since γ is an invariant curve for ψ we have $\rho \circ \psi \in (\rho)$, that is, ψ preserves the ideal I and defines a ring automorphism ψ^I of the quotient $\mathbb{R}[[x, y]]/I$, which is again tangent to the identity. Let $d_1 = \nu(\psi^I - Id)$ be the order of tangency of this automorphism with the identity. If $d_1 = \infty$, we say that I is *dynamically trivial*.

Now, we define a second invariant as follows. Denote by $\pi : \mathbb{R}[[x, y]] \to \mathbb{R}[[x, y]]/I \cong \mathbb{R}[[t]]$ the projection onto the quotient. Let $\chi = \frac{\rho\circ\psi}{\rho} - 1$; note that the division is well defined in $\mathbb{R}[[x, y]]$ since $\rho \circ \psi \in (\rho)$. We put $d_2 = \min\{\nu(\pi(\chi)), d_1 - 1\}$. Note that the order of a formal power series is invariant under (formal) changes of coordinates, hence the same is true for d_2 if ρ is given. What we need to check is thus the following:

LEMMA 4.1. *The definition of d_2 only depends on the formal curve ideal I and not on the choice of the generator ρ.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{R}[[x, y]]$ and fix $\xi \in \mathfrak{m} \setminus I$ of order 1. Then by definition of d_1 we have $\xi \circ \psi = \xi + \xi^{d_1}\alpha + \rho\beta$, where α is some invertible element of $\mathbb{R}[[x, y]]$. It also follows that $\sigma \circ \psi = \sigma + O(\xi^{d_1}) + O(\rho)$ for all $\sigma \in \mathbb{R}[[x, y]]$.

Let $\rho' = \eta\rho$ with η an invertible element in $\mathbb{R}[[x, y]]$: we can write $\eta = c + \eta_1$ for suitable $c \in \mathbb{R} \setminus \{0\}$ and $\eta_1 \in \mathfrak{m}$. Putting $k = \nu(\pi(\chi))$ we have

$$\frac{\rho' \circ \psi}{\rho'} = \frac{c + \eta_1 \circ \psi}{c + \eta_1} \cdot \frac{\rho \circ \psi}{\rho} = \frac{c + \eta_1 + O(\xi^{d_1}) + O(\rho)}{c + \eta_1} \cdot (1 + b\xi^k + O(\xi^{k+1}) + O(\rho))$$
$$= (1 + O(\xi^{d_1}) + O(\rho)) \cdot (1 + b\xi^k + O(\xi^{k+1}) + O(\rho))$$

for some $b \in \mathbb{R} \setminus \{0\}$. Setting $\chi' = (\rho' \circ \psi/\rho') - 1$, if $k < d_1$, we get $\chi' = b\xi^k + O(\xi^{k+1}) + O(\rho)$, otherwise $\chi' = O(\xi^{d_1}) + O(\rho)$. In both cases we conclude that $\min\{\nu(\pi(\chi')), d_1 - 1\} = \min\{k, d_1 - 1\} = d_2$. □

THEOREM 4.2. *Let $\psi \in \mathrm{Diff}^{\omega}(\mathbb{R}^2, 0)$ be tangent to the identity, and let $\gamma = \{r = 0\}$ be an invariant curve for ψ of class C^{∞}, $0 \in \gamma$. Let d_1, d_2 be the invariants defined above, and assume that d_1 is finite and $d_2 = d_1 - 1$. Then $\gamma \setminus \{0\}$ is of class C^{ω}.*

Furthermore, if the Taylor expansion of r at 0 has a positive radius of convergence, then γ is real analytic.

4.2. *Chain of blow-ups.* The proof of Theorem 4.2 will follow from the theory of local two-dimensional complex discrete dynamical systems, in particular from the results by Hakim [12, 13] on the existence of parabolic curves. We will first recall some notions from this theory.

Let $\psi \in \mathrm{Diff}^{\omega}(\mathbb{R}^2, 0)$ be a germ of diffeomorphism tangent to the identity, $\psi(x, y) = (g(x, y), h(x, y))$, $\psi \neq Id$. We will write

$$g(x, y) = x + \sum_{j=2}^{\infty} P_j(x, y), \quad h(x, y) = y + \sum_{j=2}^{\infty} Q_j(x, y),$$

where, for all $j \in \mathbb{N}$, P_j and Q_j are homogeneous polynomials of degree j. We will denote by $\nu = \nu(\psi - Id) = \min\{\nu(g - x), \nu(h - y)\}$ the order of tangency of ψ with the identity. This corresponds to the smallest integer $j_0 \geqslant 2$ such that either $P_{j_0} \neq 0$ or $Q_{j_0} \neq 0$. Since $\psi \neq Id$, we have $\nu < +\infty$.

Let $\psi_\nu : \mathbb{R}^2 \to \mathbb{R}^2$ be the homogeneous polynomial map $\psi_\nu(x, y) = (P_\nu(x, y), Q_\nu(x, y))$. One says that $[u : v] \in \mathbb{R}E\mathbb{P}^1$ is a *characteristic direction* for ψ if $\psi_\nu(u, v) = (\lambda u, \lambda v)$ for some $\lambda \in \mathbb{R}$. If $\lambda = 0$, $[u : v]$ is called a *degenerate* characteristic direction, *non-degenerate* otherwise.

Let $\pi : \widetilde{\mathbb{R}}^2 \to \mathbb{R}^2$ be the blow-up at 0, let $S = \pi^{-1}(0) \cong \mathbb{RP}^1$ be the exceptional divisor, and let $\widetilde{\psi}$ be the lift of ψ by the blow-up. Since $d\psi(0)$ induces the identity on $\mathbb{RP}^1$, $\widetilde{\psi}|_S \equiv Id$. For every direction $\xi \in \mathbb{RP}^1$ we have a corresponding point $p_\xi \in S$, and since $\widetilde{\psi}(p_\xi) = p_\xi$ we can consider the germ $(\widetilde{\psi}, p_\xi)$ induced by $\widetilde{\psi}$ at p_ξ. We have the following (see also [6, Proposition 2.2]):

LEMMA 4.3. *If ξ is a characteristic direction, $(\widetilde{\psi}, p_\xi)$ is again tangent to the identity and its order is at least ν.*

Proof. Up to a linear change of coordinates, we can assume $\xi = [1{:}0]$; in the chart of $\widetilde{\mathbb{R}}^2$ such that $p_\xi \cong (0,0)$ we have $\pi(x_1, y_1) = (x_1, y_1 x_1)$. If $(\widetilde{g}, \widetilde{h})$ are the components of $\widetilde{\psi}$, an easy computation gives $\widetilde{g}(x_1, y_1) = x_1 + P_\nu(1,0)x_1^\nu + O_{\nu+1} = x_1 + O_\nu$, $\widetilde{h}(x_1, y_1) = y_1 + (Q_\nu(1, y_1) - P_\nu(1,0)y_1)x^{\nu-1} + O_{\nu+1}$. Since [1:0] is a characteristic direction, we have $Q_\nu(1,0) = 0$, hence $Q_\nu(x,y) = yR_{\nu-1}(x,y)$ for some homogeneous polynomial $R_{\nu-1}$ of degree $\nu - 1$; this implies that $\widetilde{h}(x_1, y_1) = y_1 + O_\nu$. □

Let now γ be an invariant curve for ψ of class C^∞. Then $\xi = T_0(\gamma)$ defines a characteristic direction for ψ (this is valid more in general for formal invariant curves, see, for instance, [8, Proposition 2.8]). The lift $\widetilde{\gamma} = \overline{\pi^{-1}(\gamma \setminus \{0\})}$ is again a curve of class C^∞, invariant under $\widetilde{\psi}$ and passing through $p_\xi \in S$. Choosing coordinates in which $p_\xi \equiv 0$, we can identify $(\widetilde{\psi}, p_\xi)$ with an element of $\mathrm{Diff}^\omega(\mathbb{R}^2, 0)$. We set $\psi^{(1)} := (\widetilde{\psi}, p_\xi)$ and $\gamma^{(1)} := (\widetilde{\gamma}, p_\xi)$.

Iterating the process, we obtain two sequences $\{\psi^{(j)}\}_{j\in\mathbb{N}}$, $\{\gamma^{(j)}\}_{j\in\mathbb{N}}$ such that $\psi^{(j)} \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ and $\gamma^{(j)}$ is invariant by $\psi^{(j)}$ for all $j \in \mathbb{N}$.

LEMMA 4.4. *Let ψ, γ, d_1, d_2 be as in Theorem 4.2. Then the following conditions are equivalent.*

- *$d_2 = d_1 - 1$.*
- *For some $n \in \mathbb{N}$, $T_0\gamma^{(n)}$ defines a non-degenerate characteristic direction for $\psi^{(n)}$.*

Proof. Suppose that γ is defined as $\{y = f(x)\}$ for some $f \in C^\infty(\mathbb{R}, 0)$, and let $Tf \in \mathbb{R}[[x]]$ be the formal power series corresponding to the Taylor development of f. Furthermore, for $k \in \mathbb{N}$ we denote by $T_k f$ the truncation of T to the kth order, and by ϕ_k the local change of coordinates defined as $\phi_k(x,y) = (x, y - T_k f(x))$.

Since γ is an invariant curve for ψ, the formal identity

$$h(x,y) - Tf(g(x,y)) = \alpha(x,y)(y - Tf(x)) \tag{7}$$

is satisfied in $\mathbb{R}[[x,y]]$ for some unit $\alpha \in \mathbb{R}^*[[x,y]]$. Since $Tf(x) = T_k f(x) + O(x^{k+1})$,

$$h(x,y) - T_k f(g(x,y)) = \alpha(x,y)(y - T_k f(x)) + O(x^{k+1}). \tag{8}$$

Let $\psi_k = (g_k, h_k)$ be the conjugate of ψ through the change of coordinates ϕ_k. We have $\psi_k = \phi_k \circ \psi \circ \phi_k^{-1}$, that is,

$$g_k(x,y) = g(x, y + T_k f(x)), \quad h_k(x,y) = h(x, y + T_k f(x)) - T_k f(g(x, y + T_k f(x))).$$

Replacing y by $y + T_k f(x)$ in (8) we see that $h_k(x,y) = O(y) + O(x^{k+1})$.

Now, by definition of d_1 we have $g(x,y) = x + cx^{d_1} + O(x^{d_1+1}) + \beta(x,y)(y - Tf(x))$ for some $c \in \mathbb{R} \setminus \{0\}$, $\beta \in \mathbb{R}[[x,y]]$. Replacing again y by $y + T_k f(x)$, we get

$$g_k(x,y) = g(x, y + T_k f(x)) = x + cx^{d_1} + O(x^{d_1+1}) + O(y)$$

as long as $k > d_1$. On the other hand, the definition of d_2 corresponds to the following formal identity:

$$h(x,y) - Tf(g(x,y)) = (1 + bx^\ell + O(x^{\ell+1}))(y - Tf(x)) + (y - Tf(x))^2\theta(x,y)$$

for suitable $b \in \mathbb{R} \setminus \{0\}$ and $\theta \in \mathbb{R}[[x, y]]$, where $\ell = d_2$ if $d_2 < d_1 - 1$ and $\ell \geqslant d_1 - 1$ otherwise. With the substitution $y \to y + T_k f(x)$ we get

$$\begin{aligned} h_k(x, y) &= h(x, y + T_k f(x)) - T_k f(g(x, y + T_k f(x))) \\ &= (1 + bx^{\ell} + O(x^{\ell+1}))y + O(y^2) + O(x^{k+1}) \end{aligned} \tag{9}$$

as long as $k > \ell$.

Furthermore, the curve γ get transformed through the change of coordinates ϕ_k into $\gamma_k = \{y = f_k(x) = f(x) - T_k f(x)\}$; if $T f_k \in \mathbb{R}[[x]]$ denotes the Taylor expansion of f_k, clearly we have $T f_k(x) = T f(x) - T_k f(x) = O(x^{k+1})$. It follows that, if $\{\gamma_k^{(j)}\}$ is the sequence of lifts of γ_k by iterated blow-ups defined before, the tangent line of $\gamma_k^{(j)}$ at 0 is generated by [1:0] for all $j \leqslant k$. In these coordinates the corresponding point on the exceptional divisor is going to be 0 for all $j \leqslant k$.

Let then $(g_k^{(j)}(x, y), h_k^{(j)}(x, y))$ be the components of the jth lift $\psi_k^{(j)}$ of ψ_k by the iterated blow-ups, $j \leqslant k$. For the first component, we have $g_k^{(j)}(x, y) = g_k^{(j-1)}(x, yx)$ for all $j \leqslant k$. It follows by induction that $g_k^{(j)}(x, y) = x + cx^{d_1} + O(x^{d_1+1}) + yO(j)$, hence choosing k, j large enough we get that $\nu(g_k^{(j)} - x) = d_1$ and $P_{d_1}(x, y) = cx^{d_1}$. The second component satisfies the following recursive equation:

$$h_k^{(j)}(x, y) = \frac{h_k^{(j-1)}(x, xy)}{g_k^{(j-1)}(x, xy)} \tag{10}$$

for all $j \leqslant k$. We now rewrite the components of $\psi_k^{(j)}$ as $g_k^{(j)}(x, y) = \sum_i y^i r_{i,k}^{(j)}(x)$, $h_k^{(j)}(x, y) = \sum_i y^i s_{i,k}^{(j)}(x)$ for suitable $r_{i,k}^{(j)}, s_{i,k}^{(j)} \in \mathbb{R}\{x\}$. Using (10), we obtain

$$\begin{aligned} h_k^{(j)}(x, y) &= \frac{\sum_{i \geqslant 0} y^i x^i s_{i,k}^{(j-1)}(x)}{\sum_{i \geqslant 0} y^i x^i r_{i,k}^{(j-1)}(x)} = \frac{\frac{1}{x} s_{0,k}^{(j-1)}(x) + y s_{1,k}^{(j-1)}(x) + \sum_{i \geqslant 2} y^i x^{i-1} s_{i,k}^{(j-1)}(x)}{1 + cx^{d_1-1} + O(x^{d_1}) + yO(x^{j-1})} \\ &= \left(\frac{1}{x} s_{0,k}^{(j-1)}(x) + y s_{1,k}^{(j-1)}(x) + \sum_{i \geqslant 2} y^i x^{i-1} s_{i,k}^{(j-1)}(x) \right) \\ &\quad \cdot (1 - cx^{d_1-1} + O(x^{d_1}) + yO(x^{j-1})). \end{aligned} \tag{11}$$

From the previous equation we get $s_{0,k}^{(j)}(x) = \frac{1}{x} s_{0,k}^{(j-1)}(x) \cdot (1 + O(x^{d_1-1}))$. Since by (9) we have $s_{0,k}^{(0)}(x) = O(x^{k+1})$, we deduce that $s_{0,k}^{(j)}(x) = O(x^{k+1-j})$ for all $j \leqslant k$. Using this fact alongside the previous equation, we have $s_{1,k}^{(j)}(x) = s_{1,k}^{(j-1)}(x) \cdot (1 - cx^{d_1-1} + O(x^{d_1})) + O(x^k)$.

Now, by (9) it follows that $s_{1,k}^{(0)}(x) = 1 + bx^{\ell} + O(x^{\ell+1})$. Since $k > d_1$, the previous recursion on $s_{1,k}^{(j)}$ shows that, for all $j \leqslant k$,

$$\begin{aligned} &s_{1,k}^{(j)}(x) = 1 + bx^{\ell} + O(x^{\ell+1}) \text{ when } \ell = d_2 < d_1 - 1, \\ &s_{1,k}^{(j)}(x) = 1 + O(x^{d_1-1}) \ \text{ otherwise.} \end{aligned} \tag{12}$$

On the other hand, looking again at (11) one can check inductively that $s_{i,k}^{(j)}(x) = O(x^{j-1})$ for all $i \geqslant 2$, $j \leqslant k$. This fact, together with (12), shows the following: if $\ell = d_2 < d_1 - 1$, then for all $j \leqslant k$ we have $\nu(h_k^{(j)} - y) \leqslant d_2 + 1$ and actually $\nu(h_k^{(j)} - y) = d_2 + 1$, $Q_{d_2+1}(x, y) = bx^{d_2}y$ as soon as $j \geqslant d_2 + 1$. If instead $d_2 = d_1 - 1$, choosing $k = 2d_1 + 1$, $j = d_1 + 1$ (so that $s_{0,k}^{(j)}(x) = O(x^{k+1-j}) = O(x^{d_1+1})$ and $s_{i,k}^{(j)}(x) = O(x^{j-1}) = O(x^{d_1})$ for all $i \geqslant 2$) we can achieve that $\nu(h_k^j - y) \geqslant d_1$ and $Q_{d_1}(x, y) = ax^{d_1-1}y$ for some $a \in \mathbb{R}$ (where possibly $a = 0$).

In summary, if $d_2 = d_1 - 1$ we have that, for a suitable choice of k and j, $\nu(\psi_k^{(j)} - Id) = d_1$ and the homogeneous map $(\psi_k^{(j)})_{d_1}$ is given by $(x, y) \to (cx^{d_1}, ax^{d_1-1}y)$ with $a, c \in \mathbb{R}$, $c \neq 0$. This immediately implies that [1:0] is a non-degenerate characteristic direction for $\psi_k^{(j)}$. If instead $d_2 < d_1 - 1$, one has $\nu = \nu(\psi_k^{(j)} - Id) \leqslant d_1 - 1$ for any choice of $k \geqslant d_1$, $j \leqslant k$. Since, as seen before, $g_k^{(j)}(x, y) = x + cx^{d_1} + O(x^{d_1+1}) + yO(j)$ for all $j \leqslant k$, we have that the homogeneous polynomial $P_m(x, y)$ of degree m in the expansion of $g_k^{(j)}$ is always divisible by y for all $2 \leqslant m \leqslant d_1 - 1$, that is, $P_m(1, 0) = 0$. We see then that the homogeneous map $(\psi_k^{(j)})_\nu(x, y) = (P_\nu(x, y), Q_\nu(x, y))$ vanishes at [1:0], which is thus a degenerate characteristic direction of $\psi_k^{(j)}$ for any choice of $k \geqslant d_1$, $j \leqslant k$. $\square$

Remark 2. From the proof of the previous lemma one can actually derive the following: if $d_2 = d_1 - 1$, there exists $j_0 \in \mathbb{N}$ such that the tangent line of $\gamma^{(j)}$ at 0 is a non-degenerate characteristic direction for $\psi^{(j)}$ for all $j \geqslant j_0$. This can be checked by taking k arbitrarily large in the proof, since then $\nu(\psi_k^{(j)} - Id) = d_1$ for all large enough j (in fact $j > d_1$ is sufficient, independently of the choice of k).

The following lemma is a finite jet determination statement for formal curves which are invariant by a local diffeomorphism ψ:

LEMMA 4.5. *Let $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ be tangent to the identity and let $I \subset \mathbb{R}[[x, y]]$ be a formal curve ideal* (*of dimension* 1). *Assume that I is invariant by ψ and not dynamically trivial. Then there exists $m \in \mathbb{N}$ such that for every formal curve ideal J which is invariant by ψ and tangent to I up to order m we have $J = I$.*

Proof. Up to a formal change of coordinates, we can assume that I is generated by y. The invariance and non-dynamical triviality of (y) translate, in these coordinates, into the following expressions for the (formal) components (g, h) of ψ:

$$g(x, y) = x + x^{d_1} + O(x^{d_1+1}) + O(y),$$

$$h(x, y) = y(1 + O_1).$$

Setting $g(x, y) = \sum_{i,j} g_{i,j}x^iy^j$, $h(x, y) = \sum_{i,j} h_{i,j}x^iy^j$, this means that $h_{i,0} = 0$ for all $i \in \mathbb{N}$ and $g_{i,0} = 0$ for $2 \leqslant i \leqslant d_1 - 1$. Without loss of generality, suppose now that the ideal $J \neq I$ is generated by $y - f(x)$ for some $f(x) \in \mathbb{R}[[x]]$. The tangency condition on I and J amounts to $f(x) = O(x^m)$ for some large enough m. We can assume that $m > d_1$, $m \neq h_{d_1-1,1}$ and $f = \sum_{j \geqslant m} a_jx^j$ for suitable $a_j \in \mathbb{R}$: we will obtain a contradiction by showing that necessarily $a_m = 0$.

The invariance condition by ψ for the formal curve $\{y = f(x)\}$ is expressed by the equation $f(g(x, f(x))) = h(x, f(x))$. Both the left- and the right-hand side of this equation are $O(x^m)$: for any $j \geqslant m$, collecting the terms in x^j on both sides gives an algebraic equation involving the coefficients of f, g, h, which we denote by $[j]$. First, $[m]$ amounts to the identity $a_m = a_m$. For $[m + 1]$ we have $a_{m+1} = a_{m+1} + h_{1,1}a_m$ if $d_1 > 2$, and $a_{m+1} + ma_m = a_{m+1} + h_{1,1}a_m$ if $d_1 = 2$. In the latter case we conclude immediately $a_m = 0$ since by assumption $m \neq h_{1,1}$. In the first case, we either get $a_m = 0$ or $h_{1,1} = 0$.

In general, the equation $[m+j]$ reads

$$[m+j]: \quad a_{m+j} = a_{m+j} + \sum_{i=0}^{j-1} h_{j-i,1} a_{m+i}$$

for $1 \leqslant j \leqslant d_1 - 2$, and

$$[m+d_1-1]: \quad a_{m+d_1-1} + m a_m = a_{m+d_1-1} + \sum_{i=0}^{d_1-2} h_{d_1-(i+1),1} a_{m+i}.$$

Suppose now, inductively, that for some $j \leqslant d_1 - 2$ we have $h_{1,\ell} = 0$ for every $\ell < j$. Then $[m+j]$ becomes $a_{m+j} = a_{m+j} + h_{j,1} a_m$, from which we either conclude that $a_m = 0$ (and we are done) or that $h_{j,1} = 0$. We see then that $[m+d_1-1]$ reduces to the equation $a_{m+d_1-1} + m a_m = a_{m+d_1-1} + h_{d_1-1,1} a_m$: from the assumption $m \neq h_{d_1-1,1}$ follows that $a_m = 0$. □

Remark 3. It might be possible to prove the previous lemma by applying the reduction of singularities of formal vector fields shown in [**19**], passing through the formal logarithm of ψ as considered in [**6, 8**]. In our case, though, the particular form of the formal ideal I allows a more direct approach.

Remark 4. In the statement of Lemma 4.5 we can take any $m > \max\{d_1, h_{d_1-1,1}\}$. However, depending on ψ and I the m-jet determination might hold for a smaller m. This is always the case after a certain number of blow-ups, as shown in the following corollary.

COROLLARY 4.6. *Let $\psi^{(j)}$, respectively, $I^{(j)}$, be the lift of ψ, respectively, I, after a sequence of j blow-ups ($I^{(j)}$ is the formal ideal associated to the curve $\gamma^{(j)}$ at* 0). *Then there exists $j_0 \in \mathbb{N}$ such that, for all $j \geqslant j_0$, the following holds: any $\psi^{(j)}$-invariant formal curve ideal generated by $y - u(x)$ for some $u \in \mathbb{R}[[x]]$ coincides with $I^{(j)}$.*

Proof. We will prove the claim by induction: assume that the conclusion of Lemma 4.5 holds for ψ, I for a certain $m_0 \in \mathbb{N}$. We will show that the lift of ψ and I by a blow-up satisfy the same property for the order $m_0 - 1$.

Write a generator of I as $y - f(x)$ for some $f \in \mathbb{R}[[x]]$; we can assume up to a linear change of coordinates that $f(x) = O(x^2)$. A generator for the lift $\widetilde{I}$ of I through the blow-up $\pi(x,y) = (x, xy)$ is then given by $y - f_1(x)$ with $f_1(x) = f(x)/x$. Assume that there exists a formal curve (corresponding to an ideal $\widetilde{J} \subset \mathbb{R}[[x,y]]$), invariant for $\widetilde{\psi}$, admitting the equation $\{y = u(x)\}$ for $u \in \mathbb{R}[[x]]$, $u(0) = 0$: if $m_0 = 1$, we do not make any further assumption on u, otherwise we suppose $u(x) = O(x^{m_0-1})$. Then $\widetilde{J}$ is the blow-up of the formal curve J generated by $y - xu(x)$. Denote by Π, $\widetilde{\Psi}$, Ψ the ring homomorphisms $\mathbb{R}[[x,y]] \to \mathbb{R}[[x,y]]$ induced by the maps π, $\widetilde{\psi}$ and ψ: we remark that $\widetilde{\Psi}$, Ψ are automorphisms and Π is injective. Then we have $\Pi(J) = \widetilde{J}$ and $\Pi \circ \Psi = \widetilde{\Psi} \circ \Pi$. Since $\widetilde{\Psi}(\widetilde{J}) = \widetilde{J}$, this implies that $\Pi \circ \Psi(J) = \widetilde{\Psi} \circ \Pi(J) = \widetilde{\Psi}(\widetilde{J}) = \widetilde{J}$, which by injectivity implies $\Psi(J) = J$, that is, J is a formal invariant curve for ψ. Moreover, since $xu(x) = O(x^{m_0})$ we have that J is tangent to I to order m_0, hence it must coincide with I by assumption. It follows that $\widetilde{J} = \widetilde{I}$. □

4.3. *Curves tangent to a non-degenerate characteristic direction.* Suppose that ψ, γ are as in Theorem 4.2, and that $d_2 = d_1 - 1$. From the lemmas of the previous section (and in particular from Remark 2 and Corollary 4.6) it follows that, after performing a certain number of blow-ups, we can assume that the tangent line [1:0] of γ at 0 is a non-degenerate characteristic direction for ψ and, moreover, every formal curve which is tangent to [1:0] and invariant by ψ coincides with the one induced by (the Taylor expansion of a defining function for) γ.

As mentioned before, to prove the conclusion of Theorem 4.2 we are going to employ the theory of parabolic curves developed by Hakim in [12, 13] and subsequently extended in [2 6]. In what follows, we will refer to the account of Hakim's work presented in [4], where detailed proofs are also provided for the case of diffeomorphisms of order greater than 2. Let then (g, h) be the components of $\psi \in \mathrm{Diff}^{\omega}(\mathbb{R}^2, 0)$. Applying Proposition 4.3 and Lemma 4.4 in [4] (cf. equation (5.2)), by performing a blow-up and a polynomial change of coordinates we can assume that the components are (with our notation) of the following form:

$$\begin{aligned} g(x, y) &= x - x^{d_1} + O(x^{2d_1-1}) + O(yx^{d_1}), \\ h(x, y) &= y(1 - \beta x^{d_1-1}) + x^{d_1}h_1(x) + O(yx^{d_1}) + O(y^2x^{d_1-1}), \end{aligned} \tag{13}$$

for certain $\beta \in \mathbb{R}$ and $h_1 \in \mathbb{R}\{x\}$. Note that the polynomial changes of coordinates performed in Lemma 4.4 of [4] all have real coefficients except possibly for the last one, so actually by changing coordinates one can only ensure that the coefficient of x^{d_1} in $g(x, y)$ is ±1. If needed, though, we can replace ψ by ψ^{-1} to achieve (13). We also remark that the coefficient β (which is related to the coefficient α in [4, Equation (5.2)] by $\beta = (d_1 - 1)\alpha$) has an invariant, geometrical meaning, explicitly described in [4, Lemma 4.5].

The main relevant distinction to be made on β is whether it is a positive integer or not. In the former case, ψ might not admit any formal invariant curve of the form $\{y = u(x)\}$, or possibly admit infinitely many ones. Since in our situation we have exactly one formal curve which is invariant by ψ (the one associated to γ), we can show the following:

LEMMA 4.7. *Let ψ, γ be as above, with the components of ψ expressed as in* (13). *Then β is not a positive integer.*

Proof. As usual, we are going to write a defining equation of γ of the form $\{y = f(x)\}$ with $f \in C^{\infty}(\mathbb{R}, 0)$, and consider the corresponding formal curve $\{y = Tf(x)\}$.

We will derive the conclusion of the lemma by following the proof of [4, Proposition 5.2]. In particular, we fix our attention to the inductive step used in that proof, which is obtained through the following computation. For some $j \in \mathbb{N}$, let $Q_{j+1}(x)$ be a polynomial of degree $j + 1$, and write $Q_{j+1}(x) = q_j(x) + b_{j+1}x^{j+1}$ for some $b_{j+1} \in \mathbb{R}$ and a polynomial $q_j(x)$ of degree j. Then one has ([4, Equation (5.5)])

$$\begin{aligned} &h(x, Q_{j+1}(x)) - Q_{j+1}(g(x, Q_{j+1}(x))) \\ &\quad = h(x, q_j(x)) - q_j(g(x, q_j(x))) + b_{j+1}(j + 1 - \beta)x^{d_1+j} + O(x^{d_1+j+1}). \end{aligned} \tag{14}$$

Suppose now that, for some $u \in \mathbb{R}[[x]]$, the formal curve $\{y = u(x)\}$ is invariant by ψ, and for $k \in \mathbb{N}$ let u_k be the truncation of u to the degree k. Then u satisfies the formal identity $h(x, u(x)) = u(g(x, u(x)))$. Equivalently, for any $n \in \mathbb{N}$ there exists k_0 such that $h(x, u_k(x)) - u_k(g(x, u_k(x))) = O(x^{d_1+n})$ for all $k \geqslant k_0$. However, an inductive application of (14) gives that $h(x, u_i(x)) - u_i(g(x, u_i(x))) = h(x, u_j(x)) - u_j(g(x, u_j(x))) + O(x^{d_1+j})$ for all $i, j \in \mathbb{N}$, $j < i$. Thus we can actually take $k_0 = n$, which means that $\{y = u(x)\}$ is formally invariant under ψ if and only if $h(x, u_n(x)) - u_n(g(x, u_n(x))) = O(x^{d_1+n})$ for all $n \in \mathbb{N}$.

On the other hand, for some $j \in \mathbb{N}$, assume that there is a polynomial P_j, of degree j, such that $h(x, P_j(x)) = P_j(g(x, P_j(x))) + x^{d_1+j} h_{j+1}(x)$ for some $h_{j+1} \in \mathbb{R}\{x\}$. Suppose, further, that $\beta \neq j+1$. By (14), defining

$$P_{j+1}(x) = P_j(x) + \frac{h_{j+1}(0)}{\beta - (j+1)} x^{j+1}, \tag{15}$$

we have that $h(x, P_{j+1}(x)) = P_{j+1}(g(x, P_{j+1}(x))) + x^{d_1+j+1} h_{j+2}(x)$ for some $h_{j+2} \in \mathbb{R}\{x\}$.

Now assume by contradiction that β is a positive integer, $\beta = k+1$ for some $k \in \mathbb{N}$. Then for all $j \leqslant k$ equation (15) allows to construct a (uniquely determined) polynomial $P_j(x)$ satisfying $h(x, P_j(x)) - P_j(g(x, P_j(x))) = x^{d_1+j} h_{j+1}(x) = O(x^{d_1+j})$. We remark that since $\{y = Tf(x)\}$ is an invariant formal curve, as seen above the polynomials, $T_j f$ satisfy the same relation and thus we must have $P_j = T_j f$ for all $j \leqslant k$. On the other hand, using (14) with $j = k$ we get

$$\begin{aligned} &h(x, T_{k+1} f(x)) - T_{k+1} f(g(x, T_{k+1} f(x))) \\ &\quad = h(x, T_k f(x)) - T_k f(g(x, T_k f(x))) + O(x^{d_1+k+1}) = x^{d_1+k} h_{k+1}(0) + O(x^{d_1+k+1}). \end{aligned}$$

We already know, however, that $h(x, T_{k+1} f(x)) - T_{k+1} f(g(x, T_{k+1} f(x))) = O(x^{d_1+k+1})$: this is only possible if $h_{k+1}(0) = 0$. By (14), then, we deduce that for any $b \in \mathbb{R}$ the polynomial $P^b_{k+1}(x) = T_k f(x) + b x^{k+1}$ satisfies the relation $h(x, P^b_{k+1}(x)) - P^b_{k+1}(g(x, P^b_{k+1}(x))) = O(x^{d_1+k+1})$.

For all $b \in \mathbb{R}$, since $\beta \neq j+1$ for all $j \geqslant k+1$, (15) allows to construct inductively a sequence of polynomials $\{P^b_{j+1}\}_{j \geqslant k+1}$ and indeed a series $P^b \in \mathbb{R}[[x]]$ such that the formal curve $\{y = P^b(x)\}$ is invariant by ψ. These series all differ on the coefficient of x^{k+1}, which contradicts our assumption that the only element $v \in \mathbb{R}[[x]]$ such that the formal curve $\{y = v(x)\}$ is invariant with respect to ψ is $v = Tf$. □

Proof. (Theorem 4.2). Using the previous lemma, we will now follow the simplified approach to Hakim's result which is employed in [**2**, Section 3]. After yet another sequence of blow-ups and coordinate changes, we can put the components of ψ in the form (13) with $\beta < 0$ and $h_1(x) = O(x^2)$ (see [**2**, Equation (3.5) and the preceding discussion]). If $\gamma = \{y = f(x)\}$, one can check (for example by using the computations in the proof of Lemma 4.7) that in these coordinates $f(x) = O(x^3)$.

With ψ written in this form, we will prove that $\gamma \cap \{x > 0\}$ is of class C^ω. To prove the same for $\gamma \cap \{x < 0\}$ (and thus that $\gamma \setminus \{0\}$ is real analytic) one can pass through the change of coordinates $(x, y) \to (-x, y)$ and, if needed, replace ψ by its inverse to write the components again as in (13).

By Hakim's result, ψ (seen as an element of $\mathrm{Hol}(\mathbb{C}^2, 0)$) admits $d_1 - 1$ parabolic curves tangent to the direction [1:0]. The analyticity of $\gamma \cap \{x > 0\}$ will follow from two facts. The first one is that the intersection of one of the parabolic curves with $\mathbb{R}^2 \cap \{x > 0\}$ is non-empty, and gives a (real, one-sided) ψ-invariant curve of equation $\{y = u(x)\}$ with $u \in C^\omega((0, \delta)) \cap C^1([0, \delta])$ for some small $\delta > 0$. The second fact is that there can be at most one real ψ-invariant curve of class C^1 in a one-sided neighborhood of 0, hence γ must locally coincide with $\{y = u(x)\}$. Note that if the Taylor series Tf of f has a positive radius of convergence, then $\{y = Tf(x)\}$ is a ψ-invariant curve of class C^ω, and by the same uniqueness argument it must coincide with $\{y = f(x)\}$, from which follows the last statement in Theorem 4.2.

Following [**2**], let $r = d_1 - 1$ and define $D_\delta = \{\zeta \in \mathbb{C} : |\zeta^r - \delta| < \delta\}$. Let D'_δ be the component of D_δ that intersects the real axis, and let $\mathcal{E}(\delta) = \{u \in \mathrm{Hol}(D'_\delta, \mathbb{C}) : u(\zeta) = \zeta^2 u^0(\zeta), \|u^0\|_\infty < \infty\}$, which is a Banach space with the norm $\|u\|_{\mathcal{E}(\delta)} = \|u^0\|_\infty$. The parabolic curve $\mathcal{P}$ is obtained by finding, for some small $\delta > 0$, $U \in \mathcal{E}(\delta)$ such that $U(g(\zeta, U(\zeta))) = h(\zeta, U(\zeta))$ for all $\zeta \in D'_\delta$, and setting $\mathcal{P} = \{(z, w) \in D'_\delta \times \mathbb{C} : w = U(z)\}$. Additionally, U belongs to the set $\mathcal{F}(\delta) = \{v \in \mathcal{E}(\delta) : \|v^0\|_\infty \leqslant 1, |v'(\zeta)| \leqslant |\zeta|\}$ and it is uniquely determined as the fixed point of a contraction operator acting on $\mathcal{F}(\delta)$.

To prove our first claim it is sufficient to show that the restriction of U to the real axis is real-valued. Define $V(\zeta) := \overline{U(\overline{\zeta})}$, so that $\overline{U(\zeta)} = V(\overline{\zeta})$: then $V \in \mathcal{F}(\delta)$ and we need to show that $U = V$. Note that, since g, h have real coefficients, we have $\overline{g(z, w)} = g(\overline{z}, \overline{w})$, $\overline{h(z, w)} = h(\overline{z}, \overline{w})$. Conjugating the mapping equation and using the definition of V, it follows that $V(g(\overline{\zeta}, V(\overline{\zeta})) = h(\overline{\zeta}, V(\overline{\zeta}))$ for all $\zeta \in D'_\delta$, hence $\{w = V(z)\}$ is a parabolic curve. By uniqueness, $U = V$.

Let us turn to the second claim. For $\delta > 0$, define

$$\mathcal{E}'(\delta) = \{u \in C^1([0, \delta]) : u(x) = x^2 u^0(x), \|u^0\|_\infty < \infty\},$$

endowed with the norm $\|u\|_{\mathcal{E}'(\delta)} = \|u^0\|_\infty$, and

$$\mathcal{F}'(\delta) = \{u \in \mathcal{E}'(\delta) : \|u^0\|_\infty \leqslant 1, |u'(x)| \leqslant |x|\}$$

with its metric as a subspace of $\mathcal{E}'(\delta)$. Note that, since $f(x) = O(x^3)$, the restriction of f to $[0, \delta]$ certainly belongs to $\mathcal{F}'(\delta)$ for some small $\delta > 0$. By construction, the same holds for the restriction of U to the real axis. To see that f and U coincide, we will show that they are fixed points of a contraction T acting on $\mathcal{F}'(\delta)$. Of course, with the metric we are considering $\mathcal{F}'(\delta)$ is not a complete space; nevertheless, we only need the fact that if T admits a fixed point, this is uniquely determined.

We keep following [2]: for $\delta > 0$ small enough and $u \in \mathcal{E}'(\delta)$, define $g^u \in C^1([0, \delta])$ as $g^u(x) = g(x, u(x))$. More precisely, for all $C > 0$ there exists $\delta_0(C) > 0$ such that, fixed any $0 < \delta \leqslant \delta_0(C)$, for all $u \in \mathcal{E}'(\delta)$ with $\|u\|_{\mathcal{E}'(\delta)} < C$ we have that g^u is well defined, $g^u((0, \delta)) \subset (0, \delta)$ and $|(g^u)^{\circ n}(x)| = O(1/n^{1/r})$. Indeed, by (13) we have $g^u(x) = x - x^{r+1} - x^{r+2}\ell_u(x)$, where (for some $C' > 0$) $\|\ell_u\|_\infty < C'$ for all u such that $\|u^0\| < C$. It follows that $x - x^{r+1} - C'x^{r+2} \leqslant g^u(x) \leqslant x - x^{r+1} + C'x^{r+2}$ for all such u, from which we derive the previous claims. Observe that, defining (for $0 < \zeta_0 < \delta$, $u \in \mathcal{E}'(\delta)$ with $\|u\|_{\mathcal{E}'(\delta)} < C$) $\zeta_n = (g^u)^{\circ n}(\zeta_0)$, we have $\zeta_n \leqslant \frac{D}{n^{1/r}}$ for some $D > 0$ independent of u and ζ_0.

Putting $x_1 = g(x, y)$, $y_1 = h(x, y)$ define now

$$H(x, y) = y - \frac{x^\beta}{x_1^\beta} y_1 = O(x^{r+1}y) + O(x^r y^2) + O(x^{r+3})$$

and

$$Tu(\zeta_0) = \zeta_0^\beta \sum_{n=0}^{\infty} \zeta_n^{-\beta} H(\zeta_n, u(\zeta_n))$$

(see [2, Equation (3.7)]). For any $0 < \zeta_0 < \delta_0$, we have $|\zeta_n| \leqslant \frac{D}{n^{1/r}}$ and $|u(\zeta_n)| \leqslant \frac{D'}{n^{2/r}}$, so that $|H(\zeta_n, u(\zeta_n))| \leqslant \frac{D''}{n^{1+3/r}}$ for all $n \in \mathbb{N}$. These estimates are uniform for all u, ζ_0 such that $\|u\|_{\mathcal{E}'(\delta)} \leqslant C$ and $\zeta_0 \in (0, \delta_0)$. Since $\beta < 0$, it follows that the series defining $Tu(\zeta_0)$ converges for all $0 < \zeta_0 < \delta_0$, and $Tu \in C^0((0, \delta_0))$. We will see that in fact $Tu \in \mathcal{F}'(\delta)$ for all $u \in \mathcal{F}'(\delta)$.

We first check that, if $\{y = u(x)\}$ is an invariant curve for ψ, then u is a fixed point for T (see also [4, Corollary 6.13]). Denote by $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ the orbit of (x_0, y_0) under the action of ψ. If $y_0 = u(x_0)$, we also have $y_n = u(x_n)$ for all n. Then

$$\frac{Tu(x_0)}{x_0^\beta} = \sum_{n=0}^{\infty}\left(\frac{u(x_n)}{x_n^\beta} - \frac{y_{n+1}}{x_{n+1}^\beta}\right) = \sum_{n=0}^{\infty}\left(\frac{u(x_n)}{x_n^\beta} - \frac{u(x_{n+1})}{x_{n+1}^\beta}\right) = \frac{u(x_0)}{x_0^\beta}$$

which shows that $Tu(x_0) = u(x_0)$ for all $0 < x_0 < \delta_0$. Thus both f and the restriction of U to $(0, \delta_0)$ are fixed points for T.

Using (13), we can write $g(x, y) = x - x^{r+1} - x^{r+1}(xs(x) + yp(x, y))$ for suitable analytic functions s and p. Thus, $\zeta_1 = g^u(\zeta_0, u(\zeta_0)) = \zeta_0 - \zeta_0^{r+1} - \zeta_0^{r+2}\ell_u(\zeta_0)$, where

$$\ell_u(x) = s(x) + xu^0(x)p(x, x^2u^0(x))$$

is the function defined above. It follows that (as pointed out before) $\|\ell_u\|_\infty$ is bounded whenever $\|u^0\|_\infty$ is bounded. Furthermore, $\|\ell_u'\|$ is bounded whenever $\|(x(u^0)')\|_\infty$ is bounded. In particular, if $u \in \mathcal{F}'(\delta)$, we get $|(u^0)'(x)| = |\frac{u'(x)}{x^2} - 2\frac{u(x)}{x^3}| \leqslant |\frac{1}{x}| + 2|\frac{u^0(x)}{x}|$, hence $|x(u_0)'(x)| \leqslant 3$ and $\|\ell_u'\|$ is uniformly bounded.

Since $(1 - x)^{-r} = 1 + rx + x^2q(x)$ for an analytic function q, we have ([2, Equation (3.8)])

$$\frac{1}{\zeta_1^r} = \frac{1}{\zeta_0^r(1 - \zeta_0^r - \zeta_0^{r+1}\ell_u(\zeta_0))^r} = \frac{1}{\zeta_0^r}(1 + r\zeta_0^r + \zeta_0^{r+1}\theta_u(\zeta_0)) = \frac{1}{\zeta_0^r} + r + \zeta_0\theta_u(\zeta_0) \qquad (16)$$

where $\theta_u(\zeta_0) = \ell_u(\zeta_0) + \zeta_0^r(1 + \zeta_0\ell_u(\zeta_0))^2q(-\zeta_0^r - \zeta_0^{r+1}\ell_u(\zeta_0))$. It follows that $\|\theta_u\|_\infty$, $\|\theta_u'\|_\infty$ are uniformly bounded whenever that is true for $\|\ell_u\|_\infty$ and $\|\ell_u'\|_\infty$, in particular for $u \in \mathcal{F}'(\delta)$. Next, using only (16) and the boundedness of θ_u (and not the fact that u belongs to a space of holomorphic functions rather than to $\mathcal{E}'(\delta)$), one can prove the following estimate: for every $s > r$ there exists $C_s(\delta) \geqslant 1$ such that

$$\sum_{n=0}^{\infty}|\zeta_n|^s \leqslant C_s|\zeta_0|^{s-r} \qquad (17)$$

for $0 < \zeta_0 < \delta$, see [2, Equation (3.10)]. This estimate is then employed in [2] to show that, for $u \in \mathcal{F}'(\delta)$ with δ small enough, one has

$$|Tu(\zeta)| \leqslant K_1|\zeta|^3, \quad \left|\frac{dTu}{d\zeta}(\zeta)\right| \leqslant K_2|\zeta|^2 \qquad (18)$$

where the constants $K_1, K_2 > 0$ are independent of δ. Once again, only (16), (17) and the boundedness of $\|\theta_u\|_\infty$, $\|\theta_u'\|_\infty$ are needed to obtain (18). It follows in particular that Tu is differentiable when $u \in \mathcal{F}'(\delta)$. Moreover, choosing $\delta < \{1/K_1, 1/K_2\}$ one has that T sends $\mathcal{F}'(\delta)$ into itself.

The proof that $T : \mathcal{F}'(\delta) \to \mathcal{F}'(\delta)$ is a contraction depends on the same estimates. Referring again to the treatment of Hakim's result given in [4], (17) corresponds to [4, Corollary 6.3] (with $q = 0$) and (18) to [4, Lemma 6.9 and Lemma 6.10] (with $p_h = 0$). Let $u, v \in \mathcal{F}'(\delta)$, and for $0 < x < \delta$ define $x_n = (g^u)^{\circ n}(x)$, $x_n' = (g^v)^{\circ n}(x)$. Then one can use the previous inequalities as in [4, Lemma 6.11] to give an estimate of $|x_n' - x_n|$ in terms of $|x|^3\|v^0 - u^0\|_\infty$. Only the smoothness properties of u, v are important (see [4, Equation (6.15)]) and not holomorphicity.

Finally, these ingredients can be used in the same way as [4, Proposition 6.12] to show that $T : \mathcal{F}'(\delta) \to \mathcal{F}'(\delta)$ is a contraction. □

4.4. *The case of* $d_2 < d_1 - 1$. The diffeomorphisms $\psi_{n,t}$, $n \geqslant 2$, in Example 2.2 show that when $d_2 < d_1 - 1$ the conclusion of Theorem 4.2 might fail to hold: indeed, in that case we have that the invariant formal curve $\{y = 0\}$ is convergent, but $d_1 = n + 1$ and $d_2 = n - 1$. We will now show that, if γ is an invariant curve of class C^∞ for a germ $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ tangent to the identity and $d_2 < d_1 - 1$, then γ can actually be nowhere-analytic, and additionally the Taylor development of its defining function can be divergent at 0.

Example 4.8. There exists a family of germs $\psi_t \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ tangent to the identity, $t \in [0, 1]$, and a family of nowhere-analytic curves $\gamma_t = \{y = f_t(x)\}$ of class C^∞, such that γ_t is invariant under ψ_t for any t. Moreover, the Taylor expansion of f_t at 0 is divergent for all $t > 0$ and identically vanishing for $t = 0$, and $\psi_t \to \psi_0$, $f_t \to f_0$ uniformly for $t \to 0$.

Proof. Fixed any $0 \leqslant t \leqslant 1$, let $\phi_t \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ be defined as

$$\phi_t(x, y) = \left(\frac{x}{1 - x^4}, (1 + x^2)y - tx^4\right).$$

If $g(x) = x - x^5 + O(x^6)$ is the local inverse of $x \to \frac{x}{1-x^4}$, we can write $\psi_t = \phi_t^{-1}$ as

$$\psi_t(x, y) = \left(g(x), \frac{y}{1 + g(x)^2} + \frac{tg(x)^4}{1 + g(x)^2}\right) = (g(x), q(x)y + r_t(x));$$

note that $q(x) = 1 - x^2 + O(x^3)$ and $r_t(x) = tO(x^4)$.

LEMMA 4.9. *Let* ϕ_t *be as above: then there is* $u_1 \in \mathbb{R}[[x]] \setminus \mathbb{R}\{x\}$, $u_1(0) = 0$ *such that the uniquely determined* ϕ_t*-invariant formal curve* $I_t \subset \mathbb{R}[[x, y]]$ *of the form* $I_t = (y - u_t(x))$ *is given by* $u_t = tu_1$. *In particular,* I_t *is purely formal for* $t > 0$, *and* $u_0 = 0$.

Proof 31. Writing $u_t(x) = \sum_{j \geqslant 1} u_{t,j} x^j$, the formal mapping equation is

$$u_t\left(\frac{x}{1 - x^4}\right) = -tx^4 + (1 + x^2)u_t(x). \tag{19}$$

Note that, if we prove that $u_1 \in \mathbb{R}[[x]]$ is the (as we will show, unique) formal solution of the previous equation for $t = 1$, then $u_t = tu_1$ is the only formal solution for any t: in the following we can then assume $t = 1$, and rewrite (19) as

$$\sum_{j \geqslant 1} u_{1,j} \left(\sum_{\ell \geqslant 0} x^{4\ell+1}\right)^j = -x^4 + \sum_{j \geqslant 1} (u_{1,j} + u_{1,j-2})x^j, \tag{20}$$

where we set $u_{1,\ell} = 0$ for $\ell \leqslant 0$. We also write $(\sum_{\ell \geqslant 0} x^{4\ell+1})^j = \sum_{i \geqslant j} \alpha_i^j x^i$. It is clear that the coefficients α_i^j are non-negative, and that $\alpha_i^j = 0$ whenever $i - j$ is odd (in fact whenever $i \neq j \mod 4$). It is easy to compute that $\alpha_j^j = 1$ and $\alpha_{j+4}^j = j$ for all $j \geqslant 1$. Developing the left-hand side of (20) and looking at the coefficient of x^ℓ we obtain the equations

$$\sum_{j=1}^{\ell} \alpha_\ell^j u_{1,j} = u_{1,\ell} + u_{1,\ell-2}$$

for $\ell \neq 4$ and $u_{1,4} = u_{1,4} + u_{1,2} - 1$ for $\ell = 4$, that is, $u_{1,2} = 1$. Using the previous observations about the α_i^j we can rewrite the equation above as

$$u_{1,\ell-2} = (\ell - 4)u_{1,\ell-4} + \sum_{j=1}^{\ell-8} \alpha_\ell^j u_{1,j}$$

for $\ell \geqslant 3$, $\ell \neq 4$. This shows in particular that u_t is uniquely determined. For $\ell = 3$ we get $u_{1,1} = 0$, from which follows immediately that $u_{1,2k+1} = 0$ for all $k \geqslant 0$. On the other hand, from the previous recursion we obtain inductively that $u_{1,2k} \geqslant 0$ and $u_{1,2k+2} \geqslant 2k(u_{1,2k})$ for all $k \geqslant 1$. We deduce that $u_{1,2k} \geqslant 2^{k-1}(k-1)!$ for all $k \geqslant 1$, hence $u_1 \neq \mathbb{R}\{x\}$ (and $u_t = tu_1 \neq \mathbb{R}\{x\}$ for $t > 0$). □

LEMMA 4.10. *Fix $C, D > 1$, $R > 0$ and $m, j \in \mathbb{N}$. There exists $E > 0$ with the following property. For any sequences $\{c_n\}_{n\geqslant 1}$, $\{d_n\}_{n\geqslant 1}$ such that $c_n \geqslant 1/C\sqrt{n}$ for $n \geqslant m$, $|c_n| \leqslant C/\sqrt{n}$ for $n \geqslant 1$ and*

(1) *$d_n = 0$, respectively,*
(2) *$|d_n| \leqslant Dn^j e^{-\frac{\sqrt{n}}{C}}$, respectively,*
(3) *$|d_n| \leqslant D/\sqrt[4]{n^{2+j}}$,*

for all $n \geqslant 1$, define a sequence $\{y_n\}_{n\geqslant 1}$ recursively by choosing $y_1 \in [-R, R]$ and letting $y_{n+1} = (1 - c_n)y_n + d_n$ for $n \geqslant 1$. Then

(1) *$|y_n| \leqslant Ee^{-\frac{\sqrt{n}}{C}}$, respectively,*
(2) *$|y_n| \leqslant En^{j+1}e^{-\frac{\sqrt{n}}{C}}$, respectively,*
(3) *$|y_n| \leqslant E/\sqrt[4]{n^j}$,*

for all $n \geqslant 1$.

Proof. Choose $\mathbb{N} \ni n_0 > \max\{m, C^2\}$, so that $(1 - c_n) > 0$ for all $n \geqslant n_0$. We define a positive sequence $\{z_n\}$ recursively by putting $z_1 = R$ and

$$z_{n+1} = \begin{cases} \left(1 + \frac{C}{\sqrt{n}}\right)z_n + w_n & \text{if } n < n_0, \\ \left(1 - \frac{1}{C\sqrt{n}}\right)z_n + w_n & \text{if } n \geqslant n_0, \end{cases}$$

where $w_n = 0$ in case (1), $w_n = Dn^j e^{-\frac{\sqrt{n}}{C}}$ in case (2) and $w_n = D/\sqrt[4]{n^{2+j}}$ in case (3). It is clear that $\{z_n\}$ is a majorant sequence for $\{|y_n|\}$, and its definition only depends on C, D, R, m and j.

To prove (1) and (2) we choose $E \geqslant \max_{n\leqslant n_0}\{z_n/e^{-\sqrt{n}/C}\}$, and also $E \geqslant D/(1 - \frac{1}{C\sqrt{n_0}})$. Observe that for any $n \geqslant 1$

$$\frac{e^{-\sqrt{n+1}/C}}{e^{-\sqrt{n}/C}} = e^{(\sqrt{n}-\sqrt{n+1})/C} \geqslant e^{-\frac{1}{C\sqrt{n}}} \geqslant 1 - \frac{1}{C\sqrt{n}}, \text{ i.e.}$$

$$e^{-\sqrt{n}/C}\left(1 - \frac{1}{C\sqrt{n}}\right) \leqslant e^{-\sqrt{n+1}/C},$$

hence for $n \geqslant n_0$ we can estimate z_{n+1} inductively as

$$z_{n+1} = \left(1 - \frac{1}{C\sqrt{n}}\right)z_n \leqslant \left(1 - \frac{1}{C\sqrt{n}}\right)Ee^{-\frac{\sqrt{n}}{C}} \leqslant Ee^{-\frac{\sqrt{n+1}}{C}}$$

in case (1) and

$$z_{n+1} \leqslant \left(1-\frac{1}{C\sqrt{n}}\right)En^{j+1}e^{-\frac{\sqrt{n}}{C}} + Dn^{j}e^{-\frac{\sqrt{n}}{C}} \leqslant En^{j+1}e^{-\frac{\sqrt{n+1}}{C}} + En^{j}e^{-\frac{\sqrt{n+1}}{C}}$$

$$= En^{j}(n+1)e^{-\frac{\sqrt{n+1}}{C}} \leqslant E(n+1)^{j+1}e^{-\frac{\sqrt{n+1}}{C}}$$

in case (2). In order to prove (3), we choose $n_1 \geqslant n_0$ such that $n_1 \geqslant (jC)^2$. Pick $E \geqslant \max_{n \leqslant n_1}\{z_n/\sqrt[4]{n^j}\}$ and furthermore $E \geqslant 2CD$. A straightforward computation shows that

$$\frac{1}{\sqrt[4]{n^j}} \leqslant \frac{1}{\sqrt[4]{(n+1)^j}} + \frac{j}{2n\sqrt[4]{n^j}}$$

for $n \geqslant 1$, hence we have

$$z_{n+1} \leqslant \left(1-\frac{1}{C\sqrt{n}}\right)\frac{E}{\sqrt[4]{n^j}} + \frac{D}{\sqrt[4]{n^{j+2}}} \leqslant \frac{E}{\sqrt[4]{(n+1)^j}} + \frac{CD-E}{C\sqrt[4]{n^{j+2}}} + \frac{jE}{2n\sqrt[4]{n^j}}$$

$$\leqslant \frac{E}{\sqrt[4]{(n+1)^j}} + \frac{CD-E/2}{C\sqrt[4]{n^{j+2}}} \leqslant \frac{E}{\sqrt[4]{(n+1)^j}}$$

for $n \geqslant n_1$, which concludes the proof. □

With u_t as given by Lemma 4.9, we put $\rho_t(x, y) = y - u_t(x) \in \mathbb{R}[[x, y]]$. Since $(x, y) \to (x, \rho_t(x, y))$ is a non-singular formal change of coordinates in the ring $\mathbb{R}[[x, y]]$, we have that $x \mod (\rho_t)$ generates the maximal ideal of $\mathbb{R}[[x, y]]/(\rho_t)$. The homomorphism induced by ϕ_t on $\mathbb{R}[[x, y]]/(\rho_t)$ sends x to $x \circ \phi_t(x, y) = \frac{x}{1-x^4} = x + x^5 + O(x^6)$, hence it is tangent to the identity to order 5, that is, $d_1 = 5$. On the other hand, we have by (19)

$$\rho_t \circ \phi_t(x, y) = (1+x^2)y - tx^4 - u_t\left(\frac{x}{1-x^4}\right)$$

$$= (1+x^2)y - tx^4 - (1+x^2)u_t(x) + tx^4 = (1+x^2)\rho_t(x, y),$$

thus $\chi_t = \frac{\rho_t \circ \phi_t}{\rho_t} - 1 = x^2$. By the definition given at the beginning of § 4, this implies $d_2 = 2$, hence $d_2 < d_1 - 1$ for all pairs (ϕ_t, ρ_t), $t \in [0, 1]$.

In order to construct invariant curves $\gamma_t = \{y = f_t(x)\}$, we start by choosing $\xi_0 > 0$ such that $g(x) < x$ for all $0 < x \leqslant \xi_0$. For $n \in \mathbb{N}$ we put $\xi_n = g^{\circ n}(\xi_0)$ and we choose a nowhere-analytic function $f \in C^\infty([\xi_1, \xi_0])$ such that $f_t^{(j)}(\xi_0) = 0$ and $f_t^{(j)}(\xi_1) = \frac{d^j}{dx^j}(\frac{tx^4}{1+x^2})(\xi_1)$ for all $j \geqslant 0$ (see Lemma 2.7). From the proof of Lemma 2.7 it is easy to see that we can choose the family $\{f_t\}$ to vary continuously in $C^\infty([\xi_1, \xi_0])$ (this can be accomplished by choosing the function g^1_ϵ continuously in the proof of the lemma). As usual, we extend f_t to the interval $(-\xi_0, \xi_0)$ by setting recursively

$$f_t(x) = \frac{f_t(\frac{x}{1-x^4})}{1+x^2} + \frac{tx^4}{1+x^2}$$

for $x \in [\xi_{n+1}, \xi_n]$, $n \in \mathbb{N}$, and putting $f(0) = 0$, $f(x) = f(-x)$ for $x < 0$. The smoothness of f_t at ξ_1 follows from the relation above and the choice of the derivatives of f_t at ξ_1, ξ_0; indeed, since f_t is flat at ξ_0, the function $f_t(\frac{x}{1-x^4})/(1+x^2)$ is flat at ξ_1. The smoothness of f_t on $[-\xi_0, \xi_0] \setminus \{0\}$ can then be proved inductively by using again the relation above.

Since the curve $\gamma_t = \{y = f_t(x)\}$ is ψ_t-invariant by construction, the only thing to prove is the following:

LEMMA 4.11. *For any* $t \in [0,1]$ *we have* $f_t \in C^\infty([-\xi_0, \xi_0])$, *and* $f_t^{(j)}(0) = j! u_{t,j}$ *for all* $j \in \mathbb{N}$ *where* $u_t(x) = \sum_{j \geqslant 1} u_{t,j} x^j$ *is given by Lemma* 4.9.

Proof. Fix an arbitrary $\ell \in \mathbb{N}$ and put $k = 2\ell + 2$. We are going to show that f_t is of class C^ℓ for all $t \in [0,1]$. In order to do so, it is convenient to pass through the polynomial change of coordinates $\varphi_{t,k} \in \mathrm{Diff}^\omega(\mathbb{R}^2, 0)$ defined by $\varphi_{t,k}(x,y) = (x, y - \sum_{j \leqslant k} u_{t,j} x^j) = (x, y - j^k u_t(x))$, where $j^k : \mathbb{R}[[x]] \to \mathbb{R}[[x]]$ is the kth order truncation map. We remark that $a j^k b = j^k(ab) + O(x^{k+1})$ for all $a, b \in \mathbb{R}[[x]]$. From the proof of Lemma 4.9 it is easy to see that $u_t = t u_1$, thus $j^k u_t = t(j^k u_1)$. We claim that in the new coordinates the map $\widehat{\psi}_t = \varphi_{t,k} \circ \psi_t \circ \varphi_{t,k}^{-1}$ is expressed as $\widehat{\psi}_t(x,y) = (g(x), q_0(x)y + r_{t,0}(x))$ with $q_0(x) = q(x) = 1/(1+g(x)^2)$ and $r_{t,0}(x) = tO(|x|^{k+1})$. In fact, we have

$$\widehat{\psi}_t(x,y) = \left(g(x), \frac{1}{1+g(x)^2}(y + j^k u_t(x)) + \frac{t g(x)^4}{1+g(x)^2} - j^k u_t(g(x)) \right)$$

$$= \big(g(x), q(x)y - j^k u_t(g(x)) + r_t(x) + q(x) j^k u_t(x) \big).$$

Note that the ψ_t-invariance of the formal curve $\{y = u_t(x)\}$ translates into the equation (valid in $\mathbb{R}[[x]]$) $u_t \circ g = q u_t + r_t$, which implies $j^k(q u_t) + j^k r_t - j^k(u_t \circ g) = 0$. Since $u_t = t u_1 = t j^k u_1 + tO(|x|^{k+1})$ we have $j^k(u_t \circ g) = (j^k u_t) \circ g + tO(|x|^{k+1})$; furthermore $q j^k u_t = t q j^k u_1 = t j^k(q u_1) + tO(|x|^{k+1}) = j^k(q u_t) + tO(|x|^{k+1})$ and $r_t = t r_1 = j^k r_t + tO(|x|^{k+1})$. It follows that

$$r_{t,0} = q j^k u_t + r_t - (j^k u) \circ g = j^k(q u_t) + j^k r_t - j^k(u_t \circ g) + tO(|x|^{k+1}) = tO(|x|^{k+1})$$

as claimed. The previous computation also shows that $r_{t,0} = t r_{1,0}$.

Let now $\widehat{\gamma}_t = \{y = \widehat{f}_t(x)\}$ be the image of γ_t through the change of coordinates $\varphi_{t,k}$. We have $\widehat{f}_t(x) = f_t(x) - j^k u_t(x)$, and the $\widehat{\psi}_t$-invariance equation is

$$\widehat{f}_t(g(x)) = q_0(x) \widehat{f}_t(x) + r_{t,0}(x). \tag{21}$$

In a way analogous to Lemma 3.9, we will prove the following claim: for all $0 \leqslant j \leqslant k$ there exist functions $r_{t,j}(x) = t r_{1,j}(x)$, $q_{j,i}(x)$ $(0 \leqslant i \leqslant j)$, real analytic in a neighborhood of 0, such that $q_{j,j}(x) = 1 - x^2 + O(|x|^3)$, $r_{1,j} = O(|x|^{k+1-j})$ and

$$\widehat{f}_t^{(j)}(g(x)) = r_{t,j}(x) + \sum_{i=0}^{j} q_{j,i}(x) \widehat{f}_t^{(i)}(x) \tag{22}$$

for all $0 < x \leqslant \xi_0$, where we set $\widehat{f}_t^{(0)}(x) := \widehat{f}_t(x)$. The case $j = 0$ is given by (21) with $q_{0,0} = q_0$. The inductive step is obtained by differentiating (22):

$$\widehat{f}_t^{(j+1)}(g(x)) g'(x) = t r'_{1,j}(x) + \sum_{i=0}^{j+1} (q'_{j,i}(x) + q_{j,i-1}(x)) \widehat{f}_t^{(k)}(x)$$

(where we set $q_{j,j+1} = q_{j,-1} \equiv 0$), and since $g'(x) = 1 + O(x^4)$ is invertible we have that (22) holds for $j+1$ with $q_{j+1,i}(x) = (q'_{j,i}(x) + q_{j,i-1}(x))/g'(x)$ for all $0 \leqslant i \leqslant j+1$ and $r_{1,j+1}(x) = r'_{1,j}(x)/g'(x) = O(|x|^{k-j})$. Furthermore

$$q_{j+1,j+1}(x) = \frac{q_{j,j}(x)}{g'(x)} = \frac{1 - x^2 + O(x^3)}{1 + O(x^4)} = 1 - x^2 + O(x^3).$$

For $x_1 \in [\xi_1, \xi_0]$ and $n \geqslant 1$, we put $x_n = g^{\circ(n-1)}(x_1)$. We can choose $B > 0$ such that $1/B\sqrt[4]{n} \leqslant x_n \leqslant B/\sqrt[4]{n}$ for all $x_1 \in [\xi_1, \xi_0]$, $n \geqslant 1$. Keeping the analogy with Lemma 3.9, our next step is to show that for any $0 \leqslant j \leqslant k/2-1$ there exists $C_j > 0$ such that the estimates $|\widehat{f}_t^{(j)}(x_n)| \leqslant C_j/\sqrt[4]{n^{k-1-2j}}$ (for $t>0$) and $|\widehat{f}_0^{(j)}(x_n)| \leqslant C_j n^j e^{-\sqrt{n}}$ hold for any $x_1 \in [\xi_1, \xi_0]$, $n \geqslant 1$. Let us first suppose $j=0$, and consider (21) for $0 \leqslant t \leqslant 1$ at $x = x_n$, that is, $\widehat{f}_t(x_{n+1}) = q_0(x_n)\widehat{f}_t(x_n) + r_{t,0}(x_n)$. Since $r_{t,0} = tr_{1,0} = tO(|x|^{k+1})$, we can find $K>0$, independent of t, such that $|r_{t,0}(x_n)| \leqslant K/\sqrt[4]{n^{k+1}}$ for all $x_1 \in [\xi_1, \xi_0]$, $n \geqslant 1$. Similarly, since $1 - q_0(x) = x^2 + O(x^3)$ there exist $m_0, H > 0$ such that $1 - q_0(x_n) \geqslant 1/H\sqrt{n}$ if $n \geqslant m_0$, $|1-q_0(x_n)| \leqslant H/\sqrt{n}$ for $n \geqslant 1$, for all $x_1 \in [\xi_1, \xi_0]$. Applying Lemma 4.10 (points 1) and 3)) with $C = H$, $D = K$, $R = \max_{x\in[\xi_1,\xi_0], t\in[0,1]} |\widehat{f}_t(x)|$, $m = m_0$, $j = k+1$, $c_n = 1 - q_0(x_n)$, $d_n = r_{t,0}(x_n)$, $y_n = \widehat{f}_t(x_n)$, we obtain the estimates $|\widehat{f}_t(x_n)| \leqslant C_0/\sqrt[4]{n^{k-1}}$ (for $0 < t \leqslant 1$) and $|\widehat{f}_0(x_n)| \leqslant C_0 e^{-\sqrt{n}}$. Note that C_0 does not depend on $t \in [0,1]$.

Assume now that the estimates are verified for the derivatives of order less than $j \leqslant k/2-2$. With $r_{t,j}$ and $q_{j,i}$ $(0 \leqslant i \leqslant j)$ as in (22), we can choose $M' > 0$ such that $|q_{j,i}(x)| \leqslant M'$ for $x \in [0, \xi_0]$. Furthermore, we can select $m_j, H_j, K_j > 0$ such that $1 - q_{j,j}(x_n)$ and $r_{t,j}(x_n)$ satisfy estimates analogous to the ones of $1-q_0(x_n)$ and $r_{t,0}$ above. Putting $s_t(x) = r_{t,j}(x) + \sum_{i=0}^{j-1} q_{j,i}(x)\widehat{f}_t^{(i)}(x)$, choosing $M>0$ large enough, we have

$$|s_t(x_n)| \leqslant \frac{K_j}{\sqrt[4]{n^{k+1-j}}} + M' \sum_{i=0}^{j-1} |\widehat{f}_t^{(i)}(x_n)| \leqslant \frac{K_j}{\sqrt[4]{n^{k+1-j}}} + M' \sum_{i=0}^{j-1} \frac{C_i}{\sqrt[4]{n^{k-1-2i}}} \leqslant \frac{M}{\sqrt[4]{n^{k+1-2j}}}$$

for $0 < t \leqslant 1$ and

$$|s_0(x_n)| \leqslant M' \sum_{i=0}^{j-1} |\widehat{f}_0^{(i)}(x_n)| \leqslant M' \sum_{i=0}^{j-1} C_i n^i e^{-\sqrt{n}} \leqslant M n^{j-1} e^{-\sqrt{n}}$$

for $t = 0$. Again, this allows to apply Lemma 4.10 (points 2) and 3)) with $c_n = q_{j,j}(x_n)$, $d_n = s_t(x_n)$ and $y_n = \widehat{f}_t^{(j)}(x_n)$ to obtain the desired estimates $|\widehat{f}_t^{(j)}(x_n)| \leqslant C_j/\sqrt[4]{n^{k-1-2j}}$ for $0 < t \leqslant 1$ and $|\widehat{f}_0^{(j)}(x_n)| \leqslant C_j n^j e^{-\sqrt{n}}$.

For all $t \in [0,1]$, $0 \leqslant j \leqslant (k/2) - 1 = \ell$, the fact that $\widehat{f}_t^{(j)}(x) \to 0$ as $x \to 0^+$ now follows with a proof very similar to the one employed in Lemma 3.9. Since $\widehat{f}_t$ is an even function, it follows that $\widehat{f}_t$, and thus f_t, is of class C^ℓ, and in fact of class C^∞ since ℓ is arbitrary. The fact that $f_t^{(j)}(0) = j! u_{t,j}$ is a consequence of the uniqueness statement in Lemma 4.9. ☐

To prove the last claim in Example 4.8, we first note that for any $\varrho > 0$ the family $\{f_t\}_{t\in[0,1]}$ is continuous with respect to t on the set $[-\xi_0, \xi_0] \setminus (-\varrho, \varrho)$ by construction: this follows immediately from the choice of f_t over $[\xi_1, \xi_0]$ and from the continuity of $\{\phi_t\}$ with respect to t. On the other hand, the uniformity of the estimates in Lemma 4.11 show that for any $\varepsilon > 0$ there exists $\varrho > 0$ such that $|f_t| < \varepsilon$ on $[-\varrho, \varrho]$ for all $t \in [0,1]$: together with the previous observation, this implies that $f_t \to f_0$ as $t \to 0$. ☐

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