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Construction of vector-valued modular integrals and vector-valued mock modular forms

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Abstract It is known that, given a vector-valued modular form of negative weight, its Fourier coefficients can be calculated based on the principal part of the form. In this paper we start with an arbitrary principal part and complete the Fourier expansion using the calculation. We show that the so-obtained function is a vector-valued modular integral of negative weight on the full modular group. Next, we construct the supplementary function associated to a vector-valued modular cusp form of positive weight. The constructions are inspired by the construction of Eichler integrals by Knopp. We conclude with a comparison of these forms and their integrals to vector-valued weak harmonic Maass forms.

Keywords Modular forms · Fourier coefficients · Modular integrals · Supplementary function

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Dedicated to the memory of Marvin Knopp.

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1 Introduction

Consider a vector-valued function $F = \big(F_1, \ldots, F_p\big) : \mathbb{H} \to \mathbb{C}^p$ from the upper half plane $\mathbb{H}$ to the p-dimensional complex vector space $\mathbb{C}^p$. We call such a function F a *vector-valued modular integral* of integer weight k on the full modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ if it (essentially) satisfies the following two conditions: (a) For given (compatible) multiplier system ε and p-dimensional complex representation $\rho : \Gamma \to \mathrm{GL}(p, \mathbb{C})$ the function F satisfies

$$F(\tau) - \big(\varepsilon(\gamma)\big)^{-1} (c\tau + d)^k \big(\rho(\gamma)\big)^{-1} F\left(\frac{a\tau + b}{c\tau + d}\right) = P_\gamma(\tau)$$

for all $\tau \in \mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, where $P_\gamma(\tau) \in \big(\mathbb{C}_k[\tau]\big)^p$ is a column vector of polynomials in τ of degree at most k. (b) Each component function $F_j(\tau)$ has a convergent q-expansion meromorphic at infinity:

$$F_j(\tau) = q^{m_j} \sum_{\nu \geq \mu_j} a_\nu(j) q^\nu$$

with $0 \leq m_j < 1$ a rational number, μ_j an integer, and $q = e^{2\pi i \tau}$. The *principal part* of the q-expansion is the part with $\mu_j \leq \nu < 0$, similar to Laurent expansions of meromorphic functions. We call F a *vector-valued modular form* of integer weight $-k$ if $P_\gamma = 0$ holds for all $\gamma \in \Gamma$. We call F a *vector-valued cusp form* of integer weight $-k$ if F is a vector-valued modular form of weight $-k$ with $\mu_j > 0$ for all $j \in \{1, \ldots, p\}$. The precise definitions of vector-valued modular forms and integrals are given in Sect. 2.3.

The first and third authors show in [11] that if we consider a vector-valued modular form $F = \big(F_1, \ldots, F_p\big)$ of large negative weight on the full modular group, then its Fourier coefficients $a_m(j)$ for $m \geq 0$ are given by linear combinations of the familiar Rademacher–Petersson series that are expressed in terms of Kloosterman sums and the modified Bessel function of the first kind. This means that the coefficients $a_m(j)$ depend only on the principal part of F:

$$a_m(j) = f\big(m, j, \{a_\nu(l);\ \nu < 0, 0 \leq l \leq p\}\big)$$

for all integers $m \geq 0$ and $1 \leq j \leq p$. The function f is known explicitly. See Theorem 1 in Sect. 3 for the complete statement.

Our first result, see Theorem 2 in Sect. 3 for the detailed version, is in some sense a converse statement. It says that given an arbitrary principal part, i.e., an arbitrary set of coefficients $\{b_\nu(j);\ \mu \leq \nu < 0, 0 \leq j \leq p\}$ of complex numbers (for some integer $\mu < 0$), we can define a vector-valued function $F = \big(F_1, \ldots, F_p\big)$ on the complex half-plane using the q-expansion of each component:

$$F_j(\tau) = q^{m_j} \left(\sum_{\mu \le \nu < 0} b_\nu(j) q^\nu + \sum_{m \ge 0} b_m(j) q^m \right)$$

with coefficients

$$b_m(j) := f\big(m, j, \{a_\nu(l);\ \nu < 0, 0 \le l \le p\}\big)$$

given by the same explicit function f as above. We prove that such an F is in fact a modular integral.

As a corollary, we also characterize the case that the modular integral F is indeed a modular form.

Our second main result starts with a vector-valued cusp form g of positive integer weight $k+2$ (large enough), multiplier system ε, and representation ρ. It is known that we can write g as a linear combination of vector-valued Poincaré series

$$g = \sum_{i=1}^{s} b_i \, P(\rho, k+2, \varepsilon, \mu_i, \Gamma, r_i; \cdot)$$

for suitable $s \in \mathbb{N}$, $r_i \in \{1, \dots, p\}$ and nonnegative $\mu_i \in \mathbb{Z} \ge 0$ with $P(\rho, k+2, \varepsilon, \mu_i, \Gamma, r_i; \cdot)$ denoting the Poincaré series of weight $k+2$, multiplier ε, and representation ρ. (See Sect. 5.1 for a precise introduction of Poincaré series.) The *supplementary series* $\widehat{g}$ of g is the function given by

$$\widehat{g}(\tau) = \sum_{i=1}^{s} \overline{b_i} \, P(\rho', k+2, \varepsilon', \mu_i', \Gamma, r_i; \tau)$$

with

$$\mu_i' := \begin{cases} -1 - \mu_i & \text{if } m_j > 0, \\ -\mu_i & \text{if } m_j = 0. \end{cases}, \quad \varepsilon'(\gamma) := \overline{\varepsilon(\gamma)} \quad \text{and} \quad \rho'(\gamma) := \overline{\rho(\gamma)}.$$

Our second result describes $(k+1)$-fold integrals G of g and $\widehat{G}$ of $\widehat{g}$, the latter is the *supplementary function* of g. We show in Theorem 3 that the functions G and $\widehat{G}$ in fact satisfy the transformation property of modular integrals of weight $-k$. The associated polynomial terms are closely related. We refer to Theorem 3 in Sect. 5 for the precise statement.

We also introduce the *auxiliary integral* $G_2 : \mathbb{H} \to \mathbb{C}^p$ to the cusp form g, given by

$$G_2(\tau) := \overline{\int_\tau^{i\infty} g(z) \, (z - \overline{\tau})^k \, \mathrm{d}z} \qquad (\tau \in \mathbb{H}).$$

This function also satisfies the transformation property of modular integrals of weight $-k$ and its associated polynomials are again closely related to the ones associated to G and $\widehat{G}$. The precise statement is given in Theorem 4 in Sect. 5.

As in the case of classical modular forms, vector-valued supplementary series and supplementary functions play a role in the proof of Eichler isomorphism theorem for vector-valued modular forms on the full group in a recent paper of Gimenez [10]. The scalar- valued case was discussed in [12]. The main idea is that the polynomials associated to the supplementary function are in fact period polynomials of the cusp form g, see e.g., [13, Theorem 1].

We conclude the paper by describing our second result in terms of the harmonic weak Maass waveforms. Vector-valued modular forms has been extensively studied in connection to metaplectic cover of $\mathrm{SL}(2,\mathbb{R})$ along with Weil representations [2]. In addition, a lot in their connection to harmonic weak Maass forms is known as well [4].

The two main results, Theorems 2 and 3, are based on Gimenez' thesis work, see [9, Theorem 2.1 and §3.2]. The remaining results, Theorems 4 and 5, are new results.

The paper is organized as follows: The next section covers basic notations and definitions. Section 3 contains the first main theorem and its motivating question, along with two corollaries. The proof of this theorem is given in Sect. 4. We continue with the construction of the supplementary function to a cusp form in Sect. 5, our second main result. We conclude our paper by sorting our result on the supplementary function into the modern setting of weak harmonic Maass waveforms in Sect. 6.

2 Preliminaries and notations

2.1 The group, its action, and the slash operator

Let $\mathrm{GL}(p,\mathbb{C})$ the group of $p \times p$-dimensional matrices with complex entries and nonvanishing determinant and let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ denote the group of 2×2 matrices with integers entries and determinant 1 called the *full modular group*. Γ is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

These satisfy

$$S^2 = (ST)^3 = (-\mathbf{1}), \tag{2}$$

where $(-\mathbf{1}) \in \Gamma$ is the matrix with -1 on the diagonal and 0 on the off-diagonal entries. Here $\mathbf{1}$ denotes the identity matrix.

The group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ acts on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C};\ \mathrm{Im}\,(z) > 0\}$ by *fractional linear transformations*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}. \tag{3}$$

The action extends naturally to the boundary $\mathbb{R} \cup \{i\infty\}$ of $\mathbb{H}$.

The subgroup Γ_∞ of Γ generated by

$$\Gamma_\infty := \langle (-\mathbf{1}), T \rangle = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & m \\ 0 & -1 \end{pmatrix};\ m \in \mathbb{Z} \right\} \tag{4}$$

denotes the stabilizer of the boundary point $i\infty$.

We also need the following j-function:

$$j : \mathrm{SL}(2, \mathbb{R}) \times \mathbb{H} \to \mathbb{H}; \qquad j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) := c\tau + d. \tag{5}$$

We call a function $\varepsilon : \mathrm{SL}(2, \mathbb{Z}) \to \mathbb{C}_{\neq 0}$ *multiplier* or *multiplier system* compatible with real *weight* k if ε satisfies

$$\varepsilon(\gamma\delta)\, j(\gamma\delta, z)^k = \varepsilon(\gamma)\varepsilon(\delta)\, j(\gamma, \delta\, z)^k j(\delta, z)^k \tag{6}$$

and

$$|\varepsilon(\gamma)| = 1 \tag{7}$$

for every $\gamma, \delta \in \mathrm{SL}(2, \mathbb{Z})$ and $z \in \mathbb{H}$. For $k \in \mathbb{Z}$ (6) simplifies to

$$\varepsilon(\gamma\delta) = \varepsilon(\gamma)\varepsilon(\delta).$$

Let $p \in \mathbb{N}$ be a positive integer. A *p-dimensional complex representation ρ of Γ* is a function

$$\rho : \Gamma \to \mathrm{GL}(p, \mathbb{C})$$

satisfying

$$\rho(\gamma\, \delta) = \rho(\gamma)\, \rho(\delta) \tag{8}$$

for all $\gamma, \delta \in \Gamma$. We denote by $\rho_{j,m}(\gamma)$ the complex entry in the jth row and mth column of the matrix $\rho(\gamma)$. As notation, we use $\big(\rho(\gamma)\big)^{-1}_{j,m}$ to denote (j, m)th entry of the matrix $\big(\rho(\gamma)\big)^{-1}$.

In [15], Knopp and Mason show that sometimes the representation ρ can be normalized such that

$$\varepsilon(T)\, \rho(T) = \begin{pmatrix} e^{2\pi i m_1} & & \\ & \ddots & \\ & & e^{2\pi i m_p} \end{pmatrix} \tag{9}$$

holds with $0 \leq m_j < 1$, $m_j \in \mathbb{Q}$ for $1 \leq j \leq p$ and the m_j's on the diagonal are given in the Fourier expansion (17). They argue that an assumption 'ρ is normalized' is reasonable and does not lead to a significant loss of generality [15, above (10)]. *We also will always make this assumption throughout the rest of the paper.*

Remark 1 In [15] Knopp and Mason assume $0 < m_j \leq 1$. We prefer to change the interval for convenience.

We assume the argument convention $-\pi \le \arg z < \pi$ for all $z \in \mathbb{C}_{\neq 0}$ throughout the paper.

We introduce the *slash notation*. Let $k \in \mathbb{Z}$ a weight and ε be a multiplier. For $f : \mathbb{H} \to \mathbb{C}$ and $\gamma \in \mathrm{SL}(2, \mathbb{Z})$, we define

$$\left(f|_k^{\varepsilon}\gamma\right)(\tau) := \left(\varepsilon(\gamma)\right)^{-1} j(\gamma, \tau)^{-k} f(\gamma\,\tau) \tag{10}$$

for every $\tau \in \mathbb{H}$. The slash notation extends naturally to vector-valued functions:

$$\left(f_1, \ldots, f_p\right)^{\mathrm{tr}}|_k^{\varepsilon}\gamma := \left(f_1|_k^{\varepsilon}\gamma, \ldots, f_p|_k^{\varepsilon}\gamma\right)^{\mathrm{tr}}.$$

2.2 The Eichler length and bounds on ρ

To bound $\rho(\gamma)$, we introduce the *Eichler length* of γ [8] with respect to the generators S and T of Γ given in (1). Namely, we write V as a product

$$\gamma = \pm\gamma_1 \ldots \gamma_L,$$

where each γ_j is equal to either S or T^{n_j} for some integer n_j, no two consecutive γ_j are both equal to S or a power of T, and where $L = L(\gamma) \in \mathbb{N}$ is minimal. Eichler proved that

$$L(\gamma) \le n_1 \log \eta(\gamma) + n_2, \tag{11}$$

where

$$\eta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a^2 + b^2 + c^2 + d^2,$$

and n_1, n_2 are constants independent of γ.

Let ρ denote a p-dimensional complex representation. As introduced above, denote by $\rho_{j,m}(\gamma)$ the complex entry in the jth row and mth column of the matrix $\rho(\gamma)$. Knopp and Mason [16] showed that

$$\left|\rho_{j,m}(\gamma)\right| \le p^{L(\gamma)-1}\, K_1^{L(\gamma)} \qquad \text{for } 1 \le m, j \le p,$$

where K_1 is a constant that satisfies $\left|\rho_{j,m}(S)\right| \le K_1$ for all $1 \le m, j \le p$.

Therefore, if we use the usual norm

$$|\rho(\gamma)| = \sqrt{\sum_{1 \le m, j \le p} \left|\rho_{j,m}(\gamma)\right|^2}$$

we see that

$$|\rho(\gamma)| \le K_2\, \eta(\gamma)^{\alpha} \qquad \text{with } \alpha = n_1 \log p K_1, \tag{12}$$

where the constant K_2 is independent of γ. Note that since the length of γ^{-1} is the same as the length of γ, we can use the same bound for $\left|\left(\rho(\gamma)\right)^{-1}\right| = \left|\rho\left(\gamma^{-1}\right)\right|$.

The proof of the following lemma uses standard arguments and is omitted.

Lemma 1 *For given $c \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}$ such that* $\gcd(c, d) = 1$ *define the matrix*

$$\gamma_{c,d} = \begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad \text{satisfying } c > a \geq 0 & \text{if } c > 0 \text{ and} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma & \text{if } c = 0. \end{cases} \tag{13}$$

This matrix is uniquely defined.

Let $c, d \in \mathbb{Z}$ be two coprime integers, satisfying $N \geq c > -d \geq 0$ and $c > a \geq 0$. By [9, Lemma 1.2], we see that for such a $\gamma_{c,d}$ we have

$$\left|\big(\rho(\gamma_{c,d})\big)^{-1}_{j,m}\right| \leq K_2\, c^{2\delta} \qquad \text{with } \delta = \frac{2}{\log \alpha} \log p K_1. \tag{14}$$

Here, K_1 is a constant satisfying $\left|\big(\rho(S)\big)^{-1}_{j,m}\right| < K_1$ for all $1 \leq j, m \leq p$. Thus the estimate

$$\left|\big(\rho(\gamma_{c,d})\big)^{-1}_{j,m}\right| \leq K\, N^{2\delta} \tag{15}$$

holds with constants δ and K independent of γ.

2.3 Modular forms and modular integrals

Now, we continue to define p-dimensional forms.

Definition 1 Let k be an integer weight, ε be a compatible multiplier system, and $\rho : \Gamma \to \mathrm{GL}(p, \mathbb{C})$ a p-dimensional complex representation. Let

$$\mathbb{H} \to \mathbb{C}^p; \qquad \tau \mapsto F(\tau) := \big(F_1(\tau), \ldots, F_p(\tau)\big)^{\mathrm{tr}}$$

be a p-tuple of meromorphic functions in the complex upper half-plane $\mathbb{H}$. We call the pair (F, ρ) or simply F a *vector-valued meromorphic modular form of integer weight k on the modular group Γ* if

1. We have

$$\begin{aligned} & F\big|_k^{\varepsilon}\gamma = \rho(\gamma)\, F \\ \iff \quad & \big(F_1(\tau), \ldots, F_p(\tau)\big)^{\mathrm{tr}}\big|_k^{\varepsilon}\gamma(\tau) = \rho(\gamma)\big(F_1(\tau), \ldots, F_p(\tau)\big)^{\mathrm{tr}} \end{aligned} \tag{16}$$

   for all $\gamma \in \Gamma$ and $\tau \in \mathbb{H}$.
2. Each component function $F_j(\tau)$ has a convergent q-expansion meromorphic at infinity:

$$F_j(\tau) = q^{m_j} \sum_{\nu \geq \mu_j} a_\nu(j) q^\nu \tag{17}$$

with $0 \leq m_j < 1$ a rational number, μ_j an integer, and $q = e^{2\pi i\tau}$.

We denote the minimum of the μ_j's by μ:

$$\mu := \min(\mu_1, \ldots, \mu_p). \tag{18}$$

The space of such forms is denoted by $\mathscr{F}(k, \varepsilon, \rho)$.

We call a modular form F *cuspidal* or *cusp form* if each μ_j is positive. The space of cusp forms for given weight k, compatible multiplier ε, and representation ρ is denoted by $\mathscr{S}(k, \varepsilon, \rho)$.

The modular integral is inspired by the definition of the modular form above where we relax the transformation property (16).

Definition 2 Let k be an integer weight, ε be a compatible multiplier system, and $\rho : \Gamma \to \mathrm{GL}(p, \mathbb{C})$ a p-dimensional complex representation. Let

$$\mathbb{H} \to \mathbb{C}^p; \qquad \tau \mapsto F(\tau) := \big(F_1(\tau), \ldots, F_p(\tau)\big)^{\mathrm{tr}}$$

be a p-tuple of meromorphic functions in the complex upper half-plane $\mathbb{H}$. We call the pair (F, ρ) or simply F a *vector-valued modular integral of integer weight k on the modular group Γ* if

1. The function F satisfies

$$\begin{aligned} & F - \big(\rho(\gamma)\big)^{-1} F\big|_{-k}^{\varepsilon}\gamma = P_\gamma(k, \varepsilon, \rho; \cdot) \\ \iff \quad & F(\tau) - \big(\varepsilon(\gamma)\big)^{-1} j(\gamma, \tau)^k \big(\rho(\gamma)\big)^{-1} F(\gamma\tau) = P_\gamma(k, \varepsilon, \rho; \tau) \end{aligned} \tag{19}$$

   for all $\tau \in \mathbb{H}$ and $\gamma \in \Gamma$ where $P_\gamma(k, \varepsilon, \rho; \tau) \in \big(\mathbb{C}_k[\tau]\big)^p$ is a column vector of polynomials in τ of degree at most k.
2. Each component function $F_j(\tau)$ has a convergent q-expansion meromorphic at infinity:

$$F_j(\tau) = q^{m_j} \sum_{\nu \geq \mu_j} a_\nu(j) q^\nu \tag{20}$$

   with $0 \leq m_j < 1$ a rational number, μ_j an integer, and $q = e^{2\pi i\tau}$.

We denote the minimum of the μ_j's by μ:

$$\mu := \min(\mu_1, \ldots, \mu_p). \tag{21}$$

Obviously, every modular form is a modular integral.

3 Construction of vector-valued modular integrals

Let us introduce our first question about modular integrals by stating Theorem 1 below. It tells us the following: Start with a vector-valued modular form F which has a pole

at infinity and consider its Fourier expansions in (17). Then the Fourier coefficients $a_m(j)$ for $m \in \mathbb{Z}_{\geq 0}$ and $j \in \{1, \ldots, p\}$ of the vector-valued modular form F, see (17), are determined by the finite number of coefficients $a_\nu(j)$ with $\mu_j \leq \nu < 0$ and $j \in \{1, \ldots, p\}$.

Theorem 1 ([11]) *Let F be a vector-valued modular form of weight $-k$ with $k > 2\delta > 0$ with δ given in the estimate (15). Let ε denote the multiplier system and ρ the p-dimensional complex representation associated to F, as used in Definition 1. Pick any $j \in \{1, \ldots, p\}$ and consider the Fourier expansion of F as given in (17).*

For $m \in \mathbb{Z}_{\geq 0}$, the Fourier coefficients $a_m(j)$ in (17) are given by the formula:

$$a_m(j) = 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} \sum_{l=1}^{p} \sum_{\mu \leq \nu < 0} a_\nu(l)\, A_{c,\nu,m,j,l}\, B_{c,\nu,m,j,l}. \tag{22}$$

The terms $A_{c,\nu,m,j,l} = \left(A_{c,\nu,m}\right)_{j,l}$ are given by

$$A_{c,\nu,m,j,l} = \sum_{\substack{d \\ 0 \leq -d < c \\ \gcd(c,d)=1}} \Omega_{c,d,l,j}\, e^{2\pi i \frac{dm + a\nu}{c}}, \tag{23}$$

with

$$\begin{aligned} \Omega_{c,d,l,j} &= \left(\Omega_{c,d}\right)^{(j,l)} \\ &= \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l}\, e^{2\pi i \frac{m_l a + m_j d}{c}}. \end{aligned} \tag{24}$$

The coefficient a in (23) is given by the matrix $\gamma_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in (13)

The terms $B_{c,\nu,m,j,l}$ are given by

$$B_{c,\nu,m,j,l} = \left(\frac{-\nu - m_l}{m + m_j}\right)^{\frac{k+1}{2}} I_{k+1}\left(\frac{4\pi}{c}(-\nu - m_l)^{\frac{1}{2}}(m + m_j)^{\frac{1}{2}}\right) \tag{25}$$

for $m + m_j > 0$ and

$$B_{c,\nu,0,j,l} = \frac{1}{(k+1)!}\left(\frac{2\pi(-\nu - m_l)}{c}\right)^{k+1} \tag{26}$$

for $m = m_j = 0$. The $I_{k+1}(z)$ in (25) is the usual I_{k+1}-modified Bessel function of the first kind given by

$$I_{k+1}(z) = \frac{(\frac{z}{2})^{k+1}}{2\pi i} \int_{-\infty}^{(0+)} t^{-k} e^{t + \frac{z^2}{4t}}\, \mathrm{d}t$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n+k+1}}{n!(n+k+1)!}, \qquad z \in \mathbb{R}, \tag{27}$$

see e.g., [7, (10.25.2)].

In the theorem above, we started with a modular form for negative integer weight. Then, the principal part of its Fourier expansion determines the rest of its expansion. Is the converse also true? This means: If we start with an arbitrary principal part and consider the function defined by the principal part added to an expansion defined by Formula (22) stated above, is this function a modular form? The answer is "No"; such a function is just a modular integral.

Theorem 2 *For $\nu \in \mathbb{Z}_{<0}$ a given negative integer, let $b_{-1}, \ldots, b_\nu$ be a set of column vectors such that $b_\nu = \big(b_\nu(1), \ldots, b_\nu(p)\big)^{\mathrm{tr}} \in \mathbb{C}^p$, $b_\nu \neq 0$. Moreover, let $\rho : \Gamma \to \mathrm{GL}(p, \mathbb{C})$ be a p-dimensional complex representation, ε be a multiplier system compatible with integral weight $-k$ such that k satisfies $k > 2\delta$ (where δ is the same as the one in the estimate (14)). Define*

$$F(\tau) = \begin{pmatrix} \sum_{\mu \le \nu < 0} b_\nu(1)\, e^{2\pi i(m_1+\nu)\tau} + \sum_{m=0}^{\infty} b_m(1)\, e^{2\pi i(m+m_1)\tau} \\ \vdots \\ \sum_{\mu \le \nu < 0} b_\nu(p)\, e^{2\pi i(m_p+\nu)\tau} + \sum_{m=0}^{\infty} b_m(p)\, e^{2\pi i(m+m_p)\tau} \end{pmatrix}, \tag{28}$$

where we define the coefficients $b_m = \big(b_m(1), \ldots, b_m(p)\big)^{\mathrm{tr}}$, $m \in \mathbb{Z}_{\ge 0}$, as the column vectors with components given by

$$b_m(j) = 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} \sum_{l=1}^{p} \sum_{\mu \le \nu < 0} b_\nu(l) A_{c,\nu,m,j,l}\, B_{c,\nu,m,j,l}, \tag{29}$$

with $A_{c,\nu,m,j,l}$ given by (23) and $B_{c,\nu,m,j,l}$ by (25).

The function F satisfies the following properties:

1. *F is holomorphic on $\mathbb{H}$.*
2. *F is a modular integral: It satisfies*

$$\begin{aligned} & F - \big(\rho(\gamma)\big)^{-1} F\big|_{-k}^{\varepsilon} \gamma = Q_\gamma(k, \varepsilon, \rho; \cdot) \\ \iff \quad & F(\tau) - \big(\varepsilon(\gamma)\big)^{-1} j(\gamma, \tau)^k \big(\rho(\gamma)\big)^{-1} F(\gamma\tau) = Q_\gamma(k, \varepsilon, \rho; \tau), \end{aligned} \tag{30}$$

for all $\gamma \in \Gamma$ and $\tau \in \mathbb{H}$. The function $Q_\gamma(k, \varepsilon, \rho; \tau) \in \big(\mathbb{C}_k[\tau]\big)^p$ is a column vector of polynomials in τ of degree at most k.

We prove this theorem in Sect. 4.

Remark 2 The formula in (29) for calculating the Fourier coefficients $b_m(j), m \in \mathbb{Z}_{\ge 0}$ is the same as (22) in Theorem 1. In some sense, Theorem 1 can be viewed as a special case of Theorem 2, where F in (28) is a priori known to be a vector-valued modular form.

Remark 3 The term $A_{c,v,m,j,l}$ (23) is a generalization of Kloosterman sums in the following sense: Consider the setting of the trivial 1-dimensional representation: $p = 1$ and $\rho(\gamma) = 1$ for all $\gamma \in \Gamma$. If we assume that the factor $\Omega_{c,d,l,j}$ in (24) is trivial, i.e., if we assume $\Omega_{c,d,l,j} = 1$, then the term $A_{c,v,m,j,l}$ (23) is just a usual Kloosterman sum. We have

$$A_{c,v,m,j,l} = \sum_{\substack{d \\ 0 \le -d < c \\ \gcd(c,d)=1}} e^{2\pi i \frac{dm+av}{c}} = K(m, v, c),$$

where $K(m, v, c)$ is given by

$$K(m, v, c) := \sum_{d \pmod{c}^\star} e^{2\pi i \frac{md+av}{c}}.$$

Here d runs through the prime residue classes modulo c, and a is defined by the congruence $ad \equiv 1 \pmod{c}$.

Remark 4 Theorem 2 generalizes [17, Theorem 3.3].

Two simple corollaries of Theorem 2 are given below.

Corollary 1 *Assume the setting of Theorem 2. The following two statements are equivalent:*

1. *The term $Q_S(k, \varepsilon, \rho; \tau)$ in (30) vanishes for all $\tau \in \mathbb{H}$.*
2. *F defined in (28) is a vector-valued modular form of weight $-k$, multiplier ε, and representation ρ for Γ.*

Proof Assume F to be a vector-valued modular form of weight $-k$, multiplier ε, and representation ρ for Γ. Then Theorem 2, in particular (30), implies that $Q_S(\tau, k, \varepsilon, \cdot)$ vanishes since F satisfies (16).

For the converse direction let $b_{-1}, \ldots, b_\mu$ be a set of column vectors in $\mathbb{C}^p$ with $b_\mu \neq 0$, $\rho : \Gamma \to \mathrm{GL}(p, \mathbb{C})$ a p-dimensional complex representation, ε a multiplier system for weight $-k$ with $k > 2\delta$, $k, -\mu \in \mathbb{N}$. Let $\tau \mapsto F(\tau)$ be defined as in (28). Hence F has a Fourier expansion of the form specified in (17).

The assumption $Q_S(k, \varepsilon, \rho; \cdot)$ vanishes everywhere implies that F satisfies the transformation property

$$F = \big(\rho(S)\big)\big(F\big|_{-k}^{\varepsilon} S\big).$$

The Fourier expansion (28) together with the normalization (9) implies

$$F = \big(\rho(T)\big)^{-1}\big(F\big|_{-k}^{\varepsilon} T\big).$$

Hence F satisfies (17) for all generators of Γ. The cocycle condition (6) and induction in the Eichler length of each $\gamma \in \Gamma$ shows (17) for all elements of Γ. □

Corollary 2 *The v.–v. polynomial $Q_g(k, \varepsilon, \rho; \cdot)$ in Theorem 2 satisfies the cocycle relation*

$$\begin{aligned} &Q_{gh}(k, \varepsilon, \rho; \tau) \\ &\quad = \big(\varepsilon(h)\big)^{-1} \big(\rho(h)\big)^{-1} j(h, \tau)^k \, Q_g(k, \varepsilon, \rho; h\tau) + Q_h(k, \varepsilon, \rho; \tau) \end{aligned} \tag{31}$$

for all $g, h \in \Gamma$ and $\tau \in \mathbb{H}$.

Proof This can be easily seen using (30) together with (6) and (8):

$$\begin{aligned} &Q_{gh}(k, \varepsilon, \rho; \tau) \\ &\overset{(30)}{=} F(\tau) - \big(\varepsilon(gh)\big)^{-1} \big(\rho(gh)\big)^{-1} j(gh, \tau)^k \, F(gh\tau) \\ &\overset{(6),(8)}{=} \big(\varepsilon(h)\big)^{-1} \big(\rho(h)\big)^{-1} j(h, \tau)^k \\ &\qquad \cdot \left(F(h\tau) - \big(\varepsilon(g)\big)^{-1} \big(\rho(g)\big)^{-1} j(g, h\tau)^k \, F(gh\tau) \right) \\ &\qquad + F(\tau) - \big(\varepsilon(h)\big)^{-1} \big(\rho(h)\big)^{-1} j(h, \tau)^k \, F(h\tau) \\ &\overset{(30)}{=} \big(\varepsilon(h)\big)^{-1} \big(\rho(h)\big)^{-1} j(h, \tau)^k \, Q_g(k, \varepsilon, \rho; h\tau) + Q_h(k, \varepsilon, \rho; \tau) \end{aligned}$$

for all $g, h \in \Gamma$ and $\tau \in \mathbb{H}$. □

4 Proof of Theorem 2

This section contains the proof of Theorem 2 which is structured into several steps. These steps are given in the following subsections:

Step 1: We define and discuss the sets D_c and D^c in Sect. 4.1. They are closely related to the map $(c, d) \mapsto \gamma_{c,d}$ in Lemma 1. These sets are needed as index sets for the matrices $\gamma_{c,d}$ throughout the proof.
Step 2: In Sect. 4.2 we introduce the matrix-valued function $\mathscr{R}_\mu(\tau)$ related to the Fourier expansion of F. This leads to the proof of F being holomorphic on $\mathbb{H}$.
Auxiliary Step I: We prove in Sect. 4.3 the Lemmas 4 and 5 which we used in the step above.
Step 3: We recall two variants of the the Lipschitz summation formula. They are used in the following steps.
Step 4: In Sect. 4.5 we introduce and discuss matrix-valued functions $\mathscr{T}_\nu(\tau)$, $\mathscr{K}_\mu$, and $\mathscr{W}_\mu(\tau)$. They are used to rewrite parts of the Fourier expansion of F such that we can understand the action of Γ on F in the steps below.
Auxiliary Step II: We prove Lemma 6 and modified versions of [17, Lemma 2.5 and 2.13], the Lemmas 7 and 8, in Sect. 4.6. They are used in the step above.
Step 5: We define and discuss the matrix-valued function $\mathscr{S}_{\nu,K}$ and its image under S: $\mathscr{M}_{S,\nu,K}$ in Sect. 4.7. Both functions are compared in Lemma 9. This forms the basis for the behavior of F under the action of the group element $S \in \Gamma$.

Step 6: We discuss matrix-valued functions $\mathscr{U}_{S,v}$, $\mathscr{Q}_{S,v}$ and $\mathscr{Y}_{S,v}$ in see Sect. 4.8. This explains basically how the Fourier expansion of F behave under $S \in \Gamma$.
Step 7: In Sect. 4.9 we summarize the results in the step above. This shows explicitly the transformation behavior of F under S.
Step 8: Extend the transformation property of F under S to all elements $g \in \Gamma$ in Sect. 4.10. This concludes the proof.

4.1 On sets of coprime pairs of integers

Next, we introduce and discuss some sets which are needed in Sect. 3.

Let $c \in \mathbb{N}$ be a positive integer for the moment. We define the set

$$D_c := \big\{d \in \mathbb{Z};\ 0 \le -d < c \text{ and } \gcd(c, d) = 1\big\}, \tag{32}$$

where $\gcd(c, d)$ denotes the greatest common divisor of the integers c and d.

The next step is to define the set D^c for integer $c \in \mathbb{Z}$. We do this by defining three cases $c > 0$, $c < 0$, and $c = 0$.

Consider first the case of positive $c \in \mathbb{N}$. We define the set D^c_+ as the one containing all integers in the c-equivalence classes of all $d \in D_c$:

$$D^c_+ := \big\{d' \in \mathbb{Z};\ \exists d \in D_c \text{ such that } d' \equiv d \pmod{c}\big\}. \tag{33}$$

The set D^c_+ is isomorphic to $\bigcup_{d \in D_c} d + c\mathbb{Z}$.

Next we consider negative c's. We define the set

$$D^c_- := \big\{-d' \in \mathbb{Z};\ \exists d \in D_{-c} \text{ such that } d' \equiv d \pmod{-c}\big\}. \tag{34}$$

We have $D^c_- = -\big(D^{-c}_+\big)$.

As a final extension step, we define the set for $c = 0$:

$$D^0_0 := \big\{(0, 1), (0, -1)\big\}. \tag{35}$$

Summarizing all cases above, we define for integer $c \in \mathbb{Z}$

$$D^c := \begin{cases} D^c_+ & \text{if } c > 0, \\ D^c_- & \text{if } c < 0 \text{ and,} \\ D^0_0 & \text{if } c = 0. \end{cases} \tag{36}$$

The following lemma follows immediately from the inversion on $\Gamma_\infty \backslash \mathrm{SL}(2, \mathbb{Z})$:

Lemma 2 *The map* $(c, d') \mapsto (d', -c)$ *is an automorphism of the set* $\big\{(c, d');\ c \in \mathbb{Z}, d' \in D^c\big\}$.

4.2 $\mathscr{R}_\mu(\tau)$ and the holomorphicity of F

Let

$$\mathscr{R} : \mathbb{H} \to \mathbb{C}^{p\times p}; \quad \tau \mapsto \mathscr{R}_\nu(\tau) = \left(\mathscr{R}_\nu^{(j,l)}(\tau)\right)_{1\le j,l\le p}$$

be the matrix-valued function defined by its entries

$$\mathscr{R}_\nu^{(j,l)}(\tau) = \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l} e^{2\pi i(m+m_j)\tau}. \tag{37}$$

Lemmas 4 and 5 show that $\mathscr{R}_\nu^{(j,l)}(\tau)$ converges absolutely in m and in c. Therefore, we can change the order of summation in (28) and rewrite the jth component of $F(\tau)$ in the form

$$\begin{aligned} F_j(\tau) &= \sum_{\mu\le\nu<0} b_\nu(j)\, e^{2\pi i(m_j+\nu)\tau} \\ &\quad + 2\pi \sum_{m=0}^{\infty}\sum_{c=1}^{\infty} \frac{i^k}{c} \sum_{l=1}^{p} \sum_{\mu\le\nu<0} b_\nu(l) A_{c,\nu,m,j,l}\, B_{c,\nu,m,j,l}\, e^{2\pi i(m+m_j)\tau} \\ &= \sum_{\mu\le\nu<0} b_\nu(j)\, e^{2\pi i(m_j+\nu)\tau} \\ &\quad + \sum_{\mu\le\nu<0} \sum_{l=1}^{p} b_\nu(l) \mathscr{R}_\nu^{(j,l)}(\tau). \end{aligned} \tag{38}$$

Since the series converges uniformly on compact sets of $\mathbb{H}$ with respect to the hyperbolic metric by Lemma 5, $F(\tau)$ is holomorphic on $\mathbb{H}$.

4.3 Auxiliary lemmas I

Lemma 3 *For $k \in \mathbb{Z}_{\ge 0}$ and $z > 0$ we have the estimate*

$$I_{k+1}(z) \le z^k \sinh z. \tag{39}$$

Proof Using the power series expansion for e^z we get

$$\begin{aligned} z^k \sinh z = \frac{e^z - e^{-z}}{2} &= z^k \sum_{m=0}^{\infty} \frac{z^m}{m!} - \frac{(-z)^m}{m!} \\ &= 2z^k \sum_{\substack{m=0 \\ m \text{ odd}}}^{\infty} \frac{z^m}{m!} = 2\sum_{n=0}^{\infty} \frac{z^{2n+k+1}}{(2n+1)!} \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n+k+1}}{(n+1)!\,(n+k+1)!} \frac{2^{2n+k+2}\,(n+k+1)!}{(n+2)\cdots(2n+1)}.$$

Comparing the last expression termwise to (27),

$$I_{k+1}(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n+k+1}}{n!(n+k+1)!},$$

we see that (39) follows if

$$\left| \frac{2^{2n+k+2}\,(n+k+1)!}{(n+2)\cdots(2n+1)} \right| \geq 1$$

holds. Since the left-hand term is monotone in k, it suffices to prove the case $k = 0$:

$$\frac{2^{2n+2}}{2n+2} \frac{2n+2}{\binom{2n+2}{n+1}} = \left| \frac{2^{2n+2}\,(n+1)!}{(n+2)\cdots(2n+1)} \right| \geq 1,$$

which follows directly from the upper bound on the central binomial coefficient

$$\binom{2n}{n} < \frac{2^{2n}}{\sqrt{\pi n}} \leq 2^{2n},$$

see e.g., the upper bound in [21, Corollary 2.3]. □

Lemma 4 *For $k > 2\delta$, the series*

$$2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\; B_{c,\nu,m,j,l} \tag{40}$$

converges absolutely.

Defining

$$\kappa_m = \min(m_1, \ldots, m_p) \quad \textit{and} \quad \kappa_M = \max(m_1, \ldots, m_p), \tag{41}$$

we have additionally

$$2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\; B_{c,\nu,m,j,l} = \mathscr{O}\left((m+\kappa_m)^{-\frac{3}{4}-\frac{k}{2}} e^{4\pi(\nu-\kappa_m)^{\frac{1}{2}}(m+\kappa_M)^{\frac{1}{2}}} \right) \tag{42}$$

as $m \to \infty$.

Proof The strategy is the same as the arguments leading to [15, Theorem 3.2]. First, we will show that

$$\left|2\pi \sum_{c=2}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l}\right| \leq C(m+m_j)^{-\frac{1}{2}} e^{2\pi(m+m_j)^{\frac{1}{2}}(-\nu-m_l)^{\frac{1}{2}}}. \tag{43}$$

Then we show that as $m \to \infty$ the summation on c is dominated by the term for $c = 1$.

In order to bound $A_{c,\nu,m,j,l}$ in (23), we use (14) to get

$$\begin{aligned}
&\left|A_{c,\nu,m,j,l}\right| \\
&\overset{(23)}{=} \left| \sum_{\substack{d \\ 0\leq -d<c \\ \gcd(c,d)=1}} \Omega_{c,d,l,j}\ e^{2\pi i \frac{dm+a\nu}{c}} \right| \\
&\overset{(24)}{=} \left| \sum_{\substack{d \\ 0\leq -d<c \\ \gcd(c,d)=1}} \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l}\ e^{2\pi i \frac{m_l a+m_j d}{c}}\ e^{2\pi i \frac{dm+a\nu}{c}} \right| \\
&\leq \sum_{\substack{d \\ 0\leq -d<c \\ \gcd(c,d)=1}} \left| \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l}\ e^{2\pi i \frac{m_l a+m_j d}{c}}\ e^{2\pi i \frac{dm+a\nu}{c}} \right| \\
&\overset{(7)}{=} \sum_{\substack{d \\ 0\leq -d<c \\ \gcd(c,d)=1}} \left| \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} \right| \\
&\overset{(14)}{=} \mathcal{O}\left(c^{2\delta+1}\right).
\end{aligned} \tag{44}$$

On the other hand, from the power series definition (27) of $I_{k+1}(z)$ we have (39). Also, we have

$$\sinh z \leq \frac{z}{B} \sinh B, \quad \text{for } 0 \leq z \leq B, \tag{45}$$

since $\sinh : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, restricted to the positive real line, is a convex function.

Now by (44), (39), and (45) above, the definition of $B_{c,\nu,m,j,l}$ in (25), and the assumption $k > 2\delta$, we find the following two cases: $m + m_j > 0$ and $m + m_j = 0$.

For $m + m_j > 0$ and for suitable positive constants C_1, C_2 we have

$$\begin{aligned}
&\left|2\pi \sum_{c=2}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l}\right| \\
&\leq 2\pi \sum_{c=2}^{\infty} \frac{1}{c} \left|A_{c,\nu,m,j,l}\right| \left|B_{c,\nu,m,j,l}\right|
\end{aligned}$$

$$
\begin{aligned}
&\overset{(44)}{\leq} 2\pi \sum_{c=2}^{\infty} \frac{1}{c}\, C_1\, c^{2\delta+1} \left| B_{c,\nu,m,j,l} \right| \\
&\overset{(25),(39)}{\leq} 2\pi\, C_1 \sum_{c=2}^{\infty} c^{2\delta} \frac{\left(-\nu - m_l\right)^{k+\frac{1}{2}}}{\left(m+m_j\right)^{\frac{1}{2}}} \left(\frac{4\pi}{c}\right)^{k} \\
&\qquad \cdot \sinh\left(\frac{4\pi}{c}(-\nu-m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}\right) \\
&\overset{(45)}{\leq} \frac{1}{2}(4\pi)^{k+1}\, C_1 \sum_{c=2}^{\infty} c^{2\delta-k} \frac{\left(-\nu - m_l\right)^{k+\frac{1}{2}}}{\left(m+m_j\right)^{\frac{1}{2}}} \frac{2}{c} \\
&\qquad \cdot \sinh\left(2\pi(-\nu-m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}\right) \\
&= (4\pi)^{k+1}\, C_1 \frac{\left(-\nu - m_l\right)^{k+\frac{1}{2}}}{\left(m+m_j\right)^{\frac{1}{2}}} \\
&\qquad \cdot \sinh\left(2\pi(-\nu-m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}\right) \sum_{c=2}^{\infty} c^{2\delta-k-1} \\
&\leq C_2 \left(m+m_j\right)^{-\frac{1}{2}} e^{2\pi(-\nu-m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}}.
\end{aligned}
$$

We also used the trivial estimate $\sinh z \leq e^z$ in the last line. We also remark that the term $-\nu - m_l$ is positive since $\nu \in \mathbb{Z}_{<0}$ and $0 \leq m_l < 1$ by the assumptions around (17).
For $m + m_j = 0$ and for suitable constants C_1, C_3 we have

$$
\begin{aligned}
&\left| 2\pi \sum_{c=2}^{\infty} \frac{i^k}{c} A_{c,\nu,0,j,l}\, B_{c,\nu,0,j,l} \right| \\
&\leq 2\pi \sum_{c=2}^{\infty} \frac{1}{c} \left| A_{c,\nu,0,j,l} \right| \left| B_{c,\nu,0,j,l} \right| \\
&\overset{(44)}{\leq} 2\pi \sum_{c=2}^{\infty} \frac{1}{c}\, C_1\, c^{2\delta+1} \left| B_{c,\nu,0,j,l} \right| \\
&\overset{(26)}{\leq} 2\pi\, C_1 \sum_{c=2}^{\infty} c^{2\delta} \left| \frac{1}{(k+1)!} \left(\frac{2\pi(-\nu-m_l)}{c}\right)^{k+1} \right| \\
&= \frac{2\pi\, C_1}{(k+1)!} \left(2\pi(-\nu-m_l)\right)^{k+1} \sum_{c=2}^{\infty} c^{2\delta-k-1} \\
&\leq \frac{2\pi\, C_3 \left(2\pi(-\nu-m_l)\right)^{k+1}}{(k+1)!}.
\end{aligned}
$$

Note that the constants C_1, C_2, and C_3 depend on k.

Now, we discuss the remaining term $c = 1$ of the series in (43). The defining equations (23) and (24) with $c = 1$ imply

$$\begin{aligned} A_{1,\nu,m,j,l} &\overset{(23)}{=} \Omega_{1,0,l,j}\, e^{2\pi i\, a\nu} \\ &\overset{(24)}{=} \Big(\varepsilon\big(0,-1,1,0\big)\Big)^{-1} \Big(\rho\big(0,-1,1,0\big)\Big)^{-1}_{j,l}\, e^{2\pi i\, m_l a} e^{2\pi i\, a\nu} \end{aligned}$$

since $1 = c > -d \geq 0$ and $d = 0$ and $0 \leq a < c$ implies $a = d = 0$. Hence $\gamma_{1,0} = S$ holds. However, we will keep writing a in our formulae instead of simplifying the terms by omitting vanishing terms. We have

$$\begin{aligned} &2\pi\, i^k\, A_{1,\nu,m,j,l}\, B_{1,\nu,m,j,l} \\ &\quad = 2\pi\, i^k \left(\varepsilon(S)\right)^{-1} \left(\rho(S)\right)^{-1}_{j,l} e^{2\pi i\, m_l a} e^{2\pi i\, a\nu}\,. \\ &\quad\cdot \begin{cases} \left(\frac{-\nu-m_l}{m+m_j}\right)^{\frac{k+1}{2}} I_{k+1}\left(4\pi(-\nu-m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}\right) & \text{if } m+m_j > 0 \text{ and} \\ \frac{1}{(k+1)!}\big(2\pi(-\nu-m_l)\big)^{k+1} & \text{if } m+m_j = 0. \end{cases} \end{aligned} \tag{46}$$

We consider the cases $m + m_j > 0$ and $m + m_j = 0$ separately.

Assume $m + m_j > 0$. Using

$$I_{k+1}(z) \sim \frac{e^z}{\sqrt{2\pi z}}, \tag{47}$$

see [22] or alternatively [7, (10.30.4)], we have

$$\begin{aligned} &2\pi\, i^k\, A_{1,\nu,m,j,l}\, B_{1,\nu,m,j,l} \\ &\overset{(46)}{=} 2\pi\, i^k \left(\varepsilon(S)\right)^{-1} \left(\rho(S)\right)^{-1}_{j,l} e^{2\pi i\, m_l a} e^{2\pi i\, a\nu} \\ &\qquad\cdot \left(\frac{-\nu-m_l}{m+m_j}\right)^{\frac{k+1}{2}} I_{k+1}\left(4\pi(-\nu-m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}\right) \\ &\overset{(47)}{=} \mathcal{O}\left(\left(\rho(S)\right)^{-1}_{j,l} \left(-\nu-m_l\right)^{\frac{k}{2}+\frac{1}{4}} \left(m+m_j\right)^{-\frac{k}{2}-\frac{3}{4}} e^{\left(4\pi(-\nu-m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}\right)}\right) \\ &\overset{(14)}{=} \mathcal{O}\left(\left(m+m_j\right)^{-\frac{k}{2}-\frac{3}{4}} e^{\left(4\pi(-\nu-m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}\right)}\right), \end{aligned}$$

where the implicit constant may depend on k, ν, and m_l.
Assume $m + m_j = 0$. In this case we get

$$\begin{aligned} &2\pi\, i^k\, A_{1,\nu,m,j,l}\, B_{1,\nu,m,j,l} \\ &\overset{(46)}{=} 2\pi\, i^k \left(\varepsilon(S)\right)^{-1} \left(\rho(S)\right)^{-1}_{j,l} e^{2\pi i\, m_l a} e^{2\pi i\, a\nu} \end{aligned}$$

$$\cdot \frac{1}{(k+1)!}\left(2\pi(-v-m_l)\right)^{k+1}$$
$$= \mathscr{O}\left(\left(\rho(S)\right)^{-1}_{j,l}\left(-v-m_l\right)^{k+1}\right)$$
$$= \mathscr{O}\left(\left(-v-m_l\right)^{k+1}\right).$$

Summarizing, we see that the series

$$2\pi\sum_{c=1}^{\infty}\frac{i^k}{c}A_{c,v,m,j,l}\ B_{c,v,m,j,l}$$

in (43) converges absolutely. We have

$$2\pi\sum_{c=1}^{\infty}\frac{i^k}{c}A_{c,v,m,j,l}\ B_{c,v,m,j,l} = \mathscr{O}\left((m+\kappa_m)^{-\frac{3}{4}-\frac{k}{2}}e^{4\pi(-v-\kappa_m)^{\frac{1}{2}}(m+\kappa_M)^{\frac{1}{2}}}\right).$$

This concludes the proof of Lemma 4. □

A direct consequence of the lemma above is the following

Lemma 5 *Let $w > 0$ be a given positive constant. The series*

$$\sum_{m=0}^{\infty}\left|2\pi\sum_{c=1}^{\infty}\frac{i^k}{c}A_{c,v,m,j,l}\ B_{c,v,m,j,l}e^{2\pi i(m+m_j)\tau}\right| \tag{48}$$

converges absolutely for $\tau \in \mathbb{H}$.

The convergence is uniform for $\tau \in I_w = \{\tau \in \mathbb{H};\ \mathrm{Im}\,(\tau) > w\}$.

Proof We only have to prove that the series in (48) converges for $m \in \mathbb{N}$ (and, therefore, omitting the term $m = 0$).

For $m > 0$ we have

$$\left|2\pi\sum_{c=1}^{\infty}\frac{i^k}{c}A_{c,v,m,j,l}\ B_{c,v,m,j,l}e^{2\pi i(m+m_j)\tau}\right|$$
$$\overset{(42)}{=} \mathscr{O}\left((m+\kappa_m)^{-\frac{3}{4}-\frac{k}{2}}\,e^{4\pi(-v-\kappa_m)^{\frac{1}{2}}(m+\kappa_M)^{\frac{1}{2}}}\,e^{2\pi i(m+m_j)\tau}\right)$$
$$\overset{\mathrm{Im}(\tau)>w}{=} \mathscr{O}\left((m+\kappa_m)^{-\frac{3}{4}-\frac{k}{2}}\,e^{4\pi(-v-\kappa_m)^{\frac{1}{2}}(m+\kappa_M)^{\frac{1}{2}}}\,e^{-2\pi(m+m_j)\mathrm{Im}(\tau)}\right)$$
$$\overset{m_j\geq 0}{=} \mathscr{O}\left(m^{-\frac{3}{4}-\frac{k}{2}}\,e^{4\pi(m+1)^{\frac{1}{2}}}\,e^{-2\pi\mathrm{Im}(\tau)m}\right).$$

We have used that v, m_j, κ_m, and κ_M are constants regarding the m-dependency. The last estimate implies that the series in (48) converges absolutely, since the term $e^{-2\pi m\mathrm{Im}(\tau)}$ dominates.

The trivial estimate $e^{-2\pi m \mathrm{Im}(\tau)} \leq e^{-2\pi m w}$ for $\tau \in I_w$ implies that the convergence is uniform on I_w. □

4.4 Variants of the Lipschitz summation formula

The *sum-star* notation is defined as the symmetric limit

$$\sum_{q\in\mathbb{Z}}^{\star} := \lim_{N\to\infty} \sum_{q=-N}^{N}. \tag{49}$$

Recall the following variants of the Lipschitz summation formula.

Variant 1: Let $\tau \in \mathbb{H}$ and $n \in \mathbb{Z}_{\geq 0}$ if $m_j \notin \mathbb{Z}$, respectively, $n \in \mathbb{N}$ if $m_j \in \mathbb{Z}$ be a nonnegative, respectively, positive integer. The first variant of the Lipschitz summation formula is

$$\frac{(-2\pi i)^{n+1}}{n!} \sum_{m+m_j>0} (m+m_j)^n\, e^{2\pi i(m+m_j)\tau} = \begin{cases} \sum_{q\in\mathbb{Z}} \frac{e^{-2\pi i m_j q}}{(\tau+q)^{n+1}} & \text{if } n \in \mathbb{N} \text{ and} \\ \sum_{q\in\mathbb{Z}}^{\star} \frac{e^{-2\pi i m_j q}}{\tau+q} & \text{if } n = 0, \end{cases} \tag{50}$$

where the summation index m runs either through $m \in \mathbb{N}$ if $m_j \in \mathbb{Z}$ or through $m \in \mathbb{Z}_{\geq 0}$ if $m_j \in \mathbb{R} \smallsetminus \mathbb{Z}$, see e.g., [13].
Variant 2: We have

$$-2\pi i \sum_{m=1}^{\infty} e^{2\pi i m\tau} = \pi i + \sum_{q\in\mathbb{Z}}^{\star} \frac{1}{\tau+q} \tag{51}$$

for $n = 0$ and $m_j = 0$, see [9, (2.29)] and [17, (2.9)].

4.5 $\mathscr{T}_\nu(\tau)$, $\mathscr{K}_\mu$, and $\mathscr{W}_\mu(\tau)$

As (38) indicates we may rewrite $F(\tau)$ as

$$F(\tau) = \sum_{\mu\leq\nu<0} \mathscr{T}_\nu(\tau) \begin{pmatrix} b_\nu(1) \\ \vdots \\ b_\nu(p) \end{pmatrix}, \tag{52}$$

where $\mathscr{T}_\nu(\tau)$ is the matrix-valued function

$$\mathscr{T} : \mathbb{H} \to \mathbb{C}^{p\times p}; \quad \tau \mapsto \mathscr{T}_\nu(\tau) = \left(\mathscr{T}_\nu^{(j,l)}(\tau)\right)_{1\leq j,l\leq p}$$

given by

$$\mathscr{T}_\nu^{(j,l)}(\tau) = \delta_{j,l} e^{2\pi i(m_j+\nu)\tau} + \mathscr{R}_\nu^{(j,l)}(\tau), \tag{53}$$

where $\delta_{j,l}$ denotes the Kronecker delta.

For given coprime $c, d \in \mathbb{Z}$ satisfying $0 \le -d < c$ we denote by $\gamma_{c,d}$ the unique matrix given in (13):

$$\gamma_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad \text{satisfying } 0 \le -d < c \text{ and } 0 \le a < c.$$

Recall D_c in (32). We have obviously

$$\begin{aligned} D_c &= \left\{ d \in \mathbb{Z};\ 0 \le -d < c \text{ and } \gcd(c,d) = 1 \right\} \qquad (54) \\ &= \left\{ d;\ \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma;\ 0 \le -d < c, 0 \le a < c \right\} \\ &= \left\{ d;\ \exists \gamma_{c,d} \text{ given in (13)} \right\}, \qquad (55) \end{aligned}$$

which is a particular set of representatives of $d \mod c$ with $\gcd(c,d) = 1$ and $0 \le -d < c$.

From now on, we assume that τ lies on the upper imaginary ray: $\tau \in i\mathbb{R}_{>0} \subset \mathbb{H}$. Later on, we extend the obtained results by analytic continuation to all $\tau \in \mathbb{H}$.

Using (37), (23) and the absolute convergence of the double series $\mathscr{R}_v^{(j,l)}$ in m and c we find

$$\begin{aligned} \mathscr{R}_v^{(j,l)}(\tau) &= \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,v,m,j,l}\, B_{c,v,m,j,l}\, e^{2\pi i (m+m_j)\tau} \\ &= \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} \sum_{\substack{d \\ 0 \le -d < c \\ \gcd(c,d)=1}} \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} \cdot \\ &\qquad \cdot e^{2\pi i \frac{m_l a + m_j d}{c}}\, e^{2\pi i \frac{dm + av}{c}}\, B_{c,v,m,j,l}\, e^{2\pi i (m+m_j)\tau} \\ &= 2\pi \sum_{c=1}^{\infty} \frac{1}{c} \sum_{\substack{d \\ 0 \le -d < c \\ \gcd(c,d)=1}} \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} \cdot \qquad (56) \\ &\qquad \cdot e^{2\pi i a \frac{m_l + v}{c}}\, i^k \sum_{m=0}^{\infty} B_{c,v,m,j,l}\, e^{2\pi i (m+m_j)(\tau + \frac{d}{c})}. \end{aligned}$$

(56) and Lemma 6 imply that

$$\begin{aligned} \mathscr{R}_v^{(j,l)}(\tau) = \mathscr{K}_v^{(j,l)} + \sum_{c=1}^{\infty} \sum_{d \in D_c} \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l + v}{c}} \cdot \\ \cdot \sum_{q \in \mathbb{Z}}^{\star} e^{2\pi i m_j q} (c\tau + d - cq)^k \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-v - m_l)}{c(c\tau + d - cq)} \right)^n, \qquad (57) \end{aligned}$$

where

$$\mathscr{K}_\nu^{(j,l)} = \begin{cases} \sum_{c=1}^{\infty} \frac{\pi i^k}{c} A_{c,\nu,0,j,l} B_{c,\nu,0,j,l} & \text{if } m_j = 0 \text{ and} \\ 0 & \text{otherwise} \end{cases} \tag{58}$$

Lemma 4 shows that the series in (58) converges absolutely.

Now, for $q \in \mathbb{Z}$ define $d' = d - cq$ and

$$\gamma_{c,d'} := \gamma_{c,d}\, T^{-q} = \begin{pmatrix} a & \star \\ c & d' \end{pmatrix}. \tag{59}$$

As q runs through all integers and as d runs through the set D_c, d' assumes exactly once each value in the set D^c defined in (36). This leads to the set identity

$$\begin{aligned} D^c &\overset{(33)}{=} \left\{ d' \in \mathbb{Z};\ \exists d \in D_c \text{ such that } d' \equiv d \pmod{c} \right\} \\ &\overset{(59)}{=} \left\{ d';\quad \exists \gamma_{c,d'} = \begin{pmatrix} a & \star \\ c & d' \end{pmatrix} \in \Gamma,\ 0 \le a < c \right\}. \end{aligned} \tag{60}$$

The definition of $\gamma_{c,d'}$ in (59) implies

$$\begin{aligned} &\left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1} \\ &\overset{(59)}{=} \left(\varepsilon(\gamma_{c,d'} T^q)\right)^{-1} \left(\rho(\gamma_{c,d'} T^q)\right)^{-1} \\ &\overset{(6),(8)}{=} \left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(\varepsilon(T^q)\right)^{-1} \left(\rho(T^q)\right)^{-1} \left(\rho(\gamma_{c,d'})\right)^{-1} \\ &\overset{(9)}{=} \left(\varepsilon(\gamma_{c,d'})\right)^{-1} \begin{pmatrix} e^{-2\pi i m_1 q} & & \\ & \ddots & \\ & & e^{-2\pi i m_p q} \end{pmatrix} \left(\rho(\gamma_{c,d'})\right)^{-1}. \end{aligned}$$

Therefore, we find the identity

$$\left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)_{j,l}^{-1} = \left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(\rho(\gamma_{c,d'})\right)_{j,l}^{-1} e^{-2\pi i m_j q} \tag{61}$$

for the (j,l)th entry.

Now, define the matrix-valued function $\mathscr{W}_\nu(\tau) = \left(\mathscr{W}_\nu^{(j,l)}(\tau)\right)_{1 \le j,l \le p}$ by its entries

$$\begin{aligned} \mathscr{W}_\nu^{(j,l)}(\tau) &:= \delta_{j,l}\, e^{2\pi i (m_j + \nu)\tau} + \mathscr{R}_\nu^{(j,l)}(\tau) - \mathscr{K}_\nu^{(j,l)} \\ &\overset{(53)}{=} \mathscr{T}_\nu^{(j,l)}(\tau) - \mathscr{K}_\nu^{(j,l)}, \end{aligned} \tag{62}$$

and write $\mathscr{W}_\nu^{(j,l)}(\tau)$ using (57):

$$\mathscr{W}_\nu^{(j,l)}(\tau) - \delta_{j,l}\, e^{2\pi i (m_j + \nu)\tau} \tag{63}$$

$$\overset{(57)}{=} \sum_{c=1}^{\infty} \sum_{d\in D_c} \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}}$$

$$\cdot \sum_{q\in\mathbb{Z}}^{\star} e^{2\pi i m_j q} (c\tau + d - cq)^k \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(c\tau + d - cq)}\right)^n$$

$$\overset{(59),(61)}{=} \sum_{c=1}^{\infty} \lim_{N\to\infty} \sum_{\substack{d'\in D^c \\ |d'|\le N}} \left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(rho(\gamma_{c,d'})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}} \left(c\tau + d'\right)^k$$

$$\times \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(c\tau + d')}\right)^n.$$

To continue, recall that we assumed $\tau = iy$ with $y > 0$. We will show that the series

$$\sum_{c=1}^{\infty} \lim_{N\to\infty} \sum_{\substack{d'\in D^c \\ |d'|\le N}} \frac{\left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(\rho(\gamma_{c,d'})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}}}{c^{k+1}\,(ciy + d')} \tag{64}$$

converges. (Note that (64) is in fact the term with $n = k + 1$ in (63).) To do so we write (64) as

$$\sum_{c=1}^{\infty} \lim_{N\to\infty} \sum_{\substack{d'\in D^c \\ |d'|\le N}} \frac{\left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(\rho(\gamma_{c,d'})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}}}{c^{k+1}\,(ciy + d')}$$

$$= \sum_{c=1}^{\infty} \sum_{d\in D_c} \sum_{q\in\mathbb{Z}}^{\star} \frac{\left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}} e^{2\pi i m_j q}}{c^{k+2}\left(iy + \frac{d}{c} - q\right)}$$

$$= \sum_{c=1}^{\infty} \sum_{d\in D_c} \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}} \frac{1}{c^{k+2}} \sum_{q\in\mathbb{Z}}^{\star} \frac{e^{2\pi i m_j q}}{iy + \frac{d}{c} - q}, \tag{65}$$

where we used (59), (61), and the relation between the sets D^c and D_c, which is mentioned in (33) and (36).

Next, we apply the Lipschitz summation formula (50), respectively, (51) to the cases $m_j > 0$ and $m_j = 0$.

Assume $m_j > 0$. Applying (50) for $n = 0$ and $\tau = iy + \frac{d}{c}$ to (65) we get

(65)

$$= \sum_{c=1}^{\infty} \sum_{d\in D_c} \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}} \frac{1}{c^{k+2}} \sum_{q\in\mathbb{Z}}^{\star} \frac{e^{2\pi i m_j q}}{iy + \frac{d}{c} - q}$$

$$\overset{(50)}{=} \sum_{c=1}^{\infty} \sum_{d \in D_c} \frac{-2\pi i}{c^{k+2}} \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}} \sum_{m=0}^{\infty} e^{2\pi i (m+m_j)\left(iy+\frac{d}{c}\right)}.$$

In the case of $m_j = 0$ we use the variant (51) of Lipschitz summation formula. Starting again from (65), we get

(65)

$$\begin{aligned}
&= \sum_{c=1}^{\infty} \sum_{d \in D_c} \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}} \frac{1}{c^{k+2}} \sum_{q \in \mathbb{Z}}^{\star} \frac{e^{2\pi i m_j q}}{iy + \frac{d}{c} - q} \\
&\overset{(51)}{=} \sum_{c=1}^{\infty} \sum_{d \in D_c} \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}} \frac{1}{c^{k+2}} \\
&\quad \cdot \left(-\pi i - 2\pi i \sum_{m=1}^{\infty} e^{2\pi i m \left(iy+\frac{d}{c}\right)}\right) \\
&= \left(\sum_{c=1}^{\infty} \sum_{d \in D_c} \frac{\pi i}{c^{k+2}} \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}}\right) \\
&\quad + \left(\sum_{c=1}^{\infty} \sum_{d \in D_c} \frac{-2\pi i}{c^{k+2}} \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}} \sum_{m=0}^{\infty} e^{2\pi i m \left(iy+\frac{d}{c}\right)}\right).
\end{aligned}$$

Next, we will show that the following series

$$\begin{aligned}
&\sum_{n=k+2}^{\infty} \sum_{c=1}^{\infty} \lim_{N \to \infty} \sum_{\substack{d' \in D^c \\ |d'| \le N}} \left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(\rho(\gamma_{c,d'})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}} \\
&\quad \cdot \left(c\tau + d'\right)^k \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(c\tau + d')}\right)^n
\end{aligned} \tag{66}$$

is an absolutely convergent triple sum. (Note that (66) is in fact the terms with $n > k+1$ in (63).) To do so, we rewrite (66) as

$$\begin{aligned}
(66) &\overset{(59)}{=} \sum_{n=k+2}^{\infty} \frac{\left(2\pi i(-\nu - m_l)\right)^n}{n!} \\
&\quad \cdot \sum_{c=1}^{\infty} \sum_{d \in D_c} \frac{\left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}}}{c^{2n-k}} \sum_{q \in \mathbb{Z}} \frac{e^{2\pi i m_j q}}{\left(\tau + \frac{d}{c} - q\right)^{n-k}}.
\end{aligned}$$

Lemma 7 implies that the above representation converges absolutely. Hence (66) is indeed an absolutely convergent triple series.

Combining the absolute convergence of the expressions in (64) and (66) we may rewrite the expression of $\mathscr{W}_{\nu}^{(j,l)}(\tau) - \delta_{j,l}\, e^{2\pi i(m_j+\nu)\tau}$ in (63) as "(64) + (66)":

$$\begin{aligned}
&\mathscr{W}_{\nu}^{(j,l)}(\tau) - \delta_{jl} e^{2\pi i(m_j+\nu)\tau} \qquad (67)\\
&= \Bigg(\frac{\big(2\pi i(-\nu - m_l)\big)^{k+1}}{(k+1)!} \\
&\qquad \cdot \sum_{c=1}^{\infty} \lim_{N\to\infty} \sum_{\substack{d'\in D^c \\ |d'|\le N}} \frac{\big(\varepsilon(\gamma_{c,d'})\big)^{-1} \big(\rho(\gamma_{c,d'})\big)^{-1}_{j,l}\, e^{2\pi i a \frac{m_l+\nu}{c}}}{c^{k+1}\big(ciy + d'\big)} \Bigg) \\
&\qquad + \sum_{c=1}^{\infty} \lim_{N\to\infty} \sum_{\substack{d'\in D^c \\ |d'|\le N}} \big(\varepsilon(\gamma_{c,d'})\big)^{-1} \big(\rho(\gamma_{c,d'})\big)^{-1}_{j,l}\, e^{2\pi i a \frac{m_l+\nu}{c}} \big(c\tau + d'\big)^k \\
&\qquad \cdot \sum_{n=k+2}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c\big(c\tau + d'\big)} \right)^n .
\end{aligned}$$

Now, using Lemma 8 with $t = 1$, we rewrite the expression in (67) as

$$\begin{aligned}
&\mathscr{W}_{\nu}^{(j,l)}(\tau) - \delta_{j,l}\, e^{2\pi i(m_j+\nu)\tau} \\
&\overset{(71)}{=} \lim_{K\to\infty} \sum_{\substack{c\in\mathbb{Z} \\ 0<c\le K}} \sum_{\substack{d'\in D^c \\ |d'|\le K}} \frac{\big(\varepsilon(\gamma_{c,d'})\big)^{-1} \big(\rho(\gamma_{c,d'})\big)^{-1}_{j,l}\, e^{2\pi i a \frac{m_l+\nu}{c}}}{c^{k+1}\,(c\tau + d')} \\
&\qquad \cdot \frac{\big(2\pi i(-\nu - m_l)\big)^{k+1}}{(k+1)!} \\
&\qquad + \lim_{K\to\infty} \sum_{\substack{c\in\mathbb{Z} \\ 0<c\le K}} \sum_{\substack{d'\in D^c \\ |d'|\le K}} \big(\varepsilon(\gamma_{c,d'})\big)^{-1} \big(rho(\gamma_{c,d})\big)^{-1}_{j,l}\, e^{2\pi i a \frac{m_l+\nu}{c}} (c\tau + d')^k \\
&\qquad \cdot \sum_{n=k+2}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(c\tau + d')} \right)^n .
\end{aligned}$$

We collect the summands and get

$$\begin{aligned}
&\mathscr{W}_{\nu}^{(j,l)}(\tau) - \delta_{jl} e^{2\pi i(m_j+\nu)\tau} \\
&= \lim_{K\to\infty} \sum_{\substack{c\in\mathbb{Z} \\ 0<c\le K}} \sum_{\substack{d'\in D^c \\ |d'|\le K}} \big(\varepsilon(\gamma_{c,d'})\big)^{-1} \big(\rho(\gamma_{c,d'})\big)^{-1}_{j,l}\, e^{2\pi i a \frac{m_l+\nu}{c}} (c\tau + d')^k \\
&\qquad \cdot \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(c\tau + d')} \right)^n
\end{aligned}$$

$$= \lim_{K\to\infty} \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le K}} \sum_{\substack{d'\in D^c\\ |d'|\le K}} \left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(\rho(\gamma_{c,d'})\right)^{-1}_{j,l} e^{2\pi i a\frac{m_l+\nu}{c}} \left(c\tau+d'\right)^k$$
$$\cdot \left(e^{\frac{2\pi i(-\nu-m_l)}{c(c\tau+d')}} - \sum_{n=0}^{k} \frac{1}{n!} \left(\frac{2\pi i\,(-\nu-m_l)}{c(c\tau+d')} \right)^n \right),$$

where we used the power series expansion of the exponential function

$$e^x - \sum_{n=0}^{k} \frac{x^k}{n!} = \sum_{n=k+1}^{\infty} \frac{x^n}{n!}.$$

4.6 Auxiliary lemmas II

The following lemma gives a value for the summation of the special functions defined in Theorem 1. It was used in the previous part.

Lemma 6 *Depending on whether* $0 \le m_j < 1$ *vanishes or not the term*

$$\frac{2\pi}{c} i^k \sum_{m=0}^{\infty} B_{c,\nu,m,j,l}\, e^{2\pi i(m+m_j)(\tau+\frac{d}{c})}$$

If $0 < m_j < 1$ *holds we have*

$$\frac{2\pi}{c} i^k \sum_{m=0}^{\infty} B_{c,\nu,m,j,l}\, e^{2\pi i(m+m_j)(\tau+\frac{d}{c})}$$
$$= \sum_{q\in\mathbb{Z}} e^{2\pi i m_j q}\ (c\tau+d-cq)^k \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu-m_l)}{c(c\tau+d-cq)} \right)^n. \quad (68)$$

If we assume $m_j = 0$ *we have*

$$\frac{2\pi}{c} i^k \sum_{m=0}^{\infty} B_{c,\nu,m,j,l} e^{2\pi i(m+m_j)(\tau+\frac{d}{c})}$$
$$= \frac{-\pi i^k}{c} B_{c,\nu,0,j,l}$$
$$+ \sum_{q\in\mathbb{Z}}^{\star} e^{2\pi i q m_j}\ (c\tau+d-cq)^k \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu-m_l)}{c(c\tau+d-cq)} \right)^n. \quad (69)$$

The used sum-star notation is defined in (49).

Proof The proof is a simple application of the Lipschitz summation formula given in (50) and in (51). Furthermore, we need the definition of $B_{c,\nu,m,j,l}$ in (25) and the power series expansion of $I_{k+1}(z)$ in (27). We consider the cases $m_j > 0$ and $m_j = 0$ separately.

1. We assume $m_j > 0$ which implies that $m + m_j > 0$ for all $m \in \mathbb{Z}_{\geq 0}$. Hence we have

$$\begin{aligned}
&\frac{2\pi}{c} i^k \sum_{m=0}^{\infty} B_{c,\nu,m,j,l}\, e^{2\pi i(m+m_j)(\tau+\frac{d}{c})} \\
&\overset{(25),(27)}{=} \frac{2\pi}{c} i^k \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+1} \frac{(-\nu-m_l)^{n+k+1}}{n!(n+k+1)!} \\
&\qquad \cdot (m+m_j)^n\, e^{2\pi i(m+m_j)(\tau+\frac{d}{c})} \\
&= i^k \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+2} \frac{(-\nu-m_l)^{n+k+1}}{(n+k+1)!\,(-2\pi i)^{n+1}} \\
&\qquad \cdot \frac{(-2\pi i)^{n+1}}{n!} \sum_{m=0}^{\infty} (m+m_j)^n\, e^{2\pi i(m+m_j)(\tau+\frac{d}{c})} \\
&\overset{(50)}{=} i^k \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+2} \frac{(-\nu-m_l)^{n+k+1}}{(n+k+1)!\,(-2\pi i)^{n+1}} \\
&\qquad \cdot \sum_{q\in\mathbb{Z}} \frac{e^{-2\pi i m_j q}}{\left(\tau+\frac{d}{c}+q\right)^{n+1}} \\
&= \sum_{n=0}^{\infty} \left(\frac{2\pi i}{c}\right)^{n+k+1} \frac{(-\nu-m_l)^{n+k+1}}{(n+k+1)!} \sum_{q\in\mathbb{Z}} \frac{e^{-2\pi i m_j q}}{(c\tau+d+cq)^{n+1}} \\
&\overset{n\mapsto n-k-1}{=} \sum_{n=k+1}^{\infty} \left(\frac{2\pi i}{c}\right)^{n} \frac{(-\nu-m_l)^{n}}{n!} \sum_{q\in\mathbb{Z}} \frac{e^{-2\pi i m_j q}}{(c\tau+d+cq)^{n-k}} \\
&\overset{q\mapsto -q}{=} \sum_{q\in\mathbb{Z}} (c\tau+d-cq)^k\; e^{2\pi i m_j q} \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu-m_l)}{c(c\tau+d-cq)}\right)^n.
\end{aligned}$$

2. Assume $m_j = 0$. However, we still write the symbol m_j to be able to compare the formulae below to the ones derived in the case above. Doing this allows us to indicate the influence of the constant m_j to the two cases.
By the same arguments as above, we have

$$\begin{aligned}
&\frac{2\pi}{c} i^k \sum_{m=1}^{\infty} B_{c,\nu,m,j,l}\, e^{2\pi i(m+m_j)(\tau+\frac{d}{c})} \\
&\overset{(25),(27)}{=} i^k \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+2} \frac{(-\nu-m_l)^{n+k+1}}{(n+k+1)!\,(-2\pi i)^{n+1}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \frac{(-2\pi i)^{n+1}}{n!} \sum_{m=1}^{\infty} (m+m_j)^n \, e^{2\pi i (m+m_j)(\tau+\frac{d}{c})} \\
&= i^k \left(\frac{2\pi}{c}\right)^{k+2} \frac{(-\nu-m_l)^{k+1}}{(k+1)!\,(-2\pi i)} \\
&\quad \cdot \frac{(-2\pi i)}{1} \sum_{m=1}^{\infty} (m+m_j)^0 \, e^{2\pi i (m+m_j)(\tau+\frac{d}{c})} \\
&\quad + i^k \sum_{n=1}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+2} \frac{(-\nu-m_l)^{n+k+1}}{(n+k+1)!\,(-2\pi i)^{n+1}} \\
&\quad \cdot \frac{(-2\pi i)^{n+1}}{n!} \sum_{m=1}^{\infty} (m+m_j)^n \, e^{2\pi i (m+m_j)(\tau+\frac{d}{c})} \\
&\overset{(50),(51)}{=} \frac{(2\pi i)^{k+1}}{c^{k+2}} \frac{(-\nu-m_l)^{k+1}}{(k+1)!} \left(\pi i + \sum_{q\in\mathbb{Z}}{}^{\star} \frac{e^{-2\pi i m_j q}}{\tau+\frac{d}{c}+q}\right) \\
&\quad + \sum_{n=1}^{\infty} \left(\frac{2\pi i}{c}\right)^{n+k+1} \frac{(-\nu-m_l)^{n+k+1}}{(n+k+1)!} \sum_{q\in\mathbb{Z}} \frac{e^{-2\pi i m_j q}}{(c\tau+d+cq)^{n+1}} \\
&\overset{(26)}{=} -\frac{\pi i^{k+1}}{c} B_{c,\nu,0,j,l} + \left(\frac{2\pi i}{c}\right)^{k+1} \frac{(-\nu-m_l)^{k+1}}{(k+1)!} \sum_{q\in\mathbb{Z}}{}^{\star} \frac{e^{-2\pi i m_j q}}{(c\tau+d+cq)} \\
&\quad + \sum_{n=1}^{\infty} \left(\frac{2\pi i}{c}\right)^{n+k+1} \frac{(-\nu-m_l)^{n+k+1}}{(n+k+1)!} \sum_{q\in\mathbb{Z}} \frac{e^{-2\pi i m_j q}}{(c\tau+d+cq)^{n+1}}.
\end{aligned}
$$

Using $\sum_{q\in\mathbb{Z}} \frac{e^{-2\pi i m_j q}}{(c\tau+d+cq)^{n+1}} = \sum_{q\in\mathbb{Z}}{}^{\star} \frac{e^{-2\pi i m_j q}}{(c\tau+d+cq)^{n+1}}$ we get furthermore

$$
\begin{aligned}
&\frac{2\pi}{c} i^k \sum_{m=1}^{\infty} B_{c,\nu,m,j,l} \, e^{2\pi i (m+m_j)(\tau+\frac{d}{c})} \\
&= -\frac{\pi i^{k+1}}{c} B_{c,\nu,0,j,l} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{2\pi i}{c}\right)^{n+k+1} \frac{(-\nu-m_l)^{n+k+1}}{(n+k+1)!} \sum_{q\in\mathbb{Z}}{}^{\star} \frac{e^{-2\pi i m_j q}}{(c\tau+d+cq)^{n+1}} \\
&\overset{n+k+1\mapsto n}{=} -\frac{\pi i^{k+1}}{c} B_{c,\nu,0,j,l} \\
&\quad + \sum_{n=k+1}^{\infty} \left(\frac{2\pi i}{c}\right)^{n} \frac{(-\nu-m_l)^{n}}{n!} \sum_{q\in\mathbb{Z}}{}^{\star} \frac{e^{-2\pi i m_j q}}{(c\tau+d+cq)^{n-k}} \\
&\overset{q\mapsto -q}{=} -\frac{\pi i^{k+1}}{c} B_{c,\nu,0,j,l}
\end{aligned}
$$

$$+ \sum_{q=\in\mathbb{Z}}^{\star} (c\tau + d - cq)^k \; e^{2\pi i m_j q} \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(c\tau + d - cq)} \right)^n .$$

Therefore, we have

$$\begin{aligned} &\frac{2\pi}{c} i^k \sum_{m=0}^{\infty} B_{c,\nu,m,j,l} e^{2\pi i (m+m_j)\left(\tau + \frac{d}{c}\right)} \\ &\quad = \frac{\pi i^k}{c} B_{c,\nu,0,j,l} \\ &\qquad + \sum_{q\in\mathbb{Z}}^{\star} e^{2\pi i q m_j} (c\tau + d - cq)^k \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(c\tau + d - cq)} \right)^n . \end{aligned}$$

This concludes the proof. □

We now prove a modified version of Lemma 2.5 in [17] in order to show that a certain sum in proof of Theorem 2 is convergent.

Lemma 7 *Let k be a weight satisfying $k > 2\delta > 0$ (with δ as in (14)). The series*

$$\sum_{c=1}^{\infty} \left| \sum_{d\in D_c} \left(\varepsilon(\gamma_{c,d})\right)^{-1} \left(\rho(\gamma_{c,d})\right)_{j,l}^{-1} e^{2\pi i a \frac{m_l+\nu}{c}} \right| c^{-k-2} \tag{70}$$

converges. Here, the inner sum runs through the elements of the set D_c given in (54), and the matrix element $\gamma_{c,d}$ is given in (13) and determines a.

Proof The definition of the set D_c in (32) implies $\sharp D_c \leq c$. Hence we see that

$$\sum_{c=1}^{\infty} \sum_{d\in D_c} c^{-k-2} \leq \sum_{c=1}^{\infty} c^{-k-1}$$

converges for all $k > 0$. Combine this with $k > 2\delta$ and (14) proves the lemma. □

To continue we need an adapted version of Lemma 2.13 in [17] (and its proof in [18]), with some modifications to be applicable in the vector-valued case.

Lemma 8 *Let $\tau = iy$, with $y > 0$, $k > 2\delta$, ν a negative integer, and t a positive integer. Then*

$$\sum_{c=1}^{\infty} \lim_{N\to\infty} \sum_{\substack{d'\in D^c \\ |d'|\leq N}} \frac{\left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(\rho(\gamma_{c,d'})\right)_{j,l}^{-1} e^{2\pi i \frac{a(m_l+\nu)}{c}}}{c^{k+1}(c\tau + d')}$$

$$= \lim_{K\to\infty} \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le tK}} \sum_{\substack{d'\in D^c\\ |d'|\le K}} \frac{\left(\varepsilon(\gamma_{c,d'})\right)^{-1}\left(\rho(\gamma_{c,d'})\right)^{-1}_{j,l} e^{2\pi i \frac{a(m_l+\nu)}{c}}}{c^{k+1}(c\tau+d')}. \tag{71}$$

Here, the inner sums run through the entries of the set D^c given in (60).

Proof Following Rademacher's proof [20], we will show

$$\lim_{K\to\infty} \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le tK}} \lim_{N\to\infty} \sum_{\substack{d\in D^c\\ K<|d|\le N}} \frac{\left(\varepsilon(\gamma_{c,d'})\right)^{-1}\left(\rho(\gamma_{c,d'})\right)^{-1}_{j,l} e^{2\pi i \frac{a(m_l+\nu)}{c}}}{c^{k+1}(c\tau+d)} = 0. \tag{72}$$

Recall that (61) implies

$$\left(\varepsilon(\gamma_{c,d-cq})\right)^{-1}\left(\rho(\gamma_{c,d-cq})\right)^{-1}_{j,l} = \left(\varepsilon(\gamma_{c,d})\right)^{-1}\left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i q m_j}$$

with $\gamma_{c,d} \in D_c$ (and $d' = d - cq$ such that $\gamma_{c,d'} \in D^c$ as defined in (59) and (60)).

First, we will show the result for $m_l > 0$. Rewrite d' as $d' = d - cq$, see (59), and rewrite the inner sum of (61) as

$$\frac{1}{c^{k+1}} \sum_{\substack{d\in D_c\\ K<|d|\le N}} \left(\varepsilon(\gamma_{c,d})\right)^{-1}\left(\rho(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i \frac{a(m_l+\nu)}{c}} \left[S_1 + S_2\right], \tag{73}$$

with

$$S_1 := \sum_{\substack{q\in\mathbb{Z}\\ q<\frac{-K+d}{c}}} \frac{e^{2\pi i q m_j}}{c\tau+d-cq}$$

and

$$S_2 := \sum_{\substack{q\in\mathbb{Z}\\ \frac{K+d}{c}<q}} \frac{e^{2\pi i q m_j}}{c\tau+d-cq}.$$

As in [18, p. 343] we have

$$|S_1| + |S_2| < 2\frac{(t+1)t^{1-s}}{\left|\sin \pi m_j\right| c^{1-s}} K^{-s}, \tag{74}$$

with s given by

$$s := \min\left\{1, \frac{k-2\delta}{2}\right\} > 0$$

and t a positive integer appearing in (71) (and given a priori as stated in the assumption of Lemma 8).

Following [18, p. 343] we obtain the estimate

$$\lim_{N\to\infty}\left|\sum_{\substack{d'\in D^c\\ K<|d|\le N}}\frac{\left(\varepsilon(\gamma_{c,d'})\right)^{-1}\left(\rho(\gamma_{c,d'})\right)^{-1}_{j,l}\,e^{2\pi i\frac{a(m_l+\nu)}{c}}}{c^{k+1}(c\tau+d)}\right|<C^\star K^{-s}\sum_{d\in D_c}\frac{1}{c^{2+s}},$$

where $C^\star$ is a suitable positive constant (taking care of the factor $\frac{2(t+1)t^{1-s}}{|\sin\pi m_j|}$ and the implicit constant K_2 in (14)). The rest of the argument is the same as in [18].

Next, consider the case $m_l=0$. Similar to the previous case, we rewrite the inner limit of (72) as

$$\begin{aligned}&\lim_{N\to\infty}\sum_{\substack{d\in D^c\\ K<|d|\le N}}\frac{\left(\varepsilon(\gamma_{c,d})\right)^{-1}\left(\rho(\gamma_{c,d})\right)^{-1}_{j,l}\,e^{2\pi i\frac{a(m_l+\nu)}{c}}}{c^{k+1}\,(c\tau+d)}\\ &=\frac{1}{c^{k+1}}\sum_{d\in D_c}\left(\varepsilon(\gamma_{c,d})\right)^{-1}\left(\rho(\gamma_{c,d})\right)^{-1}_{j,l}\,e^{2\pi i\frac{a(m_l+\nu)}{c}}\left[S_1'+S_2'+S_3'\right],\end{aligned}\tag{75}$$

with

$$\begin{aligned}S_1'&:=\lim_{N\to\infty}\sum_{\frac{K-d}{c}\le|q|\le\frac{N+d}{c}}(c\tau+d-cq)^{-1},\\ S_2'&:=\lim_{N\to\infty}\sum_{\frac{-N+d}{c}\le q<\frac{-N-d}{c}}(c\tau+d-cq)^{-1}\end{aligned}$$

and

$$S_3':=\sum_{\frac{K+d}{c}<q\le\frac{K-d}{c}}(c\tau+d-cq)^{-1}.$$

Now apply the same argument as in [18] using the estimate (14) and the fact that $k>2\delta$. □

4.7 $\mathscr{S}_{\nu,K}(\tau)$ as approximation of $\mathscr{W}_\nu(\tau)$ and $\mathscr{M}_{S,\nu,K}$

Put

$$\mathscr{S}_{\nu,K}^{(j,l)}(\tau) := \delta_{jl} e^{2\pi i (m_j+\nu)\tau} + \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le K}} \sum_{\substack{d'\in D^c\\ |d'|\le K}} \left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(\rho(\gamma_{c,d'})\right)^{-1}_{j,l} e^{2\pi i a \frac{m_l+\nu}{c}} \left(c\tau+d'\right)^k e^{\frac{2\pi i(-\nu-m_l)}{c(c\tau+d')}}. \tag{76}$$

In other words, we have

$$\begin{aligned} &\mathscr{W}_{\nu}^{(j,l)}(\tau) \\ &= \lim_{K\to\infty} \left(\mathscr{S}_{\nu,K}^{(j,l)}(\tau) - \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le K}} \sum_{\substack{d'\in D^c\\ |d'|\le K}} \left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(\rho(\gamma_{c,d'})\right)^{-1}_{j,l} \right. \\ &\qquad \left. \cdot\ e^{2\pi i a \frac{m_l+\nu}{c}} \left(c\tau+d'\right)^k \sum_{n=0}^{k} \frac{1}{n!} \left(\frac{2\pi i\,(-\nu-m_l)}{c(c\tau+d')} \right)^n \right). \end{aligned} \tag{77}$$

Using $\gamma_{c,d'} = \begin{pmatrix} a & b \\ c & d' \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, see (59), we have

$$-\frac{a}{c} + \frac{1}{c\,(c\tau+d')} = \frac{-ac\tau - ad' + ad' - bc}{c\,(c\tau+d')} = -\frac{a\tau+b}{c\tau+d'} = -\gamma_{c,d'}\tau$$

and therefore

$$e^{2\pi i a \frac{m_l+\nu}{c}}\, e^{\frac{2\pi i(-\nu-m_l)}{c(c\tau+d')}} = e^{2\pi i(\nu+m_l)\,\gamma_{c,d'}\tau}.$$

This implies that we can write $\mathscr{S}_{\nu,K}^{(j,l)}(\tau)$ as

$$\mathscr{S}_{\nu,K}^{(j,l)}(\tau) = \delta_{j,l}\, e^{2\pi i(m_l+\nu)\tau} + \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le K}} \sum_{\substack{d'\in D^c\\ |d'|\le K}} \left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(\rho(\gamma_{c,d'})\right)^{-1}_{j,l} \left(c\tau+d'\right)^k\, e^{2\pi i(\nu+m_l)\,\gamma_{c,d'}\tau}. \tag{78}$$

The next step includes the first term in the summation. The trivial identity $\varepsilon(\mathbf{1})\,\rho(\mathbf{1}) = \mathbf{1}_{p\times p}$, where $\mathbf{1}_{p\times p}$ denotes the $p \times p$ unit matrix, implies

$$\delta_{j,l}\, e^{2\pi i(m_l+\nu)\tau} = \left(\varepsilon(\mathbf{1})\right)^{-1} \left(\rho(\gamma_{0,1})\right)^{-1}_{j,l} e^{2\pi i(m_l+\nu)\,\mathbf{1}\tau}.$$

This allows us to include the $(c,d)=(0,1)$ into the expression of $\mathscr{S}_{\nu,K}^{(j,l)}(\tau)$:

$$\mathscr{S}_{\nu,K}^{(j,l)}(\tau)=\sum_{\substack{c\in\mathbb{Z}\\0\le c\le K}}\sum_{\substack{d'\in D^c\\|d'|\le K\\(c,d')\neq(0,-1)}}\big(\varepsilon(\gamma_{c,d'})\big)^{-1}\big(\rho(\gamma_{c,d'})\big)_{j,l}^{-1}\big(c\tau+d'\big)^k\,e^{2\pi i(\nu+m_l)\,\gamma_{c,d'}\tau}. \tag{79}$$

We can write each $\gamma_{c,d'}\tau$ as

$$\gamma_{c,d'}\tau=\frac{1}{2}\big(\gamma_{c,d'}\tau+\gamma_{-c,-d'}\tau\big)$$

with $\gamma_{-c,-d'}:=-\gamma_{c,d'}$. This implies

$$e^{2\pi i(m_l+\nu)\,\gamma_{c,d'}\tau}=e^{2\pi i(m_l+\nu)\,\left(-\gamma_{c,\prime}\tau\right)},$$

and also implies together with (6)

$$\big(\varepsilon(\gamma_{c,d'})\big)^{-1}\big(\rho(\gamma_{c,d'})\big)^{-1}\big(c\tau+d\big)^k=\big(\varepsilon(-\gamma_{c,d})\big)^{-1}\big(\rho(-\gamma_{c,d})\big)^{-1}\big(-c\tau-d\big)^k.$$

This shows that, if we include the transformations $-\gamma_{c,d}=\gamma_{-c,-d}$ in the summation in (79), every term of $\mathscr{S}_{\nu,K}^{(j,l)}(\tau)$ occurs twice:

$$\mathscr{S}_{\nu,K}^{(j,l)}(\tau)=\frac{1}{2}\sum_{\substack{c\in\mathbb{Z}\\|c|\le K}}\sum_{\substack{d'\in D^c\\|d'|\le K}}\big(\varepsilon(\gamma_{c,d'})\big)^{-1}\big(\rho(\gamma_{c,d'})\big)_{j,l}^{-1}\big(c\tau+d'\big)^k\,e^{2\pi i(\nu+m_l)\,\gamma_{c,d'}\tau}. \tag{80}$$

We use this representation of the component function $\tau\mapsto\mathscr{S}_{\nu,K}^{(j,l)}(\tau)$ to define the the matrix-valued function

$$\mathscr{S}_{\nu,K}:\mathbb{H}\to\mathbb{C}^{p\times p};\quad \tau\mapsto\left(\mathscr{S}_{\nu,K}^{(j,l)}(\tau)\right)_{1\le j,l\le p}.$$

This allows us to let Γ act on the argument of $\mathscr{S}_{\nu,K}$. In particular, we define

$$\begin{aligned}&\mathscr{M}_{S,\nu,K}:=\big(\rho(S)\big)^{-1}\big(\mathscr{S}_{\nu,K}\big|_{-k}^{\varepsilon}S\big)\\ \iff\quad&\mathscr{M}_{S,\nu,K}(\tau):=\big(\varepsilon(S)\big)^{-1}\big(\rho(S)\big)^{-1}\tau^k\,\mathscr{S}_{\nu,K}(S\tau)\qquad(\tau\in\mathbb{H}).\end{aligned} \tag{81}$$

The components of $\mathscr{M}_{S,\nu,K}$ are given by

$$\begin{aligned}&\mathscr{M}_{S,\nu,K}^{(j,l)}(\tau)\\&\quad=\frac{1}{2}\sum_{\substack{c\in\mathbb{Z}\\|c|\le K}}\sum_{\substack{d'\in D^c\\|d'|\le K}}\big(\varepsilon(S)\big)^{-1}\big(\varepsilon(\gamma_{c,d'})\big)^{-1}\end{aligned}$$

$$\cdot \sum_{s=1}^{p} \left(\rho(S)\right)^{-1}_{j,s} \left(\rho(\gamma_{c,d'})\right)^{-1}_{s,l} \tau^k \left(cS\tau + d'\right)^k e^{2\pi i (m_l+\nu)\gamma_{c,d'}S\tau}$$
$$\overset{(6)}{=} \frac{1}{2} \sum_{\substack{c\in\mathbb{Z}\\ |c|\le K}} \sum_{\substack{d'\in D^c\\ |d'|\le K}} \left(\varepsilon(\gamma_{c,d'}S)\right)^{-1} \left(\rho(\gamma_{c,d'}S)\right)^{-1}_{j,l} (d'\tau - c)^k e^{2\pi i (m_l+\nu)\gamma_{c,d'}S\tau}. \tag{82}$$

We used the cocycle relation (6) for the last identity.

Lemma 9 *The limit functions of* $\mathscr{M}^{(j,l)}_{S,\nu,K}(\tau)$ *and* $\mathscr{S}^{(j,l)}_{\nu,K}(\tau)$ *for* $K \to \infty$ *coincide*

$$\lim_{K\to\infty} \mathscr{M}^{(j,l)}_{S,\nu,K}(\tau) = \lim_{K\to\infty} \mathscr{S}^{(j,l)}_{\nu,K}(\tau) \tag{83}$$

for all $\tau \in \mathbb{H}$.

In other words and using (81), Lemma 9 shows

$$\lim_{K\to\infty} \mathscr{S}^{(j,l)}_{\nu,K} = \lim_{K\to\infty} \left(\rho(S)\right)^{-1} \left(\mathscr{S}_{\nu,K}\big|^{\varepsilon}_{-k}\right).$$

Proof (of Lemma 9)

1. Let $CP := \{(c,d') \in \mathbb{Z}^2;\ \gcd(c,d') = 1\}$ be the set of all coprime pairs of integers. Then

$$CP = \left\{(c,d');\ |c| \le K, \left|d'\right| \le K\right\} = \lim_{K\to\infty} \bigcup_{\substack{c\in\mathbb{Z}\\ |c|\le K}} \bigcup_{\substack{d'\in D^c\\ |d'|\le K}} \left\{(c,d')\right\}$$

   holds and $(c,d') \mapsto (d',-c)$ is an automorphism on CP, see Lemma 2.
2. The cocycle condition (6) and the multiplicity of ρ in (8) imply

$$\begin{aligned}
&\varepsilon(T^n\gamma_{c,d'})\,\rho(T^n\gamma_{c,d'})\,j(T^n\gamma_{c,d'},\tau)^k\\
&\quad= \varepsilon(T)^n\varepsilon(\gamma_{c,d'})\,\rho(T)^n\rho(\gamma_{c,d'})\,j(T^n,\gamma_{c,d'},\tau)\,j(\gamma_{c,d'},\tau)^k\\
&\quad= \left(\varepsilon(T)\,\rho(T)\right)^n\,\varepsilon(\gamma_{c,d'})\,\rho(\gamma_{c,d'})\,j(\gamma_{c,d'},\tau)^k
\end{aligned}$$

   using $j(T^n,\gamma_{c,d'},\tau) = 1$ and $\varepsilon(T^n) = \varepsilon(T)^n$. Since

$$j(T^n\gamma_{c,d'},\tau) = c\tau + d' = j(\gamma_{c,d'},\tau)$$

   holds we have the identity

$$\varepsilon^{-1}(T^n\gamma_{c,d'})\left(\rho(T^n\gamma_{c,d'})\right)^{-1} = \varepsilon^{-1}(\gamma_{c,d'})\left(\rho(\gamma_{c,d'})\right)^{-1}\left(\left(\varepsilon(T)\right)^{-1}\left(\rho(T)\right)^{-1}\right)^n.$$

3. We have by (9)

$$\Big(\big(\varepsilon(T)\big)^{-1}\big(\rho(T)\big)^{-1}\Big)^n = \operatorname{diag}\big(e^{2\pi i m_1},\dots,e^{2\pi i m_p}\big)^{-n} = \operatorname{diag}\big(e^{-2\pi i n m_1},\dots,e^{-2\pi i n m_p}\big).$$

4. If $\gamma_{c,d'} = \begin{pmatrix} a & b \\ c & d' \end{pmatrix}$ we can write

$$\gamma_{c,d'} S = \begin{pmatrix} b & -a \\ d' & -c \end{pmatrix} = T^n \gamma_{d',-c}$$

for suitable $n \in \mathbb{Z}$.

Taking the arguments together we get

$$\begin{aligned}
&\lim_{K\to\infty} \mathscr{M}^{(j,l)}_{S,\nu,K}(\tau) \\
&\overset{(82)}{=} \frac{1}{2} \sum_{(c,d')\in CP} \big(\varepsilon(\gamma_{c,d'}S)\big)^{-1} \big(\rho(\gamma_{c,d'}S)\big)^{-1}_{j,l} (d'\tau - c)^k \, e^{2\pi i (m_l+\nu)\gamma_{c,d'}S\tau}
\end{aligned}$$

(We use the substitution $(c, d') \mapsto (d', -c)$.)

$$\begin{aligned}
&= \frac{1}{2} \sum_{(c,d')\in CP} \big(\varepsilon(T^n\gamma_{c,d'})\big)^{-1} \big(\rho(T^n\gamma_{c,d'})\big)^{-1}_{j,l} (c\tau + d')^k \, e^{2\pi i (m_l+\nu) T^n\gamma_{c,d'}\tau} \\
&= \frac{1}{2} \sum_{(c,d')\in CP} \big(\varepsilon(\gamma_{c,d'})\big)^{-1} \big(\rho(\gamma_{c,d'})\big)^{-1}_{j,l} e^{-2\pi i n m_l} (c\tau + d')^k \, e^{2\pi i (m_l+\nu) T^n\gamma_{c,d'}\tau}
\end{aligned}$$

(Recall $T\gamma_{c,d'}\tau = \gamma_{c,d'}\tau + 1$.)

$$\begin{aligned}
&= \frac{1}{2} \sum_{(c,d')\in CP} \big(\varepsilon(\gamma_{c,d'})\big)^{-1} \big(\rho(\gamma_{c,d'})\big)^{-1}_{j,l} e^{-2\pi i n m_l} \\
&\qquad \cdot (c\tau + d')^k \, e^{2\pi i (m_l+\nu)\gamma_{c,d'}\tau} \, e^{2\pi i n (m_l+\nu)} \\
&= \frac{1}{2} \sum_{(c,d')\in CP} \big(\varepsilon(\gamma_{c,d'})\big)^{-1} \big(\rho(\gamma_{c,d'})\big)^{-1}_{j,l} (c\tau + d')^k \, e^{2\pi i (m_l+\nu)\gamma_{c,d'}\tau} \, e^{2\pi i n \nu}
\end{aligned}$$

(Recall $e^{2\pi i n\nu} = 1$ for all $n \in \mathbb{Z}$ since $\nu \in \mathbb{Z}$ by assumption of Theorem 2.)

$$\overset{(80)}{=} \lim_{K\to\infty} \mathscr{S}^{(j,l)}_{S,\nu,K}(\tau).$$

Hence, we have proven

$$\lim_{K\to\infty} \mathscr{M}^{(j,l)}_{S,\nu,K}(\tau) = \lim_{K\to\infty} \mathscr{S}^{(j,l)}_{\nu,K}(\tau).$$

□

4.8 The matrix-valued functions $\mathscr{U}_{S,\nu}$, $\mathscr{Q}_{S,\nu}$, and $\mathscr{Y}_{S,\nu}$

Now, define the matrix-valued function

$$\mathscr{U}_{S,\nu} : \mathbb{H} \to \mathbb{C}^{p\times p}$$

by

$$\mathscr{U}_{S,\nu}(\tau) := \big(\varepsilon(S)\big)^{-1}\big(\rho(S)\big)^{-1}\tau^k\,\mathscr{W}_\nu\big(S\tau\big) \tag{84}$$

for all $\tau \in \mathbb{H}$, where $\mathscr{W}_\nu(\tau)$ is given in (62). Using the alternate expression in (77) for $\mathscr{W}_\nu(\tau)$ and the definition of $\mathscr{M}^{(j,l)}_{S,\nu,K}(\tau)$ in (81), we find

$$\begin{aligned}
&\mathscr{U}^{(j,l)}_{S,\nu}(\tau)\\
&\quad= \lim_{K\to\infty}\Bigg\{\mathscr{M}^{(j,l)}_{S,\nu,K}(\tau) - \sum_{\substack{c\in\mathbb{Z}\\0<c\le K}}\sum_{\substack{d'\in D^c\\|d'|\le K}}\big(\varepsilon(\gamma_{c,d'}S)\big)^{-1}\big(\rho(\gamma_{c,d'}S)\big)^{-1}_{j,l}\,\tau^k\\
&\qquad\cdot\big(cS\tau+d'\big)^k\,e^{2\pi i\frac{a(m_l+\nu)}{c}}\sum_{n=0}^{k}\frac{1}{n!}\left(\frac{2\pi i(-\nu-m_l)}{c(cS\tau+d')}\right)^n\Bigg\}
\end{aligned}$$

(using Lemma 9 and the respresentation of $\mathscr{S}_{\nu,K}$ in (78))

$$\begin{aligned}
&= \delta_{j,l}\,e^{2\pi i(m_l+\nu)\tau}\\
&\quad+\lim_{K\to\infty}\Bigg\{\sum_{\substack{c\in\mathbb{Z}\\0<c\le K}}\sum_{\substack{d'\in D^c\\|d'|\le K}}\big(\varepsilon(\gamma_{c,d'})\big)^{-1}\big(\rho(\gamma_{c,d'})\big)^{-1}_{j,l}\big(c\tau+d'\big)^k\,e^{2\pi i(m_l+\nu)\,\gamma_{c,d'}\tau}\\
&\qquad-\sum_{\substack{c\in\mathbb{Z}\\0<c\le K}}\sum_{\substack{d'\in D^c\\|d'|\le K}}\big(\varepsilon(\gamma_{c,d}S)\big)^{-1}\big(\rho(\gamma_{c,d'}S)\big)^{-1}_{j,l}\,\tau^k\big(cS\tau+d'\big)^k\\
&\qquad\cdot\,e^{2\pi i\frac{a(m_l+\nu)}{c}}\sum_{n=0}^{k}\frac{1}{n!}\left(\frac{2\pi i(-\nu-m_l)}{c(cS\tau+d')}\right)^n\Bigg\}.
\end{aligned}$$

On the other hand, the alternate expression of $\mathscr{W}_\nu$ in (77) together with the representation of $\mathscr{S}_{\nu,K}$ in (78) gives analogously

$$\begin{aligned}
&\mathscr{W}^{(j,l)}_{\nu}(\tau)\\
&\quad= \delta_{j,l}\,e^{2\pi i(m_l+\nu)\tau}\\
&\qquad+\lim_{K\to\infty}\Bigg\{\sum_{\substack{c\in\mathbb{Z}\\0<c\le K}}\sum_{\substack{d'\in D^c\\|d'|\le K}}\big(\varepsilon(\gamma_{c,d'})\big)^{-1}\big(\rho(\gamma_{c,d'})\big)^{-1}_{j,l}\big(c\tau+d'\big)^k\,e^{2\pi i(m_l+\nu)\,\gamma_{c,d'}\tau}
\end{aligned}$$

$$- \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le K}} \sum_{\substack{d'\in D^c\\ |d'|\le K}} \left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(\rho(\gamma_{c,d'})\right)^{-1}_{j,l} \left(c\tau+d'\right)^k e^{2\pi i \frac{a(m_l+\nu)}{c}}$$

$$\cdot \sum_{n=0}^{k} \frac{1}{n!} \left(\frac{2\pi i(-\nu-m_l)}{c(c\tau+d')}\right)^n \Bigg\}.$$

Comparing these two expressions above gives

$$\begin{aligned}
&\mathscr{W}_{\nu}^{(j,l)}(\tau) - \mathscr{U}_{S,\nu}^{(j,l)}(\tau) \qquad (85)\\
&= \lim_{K\to\infty} \Bigg\{ \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le K}} \sum_{\substack{d'\in D^c\\ |d'|\le K}} \left(\varepsilon(\gamma_{c,d'}S)\right)^{-1} \left(\rho(\gamma_{c,d'}S)\right)^{-1}_{j,l} \tau^k \left(cS\tau+d'\right)^k\\
&\qquad \cdot e^{2\pi i \frac{a(m_l+\nu)}{c}} \sum_{n=0}^{k} \frac{1}{n!} \left(\frac{2\pi i(-\nu-m_l)}{c(cS\tau+d')}\right)^n\\
&\qquad - \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le K}} \sum_{\substack{d'\in D^c\\ |d'|\le K}} \left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(\rho(\gamma_{c,d'})\right)^{-1}_{j,l} \left(c\tau+d'\right)^k\\
&\qquad \cdot e^{2\pi i \frac{a(m_l+\nu)}{c}} \sum_{n=0}^{k} \frac{1}{n!} \left(\frac{2\pi i(-\nu-m_l)}{c(c\tau+d')}\right)^n \Bigg\}\\
&= \lim_{K\to\infty} \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le K}} \sum_{\substack{d'\in D^c\\ |d'|\le K}} e^{2\pi i \frac{a(m_l+\nu)}{c}} \Bigg\{ \left(\varepsilon(\gamma_{c,d'}S)\right)^{-1} \left(\rho(\gamma_{c,d'}S)\right)^{-1}_{j,l} \qquad (86)\\
&\qquad \cdot \sum_{n=0}^{k} \frac{1}{n!} \left(\frac{2\pi i(-\nu-m_l)}{c}\right)^n \left(d'\tau-c\right)^{k-n} \tau^n\\
&\qquad - \left(\varepsilon(\gamma_{c,d'})\right)^{-1} \left(\rho(\gamma_{c,d'})\right)^{-1}_{j,l}\\
&\qquad \cdot \sum_{n=0}^{k} \frac{1}{n!} \left(\frac{2\pi i(-\nu-m_l)}{c}\right)^n \left(c\tau+d'\right)^{k-n} \Bigg\},
\end{aligned}$$

where we used the trivial identity $(cS\tau+d')^{k-n}\,\tau^k = (d'\tau-c)^{k-n}\tau^n$. We easily see that the term inside the limit is a polynomial of degree at most k. This implies that $\mathscr{W}_{\nu}^{(j,l)}(\tau) - \mathscr{U}_{S,\nu}^{(j,l)}(\tau) \in \mathbb{C}_{\le k}[\tau]$ is a polynomial of degree at most k, since the limit of a sequence of polynomials of degree at most k converging at $k+1$ points is a polynomial of degree at most k.

Next, we check the constant term $\mathscr{K}_{\nu}$ in (58) and its image under S. Define

$$\begin{aligned}
&\mathscr{Y}_{S,\nu} := \mathscr{K}_{\nu} - \left(\rho(S)\right)^{-1} \mathscr{K}_{\nu}\big|^{\varepsilon}_{-k} S\\
\iff\quad &\mathscr{Y}_{S,\nu}(\tau) := \mathscr{K}_{\nu}(\tau) - \left(\varepsilon(S)\right)^{-1} \tau^k \left(\rho(S)\right)^{-1} \mathscr{K}_{\nu}(\tau) \qquad \text{for all } \tau\in\mathbb{H}. \qquad (87)
\end{aligned}$$

Clearly $\mathscr{Y}_{S,\nu}^{(j,l)}(\tau)$ is a polynomial in τ of degree at most k since $\mathscr{K}_\nu^{(j,l)}$ is a constant, see (58).

Adding the two polynomials in (85) and (87) shows that

$$\begin{aligned}\mathscr{Q}_{S,\nu} &:= \Big(\mathscr{W}_\nu + \mathscr{K}_\nu\Big) - \big(\rho(S)\big)^{-1}\Big(\mathscr{W}_\nu + \mathscr{K}_\nu\Big)\big|_{-k}^{\varepsilon} S \\ &= \mathscr{W}_\nu - \mathscr{U}_{S,\nu} + \mathscr{Y}_{S,\nu}\end{aligned} \tag{88}$$

is a matrix-valued function whose entries are polynomials of degree at most k.

Using the defining identities (87), (84), and (62) we find

$$\begin{aligned}\mathscr{Q}_{S,\nu}(\tau) &\overset{(88)}{=} \mathscr{W}_\nu^{(j,l)}(\tau) - \mathscr{U}_{S,\nu}^{(j,l)}(\tau) + \mathscr{Y}_{S,\nu}(\tau) \\ &\overset{(87)}{=} \big(\mathscr{W}_\nu(\tau) + \mathscr{K}_\nu\big) - \big(\rho(S)\big)^{-1}\big(\mathscr{W}_\nu(\cdot) + \mathscr{K}_\nu\big)\big|_{-k}^{\varepsilon} S(\tau) \\ &\overset{(62)}{=} \mathscr{T}_\nu(\tau) - \big(\rho(S)\big)^{-1}\big(\mathscr{T}_\nu\big|_{-k}^{\varepsilon} S\big)(\tau)\end{aligned}$$

is also a matrix-valued function whose entries are polynomials of degree at most k. Each entry is given by

$$\begin{aligned}\mathscr{Q}_{S,\nu}^{(j,l)}(\tau) &= \mathscr{W}_\nu^{(j,l)}(\tau) - \mathscr{U}_{S,\nu}^{(j,l)}(\tau) + \mathscr{Y}_{S,\nu}^{(j,l)}(\tau) \\ &= \mathscr{T}_\nu^{(j,l)}(\tau) - \big(\varepsilon(S)\big)^{-1}\tau^k \sum_{s=1}^{p} \big(\rho(S)\big)_{j,s}^{-1} \mathscr{T}_\nu^{(s,l)}(S\tau).\end{aligned}$$

4.9 The transformation of F under S

The defining property (52) of $\mathscr{T}_\nu$ and (92) implies that

$$\begin{aligned}&F(\tau) - \big(\rho(S)\big)^{-1}\big(F\big|_{-k}^{\varepsilon} S\big)(\tau) \\ &= \sum_{\mu \le \nu < 0} \mathscr{Q}_{S,\nu}(\tau) \begin{pmatrix} b_\nu(1) \\ \vdots \\ b_\nu(p) \end{pmatrix} \\ &= \sum_{\mu \le \nu < 0} \Big(\mathscr{T}_\nu(\tau) - \big(\rho(S)\big)^{-1}\big(\mathscr{T}_\nu\big|_{-k}^{\varepsilon} S\big)\Big) \begin{pmatrix} b_\nu(1) \\ \vdots \\ b_\nu(p) \end{pmatrix} \\ &\quad + \sum_{\mu \le \nu < 0} \mathscr{Y}_{S,\nu}(\tau) \begin{pmatrix} b_\nu(1) \\ \vdots \\ b_\nu(p) \end{pmatrix}\end{aligned} \tag{89}$$

is a column vector of polynomials of degree at most k.

4.10 The transformation of F under $g \in \Gamma$

We just showed that $F - \big(\rho(S)\big)^{-1} F\big|_{-k}^{\varepsilon} S$ is a column vector of polynomials in the previous step. Next, we extend this result to all elements of Γ to prove Theorem 2.

We need a bit of notation for this. For $g \in \Gamma$ define

$$\mathscr{U}_{g,\nu} := \big(\rho(g)\big)^{-1} \mathscr{W}_\nu\big|_{-k}^{\varepsilon} g, \tag{90}$$

$$\mathscr{Y}_{g,\nu} := \mathscr{K}_\nu - \big(\rho(g)\big)^{-1} \mathscr{K}_\nu\big|_{-k}^{\varepsilon} g \quad \text{and} \tag{91}$$

$$\begin{aligned} \mathscr{Q}_{g,\nu} &:= \mathscr{W}_\nu - \mathscr{U}_{g,\nu} + \mathscr{Y}_{g,\nu} \\ &= \mathscr{T}_\nu - \big(\rho(g)\big)^{-1} \mathscr{T}_\nu\big|_{-k}^{\varepsilon} g. \end{aligned} \tag{92}$$

Obviously, $\mathscr{Y}_{g,\nu}$ is a polynomial of degree at most k since $\mathscr{K}_\nu$ is constant.

We want to show that $\mathscr{Q}_{g,\nu}$ is a matrix of polynomials of degree at most k.

We prove this by induction on $L(g)$, the Eichler length of g viewed as word in the generators S and T. To do so, we discuss the cases $g = S$, $g = T$ and the induction steps $g \mapsto gS$ and $g \mapsto gT$:

Assume $g = S$. We know it is true for $g = S$ by (88).

Assume $g = T$. The normalization (9) of the representation $\rho(T)$ implies that the Fourier-type series of $\mathscr{T}_\nu$ (53) is compatible with the action of $T \in \Gamma$:

$$\mathscr{T}_\nu = \big(\rho(T)\big)^{-1} \mathscr{T}_\nu\big|_{-k}^{\varepsilon} T. \tag{93}$$

Now, consider the induction step $g \mapsto gS$. We assume that $\mathscr{Q}_{g,\nu}(\tau)$ is a matrix of polynomials of degree at most k. Since $\mathscr{Q}_{S,\nu}(\tau)$ is also a matrix of polynomials of degree $\le k$, we have that

$$\begin{aligned} &\mathscr{T}_\nu(\tau) - \big(\varepsilon(gS)\big)^{-1} \big(\rho(gS)\big)^{-1} j(gS,\tau)^k \, \mathscr{T}_\nu(gS\tau) \\ &\overset{(6)}{=} \mathscr{T}_\nu(\tau) - \big(\varepsilon(S)\big)^{-1} \big(\rho(S)\big)^{-1} j(S,\tau)^k \\ &\qquad \cdot \Big(\big(\varepsilon(g)\big)^{-1} \big(\rho(g)\big)^{-1} j(g,S\tau)^k \, \mathscr{T}_\nu\big(g(S\tau)\big) \Big) \\ &= \mathscr{T}_\nu(\tau) - \big(\varepsilon(S)\big)^{-1} \big(\rho(S)\big)^{-1} j(S,\tau)^k \Big(\mathscr{T}_\nu(S\tau) - \mathscr{Q}_{g,\nu}(S\tau) \Big) \\ &= \mathscr{Q}_{S,\nu}(\tau) + \big(\varepsilon(S)\big)^{-1} \big(\rho(S)\big)^{-1} j(S,\tau)^k \, \mathscr{Q}_{g,\nu}(S\tau). \end{aligned}$$

The first term is a matrix of polynomials. The second term also contains polynomials since the factor τ^k combines with $\big(S\tau\big)^{k-n}$, $0 \le n \le k$.

Next, consider the induction step $g \mapsto gT$. Similarly, using the fact that $\mathscr{Q}_{T,\nu}(\tau) = 0$, we have

$$\mathscr{T}_\nu(\tau) - \big(\varepsilon(gT)\big)^{-1} \big(\rho(gT)\big)^{-1} j(gT,\tau)^k \, \mathscr{T}_\nu(gT\tau) \tag{94}$$

$$
\begin{aligned}
&= \mathscr{T}_\nu(\tau) - \big(\varepsilon(T)\big)^{-1} \big(\rho(T)\big)^{-1} \Big(\big(\varepsilon(g)\big)^{-1} \big(\rho(g)\big)^{-1} j(g, T\tau)^k \, \mathscr{T}_\nu\big(g(T\tau)\big) \Big) \\
&= \mathscr{T}_\nu(\tau) - \big(\varepsilon(T)\big)^{-1} \big(\rho(T)\big)^{-1} \Big(\mathscr{T}_\nu(T\tau) - \mathscr{Q}_{g,\nu}(T\tau) \Big) \\
&= \big(\varepsilon(T)\big)^{-1} \big(\rho(T)\big)^{-1} \mathscr{Q}_{g,\nu}(T\tau),
\end{aligned}
$$

which is a matrix of polynomials of degree at most k.

Therefore, we have shown that if $\mathscr{Q}_{M,\nu}(\tau)$ is matrix of polynomials in τ of degree at most k, so are $\mathscr{Q}_{MS,\nu}(\tau)$ and $\mathscr{Q}_{MT,\nu}(\tau)$.

Thus, (52) and (92) imply that $F(\tau) - \big(\varepsilon(g)\big)^{-1} \big(\rho(g)\big)^{-1} j(g, \tau)^k \, F(g\tau)$ is a vector of polynomials of degree at most k. This concludes the proof of Theorem 2.

5 The supplementary function

Let $k+2$ be a weight with $k > \max\{2\delta, 2\alpha\}$ with δ as in estimate (15) and α given in (12). Moreover, let ε be a multiplier and ρ be complex p-dimensional normal representation ρ. (The normal representation will be defined in Definition 5 in Sect. 5.1.) Let $g \in \mathscr{S}(k+2, \varepsilon, \rho)$ be a cusp form of weight $k+2$, multiplier ε, and representation ρ.

It is known that g can be written as a linear combination of vector-valued Poincaré series $P(\rho, k, \varepsilon, \mu, \Gamma, r; \cdot)$ for suitable parameters $\mu \in \mathbb{Z}$ and $r \in \{1, \ldots, p\}$, see Sect. 5.1 for more details on Poincaré series. Suppose that we have

$$
g(\tau) = \sum_{i=1}^{s} b_i \, P(\rho, k+2, \varepsilon, \mu_i, \Gamma, r_i; \tau) \qquad (\tau \in \mathbb{H}) \tag{95}
$$

for suitable $s \in \mathbb{N}$, $r_i \in \{1, \ldots, p\}$ and nonnegative $\mu_i \in \mathbb{Z} \geq 0$.

Remark 5 Since g is a cusp form, the Fourier expansion of linear combination of Poincaré series in (95) has no constant term. We use this fact later to neglect the possible constant term in the Fourier expansion of the Poincaré series since we are only interested in the nonexistent linear combinations of them. More details are given below.

Before we continue, let us introduce new notations. Recall that by (9) we have rational phase factors $0 \leq m_1, \ldots, m_p < 1$. We define the "primed" factors $m'_j \in [0, 1)$, $j = 1, \ldots, p$ and $\mu'_i \in \mathbb{Z}$, $i = 1, \ldots, s$, by

$$
\begin{aligned}
m'_j &:= \begin{cases} 1 - m_j & \text{if } m_i \geq 0, \\ -m_j & \text{if } m_i = 0 \text{ and} \end{cases} \\
\mu'_i &:= \begin{cases} -1 - \mu_i & \text{if } m_j > 0, \\ -\mu_i & \text{if } m_j = 0. \end{cases}
\end{aligned} \tag{96}
$$

We obviously have

$$
m'_i + \mu'_i = -(m_i + \mu_i). \tag{97}
$$

Furthermore, we define

$$\varepsilon'(\gamma) := \overline{\varepsilon(\gamma)} \qquad \text{and} \qquad \rho'(\gamma) := \overline{\rho(\gamma)} \tag{98}$$

for all $\gamma \in \Gamma$. Recall that ε' is a multiplier system for Γ for weight $k+2$ and for $-k$ since k is an integer and ε is already a multiplier system for weight $k+2$ and Γ. Also, ρ being a representation for Γ implies that ρ' is a representation of Γ.

With these notations we define the *supplementary series* $\widehat{g} : \mathbb{H} \to \mathbb{C}^p$ to g by

$$\widehat{g}(\tau) = \sum_{i=1}^{s} \overline{b_i}\, P(\rho', k+2, \varepsilon', \mu_i', \Gamma, r_i; \tau) \tag{99}$$

for all $\tau \in \mathbb{H}$.

Remark 6

1. Note that $\widehat{g}$ is not a cusp form. This comes from the fact that the factors $\mu_i' + m_j'$ are nonpositive, see (96) and use that the $\mu_i + m_j$'s in (95) are nonnegative.
2. To our knowledge, the term "supplementary series" was introduced by Knopp in [14].
3. The map induced by $g \mapsto \widehat{g}$ corresponds to taking the "flipping operator" of g in [1, §1.1].

We now have the necessary notation to state the theorem. Our result generalizes the results obtained by Husseini and Knoop in [17] and [12].

Theorem 3 *Let $g \in \mathscr{S}(k+2, \varepsilon, \rho)$ be a cusp form of weight $k+2$, and $k > \max\{2\delta, 2\alpha\}$ with δ as in (15) and α in (12), multiplier ε, and complex p-dimensional normal representation ρ given by the linear combination (95) of Poincaré series and the supplementary series $\widehat{g}$ given by (99).*

1. *The $(k+1)$-fold integral G of the cusp form g given by the $k+1$ times termwise integrated Fourier expansion of g is a holomorphic vector-valued function on $\mathbb{H}$ and satisfies the transformation property*

$$\begin{aligned} & G - \big(\rho(\gamma)\big)^{-1} G|_{-k}^{\varepsilon}\gamma =: p_\gamma \\ \iff \quad & G(\tau) - \big(\varepsilon(\gamma)\big)^{-1} j(\gamma,\tau)^k \big(\rho(\gamma)\big)^{-1} G(\gamma\tau) =: p_\gamma(\tau) \end{aligned} \tag{100}$$

with p_γ a p-dimensional column vector of polynomials of degree at most k. Moreover, G can be written as the Eichler integral

$$G(\tau) = (-1)^{k+1} k! \int_{\tau}^{i\infty} g(z)(z-\tau)^k \,\mathrm{d}z \qquad (\tau \in \mathbb{H}). \tag{101}$$

2. *The $(k+1)$-fold integral $\widehat{G}$ of $\widehat{g}$ given by the $k+1$ times termwise integrated Fourier expansion of $\widehat{g}$ is a holomorphic vector-valued function on $\mathbb{H}$ and satisfies*

the transformation property

$$\begin{aligned} & \widehat{G} - \left(\rho'(\gamma)\right)^{-1} \widehat{G}\big|_{-k}^{\varepsilon'} \gamma =: \widehat{p}_\gamma \\ \iff \quad & \widehat{G}(\tau) - \left(\varepsilon'(\gamma)\right)^{-1} j(\gamma,\tau)^k \left(\rho'(\gamma)\right)^{-1} \widehat{G}(\gamma\tau) =: \widehat{p}_\gamma(\tau) \end{aligned} \tag{102}$$

with $\widehat{p}_\gamma$ a p-dimensional column vector of polynomials of degree at most k.

3. *We have*

$$\widehat{p}_\gamma(\tau) = \overline{p_\gamma(\overline{\tau})} \tag{103}$$

for all $\gamma \in \Gamma$ and $\tau \in \mathbb{H}$.

Definition 3 The function $\widehat{G}$ defined as the $(k+1)$-fold integral of $\widehat{g}$ and given by the $k+1$ times termwise integrated Fourier expansion of $\widehat{g}$, see the second part of Theorem 3 above, is called the *supplementary function* to g.

Following Husseini–Knopp [12] we have the following

Corollary 3 *Let $k > \max\{2\delta, 2\alpha\}$, $g(\tau) \in \mathscr{S}(k+2, \rho, \varepsilon)$, and $\widehat{G}(\tau)$ the supplementary function to g. The following statements are equivalent:*

1. *$g \equiv 0$ and*
2. *$\widehat{G} \in \mathscr{F}(-k, \varepsilon', \rho')$.*

We prove it exactly in the same way as [13, Theorem 3].

Proof (of Corollary 3) If $g(\tau) \equiv 0$, then its $(k+1)$-fold integral $G(\tau)$ is also identically zero. Therefore, $p_\gamma(\tau) = 0$ for all $\gamma \in \Gamma$. Thus, by (138), we have $\widehat{p}_\gamma(\tau) = 0$, implying

$$\left(\varepsilon'(\gamma)\right)^{-1} \left(\rho'(\gamma)\right)^{-1} j(\gamma,\tau)^k \, \widehat{G}(\gamma\tau) = \widehat{G}(\tau).$$

Since $\widehat{g}(\tau)$ is a vector-valued modular form of weight $k+2$, representation ρ', and multiplier ε' for Γ we have that $\widehat{g}(\tau)$ is holomorphic on $\mathbb{H}$ and meromorphic at $i\infty$. Therefore, $\widehat{G}(\tau) \in \mathscr{F}(\Gamma, -k, \rho', \varepsilon')$.

On the other hand, if $\widehat{G}(\tau) \in \mathscr{F}(\Gamma, -k, \rho', \varepsilon')$, then $\widehat{p}_\gamma(\tau) = 0$ for all $\gamma \in \Gamma$. Using (138) we find $p_\gamma(\tau) = 0$ for all $\gamma \in \Gamma$. This implies $G(\tau) \in \mathscr{F}(\Gamma, -k, \rho, \varepsilon)$. Since $g(\tau)$ is a cusp form, we know that g is holomorphic on $\mathbb{H}$ and also in the cusp at $i\infty$. Hence its $(k+1)$-fold integral $G(\tau)$ is holomorphic on $\mathbb{H}$ and at the cusp. Then [16, Lemma 4.1] implies $G(\tau) \equiv 0$ and hence $g(\tau) \equiv 0$. □

Remark 7 Let $g \in \mathscr{S}(k+2, \varepsilon, \rho)$ satisfy the assumptions of Theorem 3 and let G and $\widehat{G}$ be the functions given there. One might think in view of (103) to consider a vector-valued difference function $\mathbb{H} \to \mathbb{C}^p$, $\tau \mapsto \widehat{G}(\tau) - \overline{G(\overline{\tau})}$. Neglecting the issue of how to define $\overline{G(\overline{\tau})}$ for the moment, one can formally deduce

1. $\left(\widehat{G} - \overline{G(\overline{\cdot})}\right)\Big|_{-k}^{\varepsilon'} \gamma = \rho'(\gamma)\left(\widehat{G} - \overline{G(\overline{\cdot})}\right)$ for all $\gamma \in \Gamma$,
2. $\frac{\partial}{\partial\overline{\tau}}\left(\widehat{G}(\tau) - \overline{G(\overline{\tau})}\right) = 0$.

The first statement follows from the formal calculation

$$\begin{aligned}
&\big(\rho'(\gamma)\big)^{-1}\left(\widehat{G}(\cdot)-\overline{G(\overline{\cdot})}\right)\Big|_{-k}^{\varepsilon'}(\tau)\\
&=\big(\rho'(\gamma)\big)^{-1}\,\widehat{G}\big|_{-k}^{\varepsilon'}(\tau)-\overline{\big(\rho(\gamma)\big)^{-1}\,G\big|_{-k}^{\varepsilon}(\overline{\tau})}\\
&\overset{(100),(102)}{=}\big(\widehat{G}(\tau)-\widehat{p}_\gamma(\tau)\big)-\big(\overline{G(\overline{\tau})}-\overline{p_\gamma(\overline{\tau})}\big)\\
&\overset{(103)}{=}\widehat{G}(\tau)-\overline{G(\overline{\tau})}
\end{aligned}$$

for all $\gamma\in\Gamma$ and $\tau\in\mathbb{H}$. The second part can be deduced from the Fourier expansions of G and $\widehat{G}$ in (133) and (134), respectively, (129) and (130) and the fact that $\overline{e^{2\pi i a\overline{\tau}}}=e^{-2\pi i a\tau}$ holds for every real $a\in\mathbb{R}$.

However, we cannot define $\overline{G(\overline{\tau})}$ even if we just take the Fourier expansion of G in (133) and replace τ with $\overline{\tau}$ which is then just a formal series and does not converge.

One of the most important results of Theorem 3 is, in our opinion, that the $(k+1)$-fold integrals G and $\widehat{G}$ have essentially the same transformation property stated in (100) and (102). In particular, the correction polynomial p_γ, respectively, $\widehat{p}_\gamma$ satisfy the identity (103).

There is also another way to construct a function G_2 replacing the $(k+1)$-fold integral G which has a similar transformation property.

Definition 4 Let $g\in\mathscr{S}(k+2,\varepsilon,\rho)$ be a cusp form of weight $k+2>2$ and p-dimensional complex representation ρ. The *auxiliary integral* $G_2:\mathbb{H}\to\mathbb{C}^p$ of the cusp form g is given by

$$G_2(\tau):=\overline{\int_\tau^{i\infty}g(z)\,(z-\overline{\tau})^k\,\mathrm{d}z}\qquad(\tau\in\mathbb{H}).\tag{104}$$

Remark 8 Sometimes, G_2 as defined above is called a *Niebur integral*, see e.g., [6, p.15].

Some properties of the auxiliary integral are given in the following.

Theorem 4 *Let $g\in\mathscr{S}(k+2,\varepsilon,\rho)$ be a cusp form of weight $k+2$, and $k>2\delta$ a positive integer with δ given in (15), multiplier ε, and complex p-dimensional normal representation ρ given by the linear combination (95) of Poincaré series and the supplementary series $\widehat{g}$ given by (99).*

1. *The auxiliary integral G_2 of the cusp form g satisfies the transformation property*

$$\begin{aligned}
&G_2-\big(\rho'(\gamma)\big)^{-1}\,G_2\big|_{-k}^{\varepsilon'}\gamma=:p_{2,\gamma}\\
\iff\quad&G_2(\tau)-\big(\varepsilon'(\gamma)\big)^{-1}\,j(\gamma,\tau)^k\,\big(\rho'(\gamma)\big)^{-1}\,G_2(\gamma\tau)=:p_{2,\gamma}(\tau)
\end{aligned}\tag{105}$$

such that $p_{2,\gamma}$ is a p-dimensional column vector of polynomials of degree at most k.

2. *The vector-valued polynomial* $p_{2,\gamma}$ *in* (105) *satisfies*

$$\frac{(-1)^{k+1}}{\Gamma(k+1)}\,\overline{p_{2,\gamma}(\overline{\tau})} = p_\gamma(\tau) \in \mathbb{C}[\tau] \tag{106}$$

for all $\tau \in \mathbb{H}$*, where* p_γ *is given in* (100) *of Theorem* 3.

3. *Each component of* $G_2 = \big((G_2)_1, \ldots, (G_2)_p\big)^{\mathrm{tr}}$ *has a Fourier expansion of the form*

$$\big(G_2(\tau)\big)_j = \sum_{m=1}^{\infty} C_{m,j} \left(\frac{1}{2\pi i(m+m_j)}\right)^{k+1} \cdot \Gamma\big(k+1, 4\pi(m+m_j)\mathrm{Im}\,(\tau)\big)\, e^{-2\pi i(m+m_j)\tau} \tag{107}$$

for all $\tau \in \mathbb{H}$ *and for each* $j \in \{1, \ldots, p\}$*. The coefficients* $C_{m,j}$ *do not depend on* τ*.*

The remainder of the section is organized as follows: We recall vector-valued Poincaré series in Sect. 5.1 followed by a short remainder about $(k+1)$-fold integrals in Sect. 5.2. The part Sect. 5.4 proves Theorem 3 with help of some auxiliary Lemmas presented in Sect. 5.3. The remaining part Sect. 5.5 proves Theorem 4.

5.1 On Poincaré series

Before we introduce the Poincaré series we need to define normal representations.

Definition 5 ([15, p.1351]) Let $\rho : \Gamma \to \mathbb{C}^{p\times p}$ be a representation of Γ. ρ is *normal* if it satisfies two conditions:

1. $\rho(T)$ is diagonal, and
2. $\rho(S^2) = \mathbf{1}$.

We introduce the vector-valued Poincaré series $P(\rho, k, \varepsilon, \mu, \Gamma, r; \tau)$ following Knopp and Mason [15, §3]. Fix integers $\mu \in \mathbb{Z}$ and $r \in \{1, \ldots, p\}$ and consider a normal representation ρ of Γ. The *vector-valued Poincaré series* is (formally) defined as

$$P(\rho, k, \varepsilon, \mu, \Gamma, r; \tau) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \big(\varepsilon(\gamma)\big)\, j(\gamma, \tau)^{-k}\, e^{2\pi i(\mu+m_r)\gamma\tau} \big(\rho(\gamma)\big)^{-1} e_r, \tag{108}$$

where e_r rth unit vector, i.e., the column vector consisting of zeros except for the rth component which is a 1. We usually shorten the notation of Poincaré series to

$$P(\tau) := P(\rho, k, \varepsilon, \mu, \Gamma, r; \tau) \qquad (\tau \in \mathbb{H})$$

if the parameters $\rho, k, \varepsilon, \mu, \Gamma$, and r are clear from the context.

Let α be the parameter introduced in (12) which depends only on the representation ρ. (Knopp and Marson write in [16, p. 118] that α is equal to const $\cdot$ log $\left(p\|\rho(S)\|\right)$ and the constant is independent of ρ.)

Proposition 1 ([15, Proposition 3.1]) *Let ε be a multiplier, ρ be a normal representation of Γ, $\mu \in \mathbb{Z}$, and $r \in \{1, \ldots, p\}$.*

If $k > 2 + 2\alpha$ then the vector-valued Poincaré series $P(\rho, k, \varepsilon, \mu, \Gamma, r; \tau)$ defined in (108) converges absolutely and uniformly on compact subsets of $\mathbb{H}$. In particular, each component of $P(\tau)$ is holomorphic in $\mathbb{H}$.

Remark 9 The sum in (108) above runs through some set of coset representatives for $\Gamma_\infty \backslash \Gamma$ and Γ_∞ stabilizes the cusp $i\infty$. One particular set of representatives is the set of matrices

$$\{\gamma_{c,d'};\ c \in \mathbb{Z}_{\geq 0}, d \in \mathbb{Z} \text{ with } \gcd(c,d) = 1\} = \{\gamma_{c,d};\ c \in \mathbb{Z}_{\geq 0}, d \in D^c\},$$

where $\gamma_{c,d'}$ are given by (13). We may, therefore, rewrite the defining formula of the vector-valued Poincaré $P(\rho, k, \varepsilon, \mu, \Gamma, r; \tau)$ as

$$P(\tau) = \frac{1}{2} \sum_{c \in \mathbb{Z}_{\geq 0}} \sum_{\substack{d \in \mathbb{Z} \\ \gcd(c,d)=1}} \left(\varepsilon(\gamma_{c,d})\right)^{-1} j(\gamma_{c,d}, \tau)^{-k}\, e^{2\pi i(\mu + m_r)\gamma_{c,d}\tau} \left(\rho(\gamma_{c,d})\right)^{-1} e_r. \tag{109}$$

Assume that ρ is a normal representation of Γ and $k > \max\left\{2\delta, 2\alpha\right\}$ with δ as in (15) and α as in (12). We list some facts about vector-valued Poincaré series:

1. $P(\rho, k+2, \varepsilon, \mu, \Gamma, r; \tau)$ is a vector-valued modular form of weight $k+2$, see [15, p.1355]. It is a cusp form if μ is positive.
2. The space of cusp forms of weight k, multiplier ε, and representation ρ is spanned by Poincaré series [15, Theorem 5.4].
3. The Fourier expansion for $P(\tau) = P(\rho, k+2, \varepsilon, \mu, \Gamma, r; \tau)$ is given by

$$P(\tau) = \mathscr{T}^\star_\mu(\tau)\, e_r, \tag{110}$$

see [15, Theorem 3.2]. Here $\mathscr{T}^\star_\mu(\tau)$ is a matrix-valued function defined by its components

$$\mathscr{T}^{\star\,(j,l)}_{\mu}(\tau) := \delta_{j,l}\, e^{2\pi i(m_j + \mu)\tau} + \mathscr{R}^{\star\,(j,l)}_{\mu}(\tau) \tag{111}$$

and

$$\mathscr{R}^{\star\,(j,l)}_{\mu}(\tau) := \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^{-k}}{c} A_{c,\mu,m,j,l}\, B^\star_{c,\mu,m,j,l}\, e^{2\pi i(m + m_j)\tau}. \tag{112}$$

In (112), the term $B^\star_{c,\mu,m,j,l}$ is given by

$$B^\star_{c,\mu,m,j,l} =$$

$$\begin{cases} \left(\frac{-\mu-m_l}{m+m_j}\right)^{-\frac{k+1}{2}} I_{k+1}\left(\frac{4\pi}{c}(-\mu-m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}\right) & \text{if } \mu+m_l<0, \\ \frac{-1}{(k+1)!}\left(\frac{2\pi}{c}(m+m_j)\right)^{k+1} & \text{if } \mu=m_l=0 \text{ and} \\ \left(\frac{\mu+m_l}{m+m_j}\right)^{-\frac{k+1}{2}} J_{k+1}\left(\frac{4\pi}{c}(\mu+m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}\right) & \text{if } \mu+m_l>0 \end{cases} \tag{113}$$

with $I_{k+1}(z)$ defined in (27) and $J_{k+1}(z)$ the Bessel Function of the first kind defined by

$$J_{k+1}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{z}{2}\right)^{2n+k+1}}{n!(n+k+1)!}, \qquad \text{for all } z\in\mathbb{R}, \tag{114}$$

see [15, pp.1355–1356] or alternatively [7, (10.2.2)]. The term $A_{c,\mu,m,j,l}$ is the same as the one in (23) of Theorem 1.

5.2 $(k+1)$-fold integrals

Let $k\in\mathbb{Z}_{\geq 0}$ and ε be a multiplier compatible with k. (Recall that this means that ε is also a multiplier compatible with weight $-k-2$, and ε' is a multiplier compatible with both weights.)

Following [13, p. 566] we recall Bol's identity

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}\tau^{k+1}}\left(j(\gamma,\tau)^k H(\gamma\tau)\right)=j(\gamma,\tau)^{-k-2}\left(\frac{\mathrm{d}^{k+1}}{\mathrm{d}\tau^{k+1}}H\right)(\gamma\tau) \tag{115}$$

for any holomorphic function $H:\mathbb{H}\to\mathbb{C}$ and $\gamma\in\Gamma$. It follows immediately from (115): If H satisfies $H\big|_{-k}^{\varepsilon}\gamma=H$ for all $\gamma\in\Gamma$ then $\left(\frac{\mathrm{d}^{k+1}}{\mathrm{d}\tau^{k+1}}H\right)\big|_{k+2}^{\varepsilon}\gamma=\left(\frac{\mathrm{d}^{k+1}}{\mathrm{d}\tau^{k+1}}H\right)$. The converse is not true in general.

Definition 6 Let $k\in\mathbb{Z}_{\geq 0}$, ε be a compatible multiplier, and ρ be a p-dimensional representation. Let $h=(h_1,\ldots,h_p)^{\mathrm{tr}}:\mathbb{H}\to\mathbb{C}^p$ be a vector-valued modular form of weight $k+2$ for Γ, as in Definition 1, such that additionally each h_i is holomorphic. We call a p-tuple of holomorphic functions $H=(H_1,\ldots,H_p)^{\mathrm{tr}}:\mathbb{H}\to\mathbb{C}^p$ an $(k+1)$-*fold integral* of h if $\frac{\mathrm{d}^{k+1}}{\mathrm{d}\tau^{k+1}}H_i=h_i$ holds for all $i=1,\ldots,p$.

A simple and direct consequence is the following.

Lemma 10 ([13, (3)]) *Let h be a vector-valued modular form of weight $k+2$ and H its $(k+1)$-fold integral. It satisfies the transformation property*

$$\big(\rho(\gamma)\big)^{-1}H\big|_{-k}^{\varepsilon}\gamma-H=p_\gamma \qquad (\gamma\in\Gamma), \tag{116}$$

with each $p_\gamma(\tau)\in\big(\mathbb{C}_{\leq k}[\tau]\big)^p$, i.e., p_γ is a vector of polynomials of degree at most k.

Lemma 11 *Let h be a cusp form of weight $k+2$ and consider the associated Eichler integral*

$$H_1(\tau) := \frac{(-1)^{k+1}}{k!} \int_\tau^{i\infty} h(z)\,(z-\tau)^k \,\mathrm{d}z \qquad (\tau \in \mathbb{H} \cup \mathbb{R} \cup \{i\infty\}). \tag{117}$$

1. *H_1 is a $k+1$-fold integral of h.*
2. *H_1 satisfies the transformation property* (116)*.*

Proof Since h is a cusp form and hence vanishes in the cusp $i\infty$ we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\Big((-1)^{k+1}k!\; H_1(\tau)\Big) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\int_{\tau+\varepsilon}^{i\infty} h(z)\,(z-\tau-\varepsilon)^k\,\mathrm{d}z - \int_\tau^{i\infty} h(z)\,(z-\tau)^k\,\mathrm{d}z\right) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\int_{\tau}^{i\infty} h(z+\varepsilon)\,(z-\tau)^k\,\mathrm{d}z - \int_\tau^{i\infty} h(z)\,(z-\tau)^k\,\mathrm{d}z\right) \\
&= \lim_{\varepsilon\to 0} \int_\tau^{i\infty} \frac{h(z+\varepsilon)-h(z)}{\varepsilon}\,(z-\tau)^k\,\mathrm{d}z \\
&= \int_\tau^{i\infty} h'(z)\,(z-\tau)^k\,\mathrm{d}z \\
&= h(z)\,(z-\tau)^k\Big|_{z=\tau}^{z=i\infty} - \int_\tau^{i\infty} h(z)\,k(z-\tau)^{k-1}\,\mathrm{d}z \\
&= -k\int_\tau^{i\infty} h(z)\,(z-\tau)^{k-1}\,\mathrm{d}z.
\end{aligned}$$

The product rule implies

$$\begin{aligned}
\frac{\mathrm{d}^{k+1}}{\mathrm{d}\tau^{k+1}} H_1(\tau) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}^k}{\mathrm{d}\tau^k} H_1(\tau)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(-\int_\tau^{i\infty} h(z)\,\mathrm{d}z\right) \\
&= h(\tau).
\end{aligned}$$

Hence H_1 is a $k+1$-fold integral of h.

The transformation property of H_1 follows the usual proof. For each $\gamma \in \Gamma$ and $\tau \in \mathbb{H} \cup \mathbb{R} \cup \{i\infty\}$ we have

$$\begin{aligned}
(-1)^{k+1} k!\, H_1\big|_{-k}\gamma(\tau) &= j(\gamma,\tau)^k \int_{\gamma\tau}^{i\infty} h(z)\,(z-\gamma\tau)^k\,\mathrm{d}z \\
&= j(\gamma,\tau)^k \int_{\tau}^{\gamma^{-1}i\infty} h(\gamma z)\,(\gamma z-\gamma\tau)^k\,\mathrm{d}(\gamma z) \\
&= \int_{\tau}^{\gamma^{-1}i\infty} h(\gamma z)\, j(\gamma,z)^{-k-2}\,(z-\tau)^k\,\mathrm{d}z \\
&= \int_{\tau}^{\gamma^{-1}i\infty} h(z)\,(z-\tau)^k\,\mathrm{d}z \\
&= \int_{\tau}^{i\infty} h(z)\,(z-\tau)^k\,\mathrm{d}z + \int_{\gamma^{-1}i\infty}^{i\infty} h(z)\,(z-\tau)^k\,\mathrm{d}z \\
&= (-1)^{k+1} k!\, H_1(\tau) + \int_{\gamma^{-1}i\infty}^{i\infty} h(z)\,(z-\tau)^k\,\mathrm{d}z,
\end{aligned}$$

using $h\big|_{k+2}\gamma = h$. The integral $\int_{\gamma^{-1}i\infty}^{i\infty} h(z)\,(z-\tau)^k\,\mathrm{d}z$ at the right-hand side is obviously a polynomial in τ of degree at most k. □

Remark 10 The integral $\displaystyle\int_{\gamma^{-1}i\infty}^{i\infty} h(z)\,(z-\tau)^k\,\mathrm{d}z$ appearing in the proof above is called a period polynomial. The theory of period polynomials associated to modular cusp forms is well known, see e.g., [19] and related work.

5.3 Auxiliary lemmas

The first of the following lemmas gives a value for the summation of special functions which will be used in Sect. 5.4. The remaining lemmas deal with aspects of the auxiliary integral and will be used in Sect. 5.5.

Lemma 12 *We have the following identities:*

1. *Assume $\mu_i' + m_l' < 0$. We have*

$$\frac{(2\pi)^{-k}}{c\, i^{2k+1}} \sum_{\substack{m=0 \\ m+m_j\neq 0}}^{\infty} \frac{B^{\star}_{c,\mu_i',m,j,l}\, e^{2\pi i (m+m_j')\left(\tau+\frac{d}{c}\right)}}{(m+m_j')^{k+1}}$$

$$= \frac{(-i)\, i^k}{(2\pi)^{k+1}} \sum_{q\in\mathbb{Z}} \frac{e^{2\pi i q m'_j}\left(c\tau + d - cq\right)^k}{(-\mu'_i - m'_l)^{k+1}} \cdot \sum_{n=k+1}^{\infty} \frac{1}{n!}\left(\frac{2\pi i(-\mu'_i - m'_l)}{c(c\tau + d - cq)}\right)^n. \tag{118}$$

2. *Assume* $\mu_i + m_l > 0$. *We have*

$$\frac{(2\pi)^{-k}}{c\, i^{2k+1}} \sum_{\substack{m=0 \\ m+m_j\neq 0}}^{\infty} \frac{B^{\star}_{c,\mu_i,m,j,l}\, e^{2\pi i(m+m_j)\left(\tau+\frac{d}{c}\right)}}{(m+m_j)^{k+1}} = \frac{i\,(-i)^k}{(2\pi)^{k+1}} \sum_{q\in\mathbb{Z}} \frac{e^{2\pi i q m_j}\left(c\tau + d - cq\right)^k}{(\mu_i + m_l)^{k+1}} \cdot \sum_{n=k+1}^{\infty} \frac{1}{n!}\left(\frac{-2\pi i(\mu_i + m_l)}{c(c\tau + d - cq)}\right)^n. \tag{119}$$

3. *Assume* $-(\mu'_i + m'_l) \overset{(97)}{=} \mu_i + m_l > 0$. *We have*

$$\overline{\frac{(2\pi)^{-k}}{c\, i^{2k+1}} \sum_{\substack{m=0 \\ m+m_j\neq 0}}^{\infty} \frac{B^{\star}_{c,\mu'_i,m,j,l}\, e^{2\pi i(m+m'_j)\left(\overline{\tau}+\frac{d}{c}\right)}}{(m+m'_j)^{k+1}}} = \frac{(2\pi)^{-k}}{c\, i^{2k+1}} \sum_{\substack{m=0 \\ m+m_j\neq 0}}^{\infty} \frac{B^{\star}_{c,\mu_i,m,j,l}\, e^{2\pi i(m+m_j)\left(\tau+\frac{d}{c}\right)}}{(m+m_j)^{k+1}}. \tag{120}$$

Proof The proof of the first identity is a simple application of the Lipschitz summation formula given in (50). Furthermore, we need the definition of $B^{\star}_{c,\nu,m,j,l}$ in (113) and the power series expansion of $I_{k+1}(z)$ in (27) since we consider the cases $\mu'_i + m'_l < 0$. Hence we have

$$\frac{(2\pi)^{-k}}{c\, i^{2k+1}} \sum_{\substack{m=0 \\ m+m'_j\neq 0}}^{\infty} \frac{B^{\star}_{c,\mu'_i,m,j,l}\, e^{2\pi i(m+m'_j)\left(\tau+\frac{d}{c}\right)}}{(m+m'_j)^{k+1}} \overset{(113),(27)}{=} \frac{(2\pi)^{-k}}{c\, i^{2k+1}} \sum_{\substack{m=0 \\ m+m'_j\neq 0}}^{\infty} \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+1} \frac{(-\mu'_i - m'_l)^n\,(m+m'_j)^n}{n!(n+k+1)!} \cdot e^{2\pi i(m+m'_j)(\tau+\frac{d}{c})}$$

$$= \frac{1}{i^{2k+1}(2\pi)^{k+1}} \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+2} \frac{(-\mu_i' - m_l')^n}{(n+k+1)!\,(-2\pi i)^{n+1}}$$

$$\cdot \frac{(-2\pi i)^{n+1}}{n!} \sum_{\substack{m=0 \\ m+m_j' \neq 0}}^{\infty} (m+m_j')^n \, e^{2\pi i (m+m_j')(\tau + \frac{d}{c})}$$

$$\overset{(50)}{=} \frac{1}{i^{2k+1}(2\pi)^{k+1}} \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+2} \frac{(-\mu_i' - m_l')^n}{(n+k+1)!\,(-2\pi i)^{n+1}}$$

$$\cdot \sum_{q \in \mathbb{Z}} \frac{e^{-2\pi i m_j' q}}{\left(\tau + \frac{d}{c} + q\right)^{n+1}}$$

$$= \frac{1}{i^{2k+1}(2\pi)^{k+1}} \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+2} \frac{(-\mu_i' - m_l')^n}{(n+k+1)!}$$

$$\cdot \frac{1}{(-2\pi i)^{n+1}} \sum_{q \in \mathbb{Z}} \frac{e^{-2\pi i m_j' q}}{\left(\tau + \frac{d}{c} + q\right)^{n+1}}$$

$$\overset{n \mapsto n+k+1}{=} \frac{1}{i^{2k+1}(2\pi)^{k+1}} \sum_{n=k+1}^{\infty} \left(\frac{2\pi}{c}\right)^{2n-k} \frac{(-\mu_i' - m_l')^{n-k-1}}{n!}$$

$$\cdot \frac{1}{(-2\pi i)^{n-k}} \sum_{q \in \mathbb{Z}} \frac{e^{-2\pi i m_j' q}}{\left(\tau + \frac{d}{c} + q\right)^{n-k}}$$

$$= \frac{1}{i^{2k+1}(2\pi)^{k+1}} \sum_{q \in \mathbb{Z}} \frac{(-2\pi i)^k \, c^k \, \left(\tau + \frac{d}{c} + q\right)^k}{(2\pi)^k \, (-\mu_i' - m_l')^{k+1}} \, e^{-2\pi i m_j' q}$$

$$\cdot \sum_{n=k+1}^{\infty} \left(\frac{2\pi}{c}\right)^{2n} \frac{(-\mu_i' - m_l')^n}{n!\,(-2\pi i)^n} \frac{1}{\left(\tau + \frac{d}{c} + q\right)^n}$$

$$= \frac{1}{i^{3k+1}(2\pi)^{k+1}} , \sum_{q \in \mathbb{Z}} \frac{(c\tau + d + cq)^k}{(-\mu_i' - m_l')^{k+1}} \, e^{-2\pi i m_j' q}$$

$$\cdot \sum_{n=k+1}^{\infty} \left(\frac{2\pi}{c}\right)^{2n} \frac{(-\mu_i' - m_l')^n}{n!\,(-2\pi i)^n} \frac{1}{\left(\tau + \frac{d}{c} + q\right)^n}$$

$$= \frac{1}{i^{3k+1}(2\pi)^{k+1}} , \sum_{q \in \mathbb{Z}} \frac{(c\tau + d + cq)^k}{(-\mu_i' - m_l')^{k+1}} \, e^{-2\pi i m_j' q}$$

$$\cdot \sum_{n=k+1}^{\infty} \left(\frac{2\pi i}{c}\right)^{n} \frac{(-\mu_i' - m_l')^n}{n!} \frac{1}{(c\tau + d + cq)^n}$$

$$= \frac{(-i)\, i^k}{(2\pi)^{k+1}} \sum_{q\in\mathbb{Z}} \frac{e^{2\pi i q m'_j} \big(c\tau + d - cq\big)^k}{(-\mu'_i - m'_l)^{k+1}} \cdot \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\mu'_i - m'_l)}{c(c\tau + d - cq)}\right)^n .$$

The proof of the second identity is analogous to the one above; we just need to add an factor $(-1)^n$ coming from the power series expansion of $J_{k+1}(z)$ in (114). The factor $(-\mu'_i - m'_l)$ appearing above is just $\mu_i + m_l$ by (97). Inserting the factor $(-1)^n$ gives us

$$\begin{aligned}
&\frac{(2\pi)^{-k}}{c\, i^{2k+1}} \sum_{\substack{m=0 \\ m+m_j\neq 0}}^{\infty} \frac{B^{\star}_{c,\mu_i,m,j,l}\, e^{2\pi i(m+m_j)\left(\tau+\frac{d}{c}\right)}}{(m+m_j)^{k+1}} \\
&\overset{(113),(114)}{=} \frac{(2\pi)^{-k}}{c\, i^{2k+1}} \sum_{\substack{m=0 \\ m+m_j\neq 0}}^{\infty} \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+1} \\
&\qquad \cdot \frac{(-1)^n (\mu_i + m_l)^n (m+m_j)^n}{n!(n+k+1)!}\, e^{2\pi i(m+m'_j)(\tau+\frac{d}{c})} \\
&= \frac{(-1)^k}{i^{3k+1}(2\pi)^{k+1}} \sum_{q\in\mathbb{Z}} \frac{(c\tau + d + cq)^k}{(\mu_i + m_l)^{k+1}}\, e^{-2\pi i m_j q} \\
&\qquad \cdot \sum_{n=k+1}^{\infty} \left(\frac{2\pi i}{c}\right)^n \frac{(\mu_i + m_l)^n}{n!} \frac{(-1)^n}{(c\tau + d + cq)^n} \\
&= \frac{(-1)^k\, (-i)\, i^k}{(2\pi)^{k+1}} \sum_{q\in\mathbb{Z}} \frac{e^{2\pi i q m_j}\big(c\tau + d - cq\big)^k}{(\mu_i + m_l)^{k+1}} \\
&\qquad \cdot \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{-2\pi i(\mu_i + m_l)}{c(c\tau + d - cq)}\right)^n .
\end{aligned}$$

Using $(-1)^k\, (-i)\, i^k = i\, (-i)^k$ we have shown the Identity (119).

The last part of the lemma follows immediately, since the two expressions on the right- hand sides of Eqs. (118) and (119) are complex conjugate to each other if we replace $\tau \mapsto \overline{\tau}$ in one of them. □

Lemma 13 *For $m + m_j > 0$ and $\tau \in \mathbb{H} \cup \mathbb{R} \cup \{i\infty\}$ we have*

$$\begin{aligned}
&\int_{\tau}^{i\infty} e^{2\pi i(m+m_j)z} (z-\tau)^k \,\mathrm{d}z \\
&\quad = \left(\frac{-1}{2\pi i(m+m_j)}\right)^{k+1} e^{2\pi i(m+m_j)\tau}\, \Gamma(k+1) \qquad (121)
\end{aligned}$$

and

$$\overline{\int_\tau^{i\infty} e^{2\pi i(m+m_j)z}(z-\overline{\tau})^k\,\mathrm{d}z} = \left(\frac{1}{2\pi i(m+m_j)}\right)^{k+1} e^{-2\pi i(m+m_j)\tau}\,\Gamma\big(k+1, 4\pi(m+m_j)\mathrm{Im}\,(\tau)\big), \quad (122)$$

where $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1}e^{-t}\mathrm{d}t$ *and* $\Gamma(\alpha) = \Gamma(\alpha, 0)$ *denote the incomplete and the usual Gamma function, see e.g., [7, (8.2.2) and (5.2.1)].*

Proof The lemma can be proven by a straightforward calculation using the integral representation of the (incomplete) Gamma function and a variable substitution. □

Lemma 14 *Let* $g \in \mathscr{S}(k+2, \varepsilon, \rho)$ *be a cusp form of weight* $k+2 > 2$ *and consider the associated auxiliary integral* G_2 *defined in* (104)*. We put*

$$\widetilde{p}_{2,\gamma}(\overline{\tau}) := \int_{\gamma^{-1}i\infty}^{i\infty} g(z)\,(z-\overline{\tau})^k\,\mathrm{d}z. \quad (123)$$

Then, G_2 *satisfies the transformation property*

$$\big(\rho'(\gamma)\big)^{-1} G_2\big|_{-k}^{\varepsilon'}\gamma(\tau) - G_2(\tau) = \overline{\widetilde{p}_{2,\gamma}(\overline{\tau})}, \quad (124)$$

where the right-hand integral is a polynomial in τ*.*

Proof First, we note that $\widetilde{p}_{2,\gamma}(\overline{\tau})$ defined in (123) is obviously a polynomial in $\overline{\tau}$ of degree at most k, since integration does not depend on $\overline{\tau}$.

The transformation property of G_2 follows the usual proof. For each $\gamma \in \Gamma$ and $\tau \in \mathbb{H} \cup \mathbb{R} \cup \{i\infty\}$ we have

$$\begin{aligned}
&j(\gamma,\tau)^k\,\big(\varepsilon'(\gamma)\big)^{-1}\big(\rho'(\gamma)\big)^{-1} G_2(\gamma\tau)\\
&= j(\gamma,\tau)^k\,\big(\varepsilon(\gamma)\big)^{-1}\big(\rho(\gamma)\big)^{-1}\overline{\int_{\gamma\tau}^{i\infty} g(z)\,(z-\gamma\overline{\tau})^k\,\mathrm{d}z}\\
&\overset{(98)}{=} \overline{j(\gamma,\overline{\tau})^k\,\big(\varepsilon(\gamma)\big)^{-1}\big(\rho(\gamma)\big)^{-1}\int_\tau^{\gamma^{-1}i\infty} g(\gamma z)\,(\gamma z-\gamma\overline{\tau})^k\,\mathrm{d}(\gamma z)}\\
&= \overline{\int_\tau^{\gamma^{-1}i\infty} j(\gamma,z)^{-k-2}\,\big(\varepsilon(\gamma)\big)^{-1}\big(\rho(\gamma)\big)^{-1} g(\gamma z)\,(z-\overline{\tau})^k\,\mathrm{d}z}\\
&= \overline{\int_\tau^{\gamma^{-1}i\infty} g(z)\,(z-\overline{\tau})^k\,\mathrm{d}z}\\
&= \overline{\int_\tau^{i\infty} g(z)\,(z-\overline{\tau})^k\,\mathrm{d}z} + \overline{\int_{\gamma^{-1}i\infty}^{i\infty} h(z)\,(z-\overline{\tau})^k\,\mathrm{d}z}
\end{aligned}$$

$$= G_2(\tau) + \overline{\int_{\gamma^{-1} i\infty}^{i\infty} g(z)\,(z-\overline{\tau})^k\,\mathrm{d}z}.$$

using $g\big|_{k+2}^{\varepsilon}\gamma = \rho(\gamma)\,g$.

The integral $\overline{\int_{\gamma^{-1} i\infty}^{i\infty} g(z)\,(z-\overline{\tau})^k\,\mathrm{d}z}$ is a period polynomial in τ. □

Using the simple change of variable $\overline{\tau} \mapsto \tau$ in the defining integrals in (125) we obtain directly the following:

Lemma 15 *In the setting of Theorem 3 and Theorem 4 let $g \in \mathscr{S}(k+2, \varepsilon, \rho)$ be a cusp form. Put*

$$\widetilde{p}_{1,\gamma}(\tau) := \int_{\gamma^{-1} i\infty}^{i\infty} g(z)\,(z-\tau)^k\,\mathrm{d}z \quad \textit{and}$$

$$\widetilde{p}_{2,\gamma}(\overline{\tau}) := \int_{\gamma^{-1} i\infty}^{i\infty} g(z)\,(z-\overline{\tau})^k\,\mathrm{d}z. \tag{125}$$

Then $\widetilde{p}_{1,\gamma}$ and $\widetilde{p}_{2,\gamma}$ are p-dimensional column vectors of polynomials of degree at most k and satisfy

$$\widetilde{p}_{1,\gamma}(\tau) = \widetilde{p}_{2,\gamma}(\tau) \tag{126}$$

for all $\tau \in \mathbb{H}$.

5.4 Proof of Theorem 3

First, we discuss the Fourier expansions of g and $\widehat{g}$. By (95) and using (110) we have

$$g(\tau) = \sum_{i=1}^{s} b_i\, P(\rho, k+2, \varepsilon, \mu_i, \Gamma, r_i; \tau)$$
$$\overset{(110)}{=} \sum_{i=1}^{s} b_i\, \mathscr{T}^{\star}_{\mu_i}(\tau)\, e_{r_i} \tag{127}$$

for all $\tau \in \mathbb{H}$. The assumption that $\mu_i + m_j > 0$ may be made because $\mu_i + m_j = 0$ yields an Eisenstein series, and Eisenstein series are orthogonal to cusp forms.

Similarly, we have

$$\begin{aligned}\widehat{g}(\tau) &:= \sum_{i=1}^{s} \overline{b_i}\; P(\rho', k+2, \varepsilon', \mu_i', \Gamma, r_i; \tau) \\ &\overset{(110)}{=} \sum_{i=1}^{s} \overline{b_i}\; \mathscr{T}^{\star}_{\mu_i'}(\tau)\, e_{r_i} \end{aligned} \tag{128}$$

for all $\tau \in \mathbb{H}$. This function is holomorphic on $\mathbb{H}$ but grows exponentially in some cusps since $m_j' + \mu' \overset{(97)}{=} -(m_j + \mu) < 0$ by assumption.

Next, consider the $(k+1)$-fold integral $\widehat{G}(\tau)$ of $\widehat{g}(\tau)$ defined by integrating term-by-term in the expansion at $i\infty$: Using the Fourier expansion of $\widehat{g}$ in (128) together with (111) and (112) we find that the entries of the vector-valued Fourier expansion of $\widehat{g}$ in (128) are linear combinations of terms of the form

$$\begin{aligned}&\delta_{j,l}\, e^{2\pi i (m_j' + \mu_i')\tau} + \mathscr{R}^{\star}_{\mu_i'}{}^{(j,l)}(\tau) \\ &\overset{(112)}{=} \delta_{j,l}\, e^{2\pi i (m_j' + \mu_i')\tau} + \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^{-k}}{c} A_{c,\mu_i',m,j,l}\, B^{\star}_{c,\mu_i',m,j,l}\, e^{2\pi i (m + m_j')\tau}.\end{aligned}$$

Hence, the $(k+1)$-fold integral $\widehat{G}(\tau)$ of the function $\widehat{g}$ can be written as

$$\widehat{G}(\tau) = \sum_{i=1}^{s} \overline{b_i}\; \mathbf{T}^{\star}_{\mu_i'}(\tau)\, e_{r_i}, \tag{129}$$

where the entries of $\mathbf{T}^{\star}_{\mu_i'}(\tau)$ are given by the $k+1$ times termwise integrated entries of $\mathscr{T}^{\star}_{\mu_i'}(\tau)$ in (111) and (112):

$$\begin{aligned}\mathbf{T}^{\star}_{\mu_i'}{}^{(j,l)}(\tau) &:= \delta_{j,l}\, \frac{e^{2\pi i (m_j' + \mu_i')\tau}}{\big(2\pi i (m_j' + \mu_j')\big)^{k+1}} \\ &\quad + \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^{-k}}{c} A_{c,\mu_i',m,j,l}\, B^{\star}_{c,\mu_i',m,j,l}\, \frac{e^{2\pi i (m + m_j')\tau}}{\big(2\pi i (m + m_j')\big)^{k+1}} \\ &=: \delta_{j,l}\, \frac{e^{2\pi i (m_j' + \mu_i')\tau}}{\big(2\pi i (m_j' + \mu_j')\big)^{k+1}} + \mathbf{R}^{\star}_{\mu_i'}{}^{(j,l)}(\tau)\end{aligned} \tag{130}$$

defined by integrating $k+1$ times term-by-term in the expansion at $i\infty$. It is normalized such that we have

$$\widehat{G}\big(T\tau\big) = \varepsilon'(T)\, \rho'(T)\, \widehat{G}(\tau) \overset{(9)}{=} \begin{pmatrix} e^{2\pi i m_1'} & & \\ & \ddots & \\ & & e^{2\pi i m_p'} \end{pmatrix} \widehat{G}(\tau).$$

(This can be seen easily by inserting $\tau \mapsto \tau + 1$ in (130) or alternatively by taking the 'primed' version of $\varepsilon(T)\rho(T)$ in (9).) Lemma 10 implies the transformation property (102).

In order to establish the third statement of Theorem 3 later on, we need to discuss the Fourier expansion of $\widehat{G}$ in some more detail. Recall $\mathbf{R}^{\star}_{\mu'}{}^{(j,l)}_{i}(\tau)$ in (129) and (130). After changing the order of the summation as we did in (56), we have

$$\begin{aligned}
\mathbf{R}^{\star}_{\mu'_i}{}^{(j,l)} &\overset{(130)}{=} \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^{-k}}{c} A_{c,\mu'_i,m,j,l}\, B^{\star}_{c,\mu'_i,m,j,l}\, \frac{e^{2\pi i(m+m'_j)\tau}}{\left(2\pi i(m+m'_j)\right)^{k+1}} \\
&\overset{(23)}{=} \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^{-k}}{c} \sum_{\substack{d \\ 0\le -d<c \\ \gcd(c,d)=1}} \left(\varepsilon'(\gamma_{c,d})\right)^{-1} \left(\rho'(\gamma_{c,d})\right)^{-1}_{j,l} \cdot \\
&\qquad \cdot e^{2\pi i \frac{m'_l a + m'_j d}{c}}\, e^{2\pi i \frac{dm + a\mu'_i}{c}}\, B^{\star}_{c,\mu'_i,m,j,l}\, \frac{e^{2\pi i(m+m'_j)\tau}}{\left(2\pi i(m+m'_j)\right)^{k+1}} \\
&= \frac{(2\pi)^{-k}}{i^{2k+1}} \sum_{c=1}^{\infty} \frac{1}{c} \sum_{\substack{d \\ 0\le -d<c \\ \gcd(c,d)=1}} \left(\varepsilon'(\gamma_{c,d})\right)^{-1} \left(\rho'_{j,l}(\gamma_{c,d})\right)^{-1} e^{2\pi i a \frac{m'_l + \mu'_i}{c}} \cdot \\
&\qquad \cdot \sum_{m=0}^{\infty} \frac{B^{\star}_{c,\mu'_i,m,j,l}\, e^{2\pi i(m+m'_j)\left(\tau + \frac{d}{c}\right)}}{(m+m'_j)^{k+1}}. \qquad (131)
\end{aligned}$$

Remark 11 We would like to point out that the above Fourier expansion does not contain the constant term, i.e., the term with $m + m'_j = 0$. The assumption in (95) is that g is a cusp form and hence vanishes in each cusp. This implies that the Fourier coefficient for $m + m'_j = 0$ vanishes. Hence we may write the summation over the index m in (131) above as

$$\sum_{m=0}^{\infty} \frac{B^{\star}_{c,\mu'_i,m,j,l}\, e^{2\pi i(m+m'_j)\left(\tau+\frac{d}{c}\right)}}{(m+m'_j)^{k+1}} = \sum_{\substack{m=0 \\ m+m'_j \neq 0}}^{\infty} \frac{B^{\star}_{c,\mu'_i,m,j,l}\, e^{2\pi i(m+m'_j)\left(\tau+\frac{d}{c}\right)}}{(m+m'_j)^{k+1}}.$$

Now, applying Lemma 12, we continue our discussion of $\mathbf{R}^{\star}_{\mu'_i}{}^{(j,l)}$. We have

$$\mathbf{R}^{\star}_{\mu'_i}{}^{(j,l)} \overset{(131)}{=} \sum_{c=1}^{\infty} \sum_{\substack{d \\ 0\le -d<c \\ \gcd(c,d)=1}} \left(\varepsilon'(\gamma_{c,d})\right)^{-1} \left(\rho'(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m'_l + \mu'_i}{c}}$$

$$
\begin{aligned}
&\cdot \frac{(2\pi)^{-k}}{c\, i^{2k+1}} \sum_{\substack{m=0 \\ m+m'_j \neq 0}}^{\infty} \frac{B^{\star}_{c,\mu'_i,m,j,l}\, e^{2\pi i (m+m'_j)\left(\tau+\frac{d}{c}\right)}}{(m+m'_j)^{k+1}} \\
&\overset{(118)}{=} \sum_{c=1}^{\infty} \sum_{\substack{d \\ 0\leq -d<c \\ \gcd(c,d)=1}} \left(\varepsilon'(\gamma_{c,d})\right)^{-1} \left(\rho'(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m'_l+\mu'_i}{c}} \\
&\quad \frac{-i\, i^k}{(2\pi)^{k+1}} \sum_{q\in\mathbb{Z}} \frac{e^{2\pi i q m'_j} \left(c\tau+d-cq\right)^k}{(-\mu'_i-m'_l)^{k+1}} \\
&\qquad \cdot \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\mu'-m'_l)}{c(c\tau+d-cq)}\right)^n \\
&= \frac{-i\, i^k}{(2\pi)^{k+1}\,(-\mu'_i-m'_l)^{k+1}} \\
&\qquad \cdot \Bigg[\sum_{c=1}^{\infty} \sum_{\substack{d \\ 0\leq -d<c \\ \gcd(c,d)=1}} \left(\varepsilon'(\gamma_{c,d})\right)^{-1} \left(\rho'(\gamma_{c,d})\right)^{-1}_{j,l} e^{2\pi i a \frac{m'_l+\mu'_i}{c}} \\
&\qquad\quad \cdot \sum_{q\in\mathbb{Z}} e^{2\pi i q m'_j} \left(c\tau+d-cq\right)^k \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\mu'-m'_l)}{c(c\tau+d-cq)}\right)^n \Bigg].
\end{aligned}
\tag{132}
$$

Comparing the right-hand side in (132) to the right-hand side in (57) in Sect. 4.5 we see that the term $\mathscr{R}^{(j,l)}_{\mu'_i}(\tau)$ in (57) and $\mathbf{R}^{\star\ (j,l)}_{\mu'_i}(\tau)$ in (132) above only differ by the multiplicative factor $\frac{-i\, i^k}{(2\pi)^{k+1}\,(-\mu'-m'_l)^{k+1}}$. The zeroth coefficient ($\mathscr{K}^{(j,l)}_{\mu'_i}$ in (57)) vanishes in our setting, see Remark 11. Hence, we may continue the exposition as in Sect. 4.5, from (57) onward, and show that $\widehat{G}$ defines a vector-valued polynomial $\widehat{p}_\gamma$ by

$$
\begin{aligned}
&\widehat{G} - \left(\rho'(\gamma)\right)^{-1} \widehat{G}\big|^{\varepsilon'}_{-k}\gamma =: \widehat{p}_\gamma \\
\iff &\widehat{G}(\tau) - \left(\varepsilon'(\gamma)\right)^{-1} j(\gamma,\tau)\big)^k \left(\rho'(\gamma)\right)^{-1} \widehat{G}(\gamma\tau) =: \widehat{p}_\gamma(\tau).
\end{aligned}
\tag{102}
$$

Next, let G be the $(k+1)$-fold integral of g in (95), defined by integrating term-by-term in the expansion at $i\infty$ as we did before with $\widehat{g}(\tau)$: We have

$$
G(\tau) = \sum_{i=1}^{s} b_i\, \mathbf{T}^{\star}_{\mu_i}(\tau)\, e_{r_i}, \tag{133}
$$

where the entries of $\mathbf{T}^{\star}_{\mu_i}(\tau)$ are given by the $k+1$ times termwise integrated entries of $\mathscr{T}^{\star}_{\mu_i}(\tau)$ in (111) and (112):

$$\begin{aligned}\mathbf{T}_{\mu_i}^{\star\ (j,l)}(\tau) :&= \delta_{j,l}\,\frac{e^{2\pi i(m_j+\mu_i)\tau}}{\left(2\pi i(m_j+\mu_j)\right)^{k+1}}\\ &\quad+\sum_{m=0}^{\infty}2\pi\sum_{c=1}^{\infty}\frac{i^{-k}}{c}\,A_{c,\mu_i,m,j,l}\,B^{\star}_{c,\mu_i,m,j,l}\,\frac{e^{2\pi i(m+m_j)\tau}}{\left(2\pi i(m+m_j)\right)^{k+1}}\\ &=:\delta_{j,l}\,\frac{e^{2\pi i(m_j+\mu_i)\tau}}{\left(2\pi i(m_j+\mu_j)\right)^{k+1}}+\mathbf{R}_{\mu_i}^{\star\ (j,l)}(\tau).\end{aligned}\tag{134}$$

Similar to the arguments around (131), we get

$$\begin{aligned}\mathbf{R}_{\mu_i}^{\star\ (j,l)} &\overset{(134)}{=}\sum_{m=0}^{\infty}2\pi\sum_{c=1}^{\infty}\frac{i^{-k}}{c}\,A_{c,\mu_i,m,j,l}\,B^{\star}_{c,\mu_i,m,j,l}\,\frac{e^{2\pi i(m+m_j)\tau}}{\left(2\pi i(m+m_j)\right)^{k+1}}\\ &=\frac{(2\pi)^{-k}}{i^{2k+1}}\sum_{c=1}^{\infty}\frac{1}{c}\sum_{\substack{d\\ 0\le -d<c\\ \gcd(c,d)=1}}\left(\varepsilon(\gamma_{c,d})\right)^{-1}\left(\rho(\gamma_{c,d})\right)^{-1}_{j,l}\,e^{2\pi i a\frac{m_l+\mu_i}{c}}\cdot\\ &\qquad\cdot\sum_{m=0}^{\infty}\frac{B^{\star}_{c,\mu_i,m,j,l}\,e^{2\pi i(m+m_j)\left(\tau+\frac{d}{c}\right)}}{(m+m_j)^{k+1}}.\end{aligned}\tag{135}$$

The same argument as in Remark 11 allows us to drop the constant term $m+m_j=0$ in the above expansion. We continue by applying Lemma 12:

$$\begin{aligned}\mathbf{R}_{\mu_i}^{\star\ (j,l)} &\overset{(119)}{=}\sum_{c=1}^{\infty}\sum_{\substack{d\\ 0\le -d<c\\ \gcd(c,d)=1}}\left(\varepsilon(\gamma_{c,d})\right)^{-1}\left(\rho(\gamma_{c,d})\right)^{-1}_{j,l}\,e^{2\pi i a\frac{m_l+\mu_i}{c}}\\ &\qquad\frac{i\,(-i)^k}{(2\pi)^{k+1}}\sum_{q\in\mathbb{Z}}\frac{e^{2\pi i q m_j}\left(c\tau+d-cq\right)^k}{(\mu_i+m_l)^{k+1}}\\ &\qquad\cdot\sum_{n=k+1}^{\infty}\frac{1}{n!}\left(\frac{-2\pi i(\mu_i+m_l)}{c(c\tau+d-cq)}\right)^n\\ &=\frac{i\,(-i)^k}{(2\pi)^{k+1}\,(\mu_i+m_l)^{k+1}}\\ &\qquad\cdot\Bigg[\sum_{c=1}^{\infty}\sum_{\substack{d\\ 0\le -d<c\\ \gcd(c,d)=1}}\left(\varepsilon(\gamma_{c,d})\right)^{-1}\left(\rho(\gamma_{c,d})\right)^{-1}_{j,l}\,e^{-2\pi i a\frac{m_l+\mu_i}{c}}\\ &\qquad\cdot\sum_{q\in\mathbb{Z}}e^{2\pi i q m_j}\left(c\tau+d-cq\right)^k\sum_{n=k+1}^{\infty}\frac{1}{n!}\left(\frac{-2\pi i(\mu_i-m_l)}{c(c\tau+d-cq)}\right)^n\Bigg].\end{aligned}\tag{136}$$

We may follow the arguments in Sect. 4.5, from (57) onward, and show that G defines a vector-valued polynomial p_γ by

$$\begin{aligned} & G - \big(\rho(\gamma)\big)^{-1} G\big|_{-k}^{\varepsilon}\gamma =: p_\gamma \\ \Longleftrightarrow\ & G(\tau) - \big(\varepsilon(\gamma)\big)^{-1} j(\gamma,\tau)\big)^k \big(\rho(\gamma)\big)^{-1} G(\gamma\tau) =: p_\gamma(\tau) \end{aligned} \tag{100}$$

as stated in the first part of Theorem 3.

Now, consider the *Eichler integral* G_1 of g given by

$$G_1(\tau) := \frac{(-1)^{k+1}}{k!} \int_\tau^{i\infty} g(z)\,(z-\tau)^k \,\mathrm{d}z \qquad (\tau \in \mathbb{H}). \tag{137}$$

Lemma 11 implies that G_1 satisfies the same transformation property (100) as G does. We plug in the Fourier expansion of g given by the expansions of the Poincaré series in (110)–(112) and exchange integral and summation. We find that we have to calculate

$$\int_\tau^{i\infty} e^{2\pi i(m+m_j)z}(z-\tau)^k \,\mathrm{d}z = \left(\frac{-1}{2\pi i(m+m_j)}\right)^{k+1} e^{2\pi i(m+m_j)\tau}\ \Gamma(k+1)$$

for the typical Fourier term of order $m + m_j > 0$ using a simple change of variables and the integral definition of the Gamma function. (We also collected this formula in (121) of Lemma 13.) We observe the Fourier expansions of G in (133)–(134) and of G_1 which is essentially given in the equation above. The Gamma factor appearing in (121) is absorbed in the factor $\frac{(-1)^{k+1}}{k!}$ appearing in the definition of G_1 in (137). We have

$$G(\tau) = G_1(\tau) \tag{101}$$

for all $\tau \in \mathbb{H}$. This concludes the first part of Theorem 3.

We continue with the proof of the third part in Theorem 3: We want to show the relation between the vector-valued polynomials p_γ and $\widehat{p}_\gamma$.

Lemma 12, in particular (120), implies that the right- hand sides in (131) and (136) are conjugated if we also conjugate τ. Therefore, we have

$$\widehat{p}_\gamma(\tau) = \overline{p_\gamma(\overline{\tau})} \tag{138}$$

for all $\gamma \in \Gamma$ and $\tau \in \mathbb{H}$.

This completes the proof of Theorem 3.

5.5 Proof of Theorem 4

To show the first part of Theorem 4 consider the auxiliary integral G_2 given in (104). Lemma 14 shows that G_2 has the required transformation property (105):

$$G_2(\tau) - \big(\rho'(\gamma)\big)^{-1} G_2\big|_{-k}^{\varepsilon'}\gamma(\tau) \overset{(124)}{=} -\overline{\widetilde{p}_{2,\gamma}(\overline{\tau})} =: p_{2,\gamma}(\tau).$$

Also, Lemma 14 shows that $\widetilde{p}_{2,\gamma}(\tau)$ is a vector-valued polynomial in τ of degree at most k. Hence $p_{2,\gamma}(\tau) := -\overline{\widetilde{p}_{2,\gamma}(\overline{\tau})}$ is a vector-valued polynomial in τ of degree at most k.

To show that G_2 has an expansion of the form given in (107) of Theorem 4, we plug in the Fourier expansion of g given by the expansions of the Poincaré series in (110)–(112). Hence the typical summand in the Fourier expansion is of the form

$$C_{m,j}\, e^{2\pi i(m+m_j)\tau} \qquad (\tau \in \mathbb{H})$$

with $C_{m,j}$ not depending on τ and $m + m_j > 0$. Also the leading term $e^{2\pi i(\mu_i+m_j)\tau}$ of the Poincaré series expansion is of the above form. (This follows from g being a cusp form which implies $\mu_i + m_j > 0$, see the discussion in Sect. 5.1.) After an exchange of integration and summation we find that we have to calculate the integral

$$\overline{\int_\tau^{i\infty} e^{2\pi i(m+m_j)z}(z-\overline{\tau})^k \,\mathrm{d}z} \qquad (\tau \in \mathbb{H}).$$

Applying Lemma 13 we get

$$\begin{aligned}&\overline{\int_\tau^{i\infty} e^{2\pi i(m+m_j)z}(z-\overline{\tau})^k \,\mathrm{d}z}\\ &\overset{(122)}{=} \left(\frac{1}{2\pi i(m+m_j)}\right)^{k+1} e^{-2\pi i(m+m_j)\tau}\ \Gamma\big(k+1, 4\pi(m+m_j)\mathrm{Im}\,(\tau)\big).\end{aligned}$$

Hence the components of G_2 have an expansion of the form stated in (107).

To show relation (106) we compare G_2 to the Eichler integral G_1 of g in (137). Following the same arguments as in the proof of Lemma 11 we have

$$G_1(\tau) - \big(\rho(\gamma)\big)^{-1} G_1\big|_{-k}^{\varepsilon}\gamma(\tau) = -\int_{\gamma^{-1}i\infty}^{i\infty} g(z)\,(z-\tau)^k \,\mathrm{d}z.$$

Using (101) and (100) we find

$$-\int_{\gamma^{-1}i\infty}^{i\infty} g(z)\,(z-\tau)^k \,\mathrm{d}z = G_1(\tau) - \big(\rho(\gamma)\big)^{-1} G_1\big|_{-k}^{\varepsilon}\gamma(\tau)$$

$$\overset{(101)}{=} \frac{\Gamma(k+1)}{(-1)^{k+1}} \left(G(\tau) - \big(\rho(\gamma)\big)^{-1} G\big|_{-k}^{\varepsilon} \gamma(\tau) \right)$$
$$\overset{(100)}{=} \frac{\Gamma(k+1)}{(-1)^{k+1}} \, p_\gamma(\tau).$$

The left-hand side of the above equation is denoted by $-\widetilde{p}_{1,\gamma}(\tau)$ in Lemma 15. It also shows that the left-hand side above is equal to vector-valued polynomial $-\widetilde{p}_{2,\gamma}(\tau)$ given by (125). Hence, we have

$$\frac{\Gamma(k+1)}{(-1)^{k+1}} \, p_\gamma(\tau) = - \int_{\gamma^{-1} i\infty}^{i\infty} g(z) \, (z-\tau)^k \, \mathrm{d}z = -\widetilde{p}_{2,\gamma}(\tau).$$

On the other hand, we have

$$-\overline{\widetilde{p}_{2,\gamma}(\overline{\tau})} = p_{2,\gamma}(\tau)$$

using (124) in Lemma 14 and the definition of $p_{2,\gamma}$ in (105). Combining the above two equations we get

$$\overline{p_{2,\gamma}(\overline{\tau})} = \frac{\Gamma(k+1)}{(-1)^{k+1}} \, p_\gamma(\tau).$$

This shows the identity (106) of Theorem 4 and concludes the proof.

6 Vector-valued harmonic weak Maass forms

In this section, we compare the modular integrals G and $\widehat{G}$ that we got along with the supplementary function in Sect. 5 to vector-valued weak harmonic Maass forms and their vector-valued mock modular functions, as introduced in e.g., [23], [3], and [5].

Let $k \in \mathbb{Z}$ be an integer and denote again $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. The *weight k hyperbolic Laplacian* is defined by the differential

$$\begin{aligned} \Delta_k &:= -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \\ &= R_{k-2} \, L_k, \end{aligned} \tag{139}$$

where R_k and L_k are the differential operators

$$R_k = 2i \frac{\partial}{\partial z} + \frac{k}{y} \qquad \text{and} \qquad L_k = -2iy^2 \frac{\partial}{\partial \overline{z}} \tag{140}$$

with $\tau = x + iy \in \mathbb{H}$ and $x = \mathrm{Re}\,(\tau)$ the real part and $y = \mathrm{Im}\,(\tau)$ the imaginary part of τ. They are called the *Maass raising* and *lowering operators* of nonholomorphic modular forms of weight k.

Definition 7 Let $F = \left(F_1, \ldots, F_p\right)^{\mathrm{tr}}$ be a p-tuple of smooth (in x and y) functions $F_i : \mathbb{H} \to \mathbb{C}$ and $\rho : \Gamma \to \mathrm{GL}(p, \mathbb{C})$ a p-dimensional complex representation. (F, ρ), or simply F, is a *vector-valued harmonic weak Maass form of weight k, representation ρ and multiplier ε on Γ* if F satisfies

1. For all $\gamma \in \Gamma$ we have
$$\left(F_1, \ldots, F_p\right)^{\mathrm{tr}}\big|_k^{\varepsilon}\gamma = \rho(\gamma)\left(F_1, \ldots, F_p\right)^{\mathrm{tr}}.$$
2. $\Delta_k F_j = 0$ for all $j \in \{1, \ldots, p\}$.
3. For every $j \in \{1, \ldots, p\}$ there exists a polynomial
$$P_{F_j}(q) = \sum_{n \le 0} c^+_{F_j}(n)\, q^n \in \mathbb{C}[q]$$
such that
$$F_j(\tau) - P_{F_j}(e^{2\pi i \tau}) = \mathscr{O}\left(e^{-\epsilon y}\right) \qquad \text{as} y \to \infty$$
holds for all $\epsilon > 0$.

The space of vector-valued harmonic weak Maass forms is denoted by $\mathscr{H}(k, \varepsilon, \rho)$. The polynomials P_{F_j} are called the *principal parts of F_j at infinity*.

Remark 12 It is worth mentioning that vector-valued modular forms of negative weights with poles supported at the cusps are vector-valued harmonic weak Maass forms. This implies in particular that the functions F discussed in Theorem 1 are in $\mathscr{H}(-k, \varepsilon, \rho)$.

Remark 13 Recall that we saw the formally defined function $\widehat{G} - \overline{G(\cdot)}$ in Remark 7, which satisfies the first two properties of harmonic weak Maass form in Definition 7. (It would fail the necessary grow condition which can be easily seen by its formal Fourier expansion.)

The above remark motivates the following question: Can we find a harmonic weak Maass form composed of a difference between $\widehat{G}$ and another function? We answer this question positively below.

Similar to [3], we see easily that a vector-valued harmonic weak Maass form $F(z) = \left(F_1(z), \ldots, F_p(z)\right)^{\mathrm{tr}}$ of weight $2-k$ is that it admits a Fourier expansion of the form

$$F_j(z) = q^{m_j} \sum_{n \gg -\infty} c^+_{F_j}(n)\, q^n \; + \sum_{n<0} c^-_{F_j}(n)\, \Gamma\left(k-1, 4\pi|n|y\right) q^n, \tag{141}$$

where $\Gamma(\alpha, x) = \displaystyle\int_x^\infty t^{\alpha-1} e^{-t} \mathrm{d}t$ denotes the incomplete Gamma function [7, (8.2.2)], $z = x + iy \in \mathbb{H}$, with real $x, y > 0$ and $q = e^{2\pi i z}$. We see similar to the situation in Sect. 1 that the m_j's above are rational numbers in $[0, 1)$; they come from the translation property $F(Tz) = \varepsilon(T)\, \rho(T)\, F(z)$ and the normalization (9).

The Fourier expansion of $F_j(z)$ in (141) obviously decomposes into the two summands, an holomorphic $F_j^+(z)$ part given by

$$F_j^+(z) := q^{m_j} \sum_{n \gg -\infty} c_{F_j}^+(n)\, q^n, \tag{142}$$

and the non-holomorphic part

$$F_j^-(z) := q^{m_j} \sum_{n<0} c_{F_j}^-(n)\, \Gamma\big(k-1, 4\pi|n|y\big)\, q^n. \tag{143}$$

This leads to a decomposition of the vector-valued harmonic weak Maass form $F(z)$ into

$$F(z) = F^+(z) + F^-(z) \tag{144}$$

with $F^{\pm}(z) = \big(F_1^{\pm}(z), \ldots, F_p^{\pm}(z)\big)^{\mathrm{tr}}$, respectively.

We define the anti-linear differential operator

$$\xi := \xi_k := 2iy^k \overline{\frac{\partial}{\partial \overline{z}}}. \tag{145}$$

As in the one-dimensional case [3, Proposition 3.2], we see that the operator maps a weight $2-k$ vector-valued harmonic weak Maass form into a vector-valued cusp form of weight k. It is in fact that

$$\xi_{2-k} : \mathscr{H}(2-k, \varepsilon, \rho) \to \mathscr{S}(k, \varepsilon, \rho).$$

Notice that

$$\Big(\xi_{2-k}(F_j)\Big)(z) = -(4\pi)^{k-1} \sum_{n>0} \overline{c_{F_j}^-(-n)} n^{k-1} q^n, \tag{146}$$

similar to classical case in [5] or alternatively in [1, Theorem 1.3, Item 2] and in particular the function $\mathscr{P}$ there and its expansion just above the theorem.

Definition 8 For $k > 0$ let $F = (F_1, \ldots, F_p)^{\mathrm{tr}} \in \mathscr{H}(2-k, \varepsilon, \rho)$ be a vector-valued harmonic weak Maass form. Any cusp form g satisfying $\xi_{2-k} F = g$ is called a *shadow* of F^+.

Theorem 5 *Let $g \in \mathscr{S}(k+2, \varepsilon, \rho)$ be a cusp form of weight $k+2$, and $k > 2\delta$ a positive integer with δ given in (15), multiplier ε, and complex p-dimensional normal representation ρ and consider the associated functions $\widehat{G}$ given in Theorem 3 and G_2 given in Theorem 4.*

The vector-valued function $F := \widehat{G} - \frac{(-1)^{k+1}}{\Gamma(k+1)} G_2$ satisfies

1. *$F \in \mathscr{H}(-k, \varepsilon', \rho')$ is a vector-valued harmonic weak Maass form.*
2. *$F^+ = \widehat{G}$ and $F^- = -\frac{(-1)^{k+1}}{\Gamma(k+1)} G_2$.*

3. $-\frac{(-1)^{k+1}}{\Gamma(k+1)}g$ *is, up to a scalar multiple, a shadow of* F.

Proof Put $F^+ = \widehat{G}$ and $F^- = -\frac{(-1)^{k+1}}{\Gamma(k+1)}G_2$. Theorem 3 shows that $F^+ = \widehat{G}$ is holomorphic on $\mathbb{H}$, and we see in the proof of the theorem that $F^+ = \widehat{G}$ satisfies by construction the required growth condition, i.e., it satisfies the required expansion of F^+ in (142). Theorem 4 shows that $F^- = -\frac{(-1)^{k+1}}{\Gamma(k+1)}G_2$ has the required expansion of F^- in (143). Hence $F = F^+ + F^- = \widehat{G} - \frac{(-1)^{k+1}}{\Gamma(k+1)}G_2$ satisfies the second and third conditions of Definition 7.

The transformation behavior of F follows from (102) and (105), together with the fact that the correcting polynomials in those equations cancel, see (103) and (106).

Since $\xi_{-k}F^- = \xi_{-k}\left(-\frac{(-1)^{k+1}}{\Gamma(k+1)}G_2\right)$ is a scalar multiple of the cusp form g by construction of G_2, we see immediately that g is a shadow of F. □

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